Selfish Mining in Ethereum

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***Abstract*—As the second largest cryptocurrency by market capitalization and today's biggest decentralized platform that runs smart contracts, Ethereum has received much attention from both industry and academia. Nevertheless, there exist very few studies about the security of its mining strategies, especially from the selfish mining perspective. In this paper, we aim to fill this research gap by analyzing selfish mining in Ethereum and understanding its potential threat. First, we introduce a 2-dimensional Markov process to model the behavior of a selfish mining strategy inspired by a Bitcoin mining strategy proposed by Eyal and Sirer. Second, we derive the stationary distribution of our Markov model and compute long-term average mining rewards. This allows us to determine the threshold of computational power that makes selfish mining profitable in Ethereum. We find that this threshold is lower than that in Bitcoin mining (which is 25% as discovered by Eyal and Sirer), suggesting that Ethereum is more vulnerable to selfish mining than Bitcoin.**

***Index Terms*—Blockchains, Ethereum, Selfish Mining, Rewards, Bitcoin**

I. Introduction

A. Motivation

The Proof-of-Work (PoW) is the most widely adopted consensus algorithm in blockchain platforms such as Bitcoin [1] and Ethereum [2]. By successfully solving math puzzles involving one-way hash functions, the winners of this PoW competition are allowed to generate new blocks that contain as many outstanding transactions as possible (up to the block size limit). As a return, each winner can collect all the transaction fees and earn a block reward (if its new block is accepted by other participants). This economic incentive encourages participants to contribute their computation power as much as possible in solving PoW puzzles—a process often called mining in the literature.

If all the miners follow the mining protocol, each miner will receive block rewards proportional to its computational power [1]. Interestingly, if a set of colluding miners deviate from the protocol to maximize their own profit, they may obtain a revenue larger than their fair share. Such a behavior is called *selfish mining* in the literature and has been studied in a seminal paper by Eyal and Sirer in the context of Bitcoin mining [3]. The selfish mining poses a serious threat to any blockchain platform adopting PoW. If colluding miners occupy a majority of the computational power in the system, they can launch a so-called 51% attack to control the entire system.

Selfish mining in Bitcoin has been well studied with various mining strategies proposed (e.g., [4]–[6]) and numerous defenses mechanisms suggested (e.g., [7], [8]). In sharp contrast, selfish mining in Ethereum is much less understood. Ethereum differs from Bitcoin in that it provides the so-called uncle and nephew rewards in addition to the (standard) block rewards used in Bitcoin [9]. This complicates the analysis. As a result, most existing research results on Bitcoin cannot be directly applied to Ethereum.

B. Objective and Contributions

In this paper, we aim to fill this research gap by analyzing selfish mining in Ethereum and understanding its potential threat. We believe, to the best of our knowledge, that we are among the first to develop a mathematical analysis for selfish mining in Ethereum. First, we introduce a 2-dimensional Markov process to model the behavior of a selfish mining strategy inspired by [3]. Second, we derive the stationary distribution of our Markov model and compute long-term average mining rewards. This allows us to determine the threshold of computational power which makes selfish mining profitable in Ethereum. We find that this threshold is lower than that in Bitcoin. In other words, selfish mining poses a more serious threat to Ethereum due to the presence of uncle and nephew rewards. Finally, we perform extensive simulations to verify our mathematical results and obtain several engineering insights.

Although our mining strategy is similar to that proposed by Eyal and Sirer [3], our analysis is different from theirs in two aspects. First, our Markov model is 2-dimensional whereas their model is 1-dimensional. Second, our analysis tracks block rewards in a probabilistic way whereas their analysis tracks rewards in a deterministic way. It turns out that our 2-dimensional model, combined with the probabilistic tracking, enables us to characterize the effect of uncle and nephew rewards, which is impossible with the 1-dimensional model and deterministic tracking. The main contributions of this paper are summarized as follows:

- To the best of our knowledge, this work is among the first to develop a mathematical analysis of a particular selfish mining strategy in Ethereum. We believe that our analysis can be extended to study more advanced selfish mining strategies.
- Using our theoretical results, we evaluate the threshold of making selfish mining profitable under different versions of Ethereum proposals with a particular focus on EIP100 (which is adopted by the released Byzantium [9]). We find that the threshold is lower than that in Bitcoin, suggesting that Ethereum is more vulnerable to selfish mining than Bitcoin.

II. A PRIMER ON ETHEREUM

Ethereum is a distributed blockchain-based platform that runs smart contracts. Roughly speaking, a smart contract is a set of functions defined in a Turing-complete environment. The users of Ethereum are called clients. A client can issue transactions to create new contracts, to send Ether (internal cryptocurrency of Ethereum) to contracts or to other clients, or to invoke some functions of a contract. The valid transactions are collected into blocks; blocks are chained together through each one containing a cryptographic hash value of the previous block.

There is no centralized party in Ethereum to authenticate the blocks and to execute the smart contracts. Instead, a subset of clients (called miners in the literature) verify the transactions, generate new blocks, and use the PoW algorithm to reach consensus, receiving Ethers for their effort in maintaining the network.

A. Blockchain

Each block in the Ethereum blockchain contains three components: a block header, a set of transactions, and some reference links to certain previous blocks called uncle blocks (whose role will be explained in Sec. III-B) [9]. The block header includes a Keccak 256-bit hash value of the previous block, a time stamp and a nonce (whose role will be explained shortly). See Fig. 1 for an illustration in which blocks are linked together by the hash references, forming a chain structure.

Such a chain structure has several desirable features. First, it is tamper free. Any changes of a block will lead to subsequent changes of all later blocks in the chain. Second, it prevents double-spending. All the clients will eventually have the same copy of the blockchain[1] so that any transactions involving double-spending will be detected and discarded.

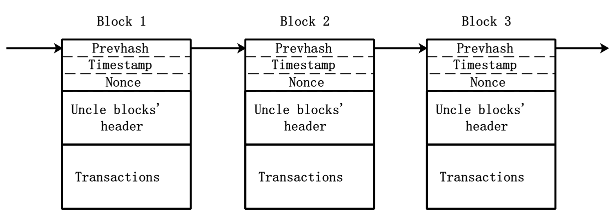


Fig. 1. An illustration of the blockchain structure in Ethereum.

B. PoW and Mining

In order for a miner to produce a new valid block in PoW, it needs to find a value of the nonce such that the hash value of the new block is below a certain threshold depending on the difficulty level—a system parameter that can be adjusted. This puzzle-solving process is often referred to as *mining*. Intuitively, the mining difficulty determines the chance of finding a new block in each try. By adjusting the mining difficulty, the blockchain system can maintain a stable chain growth.

[1]More precisely, all the clients will have a "common prefix" of the blockchain [10].

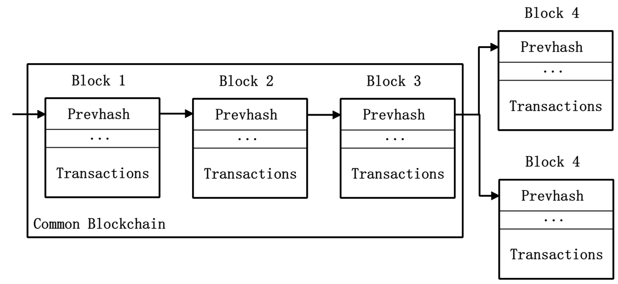


Fig. 2. An illustration of a forked blockchain.

Once a new block is produced, it will be broadcast to the entire network. In the ideal case, a block will arrive all the clients before the next block is produced. If this happens to every block, then each client in the system will have the same chain of blocks. In reality, the above ideal case doesn't always happen. For example, if a miner produces a new block *before* he or she receives the previous block, a fork will occur where two "child" blocks share a common "parent" block. See Fig. 2 for an illustration. In general, each client in the system observes a tree of blocks due to the forking. As a result, each client has to choose a main chain from the tree according to certain rules (e.g., the longest chain rule in Bitcoin and the heaviest subtree rule in the GHOST protocol)[2]. The common prefix of all the main chains is called the *system main chain*—a key concept that will be used in our analysis.

C. Ethereum Milestones

Ethereum has four milestones: Frontier, Homestead, Metropolis, and Serenity. We are now in the third milestone where the mining difficulty level depends not only on the growth of the system main chain but also on the appearance of uncle blocks, as suggested in EIP100 which is adopted by the released Byzantium [9]. By contrast, the mining difficulty level of Bitcoin only depends on the growth of the system main chain. Such a difference motivates our work.

III. SELFISH MINING ON ETHEREUM

A. Mining Model

In this paper, we consider a system of n miners. The ith miner has m_i fraction of total hash power. Clearly, we have $\sum_{i=1}^{n} m_i = 1$. We assume miners are either honest (those who follow the protocol) or selfish (those who deviate from the protocol in order to maximize their own profit). Let $\mathcal{S}$ denote the set of selfish miners and $\mathcal{H}$ denote the set of honest miners. Let α denote the fraction of total hash power controlled by selfish miners and β denote the fraction of total hash power controlled by honest miners . We have $\alpha = \sum_{i\in\mathcal{S}} m_i$ and $\beta = \sum_{i\in\mathcal{H}} m_i$. Clearly, $\alpha+\beta = 1$. Without loss of generality, we assume a single selfish mining pool with α fraction of hash power.

The PoW mining process can be viewed as a series of Bernoulli trails (as explained in Sec. II-B), each of which

[2]Although Ethereum claimed to apply the heaviest subtree rule [11], it seems to apply the longest chain rule instead [6].

independently finds a valid nonce to generate a new block with very low probability. In other words, the number of trails to mine a new block is a geometric random variable. Nowadays, miners in the network are conducting hash computations at a very large rate (e.g., $\approx 2.9 \times 10^{14}$ as of August 2018). Hence, the interval between two successfully mined blocks can be approximated by an exponential random variable. Therefore, we can model the mining process of the ith miner as a Poisson process with rate fm_i. Here, f denotes the block mining rate of the entire system (which captures the total hashing power). That is, the ith miner generates new blocks at rate fm_i. Hence, the selfish pool generates blocks at rate $f\alpha$ and the honest miners generate blocks at rate $f\beta$. This model has been widely used in the literature. See, e.g., [1], [12], [13].

B. Mining Rewards

There are three types of block rewards in Ethereum, namely, static block reward, uncle block reward and nephew block reward [2], [14], as outlined in Table I. The static reward is used in both Ethereum and Bitcoin. To explain static reward, we introduce the concepts of *regular* and *stale* blocks. A block is called regular if it is included into the system main chain, and is called stale block otherwise. Each regular block in Ethereum can bring its miner a reward of exactly 3.0 Ethers as an economic incentive.

The uncle and nephew rewards are unique in Ethereum. An uncle block is a stale block that is a "direct child" of the system main chain. In other words, the parent of an uncle block is always a regular block. An uncle block receives certain reward if it is referenced by some future regular block, called a nephew block, through the use of reference links. See Fig. 3 for an illustration of uncle and nephew blocks. The values of uncle rewards depend on the "distance" between the uncle and nephew blocks. This distance is well defined because all the blocks form a tree. For instance, in Fig. 3, the distance between uncle block $B3$ (uncle block $D2$, resp.) and its nephew block is 1 (2, resp.). In Ethereum, if the distance is 1, the uncle reward is $\frac{7}{8}$ of the (static) block reward; if the distance is 2, the uncle reward is $\frac{6}{8}$ of the block reward and so on. Once the distance is greater than 6, the uncle reward will be zero. By contrast, the nephew reward is always $\frac{1}{32}$ of the block reward. In addition to blocks rewards, miners can also receive gas cost as a reward for verifying and executing all the transactions [2]. However, gas cost is dwarf with other rewards, and so we ignore it in our analysis.

We use K_s, K_u, and K_n to denote static, uncle, and nephew rewards, respectively. Without loss of generality, we assume that $K_s = 1$ so that K_u (K_n, resp.) represents the ratio of uncle reward (nephew reward, resp.) to the static reward. As we explained before, in the current version of Ethereum, $K_n < K_u < 1$ and K_u is a function of the distance. As we will see later, our analysis allows K_u and K_n to be any functions of the distance.

TABLE I
MINING REWARDS IN ETHEREUM AND BITCOIN

	Ethereum	Bitcoin	Purpose
Static Reward	✓	✓	Compensate for miners' mining cost
Uncle Reward	✓	×	Reduce centralization trend of mining
Nephew Reward	✓	×	Encourage miners to reference uncle blocks
Transaction Fee (Gas Cost)	✓	✓	Transaction execution; Resist network attack

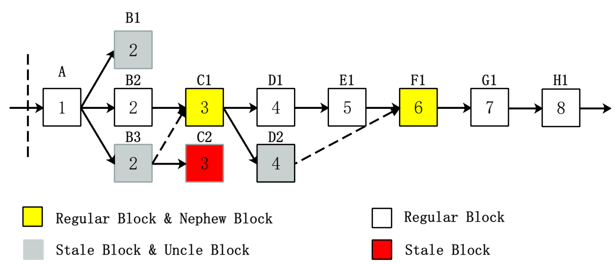


Fig. 3. Different block types in Ethereum. Here, regular blocks include $\{A, B2, C1, D1, E1, F1, G1, H1\}$ and stale blocks include $\{B1, B3, C2, D2\}$. Similarly, uncle blocks are $\{B1, B3, D2\}$ and nephew blocks are $\{C1, F1\}$. Uncle block B3 (uncle block D2, resp.) is referenced with distance one (two, resp.).

C. Mining Strategy

We now describe the mining strategies for honest and selfish miners. The honest miners follow the protocol given in Sec. II-B. Each honest miner observes a tree of blocks. It chooses a main chain from the tree and mines new blocks on its main chain. Once a new block is produced, the miner broadcasts the block to everyone in the system. Also, it includes as many reference links as possible to (unreferenced) uncle blocks in the tree.

By contrast, the selfish pool can withhold its newly mined blocks and publish them strategically to maximize its own revenue. The basic idea behind selfish mining is to increase the selfish pool's share of static rewards and, at the same time, to gain as many uncle and nephew rewards as possible. Specifically, the selfish pool keeps its newly discovered blocks private, creating a fork on purpose. The pool then continues to mine on this private branch, while honest miners still mine on public branches (which are often shorter than the private branch).

Fig. 4 gives an example in which "circle" blocks are mined by the pool and "square" blocks are mined by honest miners. In this example, the private branch consists of 4 blocks $(D1, E1, F, G)$, all of which are mined by the pool with $(D1, E1)$ published and (F, G) still private. There are two public branches, namely, $(D1, E1)$ and $(D2, E2)$, because honest miners can see both branches. Each honest miner then chooses one public branch to mine new blocks according to certain rules (e.g., the longest chain rule). Here, the two public branches are of equal length. This is not a coincidence. In fact,

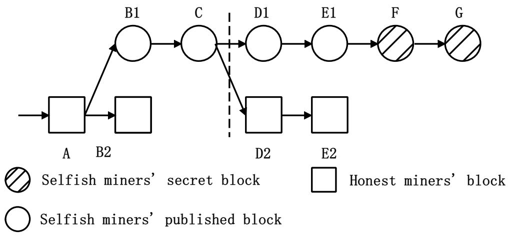


Fig. 4. An example to illustrate private branch length and public branch length.

we can show that public branches always have the same length under our selfish mining strategy.

Let $L_s(t)$ be the length of the private branch seen by the selfish pool at time t. Similarly, let $L_h(t)$ be the length of public branches seen by honest miners at time t. (Note that $L_h(t)$ is well defined because all public branches have the same length.) We are now ready to describe our selfish mining strategy which is based on the strategy in [3].

Algorithm 1 An selfish Mining Strategy in Ethereum

on The selfish pool mines a new block
1: reference all (unreferenced) uncle blocks based on its private branch
2: $L_s \leftarrow L_s + 1$
3: **if** $(L_s, L_h) = (2, 1)$ **then**
4: publish its private branch
5: $(L_s, L_h) \leftarrow (0, 0)$ (since all the miners achieve a consensus)
6: **else**
7: keep mining on its private branch

on Some honest miners mine a new block
8: The miner references all (unreferenced) uncle blocks based on its public branches
9: $L_h \leftarrow L_h + 1$
10: **if** $L_s < L_h$ **then**
11: $(L_s, L_h) \leftarrow (0, 0)$
12: keep mining on this new block
13: **else if** $L_s = L_h$ **then**
14: publish the last block of the private branch
15: **else if** $L_s = L_h + 1$ **then**
16: publish its private branch
17: $(L_s, L_h) \leftarrow (0, 0)$ (since all the miners achieve a consensus)
18: **else**
19: publish first unpublished block in its private branch
20: set $(L_s, L_h) = (L_s - L_h + 1, 1)$ if the new block is mined on a public branch that is a prefix of the private branch

Algorithm 1 presents the mining strategy. When the selfish pool mines a new block (see lines 1 to 7), it will keep this block private and continue mining on its private branch until its advantage is very limited (i.e., $(L_s, L_h) = (2, 1)$) which will be discussed later.

When some honest miners mine a new block, the length of a public branch will be increased by 1. We have the following cases. Case 1) If the new public branch is longer than the private branch, the pool will adopt the public branch and mine on it. (That is why the pool will set $(L_s, L_h) = (0, 0)$.) Case 2) If the new public branch has the same length as the private branch, the pool will publish its private block immediately hoping that as many honest miners will choose its private branch as possible (since honest miners will see two branches of the same length when the private branch is published). Case 3) If the new public branch is shorter than the private branch by just 1, the pool will publish its private branch so that all the honest miners will adopt the private branch. Case 4) If the new public branch is shorter than the private branch by at least 2, the pool will publish the first unpublished block since the pool still has a clear advantage. Moreover, if the new block is mined on a public branch that is a prefix of the private branch, the pool will set $(L_s, L_h) = (L_s - L_h + 1, 1)$ due to a new forking (caused by the honest miner).

To better illustrate the selfish mining strategy, we provide an example in Fig. 5. In Step 1, we have $(L_s, L_h) = (3, 0)$. In Step 2, some honest miner publishes block $A2$ and we have $(L_s, L_h) = (3, 1)$. This corresponds to Case 4). Hence, the pool immediately publishes block $A1$, still having an advantage of 2 blocks. In Step 3, some honest miner publishes block $B2$, leading to $(L_s, L_h) = (3, 2)$. This corresponds to Case 3). Thus, the pool publishes its private branch, making honest miners' blocks ($A2$ and $B2$) stale.

Remark 1. *The selfish mining strategy presented above isn't necessarily optimal. By studying its behavior, we hope to reveal some characteristics of the selfish mining in Ethereum, which is largely missing in the literature.*

D. Mining Pool

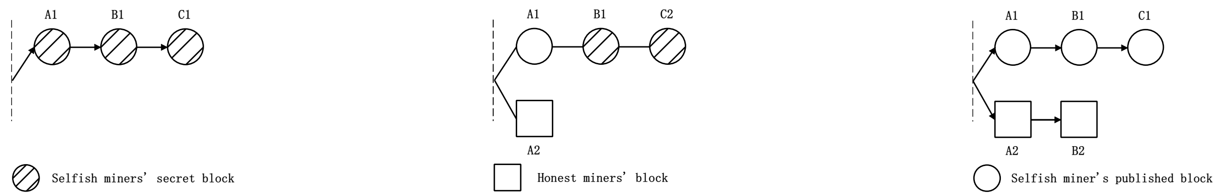
In Ethereum, individual miners can form mining pools to mine blocks together and share the revenue according to individuals' hash power. Fig. 6 presents the fractions of the hash power of various mining pools in Ethereum [15]. The largest mining pool (called Ethermine) has dominated 26.34% of the total hash power. The top two mining pools have dominated 48.8% of the total hash power. The top five mining pools have more than 81% of the total hash power. Although these mining pools are not necessarily selfish, their presence motivates us to understand the impact of selfish mining in Ethereum.

IV. Analysis of Selfish Mining

In this section, we will study the long-term behavior of the selfish mining strategy using a Markov model with a particular focus on the mining revenue.

A. Network Model

To simplify our analysis, we follow the network model of [3], [4], [6] which assumes that the time it takes to broadcast a block is negligible. In particular, we introduce the same



a) Step 1: selfish pool withholds 3 blocks b) Step 2: selfish pool publishes 1 block b) Step 3: selfish pool overrides 2 blocks

Fig. 5. A simple example of the mining strategy.

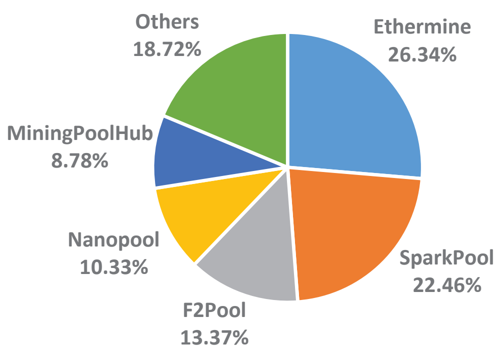


Fig. 6. The top 5 mining pools' hash power in Ethereum (2018.09).

parameter γ as in [3], which denotes the ratio of honest miners that are mining on blocks produced by the selfish pool (rather than by the honest miners) whenever they observe a fork of two branches of equal length. For example, if the honest miners apply the uniform tie-breaking rule (when they observe a fork of two branches of equal length), then $\gamma = \frac{1}{2}$. On the other hand, if the pool can launch a network attack to influence honest miners' block propagation, then only a few honest miners can see any new block produced by some honest miner. In this case, the parameter γ is close to 1. Therefore, the parameter γ captures the pool's communication capability. In this paper, we assume that γ takes values in the interval $[0, 1]$.

B. Markov Process

For ease of presentation, we re-scale the time axis so that the selfish pool generates new blocks at rate α and the honest miners generate new blocks at rate β. We are now ready to define the system state. Recall that $L_s(t)$ is the length of the private branch and $L_h(t)$ is the length of the public branches at time t. Clearly, $(L_s(t), L_h(t))$ captures the system state at time t. The state space contains the following states: $(0, 0)$, $(1, 0)$, $(1, 1)$, as well as (i, j) with $i - j \geq 2$ and $j \geq 0$. It is easy to verify that $(L_s(t), L_h(t))$ evolves as a Markov process under our selfish mining strategy and the network model, as illustrated in Fig. 7. Moreover, we can show that the process $(L_s(t), L_h(t))$ is positive recurrent and so it has a unique stationary distribution.

C. The Stationary Distribution

To compute the stationary distribution of the process $(L_s(t), L_h(t))$, we need to derive the transition rates for the state evolution. The results are provided below.

- $q_{(0,0),(0,0)} = \beta$
  This transition happens if any honest miner produces a new block, broadcasts it to everyone. Then, the selfish pool adopts the public branch and mines on it. Thus, the rate is β.
- $q_{(0,0),(1,0)} = \alpha$
  This transition happens if the pool produces a new block and keeps it private. Thus, the rate is α.
- $q_{(1,0),(2,0)} = \alpha$
  This transition happens if the pool produces a new block and keeps it private. Thus, the rate is α.
- $q_{(1,0),(1,1)} = \beta$
  This transition happens if any honest miner produces a new block and the pool immediately publishes its private block (because the new public branch has the same length as the private branch). Thus, the rate is β.
- $q_{(1,1),(0,0)} = \alpha + \beta = 1$
  This transition happens if any of the following events happens. 1) The pool produces a new block and publishes its private branch (because $(L_s, L_h) = (2, 1)$); 2) Any honest miner produces a new block and the pool has to publish its private branch (because $L_s = L_h + 1$). Thus, the rate is $\alpha + \beta = 1$.
- $q_{(i,j),(i+1,j)} = \alpha$ for $i \geq 2$ and $j \geq 0$
  This transition happens if the pool produces a new block and keeps it private. Thus, the rate is α.
- $q_{(i,j),(i-j,1)} = \beta\gamma$ for $i - j \geq 3$ and $j \geq 1$
  This transition happens if any honest miner mines a new block on a public branch which is a prefix of the private branch. Then, the pool publishes a private block accordingly. Thus, the rate is $\beta\gamma$.
- $q_{(i,j),(0,0)} = \beta$ for $i - j = 2$ and $j \geq 1$
  This transition happens if any honest miner produces a new block and then the pool publishes its private branch (because $L_s = L_h + 1$). Thus, the rate is β.
- $q_{(2,0),(0,0)} = \beta$
  This transition happens if any honest miner produces a new block and then the pool publishes its private branch (because $L_s = L_h + 1$). Thus, the rate is β.
- $q_{(i,0),(i,1)} = \beta$ for $i \geq 3$
  This transition happens if any honest miner produces a

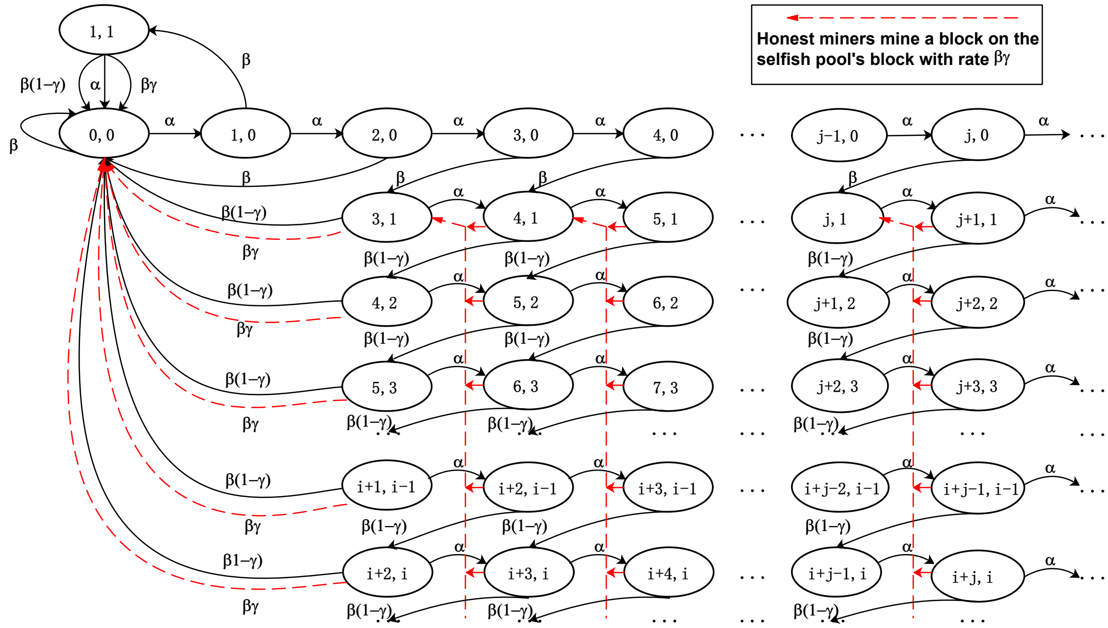


Fig. 7. The Markov process of the selfish mining in Ethereum.

new block and the pool publishes a private block. Thus, the rate is β.

- $q_{(i,j),(i,j+1)} = \beta(1-\gamma)$ for $i-j \geq 3$ and $j \geq 1$
  This transition happens if any honest miner mines a new block on a public branch which is not a prefix of the private branch. Thus, the rate is $\beta(1-\gamma)$.

Let $\{\pi_{i,j}\}$ be the steady-state distribution of the Markov process $(L_s(t), L_h(t))$. Then, the set $\{\pi_{i,j}\}$ satisfies the following global balance equations according to the above transition rates.

$$\begin{cases} \alpha\pi_{0,0} = \pi_{1,1} + \beta\sum_{j=0}^{\infty}\pi_{2+j,j}, \\ \pi_{1,1} = \beta\pi_{1,0}, \\ \pi_{3,1} = \beta\pi_{3,0} + \sum_{j=1}^{\infty}\beta\gamma\pi_{3+j,j}, \\ \pi_{i,0} = \alpha\pi_{i-1,0}, \text{for } i \geq 1, \\ \pi_{i,1} = \beta\pi_{i,0} + \alpha\pi_{i-1,1} + \sum_{j=1}^{\infty}\beta\gamma\pi_{i+j,j}, \text{for } i \geq 4, \\ \pi_{i,i-2} = \beta(1-\gamma)\pi_{i,i-1}, \text{for } i \geq 4, \\ \pi_{i,j} = \alpha\pi_{i-1,j} + \beta(1-\gamma)\pi_{i,j-1}, \text{for } j \geq 2, i \geq 5. \end{cases} \tag{1}$$

Solving the global balance equations, we obtain the following results for the stationary distribution:

$$\begin{aligned} \pi_{0,0} &= \frac{1-2\alpha}{2\alpha^3 - 4\alpha^2 + 1}, \\ \pi_{i,0} &= \alpha^i\pi_{0,0} \text{ for } i \geq 1, \\ \pi_{1,1} &= \left(\alpha - \alpha^2\right)\pi_{0,0}, \\ \pi_{i,j} &= \alpha^i(1-\alpha)^j(1-\gamma)^j f(i,j,j)\pi_{0,0} + \\ &\quad \alpha^{i-j}\gamma(1-\gamma)^{j-1}\left(\frac{1}{(1-\alpha)^{i-j-1}} - 1\right)\pi_{0,0} - \\ &\quad \gamma(1-\gamma)^{j-1}\sum_{k=1}^{j}\alpha^{i-k}(1-\alpha)^{j-k} f(i,j,j-k)\pi_{0,0} \end{aligned}$$

for $i \geq j+2$ and $j \geq 1$, where the function $f(x,y,z)$ is defined as

$$f(x,y,z) = \begin{cases} \underbrace{\sum_{s_z=y+2}^{x}\sum_{s_{z-1}=y+1}^{s_z} .. \sum_{s_1=y-z+3}^{s_2}}_{z} 1, & z \geq 1, x \geq y+2, \\ 0, & \text{otherwise}. \end{cases} \tag{2}$$

The function $f(x,y,z)$ is a multiple summations. See Appendix A for some concrete examples.

We have the following remarks for the stationary distribution in Equation (2).

Remark 2. *When $0 < \alpha < \frac{1}{2}$, we have $0 < \pi_{0,0} < 1$. Specifically, the distribution $\pi_{0,0}$ only depends on the parameter α and is monotonically decreasing. The tendency of $\pi_{0,0}$ suggests that with more hash power, the selfish pool can obtain more lead blocks and so stay in state $(0,0)$ less frequently.*

Remark 3. *The distributions $\pi_{i,0}$ with $i \geq 1$ are decreasing geometrically and are less than 10^{-6} when $i \geq 15$ when $\alpha = 0.4$. It suggests that we can truncate the states in the numerical calculation.*

D. Reward Analysis

In this subsection, we conduct reward analysis for each state transition. Our analysis differs from the previous analysis (e.g., [3], [5]) in that we track various block rewards in a probabilistic way. Recall that each state transition induces a new block (mined by a honest miner or the pool). In general, it is impossible to decide the amount of rewards associated with this new block when it is just created, because the "destiny" of this new block depends on the evolution of the system. For this reason, we will instead compute the expected rewards for the new block. In contrast, the previous analysis tracks published blocks associated with a state transition (whose destiny is already determined) rather than the new block and so it can compute the exact rewards. This gives rise to the following two questions.

1) What is wrong with tracking published blocks?
2) How shall we compute the expected rewards for a new block at the time of its creation?

To answer the first question, one shall notice that tracking published blocks doesn't provide enough information to compute the uncle and nephew rewards. Recall from Sec. III-B that a published regular block can receive nephew rewards by referencing outstanding uncle blocks. The amount of nephew rewards depends on the number of outstanding uncle blocks. As such, we need to keep track of all the outstanding uncle blocks in the system together with their depth information (which is needed to determine the amount of uncle rewards). This greatly complicates the state space.

To answer the second question, one shall notice that it suffices to compute the expected rewards for a new block by using the following information: the probability that it becomes a regular block, the probability that it becomes an uncle block, the distance to its potential nephew block (if it indeed becomes an uncle block). Perhaps a bit surprisingly, all the information can be determined when this new block is generated for our selfish mining strategy.

The complete analysis is provided in Appendix B for a better presentation flow. Here, we just provide a simple example to illustrate our analysis. Assume that the selfish pool has already mined two blocks and kept them private at time t. Then, some honest miner generates a new block. According to Algorithm 1, the pool publishes its private branch immediately. As such, this new block will become an uncle block with probability 1. Furthermore, we can show that this block will have a distance of 2 with its potential nephew block. Thus, this new block will receive an uncle reward of $K_u(2)$. Similarly, its potential nephew block will receive a nephew reward of $K_n(2)$. Moreover, this reward will belong to some honest miner with probability $\beta(1+\alpha\beta(1-\gamma))$ and belong to the pool with probability $1-\beta(1+\alpha\beta(1-\gamma))$. (See *Case 7* in Appendix B for details.) Therefore, the expected rewards associated with this new block are $K_u(2)+K_n(2)$ in total among which $K_u(2)+K_n(2)\beta(1+\alpha\beta(1-\gamma))$ rewards will belong to honest miners (and the remaining will belong to the pool).

E. Revenue Analysis

In this subsection, we apply the previous reward analysis to compute various rewards received by the selfish pool and honest miners. This calculation is straightforward.

1) Revenue Computing: First, we compute the static block rewards for the selfish pool (denoted as r_b^s) and honest miners (denoted as r_b^h). We have the following results:

$$\begin{aligned} r_b^s &= \left(\alpha(1-\pi_{0,0}) + (\alpha^2+\alpha^2\beta+\alpha\beta^2\gamma)\pi_{0,0}\right) \\ &= \alpha - \alpha\beta^2(1-\gamma)\pi_{0,0}, \\ &= \frac{\alpha(1-\alpha)^2(4\alpha+\gamma(1-2\alpha))-\alpha^3}{2\alpha^3-4\alpha^2+1} \end{aligned} \tag{3}$$

and

$$\begin{aligned} r_b^h &= \beta(\pi_{0,0}+\pi_{1,1}) + \beta^2(1-\gamma)\pi_{1,0} \\ &= \frac{(1-2\alpha)(1-\alpha)(\alpha(1-\alpha)(2-\gamma)+1)}{2\alpha^3-4\alpha^2+1}. \end{aligned} \tag{4}$$

Note that r_b^s and r_b^h represent the long-term average static rewards per time unit. Since all the miners generate new blocks at rate 1, the maximum long-term average reward is 1 per time unit. Hence, we have $r_b^s + r_b^h \leq 1$.

Remark 4. *If we only consider static rewards, the above results are the same as those in [3], though our approach is different from that in [3].*

Next, we can compute the uncle block rewards for the selfish pool (denoted as r_u^s):

$$\begin{aligned} r_u^s &= \alpha\beta^2(1-\gamma)K_u(1)\pi_{0,0} \\ &= \frac{(1-2\alpha)(1-\alpha)^2\alpha(1-\gamma)}{2\alpha^3-4\alpha^2+1}K_u(1). \end{aligned} \tag{5}$$

Remark 5. *Note that r_u^s is zero in Bitcoin. In other word, selfish pools' blocks without rewards can be viewed as the "cost" of launching selfish mining attack. Thus the additional reward in Ethereum will reduce the cost of selfish mining and make it easier. Moreover, as shown in our reward analysis, the uncle blocks of the pool are always referenced with distance 1—the minimum referencing distance possible in the system. Intuitively, this is because the pool has a global view of the system.*

Similarly, we can compute the uncle block rewards for the honest miners (denoted as r_u^h):

$$\begin{aligned} r_u^h = (\alpha\beta+\beta^2\gamma)K_u(1)\pi_{1,0} + \sum_{i=2}^{\infty}\beta K_u(i)\pi_{i,0} + \\ + \sum_{i=2}^{\infty}\sum_{j=1}^{\infty}\beta\gamma K_u(i)\pi_{i+j,j}. \end{aligned} \tag{6}$$

Remark 6. *In the current version of Ethereum, the function $K_u(\cdot)$ is given below:*

$$K_u(l) = \begin{cases} (8-l)/8, & 1 \leq l \leq 6 \\ 0, & \textit{otherwise}. \end{cases} \tag{7}$$

Our analysis applies to an arbitrary function of $K_u(\cdot)$.

Then, we can compute the nephew block rewards for the selfish pool (denoted as r_n^s) and honest miners (denoted as r_n^h):

$$r_n^s = \alpha\beta K_s(1)\pi_{1,0} + \sum_{i=2}^{\infty}\sum_{j=1}^{\infty}\beta^{i-1}\gamma(\alpha-\alpha\beta^2(1-\gamma))K_s(i)\pi_{i+j,j} \tag{8}$$

$$r_n^h = \alpha\beta^2(1-\gamma)K_s(1)\pi_{0,0} + \beta^2\gamma K_s(1)\pi_{1,0} + \\ + \sum_{i=2}^{\infty}\sum_{j=1}^{\infty}\beta^i\gamma(1+\alpha\beta(1-\gamma))K_s(i)\pi_{i+j,j}. \quad (9)$$

Remark 7. *In the current version of Ethereum, the function $K_n(\cdot)$ is always equal to $\frac{1}{32}$. Our analysis applies to an arbitrary function of $K_n(\cdot)$.*

Finally, we can obtain the total mining revenue r_{total} as

$$r_{\text{total}} = r_b^s + r_b^h + r_u^s + r_u^h + r_n^s + r_n^h. \quad (10)$$

Hence,

$$R_s = \frac{r_b^s + r_u^s + r_n^s}{r_{\text{total}}}$$

gives the share of the mining revenue by the selfish pool.

2) Absolute Revenue: We now define the absolute revenue U_s for the selfish pool. As we will soon see, although the absolute revenue is equivalent to the relative revenue (i.e., the share R_s) in Bitcoin, it is different from the relative revenue in Ethereum due to the presence of uncle and nephew rewards.

Recall that Bitcoin adjusts the mining difficulty level so that the regular blocks are generated at a stable rate, say 1 block per time unit. Thus, the long-term average total revenue is fixed to be 1 block reward per time unit with or without selfish mining. This makes the absolution revenue equivalent to the relative revenue. The situation is different in Ethereum. Even if the regular blocks are generated at a stable rate, the average total revenue still depends on the generation rate of uncle blocks, which is affected by selfish mining as we will see shortly. Indeed, Ethereum didn't take into account the generation rate of uncle blocks when adjusting the difficulty level until its third milestone. This motivates us to consider two scenarios in our analysis: 1) the regular block generation rate is 1 block per time unit, and 2) the regular and uncle block generation rate is 1 block per time unit.

In our previous analysis, the regular block generation rate is $r_b^s + r_b^h$, which is smaller than 1 as explained before. Thus, we can re-scale the time to make the regular block generation rate to be 1 block per time unit. In this scenario, the long-term absolute revenue for the selfish pool is

$$U_s = \frac{r_b^s + r_u^s + r_n^s}{r_b^s + r_b^h}, \quad (11)$$

and the long-term absolute revenue for honest miners is

$$U_h = \frac{r_b^h + r_u^h + r_n^h}{r_b^s + r_b^h}. \quad (12)$$

Similarly, we can re-scale the time to make the regular and uncle block generation rate to be 1 block per time unit and define long-term absolute revenues for the selfish pool and honest miners accordingly.

3) Threshold Analysis: First of all, if the selfish pool follows the mining protocol, its long-term average absolute revenue will be α, since the network delay is negligible (and so no stale blocks will occur). On the other hand, if the pool applies the selfish mining strategy proposed in this paper, its long-term absolute revenue is given by U_s, which can be larger than α.

Let α^* be the smallest value such that $U_s \geq \alpha$. That is, α^* is the threshold of computational power that makes selfish mining profitable in Ethereum. We can determine α^* for both scenarios through numerical calculations. The details will be presented in the next section.

V. Evaluation

In this section, we build an Ethereum selfish mining simulator to validate our theoretical analysis. In particular, we simulate a system with $n = 1000$ miners, each with the same block generation rate. In our simulations, the selfish pool controls at most 450 miners (i.e., $\alpha \leq 0.45$) and runs our Algorithm 1, while the honest miners follow the designed protocol in Sec. III-C. We consider the block propagation delay by introducing the stale rate δ in our simulation [6], [16]. Our simulation results are based on an average of 10 runs, where each run generates $100,000$ blocks.

A. Validation of the Theory Results

In this subsection, we validate the long-term average absolute revenues for the selfish pool and honest miners. Fig. 8 plots the results obtained from analysis and simulations. From the results, we can see when $\gamma = 0.5$, $K_u = 4/8K_s$ and α changes from 0 to 0.45, the simulation results match our theoretical results[3]. In addition, when α is above 0.163, the selfish pool can always gain higher revenue from selfish mining than following the protocol. More importantly, when α is below the threshold 0.163, the selfish pool loses just a small amount of revenue due to the additional uncle block rewards, which is quite different from the results in Bitcoin [3].

B. Impact of the Uncle Reward

In this subsection, we explore the impact of the uncle block rewards on the selfish pool's and honest miners' revenues. To this end, we first use the uncle reward function $K_u(\cdot)$ in Ethereum (see Sec. IV-E for details) and then set the uncle reward as a fixed value regardless of the distance, ranging from $2/8K_s$ to $7/8K_s$. Here, the fixed uncle reward value can directly show its impacts and simplify our understanding.

Fig. 9 shows that the higher uncle reward, the more absolute revenue for both the selfish pool and honest miners, which is quite intuitive. It also reveals that the total revenue increases with the selfish pool's computation power α and soars to 135% of the revenue without selfish mining, when $K_u = 7/8K_s$ and $\alpha = 0.45$. This is because, without the consideration of uncle blocks into difficulty adjustment, the selfish mining can produce additional uncle and nephew rewards, resulting in the fluctuation of total revenue. Additionally, the uncle reward function $K_u(\cdot)$ used in Ethereum has the same effect as simply setting $K_u = 7/8K_s$ for selfish pool's revenue (as explained in Sec. IV-E). In contrast, $K_u(\cdot)$ functions complicatedly for the

[3]To simplify the numerical calculations of our results, we only consider the states (i, j) with i and j less than 200. This approximation turns out to be accurate when $\alpha \leq 0.45$.

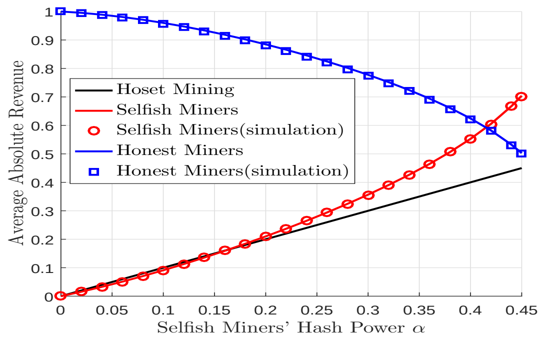


Fig. 8. Revenue rate for the selfish pool and honest miners when $\gamma = 0.5$, $K_u = 4/8K_s$ and α changes from 0 to 0.45.

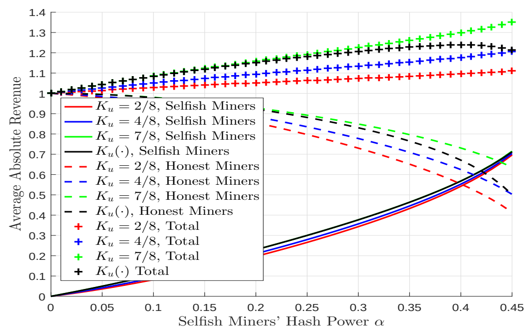


Fig. 9. Revenue rate for the selfish pool, honest miners under different uncle block reward K_u.

honest miners' revenue. When α is small, its impact is similar to the case of $K_u = 7/8K_s$, and when α is close to 0.45, its impact is similar to the case of $K_u = 4/8K_s$. This is because with the increase of α, the average referencing distances of honest miners' uncle blocks will increase, which further leads to the decrease of honest miners' average uncle rewards when using function $K_u(\cdot)$. This finding motivates our discussion in Sec. VI.

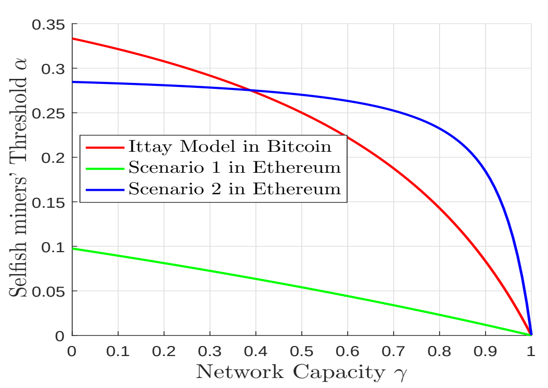


Fig. 10. The profitable threshold of hash power in Bitcoin and Ethereum.

C. Comparison with Bitcoin

In this subsection, we compare the hash power thresholds of making selfish mining profitable in Ethereum and Bitcoin under different values of γ. In Ethereum, we use function $K_u(\cdot)$ to compute the uncle rewards. Particularly, we compute the thresholds for the two scenarios described in Sec. IV-E2: 1) the regular block generation rate is 1 block per time unit, and 2) the regular and uncle block generation rate is 1 block per time unit.

From Fig. 10, we can see that the higher γ is, the lower hash power needed for a profitable selfish mining. Specially, when $\gamma = 1$, the selfish mining in Bitcoin and Ethereum can always be profitable regardless of their hash power. Besides that, the results show that the hash power thresholds of Ethereum in scenario 1 are always lower than Bitcoin. By contrast, the hash power thresholds in scenario 2 are higher than Bitcoin when $\gamma \geq 0.39$. This is because the larger γ is, the more blocks mined by honest miners are uncle blocks. However, in scenario 2 the additional referenced uncle blocks will reduce the generation rate of regular block, resulting in the decrease of selfish pools' block rewards. Thus the selfish pool needs to have higher hash power in order to make selfish mining profitable. This suggests that Ethereum should consider the uncle blocks into the difficulty adjustment under the mining strategy given in Algorithm 1.

D. Honest miners' Discounted Hash Power

In reality, new blocks cannot immediately arrive to every miners in the network, which means miners cannot always mine on top of the latest public branch [16]. In order to evaluate the impact, we introduce a parameter δ as in [6], [16] to denote the fraction of honest miners' hashing power that mines on a stale block rather than the latest public branch. Note that the parameter δ allows us to account for the impact of the network propagation delay (which is missing in our analysis). We assume that the selfish pool doesn't suffer from the propagation delay because of its powerful networking ability. Fig. 11 compares the selfish pool' revenue under different values of δ. It's clear that the higher δ is, the more honest miners' hash power is wasted and the lower threshold of making selfish mining profitable is needed.

VI. Discussion

In Ethereum, uncle and nephew rewards are initially designed to solve the mining centralization bias—miners form or join in some big mining pools (Sec. III-D). This is because, due to propagation delay, mining pools with huge hash power are less likely to generated stale blocks and can be more profitable for mining. Thus, rewarding the stale block can reduce the mining pools' advantage [17] and make them less attractive for small miners. However, as analyzed previously, the uncle reward (computed by using the function $K_u(\cdot)$) can greatly reduce the cost of launching selfish mining. To mitigate this issue, here we propose a simple uncle reward function motivated by our analysis in Sec. IV-E1. It shows that the uncle blocks mined by the selfish pool can always be

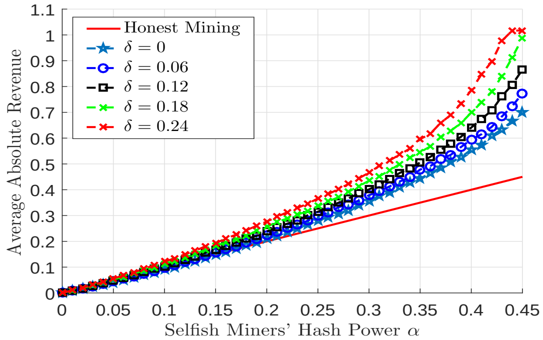


Fig. 11. The revenue rate for the selfish pool under different stale rate δ.

TABLE II
THE DISTRIBUTION OF HONEST MINERS' UNCLE BLOCK WITH DIFFERENT REFERENCING BLOCK DISTANCES

Referencing distance	$\alpha = 0.3$	$\alpha = 0.45$
1	0.527	0.284
2	0.295	0.249
3	0.111	0.171
4	0.043	0.125
5	0.017	0.096
6	0.007	0.075
Expectation	1.75	2.72

referenced with block distance one, i.e., the maximum uncle reward $7/8K_s$ using the function $K_u(\cdot)$. In contrast, the honest miners' uncle blocks cannot obtain such high reward. To illustrate the explicit situation, we provide the distribution of the honest miners' uncle block with different referencing block distances in Table II given $\gamma = 0.5$. The results show that with the increase of α, the average referencing distance of honest miners' blocks are increasing. Thus, we should decrease the reward for uncle blocks with distance one and increase the reward for the uncle blocks with longer distances. In particular, we can simply set the uncle reward function $K_u(\cdot)$) as a fixed value, say $K_u = 4/8K_s$, if uncle blocks' referencing block distance is between 1 and 6. We recompute the threshold of making selfish mining profitable using this new function and find that when $\gamma = 0.5$, the threshold increases from 0.054 to 0.163 in scenario 1, and from 0.270 to 0.356 in scenario 2. In other words, this simple change makes it harder for the selfish pool to be profitable.

VII. Related Work

The research of selfish mining is mostly focused on Bitcoin with roughly two directions: 1) optimizing the selfish mining strategies in order to increase the revenue and lower the threshold of launching selfish mining attacks; 2) proposing defense mechanisms. In [3], Eyal and Sirer developed a Markov process to model the Selfish-Mine Strategy and to evaluate the selfish pool's relative revenue. Moreover, they proposed a uniform tie-breaking defense against selfish mining, which is adopted in Ethereum. Inspired by this seminal paper, Sapirshtein et al. [4] and Nayak et al. [5] demonstrated that by adopting the optimized strategies, the threshold of the hashing power to make selfish mining profitable can be reduced to 23.2% even when honest miners adopt the uniform tie-breaking defense. Furthermore, the authors in [18] took the propagation delay into the analysis of selfish mining.

As for defense mechanisms, Heilman proposed a defense mechanism called Freshness Preferred [8], in which by using the latest unforgeable timestamp issued by a trusted party, the threshold can be increased to 32%. Bahack in [19] introduced a fork-punishment rule to make selfish mining unprofitable. Specially, each miner in the system can include a fork evidence in their block. Once confirmed, the miner can get half of the total rewards of the wining branch. Solat and Potop-Butucaru [20] propose a solution called ZeroBlock, which can make selfish miners' block expire and be rejected by all the honest miners without using forgeable timestamps. In [7], the authors proposed a backward-compatible defense mechanism called weighted FRP which considers the weights of the forked chains instead of their lengths. This is similar in spirit to the GHOST protocol [11].

However, there exist very few studies about the selfish mining in Ethereum. The author in [21] first proposed to exploit the flaw of difficulty adjustment to mine additional uncle blocks, which is shown less profitable than our selfish mining strategy. In [16] Ritz and Zugenmaier built a Monte Carlo simulation platform to quantify the security of the Ethereum after EIP100. However, their paper contains no mathematical analysis and cannot directly capture the effects of uncle block rewards and nephew rewards.

VIII. Conclusion And Future Work

In this paper, we have proposed a Markov model to analyze a selfish mining strategy in Ethereum. Our model enables us to evaluate the impact of the uncle and nephew rewards, which is generally missing in the previous analysis for selfish mining in Bitcoin. In particular, we have shown how these rewards influence the security of Ethereum mining. Additionally, we have computed the hashing power threshold of making selfish mining profitable under different scenarios, which is essential for us to evaluate the security of Ethereum mining and to design new reward functions.

As one of our major findings, we notice that it is important to consider uncle blocks when adjusting the mining difficulty level. Otherwise, Ethereum would be much more vulnerable to selfish mining than Bitcoin. This finding supports the emendation adopted by the third milestone of Ethereum. However, once the mining mechanism is changed, the selfish pool is likely to change its new mining strategies in order to maximize its own profit. We leave the design of new mining strategies as our future work. We believe that our analysis developed in this paper (especially the probabilistic tracking) would be useful in studying other mining strategies.

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APPENDIX

A. The Multiple Summations Function

The function $f(x, y, z)$ used in Sec. IV-C involves multiple summations when $z > 1$. We provide several examples to explain this function.

Example 1 **($z = 1$, $x \geq y + 2$).**

$$\begin{aligned} f(x, y, 1) &= \sum_{s_1=y+2}^{x} 1 \\ &= x - y - 1. \end{aligned}$$

Example 2 **($z = 2$, $x \geq y + 2$).**

$$\begin{aligned} f(x, y, 2) &= \sum_{s_2=y+2}^{x} \sum_{s_1=y+1}^{s_2} 1 \\ &= \sum_{s_2=y+2}^{x} (s_2 - y) \\ &= 2 + \cdots + (x - y) \\ &= \frac{(x-y-1)(x-y+2)}{2}. \end{aligned}$$

B. Reward Analysis

Lemma 1. *Consider a new block associated with a state transition from state (i, j) with $i - j \geq 2$. It will be a regular block with probability 1 if and only if it is mined by the selfish pool.*

Proof. Suppose that the current system state is (i, j) with $i - j \geq 2$. If the selfish pool mines a new block, then the state becomes $(i + 1, j)$. Now, the private branch has an advantage of at least 3 blocks (since $i + 1 - j \geq 3$) over public branches. As the system evolves, the private branch will be published with probability 1 and become part of the system main chain, according to Algorithm 1. In other words, the new block will be a regular block with probability 1. On the other hand, if some honest miners mine a new block, we consider two cases.

1) $i - j = 2$. In this case, we have $L_h = j + 1$ and $L_s = L_h + 1$. Hence, the pool will publish its private branch and the new block becomes a stale block.
2) $i - j \geq 3$. In this case, the system state becomes either $(i, j + 1)$ or $(i - j, 1)$. The private branch has an advantage of at least 3 blocks (since $i - j \geq 3$). Hence, it will be published with probability 1. In other words, the new block will be a stale block with probability 1.

□

We are now ready to analyze every state transition. We call a new block associated with a transition a *target* block.

Case 1: $(0, 0) \xrightarrow{\beta} (0, 0)$

In this case, the target block generated by some honest miners will be adopted by all the miners. Thus, it will be a regular block and receive a static reward K_s.

Case 2: $(0, 0) \xrightarrow{\alpha} (1, 0)$

In this case, the selfish pool produces the target block, keeps it private, and continues mining on it. First, we analyze the static reward by determining whether the target block will be a regular block or not. To this end, we consider the following two subcases, which are illustrated in Fig. 12.

1) *Subcase 1*: The subsequent block is mined by the pool, which happens with probability α. As a result, the pool

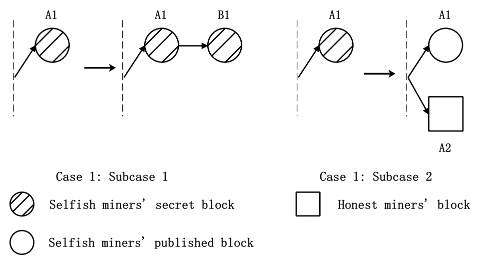

Fig. 12. The two subcases of Case 1: 1) the selfish pool mines a subsequent block; 2) some honest miners mine a new block.

owns a lead of two blocks. By Lemma 1, the target block will be a regular block and receive a static reward of K_s.

2) *Subcase 2*: The subsequent block is mined by some honest miner, which happens with probability β. Then, the pool will publish this target block. To determine whether it will be a regular block, we need to consider the following three subsubcases. See Fig. 13 for an illustration.

   Subsubcase 1: The pool mines a new block on its private branch and publishes it immediately. (This happens with probability α.) Now, the target block becomes a regular block.

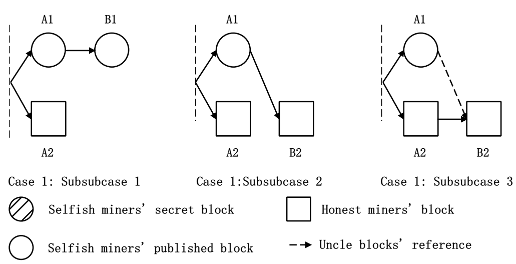

Fig. 13. The three subsubcases of Case 1: 1) the selfish pool mines a new block and publishes it; 2) the honest miners find a new block on the target block; 3) some honest miners mine a new block and reference the target block.

   Subsubcase 2: Some honest miners mine a new block on the target block. (This happens with probability $\beta\gamma$.) Now, the target block becomes a regular block.

   Subsubcase 3: Some honest miners mine a new block and references the target block. (This happens with probability $\beta(1-\gamma)$.) Now, the target block becomes an uncle block.

To sum up, the target block in *Case 2* will eventually be a regular block with probability $\alpha + \alpha\beta + \beta^2\gamma$ and be an uncle block with probability $\beta^2(1-\gamma)$. (Note that these two probabilities sum up to 1.)

Next, we analyze the uncle and nephew rewards associated with the target block. Based on our previous case-by-case discussion, only in subsubcase 3, the target block will be an uncle block. Also, the distance between the target block and its nephew block is 1. As such, the target block will bring the pool an uncle reward of $K_u(1)$ and some honest miners will receive a nephew reward of K_n. (This happens with probability $\beta^2(1-\gamma)$.)

Case 3: $(1,0) \xrightarrow{\alpha} (2,0)$

In this case , the pool produces the target block, keeps it private, and continues mining on it. By Lemma 1, the target block will be a regular block and receive a static reward of K_s.

Case 4: $(1,0) \xrightarrow{\beta} (1,1)$

In this case, some honest miners mine the target block, then the pool publishes its private block. First, we analyze the static reward by determining whether the target block is a regular block. To this end, we consider the following subcases. See Fig. 14 for an illustration.

1) *Subcase 1*: The pool mines a new block on its private branch, references the target block, and publishes its private branch. (This happens with probability α.) Now, the target block becomes an uncle block.
2) *Subcase 2*: Some honest miners mine a new block not on the target block and reference the target block. (This happens with probability $\beta\gamma$.) Now, the target block becomes an uncle block.
3) *Subcase 3*: Some honest miners mine a new block on the target block. (This happens with probability $\beta(1-\gamma)$.) Now, the target block becomes a regular block.

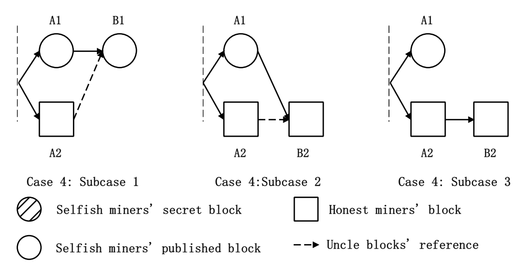

Fig. 14. The three subcases of Case 4: 1) the selfish pool mines a new block on its branch and references the target block; 2) some honest miners find a new block not on the target block and reference the target block; 3) some honest miners mine a new block on the target block.

To sum up, the target block will eventually be a regular block with probability $\beta(1-\gamma)$, and be an uncle block with probability $\alpha + \beta\gamma$.

Next, we analyze the uncle and nephew rewards associated with the target block. Based on our previous case-by-case discussion, the target block will become an uncle block only in subcases 1 and 2, where the distance is 1. Thus, the target block will bring honest miners an uncle reward of $K_u(1)$. As for the nephew reward, in *Subcase 1*, the pool receives it, and in *Subcase 2*, some honest miners receive it. To sum up, honest miners will receive an uncle block reward of $K_u(1)$ with probability $\alpha+\beta\gamma$, receive a nephew reward of K_n with probability $\beta\gamma$, and the pool will receive a nephew reward of K_n with probability α.

Case 5: $(1,1) \xrightarrow{1} (0,0)$

In this case, the target block will always be a regular block no matter who mines it. Hence, the pool receives a static reward with probability α, and some honest miners receive a static reward with probability β.

Case 6: $(i,j) \xrightarrow{\alpha} (i+1,j)$ *with* $i \geq 2$ *and* $j \geq 0$

In this case , the pool mines the target block, keeps it private, and continues mining. By Lemma 1, the target block will eventually become a regular block, receiving a static reward of K_s.

Case 7: $(i,j) \overset{\beta\gamma}{\to} (i-j,1)$ *with* $i-j \geq 3$ *and* $j \geq 1$

In this case, some honest miners mine the target block on a public branch that is a prefix of the private branch. Then, the pool publishes its first unpublished block. By Lemma 1, the target block will eventually become an uncle block.

Next, we analyze the uncle and nephew rewards associated with the target block. We begin with the special case of $(4,1) \to (3,1)$ before discussing the general case. We consider the following three subcases.

1) *Subcase 1*: The pool mines a subsequent block and references the target block. See Fig. 15 for an illustration. (This happens with probability α.) By Lemma 1, the target block will become an uncle block, receiving an uncle reward of $K_u(3)$. The pool will receive a nephew reward.

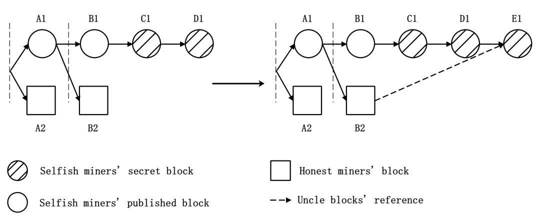

Fig. 15. The subcase 1 of Case 7, in which the pool mines a subsequent block, references the target block and eventually wins the associated nephew reward.

2) *Subcase 2*: Some honest miners mine a subsequent block on the target block. (This happens with probability $\beta(1-\gamma)$.) Then, the pool will publish its private branch. See Fig. 16 for an illustration. To determine the uncle and nephew rewards, we need to consider the following subsubcases.

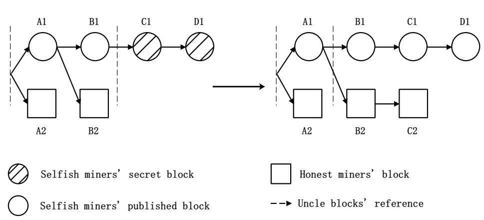

Fig. 16. The subcase 2 of Case 7, in which some honest miners mine a subsequent block on the target block.

Subsubcase 1: Some honest miners mine a new block and references the target block. See Fig. 17. This subsubcase happens with probability $\beta(1-\gamma)\beta$. The honest miner receives a nephew reward, and the target block receives an uncle reward of $K_u(3)$ since the distance is 3.

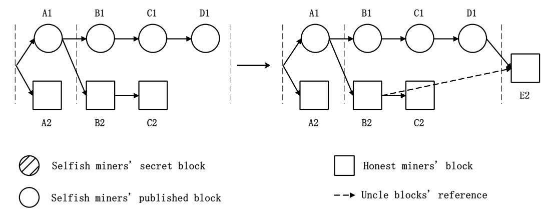

Fig. 17. The subsubcase 1 of Case 7, in which some honest miners mine a block in state $(0,0)$ and win the associated nephew reward.

Subsubcase 2: The pool mines a new block and keeps it private. This subsubcase happens with probability $\beta(1-\gamma)\alpha$. Now, if the new block later becomes a regular block (with probability $\alpha+\alpha\beta+\beta^2\gamma$ due to the discussion for *Case 2*), the pool will receive a nephew reward and the target block will receive an uncle reward of $K_u(3)$. Otherwise, if the new block later becomes a stale block (with probability $\beta^2(1-\gamma)$ due to the discussion for *Case 2*), some honest miners will receive a nephew reward and the target block will again receive an uncle reward of $K_u(3)$.

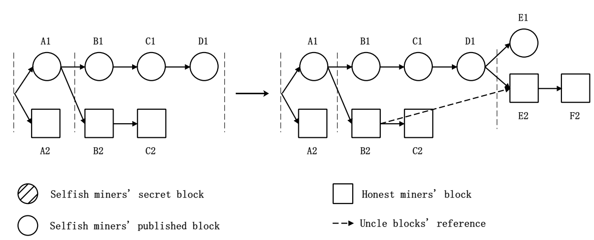

Fig. 18. The subsubcase 2 of Case 7, in which the selfish pool mines a new block in state $(0,0)$, but finally loses to the nephew reward.

3) *Subcase 3*: Some honest miners mine a subsequent block not on the target block. (This happens with probability $\beta\gamma$.) In terms of the analysis of uncle and nephew rewards, *Subcase 3* is the same as *Subcase 2*.

To sum up, for the special case of $(4,1) \to (3,1)$, the target block will always receive an uncle reward of $K_u(3)$. As for the nephew reward, one can draw a tree diagram summarizing all the subcases discussed above and conclude that it will be received by honest miners with probability $\beta^2(1+\alpha\beta(1-\gamma))$ and by the pool with probability $1-\beta^2(1+\alpha\beta(1-\gamma))$.

Here, we note that the probability $\beta^2(1+\alpha\beta(1-\gamma))$ (that the nephew reward will be received by honest miners) can be explained as follows. First, the honest miners have to "push" the system state from $(3,1)$ to $(0,0)$ while the pool mines nothing. (Otherwise, the pool will receive the nephew reward by Lemma 1.) This happens with probability β. Second, starting from $(0,0)$, the honest miners can win the nephew reward with probability $\beta(1+\alpha\beta(1-\gamma))$. This interpretation allows us to analyze the general case.

First, the honest miners have to "push" the system state from $(i-j,1)$ to $(0,0)$ while the pool mines nothing. This happens with probability β^{i-j-2}. Second, starting from $(0,0)$, the honest miners can win the nephew reward with probability $\beta(1 + \alpha\beta(1 - \gamma))$. Therefore, the nephew reward will be received by honest miners with probability $\beta^{i-j-1}(1+\alpha\beta(1-\gamma))$ and by the pool with probability $1-\beta^{i-j-1}(1+\alpha\beta(1-\gamma))$.

Case 8: $(i, j) \overset{\beta\gamma}{\to} (0, 0)$ *with* $i - j = 2$ *and* $j \geq 1$

In this case, some honest miners mine the target block. Then, the pool publishes its private branch. Now, the target block becomes an uncle block. Similar to *Case 7*, the target block will receive an uncle reward of $R_u(2)$, and the nephew reward will be received by honest miners with probability $\beta(1+\alpha\beta(1-\gamma))$ and by the pool with probability $1 - \beta(1 + \alpha\beta(1 - \gamma))$.

Case 9: $(2, 0) \overset{\beta}{\to} (0, 0)$ *with* $i \geq 2$

In this case, some honest miners mine the target block. Then, the pool publishes its private branch. The remaining discussion is the same as *Case 8*.

Case 10: $(i, 0) \overset{\beta}{\to} (i, 1)$ *with* $i \geq 3$

In this case, some honest miners mine the target block. Then, the pool publishes its first unpublished block. By Lemma 1, the target block will eventually become an uncle block. Similar to the discussion for *Case 7*, we conclude that the target block will receive an uncle reward of $K_u(i)$, and the nephew reward will be received by honest miners with probability of $\beta^{i-1}(1 + \alpha\beta(1 - \gamma))$ and by the pool with probability $1 - \beta^{i-1}(1 + \alpha\beta(1 - \gamma))$.

Case 11: $(i, j) \overset{\beta(1-\gamma)}{\to} (i, j + 1)$ *with* $i - j \geq 3$ *and* $j \geq 1$

In this case, some honest miners mine the target block. Then, the pool publishes its first unpublished block. By Lemma 1, the target block will eventually become a stale block. Since its parent block is not in the system main chain, the target block will not be an uncle block.

Case 12: $(i, j) \overset{\beta(1-\gamma)}{\to} (0, 0)$ *with* $i - j = 2$ *and* $j \geq 1$

In this case, some honest miners mine the target block. Then, the pool publishes its private branch. Similar to *Case 11*, the target block will neither be a regular block nor an uncle block.