Chapter 10.2 - The Hot Water Heater Problem Revisited

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author: Clayton Johnson

date:

autosize: true

Example 10.3: Solving Diff. Eqs with Integrating Factors

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We need to solve equation 10.7 (below):

$$

\frac{dU}{dt} = \beta - \alpha U,

$$

where

$$

\alpha = \frac{hS}{cm}, \\

\beta = \frac{q+hSu\_s}{cm}, \\

U(0) = u\_0.

$$

Example 10.3: Solving Diff. Eqs with I.F.

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$$

\frac{dU}{dt} + \alpha U = \beta,

$$

First-order linear differential equation. We can solve using integrating factor technique, producing the integrating factor,

$$

R(t) = e^{\int\_{0}^{t} \alpha dt} = e^{\alpha t}.

$$

Example 10.3: Solving Diff. Eqs with I.F.

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Rewriting the differential equation, we have that

$$

\begin{align\*}

\frac{dU}{dt} + \alpha U &= \beta, \\

\frac{dU}{dt}\times e^{\alpha t} + \alpha U \times e^{\alpha t} &= \beta \times e^{\alpha t}, \\

e^{\alpha t} \frac{dU}{dt} + \frac{d}{dt}\left( e^{\alpha t} \right) U &= \beta e^{\alpha t}, \\

\frac{d}{dt} \left( U e^{\alpha t} \right) &= \beta e^{\alpha t}, \text{ using the chain rule.}

\end{align\*}

$$

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Integrating both sides with respect to $t$,

$$

e^{\alpha t}U = \frac{\beta}{\alpha}e^{\alpha t} + K, \\

\text{and } U = \frac{\beta}{\alpha} + Ke^{-\alpha t},

$$

where $K$ is the constant of integration.

Example 10.3: Solving Diff. Eqs with I.F.

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Applying the initial condition $U(0)=u\_0$, we find that $K = u\_0 - \frac{\beta}{\alpha}$. Hence, the solution is,

$$

U = u\_0e^{-\alpha t} + \frac{\beta}{\alpha}(1-e^{-\alpha t}).

$$

Substituting $\alpha$ and $\beta$ gives equation 10.8 (below),

$$

U(t) = \left( u\_0 - u\_s - \frac{q}{hS} \right) e^{- \frac{hS}{cm} t } + u\_s + \frac{q}{hS}.

$$