

Time Value of Money and Loans

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Time Value of Money Goals

- ▶ Create a logical basis for comparing costs and benefits in different time periods.
- ▶ Introduce a common notation that engineers and many economists use for these calculations.
- ▶ Set the basis for:
 - ▶ Loans
 - ▶ Investment criteria

Assumptions

These are demonstrably false but give us tractable methods.

- ▶ Costs and benefits of equal size have equal value in all time periods. *Can evaluate without worrying about wealth or taste changes.*
- ▶ The value of costs and benefits is independent of costs and benefits in other time periods. *No habit, addiction or hangovers.*
- ▶ Benefits offset costs. *Ever fought a parking ticket?*
- ▶ Future values are known with certainty. *Math is easy*

What it looks like

$$\sum x_n d(n)$$

- ▶ $d(n)$ is a discounting function.
- ▶ Many are possible but only a few are time consistent.
 - ▶ $d(n) = \frac{1}{(1+r)^n}$ for discrete time.
 - ▶ $d(n) = \frac{1}{e^{rn}}$ for continuous time.

Example of Time Inconsistency

I did this in class

- ▶ Chose 6-pack of beer a month and a day from now or 1 beer a month from now.
- ▶ A beer right now or a 6-pack tomorrow.
- ▶ Many of you changed your mind.

Integer time

- ▶ Will say “Time 1”, “Time zero” because “1st time period is confusing”
- ▶ “Now” means time zero.
- ▶ Intervals are half open on the right, e.g., $[0, 1)$

Notes on the interest rate

- ▶ We will treat the interest rate as a simple thing until loans.
- ▶ The rate is always just the per period rate.
- ▶ Later we will get into:
 - ▶ The nominal period.
 - ▶ Compounding period.
 - ▶ Nominal statement.
 - ▶ Effective interest rates.

Time Consistent Integer Time Discounting

$$F = P(1 + i)^N$$

Symbolic notation depends on context.

- ▶ i = The interest rate
- ▶ P = Present worth = some value in time zero.
- ▶ N = N time periods from N = Time N
- ▶ F = Future Value = Present Value in time N = A value in time N .

Easy Future Value Calculation

“If you deposit \$26 into an account that earns 2% a month, how much will be in the account after 500 months?”

$$F = 26(1 + .02)^{500} = 518870.7975218$$

- ▶ “how much will be in the account after” is a good signal for future value.
- ▶ 100ths of a percent is more than adequate for calculations.
- ▶ Usually, cents is close enough.
- ▶ D2L questions have a tolerance.

Easy Present Worth Calculation

“How much would you have to deposit now into an account that earns 10% per year to have 100K in 10 years?”

$$P = \frac{100K}{(1 + .1)^{10}} = 38.5543289K$$

- ▶ “How much would you have to deposit now” is a present worth indicator.

Graphical Notation

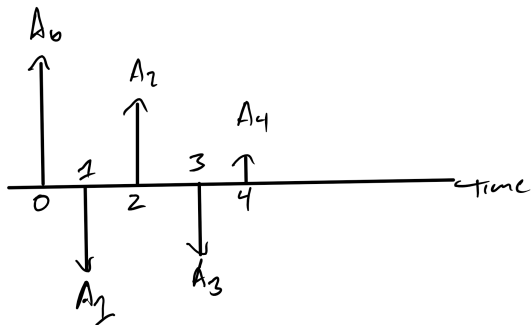


Figure 1

Comments on Graphical Notation

- ▶ Up vectors are cash inflow.
- ▶ Down are cash outflow.
- ▶ Magnitude is size of cash flows.
- ▶ Cash flow in any period is A_n .

Symbols Used in Symbolic (Factor) Notation

- ▶ A_n Cost or benefit in time n
- ▶ n Arbitrary time period
- ▶ N Usually but not always the last time period
- ▶ P Present worth in time zero
- ▶ P_n Present worth in time n .
- ▶ A Used in constant series to indicate same value for time 1 to N .
- ▶ G Used in linear gradient series to indicate the change in cash flow from time 1 to N .
- ▶ g Used in geometric gradient series to indicate the percent change in cash flow from time 1 to N .

What is factor notation?

It is a functional representation of common cash flow patterns:

- ▶ Constant Series, $A_{n+1} = A_n, \forall n = 1 \dots N$
- ▶ Linear Gradient, $A_{n+1} = A_n + G, A_1 = 0, \forall n = 1 \dots N$
- ▶ Geometric Gradient, $A_{n+1} = A_n(1 + g), \forall n = 1 \dots N$

There are other notations in use, e.g. actuarial notation.

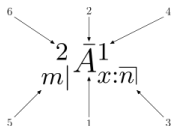


Figure 2: By Delimata - Own work, GFDL,
<https://commons.wikimedia.org/w/index.php?curid=4312450>

Factor Notation

- ▶ It is a short-hand for the closed form representations of present and future worth for the common patterns
- ▶ Example

$$10(P|A, i = 10\%, 5) = 10 \left[\frac{(1 + .1)^5 - 1}{.1(1 + .1)^5} \right] = 37.9078677$$

The things in the parentheses are a substitute for the things in the square brackets.

The Key

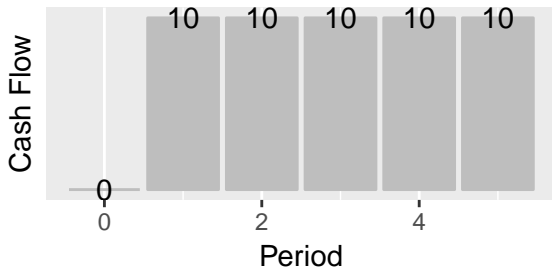
Looking For = Number(Looking For|Whatisnumber, i, HowLong)

- ▶ A big list is here <http://ec314-pdx-edu.wikidot.com/annotated-equations-sheet>
- ▶ Hint, if you know $(P|X)$, $(F|X) = (P|X)(1 + r)^N$

Wait, closed form notation?

- ▶ Remember the tricks from sequences and series? Many work for finite sequences.
- ▶ Closed form has advantage over brute force (We do that next) in that if you mess it up, it looks stupid.
- ▶ There are also tables of pre-calculated values which are useful for the PE exam.

Present Worth Three Ways

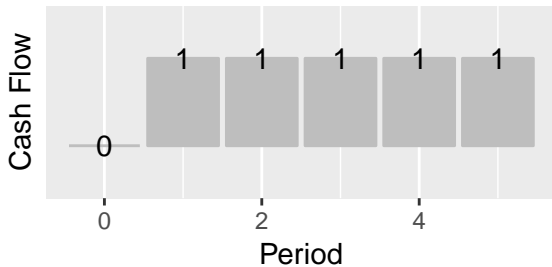


- ▶ Brute force: $\sum_{n=1}^5 \frac{10}{(1+.1)^n}$
- ▶ Tabular: $10 * 3.7907868$
- ▶ Closed form: $10(P|A, i = 10\%, 5) = 10 \frac{(1+.1)^5 - 1}{.1(1+.1)^5}$
- ▶ Either way you should get 37.9078677.

Comments on Three Methods

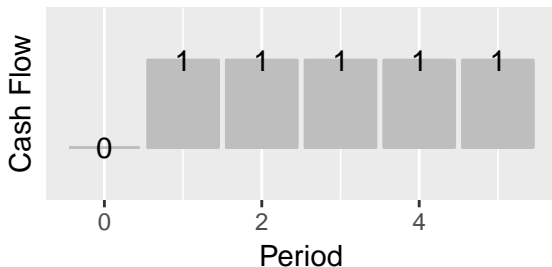
- ▶ Brute force always works but when you make a mistake it hides.
- ▶ Tables don't exist for all interest rates but it was way faster on the PE, which is a speed test.
- ▶ Closed form looks stupid if you have made a mistake, but easy to make a mistake.

Easy Cash Flow to Factor Notation



Find representation of Present Worth in factor notation

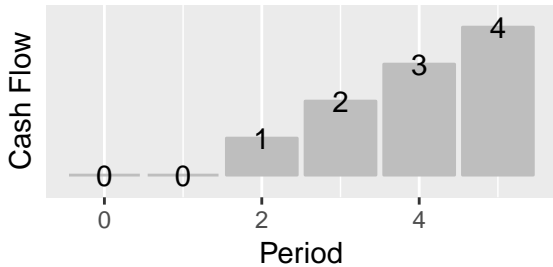
Answer



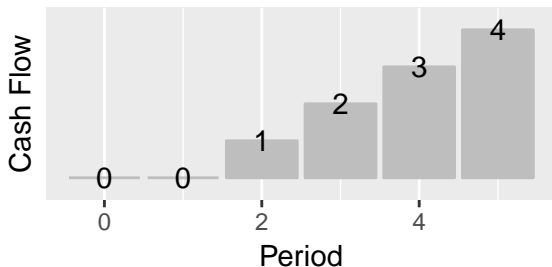
- ▶ Shape is constant series
- ▶ Length is 5, which you get by counting the number of non-zero elements.
- ▶ Height is 1, that the the A you are looking for.

$$1(P|A, i, 5)$$

Try this



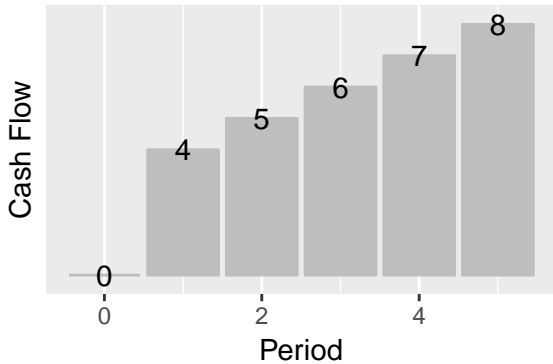
Linear Gradient is Tricky



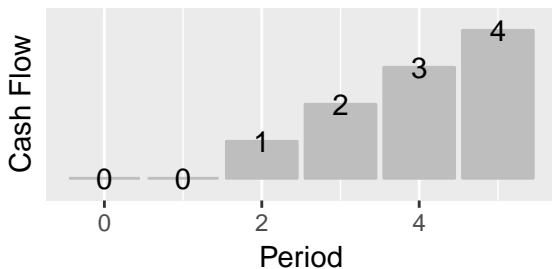
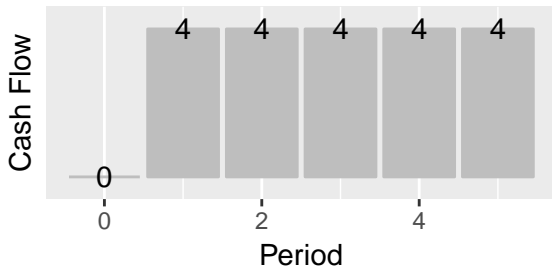
- ▶ Shape is linear gradient
- ▶ Length is 5, count non-zeros and 'add one for the pot'. If zero is the start, period two is the first time you see a non-zero value.
- ▶ G , is 1, how much it changes by from period to period

$$1(P|G, i, 5)$$

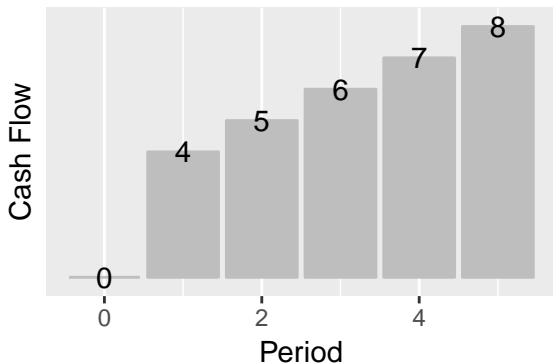
Now lets put two things together



You should see this as



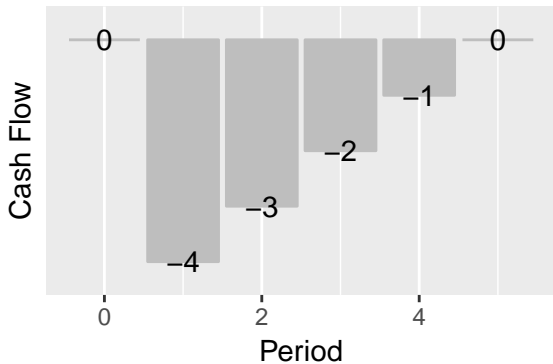
Following from above



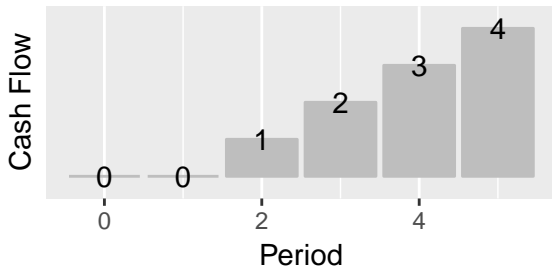
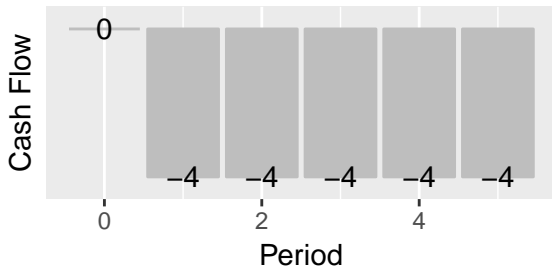
- ▶ See the constant series that is 4 high and 5 long $4(P|A, i, 5)$
- ▶ See the linear gradient that has G of 1 and is 5 long $1(P|G, i, 5)$

$$4(P|A, i, 5) + 1(P|G, i, 5)$$

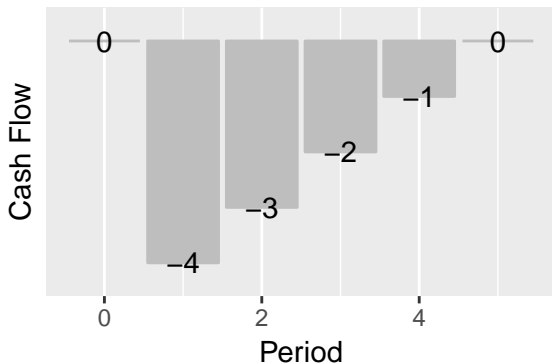
How about this. Single byte change



You should see this



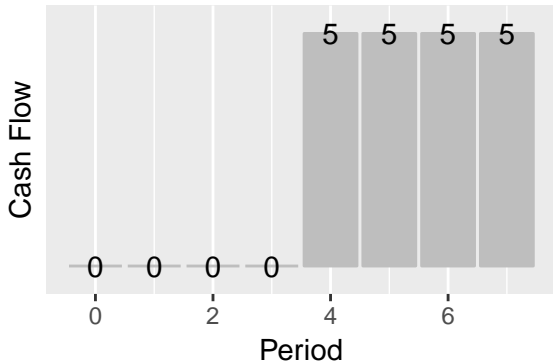
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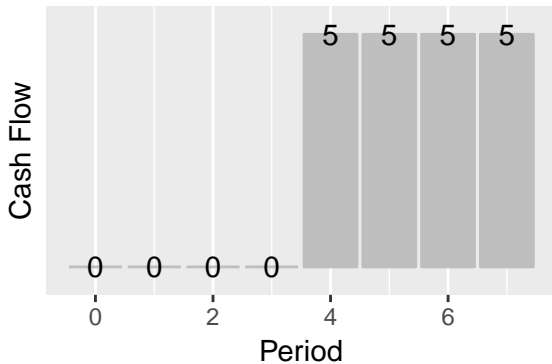
- ▶ See the constant series that is -4 high and 5 long $-4(P|A, i, 5)$
- ▶ See the linear gradient that has G of 1 and is 5 long $1(P|G, i, 5)$

$$-4(P|A, i, 5) + 1(P|G, i, 5)$$

Try this with only addition and subtraction



Should get this



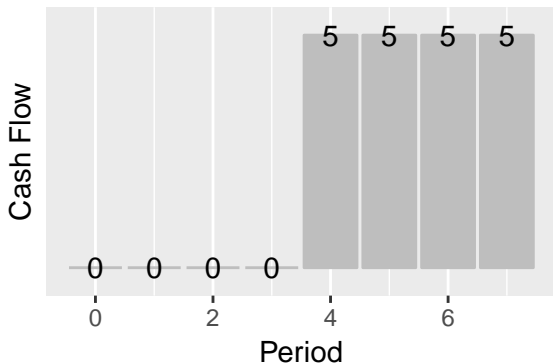
$$5(P|A, i, 7) - 5(P|A, i, 3)$$

Time Shifting is easier

$$P = \frac{F}{(1+i)^N}$$

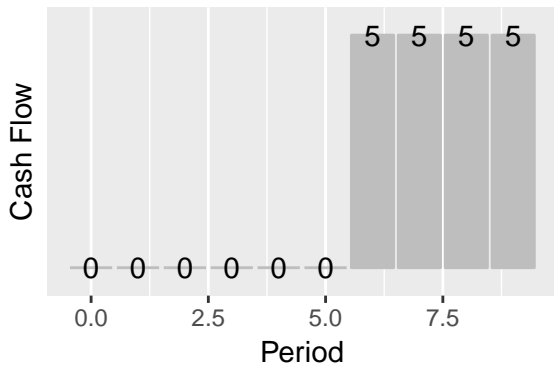
- ▶ You can replace the F with anything to move the common shapes around.

Last example as time shift



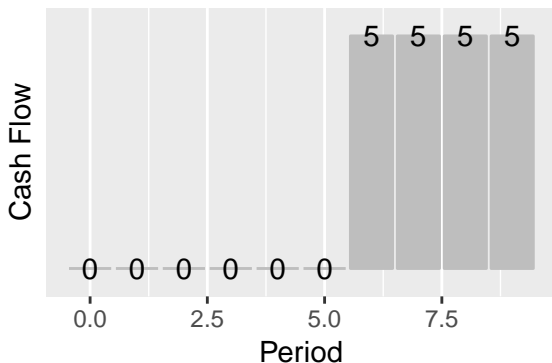
$$\frac{5(P|A, i, 4)}{(1+i)^3}$$

Further out



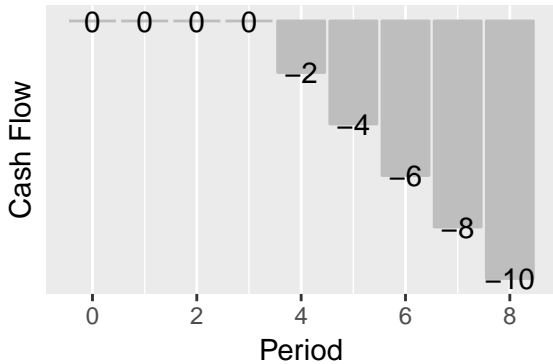
$$\frac{5(P|A, i, 4)}{(1+i)^5}$$

Two ways of thinking about time shifting

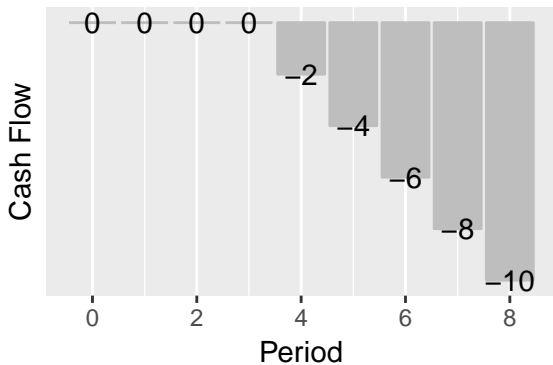


- ▶ $\frac{5(P|A,i,4)}{(1+i)^5}$
- ▶ The the exponent is the time period just to the left of the series.
- ▶ The sum of the N and the exponent is equal to the last period you see the series.

Tricky one



You should see

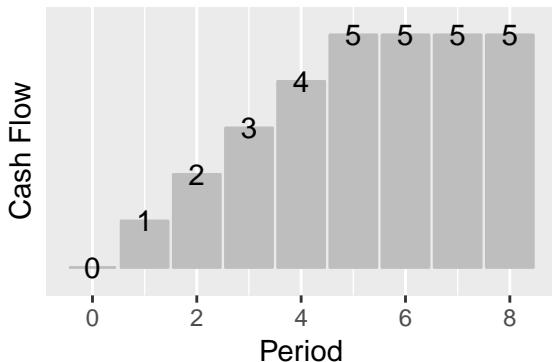


$$\frac{-2(P|G, i, 6)}{(1+i)^2}$$

Warnings

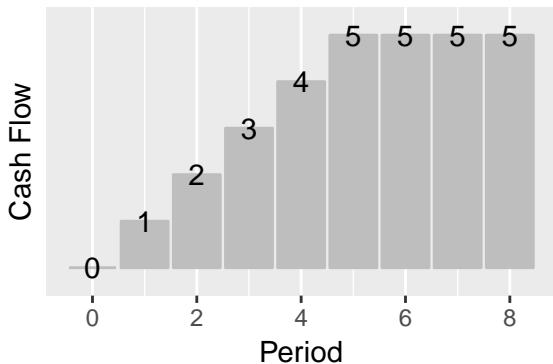
- ▶ There is no unique factor representation of a cash flow.
- ▶ Some are easier to spot than others.
- ▶ Usually the story tells you the shape.
 - ▶ Loan payments look like $(P|A)$
 - ▶ Accelerator clauses look like $(P|A_1)$

Same Problem Several Ways



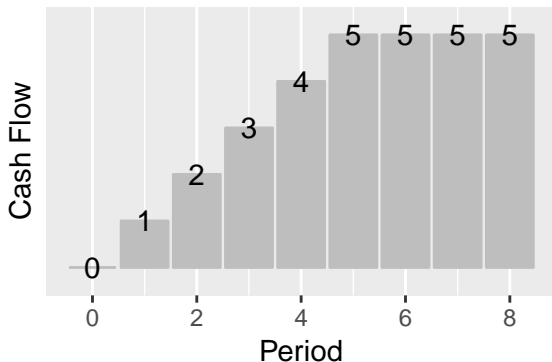
Try it with the linear gradient ending at time 4 and the constant series going from 5 to 8.

How did you do?



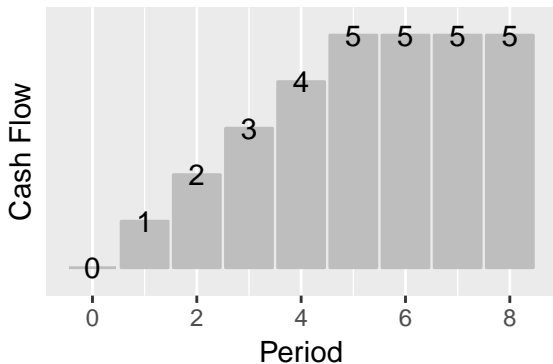
$$\frac{1(P|G, i, 5)}{(1+i)^{-1}} + \frac{5(P|A, i, 4)}{(1+i)^4}$$

Again



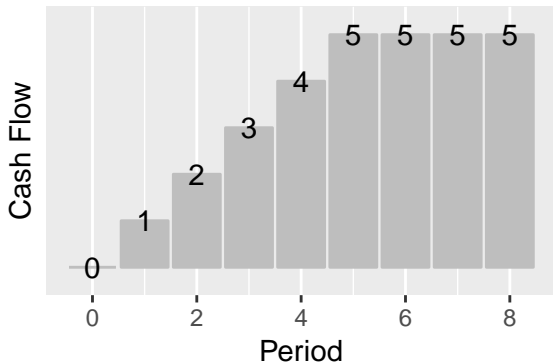
Try it with the linear gradient ending at time 5 and the constant series going from 6 to 8.

How did you do?



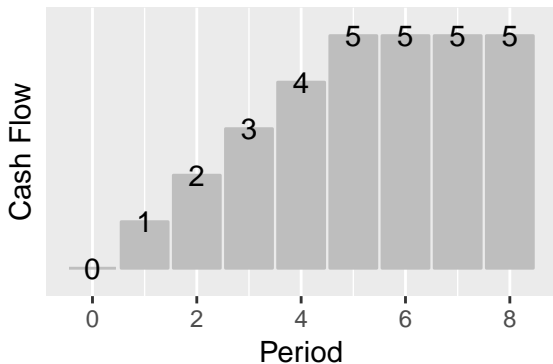
$$\frac{1(P|G, i, 6)}{(1+i)^{-1}} + \frac{5(P|A, i, 3)}{(1+i)^5}$$

Try the civil engineer's solution



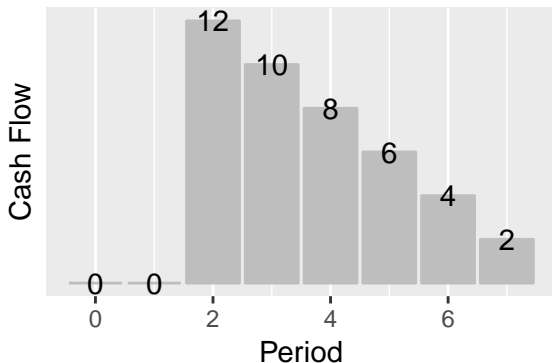
Stack constant series like a layer cake. There is one that is one high from period 1 to 8. Another that is one high from period 2 to 8 and so on.

Civil engineer's solution



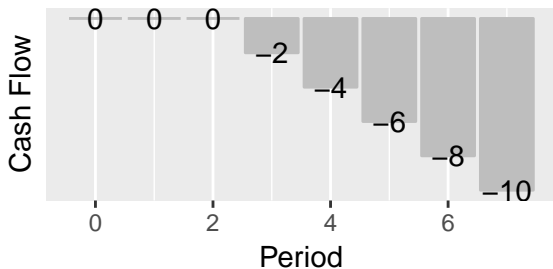
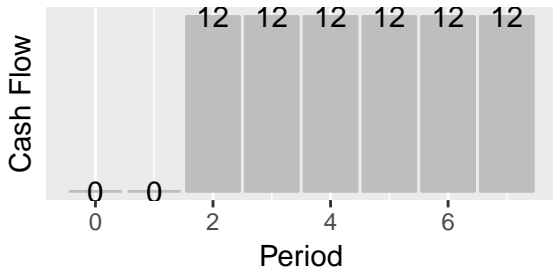
$$\frac{1(P|A, i, 8)}{(1+i)^0} + \frac{1(P|A, i, 7)}{(1+i)^1} + \frac{1(P|A, i, 6)}{(1+i)^2} + \frac{1(P|A, i, 5)}{(1+i)^3} + \frac{1(P|A, i, 4)}{(1+i)^4}$$

How about this?



Hint: $-2(P|G, i, 7)$ is wrong in many ways.

You should see it as



Solution

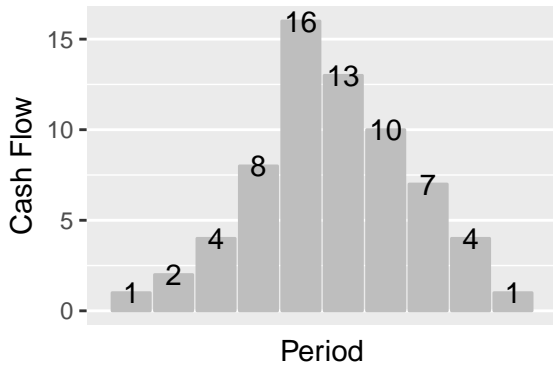
$$\frac{12(P|A, i, 6)}{(1+i)^1} + \frac{-2(P|G, i, 6)}{(1+i)^1}$$

- ▶ All Linear gradients start at zero. If they don't you have to include a constant series.
- ▶ Work by analogy from slope intercept form of a line.
 - ▶ $Y = MX + b$
 - ▶ $Y = (P|G) + (P|A)$

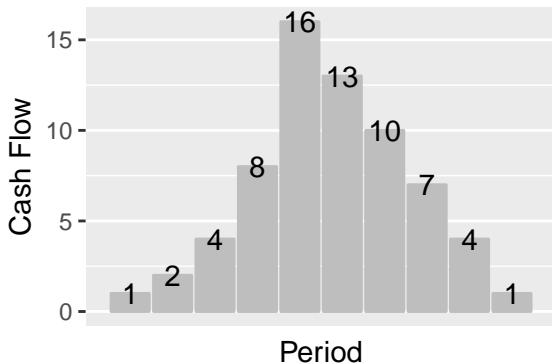
Review

- ▶ Look for shapes constant, geometric gradient and linear gradient.
- ▶ Count for the N. Remember to add one for the pot only on linear gradient.
- ▶ Sum of N and exponent is equal to the time period where you last see the cash flow
- ▶ Work the analogy with slope intercept form of a line.

Big Test

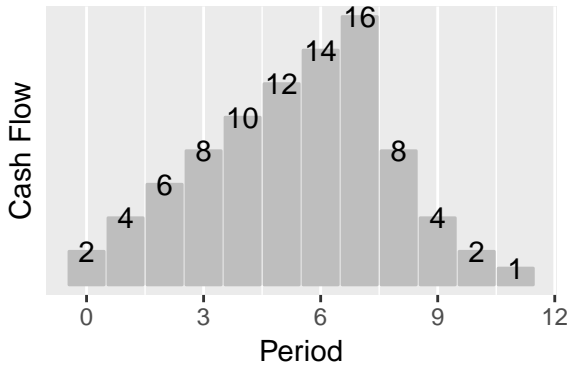


Big Test (A Solution)

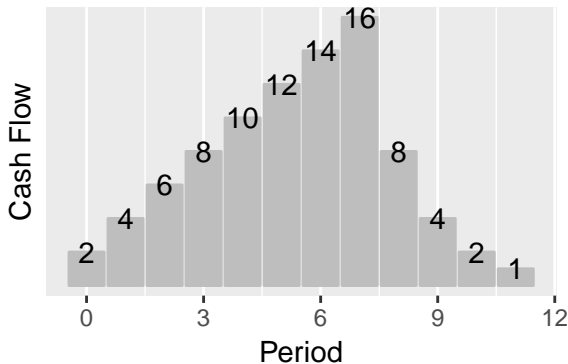


$$\frac{1(P|A_1, i, g = 100\%, 4)}{(1+i)^{-1}} + \frac{16(P|A, i, 6)}{(1+i)^3} - \frac{3(P|G, i, 6)}{(1+i)^3}$$

Similar problem



Similar problem (Solution)

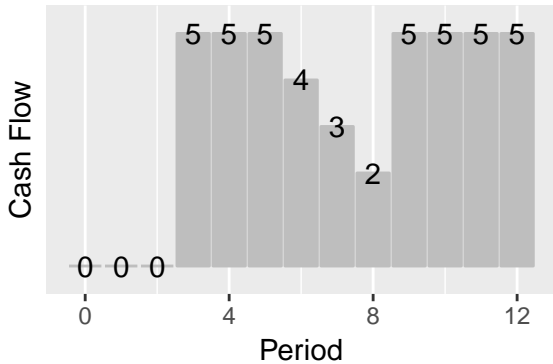


$$\frac{2(P|G, i, 8)}{(1+i)^{-2}} + \frac{16(P|A_1, i, g = -50\%, 5)}{(1+i)^6}$$

Lets go backwards. Find the intended cash flow for this.

$$\frac{5(P|A, i, 10)}{(1+i)^2} - \frac{1(P|G, i, 4)}{(1+i)^4}$$

Solution



Hints for Study

- ▶ Kill trees. Work lots of problems. Everything after this depends on understanding TVM.
- ▶ Drill and Kill the examples and the videos at bottom of the page. <http://ec314-pdx-edu.wikidot.com/q2:time-value-of-money>
- ▶ When evaluating these expressions, try to do all the work in the calculator in one operation. Sometimes rounding and truncation errors are bad with large interest rates.

First Application of TVM: Loans

- ▶ Been dealing with the interest rate as the per period rate.
- ▶ We could move value in time but could not change the unit of analysis, the definition of a period.
- ▶ Effective interest rates allows us to stretch and compress time scales, and
- ▶ Adapt the interest rate to the unit of analysis we want to use.

Basics of Effective Interest Rates

- ▶ Interest rates have three components:
 - ▶ Nominal statement
 - ▶ Nominal Period
 - ▶ Compounding period
- ▶ The final component 'Desired Effective Period' depends on the problem you are trying to solve.
 - ▶ For loans, you want the effective rate per payment period.
 - ▶ For general interest rate comparisons, you want an annual rate most of the time.
 - ▶ Others are problem specific.

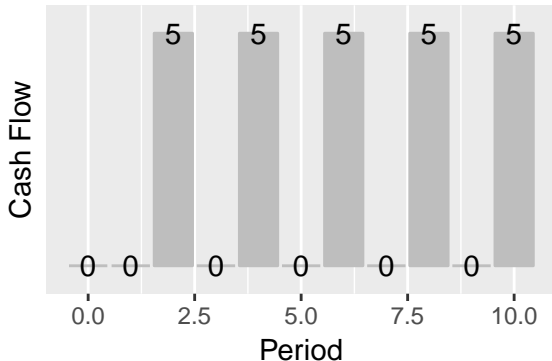
General Form

$$\left(1 + \frac{\textit{Nominal}}{\textit{\# Compounding in Nominal Period}}\right)^{\textit{\# Compounding in Desired}} - 1$$

Example: Effective annual rate for 2% per month compounded daily

$$\left(1 + \frac{.02}{30}\right)^{365} - 1 = 0.2753903$$

Application to Time Value of Money Calcs



Not a constant series but you can find PW with brute force with interest at 10% per period compounded each period.

$$P = \frac{5}{(1 + .1)^2} + \frac{5}{(1 + .1)^4} + \frac{5}{(1 + .1)^6} + \frac{5}{(1 + .1)^8} + \frac{5}{(1 + .1)^{10}}$$

Reframe the problem

Find the 2-Period interest rate

$$\left(1 + \frac{.1}{1}\right)^2 - 1 = 0.21 = i_2$$

and rewrite as a constant series

$$P = 5(P|A, i = i_2, 5)$$

Try the two calculations, brute force and this. They give the same result 14.6299217.

But Lets Get Into Loans

There is a bewildering number of parameters for loans.

- ▶ How interest is treated:
 - ▶ Simple interest – Which we will not address
 - ▶ Amortizing – which uses $(P|A)$ and is common.
- ▶ How interest is paid
 - ▶ Interest only – you owe what your borrowed at the end.
 - ▶ Fully Amortizing – you owe nothing at the end.
 - ▶ Other – Negative and partial amortizing
- ▶ Stability of interest rates and how changes are implemented
 - ▶ Fixed rate loans
 - ▶ Adjustable rate loans
 - ▶ Refinance

And many conventions on how loans and interest rates are described.

Example Convention

“I took out a 3.5% mortgage”.

- ▶ That means that they took out a 30-year fixed rate mortgage where the interest rate is 3.5% per year compounded monthly with monthly payments.
- ▶ They told you the nominal rate but all the other parameters are by convention.

Lets take out and pay off some loans

- ▶ Our loan examples will typically have high rates, so you can see the effects of changes on the interest rates.
- ▶ The terms may be goofy, weekly payments compounded daily.
- ▶ Usually a small number of payments to keep everything on one slide.

How to Calculate Payments on Amortizing Loan

$$Payment = Principal(A|P, i = i_e, \# \text{ Payments})$$

i_e is the effective interest rate per payment period.

Example

Borrow 10,000 at 7% per week compounded daily with biweekly payments for 10 months.

Effective interest rate per payment period:

$$\left(1 + \frac{.07}{7}\right)^{14} - 1 = 0.1494742 = i_e$$

Payments: $10000(A|P, i = i_e, 20) = 1592.9671609$

Simpler Example with Payment Details

\$10,000 at 10% per month, compounded monthly, for three monthly payments.

What are the payments? $10000(A|P, i = .1, 3) = 4021.1480363$

Amortization Tables

- ▶ They show for each payment
 - ▶ The payment
 - ▶ The interest expense – important since this goes on the incomes statement and reduces taxes.
 - ▶ The principal payment – Does not reduces taxes or go on income statement but it does reduce cash flow.
 - ▶ Balance – How much you owe after the payment.
- ▶ How you get them:
 - ▶ Payments are always the same
 - ▶ Interest expense is the balance remaining times the effective interest rate per payment period.
 - ▶ Principal payment is the payment less the interest expense.
 - ▶ Balance remaining is the previous balance remaining less then principal payment.

For the loan

Payment Number	Payment	Interest	Principal	Balance
0				10,000
1	4,021	1,000	3,021	6,979
2	4,021	698	3,323	3,656
3	4,021	366	3,656	0.000000000012

Yes, there is a little rounding.

You can also find balance remaining with $(P|A)$

- ▶ The balance remaining on an amortizing loan is always equal to the present worth of the remaining payments.
- ▶ From Table before, balance remaining after one payment is, 6,979.
- ▶ $4,021(P|A, i = .1, 2) = 6978.8519637$

Refi vs Interest rate change

- ▶ Refinance means taking out a whole new loan with a different interest rate.
 - ▶ If you refinance your mortgage, 360 payments, after making 120 payments, your new payments are based on the current balance remaining and the new rate, but it resets the number of payments
 - ▶ You have 360 payments to pay off the loan.
- ▶ If the interest rate changes, such as when you have an ARM loan.
 - ▶ You calculate the new payments based on the new rate and current balance remaining, but the number of remaining payments stays the same
 - ▶ You would still have only 240 payment left.

Compare a refi at 5% vs interest reduction to 5% after one payment.

- ▶ Refi: $6978.8519637(A|P, i = .05, 3) = 2562.6942124$
- ▶ ARM: $6978.8519637(A|P, i = .05, 2) = 3753.2606293$
- ▶ In short:
 - ▶ Refi can lower your rate and extend the term resulting in lower payments.
 - ▶ Lowering an interest rate reduces the rate but does not decrease the remaining term. Payments are lower but by not as much as a refi.

What about Interest Only Loans?

- ▶ There are good use cases.
- ▶ They are great if:
 - ▶ You are trying to conserve cash
 - ▶ You do not intend to make all the payments on the loan.
 - ▶ You are financing an asset that *will* appreciate in value.
- ▶ Having to pay back what you borrowed after making a lot of payments can come as a surprise.
 - ▶ Modern terminology is 'balloon payment'
 - ▶ Old terminology is 'bullet'

Compare and interest only mortgage vs amortizing

Start with the same 6% interest rate on a 30-year \$200,000 mortgage.

- ▶ The effective interest rate per payment period is $\frac{.06}{12} = .5\%$
- ▶ Payment on amortizing loan is $200000(A|P, i = .5\%, 360) = 1199.1010503$.
- ▶ Payment on interest only is $200000(.005) = 1000$.
- ▶ Note that the payments are lower for interest only, but they still owe the original \$200000 after making all those payments.

Typical Amortization Table for Interest Only Loan

Payment Number	Payment	Interest	Principal	Balance
0				10,000
1	1,000	1,000	0	10,000
2	1,000	1,000	0	10,000
3	1,000	1,000	10,000	0

There are multiple conventions on where to put the bullet. I like it there but others have a separate line or column for the principal repayment. The disadvantage is that it blows the usual definition of principal payment as the payment less interest expense.