A SYSTEM OF RETIREMENT FREQUENCIES FOR DEPRECIABLE ASSETS

Thomas A. Oates* and Milton H. Spencer*

THE purpose of this paper is to outline a general method for deriving the survival coefficients (or their counterparts, the retirement coefficients) for depreciable physical property. By a *survival coefficient* is meant the percentage of original units of a good acquired in a given period, that may be expected to survive to a given subsequent period. Of course, the difference between a survival coefficient and 100% is the *mortality coefficient*, or the percentage of original units that may be expected to "die" in a given period.

The application of the survival or mortality coefficients (the terms may be used interchangeably since they are viewed as complementary) will result in a table describing the anticipated retirement schedule of a given physical good. This type of information can obviously be of considerable value in constructing depreciation schedules, capital budgets, and similar programs that are of interest to accountants.

In general, the statistical theory underlying the development of the coefficients consists of deriving, from the truncated normal distribution, a family of probability curves which describes the retirement frequency of physical property. Two parameters are needed to select the unique curve within the family which describes the retirement characteristics of the asset in question. These parameters are based on the *average service life* and the *mortality distribution variance*. Unfortunately, the latter parameter is not easily adaptable to practical situations, for most analysts are probably not familiar with the meaning of the term, nor are they likely to have access to the statistic.

In view of this, the parameter may be expressed instead as a functional relationship between the *average service life* and the *maximum service life* of the asset in question. Cast in this framework, the analysis takes on more meaning to the typical reader, for on the basis of these two parameters the mortality coefficients may easily be computed as demonstrated below.

This paper consists of two parts. Part I illustrates a mechanical procedure for calculating the survival coefficients based on the above two parameters, average service life and maximum service life, both statistics often being obtainable from market research or engineering studies. Part II is a mathematical appendix in which the underlying formulas are developed for those who desire them.

Part I. Table 1 Worksheet

The worksheet in Table 1 provides an outline of the steps to be followed in deriving the survival coefficients. The table is self-explanatory since each column heading shows the column number, the source or nature of the operation performed in that column, and the corresponding mathematical expression or formula from Part II below. Hence the description of the procedure is given briefly.

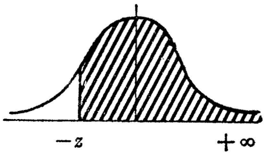
Columns 1, 2, 3. For illustrative purposes, we may assume a hypothetical pro-

* Thomas A. Oates is an industrial engineer with Ford Motor Company and is engaged in research on business games and simulation theory. Milton H. Spencer is Associate Professor of Business Administration in the School of Business Administration of Wayne State University, Detroit, and is the author of four books and various articles on economics and business. He has acted as advisor to several domestic and foreign government agencies.

TABLE 1
WORKSHEET

1	2	3	4	5	6	7	8	9	10*	11
Given	Given	2/1	Table 2	Table 2	Table 2	Given	$\frac{7}{1}(4+5)$	8−4	$\Phi(9)$	10/6
A	M	M/A	y	δ	$\Phi(-y)$	Age in Years t	$\frac{t}{A}(y+\delta)$	$z=\frac{t}{A}(y+\delta)-y$	$\Phi(z)=\Phi\left(\frac{t}{A}(y+\delta)-y\right)$	Survival Coefficient $F(x)=\frac{\Phi\left(\frac{t}{A}(y+\delta)-y\right)}{\Phi(-y)}$
6	10	1.67	5.6	0.0000	1.0000	1	0.94	−4.66	1.0000	1.0000
						2	1.86	−3.74	0.9999	0.9999
						3	2.80	−2.80	0.9974	0.9974
						4	3.74	−1.86	0.9686	0.9686
						5	4.66	−0.94	0.8264	0.8264
						6	5.60	0.00	0.5000	0.5000
						7	6.54	0.94	0.1736	0.1736
						8	7.46	1.86	0.0314	0.0314
						9	8.40	2.80	0.0026	0.0026
						10	9.34	3.73	0.0001	0.0001

* Col. 10: $\Phi(z)$ = area under normal curve from z to $+\infty$.



duct whose average service life, A, is 6 years, and whose maximum service life, M, is 10 years. As pointed out above, these figures may be based on engineering or market research data. The maximum service life is defined as the point where only 1 unit is surviving out of 1,000 original units. The parameters are entered in the first two columns of Table 1. The ratio of M to A is then recorded in column 3. In this case the ratio is 1.67.

Columns 4, 5, 6. It is now necessary to relate the M/A ratio, 1.67, to the particular probability curve in the family of probability curves mentioned earlier. In Table 1, the appropriate values are represented in column 4 by the letter y, which is the parameter describing the particular probability curve that is wanted; and in columns 5 and 6 by the value of δ (delta) and the function $\Phi(-y)$, which is read "*phi* of minus y," respectively. These are also parameters relating to the particular y curve. These values are all developed in convenient tabular form from the equations in Part II. Table 2 below provides these data. Referring to the Table 1 worksheet, it may be noted that the data in columns 4, 5, and 6 are obtained directly from Table 2 by reading off the values corresponding to the M/A ratio, 1.67

Columns 7, 8, 9. Column 7 of Table 1 contains the age of the asset, represented by the letter t, while column 8 presents, for each year of the asset's life, the ratio of its age to its average service life, multiplied by the sum of the two parameters of columns 4 and 5. Next, in column 9, the parameter y from column 4 is subtracted from the answers obtained in column 8. The results are symbolized by the letter z, the necessary values of which are readily obtained from a table of areas of the normal curve, as explained next.

Columns 10, 11. A table of areas of the normal curve, available from virtually any statistics textbook, is presented for convenience in Table 3. The z column in this table corresponds to the z values computed in column 9 of Table 1. It is evident that

TABLE 2

RELATED VALUES OF M/A, y, δ, AND $\Phi(-y)$ AT $F(x)=0.0001$

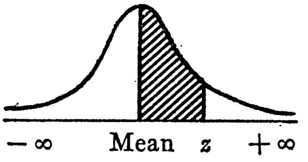
M/A	y	δ	$\Phi(-y)$
4.88	0.000	.7978	.5000
4.81	0.050	.7663	.5199
4.76	0.100	.7355	.5398
4.70	0.150	.7050	.5596
4.65	0.200	.6750	.5793
4.59	0 250	.6459	.5987
4.54	0.300	.6173	.6179
4.47	0.350	.5892	.6368
4.42	0.400	.5619	.6554
4.36	0.450	.5352	.6736
4.30	0.500	.5092	.6915
4.24	0.550	.4838	.7088
4.18	0.600	.4591	.7257
4.12	0.650	.4352	.7422
4.06	0.700	.4120	.7580
4.01	0.750	.3893	.7734
3.95	0.800	.3676	.7881
3.89	0.850	.3465	.8023
3.83	0.900	.3261	.8159
3.78	0.950	.3066	.8289
3.73	1.000	.2877	.8413
3.67	1.050	.2695	.8531
3.62	1.100	.2521	.8643
3.57	1.150	.2353	.8749
3.52	1.200	.2195	.8849
3.46	1.250	.2042	.8944
3.41	1.300	.1898	.9032
3.36	1.350	.1760	.9115
3.31	1.400	.1629	.9192
3.26	1.450	.1505	.9265
3.22	1.500	.1388	.9332
3.17	1.550	.1277	.9394
3.13	1.600	.1173	.9452
3.08	1.650	.1076	.9505
3.04	1.700	.0984	.9554
3.00	1.750	.0899	.9599
2.96	1.800	.0819	.9641
2.92	1.850	.0745	.9678
2.88	1.900	.0675	.9713
2.84	1.950	.0612	.9744
2.80	2.000	.0553	.9772
2.76	2.050	.0498	.9798
2.73	2.100	.0448	.9821
2.70	2.150	.0402	.9842
2.67	2.200	.0360	.9861
2.63	2.250	.0321	.9878
2.60	2.300	.0286	.9893
2.57	2.350	.0254	.9906
2.54	2.400	.0226	.9918
2.51	2.450	.0199	.9929
2.49	2.500	.0176	.9938
2.46	2.550	.0155	.9946
2.43	2.600	.0137	.9953
2.41	2.650	.0119	.9960
2.38	2.700	.0104	.9965
2.36	2.750	.0091	.9970
2.33	2.800	.0079	.9974
2.29	2.900	.0060	.9981
2.25	3.000	.0044	.9987
2.21	3.100	.0033	.9990
2.17	3.200	.0024	.9993
2.14	3.300	.0017	.9995
2.10	3.400	.0012	.9997
2.07	3.500	.0009	.9998
2.04	3.600	.0006	.9998
2.01	3.700	.0004	.9999
1.99	3.800	.0003	.9999
1.96	3.900	.0002	1.0000
1.92	4.100	.0000	1.0000
1.87	4.300	.0000	1.0000
1.83	4.500	.0000	1.0000
1.78	4.800	.0000	1.0000
1.72	5.200	.0000	1.0000
1.67	5.600	.0000	1.0000
1.65	5.800	.0000	1.0000
1.58	6.500	.0000	1.0000
1.54	7.000	.0000	1.0000
1.50	7.500	.0000	1.0000
1.47	8.000	.0000	1.0000
1.44	8.500	.0000	1.0000
1.40	9.500	.0000	1.0000
1.38	10.000	.0000	1.0000
1.34	11.000	.0000	1.0000
1.31	12.000	.0000	1.0000
1.29	13.000	.0000	1.0000
1.27	14.000	.0000	1.0000
1.25	15.000	.0000	1.0000
1.23	16.000	.0000	1.0000
1.22	17.000	.0000	1.0000

Source: Derived from Part II, equation (10).

the z values in column 9 can be positive or negative. Since we want the area under the normal curve from z to ∞, it means that the area obtained from Table 3 will have to be *subtracted* from .5000 if z is positive, and *added* to .5000 if z is negative. Thus, referring to the worksheet of Table 1, two illustrative calculations may be shown. For instance, in column 9, at $z=-2.80$, the area from Table 3 is seen to be .4974. *Adding* this to .5000, the result is 0.9974, which is entered on the appropriate line in column 10. Similarly, in column 9, at $z=1.86$, the area in Table 3 is found to be .4686. *Subtracting* this from .5000, the result, 0.0314, is entered on the appropriate line in column 10. The figures in column 10 thus represent the function "*phi of z*," written $\Phi(z)$, where z describes the position on the y curve as mentioned

TABLE 3

TABLE OF AREAS OF NORMAL PROBABILITY DISTRIBUTION



z	0	1	2	3	4	5	6	7	8	9
.0	.0000	.0040	.0080	.0120	.0160	.0199	.0239	.0279	.0319	.0359
.1	.0398	.0438	.0478	.0517	.0557	.0596	.0636	.0675	.0714	.0753
.2	.0793	.0832	.0871	.0910	.0948	.0987	.1026	.1064	.1103	.1141
.3	.1179	.1217	.1255	.1293	.1331	.1368	.1406	.1443	.1480	.1517
.4	.1554	.1591	.1628	.1664	.1700	.1736	.1772	.1808	.1844	.1879
.5	.1915	.1950	.1985	.2019	.2054	.2088	.2123	.2157	.2190	.2224
.6	.2257	.2291	.2324	.2357	.2389	.2422	.2454	.2486	.2517	.2549
.7	.2580	.2611	.2642	.2673	.2703	.2734	.2764	.2794	.2823	.2852
.8	.2881	.2910	.2939	.2967	.2995	.3023	.3051	.3078	.3106	.3133
.9	.3159	.3186	.3212	.3238	.3264	.3289	.3315	.3340	.3365	.3389
1.0	.3413	.3438	.3461	.3485	.3508	.3531	.3554	.3577	.3599	.3621
1.1	.3643	.3665	.3686	.3708	.3729	.3749	.3770	.3790	.3810	.3830
1.2	.3849	.3869	.3888	.3907	.3925	.3944	.3962	.3980	.3997	.4015
1.3	.4032	.4049	.4066	.4082	.4099	.4115	.4131	.4147	.4162	.4177
1.4	.4192	.4207	.4222	.4236	.4251	.4265	.4279	.4292	.4306	.4319
1.5	.4332	.4345	.4357	.4370	.4382	.4394	.4406	.4418	.4429	.4441
1.6	.4452	.4463	.4474	.4484	.4495	.4505	.4515	.4525	.4535	.4545
1.7	.4554	.4564	.4573	.4582	.4591	.4599	.4608	.4616	.4625	.4633
1.8	.4641	.4649	.4656	.4664	.4671	.4678	.4686	.4693	.4699	.4706
1.9	.4713	.4719	.4726	.4732	.4738	.4744	.4750	.4756	.4761	.4767
2.0	.4772	.4778	.4783	.4788	.4793	.4798	.4803	.4808	.4812	.4817
2.1	.4821	.4826	.4830	.4834	.4838	.4842	.4846	.4850	.4854	.4857
2.2	.4861	.4864	.4868	.4871	.4875	.4878	.4881	.4884	.4887	.4890
2.3	.4893	.4896	.4898	.4901	.4904	.4906	.4909	.4911	.4913	.4916
2.4	.4918	.4920	.4922	.4925	.4927	.4929	.4931	.4932	.4934	.4936
2.5	.4938	.4940	.4941	.4943	.4945	.4946	.4948	.4949	.4951	.4952
2.6	.4953	.4955	.4956	.4957	.4959	.4960	.4961	.4962	.4963	.4964
2.7	.4965	.4966	.4967	.4968	.4969	.4970	.4971	.4972	.4973	.4974
2.8	.4974	.4975	.4976	.4977	.4977	.4978	.4979	.4979	.4980	.4981
2.9	.4981	.4982	.4982	.4983	.4984	.4984	.4985	.4985	.4986	.4986
3.0	.4987	.4987	.4987	.4988	.4988	.4989	.4989	.4989	.4990	.4990
3.1	.4990	.4991	.4991	.4991	.4992	.4992	.4992	.4992	.4993	.4993
3.2	.4993	.4993	.4994	.4994	.4994	.4994	.4994	.4995	.4995	.4995
3.3	.4995	.4995	.4995	.4996	.4996	.4996	.4996	.4996	.4996	.4997
3.4	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4998
3.5	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998
3.6	.4998	.4998	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999
3.7	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999
3.8	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999
3.9	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000

Illustration: For $z=2.74$, shaded area is .4969 out of total area of 1. For $z=-1.65$, shaded area is .9505 out of total area of 1.

above, and is equivalent to the abscissa on a normal probability curve.

Column 11 provides the survival coefficients, which are obtained by dividing the figures in column 10 by the datum in column 6. The result, called $F(x)$ or "*F of x*," represents the proportion of original units surviving to a given year. Thus, in the first year of life, or at $t=1$ (in column 7), $100\pm$ of the units are expected to survive as shown in column 11. At $t=5$, $82.64\pm$ of the original group may be expected to survive. And at $t=10$, only $.01\pm$ should survive, which is an arbitrary but reasonably practical limit chosen for $f(x)$ as defined in Part II below. The survival coefficients in column 11 can, of course, be easily converted to mortality coefficients or retirement frequencies by simply subtracting the survival coefficients from 1.

Part II. Mathematical Appendix

The derivation of survival or retirement frequencies as described above is based on the truncated normal distribution. Earlier formulations of these ideas relied on two parameters, the *average service life* and the *mortality distribution variance*.[1] In the following paragraphs it is shown how the latter parameter may be expressed instead as a function of the former and also of the *maximum service life*, thus resulting in a more "practical" formulation.

Figure 1 below contains a family of mortality curves which takes the form of the well-known "*S*" curve. This generalized family of curves results from the truncation of a normal probability distribution which is expressed as:

$$F(x) = \frac{\Phi[x(y+\delta) - y]}{\Phi(-y)} \qquad (1)$$

where $\phi(x)$ is the ordinate of the normal curve,

$$\phi(x) = \frac{e^{-x2/2}}{\sqrt{2\pi}} \qquad (2)$$

and $\Phi(x)$ is the area under the normal curve:

$$\Phi(x) = \int_x^\infty \phi(\alpha)d\alpha \qquad (3)$$

Since the abscissa of Figure 1 is expressed in per cent of average service life, A, the area under each of the curves is equal to unity. The parameter defining each curve is y while δ is derived from y as in (4):

$$\delta = \frac{\phi(-y)}{\Phi(-y)} \qquad (4)$$

The value of δ is a constant for any specific curve described by y. Any point on the curve, $F(x)$, is determined by x which is the age, t, relative to the average service life.

Defining $f(x)$ as the mortality frequency function that describes the frequency of retirement through time, the variance may be determined by:

$$\sigma^2 = \int_0^\infty x^2 f(x)dx - A^z \quad (A^z = 1) \qquad (5)$$

and the frequency function may be written:

$$f(x) = \frac{df(x)}{-dx} = \frac{(y+\delta)\Phi[x(y+\delta) - y]}{\Phi(-y)} \qquad (6)$$

Incorporating (6) into (5) and performing the necessary integration results in the variance:

$$\sigma^2 = \frac{(1 - y\delta - \delta^2)}{(\delta + y)^2} \qquad (7)$$

The variance is thus a descriptive term instead of a well-known parameter, although y is defined by the variance. Since it is necessary to determine the value of y, this can be quickly and satisfactorily ac-

[1] See, for example, E. J. Gumbel, *Jahrbucher da Nationalthenanie und Statistik*, Jena, 1935; and an article by B. F. Kimball in *Econometrica*, October, 1947, as well as various other sources in the general field of renewal theory and related subjects.

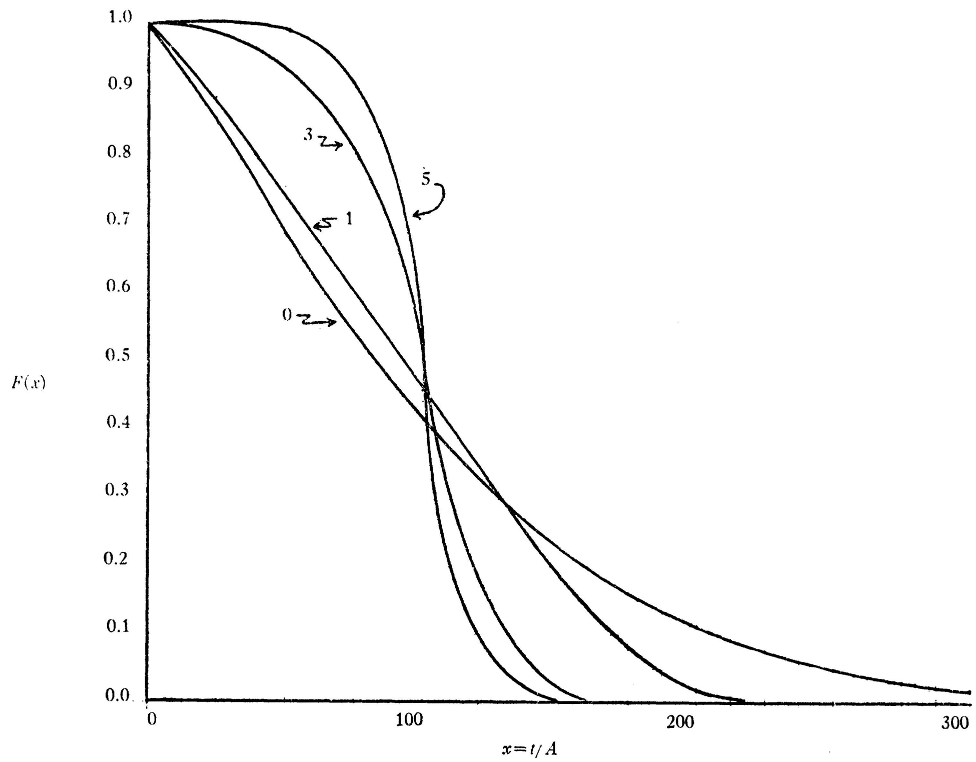


FIG. 1. Survival curves at $Y=0.0$, 1.0, 3.0, and 5.0.

complished in a graphic manner from two points within the distribution, although the theoretical fit is improved, of course, with additional points.

The average and maximum service lives, or A and M respectively, are the two points of chief importance. The latter, based on probability considerations discussed below, may be defined as the point where the proportion of original units surviving, $F(x)$, is equal to .0001.

Equation (1) may be solved for y at given values of x with $F(x)=.0001$. This procedure results in Figure 2. Table 2 has also been calulated by this method. Noting that:

$$x = t/A \tag{9}$$

where t= age of survivors expressed in the same time units as A, the distribution variance can be determined from A and M.

The survival coefficients $F(x)$ are given by the equation

$$F(x) = \frac{\Phi\left[\frac{t}{A}(y+\delta) - y\right]}{\Phi(-y)} \tag{10}$$

Of course, the overall accuracy of the results will be determined by the size of the sample.

Characteristics of the Coefficients. The coefficients have certain characteristics and properties which may be outlined as follows:

1. The chief advantage of using the mortality coefficients is the allowance that is made for "chance" retirements, i.e., retirements that are independent of age. If the mortality frequency function, equation (6), is plotted, the curves are seen to be left-moded. Values of y less than 3.0

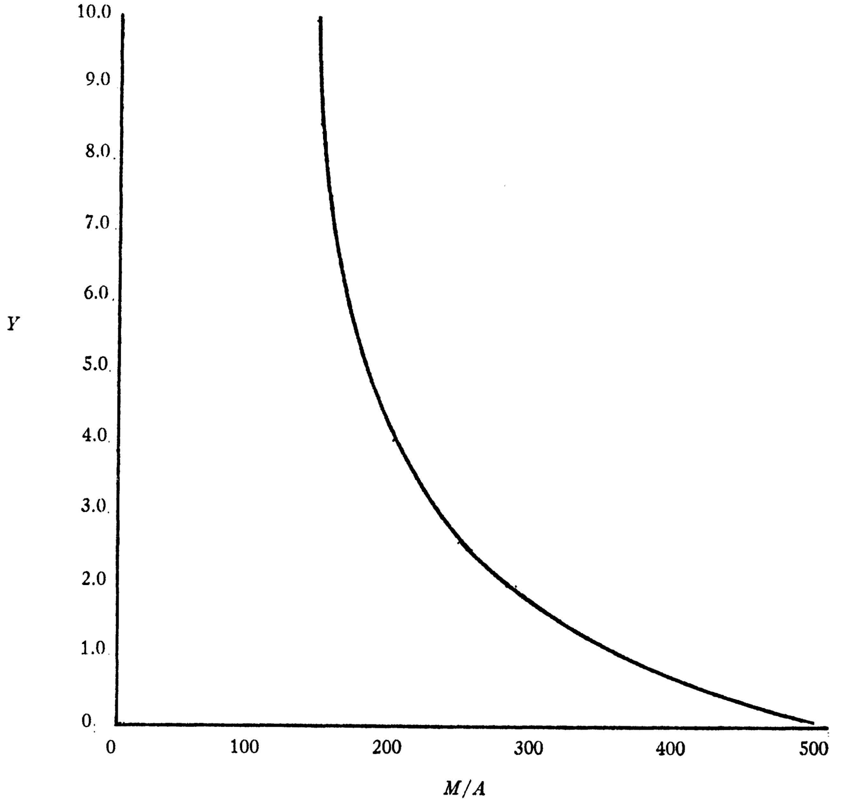


FIG. 2. Relationship between Y and M/A at $(Fx)=0.0001$.

are associated with curves that have a retirement rate greater than zero at the initial time, while values of y greater than 3.0 have no retirements in the first small portion of life. Thus, the frequency of chance retirements varies inversely with the value of y.

2. The value of y is also inversely related to the distribution variance [equation (7)], as seen in Figure 1. Note that the slope of the curve is very steep at 100% average service life with the relatively high values of y, as contrasted with the low values of y where the slope of the mortality curve is quite shallow.

3. The effect of chance retirement is further illustrated by the extreme values of the variance. In Figure 1, if the variance is zero, the curve will be a step function with the step occurring at age equal to 100% of the average service life. In that case there would be no chance retirements and the average service life would equal the maximum service life. The opposite case would be one in which there is a large variance with y equal to zero. The mortality curve would then be one of exponential decay, approaching zero. The latter case might be applicable to the mortality curve of a nondurable good. Since the average service life of such a good is usually difficult to express, an analysis of the mortality rates of nondurable goods would probably have to be conducted in terms of parameters other than average and maximum service life.

4. A further property of the family of curves is that they are left-moded. Thus in

considering the retirement frequency function, note that the mode occurs at age equal to 100% average service life when $y > 1.0$. As the value of y decreases below 1.0, the mode shifts further to the left until it reaches age zero in the case of the exponential decay. The reasons for this can be explained in terms of chance retirements.

5. It has been pointed out that chance retirements are maximized with the minimum values of y, hence the greatest retirement frequency under this condition will occur at the point of "birth." Under normal conditions, the greatest retirement frequency will occur at age equal to 100% of the average service life.

6. The majority of applications will yield a value $2.0 \leqq y \leqq 5.0$, which is the equivalent of stating that the maximum service life will be from 1.75 to 2.80 times the average service life.

7. The effectiveness of this approach obviously diminishes in those applications where retirements may be due primarily to factors other than chance, since probability considerations are then minimized.