Ch. 12.4 Lake Pollution Revisited

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author:

date:

autosize: true

Formulating the PDE

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- IvAnn's Word Equation:

$$

\begin{equation}

\begin{Bmatrix}

\mathrm{rate \, of \, change} \\

\mathrm{of \, pollutant} \\

\mathrm{mass} \\

\end{Bmatrix}

=

\begin{Bmatrix}

\mathrm{rate \, pollutant} \\

\mathrm{mass \, enters} \\

\mathrm{at\,} x \\

\end{Bmatrix}

-

\begin{Bmatrix}

\mathrm{rate \, pollutant} \\

\mathrm{mass \, leaves} \\

\mathrm{at \,} x+\delta x \\

\end{Bmatrix}

\end{equation}

$$

- Terms:

- $C(x, t)$ denotes the pollutant concentration $\left( \frac{ \text{Mass} }{ \text{Volume} } \right)$

- $J(x, t)$ denotes the pollutant mass flux $\left( \frac{ \text{Mass} }{ \text{Area} \times \text{Time} } \right)$

- $A$ is the area of the cross-section

Formulating the PDE (Cont.)

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$$

\begin{equation}

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-

\begin{Bmatrix}

\mathrm{rate \, pollutant} \\

\mathrm{mass \, leaves} \\

\mathrm{at \,} x+\delta x \\

\end{Bmatrix}

\end{equation}

$$

$$

\begin{align\*}

\frac{ \partial }{ \partial t } \left( CA \delta x \right) &= J(x, t)A - J(x + \delta x, t)A, \\

A \delta x \frac{ \partial C }{ \partial t } &= J(x, t)A - J(x + \delta x, t)A, \\

\frac{ \partial C }{ \partial t } &= - \left( \frac{ J(x + \delta x, t) - J(x, t) }{ \delta x } \right). \\

\text{Let } \delta x \rightarrow 0,\\

\frac{ \partial C }{ \partial t } &= - \frac{ \partial J }{ \partial x }.

\end{align\*}

$$

Formulating the PDE (Cont.)

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How does the pollutant travel through the lake? Advection!

We are going to assume a constant flow rate, $F$, of the pollutant/water mix. The pollutant mass flux, $J$, is the $\textbf{rate}$ of mass flow per area, $A$. Thus, we have that

$$

\text{Rate of Pollutant Mass Flow} = FC(x, t),

$$

and

$$

J(x, t) = \frac{ FC(x, t) }{ A }.

$$