Section 2.1 Quadratic Functions

General Form: $ax^2 + bx + c, \quad a \neq 0$

Standard Form: $f(x) = a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}, \quad a \neq 0$

Standard Form Expanded:

$$a\left(x^2 + \frac{2bx}{2a} + \frac{b^2}{4a^2}\right) + c - \frac{b^2}{4a} \quad \rightarrow \quad ax^2 + \frac{2abx}{2a} + \frac{ab^2}{4a^2} + c - \frac{b^2}{4a} \quad \rightarrow \quad ax^2 + bx + \frac{b^2}{4a} + c - \frac{b^2}{4a} \quad \rightarrow \quad ax^2 + bx + \frac{b^2}{4a} - \frac{b^2}{4a} + c$$

$= ax^2 + bx + c, \quad a \neq 0$

Coordinates of the vertex of the graph of f: $\left(-\frac{b}{2a},\ c - \frac{b^2}{4a}\right) \quad \rightarrow \quad \left(-\frac{b}{2a},\ f\left(-\frac{b}{2a}\right)\right)$

If a $>$ 0, f has a minimum at x $= -\frac{b}{2a}$. The minimum value is: $f\left(-\frac{b}{2a}\right)$

If a $<$ 0, f has a maximum at x $= -\frac{b}{2a}$. The minimum value is: $f\left(-\frac{b}{2a}\right)$

The equation of the axis of symmetry: $x + \frac{b}{2a} = 0$

x-interceptions:

Consider the general form: $ax^2 + bx + c, \quad a \neq 0$

[from the figure, if the quadratic function / parabola intersects the x-axis, then the y-coordinate is always zero]

thus: $ax^2 + bx + c = 0$

To find the x-coordinates, solve the equation: $ax^2 + bx + c = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Transforming $ax^2 + bx + c$ into $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$:

$ax^2 + bx + c = 0, \quad a \neq 0$

Complete the square:

$x^2 + \frac{bx}{a} + \frac{c}{a} = 0$, divide the equation by a in order to ensure that x^2 has a coefficient of 1.

$$x^2 + \frac{bx}{a} + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a} = 0$$

$$x^2 + \frac{bx}{a} + \left(\frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a}$$

$$\left(x + \frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a} \quad \rightarrow \quad \sqrt{\left(x + \frac{b}{2a}\right)^2} = \pm\sqrt{\left(\frac{b}{2a}\right)^2 - \frac{c}{a}} \quad \rightarrow \quad x + \frac{b}{2a} = \pm\sqrt{\left(\frac{b}{2a}\right)^2 - \frac{c}{a}}$$

$$x = -\frac{b}{2a} \pm \sqrt{\left(\frac{b}{2a}\right)^2 - \frac{c}{a}} \quad \rightarrow \quad x = -\frac{b}{2a} \pm \sqrt{\frac{b^2}{4a^2} - \frac{c}{a}} \quad \rightarrow \quad x = -\frac{b}{2a} \pm \sqrt{\frac{b^2}{4a^2} - \frac{4ac}{4a^2}}$$

$$x = -\frac{b}{2a} \pm \sqrt{\frac{b^2 - 4ac}{4a^2}} \quad \rightarrow \quad x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{\sqrt{4a^2}} \quad \rightarrow \quad x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Thus we have transformed $ax^2 + bx + c$ into $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad :$$

If b^2 - 4ac $<$ 0, then there is no x-interception.

If b^2 - 4ac = 0, then the parabola touches the x-axis coordinates of the point of contact at: $\left(-\dfrac{b}{2a}, 0\right)$

If b^2 - 4ac $>$ 0, then the parabola meets the x-axis at two distinct points.

The coordinates / x-interceptions are: $\left(\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a},\ 0\right)$, or $\left(\dfrac{-b + \sqrt{b^2 - 4ac}}{2a},\ 0\right)$ and $\left(\dfrac{-b - \sqrt{b^2 - 4ac}}{2a},\ 0\right)$

Example, page 120, problem 18:

General Form: $f(x) = x^2 + 16x + 61$

a.) Convert to standard form: $f(x) = a\left(x + \dfrac{b}{2a}\right)^2 + c - \dfrac{b^2}{4a},\ \ a \neq 0$

a = 1, b = 16, c = 61

$$f(x) = \left(x + \frac{16}{2}\right)^2 + 61 - \frac{16^2}{4} \quad \rightarrow \quad f(x) = (x+8)^2 + 61 - \frac{(4^2)^2}{4} \quad \rightarrow \quad f(x) = (x+8)^2 + 61 - \frac{4^4}{4}$$

$$\rightarrow f(x) = (x+8)^2 + 61 - 4^3 \quad \rightarrow \quad f(x) = (x+8)^2 + 61 - 64 \quad \rightarrow \quad f(x) = (x+8)^2 - 3$$

thus, the standard form is: $f(x) = (x+8)^2 - 3$

b.) Find the vertext coordinates: $\left(-\dfrac{b}{2a},\ \ c - \dfrac{b^2}{4a}\right)$

a = 1, b = 16, c = 61

$$\left(-\frac{16}{2},\ 61 - \frac{16^2}{4}\right) \quad \rightarrow \quad \left(-8,\ 61 - \frac{(4^2)^2}{4}\right) \quad \rightarrow \quad \left(-8,\ 61 - \frac{4^4}{4}\right) \quad \rightarrow \quad \left(-8,\ 61 - 4^3\right) \quad \rightarrow \quad (-8,\ 61 - 64)$$

$\rightarrow \quad (-8, -3)$ is the vertex

Since a = 1 $>$ 0, the vertex $(-8, -3)$ is the minimum point of the graph of f.

Equation of the axis of symmetry: $x + \dfrac{b}{2a} = 0 \quad \rightarrow \quad x + \dfrac{16}{2 \cdot 1} = 0 \quad \rightarrow \quad x + 8 = 0 \quad \rightarrow \quad x = -8$

Find the x-interceptions: $\left(\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a},\ 0 \right)$

a = 1, b = 16, c = 61

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \rightarrow \quad \frac{-16 \pm \sqrt{16^2 - 4 \cdot 1 \cdot 61}}{2 \cdot 1} \quad \rightarrow \quad \frac{-16 \pm \sqrt{256 - 244}}{2} \quad \rightarrow \quad \frac{-16 \pm \sqrt{12}}{2} \quad \rightarrow \quad \frac{-16 \pm 2\sqrt{3}}{2}$$

$x = -8 \pm \sqrt{3}$

x-interceptions: $\left(-8 - \sqrt{3},\ \ 0\right)$ and $\left(-8 + \sqrt{3},\ \ 0\right)$

The vertex $(-8, -3)$ is the point of minimum and the minimum value is - 3.

<u>Example:</u> $f(x) = -x^2 - 4x + 1$

Convert to standard form: $f(x) = a\left(x + \dfrac{b}{2a}\right)^2 + c - \dfrac{b^2}{4a},\ \ a \neq 0$

a = -1, b = -4, c = 1

$$f(x) = -\left(x + \frac{(-4)}{-2}\right)^2 + 1 - \frac{(-4)^2}{-4} \quad \rightarrow \quad -\left(x+2\right)^2 + 1 - \frac{16}{-4} \quad \rightarrow \quad -\left(x+2\right)^2 + 1 + 4 \quad \rightarrow \quad -\left(x+2\right)^2 + 5$$

thus:

$$f(x) = -\left(x+2\right)^2 + 5 \quad \text{or} \quad f(x) = -\left(x^2 + 4x + 4\right) + 5 \quad \rightarrow \quad -x^2 - 4x - 4 + 5 \quad \rightarrow \quad -x^2 - 4x + 1$$

Coordinates of the vertex of the graph of f: $\left(-\dfrac{b}{2a}\ ,\ \ c - \dfrac{b^2}{4a}\right)$ or $\left(-\dfrac{b}{2a}\ ,\ \ f\left(-\dfrac{b}{2a}\right)\right)$

$$\left(-\frac{b}{2a}\ ,\ \ c - \frac{b^2}{4a}\right) \quad \rightarrow \quad \left(-\frac{(-4)}{2(-1)}\ ,\ \ 1 - \frac{(-4)^2}{4(-1)}\right) \quad \rightarrow \quad \left(-\frac{(4)}{2}\ ,\ \ 1 - \frac{16}{-4}\right) \quad \rightarrow \quad (-2\ ,\ \ 1+4) \quad \rightarrow \quad (-2\ , 5)$$

Since a = -1 $<$ 0, the vertex $(-2, -5)$ is the maximum point of the graph of f and the maximum value is 5.

Equation of the axis of symmetry: $x + \dfrac{b}{2a} = 0 \quad \rightarrow \quad x + \dfrac{(-4)}{2(-1)} = 0 \quad \rightarrow \quad x + 2 = 0 \quad \rightarrow \quad x = -2$

Find the x-interceptions: $\left(\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a},\ 0 \right)$ of $f(x) = -x^2 - 4x + 1$

a = -1, b = -4, c = 1

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \rightarrow \quad \frac{-(-4) \pm \sqrt{(-4)^2 - 4 \cdot (-1) \cdot (1)}}{2 \cdot (-1)} \quad \rightarrow \quad \frac{4 \pm \sqrt{16 + 4}}{-2} \quad \rightarrow \quad \frac{4 \pm \sqrt{20}}{-2} \quad \rightarrow \quad \frac{4 \pm 2\sqrt{5}}{-2}$$

$x = -2 \pm -\sqrt{5}$

x-interceptions: $\left(-2 + \sqrt{5},\ \ 0\right)$ and $\left(-2 - \sqrt{5},\ \ 0\right)$ or $\left(\sqrt{5} - 2,\ \ 0\right)$ and $\left(-(2 + \sqrt{5}),\ \ 0\right)$

<u>Homework Problems:</u>

Page 120: Exercises 15, 17, 19, 20, 23, 25 and Page 121: Exercises 38, 40, 42, 44

Different Type of Problem: Solve for the value of a

Write the standard form of the quadratic function with a vertex at $(5, 12)$ and a point $(7, 15)$ on the equation

when x = 7 and y or f(x) = 15

$f(x) = a\,(x-5)^2 + 12 \quad \to \quad 15 = a\,(7-5)^2 + 12 \quad \to \quad 15 = a\,(2)^2 + 12 \quad \to \quad 15 = 4a + 12 \quad \to \quad 3 = 4a$

$a = \dfrac{3}{4}$ plug value of a back into function: f(x) = a$(x-5)^2$ + 12

$y \;=\; f(x) \;=\; \dfrac{3}{4}\,(x-5)^2 + 12$

Example:

Vertex: $(-2, -2)$ at the point: $(-1, 0)$

$f(x) = a\,(x+2)^2 - 2 \quad \to \quad 0 = a\,(-1+2)^2 - 2 \quad \to \quad 0 = a\,(1)^2 - 2 \quad \to \quad 0 = a - 2 \quad \to \quad 2 = a$

plug value of a back into function: $f(x) = a\,(x+2)^2 - 2 \quad \to \quad f(x) = 2\,(x+2)^2 - 2$

Worksheet 4:

2.) $f(x) = -\dfrac{1}{2}\,(x-3)^2 + 4$

Shift the graph horizontally 3 units to the right and and shift it vertically 4 units up.