

# Time Value of Money

James Woods

# Time Value of Money Goals

- ▶ Create a logical basis for comparing costs and benefits in different time periods.
- ▶ Introduce a common notation that engineers, and many economists use, for these calculations.
- ▶ Set the basis for:
  - ▶ Loans
  - ▶ Investment criteria

# Assumptions

These are demonstrably false but give us tractable methods.

- ▶ Costs and benefits of equal size have equal value in all time periods. *Can evaluate without worrying about wealth or taste changes.*
- ▶ The value of costs and benefits is independent of costs and benefits in other time periods. *No habit, addiction or hangovers.*
- ▶ Benefits offset costs. *Ever fought a parking ticket?*
- ▶ Future values are known with certainty. *Math is easy*

## What it looks like

$$\sum x_n d(n)$$

- ▶  $d(n)$  is a discounting function.
- ▶ Many are possible but only a few are time consistent.
  - ▶  $d(n) = \frac{1}{(1+r)^n}$  for discrete time.
  - ▶  $d(n) = \frac{1}{e^{rn}}$  for continuous time.

## Example of Time Inconsistency

Back in the day I did this in class with real beer. Now, it's a policy violation.

- ▶ Chose 6-pack of beer a month and a day from now or 1 beer a month from now.
- ▶ A beer right now or a 6-pack tomorrow.
- ▶ Many of you changed your mind.

# Integer time

- ▶ Will say “Time 1”, “Time zero” because “1st time period is confusing”
- ▶ “Now” means time zero.
- ▶ Intervals are half open on the right, e.g.,  $[0, 1)$
- ▶ Jan 1, 1908 and Dec 31, 1908 are in the same period, but Jan 1, 1909 is not, when the unit of analysis is a year.

# Notes on the interest rate

- ▶ We will treat the interest rate as a simple thing until loans.
- ▶ The rate is always just the per period rate.
- ▶ Later we will get into:
  - ▶ The nominal period.
  - ▶ Compounding period.
  - ▶ Nominal statement.
  - ▶ Effective interest rates.

# Time Consistent Integer Time Discounting

$$F = P(1 + i)^N$$

Symbolic notation depends on context.

- ▶  $i$  = The interest rate
- ▶  $P$  = Present worth *or* some value in time zero.
- ▶  $N$  =  $N$  time periods from Now *or* Time  $N$
- ▶  $F$  = Future Value *or* Present Value in time  $N$  *or* A value in time  $N$ .

We will have a full list later.



## Easy Future Value Calculation

“If you deposit \$26 into an account that earns 2% a month, how much will be in the account after 500 months?”

$$F = 26(1 + .02)^{500} = 518870.7975218$$

- ▶ “how much will be in the account after” is a good signal for future value.
- ▶ 100ths of a percent is more than adequate for calculations.
- ▶ Usually, cents is close enough.
- ▶ D2L questions have a tolerance.
- ▶ Most common mistakes:
  - ▶  $26(1 + .02)^{500/12}$
  - ▶  $26(1 + .2)^{500}$

## Easy Present Worth Calculation

“How much would you have to deposit now into an account that earns 10% per year to have 100K in 10 years?”

$$P = \frac{100K}{(1 + .1)^{10}} = 38.5543289K$$

- ▶ “How much would you have to deposit now” is a present worth indicator.

# Graphical Notation

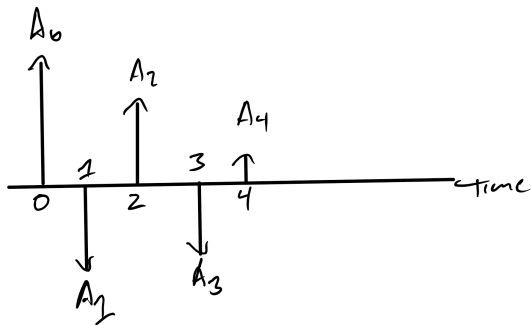


Figure 1:

## Comments on Graphical Notation

- ▶ Up vectors are cash inflow.
- ▶ Down are cash outflow.
- ▶ Magnitude is size of cash flows.
- ▶ Cash flow in any period is  $A_n$ .

## Symbols Used in Symbolic (Factor) Notation

- ▶  $A_n$  = Cost or benefit in time  $n$
- ▶  $n$  = Arbitrary time period
- ▶  $N$  = Usually but not always the last time period
- ▶  $P$  = Present worth in time zero
- ▶  $P_n$  = Present worth in time  $n$ .
- ▶  $A$  = Used in constant series to indicate same value for time 1 to  $N$ .
- ▶  $G$  = Used in linear gradient series to indicate the change in cash flow from time 1 to  $N$ .
- ▶  $g$  = Used in geometric gradient series to indicate the percent change in cash flow from time 1 to  $N$ .

# What is factor notation?

It is a functional representation of common cash flow patterns:

- ▶ Constant Series,  $A_{n+1} = A_n, \forall n = 1 \dots N$
- ▶ Linear Gradient,  $A_{n+1} = A_n + G, A_1 = 0, \forall n = 1 \dots N$
- ▶ Geometric Gradient,  $A_{n+1} = A_n(1 + g), \forall n = 1 \dots N$

There are other notations in use, e.g. actuarial notation.

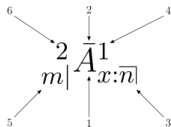


Figure 2: By Delimata - Own work, GFDL,  
<https://commons.wikimedia.org/w/index.php?curid=4312450>

# Factor Notation

- ▶ It is a short-hand for the closed form representations of present and future worth for the common patterns
- ▶ Example

$$10(P|A, i = 10\%, 5) = 10 \left[ \frac{(1 + .1)^5 - 1}{.1(1 + .1)^5} \right] = 37.9078677$$

The things in the parentheses are a substitute for the things in the square brackets.

## Mostly overkill for this class but . . .

- ▶ In the real world the problems are harder.  
[http://web.pdx.edu/~woodsaj/Teaching/PDC\\_PSU\\_revised%20final%20report\\_07-2009.pdf](http://web.pdx.edu/~woodsaj/Teaching/PDC_PSU_revised%20final%20report_07-2009.pdf) (Page 90).
- ▶ Explains the shapes better than the algebraic expression.
- ▶ Factor notation makes it easier to do parametric studies, i.e., Spider Graphs.



# Sample Spider Graph

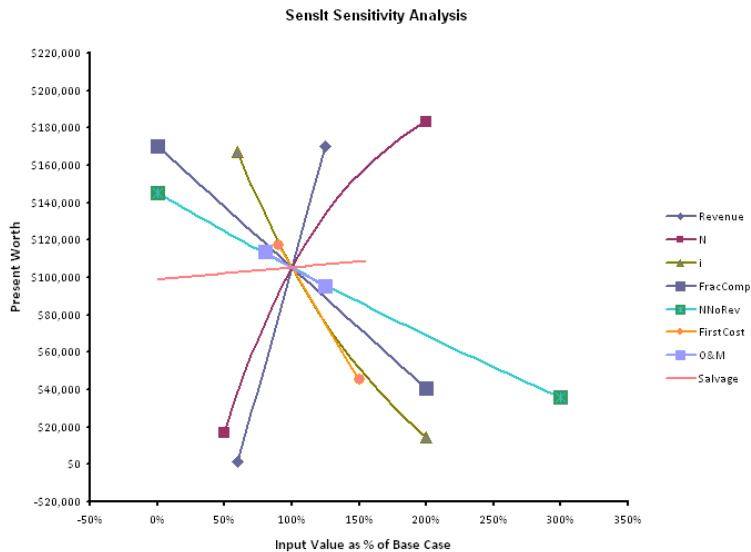


Figure 3:

# The Key

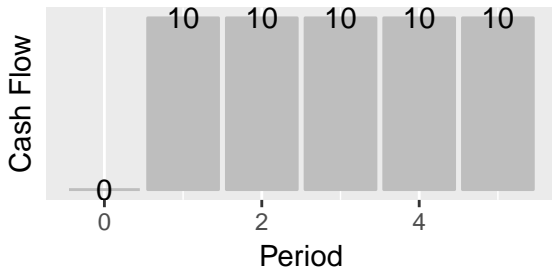
*Looking For = Number(Looking For|What is number, i, How Long)*

- ▶ A big list is here <http://ec314-pdx-edu.wikidot.com/annotated-equations-sheet>
- ▶ Hint, if you know  $(P|X)$ ,  $(F|X) = (P|X)(1 + r)^N$

## Wait, closed form notation?

- ▶ Remember the tricks from sequences and series? Many work for finite sequences.
- ▶ Closed form has advantage over brute force (We do that next) in that if you mess it up, it looks stupid.
- ▶ There are also tables of pre-calculated values which are useful for the PE exam.

## Present Worth Three Ways



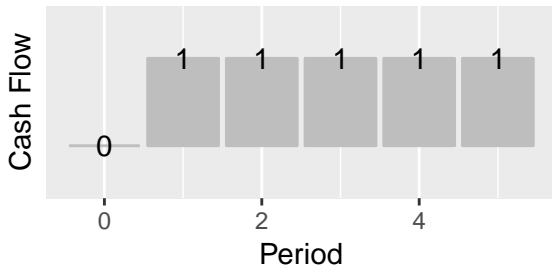
- ▶ Brute force:  $\sum_{n=1}^5 \frac{10}{(1+.1)^n}$
- ▶ Tabular:  $10 * 3.7907868$
- ▶ Closed form:  $10(P|A, i = 10\%, 5) = 10 \frac{(1+.1)^5 - 1}{.1(1+.1)^5}$
- ▶ Either way you should get 37.9078677.

## Comments on Three Methods

- ▶ Brute force always works but when you make a mistake it hides.
- ▶ Now try  $10(P|A, i = .1, 300)$  by brute force.
- ▶ Tables don't exist for all interest rates but it was way faster on the PE, which is a speed test.
- ▶ Closed form looks stupid if you have made a mistake, but easy to make a mistake.

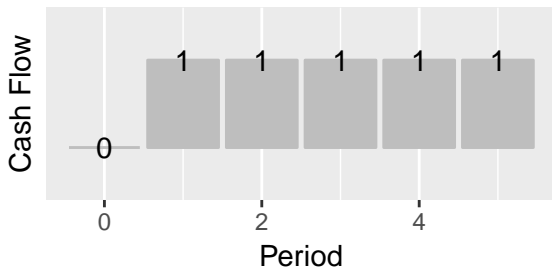
I tend to use brute force when it is five or fewer values.

## Easy Cash Flow to Factor Notation



Find representation of Present Worth in factor notation

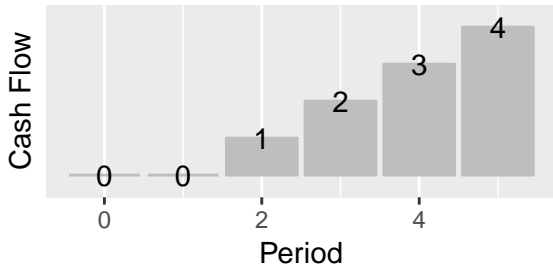
## Answer



- ▶ Shape is constant series
- ▶ Length is 5, which you get by counting the number of non-zero elements.
- ▶ Height is 1, that the the A you are looking for.

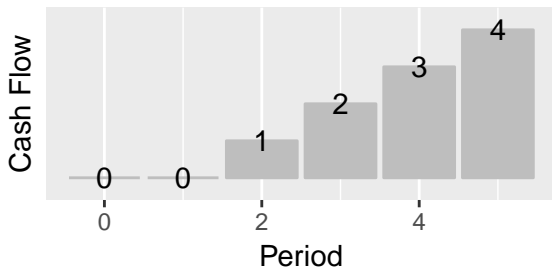
$$1(P|A, i, 5)$$

Try this





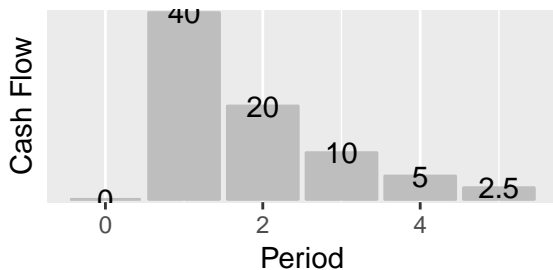
# Linear Gradient is Tricky



- ▶ Shape is linear gradient
- ▶ Length is 5, count non-zeros and 'add one for the pot'. If zero is the start, period two is the first time you see a non-zero value.
- ▶  $G$ , is 1, how much it changes by from period to period

$$1(P|G, i, 5)$$

The Geometric Gradient is for growth.



$$40(P|A_1, i, g = -50\%, 5)$$

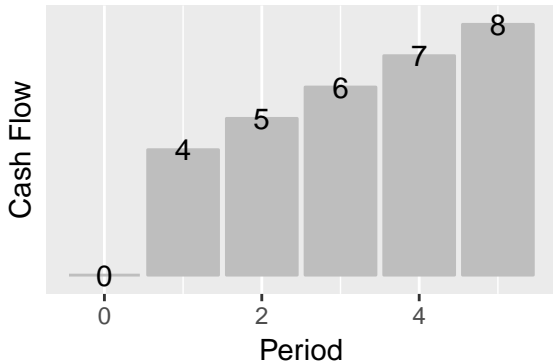
# Geometric Gradient

- ▶ Pretty common in the wild.
- ▶ I tend to give easy  $g$  values, 100%, -50%, 10%, that are easy to spot graphically.
- ▶ Two closed form equations.

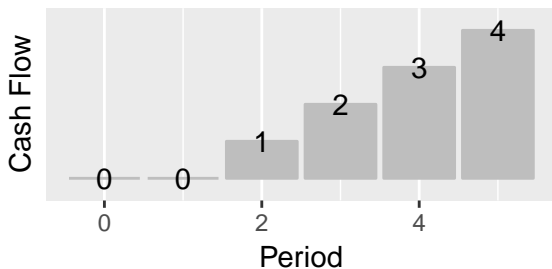
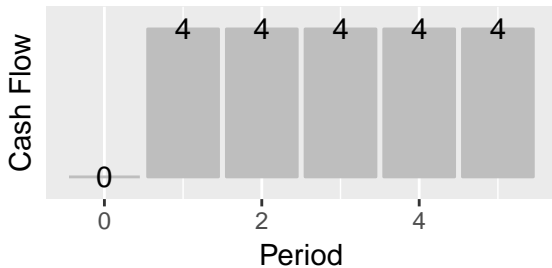
$$P = A_1 \left( \frac{1 - (1 + g)^N (1 + i)^{-N}}{i - g} \right)$$

$$P = A_1 \left( \frac{N}{1 + i} \right) \text{ (if : } i = g \text{)}$$

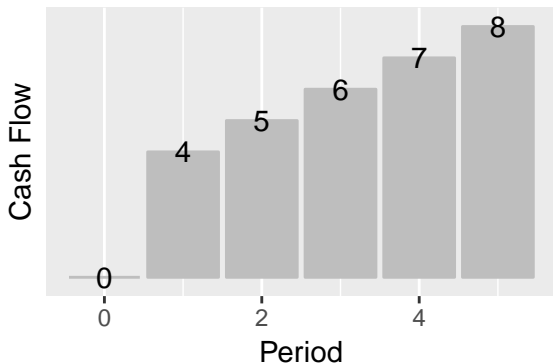
Now lets put two things together



You should see this as



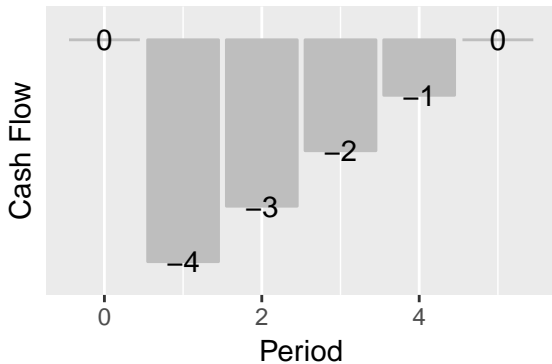
Following from above



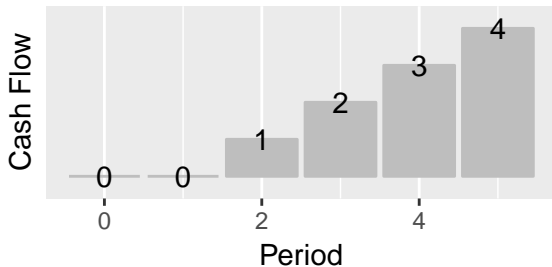
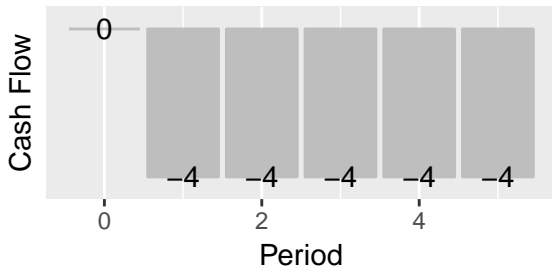
- ▶ See the constant series that is 4 high and 5 long  $4(P|A, i, 5)$
- ▶ See the linear gradient that has G of 1 and is 5 long  $1(P|G, i, 5)$

$$4(P|A, i, 5) + 1(P|G, i, 5)$$

How about this. Single byte change

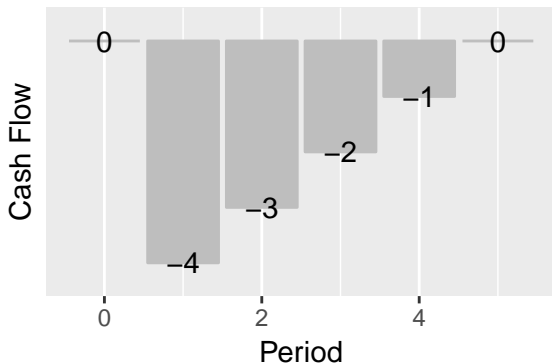


You should see this





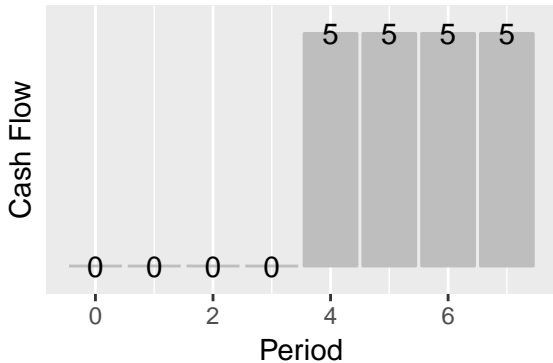
Following from above



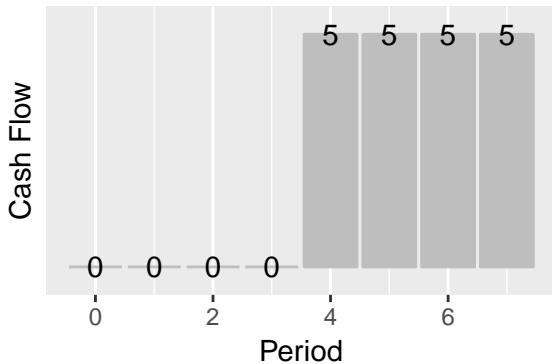
- ▶ See the constant series that is -4 high and 5 long  $-4(P|A, i, 5)$
- ▶ See the linear gradient that has G of 1 and is 5 long  $1(P|G, i, 5)$

$$-4(P|A, i, 5) + 1(P|G, i, 5)$$

Try this with only addition and subtraction



Should get this



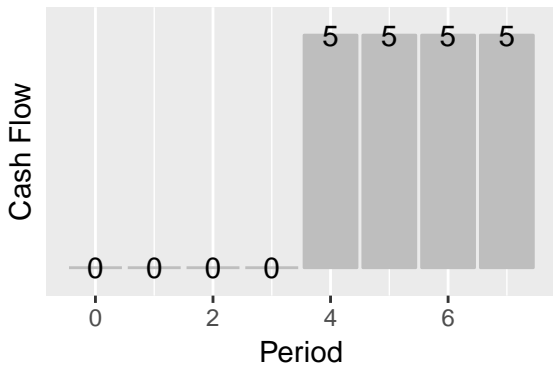
$$5(P|A, i, 7) - 5(P|A, i, 3)$$

## Time Shifting is easier

$$P = \frac{F}{(1+i)^N}$$

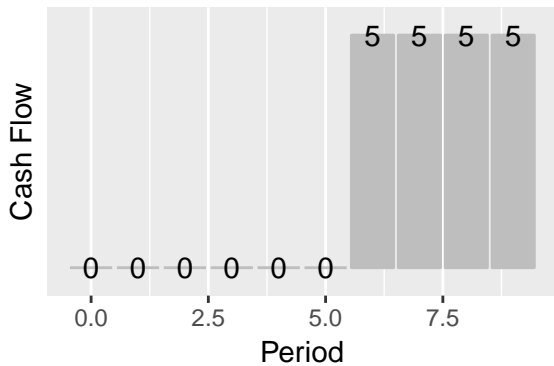
- ▶ You can replace the  $F$  with anything to move the common shapes around.

## Last example as time shift



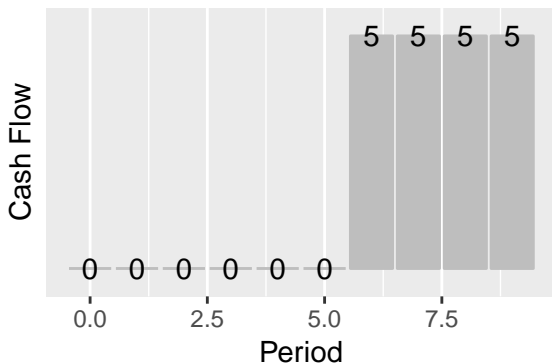
$$\frac{5(P|A, i, 4)}{(1+i)^3}$$

Further out



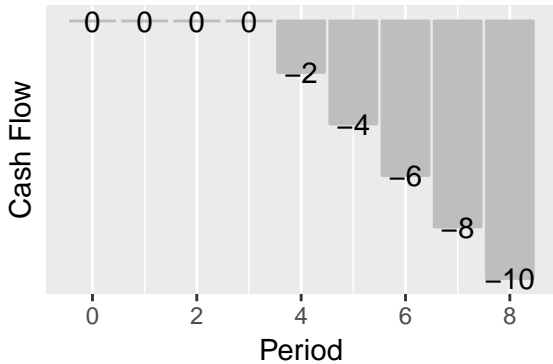
$$\frac{5(P|A, i, 4)}{(1+i)^5}$$

## Two ways of thinking about time shifting



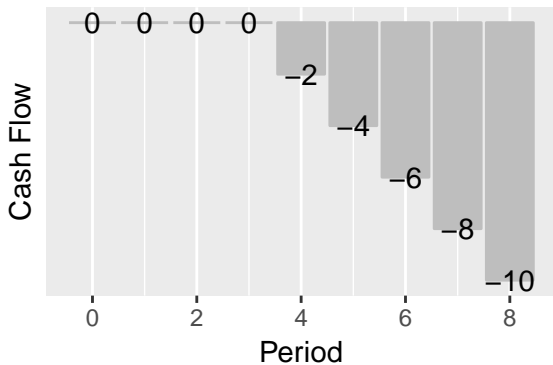
- ▶  $\frac{5(P|A,i,4)}{(1+i)^5}$
- ▶ The the exponent is the time period just to the left of the series.
- ▶ The sum of the N and the exponent is equal to the last period you see the series.

## Tricky one





You should see

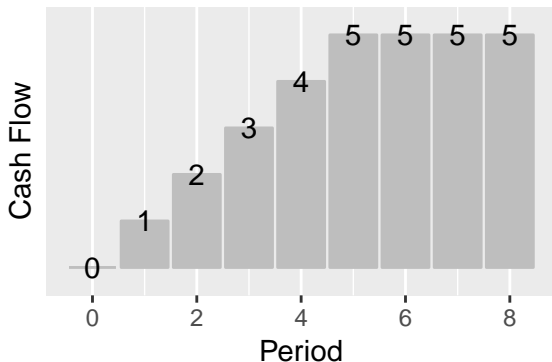


$$\frac{-2(P|G, i, 6)}{(1+i)^2}$$

# Warnings

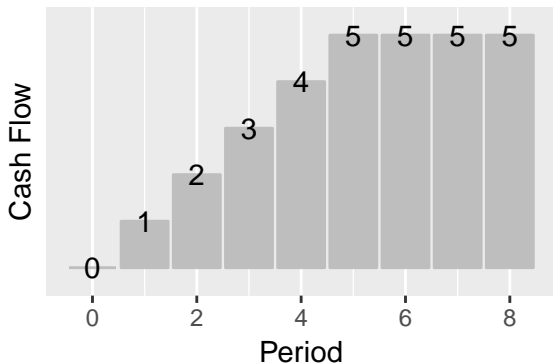
- ▶ There is no unique factor representation of a cash flow.
- ▶ Some are easier to spot than others.
- ▶ Usually the story tells you the shape.
  - ▶ Loan payments look like  $(P|A)$
  - ▶ Accelerator clauses look like  $(P|A_1)$

## Same Problem Several Ways



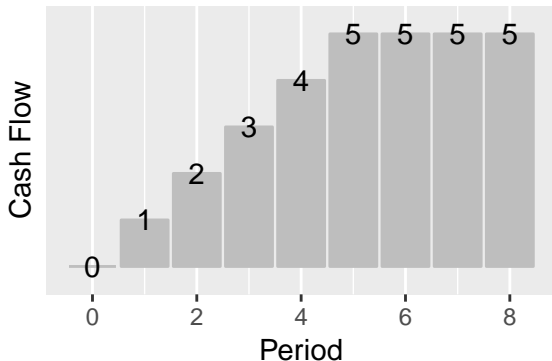
Try it with the linear gradient ending at time 4 and the constant series going from 5 to 8.

How did you do?



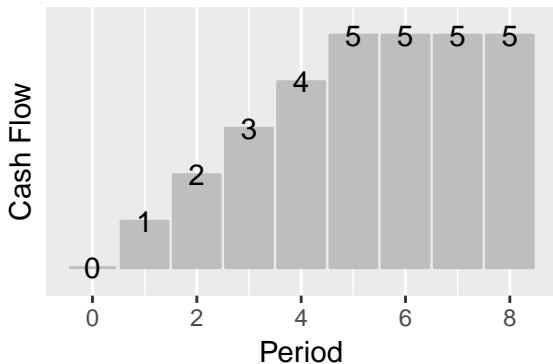
$$\frac{1(P|G, i, 5)}{(1+i)^{-1}} + \frac{5(P|A, i, 4)}{(1+i)^4}$$

Again



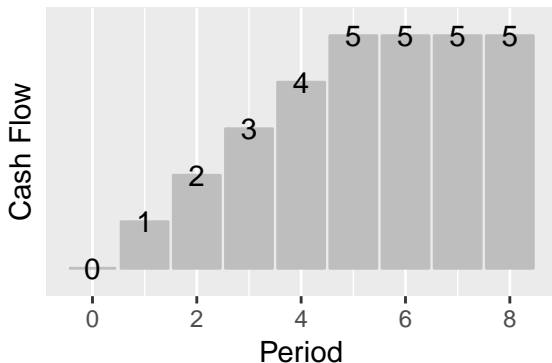
Try it with the linear gradient ending at time 5 and the constant series going from 6 to 8.

How did you do?



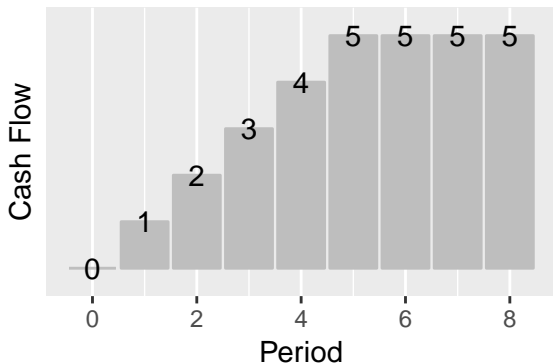
$$\frac{1(P|G, i, 6)}{(1+i)^{-1}} + \frac{5(P|A, i, 3)}{(1+i)^5}$$

## Try the civil engineer's solution



Stack constant series like a layer cake. There is one that is one high from period 1 to 8. Another that is one high from period 2 to 8 and so on.

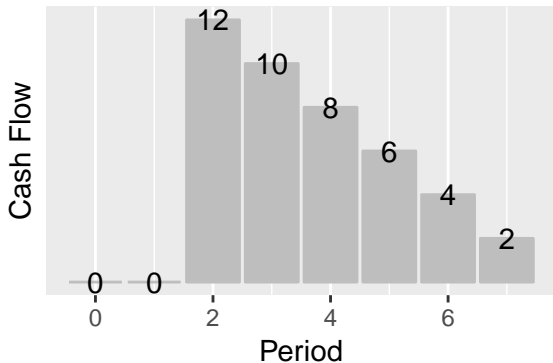
## Civil engineer's solution



$$\frac{1(P|A, i, 8)}{(1+i)^0} + \frac{1(P|A, i, 7)}{(1+i)^1} + \frac{1(P|A, i, 6)}{(1+i)^2} + \frac{1(P|A, i, 5)}{(1+i)^3} + \frac{1(P|A, i, 4)}{(1+i)^4}$$

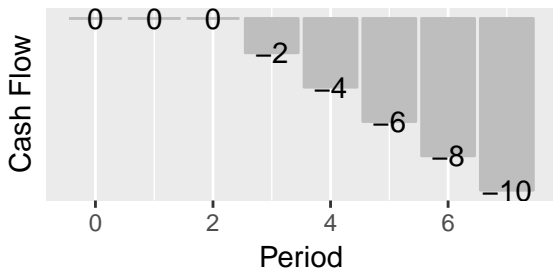
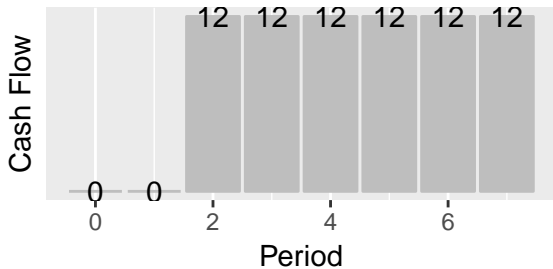


How about this?



Hint:  $-2(P|G, i, 7)$  is wrong in many ways.

You should see it as



## Solution

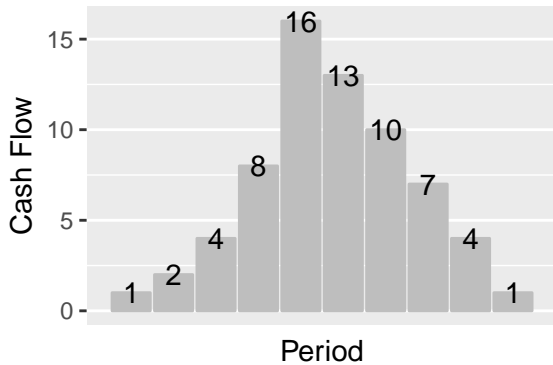
$$\frac{12(P|A, i, 6)}{(1+i)^1} + \frac{-2(P|G, i, 6)}{(1+i)^1}$$

- ▶ All Linear gradients start at zero. If they don't you have to include a constant series.
- ▶ Work by analogy from slope intercept form of a line.
  - ▶  $Y = MX + b$
  - ▶  $Y = (P|G) + (P|A)$

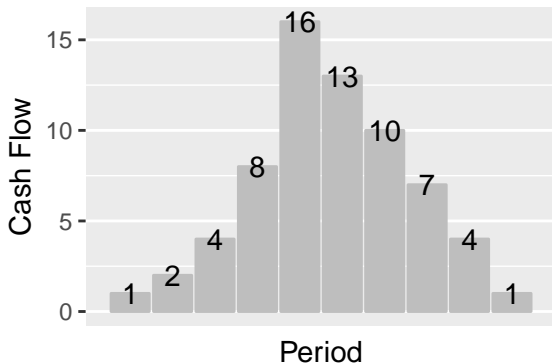
# Review

- ▶ Look for shapes constant, geometric gradient and linear gradient.
- ▶ Count for the N. Remember to add one for the pot *only* on linear gradient.
- ▶ Sum of N and exponent is equal to the time period where you last see the cash flow
- ▶ Work the analogy with slope intercept form of a line.

# Big Test

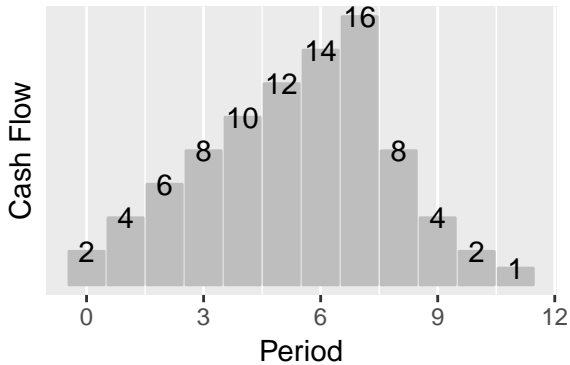


## Big Test ( A Solution)

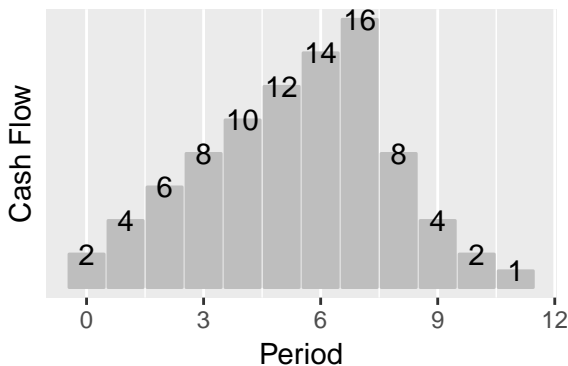


$$\frac{1(P|A_1, i, g = 100\%, 4)}{(1+i)^{-1}} + \frac{16(P|A, i, 6)}{(1+i)^3} - \frac{3(P|G, i, 6)}{(1+i)^3}$$

## Similar problem



## Similar problem (Solution)



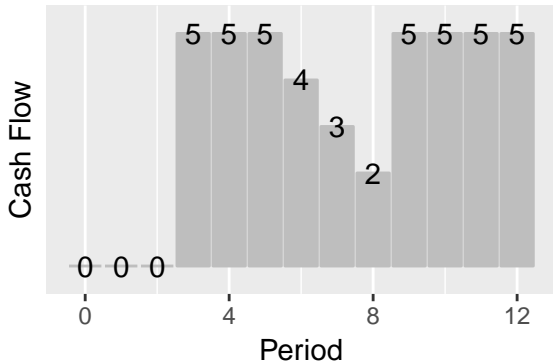
$$\frac{2(P|G, i, 8)}{(1+i)^{-2}} + \frac{16(P|A_1, i, g = -50\%, 5)}{(1+i)^6}$$



Lets go backwards. Find the intended cash flow for this.

$$\frac{5(P|A, i, 10)}{(1+i)^2} - \frac{1(P|G, i, 4)}{(1+i)^4}$$

## Solution



## Hints for Study

- ▶ Kill trees. Work lots of problems. Everything after this depends on understanding TVM.
- ▶ Drill and Kill the examples and the videos at bottom of the page. <http://ec314-pdx-edu.wikidot.com/q2:time-value-of-money>
- ▶ When evaluating these expressions, try to do all the work in the calculator in one operation. Sometimes rounding and truncation errors are bad with large interest rates.
- ▶ Go forwards, factor notation to cash flow diagram, and backwards, cash flow to factor.