Notes - On Topological Cyclic Homology

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Contents

1 Equivariant Stable Homotopy Theory

Definition 1.1. The ∞-category of genuine G-spectra is $G\mathbf{Sp} := \mathbf{Fun}^{\oplus}(\mathbf{A}^{\mathrm{eff}}(\mathbf{Fin}_G), \mathbf{Sp})$

Remark 1. In [NS] they define equivalences of G-spectra using geometrix fixed points. This is equivalent to equivalences detected by genuine fixed points.

We now construct the geometrix fixed point functors following the appendix in Denis's thesis appendix. Let $\mathcal{F}$ be a family of subgroups closed under conjugation and subgroups. Then define

$$(E\mathcal{F})^K \begin{cases} * & k \in \mathcal{F} \\ \varnothing & k \notin \mathcal{F} \end{cases} \in G\mathbf{Spc}.$$

Example 1. Fix a subgroup $H \subseteq G$, then define $\mathcal{F}_H$ to be the collection of subgroups which do not contain a conjugate of H as a subgroup.

Remark 2. We could alternatively describe such a collection of subgroups as a sieve in $\mathbf{O}_G$. In particular, the above example can be described as the maximal sieve not containing G/H.

We will also be interested in the unreduced suspension of $E\mathcal{F}$. That is

$$E\mathcal{F}_* \to S^0 \to \widetilde{E\mathcal{F}},$$

in particular

$$(\widetilde{E\mathcal{F}})^K = \begin{cases} * & K \in \mathcal{F} \\ S^0 & K \notin \mathcal{F}. \end{cases}$$

In the case $\mathcal{F}$ consists of only the trivial subgroup, then $E\mathcal{F} = EG$ and $\widetilde{E\mathcal{F}} = \widetilde{EG}$.

Lemma 1. Let $E \in G\mathbf{Spc}$, then $(E \wedge \Sigma^\infty \widetilde{E\mathcal{F}})^K \simeq 0$ if $K \in \mathcal{F}$ and $E^K \to (E \wedge \Sigma^\infty \widetilde{E\mathcal{F}})^K$ is an equivalence if $K \notin \mathcal{F}$.

Remark 3. The equivalences in the lemma above are induced by the map $S^0 \to \widetilde{E\mathcal{F}}$ from the cofiber sequence.

Theorem 1. *There is a smashing localization such that the local objects are G-spectra concentrated away from $\mathcal{F}$, i.e. $E \to E \wedge \Sigma^\infty \widetilde{E\mathcal{F}}$ is an equivalence.*

Remark 4. This is immediate from the lemma and that $\widetilde{E\mathcal{F}}$ is idempotent from looking at the fixed points.

Let $\mathbf{Fin}_{\mathcal{F}^c} \subseteq \mathbf{Fin}_G$ denote the full subcategory of those fintie G-sets whose stabilizers are not in $\mathcal{F}$.

Theorem 2. *The category of G-spectra concentrated away from $\mathcal{F}$ is equivalent to $\mathbf{Fun}^{\oplus}(\mathbf{A}^{\mathrm{eff}}(\mathbf{Fin}_{\mathcal{F}^c}), \mathbf{Sp})$ and the inclusion of the precomposition with the map $\Psi\colon \mathbf{A}^{\mathrm{eff}}(\mathbf{Fin}_G) \to \mathbf{A}^{\mathrm{eff}}(\mathbf{Fin}_{\mathcal{F}^c})$ that sends every $I \in \mathbf{Fin}_G$ to the subset of points with stabilizers in $\mathcal{F}$.*

Remark 5. Fix $H \leq G$ and consider $\mathcal{F}$ as in the previous example. Observe that $K \notin \mathcal{F}$ if and only if there is a map of G-sets $G/H \to G/K$. In the case that H is a normal subgroup of G, then $\mathbf{Fin}_{\mathcal{F}^c}$ is the category of finite $(G/H) - sets$. Indeed $\mathbf{Fin}_{\mathcal{F}^c}$ would be all finite G-sets whose stabilizers are subgroups of H.

In particular, the localization functor is given by left Kan extension along Ψ, i.e.

$$\Psi_! \colon G\mathbf{Sp} \to \mathbf{Fun}^{\oplus}(\mathbf{A}^{\text{eff}}(\mathbf{Fin}_{\mathcal{F}^c}), \mathbf{Sp}).$$

We call $\Phi_!$ the geometric H-fixed point functor and denote it by Φ^H or $(-)^{\Phi H}$. In the case H is normal, then this functor takes values in (G/H)-spectra.

Remark 6. The geometric H-fixed point functors are symmetric monoidal.

Definition 1.2. The homotopy fixed point functors are given by forgetting to $\mathbf{Sp}^{BG}$ then taking the usual homotopy H-fixed points.

Theorem 3. *The functor $G\mathbf{Sp} \to \mathbf{Sp}^{BG}$ admits a fully faithful right adjoint $B_G \colon \mathbf{Sp}^{BG} \to G\mathbf{Sp}$. The essential image of B_G are those $X \in G\mathbf{Sp}$ such that the natural map $X^H \to X^{hH}$ is an equivalence for all $H \subseteq G$.*