Time Value of Money

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Time Value of Money Goals

- Create a logical basis for comparing costs and benefits in different time periods.
- Introduce a common notation that engineers, and many economists use, for these calculations.
- Set the basis for:
  - Loans
  - Investment criteria

Assumptions

These are demonstrably false but give us tractable methods.

- Costs and benefits of equal size have equal value in all time periods. *Can evaluate without worrying about wealth or taste changes.*
- The value of costs and benefits is independent of costs and benefits in other time periods. *No habit, addiction or hangovers.*
- Benefits offset costs. *Ever fought a parking ticket?*
- Future values are known with certainty. *Math is easy*

What it looks like

$$\sum x_n d(n)$$

- $d(n)$ is a discounting function.
- Many are possible but only a few are time consistent.
  - $d(n) = \frac{1}{(1+r)^n}$ for discrete time.
  - $d(n) = \frac{1}{e^{rn}}$ for continuous time.

Example of Time Inconsistency

Back in the day I did this in class with real beer. Now, it's a policy violation.

- Chose 6-pack of beer a month and a day from now or 1 beer a month from now.
- A beer right now or a 6-pack tomorrow.
- Many of you changed your mind.

Integer time

- Will say "Time 1", "Time zero" because "1st time period is confusing"
- "Now" means time zero.
- Intervals are half open on the right, e.g., $[0, 1)$
- Jan 1, 1908 and Dec 31, 1908 are in the same period, but Jan 1, 1909 is not, when the unit of analysis is a year.

Notes on the interest rate

- We will treat the interest rate as a simple thing until loans.
- The rate is always just the per period rate.
- Later we will get into:
  - The nominal period.
  - Compounding period.
  - Nominal statement.
  - Effective interest rates.

Time Consistent Integer Time Discounting

$$F = P(1+i)^N$$

Symbolic notation depends on context.

- i = The interest rate
- P = Present worth *or* some value in time zero.
- N = N time periods from Now *or* Time N
- F = Future Value *or* Present Value in time N *or* A value in time N.

We will have a full list later.

Easy Future Value Calculation

"If you deposit \$26 into an account that earns 2% a month, how much will be in the account after 500 months?"

$$F = 26(1 + .02)^{500} = 518870.7975218$$

- "how much will be in the account after" is a good signal for future value.
- 100ths of a percent is more than adequate for calculations.
- Usually, cents is close enough.
- D2L questions have a tolerance.
- Most common mistakes:
  - $26(1 + .02)^{500/12}$
  - $26(1 + .2)^{500}$

Easy Present Worth Calculation

"How much would you have to deposit now into an account that earns 10% per year to have 100K in 10 years?"

$$P = \frac{100K}{(1+.1)^{10}} = 38.5543289K$$

- "How much would you have to deposit now" is a present worth indicator.

Graphical Notation

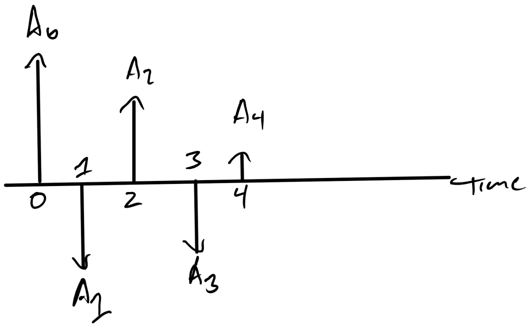


Figure 1:

Comments on Graphical Notation

- Up vectors are cash inflow.
- Down are cash outflow.
- Magnitude is size of cash flows.
- Cash flow in any period is A_n.

Symbols Used in Symbolic (Factor) Notation

- A_n = Cost or benefit in time n
- n = Arbitrary time period
- N = Usually but not always the last time period
- P = Present worth in time zero
- P_n = Present worth in time n.
- A = Used in constant series to indicate same value for time 1 to N.
- G = Used in linear gradient series to indicate the change in cash flow from time 1 to N.
- g = Used in geometric gradient series to indicate the percent change in cash flow from time 1 to N.

What is factor notation?

It is a functional representation of common cash flow patterns:

- Constant Series, $A_{n+1} = A_n, \forall n = 1 \ldots N$
- Linear Gradient, $A_{n+1} = A_n + G, A_1 = 0, \forall n = 1 \ldots N$
- Geometric Gradient, $A_{n+1} = A_n(1 + g), \forall n = 1 \ldots N$

There are other notations in use, e.g. actuarial notation.

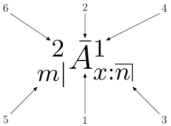


Figure 2: By Delimata - Own work, GFDL,
https://commons.wikimedia.org/w/index.php?curid=4312450

Factor Notation

- It is a short-hand for the closed form representations of present and future worth for the common patterns
- Example

$$10(P|A, i = 10\%, 5) = 10\left[\frac{(1+.1)^5 - 1}{.1(1+.1)^5}\right] = 37.9078677$$

The things in the parentheses are a substitute for the things in the square brackets.

Mostly overkill for this class but . . .

- In the real world the problems are harder. http://web.pdx.edu/~woodsj/Teaching/PDC_PSU_revised%20final%20report_07-2009.pdf (Page 90).
- Explains the shapes better than the algebraic expression.
- Factor notation makes it easier to do parametric studies, i.e., Spider Graphs.

Sample Spider Graph

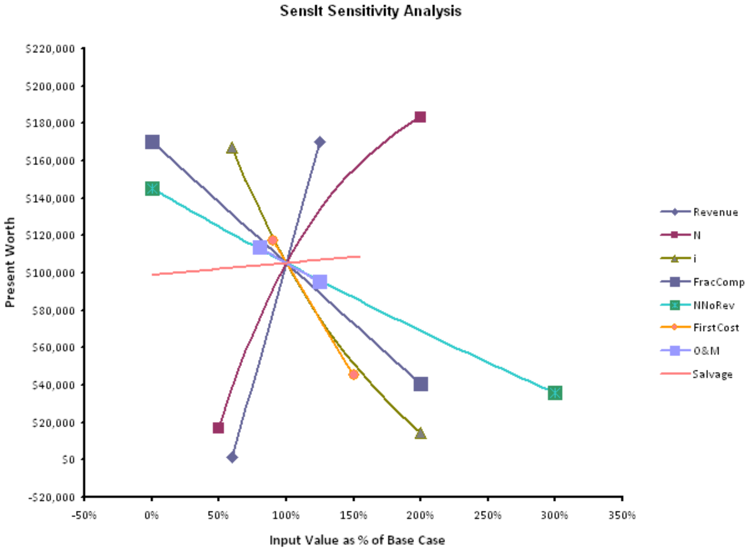


Figure 3:

The Key

$$Looking\ For = Number(Looking\ For|What\ is\ number, i, How\ Long)$$

- A big list is here http://ec314-pdx-edu.wikidot.com/annotated-equations-sheet
- Basic Present Worth Examples for Common Shapes
  - $P = 3(P|A, i = 10\%, 5)$
  - $P = 2(P|G, i = 10\%, 4)$
  - $P = 10(P|A_1, i = 10\%, g = 100\%, 10)$
- Hint, if you know $(P|X)$, $(F|X) = (P|X)(1 + r)^N$

Step Through For Basics

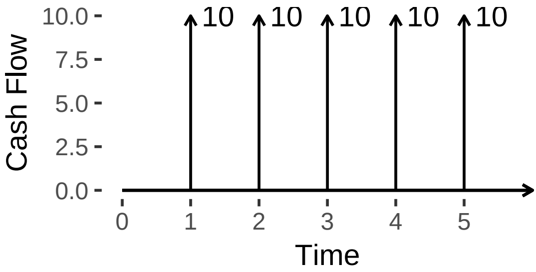
$$Looking\ For = Number(Looking\ For | What\ is\ number, i, How\ Long)$$

- $P = How\ tall(P|A, i, How\ Long)$
- $P = Change\ From\ Last(P|G, i, Non-zero + One)$
- $P = Time\ 1\ Value(P|A_1, i, Growth\ Rate, How\ Long)$

Wait, closed form notation?

- Remember the tricks from sequences and series? Many work for finite sequences.
- Closed form has advantage over brute force (We do that next) in that if you mess it up, it looks stupid.
- There are also tables of pre-calculated values which are useful for the PE exam.

Present Worth Three Ways



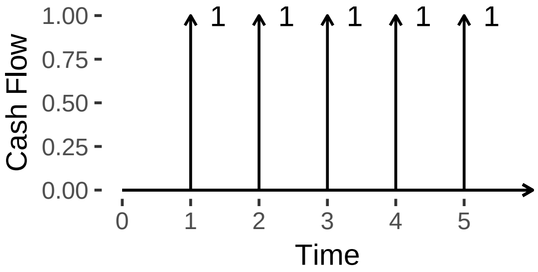
- Brute force: $\sum_{n=1}^{5} \frac{10}{(1+.1)^n}$
- Tabular: $10 * 3.7908$
- Closed form: $10(P|A, i = 10\%, 5) = 10\frac{(1+.1)^5-1}{.1(1+.1)^5}$
- Either way you should get 37.91.

Comments on Three Methods

- Brute force always works but when you make a mistake it hides.
- Now try $10(P|A, i = .1, 300)$ by brute force.
- Tables don't exist for all interest rates but it was way faster on the PE, which is a speed test.
- Tables have limited digits. Don't use them for large values
- Closed form looks stupid if you have made a mistake, but easy to make a mistake.

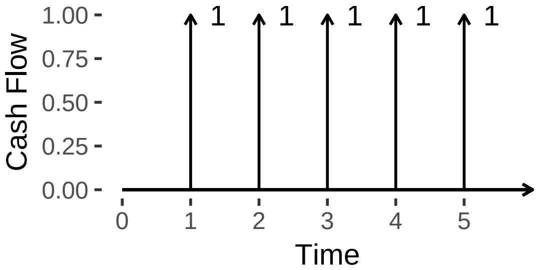
I tend to use brute force when it is five or fewer values.

Easy Cash Flow to Factor Notation



Find representation of Present Worth in factor notation

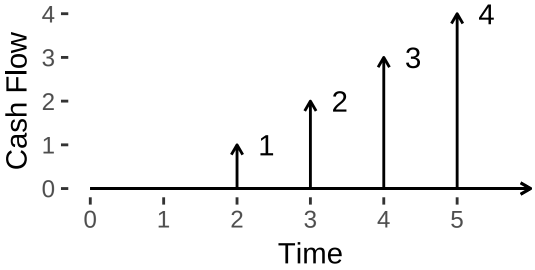
Answer



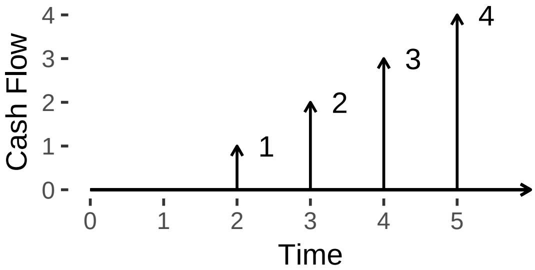
- Shape is constant series
- Length is 5, which you get by counting the number of non-zero elements.
- Height is 1, that the the A you are looking for.

$$1(P|A, i, 5)$$

Try this



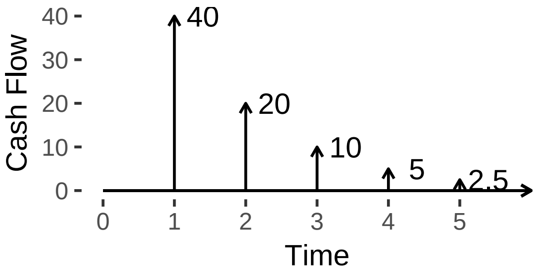
Linear Gradient is Tricky



- Shape is linear gradient
- Length is 5, count non-zeros and 'add one for the pot'. If zero is the start, period two is the first time you see a non-zero value.
- G, is 1, how much it changes by from period to period

$$1(P|G, i, 5)$$

The Geometric Gradient is for growth.



$$40(P|A_1, i, g = -50\%, 5)$$

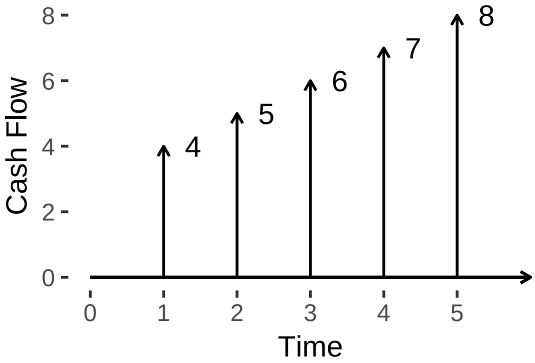
Geometric Gradient

- Pretty common in the wild.
- I tend to give easy g values, 100%, -50%, 10%, that are easy to spot graphically.
- Two closed form equations.

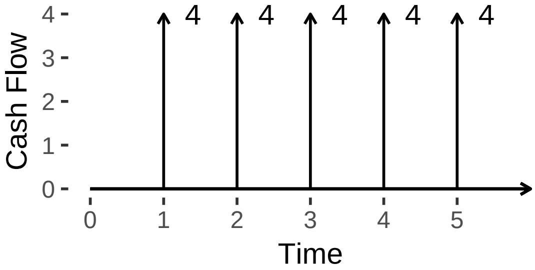
$$P = A_1 \left(\frac{1 - (1+g)^N (1+i)^{-N}}{i - g} \right)$$

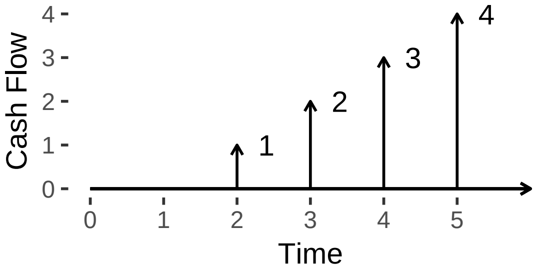
$$P = A_1 \left(\frac{N}{1+i} \right) (if : i = g)$$

Now lets put two things together



You should see this as





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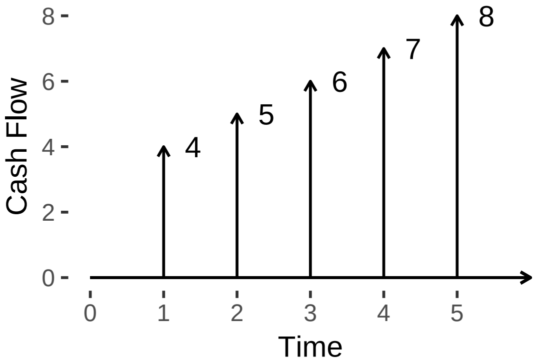
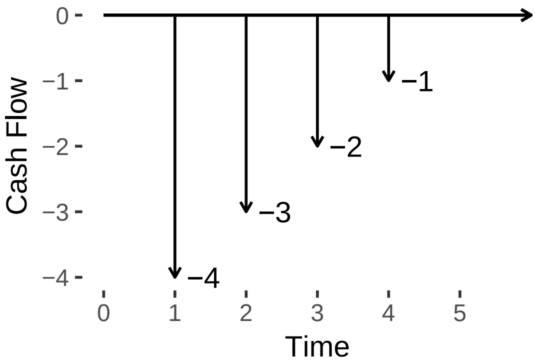


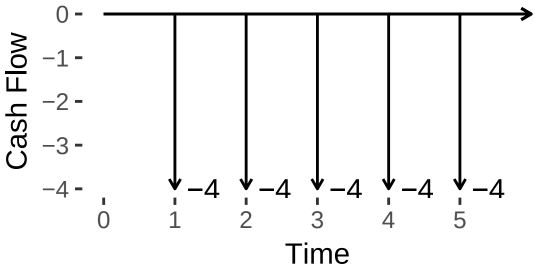
- See the constant series that is 4 high and 5 long $4(P|A, i, 5)$
- See the linear gradient that has G of 1 and is 5 long $1(P|G, i, 5)$

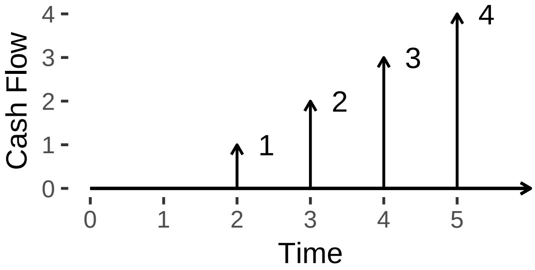
$$4(P|A, i, 5) + 1(P|G, i, 5)$$

How about this. Single byte change



You should see this





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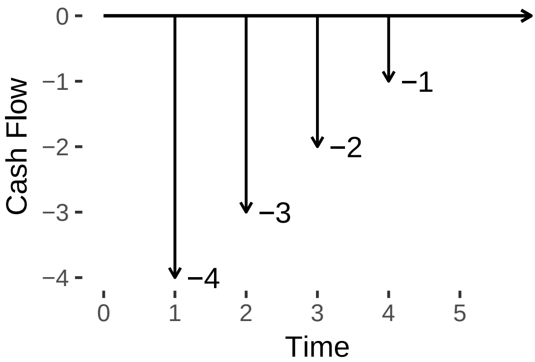
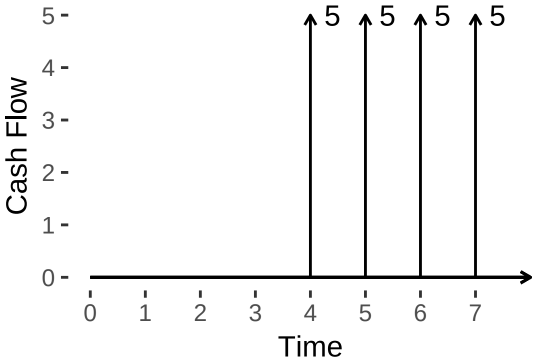


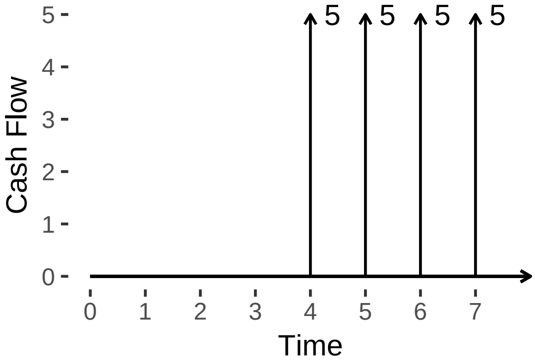
- See the constant series that is -4 high and 5 long $-4(P|A, i, 5)$
- See the linear gradient that has G of 1 and is 5 long $1(P|G, i, 5)$

$$-4(P|A, i, 5) + 1(P|G, i, 5)$$

Try this with only addition and subtraction



Should get this



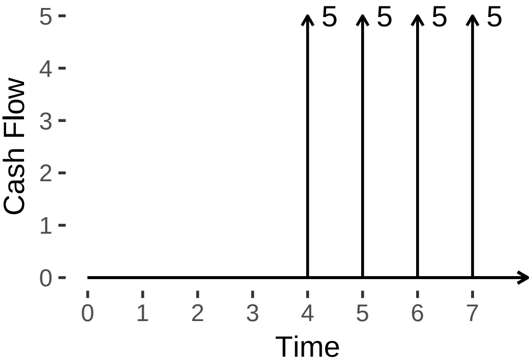
$$5(P|A, i, 7) - 5(P|A, i, 3)$$

Time Shifting is easier

$$P = \frac{F}{(1+i)^N}$$

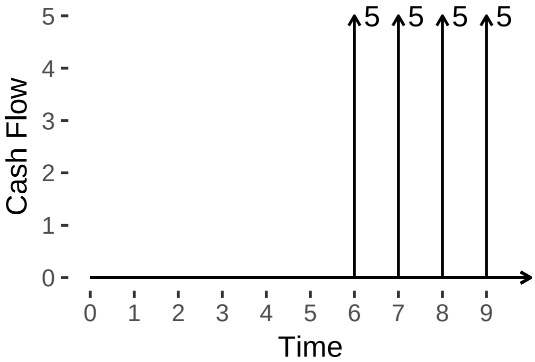
- You can replace the F with anything to move the common shapes around.

Last example as time shift



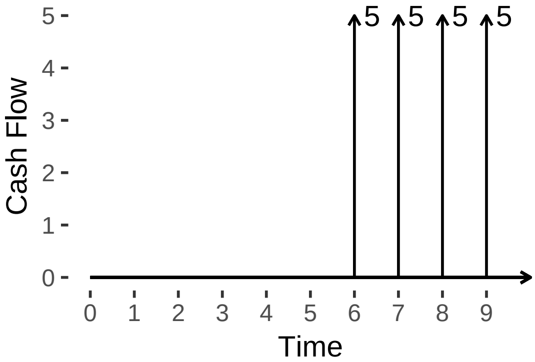
$$\frac{5(P|A, i, 4)}{(1+i)^3}$$

Further out



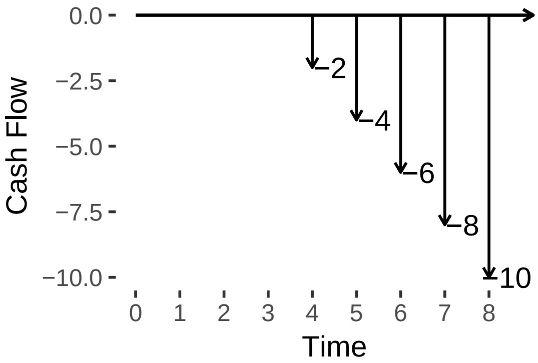
$$\frac{5(P|A,i,4)}{(1+i)^5}$$

Two ways of thinking about time shifting

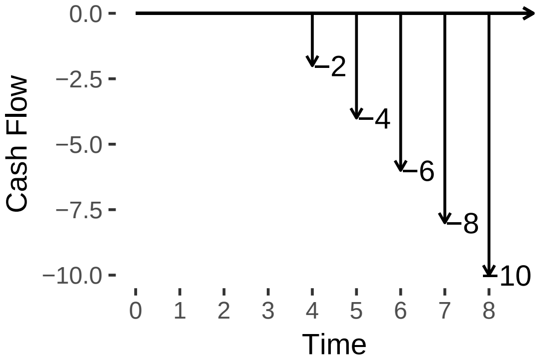


- $\frac{5(P|A,i,4)}{(1+i)^5}$
- The the exponent is the time period just to the left of the series.
- The sum of the N and the exponent is equal to the last period you see the series.

Tricky one



You should see

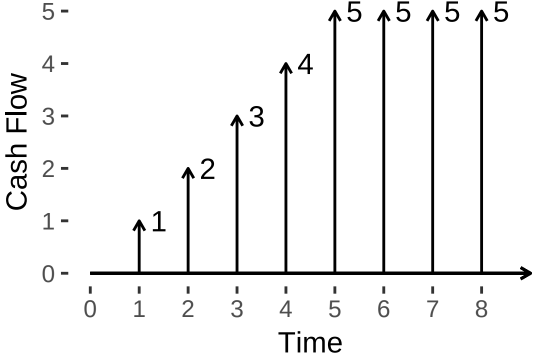


$$\frac{-2(P|G,i,6)}{(1+i)^2}$$

Warnings

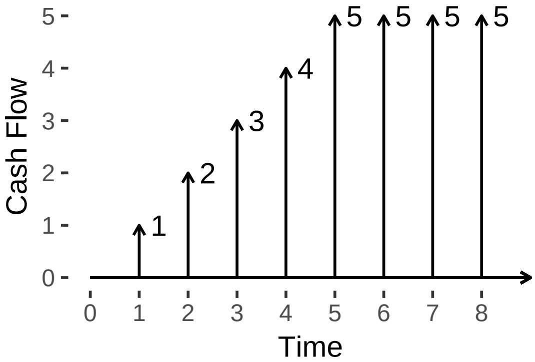
- There is no unique factor representation of a cash flow.
- Some are easier to spot than others.
- Usually the story tells you the shape.
  - Loan payments look like $(P|A)$
  - Accelerator clauses look like $(P|A_1)$

Same Problem Several Ways



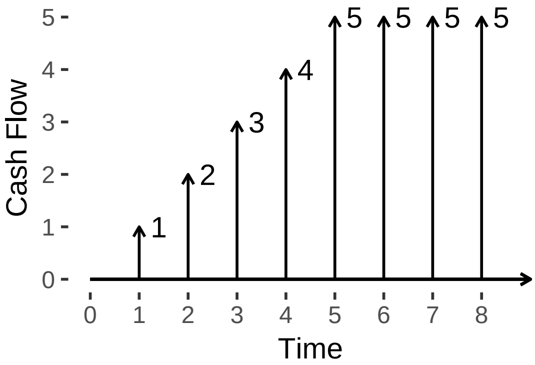
Try it with the linear gradient ending at time 4 and the constant series going from 5 to 8.

How did you do?



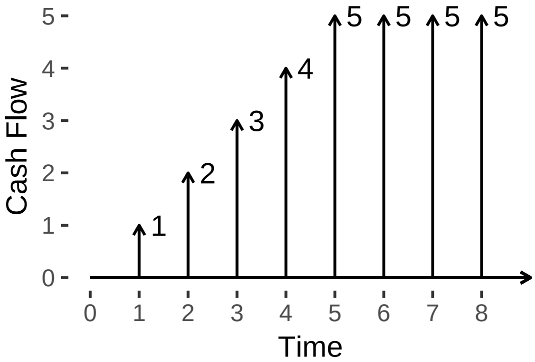
$$\frac{1(P|G,i,5)}{(1+i)^{-1}} + \frac{5(P|A,i,4)}{(1+i)^{4}}$$

Again



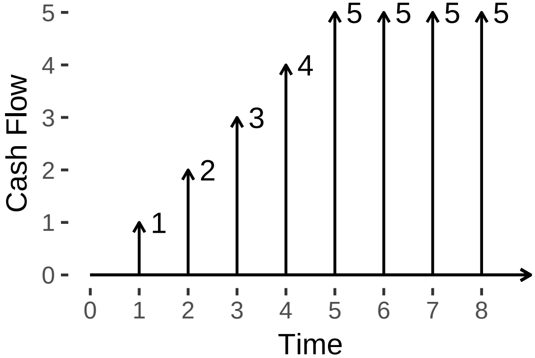
Try it with the linear gradient ending at time 5 and the constant series going from 6 to 8.

How did you do?



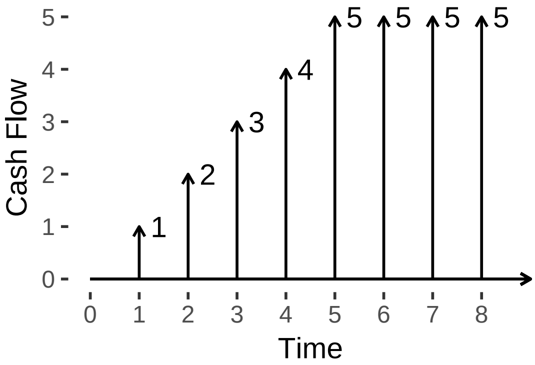
$$\frac{1(P|G, i, 6)}{(1+i)^{-1}} + \frac{5(P|A, i, 3)}{(1+i)^5}$$

Try the civil engineer's solution



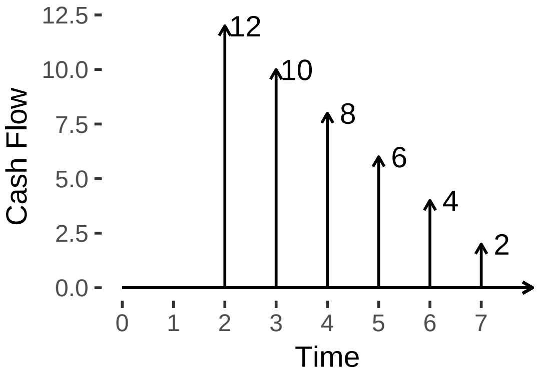
Stack constant series like a layer cake. There is one that is one high from period 1 to 8. Another that is one high from period 2 to 8 and so on.

Civil engineer's solution



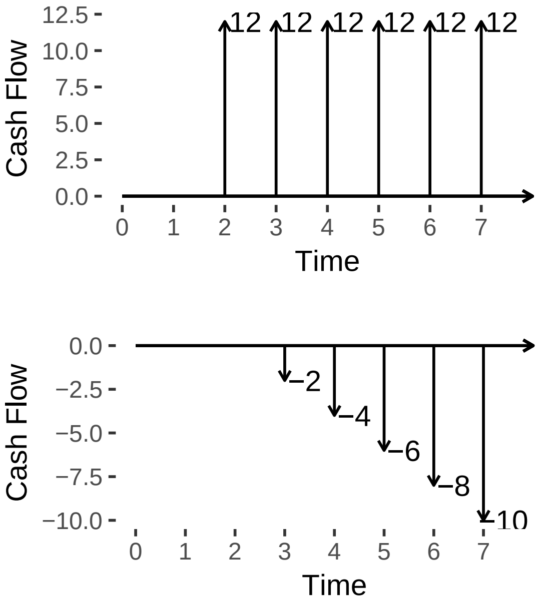
$$\frac{1(P|A,i,8)}{(1+i)^0}+\frac{1(P|A,i,7)}{(1+i)^1}+\frac{1(P|A,i,6)}{(1+i)^2}+\frac{1(P|A,i,5)}{(1+i)^3}+\frac{1(P|A,i,4)}{(1+i)^4}$$

How about this?



Hint: $-2(P|G, i, 7)$ is wrong in many ways.

You should see it as



Solution

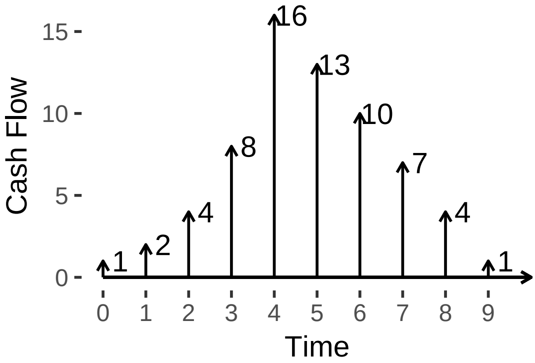
$$\frac{12(P|A,i,6)}{(1+i)^1} + \frac{-2(P|G,i,6)}{(1+i)^1}$$

- All Linear gradients start at zero. If they don't you have to include a constant series.
- Work by analogy from slope intercept form of a line.
  - $Y = MX + b$
  - $Y = (P|G) + (P|A)$

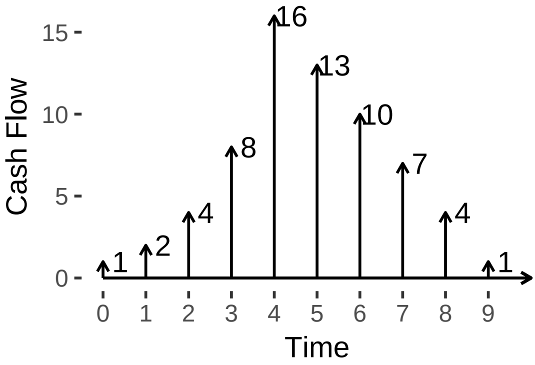
Review

- Look for shapes constant, geometric gradient and linear gradient.
- Count for the N. Remember to add one for the pot *only* on linear gradient.
- Sum of N and exponent is equal to the time period where you last see the cash flow
- Work the analogy with slope intercept form of a line.

Big Test

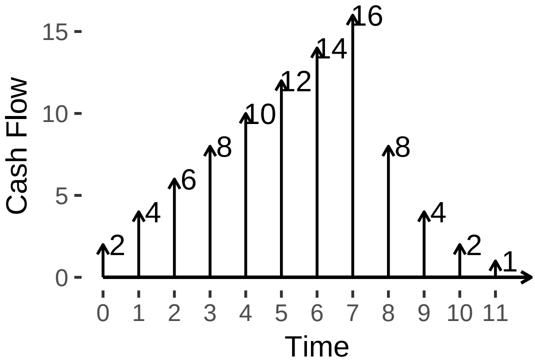


Big Test (A Solution)

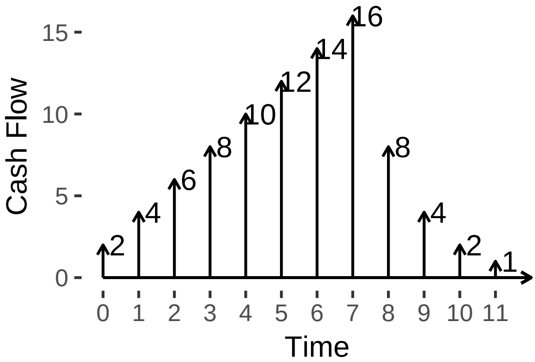


$$\frac{1(P|A_1, i, g = 100\%, 4)}{(1+i)^{-1}} + \frac{16(P|A, i, 6)}{(1+i)^3} - \frac{3(P|G, i, 6)}{(1+i)^3}$$

Similar problem



Similar problem (Solution)

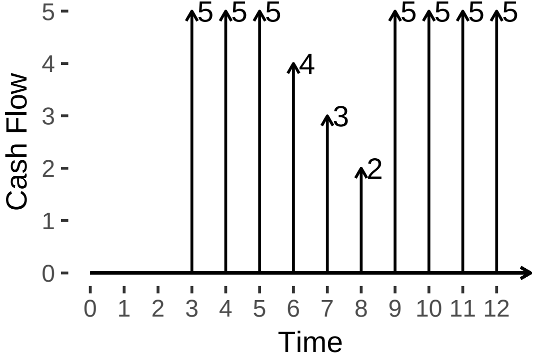


$$\frac{2(P|G,i,8)}{(1+i)^{-2}}+\frac{16(P|A_1,i,g=-50\%,5)}{(1+i)^6}$$

Lets go backwards. Find the intended cash flow for this.

$$\frac{5(P|A,i,10)}{(1+i)^2} - \frac{1(P|G,i,4)}{(1+i)^4}$$

Solution



Hints for Study

- Kill trees. Work lots of problems. Everything after this depends on understanding TVM.
- Drill and Kill the examples and the videos at bottom of the page.
  http://ec314-pdx-edu.wikidot.com/q2:time-value-of-money
- When evaluating these expressions, try to do all the work in the calculator in one operation. Sometimes rounding and truncation errors are bad with large interest rates.
- Go forwards, factor notation to cash flow diagram, and backwards, cash flow to factor.