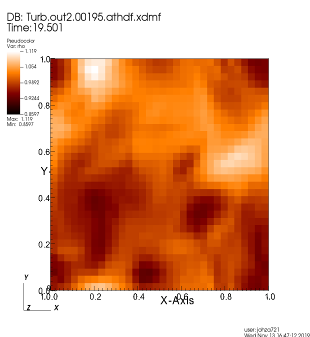
Project Writeup
Zade Johnston

Turbulence in Plasma

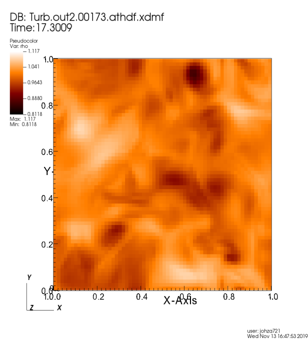
Week 1: 11 Nov - 13 Nov

Started this week learning about how to use MPI in Athena++. Followed the 3D blast wave tutorial and managed to get it working on Thunderbird with HDF5. Jono also recommended learning a text editor, so I spent an hour learning the basics of Vim. **Note:** HDF5 is better to use than .vtk with parallel computing as it allows all the processors being used to write the data in one file compared to a file for each processor that need to be joined (less hassle).

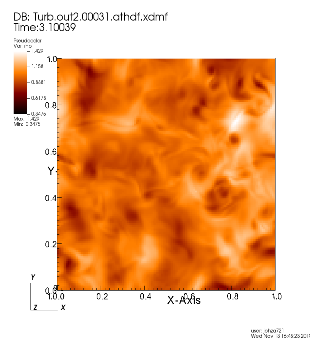
Jono then gave me an Athena++ hydrodynamic turbulence input script to play around with, and some MATLAB scripts that analyse the energy spectrum of the fluid in question. At the moment, we model the fluid in a cube. There are two different modes that we're wanting to focus on: decaying turbulence (disturb the fluid initially then leave it to its own devices) and continuously driven turbulence. Ran the Athena++ code with 3 different grid sizes for both modes; see screenshots below.



(a) Grid Size: 32



(b) Grid Size: 64



(c) Grid Size: 128

Figure 1: Face-on view of the 3D forced turbulence simulations with different grid sizes; density plotted

These were just the simulations with none of the parameters changed in the input script. Using the MATLAB scripts I also obtained the energy spectrum of the fluid; was very rough due to the low resolution. Wanted to try start a 256^3 grid size simulation before I left on Wednesday but didn't have enough time to set up; will try again later. Thursday and Friday of this week were spent at the Otago Software Carpentry Workshop.

Next Week: Play around with parameters. Want to run the larger grid size simulations to obtain a better energy spectrum that fits the $k^{-5/3}$ law, will add screenshots of spectrum then. Plot the total energy over time for both modes; should observe fluctuations in the energy for forced turbulence.

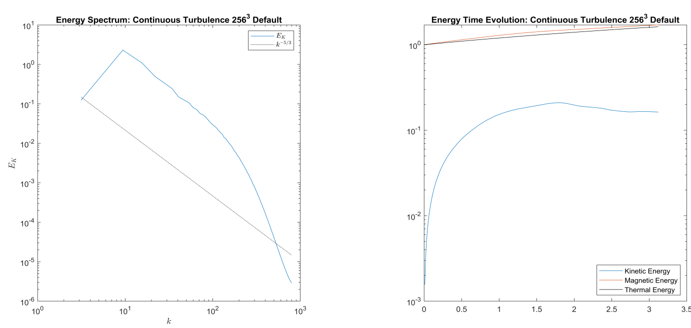
Week 2: 18 Nov - 22 Nov

This week I ran bigger simulations for both decay and forced turbulence from last week in order to be able to plot the energy spectrum and time evolution. The grid size ranged from 32 to 256, and runs over 30 seconds. All simulations left the parameters in `athinput.turb` unchanged.

Calculated the turnover time $\tau \sim L/u_l$ of eddies on the scale of the box to get an idea of the timescales involved in the energy cascade. This is important in the decaying case as all the input energy dissipates within a few turnover times, so this allowed me to get an approximate time range to average the energy spectrum over. For these simulations, I used $L = 1$ and $u_L = \sqrt{u_x^2 + u_y^2 + u_z^2}$ taken at the start of the simulation from the `Turb.hst` file.

Continuous Forcing: For the continously forced case, the 256 grid size gave the best result. This is expected as it is able to simulate smaller scale eddies compared to lower resolution simulations, allowing more of the energy cascade to be observed. We see that the energy spectrum does follow the $k^{-5/3}$ law (shown in Fig. 2a) for a given range of wavevectors.

The energy evolution (Fig. 2b) shows an increase in kinetic energy until it levels out after a few turnover times. This leveling out is due to the energy dissipation rate matching the energy input rate from the forcing. There is still some variation around the mean value, which arises from fluctuations due to turbulence.

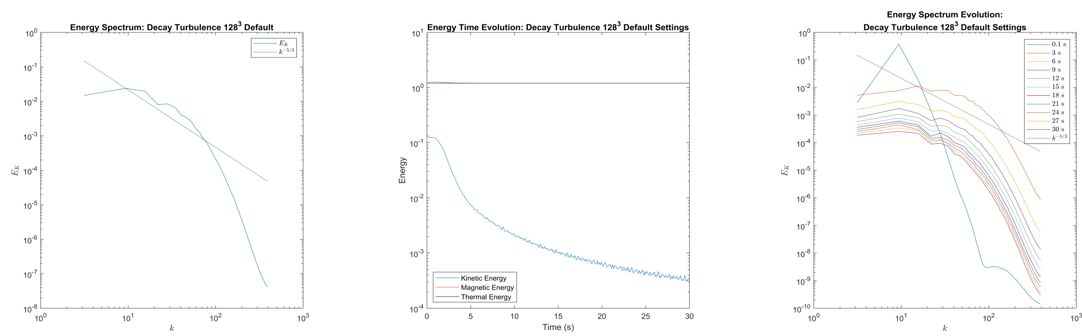


(a) Averaged Spectrum (b) Energy Evolution

Figure 2: Plots showing the energy spectrum and time evolution of the 256 grid size continuous forcing simulation

Decay: The energy spectrum (Fig 3a), averaged over the first two turnover cycles, is not as well defined as the continuous case. I think this is because the energy depends on time instead of being approximately constant. The energy evolution (Fig. 3b) shows that the total kinetic energy decreases over time due to dissipation from viscosity.

For the spectrum evolution (Figure 3c) I took snapshots of the spectrum at 3 second intervals, with a snapshot at 0.1 seconds to observe the energy input (the curve sharply peaked at $k \sim 10$). We see that the energy cascades down through the length scales directly after input (at 3 seconds), and then decreases as it is dissipated as heat at the micro scale (shown by the "sinking" of the spectrum). This is expected as we saw that the total energy in Fig. 3b is decreasing.



(a) Averaged Spectrum (b) Energy Evolution (c) Spectrum Evolution

Figure 3: Plots showing the energy spectrum and time evolution of the 128 grid size decay simulation

The magnetic and thermal energy had no relevance to these simulations; it's just part of the MATLAB script that I forgot to remove.

I learnt that it's good to run the simulations at different resolutions starting with the lowest as it allows you to get a rough idea of what is going to happen without the expense of computing time. It also helps as you can compare with the model to see whether the configuration used is worth investigating. The higher resolutions could differ as turbulence depends strongly on all length scales in the inertial subrange, which are included in the bigger simulations, but it still helps to see what could happen.

Theory

Kolmogorov energy cascade law: Energy that is put into a turbulent fluid at the largest relevant scale doesn't immediately dissipate as heat. Instead it is cascaded down the length scales of the fluid (from macro to micro) in the formation of increasingly smaller eddies, and is dissipated as heat at the microscopic scale due to viscous effects. This produces a kind of order out of the (seemingly) chaos of turbulence.

For a given eddy length scale l, we can find a relation between the wavenumber ($k = 2\pi/l$) and the energy contained in the fluid $E(k)$. This theory was developed by Kolmogorov and says $E(k) \sim k^{-5/3}$ for a given range of wavenumbers, called the **inertial subrange**. This law fails at small scales (large k) for the reason explained above, and at large scales (small k) as this is where the energy is being injected into the system.

Derivation using dimensional analysis: The Navier-Stokes equation is

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla p + \nu \nabla^2 \mathbf{u}$$

Turbulence arises from the nonlinear term $(\mathbf{u} \cdot \nabla)\mathbf{u}$ (represents the energy cascade), and energy is dissipated via the viscosity term $\nu \nabla^2 \mathbf{u}$. Using dimensional analysis, at a given scale l with bulk velocity u_l we have $|(\mathbf{u} \cdot \nabla)\mathbf{u}| \sim u_l^2/l$ and $|\nu \nabla^2 \mathbf{u}| \sim \nu u_l/l^2$ (using $\nabla \sim 1/l$).

Reynolds number Re is the ratio of the turbulence term to the viscosity term:

$$\mathrm{Re} \sim \frac{|(\mathbf{u} \cdot \nabla)\mathbf{u}|}{|\nu \nabla^2 \mathbf{u}|} = \frac{u_l l}{\nu}$$

The viscous term is much smaller than the nonlinear term for large l (due to the factor of $1/l^2$). This means that at large scales energy can't dissipate as heat, so the nonlinear term effectively cascades energy through eddies as it has nowhere else to go. Only when the two terms are comparable ($\mathrm{Re} \sim 1$) can the energy be dissipated as heat. This also explains the self-similarity in the energy spectrum across the inertial subrange.

For a given length scale l and velocity u_l (this is the **structure function**), the energy is $E \sim u_l^2$. The turn-over time for an eddy is $\tau \sim l/u_l$. The rate of energy dissipation is then $E/\tau \sim u_l^3/l = \text{const.}$ across the inertial subrange due to self-similarity.

We then have $u_l \sim l^{1/3} \sim k^{-1/3} \equiv u_k$. The energy in all eddies with wavenumber $\geq k$ is given by

$$E = u_k^2 \sim k^{-2/3} \sim \int_k^\infty E(\kappa)\, d\kappa \implies E(k) \sim k^{-5/3}$$

giving us Kolmogorov's result.

The structure function $\langle [u_l]^2 \rangle = \langle [u(x+l) - u(x)]^2 \rangle$ measures how smooth the velocity distribution is for a given length scale. It is small at large l as velocity is continous at that scale, and large at small l as velocity can change appreciatively over that scale.