Matrix Decompositions: QR & SVD Applied to Image Processing With Python

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Matrix Decompositions

A = QR → **R = Q^TA** → **A = QR**

Transpose: interchange the rows and columns of a matrix or array

A_Transpose = A.T or B = np.transpose(A)

A = Image Matrix

Q = Orthogonal Matrix --> $\mathbf{Q}^T\mathbf{Q} = I$ (Identity Matrix)

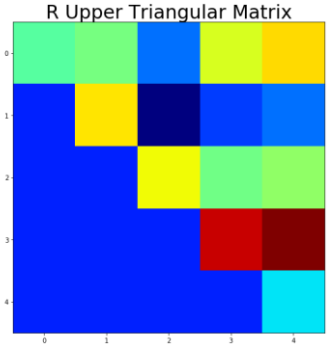
R = Upper Triangular Matrix

R = np.triu(A) or R1 = np.triu(A,0) # 0 is the index

D = Diagonal Matrix

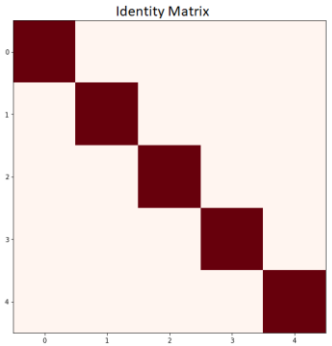
D = np.diag(A)

$$R = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$



Upper Triangular Matrix

$$I = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$



Identity Matrix

Gram–Schmidt General Formula

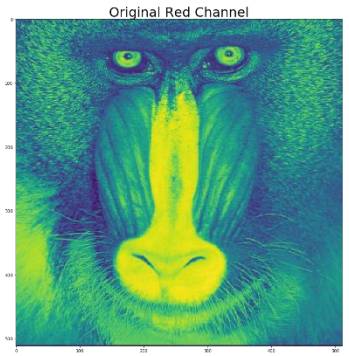
$$\mathbf{u}_\mathrm{p} = \mathbf{v}_\mathrm{p} - \sum_{i=1}^{p-1} \left(\frac{\mathbf{u}_i \cdot \mathbf{v}_p}{\| \mathbf{u}_i \|^2} \right) \mathbf{u}_i \text{ for p} = 2, 3, \ldots, \mathrm{n}$$

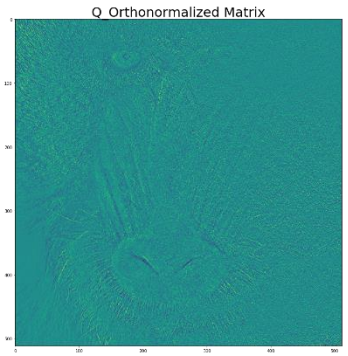
- Converts non-orthgonal matrix into an orthogonal matrix
- “Distills” non-”cost-efficient” matrix into a “cost-efficient” matrix
  - “Cost-efficient” = computationally efficient

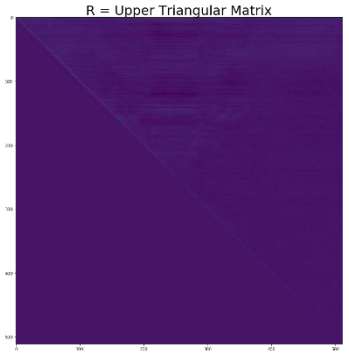
Gram–Schmidt Process – Python Output

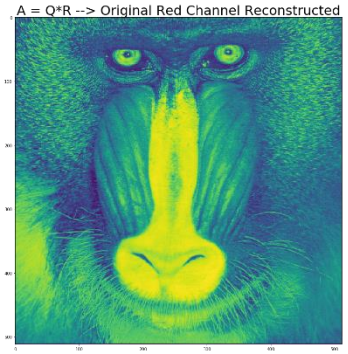
Image Decomposition: **A = QR** **&** **R = Q^TA**

Red Channel

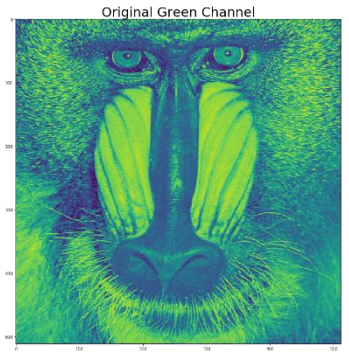


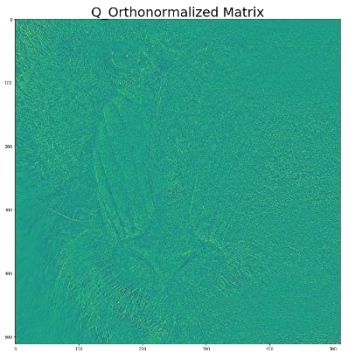


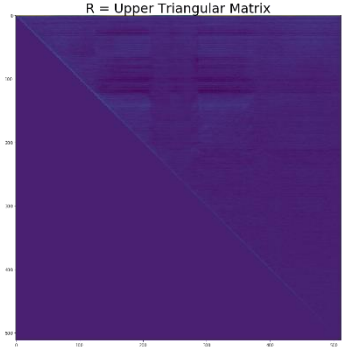


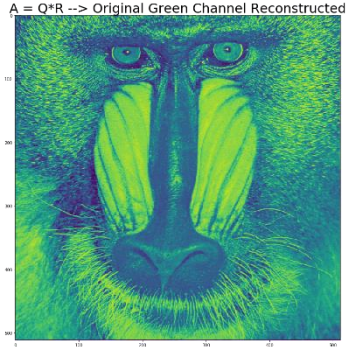


Green Channel

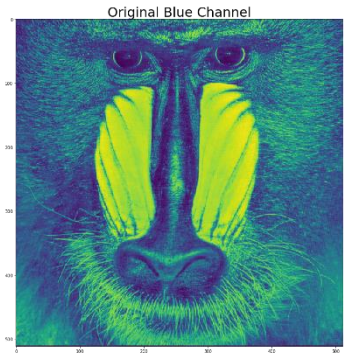


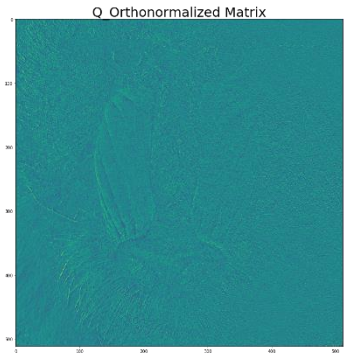


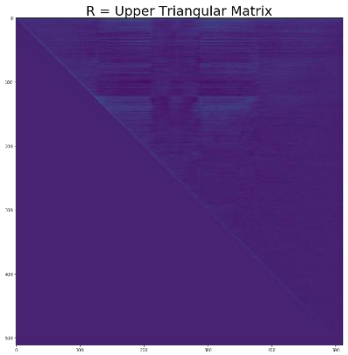


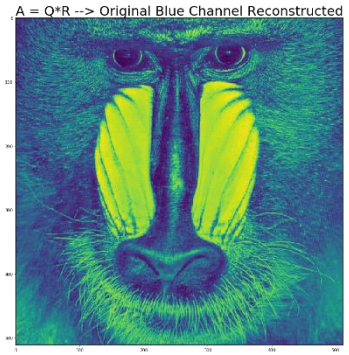


Blue Channel



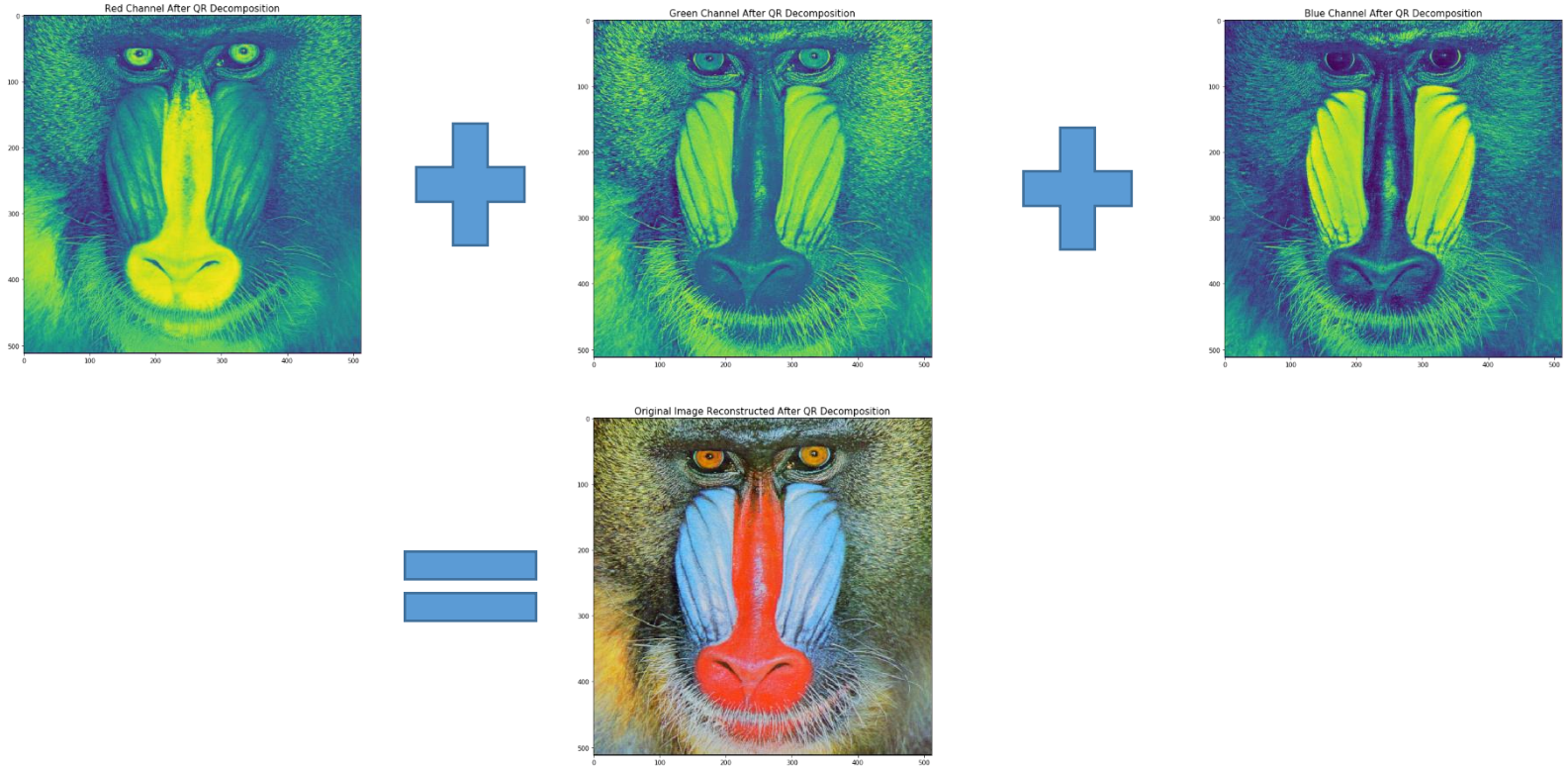






Gram–Schmidt Process – Python Output

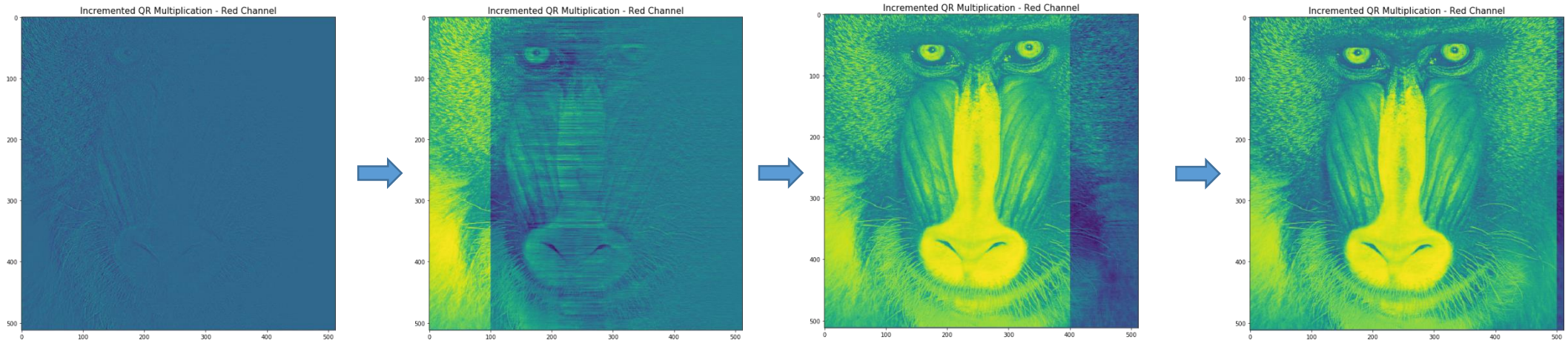
- Image Reconstruction – 3 Channels Combined to Reconstruct Original Image



Gram–Schmidt Process – Python Output

Image Reconstruction Through Incremental

A = QR Multiplication – One Channel



A = QR → **$R = Q^T A$** → **A = QR**

- Original channel incrementally reconstructed after successive QR multiplication, (501 iterations), reconstructing approximately, 97.8515625 % per color channel of each original channel.

SVD – Singular Value Decomposition

$$\mathbf{A} = U\Sigma V^{\mathsf{T}}$$

A = Original Matrix

∑ = diagonal matrix (square root of eigenvalues of AA^T & A^TA)

U = Eigenvectors of AA^T (Left singular vectors)

V = Eigenvectors of A^TA (Right singular vectors)

$U^T U = I$ (Identity Matrix)

$V^T V = I$ (Identity Matrix)

```
U, Sig, V = np.linalg.svd(A)
```

```
A_Recon = np.linalg.multi_dot([U,np.diag(Sig),V])
```

Eigenvectors and eigenvalues store key information / key features of a matrix; and help in time travel

Source: www.inverse.com

$$\begin{bmatrix} \vdots & \vdots & \vdots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \hat{u}_1 & \hat{u}_2 & \hat{u}_3 & \cdots & \hat{u}_m \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \cdots & \vdots \end{bmatrix}$$

$U_{\,m \times m}$

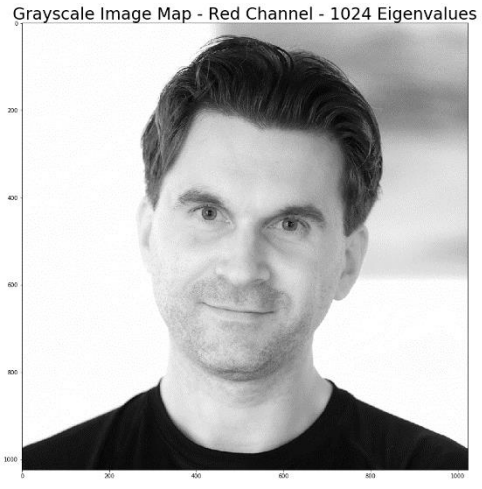
$$\begin{bmatrix} \sigma_{11} & 0 & 0 & \cdots & 0 \\ 0 & \sigma_{22} & 0 & \cdots & 0 \\ 0 & 0 & \sigma_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \sigma_{mn} \end{bmatrix}$$

$\Sigma_{\,m \times n}$

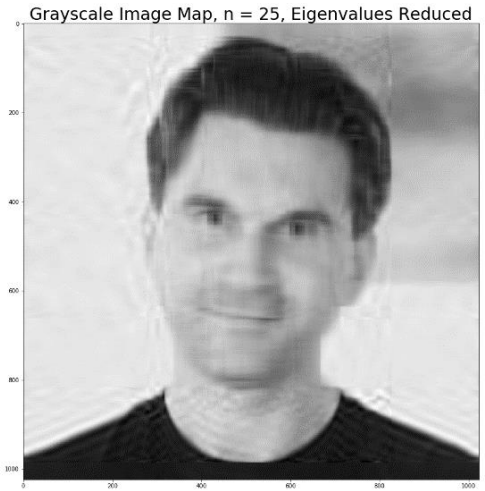
$$\begin{bmatrix} \cdots & \cdots & \hat{v}_1 & \cdots & \cdots \\ \cdots & \cdots & \hat{v}_2 & \cdots & \cdots \\ \cdots & \cdots & \hat{v}_3 & \cdots & \cdots \\ \cdots & \cdots & \vdots & \cdots & \cdots \\ \cdots & \cdots & \hat{v}_n & \cdots & \cdots \end{bmatrix}$$

$V^T_{\,n \times n}$

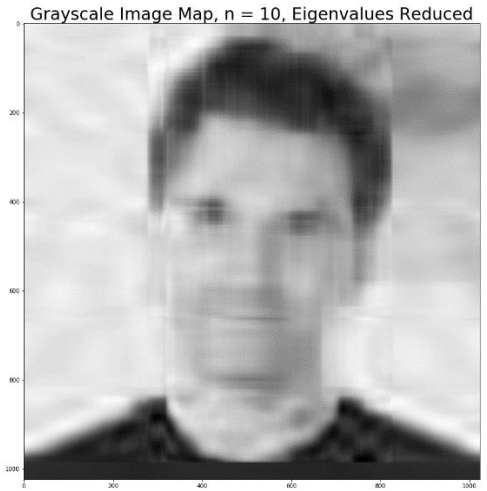
Applications of SVD – Eigenvalue Reduction



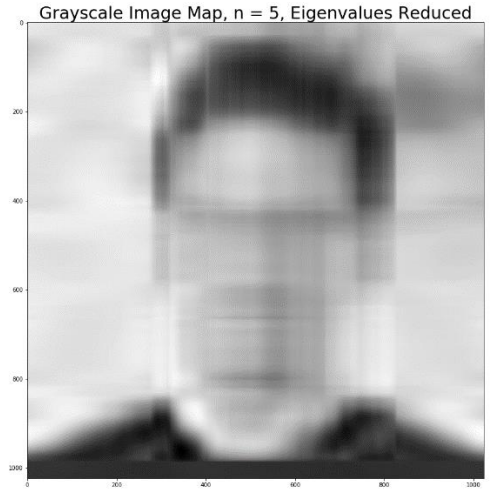
1024
Eigenvalues



25
Eigenvalues



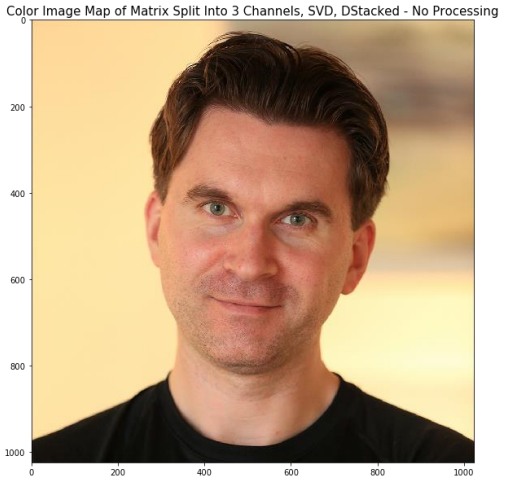
10
Eigenvalues



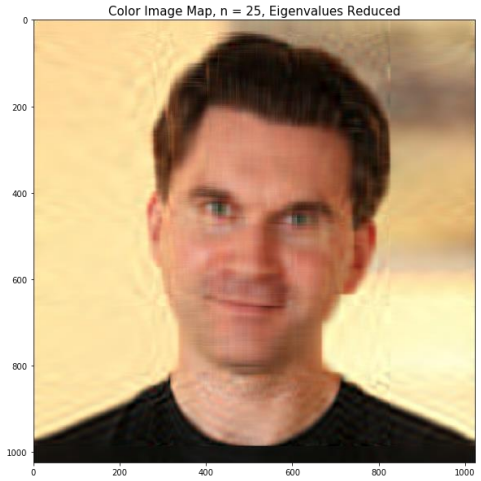
5
Eigenvalues

- Eigenvectors U and V^T remain unchanged, while eigenvalues ∑ are reduced
  - Key facial features (principal components) are still recognizable

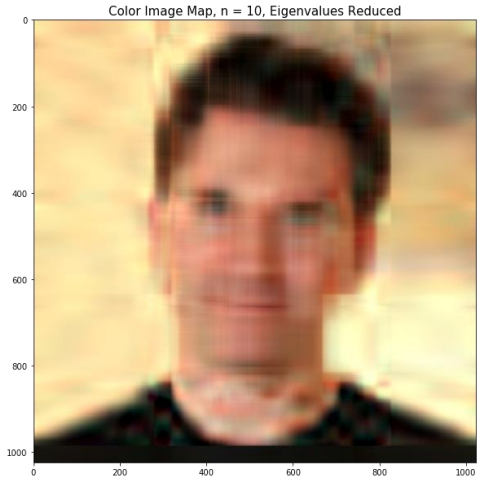
Applications of SVD – Eigenvalue Reduction



1024 Eigenvalues



25 Eigenvalues

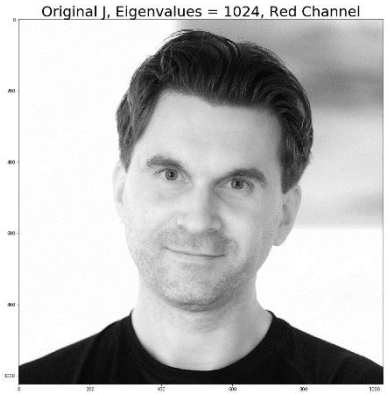


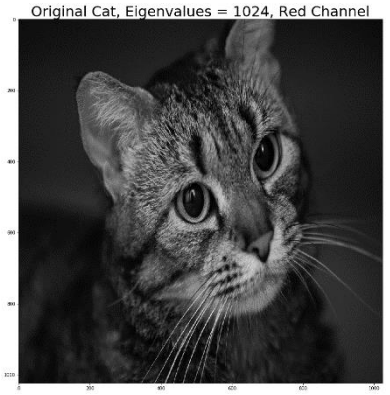
10 Eigenvalues

5 Eigenvalues

- Eigenvectors U and V^T remain unchanged, while eigenvalues ∑ are reduced
- Key facial features (principal components) are still recognizable
- SVD performed per channel - RGB

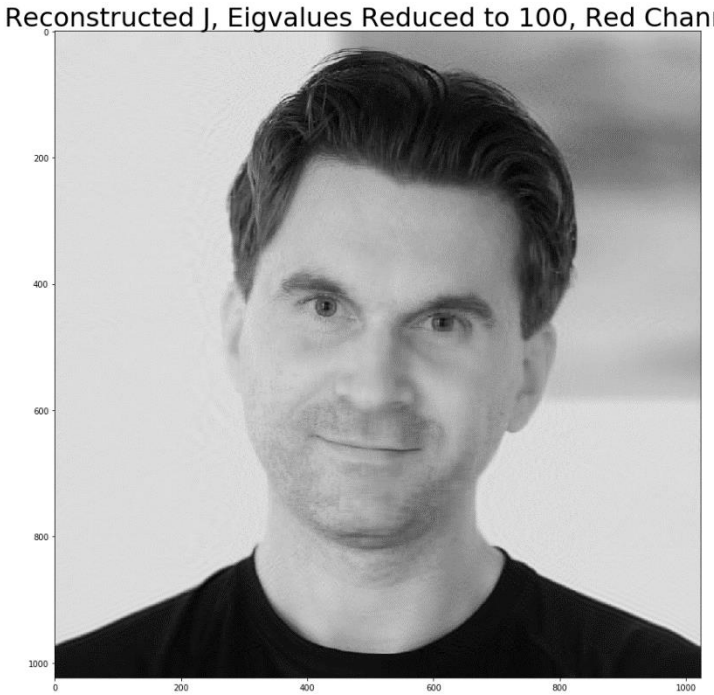
SVD – Mixing Eigenvectors and Eigenvalues

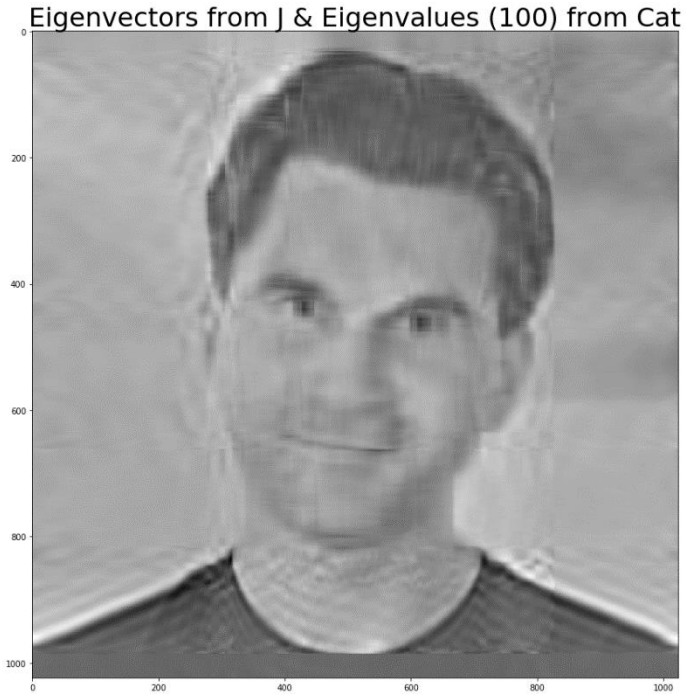




U & V^T

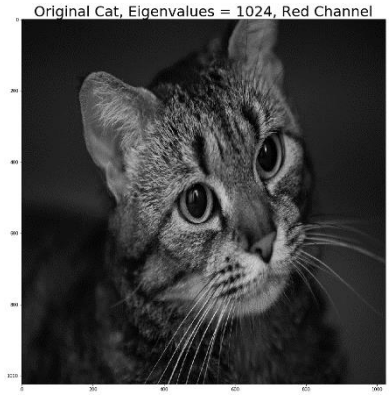
Σ

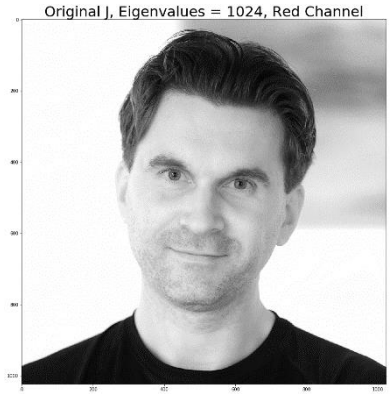




- Eigenvectors from J and Eigenvalues (reduced to 100) from Cat
  - Recombined through SVD = $U \sum V^T$
- Eigenvalues are affecting the luminance (lights and darks) values of the image

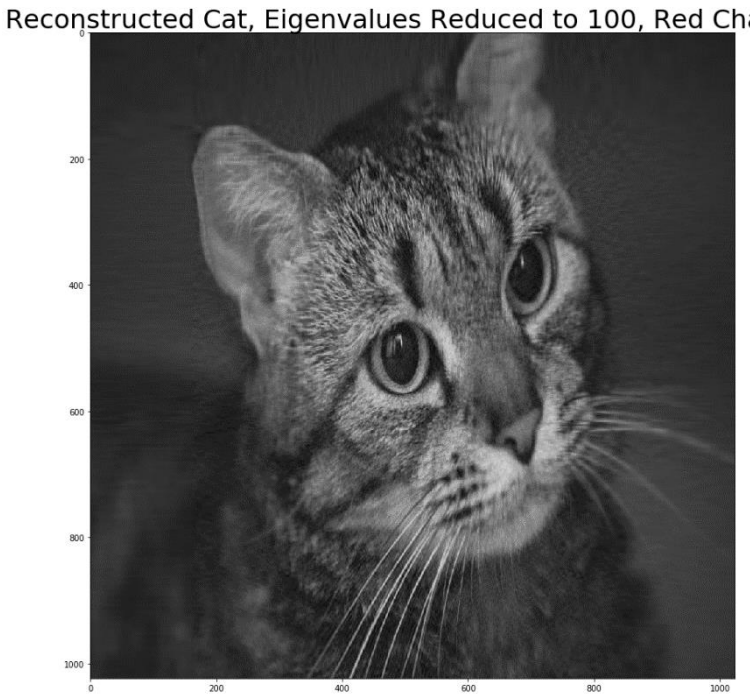
SVD – Mixing Eigenvectors and Eigenvalues

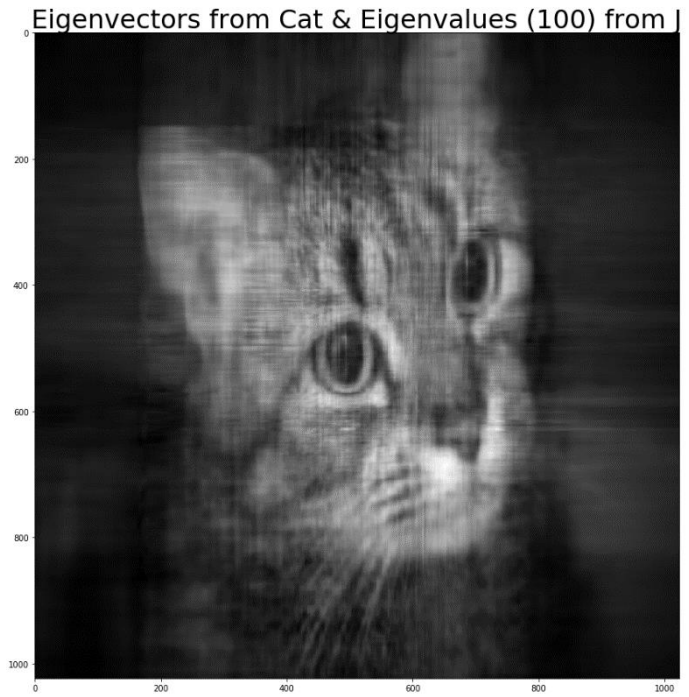




U & V^T

Σ





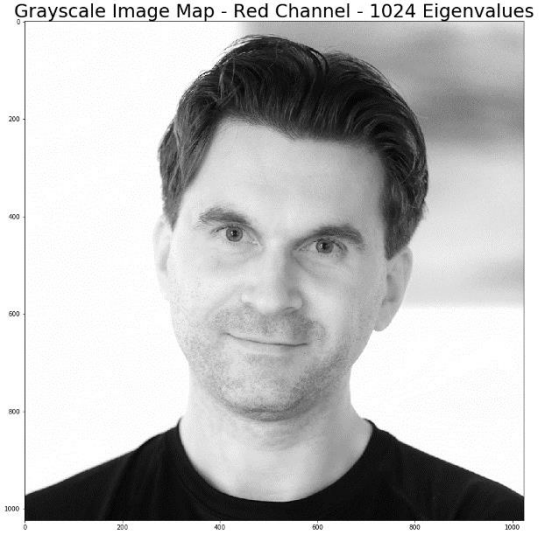
- Eigenvectors from Cat and Eigenvalues (reduced to 100) from J
  - Recombined through SVD = $U \sum V^T$
- Eigenvalues are affecting the luminance (lights and darks) values of the image

Links

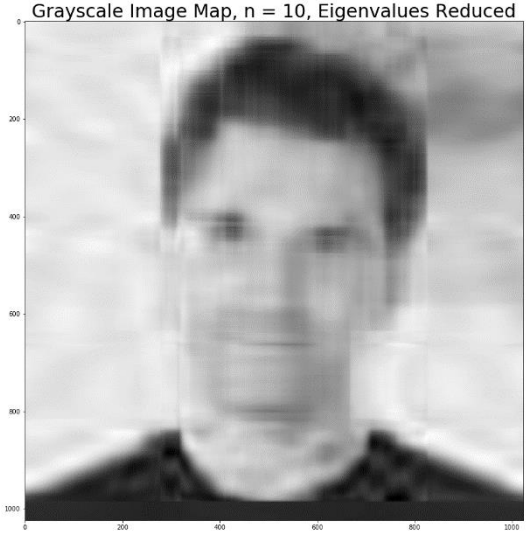
- My GitHub Page:
  - https://github.com/Johnstul
  - https://github.com/Johnstul/QR-Decomposition-Images
  - https://github.com/Johnstul/SVD-Image-Processing
- Jupyter Online Notebooks:
  - https://jupyter.org/try

Thanks

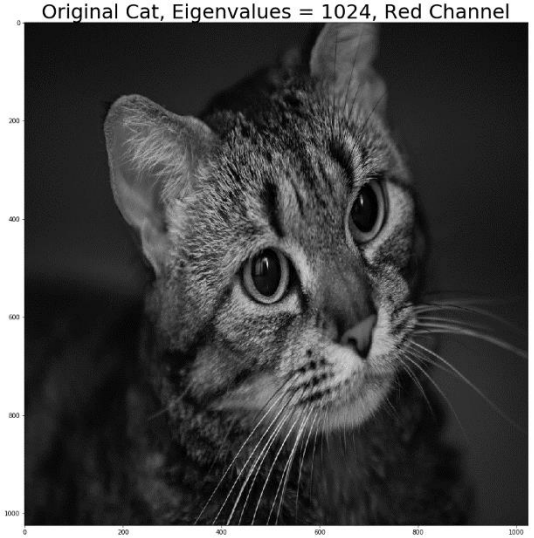
Grayscale Image Map - Red Channel - 1024 Eigenvalues



Grayscale Image Map, n = 10, Eigenvalues Reduced



Original Cat, Eigenvalues = 1024, Red Channel



ReconCat1, Eigenvalues = 10, Red Channel