

First Example - From Class:

$$f(x) = \frac{x-1}{x+5} \rightarrow y = \frac{x-1}{x+5}$$

Swap x and y:

$$x = \frac{y-1}{y+5}$$

Solve for y:

$$x(y+5) = y-1 \rightarrow xy + 5x = y-1$$

$$xy - y = -5x - 1 \rightarrow y(x-1) = -(5x+1)$$

$$y = \frac{-(5x+1)}{x-1} = \frac{-(5x+1)}{-(-x+1)} = \frac{5x+1}{1-x}$$

$$\text{Let } y = y^{-1} = f^{-1}(x)$$

$$y = \frac{5x+1}{1-x} \rightarrow y^{-1} = f^{-1}(x) = \frac{5x+1}{1-x}$$

$$\text{Verify } f(f^{-1}(x)) = x$$

$$f(f^{-1}(x)) = \frac{\left(\frac{5x+1}{1-x}\right) - 1}{\left(\frac{5x+1}{1-x}\right) + 5} = \frac{\frac{5x+1 - (1-x)}{1-x}}{\frac{5x+1 + 5(1-x)}{1-x}} = \frac{5x+1 - (1-x)}{5x+1 + 5(1-x)}$$

$$f(f^{-1}(x)) = \frac{5x+1-1+x}{5x+1+5-5x} = \frac{5x+x}{1+5} = \frac{6x}{6} = x$$

$$\text{Thus } f(f^{-1}(x)) = x$$

$$\text{Verify } f^{-1}(f(x)) = x$$

$$f^{-1}(f(x)) = \frac{5\left(\frac{x-1}{x+5}\right) + 1}{1 - \left(\frac{x-1}{x+5}\right)} = \frac{\frac{5(x-1) + x + 5}{x+5}}{\frac{(x+5) - (x-1)}{x+5}} = \frac{5(x-1) + x + 5}{(x+5) - (x-1)}$$

$$f^{-1}(f(x)) = \frac{5x-5+x+5}{x+5-x+1} = \frac{5x+x}{5+1} = \frac{6x}{6} = x$$

$$\text{Thus } f^{-1}(f(x)) = x$$

Example of Graphs:

<https://www.desmos.com/calculator/qlalxjpnaw>