Internal Rate of Return

Goals

- Present a few ways of calculating internal rate of return with cautions.
- Explain why the IRR criteria is different for loans and investments.
- Explain the quirks of the exclusive choice procedure.

What is Internal Rate of Return?

It is an interest rate such that the present worth of an asset is zero.

- PW is with known i. IRR has a known PW, zero, and you solve for i.
- It is "an" because multiple internal rates of return are fairly common.

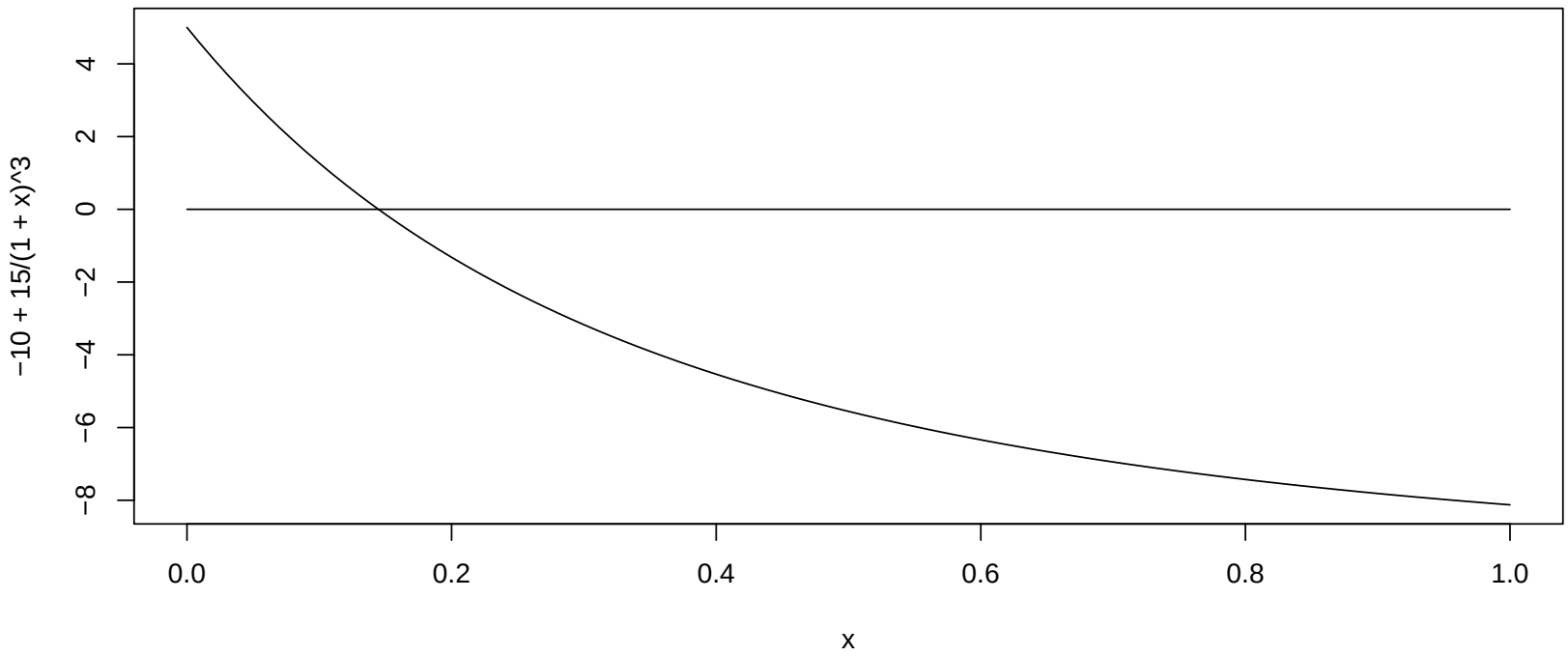
The Problem

$$PW = A_0 + \frac{A_1}{(1+i)} + \frac{A_2}{(1+i)^2} + \cdots + \frac{A_N}{(1+i)^N}$$

- Should look like a polynomial
- Internal rates of return are the roots of this polynomial
- You can have more than one root.

Example One Root

$$PW = -10 + \frac{15}{(1+i)^3}$$



Note the single root at $\left(\frac{15}{10}\right)^{\frac{1}{3}} - 1 = 0.1447142$

Easy Calculation

Simple case is when you have a value for A_0 and then one value in another time period, A_N

$$PW = 0 = A_0 + \frac{A_N}{(1+i)^N}$$

$$\Rightarrow -A_0 = \frac{A_N}{(1+i)^N}$$

$$\Rightarrow i = \left(\frac{-A_N}{A_0}\right)^{\frac{1}{N}} - 1$$

Multiple Roots

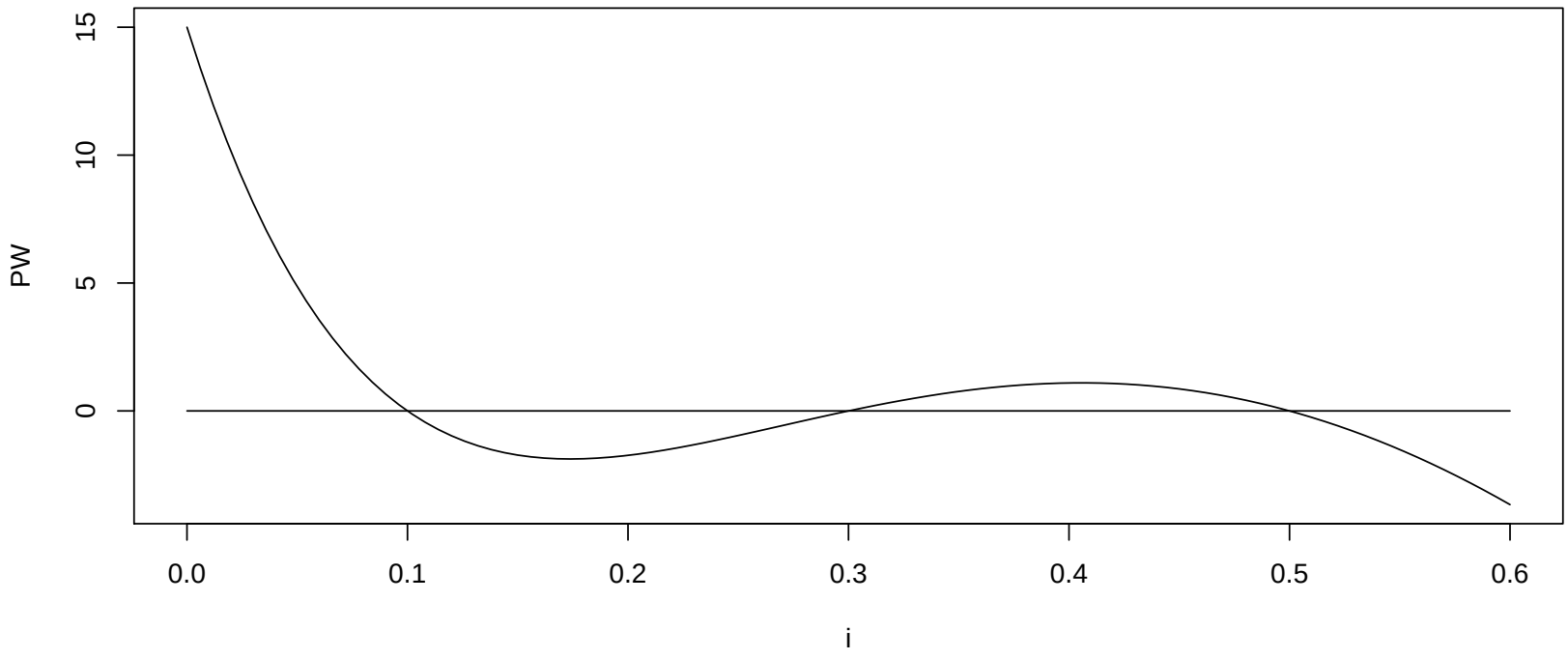
Descartes' rule of signs: The number of positive real roots is less than or equal to the number of sign changes in the coefficients of the polynomial.

Year	A	B
0	-10	-10
1	10	10
2	10	-5

- A Has only one sign change and therefore at most one IRR, i.e., one positive real root.
- B has two sign changes and therefore *at most* two positive IRRs, i.e, two positive real roots.

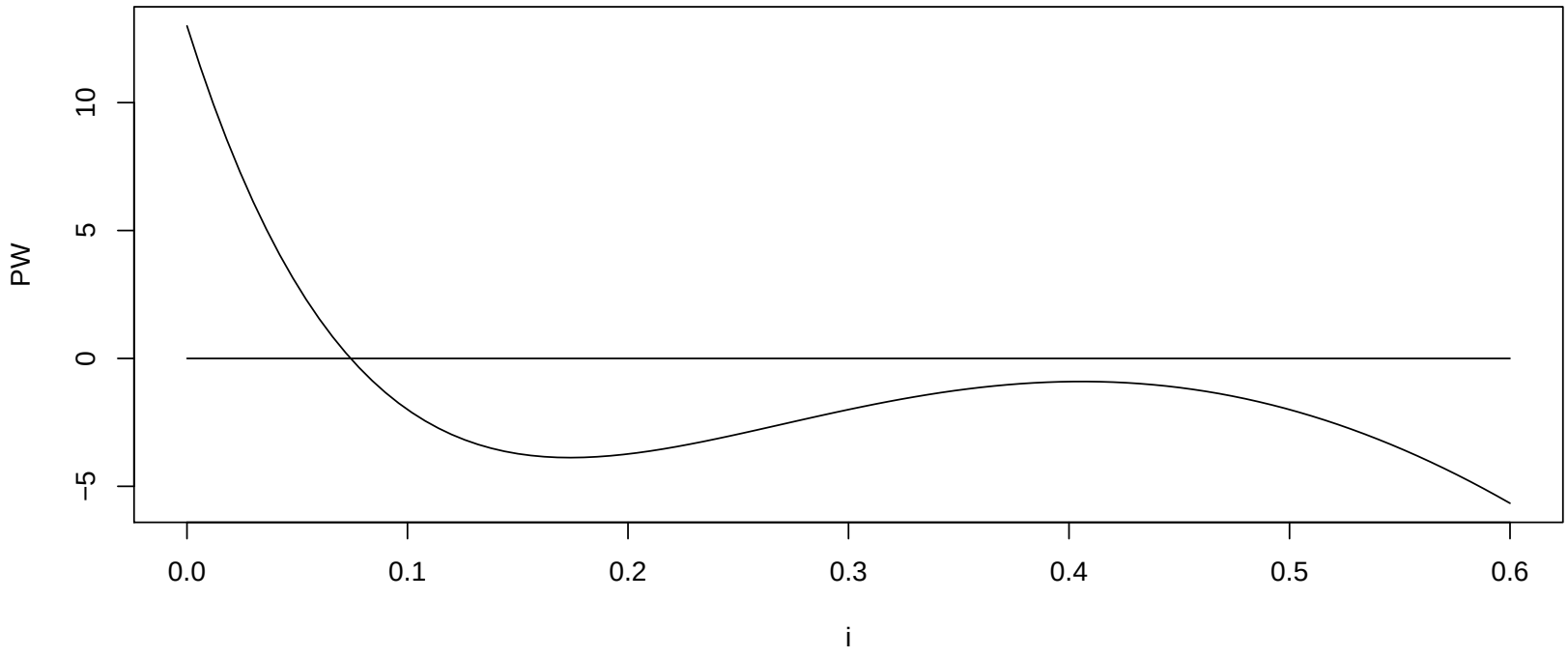
Multiple IRRs

This has IRRs at 10%, 30% and 50%



- This is -1000, 3900, -5030, 2145
- Note three sign changes.

At Most . . .



- This is -1002, 3900, -5030, 2145
- Only one root.

Calculating IRR

- Your calculator will have:
  - IRR function: Works but you need to give it a starting value if there is more than one root.
  - solve: Which is symbolic algebra and is gimped and limits number of cash flows.
  - nsolve: Similar to IRR with a starting value requirement.
- Spreadsheets
  - IRR(range, [estimated_irr])
  - Numerical root finder that requires starting value if you have more than one root.

Try a few

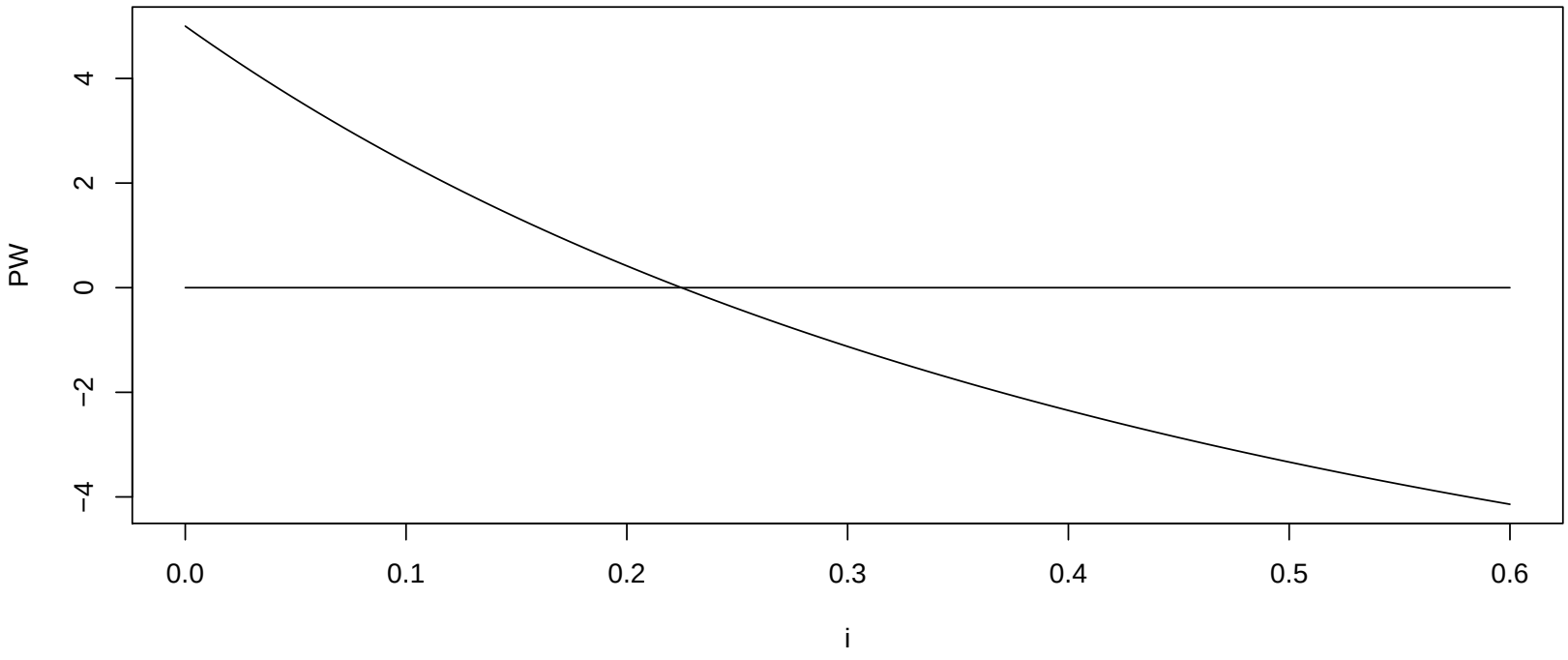
Year	A	B	C
0	-10	10	-10
1	0	0	7
2	15	-15	7

How did you do?

Year	A	B	C
0	-10	10	-10
1	0	0	7
2	15	-15	7
IRR	0.2247501	0.2247501	0.2568729

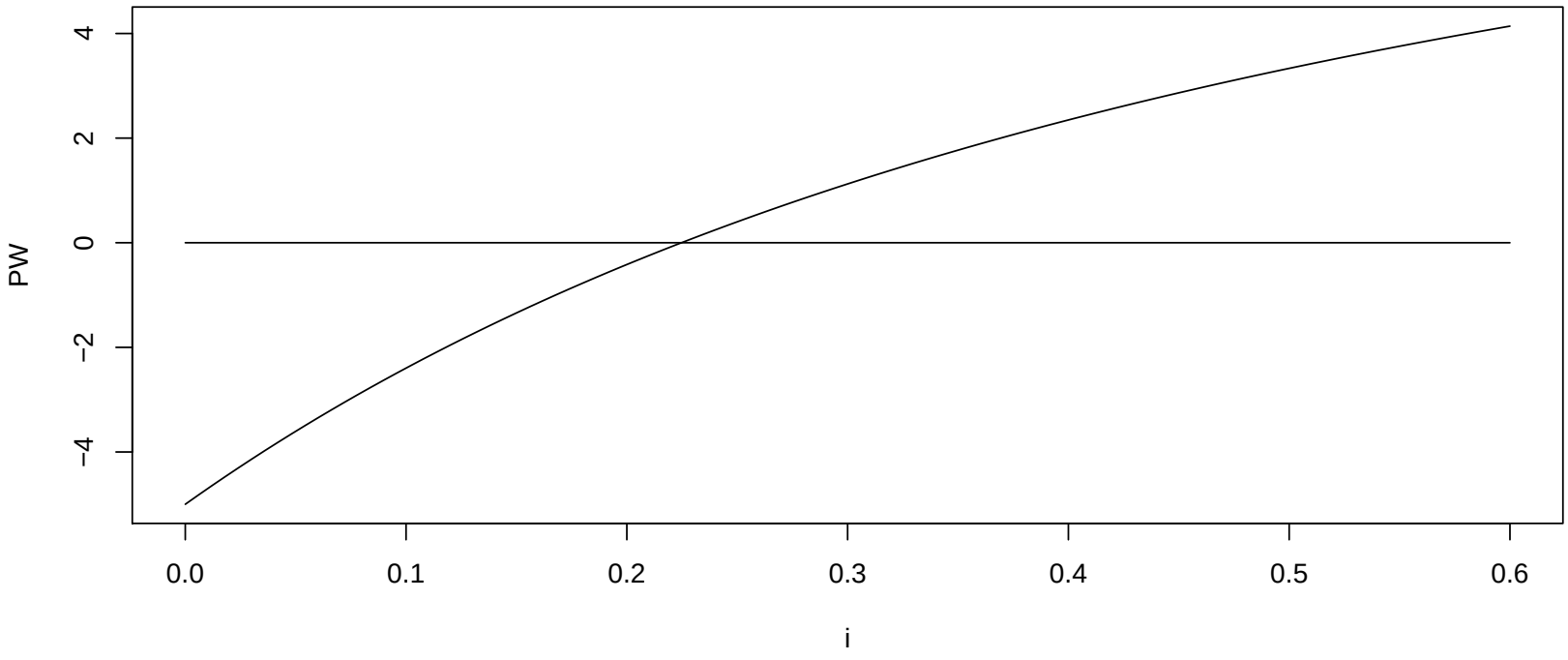
- It should not be a shock that A and B give the same result.
- BTW I used a numerical solver for this.
- Lets look at shapes starting with A

Cost 10 Now and Gives 15 in time 2



- PW is zero at 0.2247501.
- Is investment shaped, costs now and benefits later.
- If MARR is less than IRR, PW$>$0 and a good asset.
- If MARR is greater than IRR, PW$<$0 and not good.

Gives 10 Now and Costs 15 in time 2

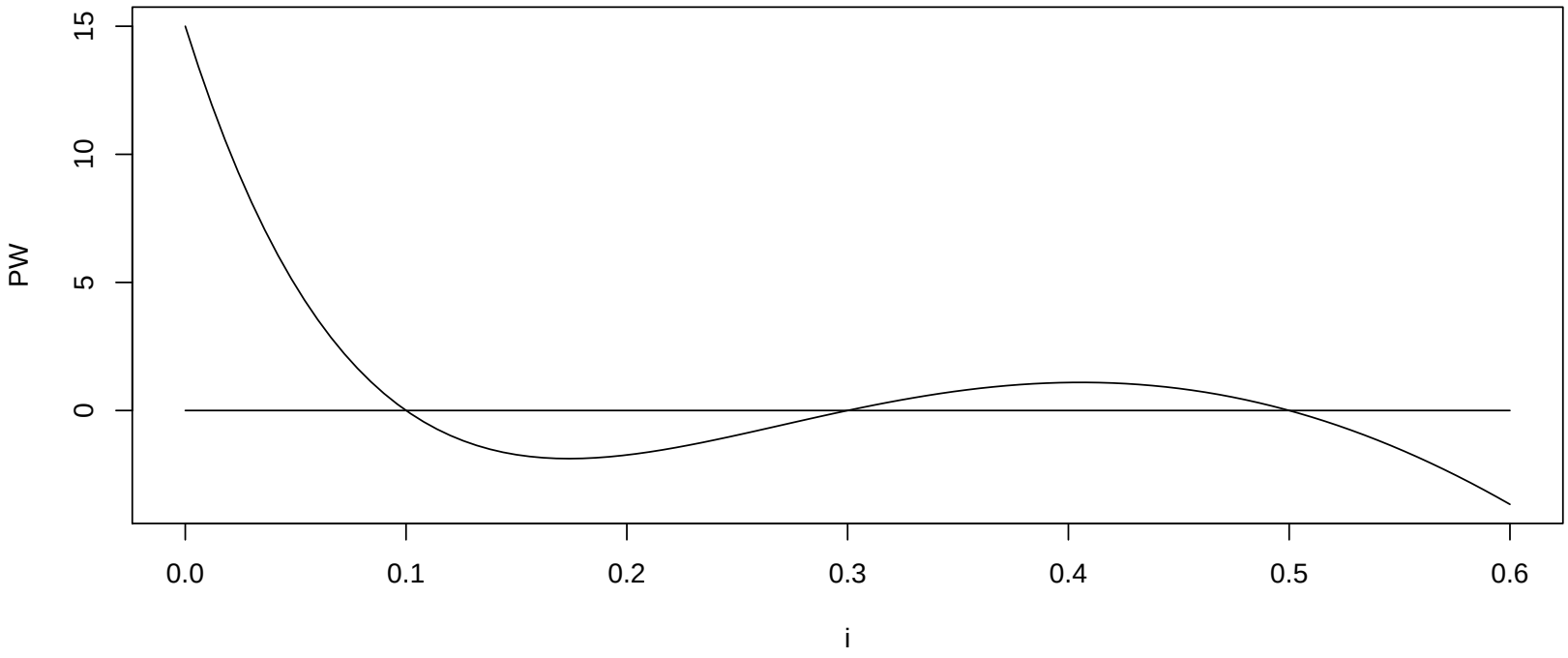


- PW is zero at 0.2247501.
- Is loan shaped, benefits now and costs later.
- If MARR is less than IRR, PW$>$0 and a bad asset.
- If MARR is greater than IRR, PW$<$0 and good asset.

Summary Unconstrained Choice Criteria

- If the asset has a single root...
  - and is an investment, buy if $IRR \geq MARR$
  - and is a loan, buy it if $MARR \geq IRR$
- In words
  - Buy assets with high returns.
  - Take out loans with low rates.

What if there is more than one IRR?



- It depends on the client
- Often the clients that ask for IRR don't understand that there can be more than one root.
- Had a student with real problem that had 100 IRRs.
- One of my first consulting oopses was about IRR.

True Story

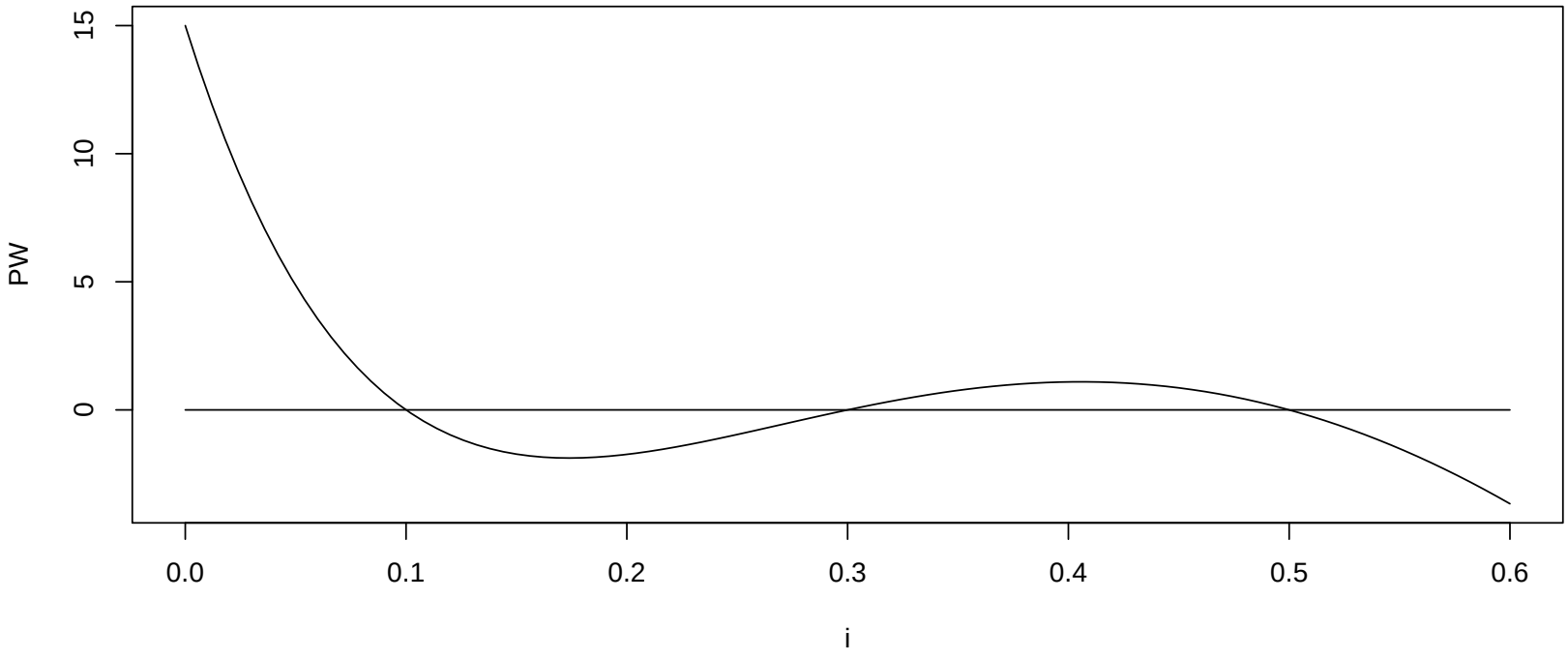
- Was presenting and asset had two roots.
- This guy, George, questioned my competence because he thought it wasn't possible.
- The Chief Economist taught me a more gentle way of treating fools. . . you lie.

A lie of omission, not commission.

An Approach

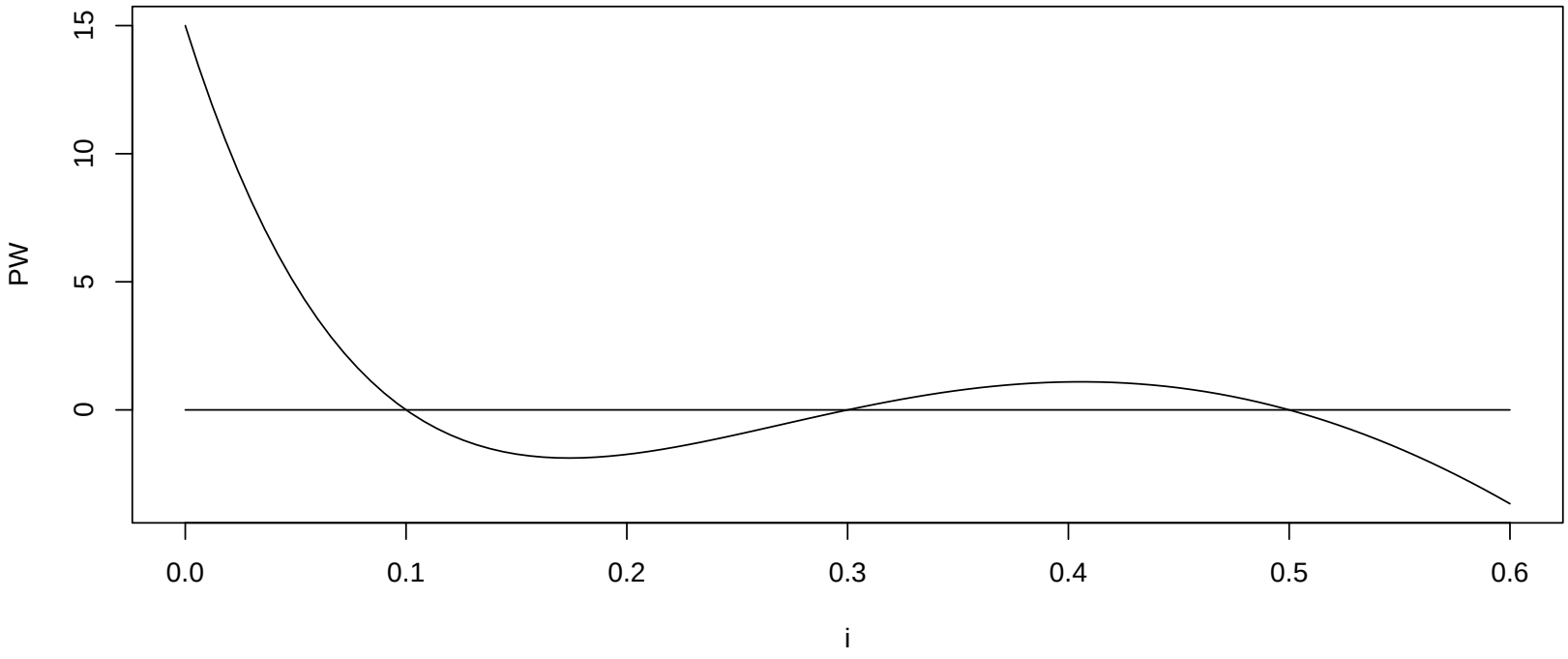
- Feel out your client. If they get that there can be more than one IRR, tell them.
- If they don't
  - Find out the range for the likely MARR.
  - Only tell them the IRR that leads them to the correct decision and is reasonably conservative.
- If they are a long-term client, you can introduce the idea later.
  - Consulting is a tough teaching gig.
  - Just because everyone nods, doesn't mean they get it. Lots off emperors new clothes effects.

Example: MARR<10%



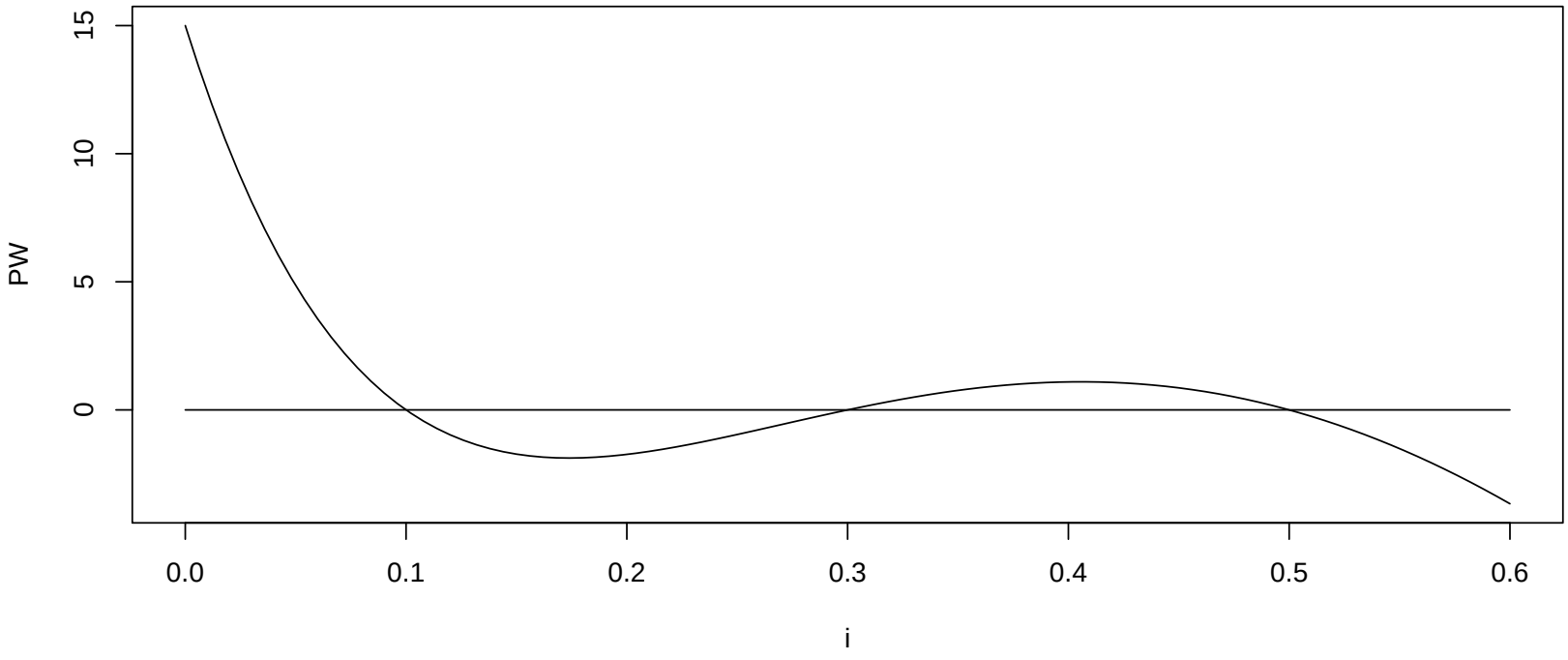
- Notice that the PW > 0 and they should buy it.
- Tell them 10%
  - Smallest number that
  - Leads them to correct conclusion, "buy", because IRR>MARR

Example: 30%>MARR>10%



- Notice that the PW < 0 and they should NOT buy it.
- Tell them 10%
  - Smallest number that
  - Leads them to correct conclusion, "Don't Buy", because IRR<MARR

Example: 50%>MARR>30%



- Notice that the PW > 0 and they should buy it.
- Tell them 50%
  - Smallest number that
  - Leads them to correct conclusion, "Buy", because IRR>MARR

Exclusive Choice

- Unconstrained choice IRR is pretty common.
- Exclusive choice is not
  - Never saw it in the real world.
  - Only on the PE exam.
- Promise me you will never use this in the real world.

Promise Me

I'm serious, never use this.

Exclusive Choice IRR

- Historical algorithm
- Great for when computation was expensive and decision making was cheap.
- Economized on computation. You did it once.
- After you had that data you could play “What-if” with the MARR as long as you wanted.

Don't pick asset with largest IRR

Year	A	B
0	-1	-1000
1	2	1110
IRR	1	0.11

A has the highest IRR but if MARR is moderate, you want B.

The Procedure

This procedure has a lot of caveats

- All assets are investments.
- All assets have a single IRR
- All incremental IRRs are unique
- Only $A_0 < 0$. No costs after time zero.

Incremental IRR

- Incremental IRR is the internal rate of return of the difference between two cash flows.
- Often though of as the IRR of upgrading to another asset.
- Example:

Year	A	B	B-A
0	-1	-1000	-999
1	2	1110	1109
IRR			0.1101101

$$IRR(B - A) \neq IRR(B) - IRR(A)$$

Common Problem with Incremental IRRs

Year	A	B	B-A
0	-1	-3	-2
1	0	6	6
2	2	0	-2

- Notice that both A and B have one sign change
- Notice that B-A has two sign changes.
  - Not two IRRs but the threat is there.
  - This causes problems for the algorithm.

The Algorithm

- Order assets from smallest initial investment to largest.
- Eliminate all assets with IRR$<$MARR.
- Set asset with smallest initial investment as Best Candidate (BC)
- While assets remain:
  - Set next asset as Challenger (C)
  - If Incremental IRR $\geq$ MARR, Eliminate BC and set C as BC.
  - Else, Eliminate C.
- Remaining Asset is the best.

Commentary on the Algorithm

- Order assets from smallest initial investment to largest.
  - Done so incremental IRRs are all interpreted as investments and you can use IRR$>$MARR as a test.
- Eliminate all assets with IRR$<$MARR.
  - Assets that are not acceptable can't be the best.

Commentary (Con't)

- Set asset with smallest initial investment as Best Candidate (BC)
- While assets remain:
  - Set next asset as Challenger (C)

This initializes a loop and creates a series of tournaments.

- If Incremental IRR $\geq$ MARR, Eliminate BC and set C as BC.
- Else, Eliminate C.

The uniqueness of IRRs and the ordering by initial investment, coupled with only allowing costs in period zero, allows the easy comparison.

Example with Computation of All Values

Year	A	B	C
0	-1	-3	-7
1	0	0	0
2	2	5	10

- Compute IRRs for all assets.
- Compute incremental IRRs for all pairs of assets.
- Use data to follow the algorithm

IRRs

Year	A	B	C
0	-1	-3	-7
1	0	0	0
2	2	5	10
IRR	0.4142136	0.2909944	0.1952286

Remember $IRR = \left(\frac{-A_N}{A_0}\right)^{\frac{1}{N}} - 1$ when you only have values in time zero and one other time period.

Try computing IRR(B-A) and IRR(C-A).

Incremental IRRs

Year	A	B	B-A
0	-1	-3	-2
1	0	0	0
2	2	5	3
IRR	0.4142136	0.2909944	0.2247449

Year	A	C	C-A
0	-1	-7	-6
1	0	0	0
2	2	10	8
IRR	0.4142136	0.1952286	0.1547005

Try IRR(C-B)

Incremental IRRs (Con't)

Year	B	C	C-B
0	-3	-7	-4
1	0	0	0
2	5	10	5
IRR	0.2909944	0.1952286	0.118034

Summary of Data

Asset	IRR	A	B	C
A	0.4142136		0.2247449	0.1547005
B	0.2909944			0.118034
C	0.1952286			

The other half of the incremental IRR matrix is often not given because it is symmetric.

Find the best asset when the MARR is 15%.

MARR of 15%

Asset	IRR	A	B	C
A	0.4142136		0.2247449	0.1547005
B	0.2909944			0.118034
C	0.1952286			

- Order assets as A, B ,C because initial costs are 1, 3, and 7 respectively.
- No assets have IRRs less than MARR
- A is BC and B is first challenger.
  - IRR(B-A) $= 0.2247449 >$ MARR, toss A and set B as BC.
  - IRR(C-B) $= 0.118034 <$ MARR, toss C.

B is the best asset at 15%.

Arrow Notation Explained

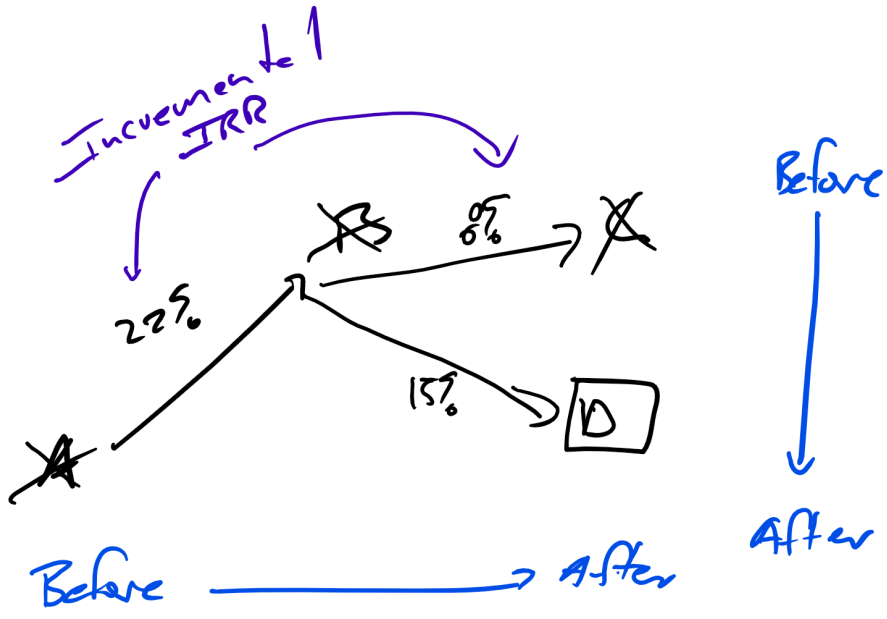
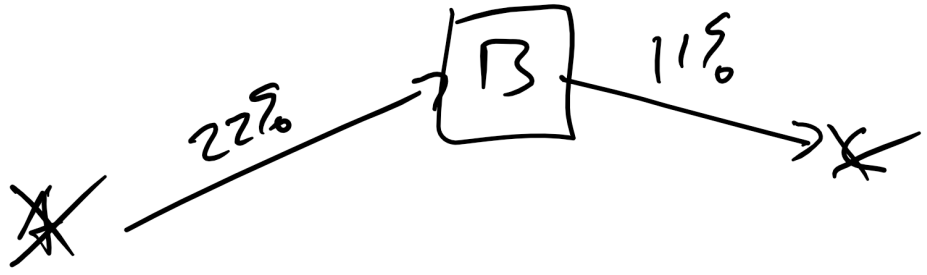


Figure 1

Arrow Notation MARR 15%

Asset	IRR	A	B	C
A	0.4142136		0.2247449	0.1547005
B	0.2909944			0.118034
C	0.1952286			



MARR of 20%

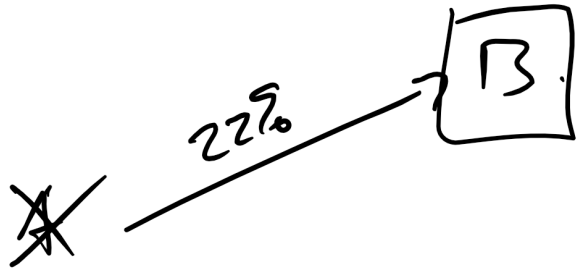
Asset	IRR	A	B	C
A	0.4142136		0.2247449	0.1547005
B	0.2909944			0.118034
C	0.1952286			

- Order assets as A, B ,C because initial costs are 1, 3, and 7 respectively.
- Eliminate C since IRR(C) = 0.1952286<MARR
- A is BC and B is first challenger.
  - IRR(B-A) = 0.2247449 > MARR, toss A.

B is the best asset at 20%.

Arrow Notation MARR 20%

Asset	IRR	A	B	C
A	0.4142136		0.2247449	0.1547005
B	0.2909944			0.118034
C	0.1952286			



Further Intuition on Why the Algorithm works.

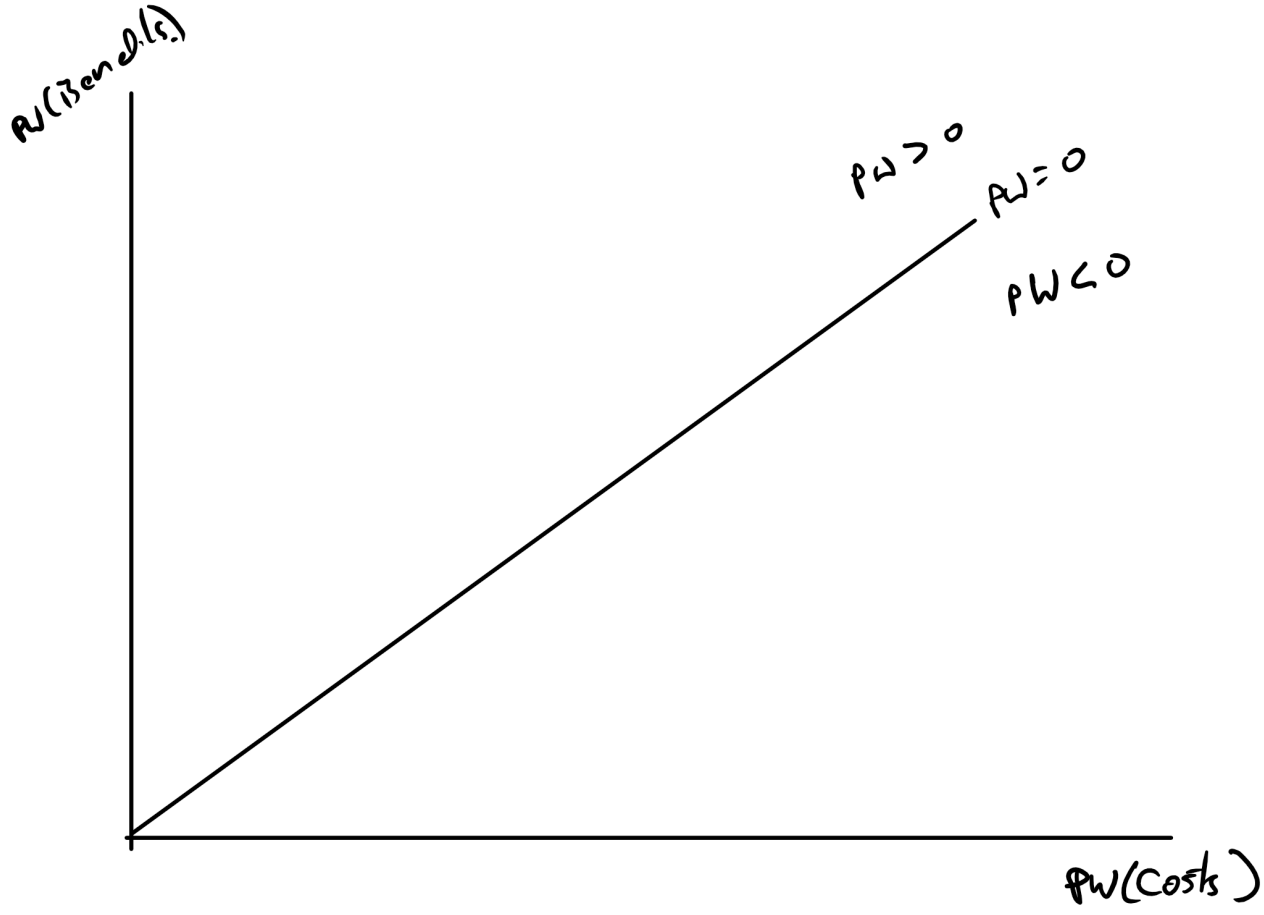


Figure 4

Describe Assets

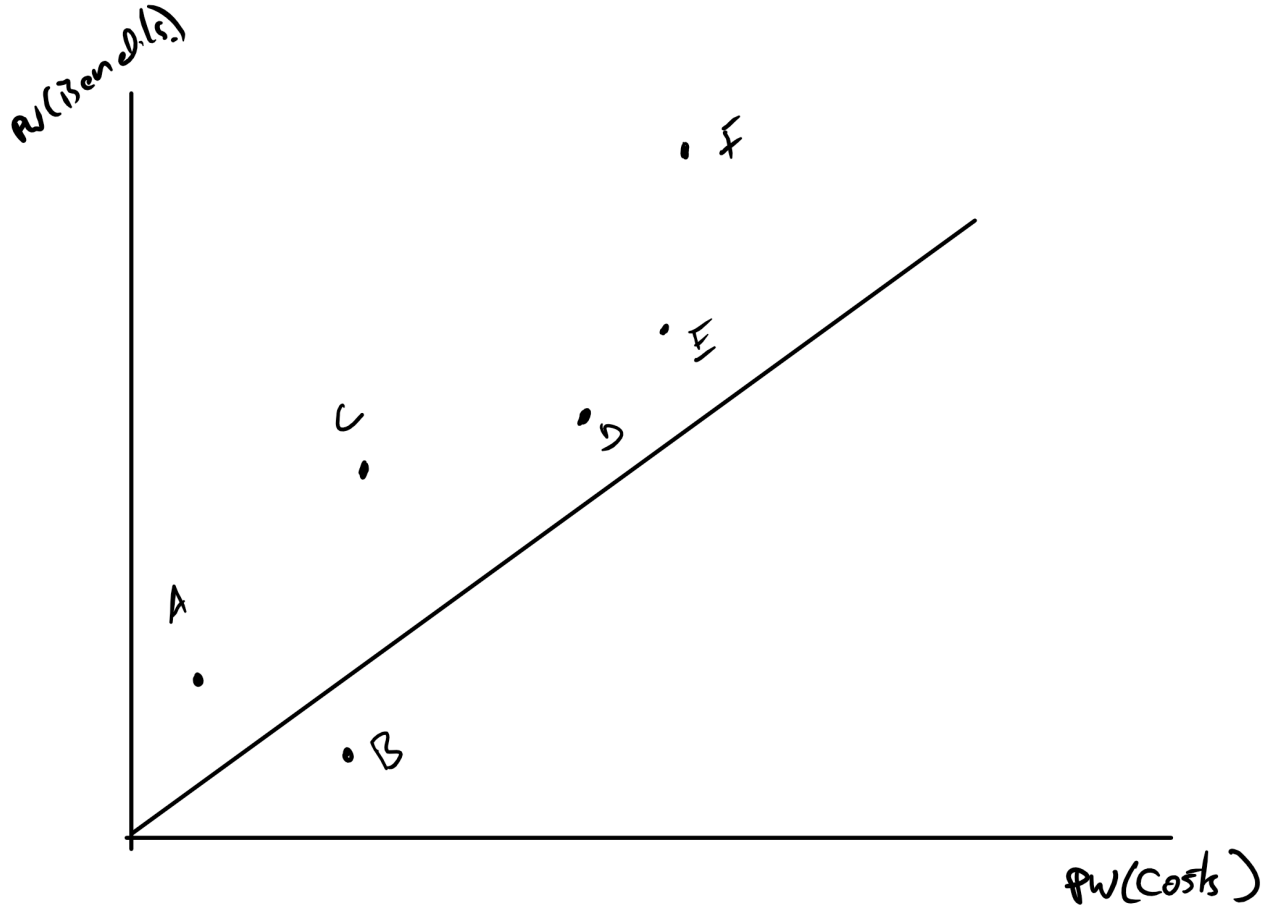


Figure 5

Eliminate Unacceptable

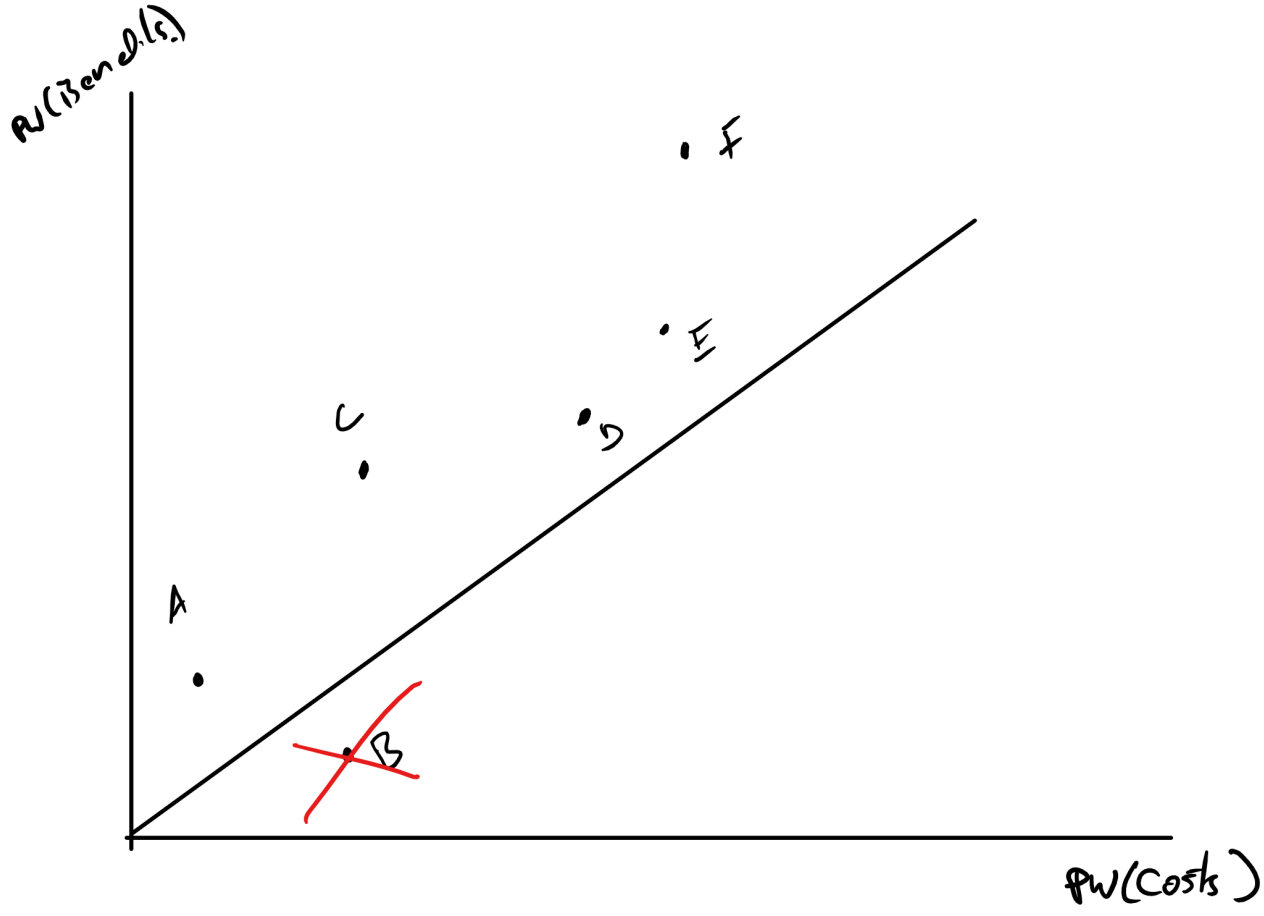


Figure 6

The Loop

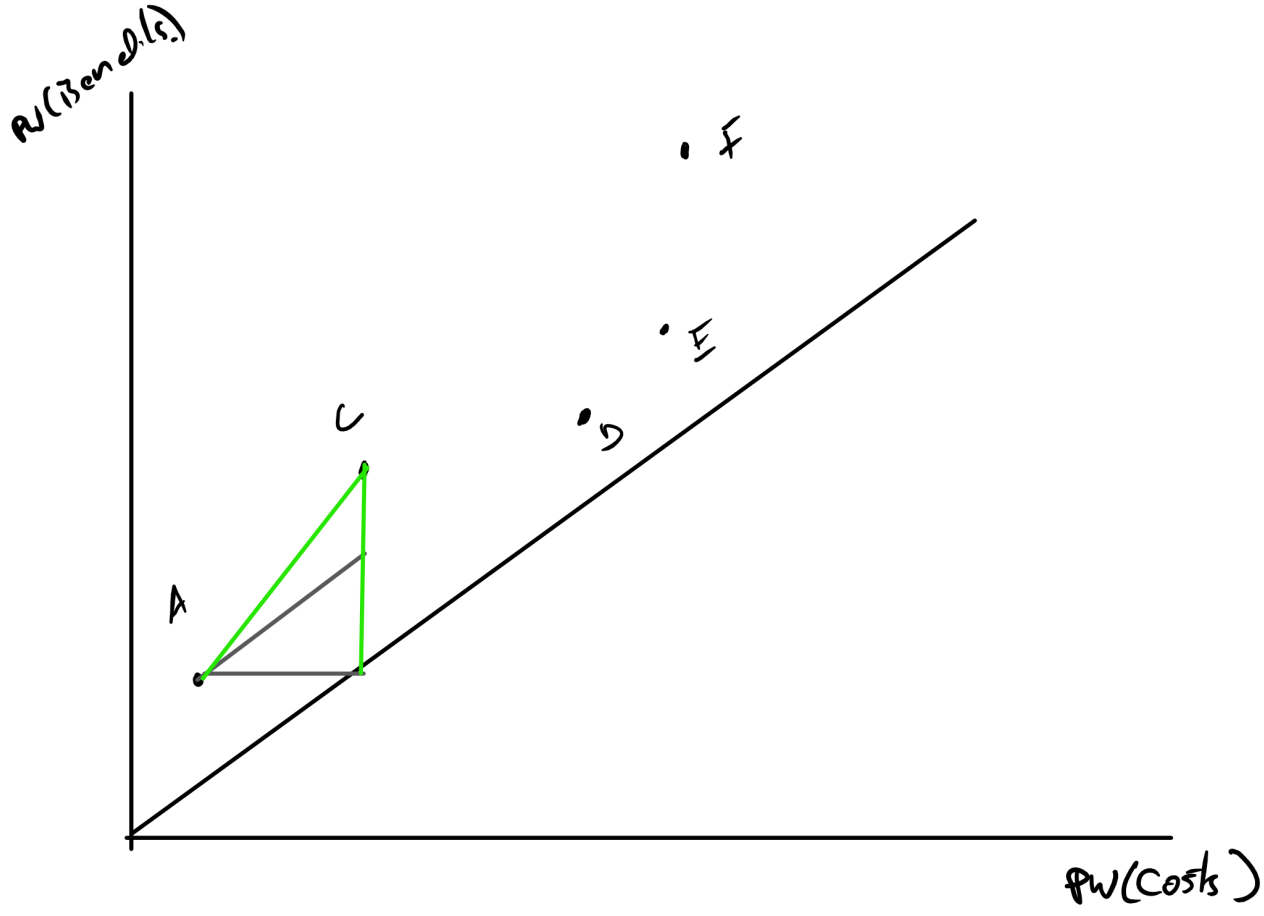


Figure 7

Loop (Cont)

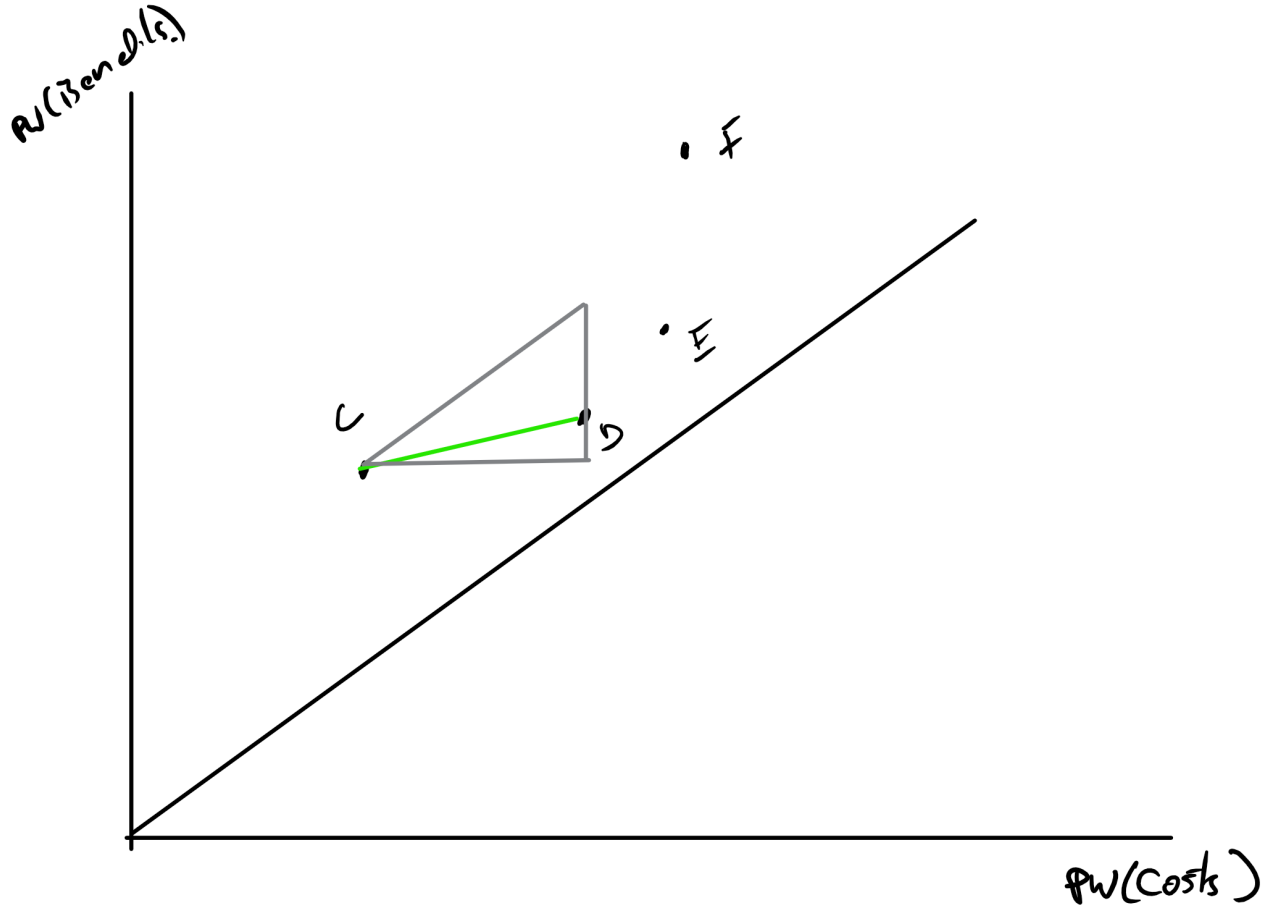


Figure 8

Loop (Cont)

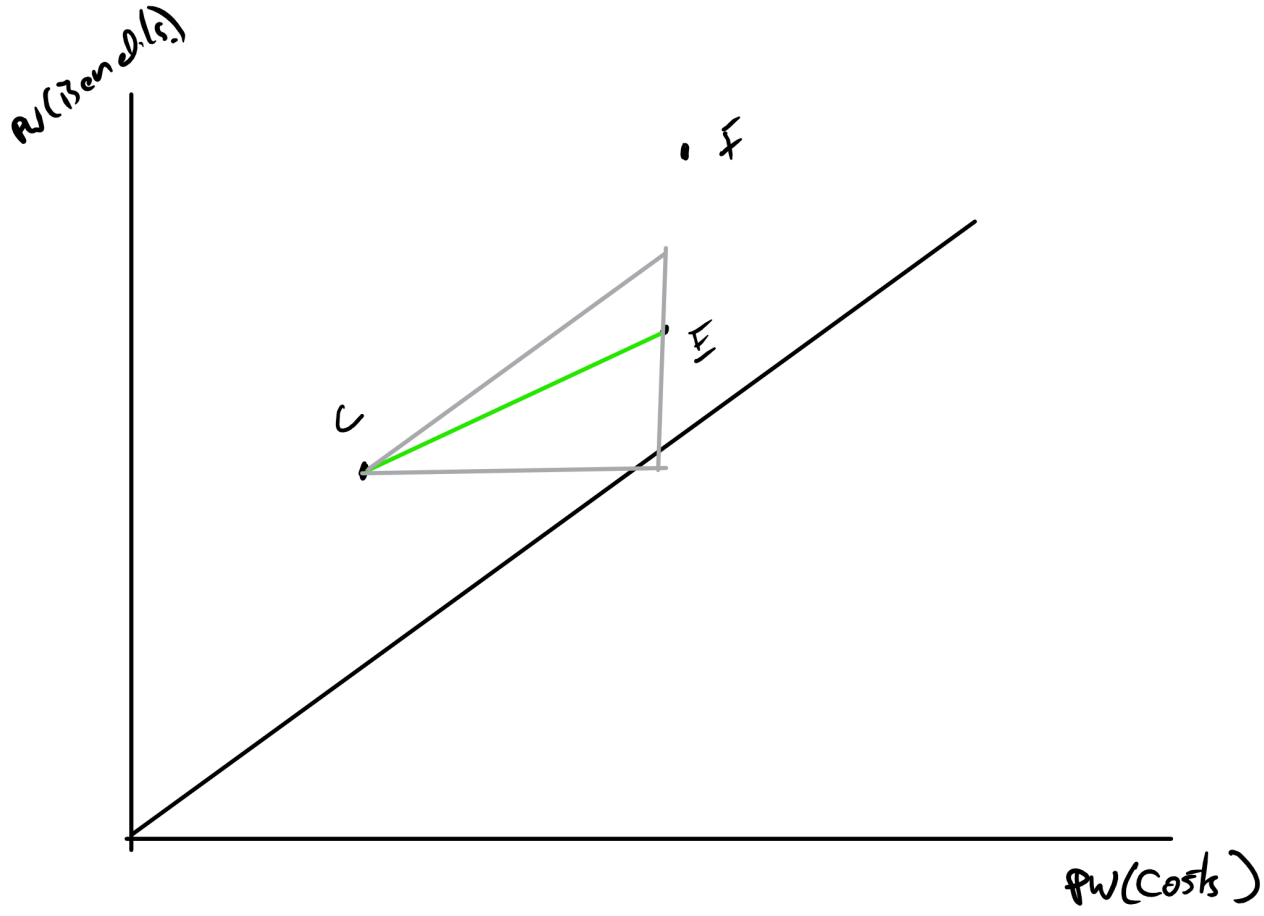


Figure 9

The Last Asset Standing

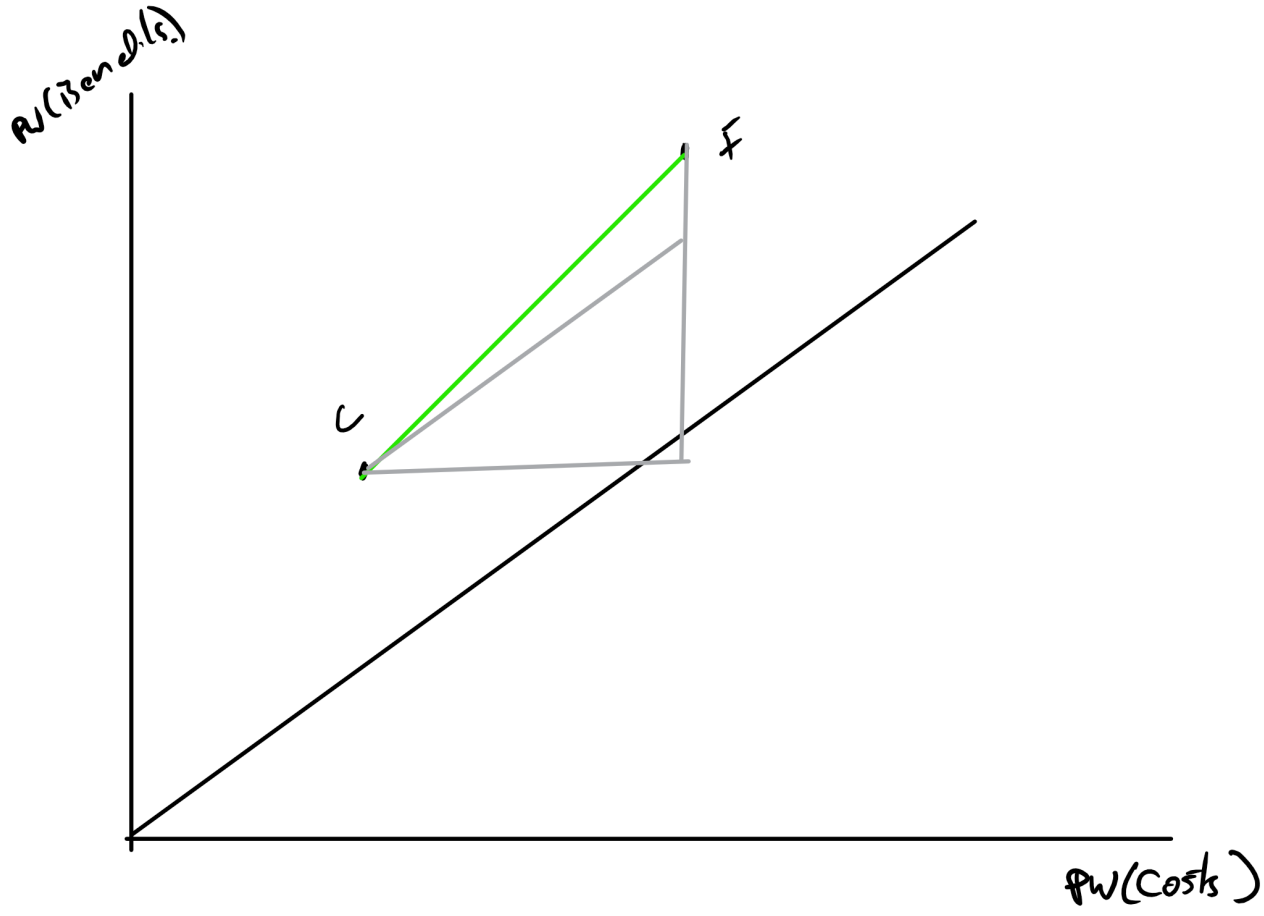


Figure 10

More Examples

- Examples are randomly generated.
- If you want, download or clone, and knit.
- Correct answer and initial ordering are produced but not the steps or arrow diagrams.

Larger Example

All values in %.

	IRR	AO	A	B	C	D	E	F
A	23.71	14	0.00	24.36	41.94	31.94	40.92	40.11
B	20.43	17	24.36	0.00	15.67	32.36	20.16	23.77
C	38.22	2	41.94	15.67	0.00	25.94	42.56	42.48
D	22.72	16	31.94	32.36	25.94	0.00	22.72	42.60
E	18.26	21	40.92	20.16	42.56	22.72	0.00	36.41
F	36.12	4	40.11	23.77	42.48	42.60	36.41	0.00

Step 1: Get the order of the assets right, lowest initial cost to highest C, F, A, D, B, E.

MARR = 19.35%

	IRR	AO	A	B	C	D	E	F
A	23.71	14	0.00	24.36	41.94	31.94	40.92	40.11
B	20.43	17	24.36	0.00	15.67	32.36	20.16	23.77
C	38.22	2	41.94	15.67	0.00	25.94	42.56	42.48
D	22.72	16	31.94	32.36	25.94	0.00	22.72	42.60
E	18.26	21	40.92	20.16	42.56	22.72	0.00	36.41
F	36.12	4	40.11	23.77	42.48	42.60	36.41	0.00

MARR = 19.35%

	IRR	AO	A	B	C	D	E	F
A	23.71	14	0.00	24.36	41.94	31.94	40.92	40.11
B	20.43	17	24.36	0.00	15.67	32.36	20.16	23.77
C	38.22	2	41.94	15.67	0.00	25.94	42.56	42.48
D	22.72	16	31.94	32.36	25.94	0.00	22.72	42.60
E	18.26	21	40.92	20.16	42.56	22.72	0.00	36.41
F	36.12	4	40.11	23.77	42.48	42.60	36.41	0.00

The acceptable assets are: A, B, C, D, F and the best is B.

MARR = 17.26%

	IRR	AO	A	B	C	D	E	F
A	23.71	14	0.00	24.36	41.94	31.94	40.92	40.11
B	20.43	17	24.36	0.00	15.67	32.36	20.16	23.77
C	38.22	2	41.94	15.67	0.00	25.94	42.56	42.48
D	22.72	16	31.94	32.36	25.94	0.00	22.72	42.60
E	18.26	21	40.92	20.16	42.56	22.72	0.00	36.41
F	36.12	4	40.11	23.77	42.48	42.60	36.41	0.00

MARR = 17.26%

	IRR	AO	A	B	C	D	E	F
A	23.71	14	0.00	24.36	41.94	31.94	40.92	40.11
B	20.43	17	24.36	0.00	15.67	32.36	20.16	23.77
C	38.22	2	41.94	15.67	0.00	25.94	42.56	42.48
D	22.72	16	31.94	32.36	25.94	0.00	22.72	42.60
E	18.26	21	40.92	20.16	42.56	22.72	0.00	36.41
F	36.12	4	40.11	23.77	42.48	42.60	36.41	0.00

The acceptable assets are: A, B, C, D, E, F and the best is E.

How to Study

- Worked Examples are at the bottom of this page.
- Examples ask you to
  - Calculate IRR for simple asset.
  - Calculate and apply AW criteria
  - Apply PW criteria
  - Use the IRR procedure