Notes - On Topological Cyclic Homology

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1 Cyclotomic Spectra

1.1 Cyclotomic spectra and TC

Definition 1.1. A cyclotomic spectrum is a spectrum X with a $\mathbb{T}$-action together with $\mathbb{T}$-equivariant maps $\varphi_p\colon X \to X^{tC_p}$. For p a fixed prime, a p-cyclotomic spectrum is a spectrum X with a C_{p^∞}-action and a C_{p^∞}-equivariant map $\varphi_p\colon X \to X^{tC_p}$.

Remark 1. Here we are using the identification $\mathbb{T} \simeq \mathbb{T}/C_p$ (resp. $C_{p^\infty} \simeq C_{p^\infty}/C_p$) via the map $x \mapsto [x^{1/p}]$ to identify the residual $\mathbb{T}/C_p$-action (resp. C_{p^∞}/C_p-action) of X^{tC_p}.

Example 1. Let $S^0 \in \mathbf{Sp}^{B\mathbb{T}}$ with trivial $\mathbb{T}$-action and $f\colon B\mathbb{T} \to B(\mathbb{T}/C_p)$. Then we the following composition of maps

$$S^0 \to f_* f^*(S^0) \to (f^*(S^0))^{tC_p}$$

in $\mathbf{Sp}^{B(\mathbb{T}/C_p)} \simeq \mathbf{Sp}^{B\mathbb{T}}$. It will later be shown that as a cyclotomic spectra, $S^0 \simeq \mathrm{THH}(S^0)$.

Remark 2. In the notes they worried about the equivariant equivalence of the map a little more. Not quite sure why they do this...

Definition 1.2. The ∞-category of cyclotomic spectra, denoted **CycSp**, is defined as the lax equalizer

$$\mathbf{CycSp} := \mathrm{LEq}((\mathrm{id})_p, ((-)^{tC_p})_p\colon \mathbf{Sp}^{B\mathbb{T}} \rightrightarrows \prod_p \mathbf{Sp}^{B\mathbb{T}}).$$

Similarly for a fixed prime p the ∞-category of p-cyclotomic spectra, denoted $\mathbf{CycSp}_p$, is defined as the lax equalizer

$$\mathbf{CycSp}_p := \mathrm{LEq}(\mathrm{id}, (-)^{tC_p}\colon \mathbf{Sp}^{BC_{p^\infty}} \rightrightarrows \mathbf{Sp}^{BC_{p^\infty}}).$$

Both **CycSp** and $\mathbf{CycSp}_p$ are presentable stable ∞-categories. The forgetful functors $\mathbf{CycSp} \to \mathbf{Sp}$ and $\mathbf{CycSp}_p \to \mathbf{Sp}$ reflect equivalences, are exact, and preserve all small colimits.

Definition 1.3. Let $(X, (\varphi_p)_p) \in \mathbf{CycSp}$ (resp. $(X, \varphi_p) \in \mathbf{CycSp}_p$) be a cyclotomic (resp. p-cyclotomic) spectrum. The integral (resp. p-typical) topological cyclic homology, denoted $\mathrm{TC}(X)$ (resp. $\mathrm{TC}(X,p)$) is the mapping spectrum

$$\mathrm{MapSp}_{\mathbf{CycSp}}(S^0, X)(resp. \mathrm{MapSp}_{\mathbf{CycSp}_p}(S^0, X)).$$

Let $R \in \mathbf{Alg}_{\mathbb{E}_1}(\mathbf{Sp})$, then $\mathrm{TC}(R) := \mathrm{TC}(\mathrm{THH}(R))$ and $\mathrm{TC}(R,p) := \mathrm{TC}(\mathrm{THH}(R), p)$.

Remark 3. It will be shown later that $\mathrm{THH}(R)$ has a canonical structure of a cyclotomic spectrum.

Proposition 1. *Let $(X, (\varphi_p)_p)$. There is a functorial fiber sequence*

$$\mathrm{TC}(X) \to X^{h\mathbb{T}} \xrightarrow{(\varphi_p^{h\mathbb{T}} - \mathrm{can})_p} \prod_p (X^{tC_p})^{h\mathbb{T}}$$

where $\mathrm{can}\colon X^{h\mathbb{T}} \simeq (X^{hC_p})^{h(\mathbb{T}/C_p)} \simeq (X^{hC_p})^{h\mathbb{T}} \to (X^{tC_p})^{h\mathbb{T}}$.

2 Topological Hochschild Homology

2.1 The Tate Diagonal

Goal. Produce a substitute for the diagonal map of spaces in the category of spectra.

Proposition 2. *Let p be a prime. The functor $T_p\colon \mathbf{Sp} \to \mathbf{Sp}$ given by $X \mapsto (X^{\otimes p})^{tC_p}$ is exact, where $X^{\otimes p}$ denotes the p-fold self tensor product with the C_p-action given by the cyclic permutation of the factors.*

Proof. □

Recall that there is an equivalence $\mathbf{Fun}^{\mathrm{Ex}}(\mathbf{Sp}, \mathbf{Sp}) \simeq \mathbf{Fun}^{\mathrm{lex}}(\mathbf{Sp}, \mathbf{Spc})$ given by composing with Σ^∞. Since $\mathrm{id}_{\mathbf{Sp}}$ corresponds to Ω^∞ under this correspondence, by the Yoneda lemma, we have an equivalence between the space of natural transformations $\mathbf{Map}_{\mathbf{Fun}^{\mathrm{Ex}}(\mathbf{Sp},\mathbf{Sp})}(\mathrm{id}_{\mathbf{Sp}}, F)$ and the mapping space $\mathbf{Map}_{\mathbf{Sp}}(S^0, F(S^0))$.

Definition 2.1. The Tate diagonal is the natural transformation

$$\Delta_p\colon \mathrm{id}_{\mathbf{Sp}} \to T_p\colon X \to (X^{\otimes p})^{tC_p}$$

corresponding to the map $S^0 \to (S^0)^{tC_p}$ associated to the free cyclotomic structure of S^0.

Theorem 1. *Let $X \in \mathbf{Sp}$ be bounded below. Then the map Δ_p exhibits $(X^{\otimes p})^{tC_p}$ as the p-completion of X.*

Remark 4. Note that the existence of Δ_p uses the universal characterization of **Sp** as the stabilization of **Spc**. In particular there is no lax symmetric monoidal transformation $C \to (C^{\otimes p})^{tC_p}$ as functors $\mathcal{D}(\mathbb{Z}) \to \mathcal{D}(\mathbb{Z})$.

A Equivariant Stable Homotopy Theory

Goal. Reproduce as much of the required theory regarding spectra with G-action and genuine G-spectra with spectral mackey functors. In particular:

- All the different kinds of fixed points in $G\mathbf{Sp}$.
- Describe the fixed point functors for $C_{p^\infty}\mathbf{Sp} := \lim_n C_{p^n}\mathbf{Sp}$ via forgetful maps.
- Similarly for $\mathbb{T}\mathbf{Sp}_{\mathcal{F}}$ and furthermore achieve this using cyclonic spectra.

A.1 Genuine G-Spectra

Definition A.1. The ∞-category of genuine G-spectra is $G\mathbf{Sp} := \mathbf{Fun}^{\oplus}(\mathbf{A}^{\mathrm{eff}}(\mathbf{Fin}_G), \mathbf{Sp})$. For $H \leq G$ and $X \in G\mathbf{Sp}$, the genuine H-fixed point spectrum of X is defined as $X^H := X(G/H) \simeq \mathrm{MapSp}(\Sigma^\infty_+ G/H, X)$.

Proposition 3. *The ∞-category $G\mathbf{Sp}$ is stable with t-structure where $X \in G\mathbf{Sp}$ is connective if it is pointwise connective. In particular $G\mathbf{Sp}^\heartsuit$ is equivalent to the classical category of mackey functors. There is an adjunction*

$$\Sigma^\infty \dashv \Omega^\infty \colon G\mathbf{Spc}_* \leftrightarrows G\mathbf{Sp}$$

where $\Omega^\infty(E) = \Omega^\infty \circ E \circ j$ and $j \colon \mathbf{O}_G^{\mathrm{op}} \to \mathbf{A}^{\mathrm{eff}}(\mathbf{Fin}_G)$ is the obvious inclusion. For $I \in \mathbf{Fin}_G$, the suspension spectra $\Sigma^\infty_+ I$ is self dual in $G\mathbf{Sp}$.

Remark 5. In [NS] they define equivalences of G-spectra using geometrix fixed points. This is equivalent to equivalences detected by genuine fixed points.

Todo

- Add in detail about $\mathbf{A}^{\mathrm{eff}}$ and its self duality

A.2 Restriction, Induction, and the Wirthmüller Isomorphism

Let H be a subgroup of G, we will construct an ambidextrous adjunction

$$\mathrm{res}^G_H \colon G\mathbf{Sp} \leftrightarrows H\mathbf{Sp} \colon \mathrm{ind}^G_H.$$

To construct the above adjunction, we define adjunctions at the following steps:

- Finite G sets $\mathbf{Fin}_G$.
- Span categories $\mathbf{A}^{\mathrm{eff}}(\mathbf{Fin}_G)$.
- Spectral Mackey Functors $\mathbf{Fun}^{\oplus}(\mathbf{A}^{\mathrm{eff}}(\mathbf{Fin}_G), \mathbf{Sp})$.

Let $\mathrm{res}^G_H \colon \mathbf{Fin}_G \to \mathbf{Fin}_H$ denote the forgetful functor. Then the left adjoint is denoted by $\mathrm{ind}^G_H \colon \mathbf{Fin}_H \to \mathbf{Fin}_G$ which sends a finite H-set X to the G-set $G \times_H X := (G \times X)/H$ where H acts by $h(g, x) = (gh^{-1}, hx)$. In particular, ind^G_H sends H/K to G/K. Since both res^G_H and ind^G_H preserve pullbacks both extend to functors $\mathrm{res}^G_H \colon \mathbf{A}^{\mathrm{eff}}(\mathbf{Fin}_G) \to \mathbf{A}^{\mathrm{eff}}(\mathbf{Fin}_H)$ and $\mathrm{ind}^G_H \colon \mathbf{A}^{\mathrm{eff}}(\mathbf{Fin}_H) \to \mathbf{A}^{\mathrm{eff}}(\mathbf{Fin}_G)$. Furthermore they are still adjoints at this level, i.e. $\mathrm{res}^G_H \dashv \mathrm{ind}^G_H$.

Remark 6. Adjointness is nontrivial and does not follow immediately from $\mathbf{A}^{\mathrm{eff}}$ being functorial.

In fact by self duality of $\mathbf{A}^{\mathrm{eff}}(\mathbf{Fin}_G)$, we have that $\mathrm{ind}^G_H \dashv \mathrm{res}^G_H$. Finally we define $\mathrm{res}^G_H \colon G\mathbf{Sp} \to H\mathbf{Sp}$ by precomposition with $\mathrm{ind}^G_H \colon \mathbf{A}^{\mathrm{eff}}(\mathbf{Fin}_H) \to \mathbf{A}^{\mathrm{eff}}(\mathbf{Fin}_G)$ and similarly for ind^G_H.

Remark 7. Observe that for $X \in G\mathbf{Sp}$ and $K \leq H \leq G$, we have that

$$\begin{aligned}(\mathrm{res}^G_H X)^K &= (\mathrm{res}^G_H X)(H/K) \\ &= X(\mathrm{ind}^G_H(H/K)) \\ &= X(G/K) \\ &= X^K.\end{aligned}$$

That is the genuine K-fixed points for $K \leq H$ do not change.

Todo

- Why do $\mathrm{ind}, \mathrm{res}$ preserve pullbacks?
- Lol if deranged enough then explaning why adjoints at the level of $\mathbf{A}^{\mathrm{eff}}$.

A.3 Homotopy Fixed Points

Recall that for a spectra with G-action $X \in \mathbf{Sp}^{BG} := \mathbf{Fun}(BG, \mathbf{Sp})$, the homotopy fixed points of X is $X^{hG} := \lim_{BG} X$. For a genuine G-spectra forgetting to the underlying spectra with G-action is encoded by the functor $u_G \colon G\mathbf{Sp} \to \mathbf{Fun}(BG, \mathbf{Sp})$ given by restriction along $BG \hookrightarrow \mathbf{A}^{\text{eff}}(\mathbf{Fin}_G)$. Then the homotopy G-fixed point functors are given by forgetting to $\mathbf{Sp}^{BG}$ then taking the usual homotopy fixed points. We can similarly define the homotopy H-fixed point functors by taking a further restriction to $\mathbf{Sp}^{BH}$ or equivalently by initially applying res^G_H to land in $H\mathbf{Sp}$.

By a result of Saul Glasman we have an alternate way of defining the homotopy fixed points. Let $\mathbf{Fin}_G^{\text{free}} \subset \mathbf{Fin}_G$ denote the full subcategory of finite free G-sets.

Theorem 2. *The functor $G\mathbf{Sp}^{\text{free}} := \mathbf{Fun}^{\oplus}(\mathbf{A}^{\text{eff}}(\mathbf{Fin}_G^{\text{free}}), \mathbf{Sp}) \to \mathbf{Fun}(BG, \mathbf{Sp})$ given by precomposition with the inclusion induces an equivalence of ∞-categories.*

Remark 8. Here the inclusion is given by sending $* \in BG$ to $G/e \in \mathbf{O}_G \subset \mathbf{Fin}_G^{\text{free}} \subset \mathbf{A}^{\text{eff}}(\mathbf{Fin}_G^{\text{free}})$. In particular, $\mathbf{A}^{\text{eff}}(\mathbf{Fin}_G^{\text{free}})$ is the free semiadditive ∞-category generated by BG.

Proof. Can be proved via Lawvere theory, we refer to notes by Maxime Ramzi. ☐

On the other hand we may also consider the restriction map $u_G \colon G\mathbf{Sp} \to G\mathbf{Sp}^{\text{free}}$ induced by the inclusion $\mathbf{Fin}_G^{\text{free}} \subset \mathbf{Fin}_G$. It is not hard to see that composing u_G with the equivalence from before is the same as directly restricting to BG. Define the Borel completion which we denote by b_G as the right adjoint to u_G.

Remark 9. Abusing notation and conflating $G\mathbf{Sp}^{\text{free}}$ with $\mathbf{Sp}^{BG}$, we may define the homotopy H-fixed point functor as $X \mapsto (u_G X)^{hH}$. Equivalently we first restrict to H-spectra than apply the corresponding functor, i.e. $X \mapsto (u_H \text{res}^G_H X)^{hH}$.

Theorem 3. *For $X \in G\mathbf{Sp}^{\text{free}}$, there is an equivalence $X^{hH} \simeq (b_G X)^H$ for all $H \leq G$.*

Proof. First suppose that this is true for $H = G$. As shown above both adjoints u_G and b_G commute with restrictions. Then

$$(b_G X)^{hH} = (\text{res}^G_H u_G b_G X)^{hH} \simeq (u_H b_H \text{res}^G_H X)^{hH} \simeq (\text{res}^G_H X)^{hH} \simeq (b_H \text{res}^G_H X)^H \simeq (\text{res}^G_H b_G X)^H \simeq (b_G X)^H.$$

Thus it remains to prove the case $H = G$. ☐

Theorem 4. *The functor b_G is fully faithful and the essential image consists of the full subcategory of $X \in G\mathbf{Sp}$ such that the natural map $X^H \to X^{hH}$ is an equivalence for all $H \leq G$ which we refer to as Borel-complete G-spectra.*

Todo

- Give proof for homotopy fixed points = genuine fixed points.
- Replace the above remark with proof that restriction and adjoints commute.
- Why is it true that $(u_H b_H \text{res}^G_H X)^{hH} \simeq (\text{res}^G_H X)^{hH}$?

A.4 Geometric Fixed Points

We now construct the geometrix fixed point functors. Let $\mathcal{F}$ be a family of subgroups closed under conjugation and subgroups. Then define

$$(E\mathcal{F})^K \begin{cases} * & k \in \mathcal{F} \\ \varnothing & k \notin \mathcal{F} \end{cases} \in G\mathbf{Spc}.$$

Example 2. Fix a subgroup $H \subseteq G$, then define $\mathcal{F}_H$ to be the collection of subgroups which do not contain a conjugate of H as a subgroup.

Remark 10. We could alternatively describe such a collection of subgroups as a sieve in $\mathbf{O}_G$. In particular, the above example can be described as the maximal sieve not containing G/H.

We will also be interested in the unreduced suspension of $E\mathcal{F}$. That is

$$E\mathcal{F}_* \to S^0 \to \widetilde{E\mathcal{F}},$$

in particular

$$(\widetilde{E\mathcal{F}})^K = \begin{cases} * & K \in \mathcal{F} \\ S^0 & K \notin \mathcal{F}. \end{cases}$$

In the case $\mathcal{F}$ consists of only the trivial subgroup, then $E\mathcal{F} = EG$ and $\widetilde{E\mathcal{F}} = \widetilde{EG}$.

Lemma 1. Let $E \in G\mathbf{Spc}$, then $(E \wedge \Sigma^\infty \widetilde{E\mathcal{F}})^K \simeq 0$ if $K \in \mathcal{F}$ and $E^K \to (E \wedge \Sigma^\infty \widetilde{E\mathcal{F}})^K$ is an equivalence if $K \notin \mathcal{F}$.

Remark 11. The equivalences in the lemma above are induced by the map $S^0 \to \widetilde{E\mathcal{F}}$ from the cofiber sequence.

Theorem 5. *There is a smashing localization such that the local objects are G-spectra concentrated away from $\mathcal{F}$, i.e. $E \to E \wedge \Sigma^\infty \widetilde{E\mathcal{F}}$ is an equivalence.*

Remark 12. This is immediate from the lemma and that $\widetilde{E\mathcal{F}}$ is idempotent from looking at the fixed points.

Let $\mathbf{Fin}_{\mathcal{F}^c} \subseteq \mathbf{Fin}_G$ denote the full subcategory of those fintie G-sets whose stabilizers are not in $\mathcal{F}$.

Theorem 6. *The category of G-spectra concentrated away from $\mathcal{F}$ is equivalent to $\mathbf{Fun}^\oplus(\mathbf{A}^{\mathrm{eff}}(\mathbf{Fin}_{\mathcal{F}^c}), \mathbf{Sp})$ and the inclusion of the precomposition with the map $\Psi\colon \mathbf{A}^{\mathrm{eff}}(\mathbf{Fin}_G) \to \mathbf{A}^{\mathrm{eff}}(\mathbf{Fin}_{\mathcal{F}^c})$ that sends every $I \in \mathbf{Fin}_G$ to the subset of points with stabilizers in $\mathcal{F}$.*

Remark 13. Fix $H \leq G$ and consider $\mathcal{F}$ as in the previous example. Observe that $K \notin \mathcal{F}$ if and only if there is a map of G-sets $G/H \to G/K$. In the case that H is a normal subgroup of G, then $\mathbf{Fin}_{\mathcal{F}^c}$ is the category of finite $(G/H) - sets$. Indeed $\mathbf{Fin}_{\mathcal{F}^c}$ would be all finite G-sets whose stabilizers are subgroups of H.

In particular, the localization functor is given by left Kan extension along Ψ, i.e.

$$\Psi_!\colon G\mathbf{Sp} \to \mathbf{Fun}^\oplus(\mathbf{A}^{\mathrm{eff}}(\mathbf{Fin}_{\mathcal{F}^c}), \mathbf{Sp}).$$

We call $\Phi_!$ the geometric H-fixed point functor and denote it by Φ^H or $(-)^{\Phi H}$. In the case H is normal, then this functor takes values in (G/H)-spectra.

Remark 14. The geometric H-fixed point functors are symmetric monoidal.

A.5 Comparison of Fixed Points

A.6 $G = C_{p^\infty}$ or $\mathbb{T}$

Todo

- Specify a bit more about genuine fixed points
  - In particular where the natural transformation $(-)^H \to (-)^{\Phi H}$ comes from.
  - Also the composability of geometric fixed points, i.e. something like $\Phi^{H'/H} \circ \Phi^H \to \Phi^{H'}$ is an equivalence.
  - Mention how the geometric fixed points form a localization and what the corresponding reflective subcategory is.
- Do the same for homotopy fixed points
  - In particular how there is NO DIRECT relation with geometric fixed points.
- Write about the C_{p^∞} and $\mathbb{T}$ equivariant variants of the theory
  - In particular what results hold "essentially" analogously.
- Also specify how we can get borel G-spectra using spectral mackey functors.