A SYSTEM OF LIFE TABLES FOR PHYSICAL PROPERTY BASED ON THE TRUNCATED NORMAL DISTRIBUTION*

By BRADFORD F. KIMBALL

1. INTRODUCTION

THE object of this paper is to introduce a system of life tables for physical property that satisfy the following criteria:

(1) It is applicable to a wide class of physical property in the sense that it offers a rough approximation to the life tables of such property by the determination of two parameters.

(2) It is subject to convenient mathematical generalization and computation.

(3) The related renewal functions are also subject to convenient mathematical formulation.

Several systems of generalized life tables for property retirements have been used. In the New York State Public Service Commission a convenient system used for rough generalization has been the Patterson system, where the "generalized" life table of unit service life (see Ref. [1], Section 10) is given by

$$(1.1)\qquad \begin{aligned} M_1(t, n) &= 1 - t^n/2, && 0 \leqq t \leqq 1, \\ &= (2 - t^n)/2, && 1 \leqq t \leqq 2. \end{aligned}$$

With the life table given with unit service life it is only necessary to use the relation

$$(1.2)\qquad M(x) = M_1(x/L, n)$$

to determine the life table with the average service life equal to L, pos-

* Just as this issue was to go to press, the following note was received from the author.—MANAGING EDITOR.

I have just learned from Professor E. J. Gumbel that he published a mathematical formulation of a system of life tables based on the truncated normal distribution in 1933: "Die Gaussische Verteilung der Gestorbenen," *Jahrbücher für Nationalökonomie und Statistik*, 138 Bd., III Folge, Bd. 83, Jena, 1933, pp. 365–389; and "Die Verteilung der Gestorbenen um das Normalalter," *Aktuarske Vedy*, T. IV, Nr. 2, Prague, 1933, pp. 65–96.

Although based on the normal distribution and set up exactly as I have set up the h-system of life tables (see latter reference, p. 67) except for the notation, the treatment for numerical application is quite different. A major emphasis is given to the modal age and the modal life-table value, since in the case of human mortality studies the system would not hold for infant mortality.

I find that the treatment from the point of view of methods of application involves very little repetition. Professor Gumbel's articles should be of especial value to persons interested in fitting the latter part of a life table to a distribution based on the normal frequency curve.

sessing the same general characteristics. The variance of a generalized retirement frequency curve of the above system (with $L=1$) can be shown to be

$$\sigma_1^2 = 2/[(n+1)(n+2)], \qquad L = 1. \tag{1.3}$$

Thus the above system (1.1) represents a two-parameter family of life tables, with the service life L acting implicitly as one such parameter, and the index n determined by the variance σ_1^2 of the generalized frequency curve, serving as the second parameter.

This system for some purposes is oversimplified, but for turnover-cycle computations it has been found very useful (see Ref. [2]).

The Iowa system of generalized life tables developed at the Iowa Engineering Experiment Station (see Ref. [3]) is more definitive in using three sets of life tables, applicable to cases of left-moded, symmetrical, and right-moded retirement frequency distributions. However, the mathematical formulas for these curves are not suited for a single mathematical generalization applicable to the whole system.

The Gompertz-Makeham curve used in the study of human mortality does not offer satisfactory generalization for property retirements (see Ref. [4]).

A system based on the Pearson Type III frequency curve offers in some mathematical respects a satisfactory system (see Ref. [5]). It has the advantage of possessing a single mathematical formula for the whole system, and in having a renewal function that can be easily computed for integral values of the parameter. However, the author has found that for the larger variances (say for $\sigma_1^2 > 0.20$) the left-moded retirement frequency curves of this system are too peaked (cf. Figure 1) to represent the usual pattern of property retirements. In such cases of a relatively large "dispersion" of retirements a considerable "chance factor" of retirement, representing retirement due to causes independent of age, is usually present. This reduces the peakedness of the retirement frequency curve (see Section 5).

A system based on the Pearson Type I frequency curve suggested by Kurtz prior to the present Iowa system (see Ref. [4], Chapter 4) is given by a single mathematical generalization. When, however, the parameters are reduced to two in number, depending upon the service life and variance, it is the writer's opinion that the system becomes too specialized to adequately represent a general system of property mortality frequency curves. Furthermore, unless the exponents in the Type I frequency function are integers, difficulty will be encountered in obtaining a formula for the renewal function (compare Ref. [6]).

The most obvious system has been passed over as far as mathematical generalization is concerned. I refer to a system based upon a normal frequency distribution of retirements. It is the particular object of this paper to set forth some of the properties of such a system in such a manner that economists and engineers with a reasonable background of mathematics can make use of these curves in the study of the behavior of plant accounts that are subject to a constant mortality law of retirement.

Individual curves of such a system have been used in the past, and such a system is implied, for example, where a mortality curve is to be determined graphically by a straight line through the points of an observed life table plotted on arithmetical probability paper (see Ref. [7] and Figure 5). However, no precise mathematical formulation has appeared heretofore in the literature as far as the writer is aware.

2. The "h-System" of Life Tables

Let $\phi(t)$ denote the normal frequency function

$$\phi(t) = e^{-t^2/2}/\sqrt{2\pi}$$

and set

$$\Phi(t) = \int_t^\infty \phi(s)ds. \tag{2.1}$$

A general family of possible life tables can be defined by

$$M(t) = \Phi(wt - h)/\Phi(-h), \tag{2.2}$$

where w and h are parameters to be discussed later, and $M(t)$ refers to the proportion of survivors remaining at time t, from an original installation of unit amount at time zero.

The associated retirement frequency function is denoted by $f(t)$,

$$f(t) = -M'(t) = w\phi(wt - h)/\Phi(-h). \tag{2.3}$$

Before further discussion it will be well to determine the average service life and the variance of $f(t)$ in terms of w and h. If the average service life is denoted by L, an integration by parts gives

$$L = \int_0^\infty M(s)ds = (1/w)[h + \phi(-h)/\Phi(-h)]. \tag{2.4}$$

Since L is the first moment of the frequency distribution, it follows that the variance σ^2 of $f(t)$ is given by

$$\sigma^2 = \int_0^\infty t^2 f(t)dt - L^2$$

$$= [1/w\,\Phi(-h)] \int_0^\infty (wt)^2 \phi(wt - h)dt - L^2.$$

Performing the integration and using the relation (2.4) we obtain

$$(2.5) \qquad \sigma^2 = (1 - hH_0 - H_0^2)/w^2, \qquad \text{with } H_0 = \phi(-h)/\Phi(-h).$$

In practice a life table referred to abscissa measured as the proportion of the average service life defines the type of life table to be used, and it is a simple matter to change the average service life as required. Hence we are particularly interested in the *one*-parameter family of life tables derived from (2.1) for which the average service life is unity. The subscript one will be used in this section to denote parameters and functions which characterize this family of life tables. From (2.4),

$$(2.6) \qquad w_1 = h + \phi(-h)/\Phi(-h),$$

and substitution in (2.5) gives

$$(2.7) \qquad \sigma_1^2 = 1/(h + H_0)^2 - H_0/(h + H_0), \qquad H_0 = \phi(-h)/\Phi(-h).$$

This last relation will be found to give a one-to-one correspondence between σ_1^2 and h on the range (see Figure 3)

$$0 \leqq \sigma_1^2 \leqq 1, \qquad -\infty \leqq h \leqq +\infty.$$

THEOREM 1. *The function of t,*

$$(2.8) \qquad M_1(h, t) = \Phi(w_1 t - h)/\Phi(-h),$$

with w_1 given by (2.6), *defines a one-parameter family of possible life tables in the time variable t, with average service life equal to unity, and with the variance of the retirement frequency distribution determined by the value of h through the relation* (2.7). *The retirement frequency distribution is given by* (2.3) *with $w = w_1$.*

This family of life tables will be referred to in this paper as a family or system of life tables of the "h-type." It is not referred to as a system of the normal type because the frequency function is a *truncated* normal curve and hence when the variance is large ($> 1/2$) and h is a small positive quantity or negative, the departure from normality is considerable (see Figure 1 and Section 5).

3. DISCUSSION OF SOME OF THE CHARACTERISTICS OF THE FAMILY OF LIFE TABLES OF THE h-TYPE

For greater convenience, for the remainder of this paper the subscript will be dropped from w_1. Also we shall use the notation

$$(3.1) \qquad \phi_0 = \phi(-h), \qquad \Phi_0 = \Phi(-h).$$

Thus

$$w = h + H_0, \qquad H_0 = \phi_0/\Phi_0, \tag{3.2}$$

and the retirement frequency curve for service life equal to unity is given by

$$f(t) = w\phi(wt - h)/\Phi_0, \qquad L = 1. \tag{3.3}$$

It will be noted that

$$f(0) = wH_0 \tag{3.4}$$

and that for h positive or zero, the mode is at

$$\text{Mode}: t_{mo} = h/w, \qquad h \geqq 0, \tag{3.5}$$

which is at a distance H_0/w to the left of the mean. Thus *the retirement frequency curves, although based on the normal distribution function, do not start from zero, and are left-moded.* This fact has possibly prevented a more widespread use of these curves. However, experience with general type mortality curves as applied to property retirements indicates that the failure of $f(t)$ to start from zero is an argument in favor of the use of these curves, for it serves to allow for the "chance" retirements that are due to causes unrelated to age (see Section 5 for a more complete discussion).

This may be recognized as a fundamental difference between frequency curves associated with life tables, and the frequency curves associated with the majority of other statistical problems.

In Figure 1 are shown the retirement frequency curves of the h-system for $h=0.5$, 1.5, 2, 2.5, 3, 4, and 5. The exponential curve $y=e^{-t}$ is marked *Exp*. It can be shown that this curve is the limiting member of the h-system for $h=-\infty$ (see Appendix). It will be noted that for values of h greater than 3, which means that the variance is less than approximately 0.10, the curves are very nearly symmetrical. As the variance becomes larger the curves tend to become more and more left-moded.

The corresponding mortality curves (=survivor curves or life-table curves) are shown in Figure 2. A life table M_1 (h, t) with unit service life becomes a life table with service life L if t be replaced by x/L. Thus we write

$$\begin{array}{c}\text{Life Table Fu.}\\ \text{with}\\ \text{Service Life } L\end{array} \qquad M(x) = \Phi(wx/L - h)/\Phi_0, \tag{3.6}$$

$$x = \text{age in years}, \qquad L = \text{average service life}.$$

With the aid of tables of the probability functions (see Ref. [8]) with

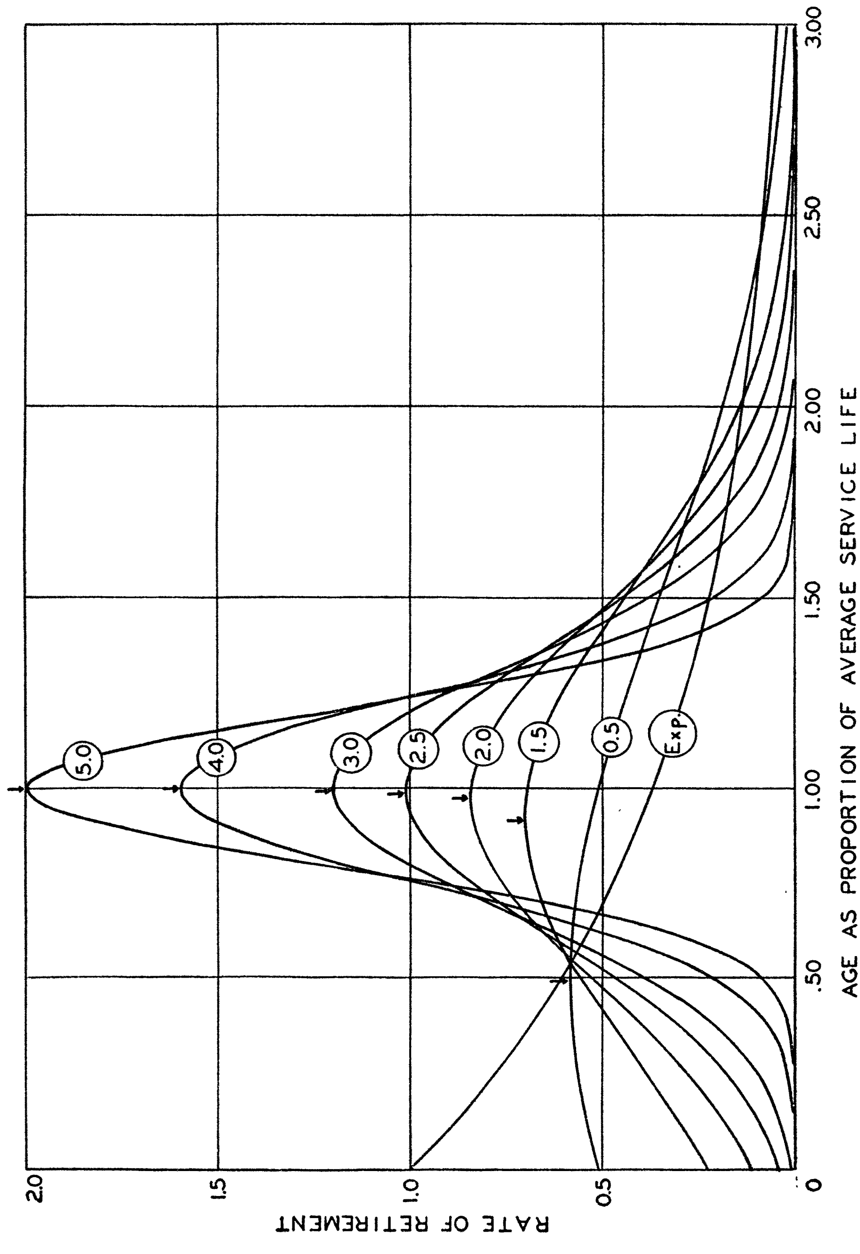

FIGURE 1.—Retirement frequency curves of h-system of life tables.

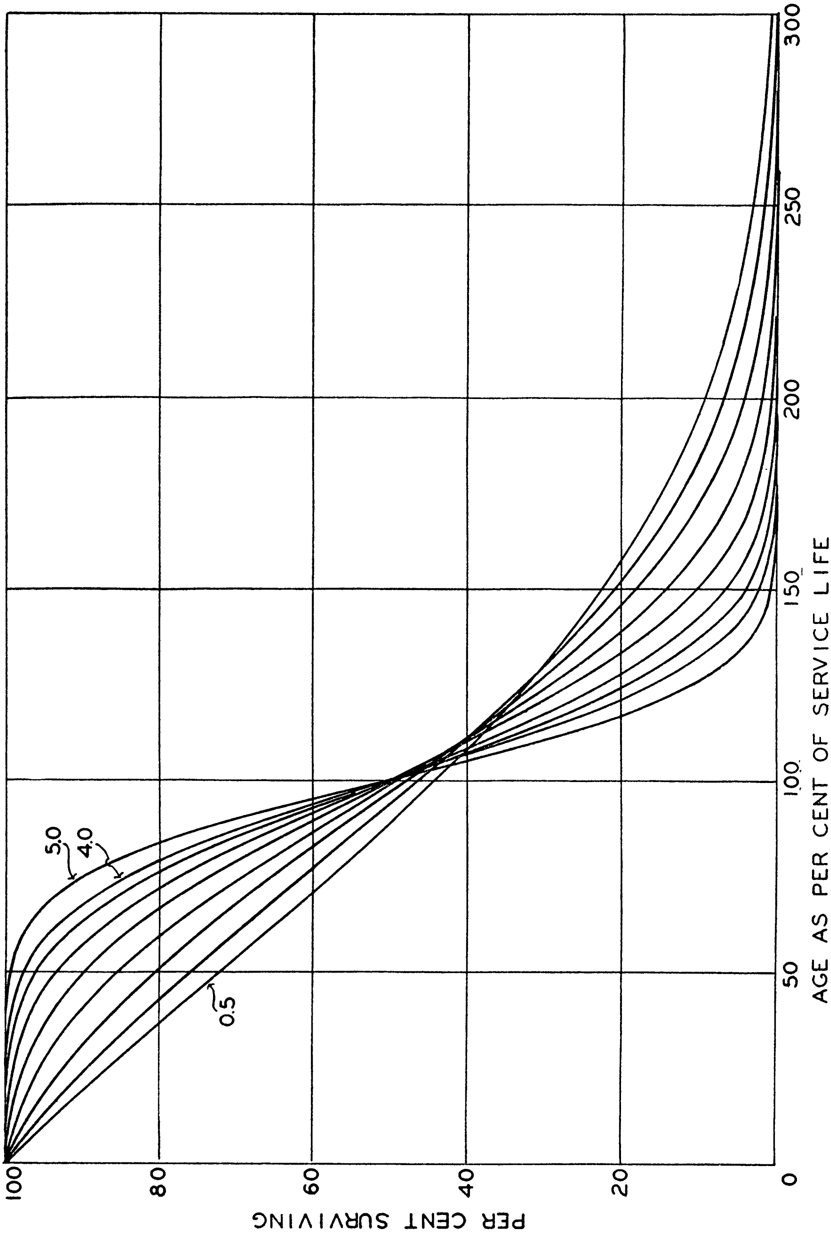

FIGURE 2.—h-system of generalized life tables.
Note: Curves intermediate between $h=0.5$ and $h=4.0$ are the life-table curves: $h=1.0$, 1.5, 2.0, 2.5, 3.0, and 3.5.

h, w, and Φ_0 known, the life table is set up without much difficulty. In Table 1 are listed the values of w, $1/\Phi_0$, and the variance index σ_1^2 corresponding to a range of h from 0 to 5 at intervals of 0.5.

TABLE 1

VALUES OF w, $1/\Phi_0$, AND VARIANCE

h	w	$1/\Phi_0$	σ_1^2
0.0	0.7978 85	2.0000 00	0.5707 96
0.5	1.0091 60	1.4462 10	.4773 89
1.0	1.2876 00	1.1885 73	.3798 06
1.5	1.6387 90	1.0715 90	.2876 61
2.0	2.0552 48	1.0232 80	.2098 59
2.5	2.5176 38	1.0062 48	.1507 60
3.0	3.0044 48	1.0013 52	.1093 06
3.5	3.5008 73	1.0002 33	.0813 43
4.0	4.0001 34	1.0000 32	.0624 62
4.5	4.5000 16	1.0000 03	.0493 44
5.0	5.0000 01	1.0000 00	0.0400 00

In Figure 3 is shown a chart relating the index h to the reciprocal of the standard deviation of the retirement frequency curve ($=1/\sigma_1$). By means of this chart h indices can be found for variances intermediate to those listed in Table 1.

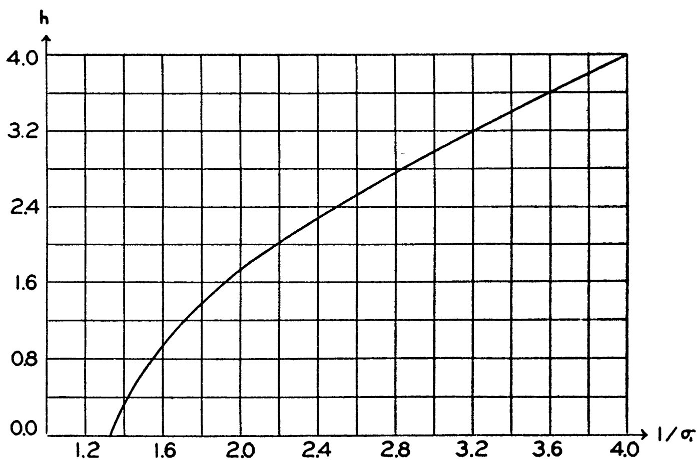

FIGURE 3.—Relation between h index and standard deviation σ_1.

Possibility of rough determination of type parameter h from chart of observed life table. A property of the h-system of life tables that makes them readily adaptable for practical use is brought out in Figure 4 where they are plotted on arithmetical probability graph paper. All of these life tables appear as approximately straight lines for ages greater than 100 per cent of average service life, for the likely range of h, greater than 0.5. Because of the considerable truncation of the normal curve of frequency retirements, these lines are curved in the earlier years for index numbers less than 4 (variance greater than 1/16). For variance less than 1/16 the life tables appear as straight lines over the

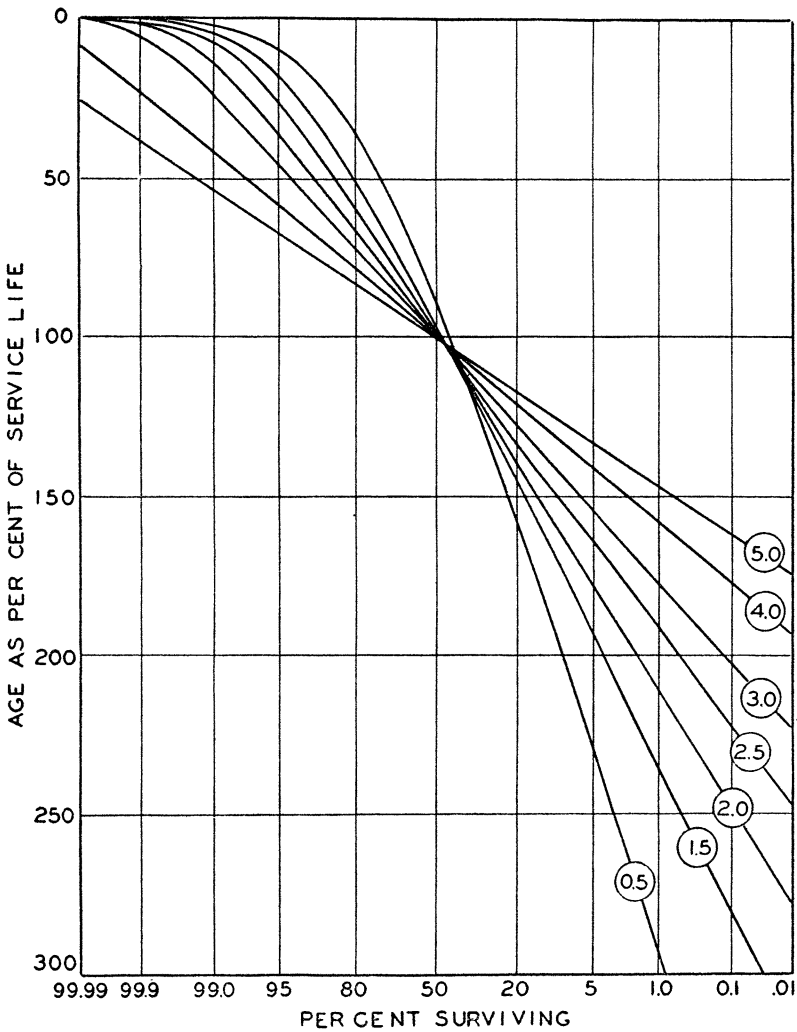

FIGURE 4.—h-system of life tables drawn to probability scale.

whole range of life of the property, provided that the chart does not carry back to survivor ratio greater than 0.999.

Thus such a chart offers a rough means for determining the value of the parameter h that applies to a given observed life table, if that table furnishes values to a point where, say, not more than 20 per cent of the property is surviving.

In order to facilitate the use of such a chart, Table 2 has been prepared, giving values of the h-system life tables at age equal to 100 per cent of the average service life, and at age equal to 150 per cent of average service life for values of h from 0 to 4 at intervals of 0.1, and for $h=4.5$ and 5.0. Also since graphs of the h-system life tables for h less than 2 are slightly curved beyond 100 per cent of age, the values at 200 per cent of average service life are shown for h from 0 to 2.

As an example of fitting, consider the observed life table given in Column 2 of Table 3. These observed values are shown by points indicated by circles on arithmetical probability graph paper (see Figure 5). With the 50-per cent point of the life table (median of retirement frequency curve) occurring at age 10 years (see Column 1 of Table 3 or Figure 5), the average service life indicated is approximately 10 yrs.[1] At age 150 per cent of this estimated service life, the points at which mortality curves of the h-system for $h=2.0$, 2.2, 2.4, 2.6, 2.8, and 3.0 would be plotted, are shown (cf. Table 2). The straight line passing through the later X points of the observed life table then indicates by the point at which it cuts the horizontal line through age 150 per cent that the best choice of h is about $h=2.72$ (interpolating between 2.6 and 2.8).

Having chosen a best value for h, we set up the life table by using formula (3.6). From table of the normal probability function

$$\Phi(-2.72) = 0.99674, \qquad \phi(-2.72) = 0.0098712.$$

Hence

$$H_0 = \phi_0/\Phi_0 = 0.009903,$$
$$w = h + H_0 = 2.729903.$$

With $L=10$ the formula for the life table is

$$M(x) = \Phi(wx/10 - h)/\Phi_0. \tag{3.7}$$

Thus to compute the life table at ages 0.5, 1.5, 2.5, . . . the values of

[1] Note from Figure 4 that for curves of small index h, the 50-per-cent point of the life table occurs somewhat *above* the horizontal line through age corresponding to 100 per cent of average service life. This fact should be taken into account in estimating service life from the 50-per-cent point of the life table.

TABLE 2

CERTAIN VALUES OF THE h-SYSTEM OF LIFE TABLES FOR GRAPHICAL DETERMINATION OF THE PARAMETER h

h	At 100% A.S.L.	Diff.	At 150% A.S.L.	Diff.	At 200% A.S.L.	Diff.	h
0.0	0.4249	31	0.23137	82	0.11054	286	0.0
0.1	.4280	33	.23055	105	.10768	314	0.1
0.2	.4313	33	.22950	131	.10454	346	0.2
0.3	.4346	34	.22819	160	.10108	376	0.3
0.4	.4380	36	.22659	192	.09732	409	0.4
0.5	.4416	35	.22467	228	.09323	440	0.5
0.6	.4451	37	.22239	268	.08883	470	0.6
0.7	.4488	37	.21971	311	.08413	499	0.7
0.8	.4525	36	.21660	356	.07914	523	0.8
0.9	.4561	37	.21304	406	.07391	544	0.9
1.0	.4598	36	.20898	456	.06847	559	1.0
1.1	.4634	35	.20442	508	.06288	568	1.1
1.2	.4669	34	.19934	559	.05720	568	1.2
1.3	.4703	32	.19375	609	.05152	562	1.3
1.4	.4735	31	.18766	658	.04590	546	1.4
1.5	.4766	30	.18108	702	.04044	524	1.5
1.6	.4796	27	.17406	741	.03520	493	1.6
1.7	.4823	25	.16665	775	.03027	457	1.7
1.8	.4848	23	.15890	801	.02570	416	1.8
1.9	.4871	20	.15089	821	.02154	373	1.9
2.0	.4891	18	.14268	832	0.01781		2.0
2.1	.4909	16	.13436	835			2.1
2.2	.4925	14	.12601	830			2.2
2.3	.4939	11	.11771	819			2.3
2.4	.4950	10	.10952	799			2.4
2.5	.4960	9	.10153	775			2.5
2.6	.4969	7	.09378	746			2.6
2.7	.4976	5	.08632	713			2.7
2.6	.4981	4	.07919	676			2.8
2.9	.4985	4	.07243	639			2.9
3.0	.4989	3	.06604	600			3.0
3.1	.4992	2	.06004	560			3.1
3.2	.4994	2	.05444	521			3.2
3.3	.4996	1	.04923	482			3.3
3.4	.4997	1	.04441	445			3.4
3.5	.4998	0	.03996	410			3.5
3.6	.4998		.03586	375			3.6
3.7	.4999		.03211	342			3.7
3.8	.4999		.02869	312			3.8
3.9	.4999		.02557	283			3.9
4.0	.5000		.02274				4.0
4.5	.5000		.01222				4.5
5.0	0.5000		0.00621				5.0

TABLE 3

COMPARISON OF NUMERICAL VALUES OF OBSERVED AND FITTED LIFE TABLES

Age in Years	Observed Life Table	Fitted Life Table
0.0	1.000	1.000
0.5	0.999	0.998
1.5	0.997	0.993
2.5	0.990	0.982
3.5	0.973	0.964
4.5	0.943	0.935
5.5	0.897	0.891
6.5	0.835	0.831
7.5	0.755	0.752
8.5	0.660	0.657
9.5	0.555	0.552
10.5	0.448	0.443
11.5	0.341	0.339
12.5	0.244	0.245
13.5		0.168
14.5		0.108
15.5		0.066
16.5		0.037
17.5		0.020
18.5		0.010
19.5		0.005
20.5		0.002
21.5		0.001

Φ are found for values of the argument: $(0.05)w-h$, $(0.15)w-h$, $(0.25)w-h$, . . . These may be found directly to four decimal places without interpolation from published tables.[2] When divided by $\Phi_0=0.99674$ the required values of the life table are obtained. The life table thus determined is given in Table 3, and is indicated by the full line in Figure 5.

As noted elsewhere in the literature (see Ref. [2], App. C) the above method does not constitute a refined method for determining the average service life and shape of life table. It merely serves as a means for picking out the member of the h-system that most nearly fits a given observed mortality curve, when it is desired that a generalized mortality curve of the h-type be used as an approximation.

4. GENERAL FORMULA FOR THE EXPECTATION OF LIFE OF THE h-SYSTEM OF LIFE TABLES

Denote the expectation of life of a member of the h-system of life

[2] For example, *The Kelley Statistical Tables*, Ref. [8].

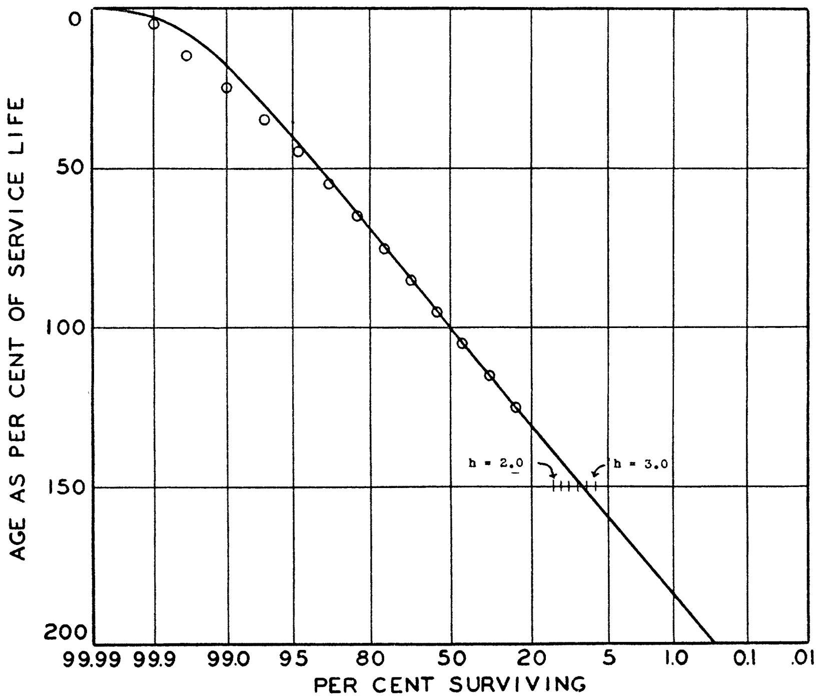

FIGURE 5.—Graphical fit of life table of h-system to an observed life table.

○○○ Observed Life Table.

——— Fitted Life Table, $h=2.72$.

| | | | Indicate where h-curves; $h=2.0$, 2.2, 2.4, 2.6, 2.8 and 3.0 cross the 150-per-cent line.

tables with average service life of unity by $E_1(h, t)$. Then by definition

$$E_1(h, t) = [1/M_1(h, t)] \int_t^\infty M_1(h, s)ds. \tag{4.1}$$

Using (2.8) and setting $y=wt$ in $\Phi(wt-h)$, we have

$$M_1(h, t)\, E_1(h, t) = [1/\Phi_0 w] \int_y^\infty \Phi(z-h)dz, \qquad y = wt.$$

Integrating by parts, we get

$$\int_y^\infty \Phi(z-h)dz = [(z-h)\,\Phi(z-h)]\Big|_y^\infty + \int_y^\infty (z-h)\phi(z-h)dz$$
$$= -(y-h)\,\Phi(y-h) + \phi(y-h), \qquad y = wt.$$

Replacing y by wt, we have

$$(4.2) \qquad E_1(h, t) = [1/w][\phi(wt - h)/\Phi(wt - h) - (wt - h)].$$

Denote the expectation of life for a life table with service life L by $LE(x)$. Thus *$E(x)$ will denote the ratio of the life expectancy at age x, to the service life L.* It is easily demonstrated that (see Ref. [1], Section 10)

$$(4.3) \qquad E(x) = E_1(h, x/L).$$

Introduce the function $E_0(u)$ defined by

$$(4.4) \qquad E_0(u) = \phi(u)/\Phi(u) - u$$

which might be called *the generalized life-expectancy function* for the h-system of life tables, since by virtue of (4.2) and (4.3) we have the general relation

$$(4.5) \quad \text{Life Exp. Ratio at Age } x\text{:} \quad E(x) = E_0(u)/w, \quad u = wx/L - h,$$

which holds for any life table of the h-system.

Charts can be prepared (see Figures 6a, 6b, and 6c) so that the generalized life-expectancy function $E_0(u)$ may be determined graphically for a range u from -4.0 to $+5.0$ to something like three significant figures. It is interesting to note that

$$(4.6) \qquad w = E_0(-h).$$

The results of this section may be summarized in

THEOREM 2. *For any member of the h-system of life tables, if $E(x)$ denotes the ratio of future life expectancy at age x to the service life L, $E(x)$ is given by the general formula* (4.5) *where the constants w, h, and L depend upon the particular life table used, and the service life L that applies* (*see Table 1 and Figure 3*). *The function $E_0(u)$ may be determined graphically from Figures 6a, 6b, and 6c.*

Example (with charts of Figure 5 double size and intervening scale lines filled in): For life table $h = 1.35$, with service life of 24 years, compute the life expectancy ratio at age 40.5.

The value of w can be found either from Figure 6a from the relation

$$w = E_0(-h)$$

or by computing $H_0 = \phi_0/\Phi_0$ and the relation

$$w = h + H_0.$$

For purposes of illustration let us use both methods and compare the accuracy of the results.

From Figure 6a we find $w = 1.35 + 0.175 = 1.525$.
From computation using Tables of Probability Functions,

$$\phi(-1.35) = 0.1603833, \quad \Phi(-1.35) = 0.9114920, \quad \text{giving}$$
$$H_0 = 0.17596, \quad w = 1.52596.$$

With w taken from chart ($w = 1.525$):

$$u = (1.525)(40.5)/24 - 1.35 = 1.223, \quad \text{and from Figure 6b}$$
$$E_0(u) = 0.483, \quad E(x) = E_0(u)/w = 0.3167.$$

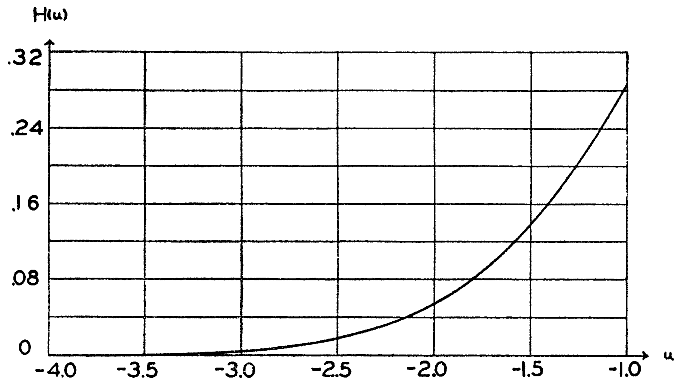

FIGURE 6a.—Generalized life-expectancy function $E_0(u)$, $-4.0 < u < -1.0$.
Note: To find $E_0(u)$ from chart add vertical reading $H(u)$ to $-u$ since
$E_0(u) = -u + H(u)$

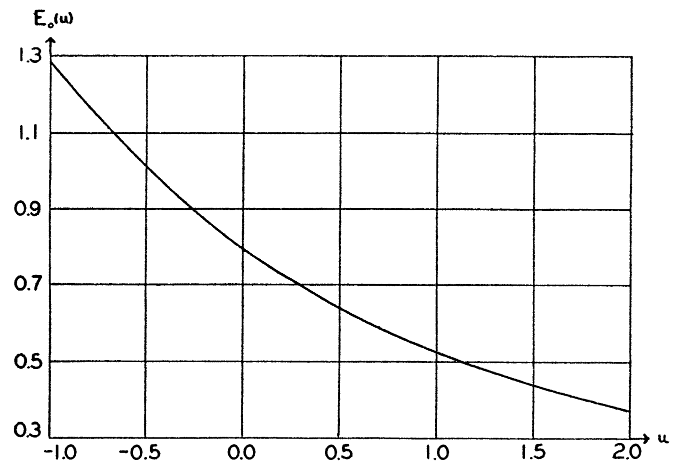

FIGURE 6b.—Generalized life-expectancy function $E_0(u)$, $-1.0 < u < +2.0$.

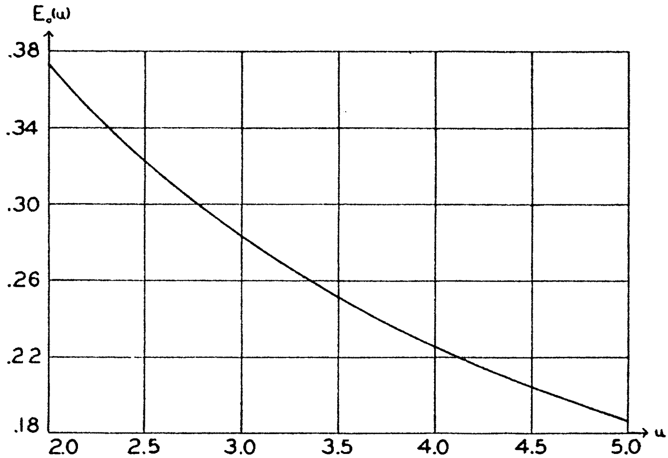

FIGURE 6c.—Generalized life-expectancy function $E_0(u)$, $+2.0<u<+5.0$.

With w computed ($w=1.52596$):

$$u = (1.52596)(40.5)/24 - 1.35 = 1.225, \quad \text{and from Figure 6b}$$

$$E_0(u) = 0.482, \quad E(x) = E_0(u)/w = 0.3159.$$

By direct computation without the use of the chart, $E(x)=0.3164$.

5. Discussion of Separation of the Retirement Frequency Function into Two Parts, One of Which Represents Retirements Due to Chance Causes, and the Other of Which Represents Retirements Related to Age

Write the retirement frequency function as

$$f(t) = f(0)M(t) + f_a(t), \qquad L = 1. \tag{5.1}$$

The term $f(0)M(t)$ represents the part of the retirement function due to retirements *independent* of age, for at a given age t there are survivors to the amount $M(t)$ and the ratio

$$f(t)/M(t) = f(0) + f_a(t)/M(t) \tag{5.2}$$

represents the "force of mortality" or retirement rate at age t. Thus the constant term in (5.2) gives the chance retirement rate. Integrating the term $f(0)M(t)$ from $t=0$, to $t=\infty$ we note that the result is $f(0)$. Hence *the proportion of total retirements from a placement of property whose retirement frequency function is $f(t)$, over the complete life of that property, will be $f(0)$.* (The above italicized statement can be easily shown to hold for any service life, provided that $f(t)$ is the generalized retirement frequency function, see Ref. [1], Section 10.)

By the use of (3.4) and (3.6) it follows from (5.1) that

$$f_a(t) = f(t) - wH_0\Phi(wt - h)/\Phi_0, \qquad L = 1, \tag{5.3}$$

where $f_a(t)$ denotes that part of the frequency function determined by causes of retirement that are related to age. The derivative of $f_a(t)$, by use of (3.2) and (3.3), reduces to

$$f_a'(t) = (w^3/\Phi_0)(1 - t)\phi(wt - h). \tag{5.4}$$

Thus the mode of $f_a(t)$ lies at $t=1$. The mean and median can be shown to lie to the right of this point.

Hence *the generalized frequency function $f(t)$ of the h-system of life tables can be thought of as composed of two parts, one of which reflects only chance retirements, given by $f(0)M(t)$, and the other of which is a left-moded curve of normal character with mode at $t=1(=100$ per cent of average life).*

Finally, with the chance retirement rate equal to $f(0)$, it can be shown that $f(0)$ is a monotonically increasing quantity as h moves from $+\infty$ to $-\infty$. At $h=-\infty$ it is proved in the appendix that the retirement frequency function becomes identical with the exponential curve. Furthermore, as is well known, the exponential curve represents a retirement frequency function for which all causes of retirement are independent of age (since the force of mortality is in this case a constant). Thus

THEOREM 3. *The generalized retirement frequency function of the h-system of life tables is such that as h increases over positive values, the retirement curve tends towards the normal frequency distribution with chance retirements playing a smaller and smaller part. As h decreases towards minus infinity, the chance retirements become more dominant, and the retirement curve takes on less and less of the normal character. In the limit at $h=-\infty$ all retirements are independent of age and the curve takes the form $y=e^{-t}$. As h increases from $-\infty$ to $+\infty$ the variance of the generalized frequency curve decreases monotonically from unity to zero.*

6. SUMMARY AND CONCLUSION

A system of life tables called the "h-system" is here presented (see Theorem 1). These life tables are based upon a retirement frequency function that is a truncated normal curve. The fact that this curve is truncated at the left does not invalidate the use of such life tables in practice. It rather increases their validity since it allows for chance retirements unrelated to age (Section 5 and Theorem 3).

A rough method for the selection of the member of the h-system that is the best fit to an observed life table is presented (Figures 4 and 5).

A general formula and charts for obtaining the life expectancy at any age, for any member of the system, are given (Theorem 2 and Figures 6a, 6b, and 6c).

Although the h-system of life tables is of course not applicable to all cases of property retirements, for purposes of the general consideration of the behavior of property retirements in the broader aspects of the problem it is very useful to have such a system of life tables available in simple mathematical form. Tests of this system against several hundred life tables based on actual experience of utility property studied in the Bureau of Valuation of the New York Commission indicate very close agreement.

The author has found it possible to set up a general formula from which it is not too difficult to compute the renewal function of any member of the h-system of life tables (see Ref. [1]), Section 4; and Refs. [9] and [10]). This will be offered for publication at a later date.

Appendix

Proof that the Limiting Life Table of the h-System as h Approaches Minus Infinity is the Exponential Curve $y=e^{-t}$

We first inquire as to the behavior of w and H_0 as h approaches minus infinity. In order to do this we use the asymptotic expansion of the probability integral in the form

$$\Phi(u) = [\phi(u)/u][1 - u^{-2} + 3u^{-4} - 3.5u^{-6} + \cdots], \quad u > 0,$$

which is a semi-convergent series. By division we obtain

$$\phi(u)/\Phi(u) = u + u^{-1} - 2u^{-3} + 10u^{-5} - \cdots, \quad u > 0, \tag{A.1}$$

which beyond the second term is an alternating semi-convergent infinite series.

Setting $u=-h$, where h is negative, and recalling the definitions of w and H_0 we see that

$$\lim_{h\to-\infty} w = 0, \qquad \lim_{h\to-\infty} wh = -1, \qquad \lim_{h\to-\infty} wH_0 = +1. \tag{A.2}$$

We shall now seek to show that the retirement frequency function $f(t)$ approaches as its limit e^{-t} as h approaches minus infinity. It will then follow that the life table will also be of this form. To that end write

$$f(t) = w\phi(wt-h)/\Phi_0 = wH_0\phi(wt-h)/\phi_0,$$

and by virtue of (A.2) it will be sufficient to show that

$$\text{(A.3)} \qquad \lim_{h \to -\infty} [\phi(wt - h)/\phi(-h)] = e^{-t}.$$

Expanding $\phi(wt-h)$ in power series about $(-h)$, we have

$$\text{(A.4)} \qquad \begin{aligned} \phi(wt - h)/\phi_0 = 1 &+ [w\phi'(-h)/\phi_0]t \\ &+ [w^2\phi''(-h)/\phi_0]t^2/2! + \cdots . \end{aligned}$$

From the properties of $\phi(u)$ it is clear that

$$\phi^{[r]}(-h)/\phi_0 = h^r + \text{terms involving lower powers of } h.$$

Hence from (A.2)

$$\text{(A.5)} \qquad \lim_{h \to -\infty} w^r\phi(-h)\ \phi_0 = \lim_{h \to -\infty} (wh)^r = (-1)^r.$$

Thus the $(r+1)$th term of the series (A.4) has the limit $(-1)^r t^r/r!$ Hence each term of the series (A.4) approaches the corresponding term of the series expansion of e^{-t}. By the use of Taylor's series with a remainder it is not difficult to show that the series (A.4) is uniformly convergent for large negative values of h including minus infinity. Hence the relation (A.3) follows.

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