7th EDITION

NEW SYLLABUS MATHEMATICS

SAMPLE COPY NOT FOR SALE

Consultant • Dr Yeap Ban Har **Authors** • Dr Joseph Yeo • Teh Keng Seng • Loh Cheng Yee • Ivy Chow

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PREFACE

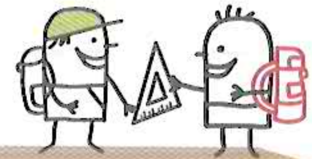
New Syllabus Mathematics (NSM)

is a series of textbooks specially designed to provide valuable learning experiences to engage the hearts and minds of students sitting for the GCE O level examination in Mathematics. Included in the textbooks are **Investigation**, **Class Discussion**, **Thinking Time**, **Journal Writing**, **Performance Task** and **Problems in Real-World Contexts** to support the teaching and learning of Mathematics.

Every chapter begins with a chapter opener which motivates students in learning the topic. Interesting stories about Mathematicians, real-life examples and applications are used to arouse students' interest and curiosity so that they can appreciate the beauty of Mathematics in their surroundings.

The use of ICT helps students to visualise and manipulate mathematical objects more easily, thus making the learning of Mathematics more interactive. Ready-to-use interactive ICT templates are available at http://www.shinglee.com.sg/StudentResources/

KEY FEATURES



CHAPTER OPENER

Each chapter begins with a chapter opener to arouse students' interest and curiosity in learning the topic.

LEARNING OBJECTIVES

Learning objectives help students to be more aware of what they are about to study so that they can monitor their own progress.

RECAP

Relevant prerequisites will be revisited at the beginning of the chapter or at appropriate junctures so that students can build upon their prior knowledge, thus creating meaningful links to their existing schema.

WORKED EXAMPLE

This shows students how to apply what they have learnt to solve related problems and how to present their working clearly. A suitable heading is included in brackets to distinguish between the different Worked Examples.

PRACTISE NOW

At the end of each Worked Example, a similar question will be provided for immediate practice. Where appropriate, this includes further questions of progressive difficulty.

SIMILAR QUESTIONS

A list of similar questions in the Exercise is given here to help teachers choose questions that their students can do on their own.

EXERCISE

The questions are classified into three levels of difficulty – Basic, Intermediate and Advanced.

SUMMARY

At the end of each chapter, a succinct summary of the key concepts is provided to help students consolidate what they have learnt.

REVIEW EXERCISE

This is included at the end of each chapter for the consolidation of learning of concepts.

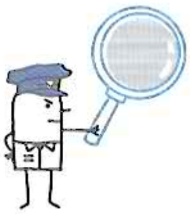
CHALLENGE YOURSELF

Optional problems are included at the end of each chapter to challenge and stretch high-ability students to their fullest potential.

REVISION EXERCISE

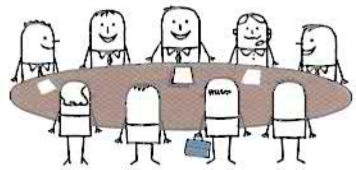
This is included after every few chapters to help students assess their learning.

Learning experiences have been infused into Investigation, Class Discussion, Thinking Time, Journal Writing and Performance Task.



Investigation

Activities are included to guide students to investigate and discover important mathematical concepts so that they can construct their own knowledge meaningfully.

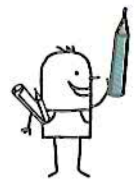


Class Discussion

Questions are provided for students to discuss in class, with the teacher acting as the facilitator. The questions will assist students to learn new knowledge, think mathematically, and enhance their reasoning and oral communication skills.

Thinking Time

Key questions are also included at appropriate junctures to check if students have grasped various concepts and to create opportunities for them to further develop their thinking.



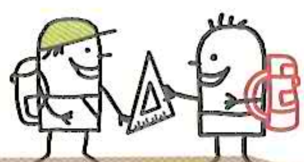
Journal Writing

Opportunities are provided for students to reflect on their learning and to communicate mathematically. It can also be used as a formative assessment to provide feedback to students to improve on their learning.

Performance Task

Mini projects are designed to develop research and presentation skills in the students.

MARGINAL NOTES



This contains important information that students should know.

This guides students on how to approach a problem.

This includes information that may be of interest to students.

This contains certain mathematical concepts or rules that students have learnt previously.

This contains puzzles, fascinating facts and interesting stories about Mathematics as enrichment for students.

This guides students to search on the Internet for valuable information or interesting online games for their independent and self-directed learning.

Contents

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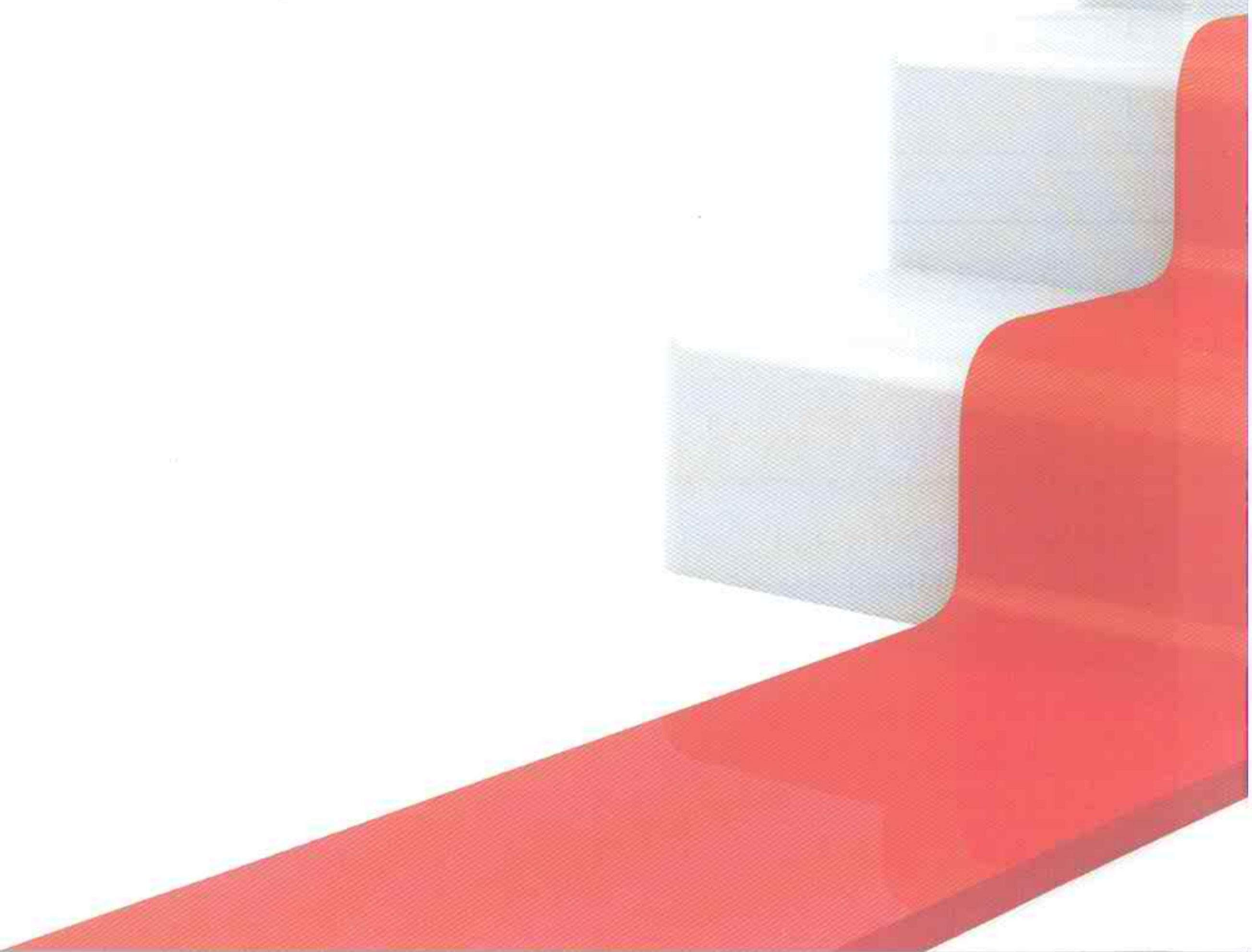
Linear Inequalities in Two Variables

Many companies are devoted to using systems of inequalities to ensure that they are making the most profit. This can be achieved by producing the right combination of items in terms of quantity or spending the least amount of money in the production of certain items. Linear programming is the mathematical process of analysing a system of inequalities to make the best decisions given the constraints of the situation.



Chapter One

LEARNING OBJECTIVES

At the end of this chapter, you should be able to:

- illustrate linear inequalities in two variables,
- differentiate linear inequalities in two variables from linear equations in two variables,
- graph linear inequalities in two variables,
- solve problems involving linear inequalities in two variables,
- solve a system of linear inequalities in two variables,
- solve problems involving systems of linear inequalities in two variables.

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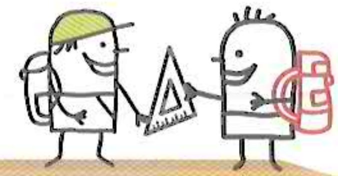
1.1 Linear Inequalities in Two Variables



Recap (Properties of Inequalities)

In Book 2 and Book 3, we have learnt some properties of inequalities.

We can add or subtract a *positive* number from both sides of an inequality *without* having to reverse the inequality sign, i.e.

$$\text{if } x \geqslant y \text{ and } a > 0, \text{ then } x + a \geqslant y + a \text{ and } x - a \geqslant y - a.$$

This is also true for a *negative* number b.

$$\text{if } x \geqslant y \text{ and } b < 0, \text{ then } x + b \geqslant y + b \text{ and } x - b \geqslant y - b.$$

We can multiply or divide both sides of an inequality by a *positive* number *without* having to *reverse* the inequality sign, i.e.

$$\text{if } x \geqslant y \text{ and } c > 0, \text{ then } cx \geqslant cy \text{ and } \frac{x}{c} \geqslant \frac{y}{c}.$$

However, if we multiply or divide both sides of an inequality by a *negative* number, we will have to *reverse* the inequality sign, i.e.

$$\text{if } x \geqslant y \text{ and } d < 0, \text{ then } dx \leqslant dy \text{ and } \frac{x}{d} \leqslant \frac{y}{d}.$$

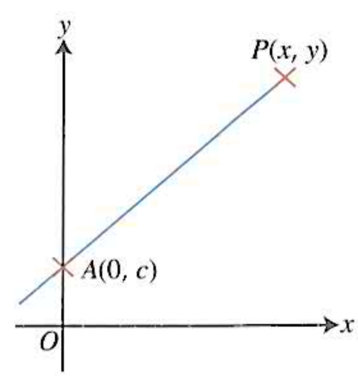
For any three numbers x, y and z,

$$\text{if } x > y \text{ and } y > z, \text{ then } x > z.$$

This is known as the **transitive** property of inequalities.

Recap (Linear Equations in Two Variables)

We have learnt linear equations in two variables and how to draw graphs of linear equations in the form $y = mx + c$, where x and y are the variables.
In general, for a straight line passing through the point $(0, c)$ and with gradient m, the equation is $y = mx + c$.
In this section, we shall learn about linear inequalities in two variables.



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Investigation

Linear Inequalities in Two Variables

Case 1: On the line

1. The graph of the linear equation $x + 2y = 4$ is shown in Fig. 1.1**(a)**. Some points are marked on the line.

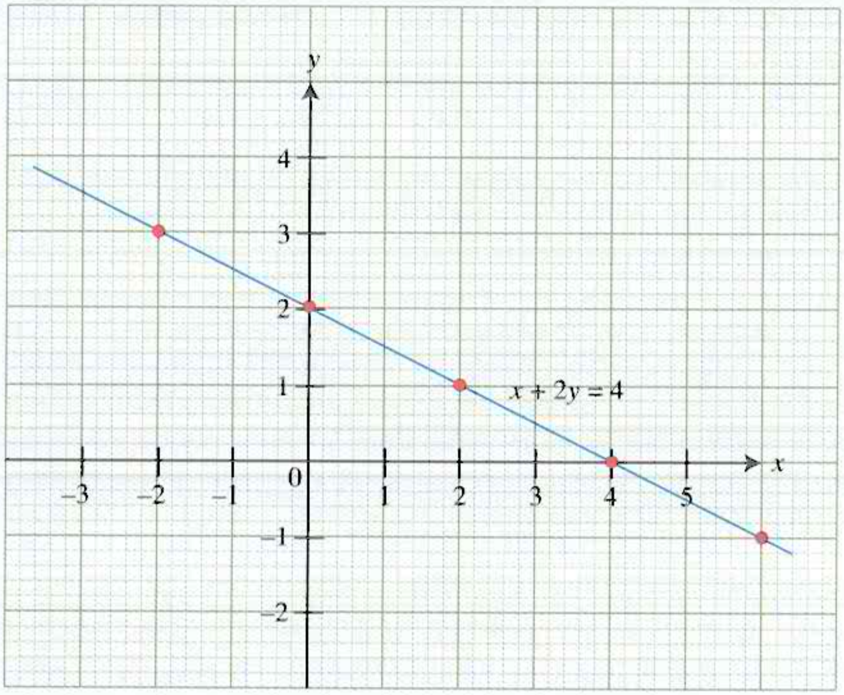


Fig. 1.1(a)

Find the values of $x + 2y$ by substituting the coordinates of each point on the line. Record the values in Table 1.1. What do you notice about the values of $x + 2y$?

Case 2: Below the line

2. The graph of the linear equation $x + 2y = 4$ is shown in Fig. 1.1**(b)**. Some points are marked below the line.

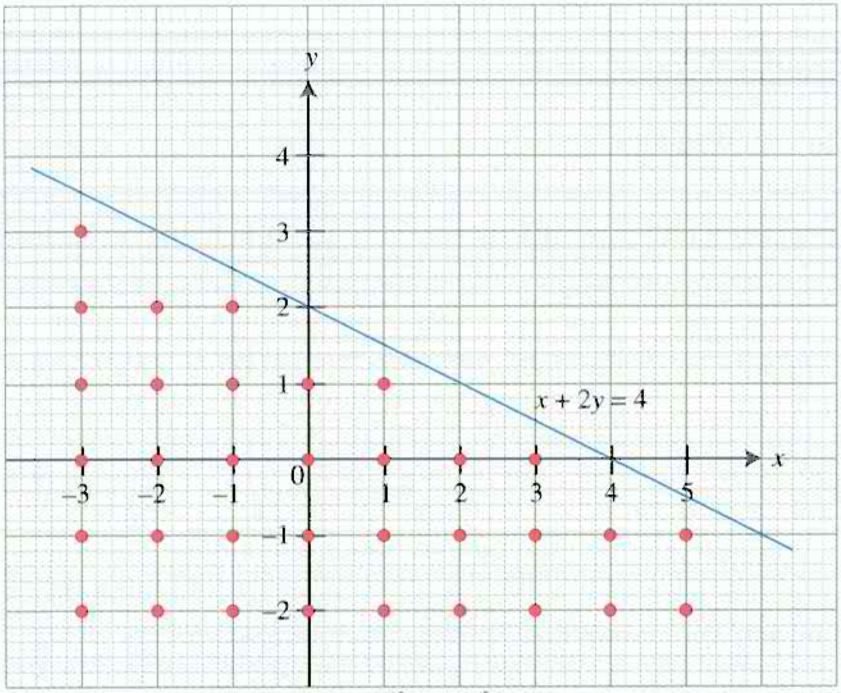


Fig. 1.1(b)

Find the values of $x + 2y$ by substituting the coordinates of each point below the line. Record the values in Table 1.1. What do you notice about the values of $x + 2y$?

Case 3: Above the line

3. The graph of the linear equation $x + 2y = 4$ is shown in Fig. 1.1**(c)**. Some points are marked above the line.

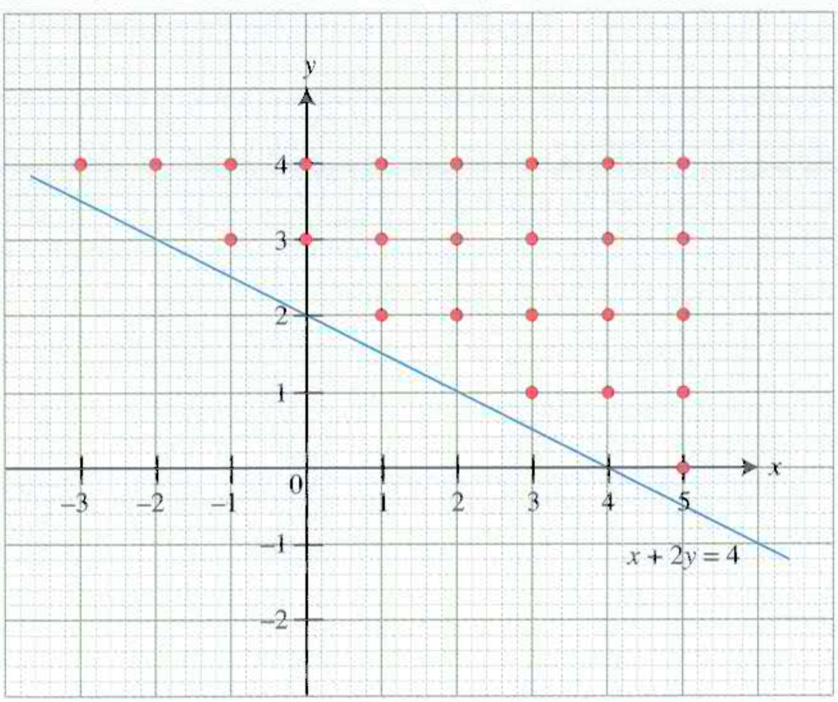


Fig. 1.1(c)

Find the values of $x + 2y$ by substituting the coordinates of each point above the line. Record the values in Table 1.1. What do you notice about the values of $x + 2y$?

	Case 1		Case 2		Case 3	
	Coordinates of point	$x + 2y$	Coordinates of point	$x + 2y$	Coordinates of point	$x + 2y$
Observation						

Table 1.1

From the investigation, we observe the following:

(i) In **Case 1**, the values of $x + 2y$ are always equal to 4;
(ii) In **Case 2**, the values of $x + 2y$ are always less than 4;
(iii) In **Case 3**, the values of $x + 2y$ are always greater than 4.

We are familiar with Case 1 as the points on the line represent a linear equation in two variables $x + 2y = 4$.

In Case 2, since the values of $x + 2y$ are always less than 4 for points below the line of linear equation $x + 2y = 4$, we can write the inequality $x + 2y < 4$ to represent the region *below* a straight line graph.

In Case 3, since the values of $x + 2y$ are always greater than 4 for points above the line of linear equation $x + 2y = 4$, we can write the inequality $x + 2y > 4$ to represent the region *above* a straight line graph.

We can say that the points in Fig. 1.1**(b)** satisfy the inequality $x + 2y < 4$ and the points in Fig. 1.1**(c)** satisfy the inequality $x + 2y > 4$.

Worked Example 1

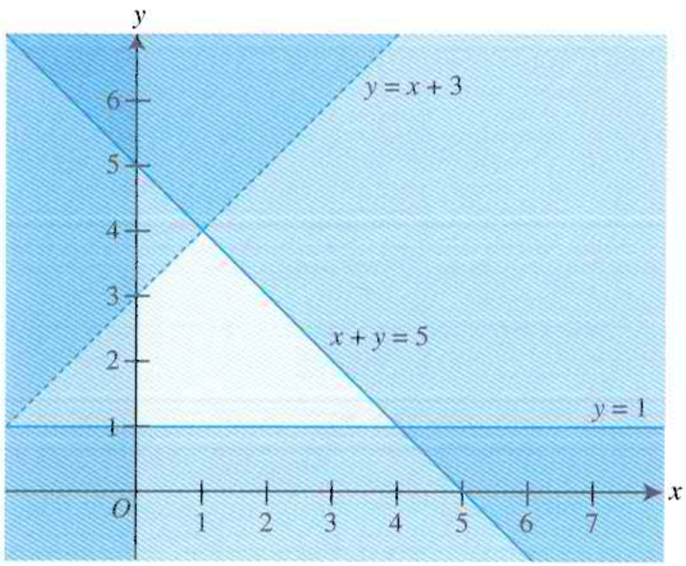
(Drawing Graphs of Linear Inequalities in Two Variables)

Show, unshaded, the region satisfied by the following inequalities:

$$x \geqslant 0,\ y \geqslant 1,\ x + y \leqslant 5,\ y < x + 3$$

Solution:

The graphs of $x = 0$, $y = 1$, $x + y = 5$ and $y = x + 3$ are drawn.



ATTENTION

We use a solid line for inequalities with $\geqslant$ or $\leqslant$ sign, and a dotted line for inequalities with $>$ or $<$ sign.

For $x \geqslant 0$, we shade the region to the left of the y-axis.

For $y \geqslant 1$, we shade the region below the line $y = 1$.

For $x + y \leqslant 5$, the region above $x + y = 5$ is shaded.

For $y < x + 3$, the region above the line $y = x + 3$ is shaded. Since $y < x + 3$, we use a dotted line for $y = x + 3$.

Problem Solving Tip

The required region is represented by the unshaded part. Check whether the unshaded region is the correct region by using a point in the unshaded region, e.g. (1, 2), to verify the inequalities.

PRACTISE NOW 1

Show, unshaded, the region satisfied by the inequalities $x + 2y \leqslant 8$ and $x > y$.

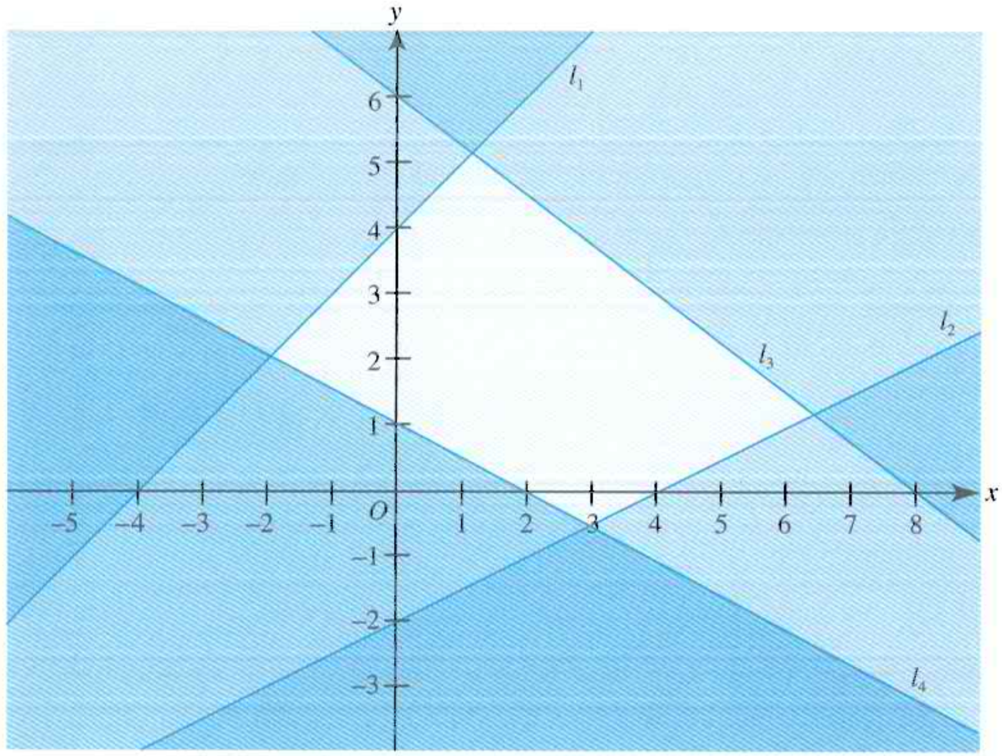
SIMILAR QUESTIONS

Exercise 1A Questions 1(a)–(f)

Worked Example 2

(Writing Linear Inequalities in Two Variables from Graphs)

Write down the inequalities which define the unshaded region.



Solution:

Equation of l_1:

$$\frac{y-4}{x-0}=\frac{4-0}{0-(-4)}$$

$$y-4=x$$

$$y=x+4$$

The unshaded region lies below l_1. Hence $y \leqslant x+4$ defines a part of the unshaded region.

Equation of l_2:

$$\frac{y-0}{x-4}=\frac{-2-0}{0-4}$$

$$\frac{y}{x-4}=\frac{1}{2}$$

$$y \geqslant \frac{1}{2}x-2$$

The unshaded region lies above l_2. Hence $y \geqslant \frac{1}{2}x-2$ defines a part of the unshaded region.

Equation of l_3:

$$\frac{y-6}{x-0}=\frac{6-0}{0-8}$$

$$\frac{y-6}{x}=-\frac{3}{4}$$

$$y=-\frac{3}{4}x+6$$

The unshaded region lies below l_3. Hence, $y \leqslant -\frac{3}{4}x + 6$ defines a part of the unshaded region.

Equation of l_4:

$$\frac{y-1}{x-0} = \frac{1-0}{0-2}$$

$$\frac{y-1}{x} = -\frac{1}{2}$$

$$y = -\frac{1}{2}x + 1$$

The unshaded region lies above l_4. Hence, $y \geqslant -\frac{1}{2}x + 1$ defines a part of the unshaded region.

∴ The unshaded region is defined by the four inequalities:

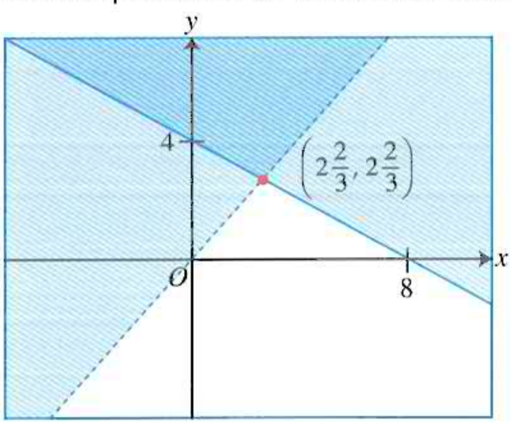
$$y \leqslant x + 4,\ y \geqslant \frac{1}{2}x - 2,\ y \leqslant -\frac{3}{4}x + 6 \text{ and } y \geqslant -\frac{1}{2}x + 1.$$

PRACTISE NOW 2

SIMILAR QUESTIONS

Exercise 1A Questions 2(a)–(d), 3(a)–(d)

Write down the inequalities which define the unshaded region.



Worked Example 3

(Finding the Greatest and Least Values from Graphs of Linear Inequalities in Two Variables)

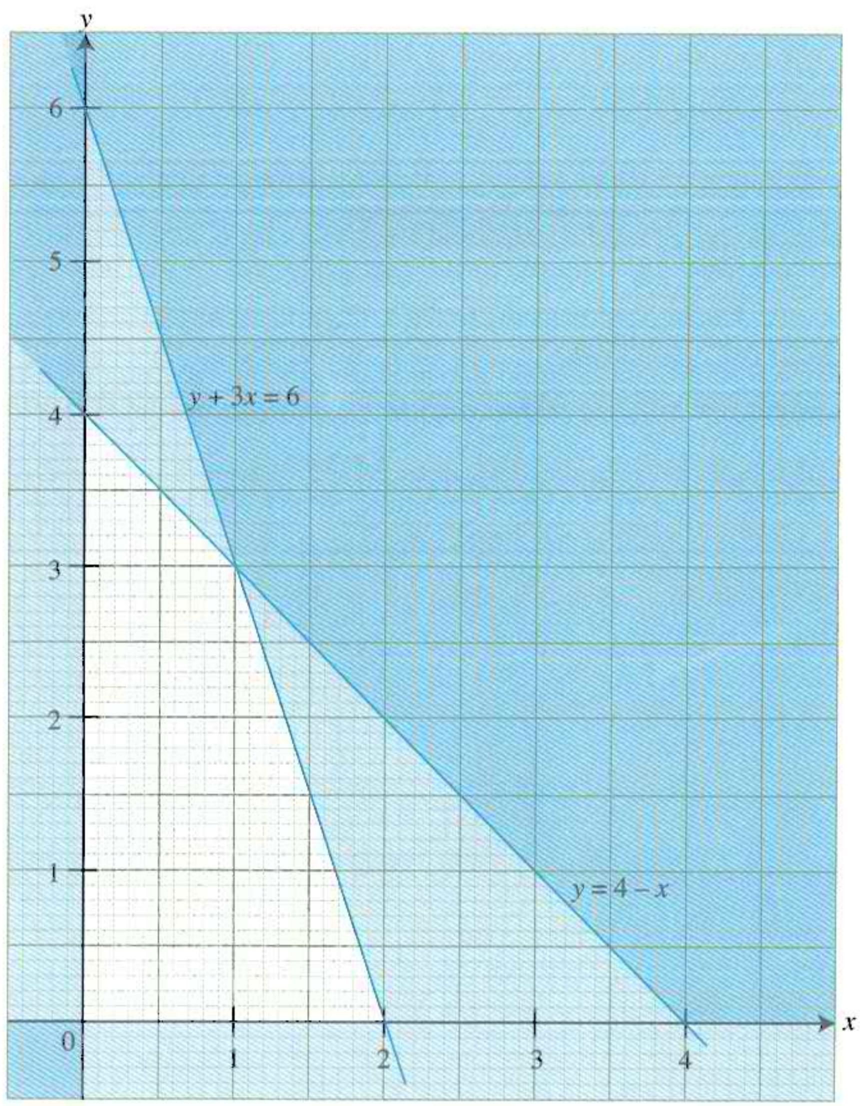
Show, unshaded, the regions satisfied by the following inequalities:

$$x \geqslant 0,\ y \geqslant 0,\ y \leqslant 4 - x,\ y + 3x \leqslant 6.$$

Hence, find the greatest and least values of $3x + 2y$ satisfying the region.

Solution:

Draw the lines $y = 4 - x$ and $y + 3x = 6$.
Shade the regions not required by the inequalities:
$x \geqslant 0$, $y \geqslant 0$, $y \leqslant 4 - x$ and $y + 3x \leqslant 6$
(i) Below the x-axis
(ii) Left of the y-axis
(iii) Above $y = 4 - x$
(iv) Above $y + 3x = 6$



Problem Solving Tip

The greatest and least values are obtained by substituting the coordinates of one of the vertices of the unshaded region on the graph.

$3x + 2y$ must be satisfied by the unshaded region.
If $x = 1$, $y = 3$, we obtain the greatest value of $3x + 2y$.
Greatest value of $3x + 2y = 3(1) + 2(3)$
$= 9$
If $x = 0$, $y = 0$, we obtain the least value of $3x + 2y$.
Least value of $3x + 2y = 3(0) + 2(0)$
$= 0$

PRACTISE NOW 3

SIMILAR QUESTIONS

Exercise 1A Questions 4–6

On a labelled diagram, leave unshaded the region defined by the following inequalities:

$$y \geqslant x,\ y \leqslant 2x \text{ and } 20 \leqslant x + y \leqslant 30.$$

Hence, find the greatest and least values of $x + 2y$ satisfying the region.

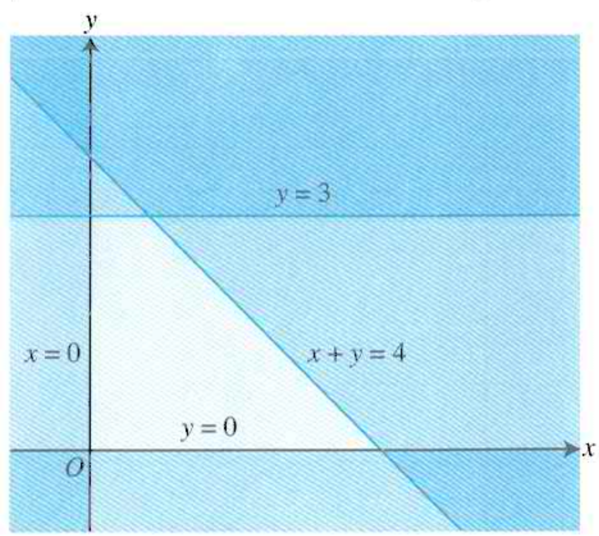
Exercise 1A

BASIC LEVEL

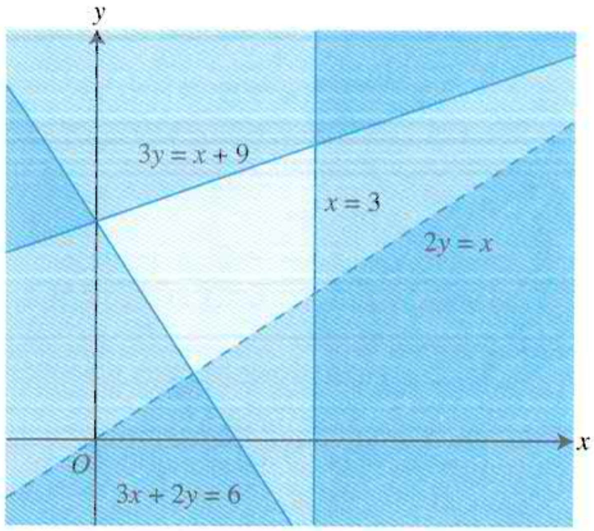
1. Show, unshaded, the regions satisfied by the following inequalities:
   (a) $x \geqslant 0$, $y \geqslant 2x$
   (b) $x > 2$, $y \geqslant x + 1$
   (c) $x + y < 4$, $y \geqslant x - 1$
   (d) $x > 0$, $y > 2$, $y \leqslant 6 - x$
   (e) $x > 0$, $2x + y \leqslant 10$, $y \geqslant 1$
   (f) $y < x + 3$, $x \leqslant 5$, $y \geqslant -1$

INTERMEDIATE LEVEL

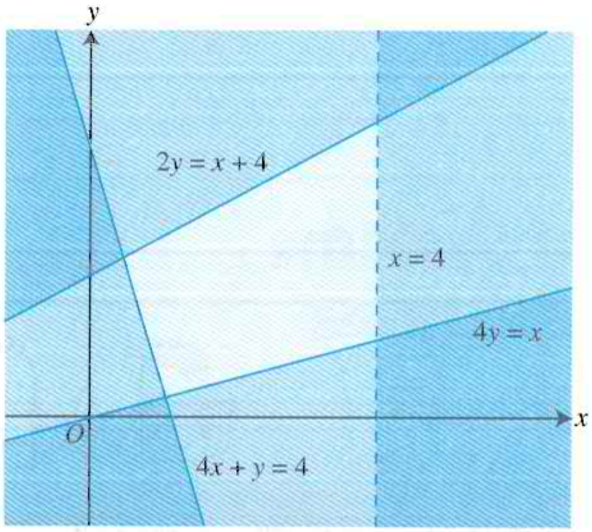
2. In each of the following cases, write down the inequalities which define the unshaded region.
   (a)



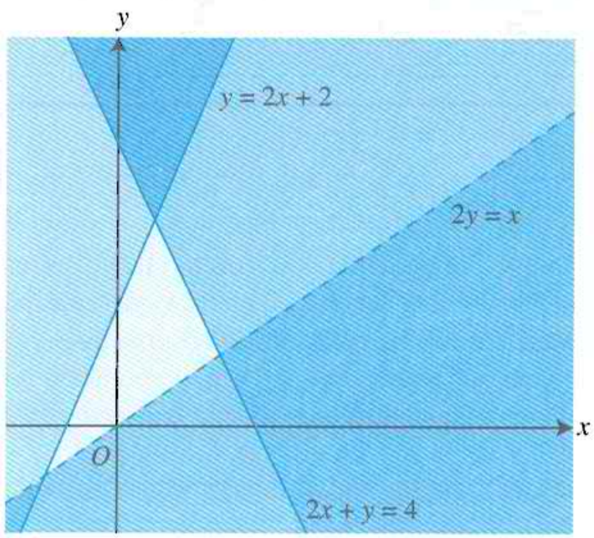
(b)



(c)



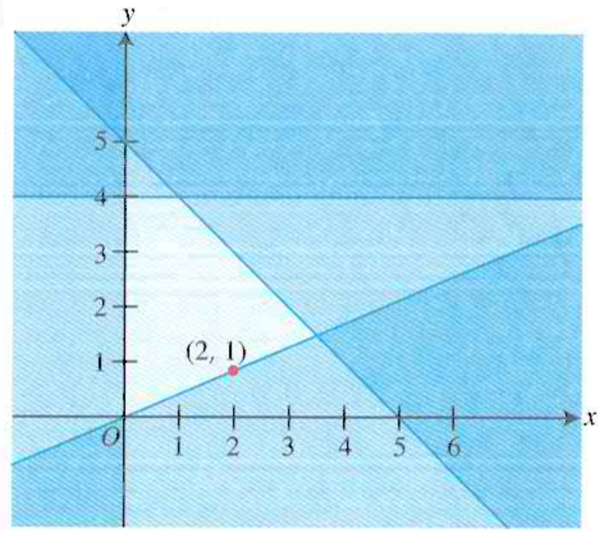
(d)



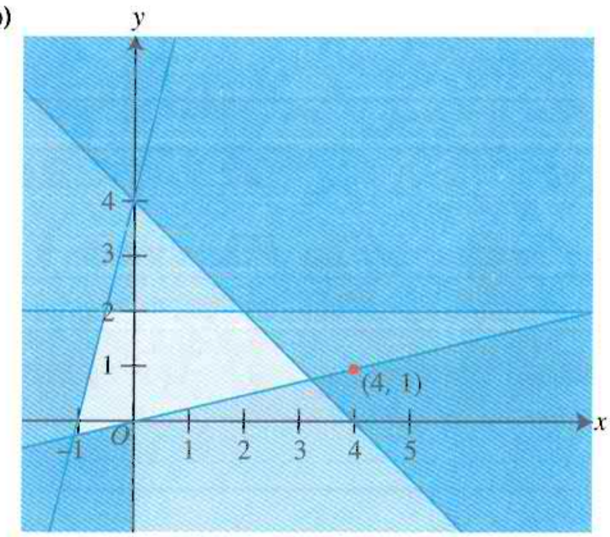
ADVANCED LEVEL

3. In each of the following cases, write down the inequalities which define the unshaded region.

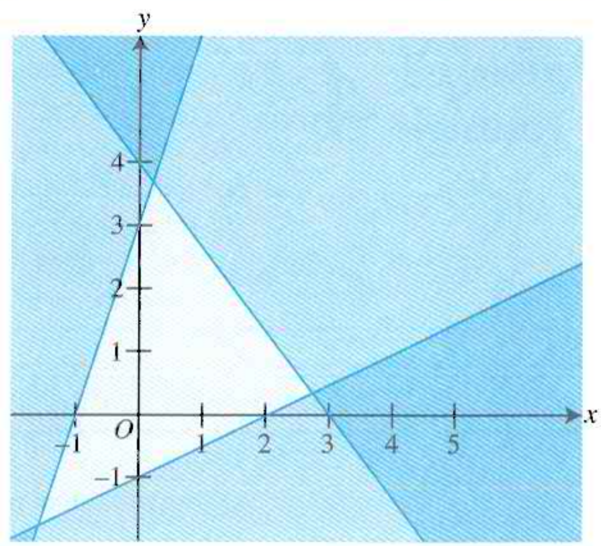
(a)



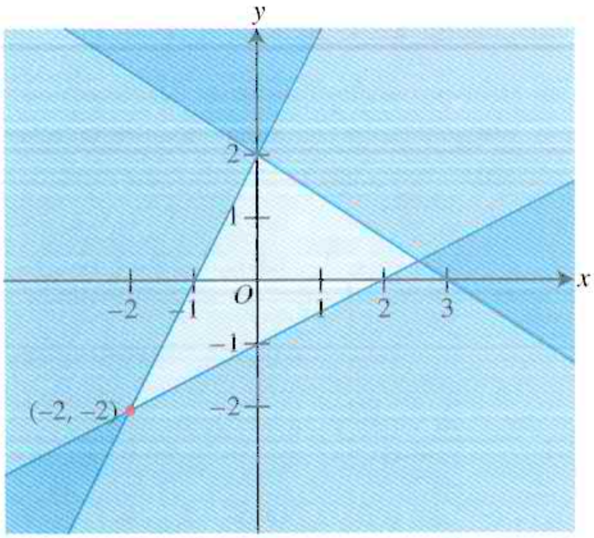
(b)



(c)



(d)



4. Leave unshaded the region defined by the following inequalities:

$$x \geqslant 0,\ y \geqslant 0,\ 3x + 4y \leqslant 24 \text{ and } 2x + y \leqslant 8.$$

Find the greatest value of $4x + y$ which satisfies the above inequalities.

5. On the same diagram, draw the graphs satisfying the following inequalities and outline the region representing them.

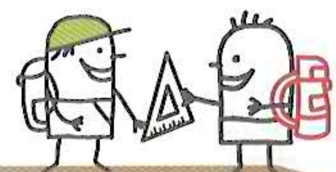
$$x \geqslant 2,\ y \geqslant 2,\ y \leqslant 5 \text{ and } x + y \leqslant 12.$$

Find the greatest and least values of P if $P = x + 3y$ passes through at least one point in the area representing the above region.

6. Find the greatest and least values of P, where $P = 3x + y$, subject to the conditions:

$$x \geqslant 1,\ y \geqslant 1,\ x + y \leqslant 6 \text{ and } 2x + y \leqslant 10.$$

1.2 Application of Systems of Linear Inequalities in Two Variables in Real-World Contexts



In this section, we will learn how to apply systems of linear inequalities in two variables to solve mathematical and real-life problems.

Worked Example 4

(Solving Real-life Problems involving Systems of Linear Inequalities in Two Variables)

A farmer plans to divide his land into not more than 36 plots to plant either a banana tree or coconut tree on each plot. He decides that he will plant at least 20 banana trees and that there will be at least twice as many banana trees as coconut trees.

(a) Taking x to represent the number of banana trees and y to represent the number of coconut trees, write down three inequalities, other than $x \geqslant 0$ and $y \geqslant 0$, which satisfy the above conditions.

(b) On a sheet of graph paper, show, unshaded, the region satisfied by the inequalities in **(a)**.

(c) A plot of land used to plant a banana tree has an area of 16 m^2 and a plot of land used to plant a coconut tree has an area of 4 m^2. Use your graph to estimate the maximum possible land area that the farmer has.

Solution:

(a) The three inequalities are $x + y \leqslant 36$, $x \geqslant 20$ and $x \geqslant 2y$.

(b) Draw the lines $x + y = 36$, $x = 20$ and $x = 2y$.

Shade the regions not required by the inequalities:

$y \leqslant 36 - x$, $x \geqslant 20$, $y \leqslant \frac{x}{2}$, $x \geqslant 0$ and $y \geqslant 0$

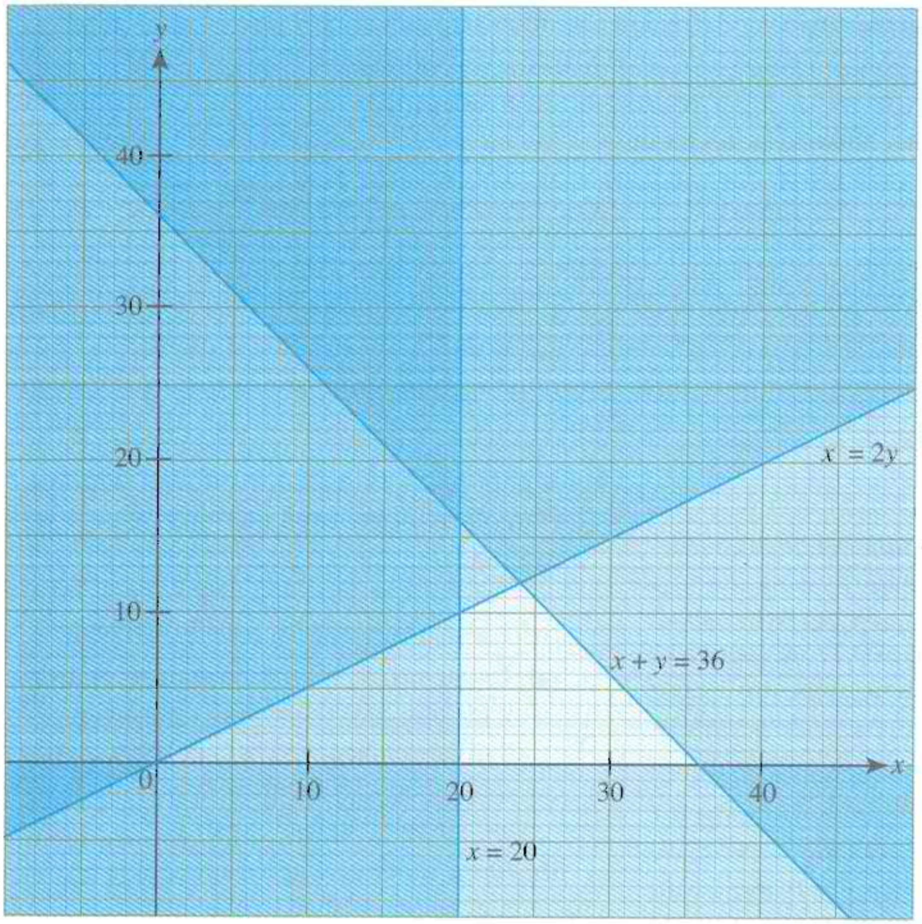
(i) Above $x + y = 36$

(ii) Left of $x = 20$

(iii) Above $x = 2y$

(iv) Below the x-axis

(v) Left of the y-axis



(c) Let the land area that the farmer has be A m^2.

A is given by $16x + 4y$ and must be satisfied by the unshaded region.

If the farmer plants 36 banana trees, i.e. $x = 36$ and $y = 0$, he has the maximum possible land area.

Maximum possible value of A

$= 16(36) + 4(0)$

$= 576$ m^2

ATTENTION

When shading the region not required by an inequality involving x and y, write the inequality such that y is on the LHS of the inequality.

Problem Solving Tip

The maximum possible land area is obtained by substituting the coordinates of one of the vertices of the unshaded region on the graph. Since the plot of land used to plant a banana tree has a greater area than that used to plant a coconut tree, the maximum possible land area occurs when as many banana trees as possible are planted. That is, $x = 36$ and $y = 0$.

PRACTISE NOW 4

SIMILAR QUESTIONS

Exercise 1B Questions 1–5

A shopkeeper stocks two brands of drinks called *Coola* and *Shiok*. He is ordering fresh supplies and finds that he has room for up to 1000 cans. He proposes to order at least twice as many cans of *Shiok* as of *Coola*. He wishes to have at least 100 cans of *Coola* and not more than 800 cans of *Shiok*.

(a) Taking x to represent the number of cans of *Coola* and y to represent the number of cans of *Shiok* that he orders, write down four inequalities involving x and/or y, other than $x \geqslant 0$ and $y \geqslant 0$, which satisfy the above conditions.

(b) The point (x, y) represents x cans of *Coola* and y cans of *Shiok* ordered. Using a scale of 1 cm to represent 100 cans on each axis, construct and indicate clearly, by shading the unwanted regions, the region in which (x, y) must lie.

(c) The profit made by selling a can of *Coola* is \$6 and that of a can of *Shiok* is \$5. Use your graph to estimate the number of cans of each brand that the shopkeeper should order to give the maximum profit.

Exercise 1B

BASIC LEVEL

1. Kate plans to divide her ribbon into not more than 30 pieces. Each piece of ribbon is to be used to wrap either a box of cookies or a packet of candies. She will wrap at least 8 boxes of cookies and there will be at least twice as many packets of candies as boxes of cookies.

   (a) Taking x to represent the number of boxes of cookies and y to represent the number of packets of candies that she wraps, write down three inequalities, other than $x \geqslant 0$ and $y \geqslant 0$, which satisfy the above conditions.

   (b) On a sheet of graph paper, show, unshaded, the region satisfied by the inequalities in **(a)**.

   (c) A piece of ribbon used to wrap a box of cookies is 30 cm long and a piece of ribbon used to wrap a packet of candies is 15 cm long. Use your graph to estimate the maximum possible length of ribbon that Kate has.

2. A chef plans to divide his dough into not more than 40 portions. Each portion of dough is to be used to make either a pizza or bread. He will make at least a dozen pizzas and there will be at least twice as many pizzas as loaves of bread.

   (a) Taking x to represent the number of pizzas and y to represent the number of loaves of bread that he makes, write down three inequalities, other than $x \geqslant 0$ and $y \geqslant 0$, which satisfy the above conditions.

   (b) On a sheet of graph paper, show, unshaded, the region satisfied by the inequalities in **(a)**.

   (c) The dough used to make a pizza weighs 6 g and the dough used to make a loaf of bread weighs 8 g. Use your graph to estimate the maximum possible weight of dough that the chef has.

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INTERMEDIATE LEVEL

3. A supermarket manager stocks two brands of detergent called *Power Clean* and *Disappear*. His stock is running low and finds that he has room for up to 200 bottles. He proposes to order at least twice as many bottles of *Power Clean* as of *Disappear*. He wishes to have at least 50 bottles of *Disappear* and not more than 140 bottles of *Power Clean*.

   (a) Taking x to represent the number of bottles of *Power Clean* and y to represent the number of bottles of *Disappear* that he orders, write down four inequalities involving x and/or y, other than $x \geqslant 0$ and $y \geqslant 0$, which satisfy the above conditions.

   (b) The point (x, y) represents x bottles of *Power Clean* and y bottles of *Disappear* ordered. Using a scale of 2 cm to represent 50 bottles on each axis, construct and indicate clearly, by shading the unwanted regions, the region in which (x, y) must lie.

   (c) The profit of a bottle of *Power Clean* is \$10 and the profit of a bottle of *Disappear* is \$8. Use your graph to estimate the number of bottles of each brand that the supermarket manager should order to give the maximum profit.

4. Two types of ship, *Gigantic* and *Jumbo*, are available to move 300 men and 20 000 kg of equipment. Each *Gigantic* ship can carry 40 men and 3000 kg of equipment. Each *Jumbo* ship can carry 50 men and 2000 kg of equipment.

   (a) If x *Gigantic* ships and y *Jumbo* ships are used, write down the inequalities, other than $x \geqslant 0$ and $y \geqslant 0$, which x and y must satisfy.

   (b) The point (x, y) represents the number of *Gigantic* ships, x, and the number of *Jumbo* ships, y. Using a suitable scale on each axis, construct and indicate clearly, by shading the unwanted regions, the region in which (x, y) must lie.

   (c) Use your graph to estimate the least number of ships that can move 300 men and 20 000 kg of equipment.

ADVANCED LEVEL

5. A dealer produces two blends of tea, *Fragrant* and *Instant*, by mixing two varieties of tea leaves, A and B.

   In *Fragrant* blend, $\frac{\text{weight of } A}{\text{weight of } B} = \frac{4}{1}$ and in *Instant* blend, $\frac{\text{weight of } A}{\text{weight of } B} = \frac{2}{3}$.

   Given that he produces x kilograms of *Fragrant* and y kilograms of *Instant*, copy and complete the following table.

	Total Weight (kg)	Weight of A (kg)	Weight of B (kg)
Fragrant	x	$\frac{4x}{5}$	
Instant	y		

   (a) The dealer has at most 3200 kg of variety A and at least 3000 kg of variety B. Write down two inequalities involving x and y which satisfy these conditions and show that they simplify to $2x + y \leqslant 8000$ and $x + 3y \leqslant 15\ 000$.

   (b) He wishes to produce less *Fragrant* than *Instant* and has sufficient containers for only 2300 kg of *Fragrant* and 5000 kg of *Instant*. Write down three inequalities which satisfy these conditions.

   (c) The point (x, y) represents x kilograms of *Fragrant* and y kilograms of *Instant*. Using a scale of 2 cm to represent 1000 kg on each axis, construct and indicate clearly, by shading the unwanted regions, the region in which (x, y) must lie.

   (d) The dealer makes the same profit per kilogram on *Fragrant* as on *Instant*. Use your graph to estimate the weight of each blend that he should produce to maximise the profit.

Summary

1. A linear inequality in two variables can be illustrated on a graph by drawing the linear equation in two variables and shading the unwanted region.

2. The region *below* $y = mx + c$ represents the inequality $y < mx + c$, and the region *above* $y = mx + c$ represents the inequality $y > mx + c$.

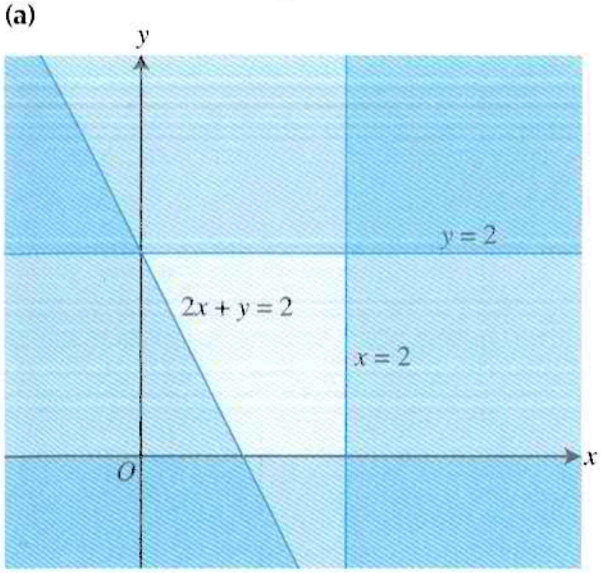
3. A system of linear inequalities in two variables can be solved by the graphical method. The solutions to the system of linear inequalities are represented by the unshaded region.

Review Exercise 1

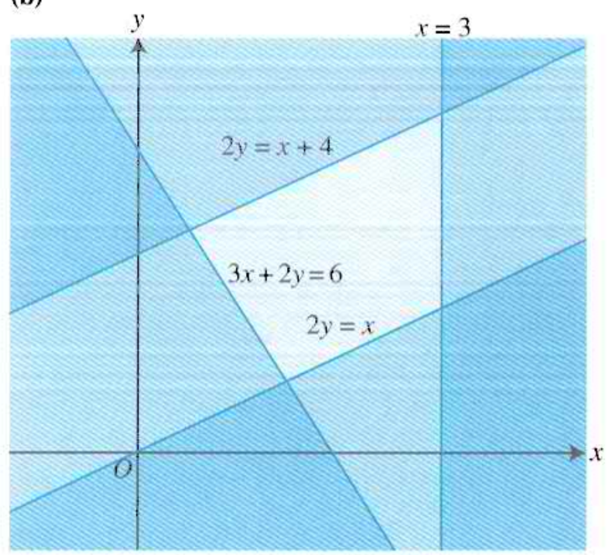
1. Show, unshaded, the regions satisfied by the following inequalities:
   (a) $y \geqslant 0,\ y \leqslant 2x$
   (b) $x > -1,\ y \geqslant 2x + 1$
   (c) $x + 2y < 2,\ y \geqslant x + 1$
   (d) $x > 1,\ y \leqslant 2,\ y \leqslant x - 6$
   (e) $x > 0,\ 2x + 2y \leqslant 9,\ y \geqslant 2$
   (f) $y < x + 1,\ y - 2x + 3 \geqslant 0,\ y \geqslant 1$

2. In each of the following cases, write down the inequalities which define the unshaded region.

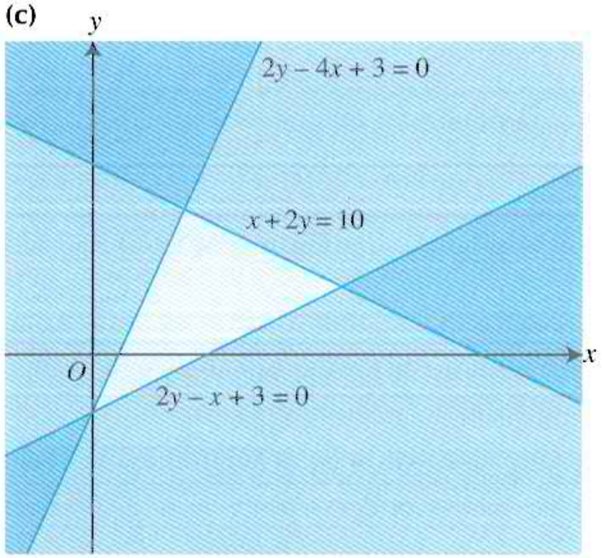
(a)



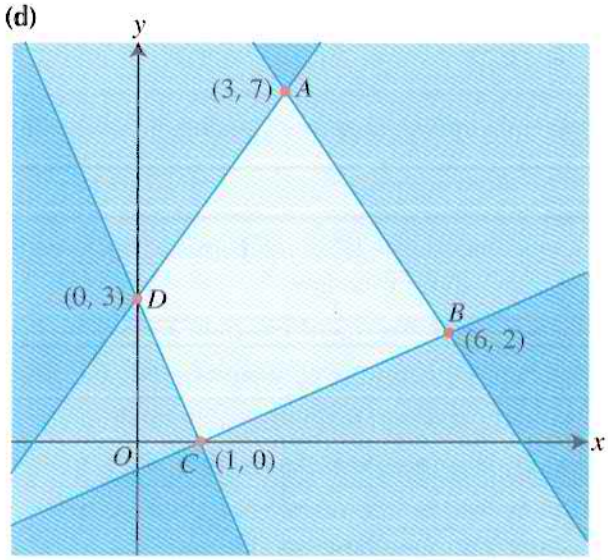
(b)



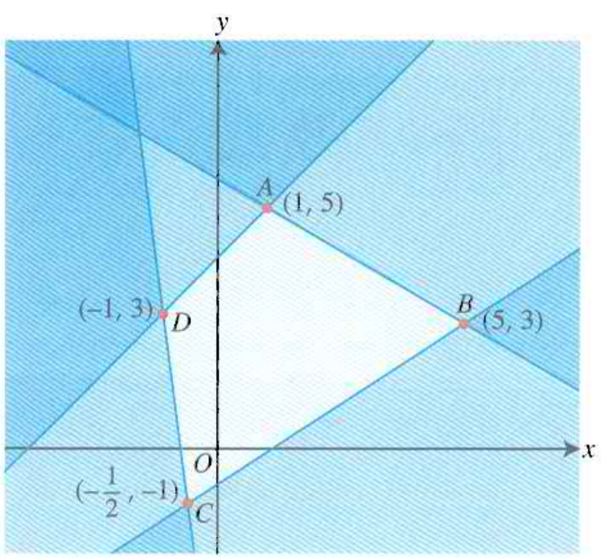
(c)



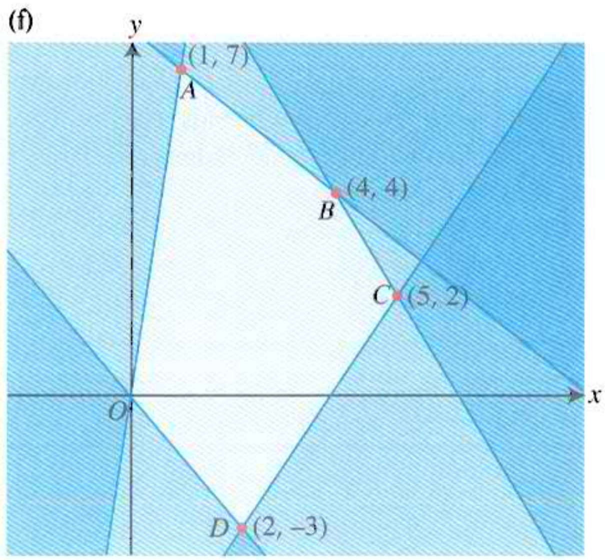
(d)



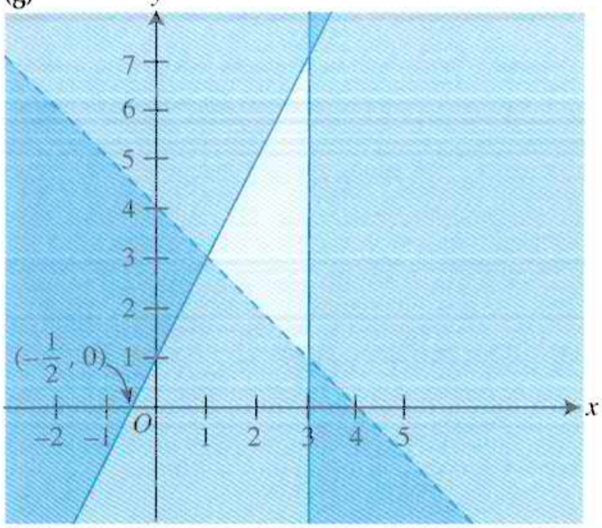
(e)



(f)



(g)



3. Leave unshaded the region defined by the following inequalities:

$$x \geqslant 0,\ y \geqslant 0,\ x \geqslant 3y \text{ and } y \geqslant 4x - 11.$$

Find the greatest value of $6y - x$ which satisfies the above inequalities.

4. Find the maximum value of $x - y$ subject to the following inequalities:

$$x + 4y \leqslant 12,\ 2y \leqslant 3x + 6,\ y \geqslant x - 2 \text{ and } 3x - y \leqslant 10.$$

5. Nora, who intended to keep chickens and ducks on her farm, asked each of her four friends how many chickens and/or ducks she should keep.

(i) Daniel suggested that she should keep more than 10 ducks.

(ii) Michael suggested that the number of chickens should be at least 20 but not more than 50.

(iii) Amirah suggested that the total number of chickens and ducks should be less than 70.

(iv) Shirley suggested that the number of chickens should be greater than or equal to the number of ducks.

(a) Taking x to be the number of chickens and y to be the number of ducks, write down the inequalities which represent these conditions.

(b) The point (x, y) represents x chickens and y ducks that Nora kept. Using a scale of 2 cm to represent 20 chickens on the x-axis and a scale of 2 cm to represent 20 ducks on the y-axis, and indicate clearly by shading the unwanted regions, the region in which (x, y) must lie.

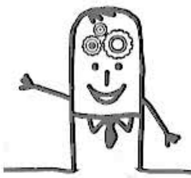
(c) Assume Nora took all her friends' suggestions. When she sold the animals, she made a profit of \$6 on each chicken and \$12 on each duck. Find the minimum number of ducks she kept on her farm to ensure a profit of at least \$480.

6. Brand A of potato chips contains 240 calories per kilogram and 200 units of vitamins per kilogram. Brand B of potato chips contains 160 calories per kilogram and 80 units of vitamins per kilogram. It is desired to have at least 10 kg mixture of brands A and B that contains not more than 2400 calories and at least 1600 units of vitamins.

   (a) If x kilograms of Brand A and y kilograms of Brand B are mixed, write down the inequalities, other than $x \geqslant 0$ and $y \geqslant 0$, which x and y must satisfy.

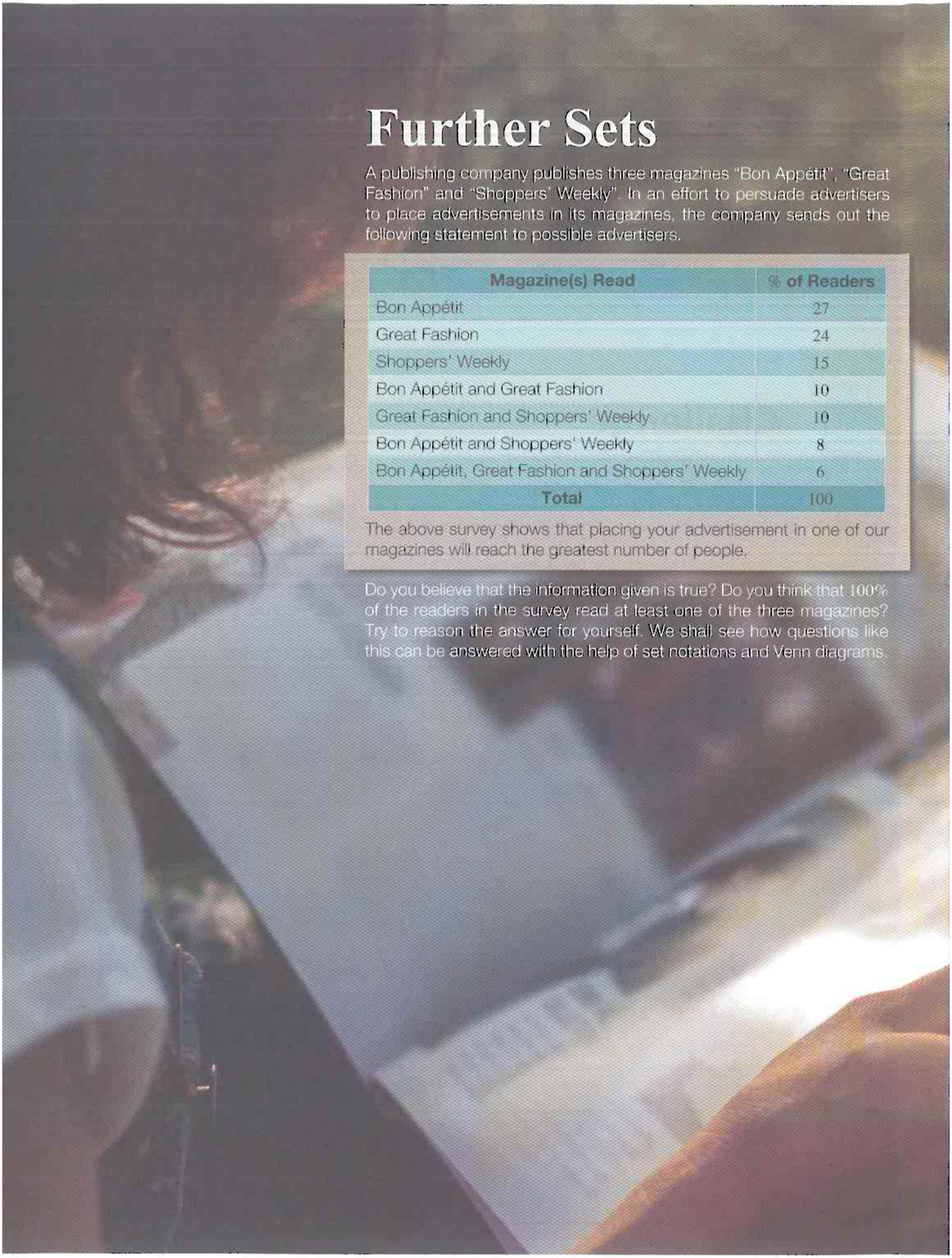
   (b) The point (x, y) represents the weight of Brand A, x kilograms, and the weight of Brand B, y kilograms. Using a suitable scale on each axis, construct and indicate clearly, by shading the unwanted regions, the region in which (x, y) must lie.

   (c) Use your graph to estimate the maximum weight of the mixture of potato chips that contains the desired amount of calories and vitamins.



Challenge Yourself

A banker has \$1 000 000 to invest in three different funds. The government bond fund has a 5% return, the local bank's fund has a 7% return, and a high-risk account has an expected 10% return. To minimise risk, the banker decides not to invest more than \$100 000 in the high-risk account. For regulation reasons, he needs to invest at least three times as much in the government bond fund as in the bank's fund. How should the money be invested to maximise the expected returns?



Further Sets

A publishing company publishes three magazines "Bon Appétit", "Great Fashion" and "Shoppers' Weekly". In an effort to persuade advertisers to place advertisements in its magazines, the company sends out the following statement to possible advertisers.

Magazine(s) Read	% of Readers
Bon Appétit	27
Great Fashion	24
Shoppers' Weekly	15
Bon Appétit and Great Fashion	10
Great Fashion and Shoppers' Weekly	10
Bon Appétit and Shoppers' Weekly	8
Bon Appétit, Great Fashion and Shoppers' Weekly	6
Total	100

The above survey shows that placing your advertisement in one of our magazines will reach the greatest number of people.

Do you believe that the information given is true? Do you think that 100% of the readers in the survey read at least one of the three magazines? Try to reason the answer for yourself. We shall see how questions like this can be answered with the help of set notations and Venn diagrams.

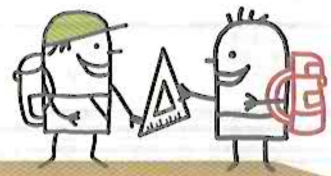
Chapter Two

LEARNING OBJECTIVES

At the end of this chapter, you should be able to:

- solve problems on classification and cataloguing,
- express problems in set notations and draw Venn diagrams to obtain solutions.

2.1 Applications of Venn Diagrams in Problem Sums



In Book 2, we have learnt how to solve problems involving set notations and Venn diagrams. In addition to the problems that we have covered in Book 2, Venn diagrams may also be used to solve problems on classification and cataloguing. In this section, we will take a look at some of these problems.

Scenarios involving Two Sets

Consider the following scenario.

In a class of 40 pupils, 27 play basketball, 25 play tennis while 17 play both.

How can we draw a Venn diagram to represent the data above?

Step 1: Let B represent the set of pupils who play basketball and T represent those playing tennis.

Step 2: The 17 pupils who play both games are represented in the region $B \cap T$.

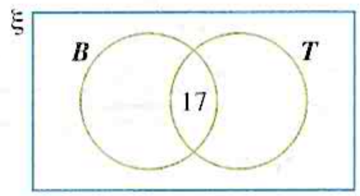


Fig. 2.1

Step 3: Since there is a total of 27 pupils in set B, the remaining $(27 - 17) = 10$ pupils are represented by the rest of the circle B. Similarly, the remaining $(25 - 17) = 8$ pupils who play tennis only are represented by the remaining part of T.

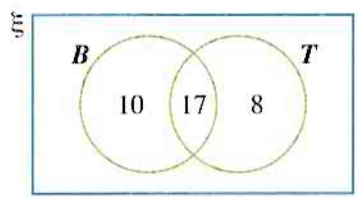


Fig. 2.2

Step 4: Now, $n(\xi) = 40$ but $n(B \cup T) = 10 + 17 + 8 = 35$.
Thus, we draw the following Venn diagram.

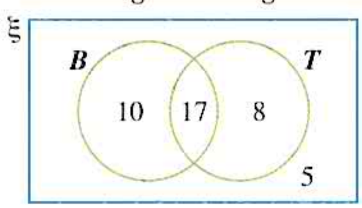


Fig. 2.3

From Fig. 2.3, we can conclude that there are 5 pupils who play neither basketball nor tennis, 10 who play basketball only, 8 who play tennis only and 17 who play both basketball and tennis.

George Boole (1815 – 1864) introduced a symbolic approach to the study of logic. This allowed him to clarify difficult logical problems in symbolic forms based on sets. The algebra of sets having union and intersection as its basic operations is known as 'Boolean Algebra'. Today, Boolean Algebra is used widely as a tool to aid sound reasoning.

Worked Example 1

(Problem involving Intersection of Two Sets)

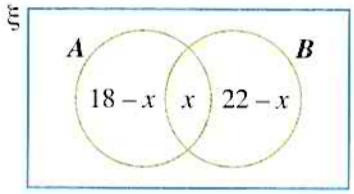
In a class of 30 pupils, 18 like folk music and 22 like classical music. If all the pupils like at least one of the two types of music, find the number of pupils who like both types.

Solution:

Let A = {pupils who like folk music} and B = {pupils who like classical music}.

Let x be the number of pupils who like both types of music.

Since there are 18 in A altogether, there must be $(18 - x)$ who are only in A.
Similarly, there must be $(22 - x)$ who are only in B.



Since all 30 pupils like at least one of the two types of music, $A \cup B = \xi$.

$$(18 - x) + x + (22 - x) = 30$$
$$x = 10$$

$\therefore$ There are 10 pupils who like both types of music.

PRACTISE NOW 1

SIMILAR QUESTIONS

Exercise 2A Questions 1–2, 6–7

50 people were asked to participate in a food survey. 24 responded that they enjoy Italian food and 33 responded that they enjoy Chinese food. If all the respondents enjoy at least one of the two types of cuisine, find the number of people who enjoy both types.

Scenarios involving Three Sets

Consider the following scenario.

In a certain school, the pupils have to study Mathematics, Science or Geography. Each pupil must study at least one of the three subjects. Among a group of 40 pupils, 20 study Mathematics, 22 Science and 28 Geography; 12 study Mathematics and Science, 14 study Science and Geography and 15 study Mathematics and Geography.

How can we draw a Venn diagram to represent the data above?

Step 1: Let M = {pupils who study Mathematics}, S = {pupils who study Science} and G = {pupils who study Geography}.

Since every pupil must study at least one of the three subjects,

$$n(M \cup S \cup G) = n(\xi)$$
$$= 40$$

Step 2: Let the number of pupils who study all three subjects be x. Hence, the central region of the Venn diagram is marked x.

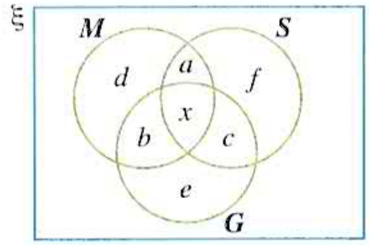


Fig. 2.4

Step 3: Next we shall find the different values of a, b, c, d, e and f to complete the Venn diagram. Since 12 pupils study Mathematics and Science, there must be $(12 - x)$ pupils who study only these two subjects.

$\therefore a = 12 - x$

By the same argument,

$b = 15 - x$ and $c = 14 - x$.

Altogether, 20 pupils study Mathematics.

$$\begin{aligned} \therefore d &= 20 - x - a - b \\ &= 20 - x - (12 - x) - (15 - x) \\ &= x - 7 \end{aligned}$$

By the same argument,

$$\begin{aligned} f &= 22 - x - (12 - x) - (14 - x) \\ &= x - 4 \end{aligned}$$

$$\begin{aligned} e &= 28 - x - (15 - x) - (14 - x) \\ &= x - 1 \end{aligned}$$

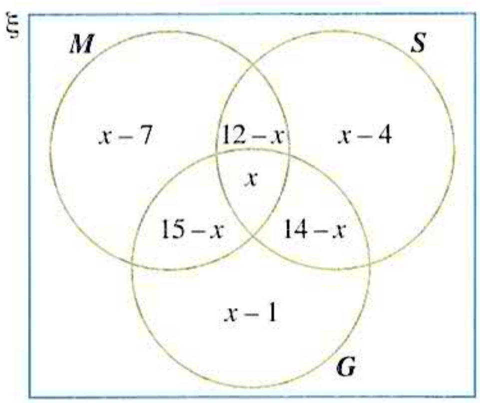


Fig. 2.5

Step 4: $n(M \cup S \cup G) = d + b + a + x + f + c + e$

$$= 40$$

$$(x - 7) + (15 - x) + (12 - x) + x + (x - 4) + (14 - x) + (x - 1) = 40$$

$$29 + x = 40$$

$$x = 11$$

Problem Solving Tip

Alternatively,

$$n(M) + (x - 4) + (14 - x) + (x - 1) = 40$$

$$20 + x + 9 = 40$$

$$x = 11$$

Substitute $x = 11$ into the Venn diagram in **Step 2** and we get the following diagram.

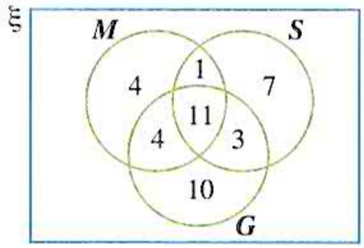


Fig. 2.6

From Fig. 2.6, we summarise the results as follows:

Subject(s)	Number of pupils
Mathematics only	4
Geography only	10
Science only	7
Mathematics and Geography only	4
Mathematics and Science only	1
Science and Geography only	3
Mathematics, Geography and Science	11

Worked Example 2

(Problem involving Intersection of Three Sets)

In a group of girls, 20 play volleyball, 21 play badminton and 18 play table tennis; 7 play volleyball only, 9 play badminton only; 6 play volleyball and badminton only and 2 play badminton and table tennis only.

(i) How many play all three games?
(ii) How many play volleyball and table tennis only?
(iii) How many play table tennis only?
(iv) How many girls are there altogether?

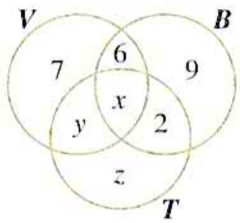
Solution:

Let V = {volleyball players},
B = {badminton players},
T = {table tennis players}.

$\therefore$ $n(V) = 20$, $n(B) = 21$, $n(T) = 18$

Let x represent the girls who play all three games, y to represent the girls who play volleyball and table tennis and z to represent the girls who play table tennis only.

With this information, we draw the following Venn diagram.



(i) $\quad \text{n}(B) = 21$

$x + 6 + 2 + 9 = 21$

$x = 4$

$\therefore$ 4 girls play all three games.

(ii) $\quad \text{n}(V) = 20$

$y + 7 + 6 + 4 = 20$

$y = 3$

$\therefore$ 3 girls play volleyball and table tennis only.

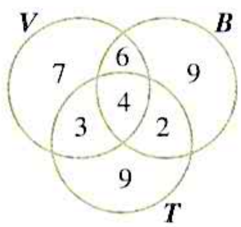
(iii) $\quad \text{n}(T) = 18$

$z + 2 + 4 + 3 = 18$

$z = 9$

$\therefore$ 9 girls play table tennis only.

(iv)



$$\begin{aligned}\text{n}(V \cup B \cup T) &= 7 + 6 + 4 + 3 + 9 + 2 + 9 \\ &= 40\end{aligned}$$

$\therefore$ There are 40 girls altogether.

PRACTICE NOW 2

SIMILAR QUESTIONS

Exercise 2A Questions 8–11, 20–21

In a class of students, 20 are members of the dance group, 26 are members of the choir and 15 are members of the band; 8 are members of the dance group only, 13 are members of the choir only; 6 are members of the dance group and the choir only and 5 are members of the choir and the band only.

(i) How many students are involved in all three activities?

(ii) How many students are members of the dance group and the band only?

(iii) How many students are members of the band only?

(iv) How many students are there altogether?

Further Applications of Sets

Worked Example 3

(Sets and Linear Inequalities)

It is given that $\xi = \{x : x \text{ is a real number}, -15 \leqslant x \leqslant 15\}$, $A = \{x : -15 \leqslant x \leqslant 15\}$, $B = \{x : 5 \leqslant x \leqslant 15\}$, $C = \{x : -10 < x \leqslant 10\}$ and $D = \{x : -15 \leqslant x \leqslant 8\}$.

Write expressions for the following, illustrating each solution on a number line.

(i) A' **(ii)** $B \cap C$

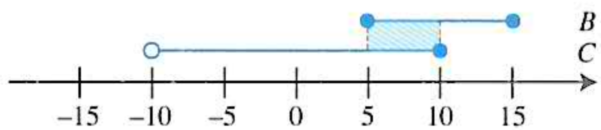
(iii) $C \cap D$ **(iv)** $B \cup D$

(v) $B \cap D'$

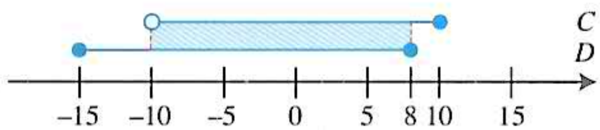
Solution:

(i) $A' = \varnothing$

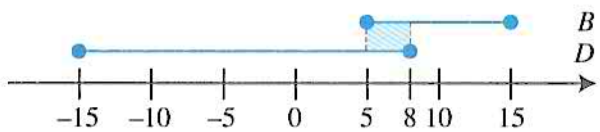
(ii) $B \cap C = \{x : 5 \leqslant x \leqslant 10\}$



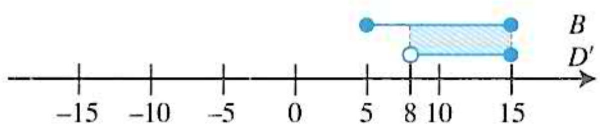
(iii) $C \cap D = \{x : -10 < x \leqslant 8\}$



(iv) $B \cup D = \{x : -15 \leqslant x \leqslant 15\}$
$= \xi$



(v) $B \cap D' = \{x : 8 < x \leqslant 15\}$



PRACTISE NOW 3

SIMILAR QUESTIONS

Exercise 2A Question 12

It is given that $\xi = \{x : -20 \leqslant x \leqslant 20\}$, $A = \{x : -20 < x < 20\}$, $B = \{x : -15 < x \leqslant 8\}$ and $C = \{x : 8 \leqslant x \leqslant 12\}$.

Write down expressions for the following, illustrating each solution on a number line.

(i) A' **(ii)** $B \cap C$ **(iii)** $B \cup C$ **(iv)** $B \cap A'$

Worked Example 4

(Sets and Coordinate Geometry)

$A = \{(x, y) : (x, y)$ lies on the line $y = 2x + 3\}$ and $B = \{(x, y) : (x, y)$ lies on the line $y = kx + h\}$. If $n(A \cap B) = 0$, state the value of k and give a possible value for h.

Solution:

A is the set of points on the straight line $y = 2x + 3$ with gradient 2 and y-intercept 3.
B is the set of points on the line $y = kx + h$.
Since $n(A \cap B) = 0$, the two lines do not intersect.
Therefore, they must be parallel, i.e. $k = 2$.
h can take any value except 3, one possible value of h is 5.

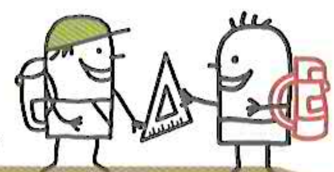
PRACTISE NOW 4

SIMILAR QUESTIONS

Exercise 2A Question 13

$A = \{(x, y) : (x, y)$ lies on the line $y = 5x - 2\}$ and $B = \{(x, y) : (x, y)$ lies on the line $y = ax + b\}$. If $n(A \cap B) = 0$, state the value of a and give a possible value for b.

2.2 Formulas in Set Theory



To find the number of elements in the set $A \cup B$, where A and B can be any two sets, we have to consider two cases.

Case 1: A and B are disjoint.

In this case $A \cap B = \varnothing$, i.e. $n(A \cap B) = 0$.

Thus, $\mathbf{n(A \cup B) = n(A) + n(B)}$

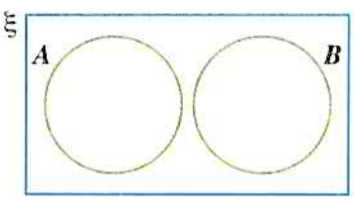


Fig. 2.7

Just For Fun

Find out how many times each of the following digits, 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9, appear in the set of integers from 1 to 100, inclusive.

Case 2: A and B intersect.

In this case $A \cap B \neq \varnothing$.

$n(A) + n(B)$ counts the common elements in both sets, i.e. $n(A \cap B)$, twice.

Thus, $\mathbf{n(A \cup B) = n(A) + n(B) - n(A \cap B)}$

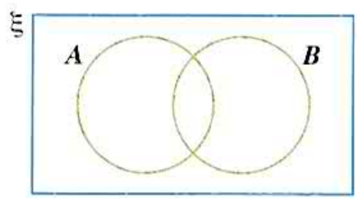


Fig. 2.8

ATTENTION

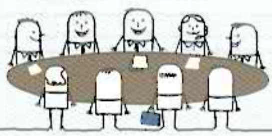
We can always change the subject of the formula to get the formula for intersection, i.e.
$\mathbf{n(A \cap B) = n(A) + n(B) - n(A \cup B)}$

We can extend the formula learnt in the two cases to data requiring classification into three or more disjoint sets but it is often easier to rely on the use of Venn diagrams to solve such problems.

In the case where the data can be classified into three intersecting sets, the overlaps need to be taken into consideration, as illustrated in Case 2, to prevent the same problem of repeated counting.

Thus,

$$\mathrm{n}(A \cup B \cup C) = \mathrm{n}(A) + \mathrm{n}(B) + \mathrm{n}(C) - \mathrm{n}(A \cap B) - \mathrm{n}(B \cap C) - \mathrm{n}(A \cap C) + \mathrm{n}(A \cap B \cap C)$$



Class Discussion

Work in pairs to verify the formula for the union of three sets.

PRACTISE NOW

If $\mathrm{n}(A) = 5$, $\mathrm{n}(B) = 10$ and $\mathrm{n}(A \cup B) = 12$, state the value of $\mathrm{n}(A \cap B)$.

SIMILAR QUESTIONS

Exercise 2A Questions 3–4, 14–17

Worked Example 5

(Application of Formula to find Extreme Values)

Given that $\mathrm{n}(A) = 15$ and $\mathrm{n}(B) = 10$, find the

(i) greatest value of $\mathrm{n}(A \cup B)$,

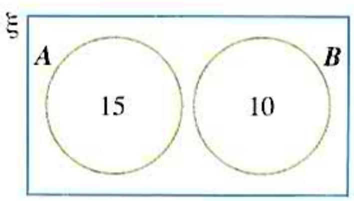
(ii) least value of $\mathrm{n}(A \cup B)$,

illustrating the cases with Venn diagrams.

Solution:

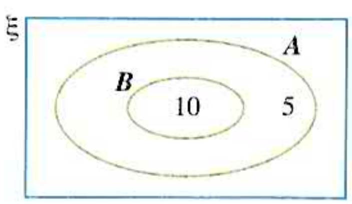
(i) $\mathrm{n}(A \cup B)$ will have the greatest value when $A \cap B = \varnothing$.

$$\begin{aligned} \mathrm{n}(A \cup B) &= \mathrm{n}(A) + \mathrm{n}(B) \\ &= 15 + 10 \\ &= 25 \end{aligned}$$



(ii) $\mathrm{n}(A \cup B)$ will have the least value when $B \subseteq A$.

$$\begin{aligned} \mathrm{n}(A \cup B) &= \mathrm{n}(A) \\ &= 15 \end{aligned}$$



PRACTISE NOW 5

SIMILAR QUESTIONS

Exercise 2A Questions 5, 18–19, 22

Given that $n(A) = 23$ and $n(B) = 9$, find the
(i) greatest value of $n(A \cup B)$,
(ii) least value of $n(A \cup B)$,
illustrating the cases with Venn diagrams.

Thinking Time

How can we find the greatest and least values of $n(A \cap B)$ in Worked Example 5?

From Practise Now 5 and Thinking Time, we observe that, given two sets A and B, we can obtain

- the greatest intersection when union is the smallest, i.e. when one set is a proper subset of another.
- the smallest intersection when union is the greatest, i.e. when two sets are disjoint.

Exercise 2A

BASIC LEVEL

1. In a group of 120 pupils, 80 can play the guitar only and 25 can play the piano only. Find the number of pupils who can play both instruments if there are 3 pupils who can play neither.

2. In a group of 40 community leaders, 35 can speak English and 12 can speak Mandarin. How many can speak both English and Mandarin if each of them can speak at least one of the two languages?

3. If $n(A \cup B) = 44$, $n(A \cap B) = 5$ and $n(A) = 20$, find the value of $n(B)$ with the help of a Venn diagram.

4. If $n(A) = 10$, $n(B) = 6$ and $n(A \cap B) = 3$, state the value of $n(A \cup B)$.

5. If $n(A) = 24$, $n(B) = 17$, $n(\xi) = 40$, find the greatest and the least values of the following.
(i) $n(A \cap B)$ **(ii)** $n(A \cup B)$

INTERMEDIATE LEVEL

6. All the 35 families in an estate possess either a car or a motorcycle, or both. 18 have cars and 27 have motorcycles.
(a) Draw a Venn diagram to represent the data.
(b) From the Venn diagram, find the number of families with
(i) both cars and motorcycles,
(ii) cars but not motorcycles.

7. In a school, 60% of the pupils have savings accounts with POSBank and 72% of them have savings accounts with commercial banks. Find the percentage of pupils who have savings accounts in both POSBank and commercial banks if all the pupils have at least one savings account.

8. In a group of 160 pupils, 46 pupils failed Mathematics, 52 pupils failed History and 50 pupils failed Geography; 31 pupils failed Mathematics and History, 33 pupils failed History and Geography, 36 failed Mathematics and Geography and 24 failed all three subjects. Draw a Venn diagram to illustrate this information and find the number of pupils who failed at least one subject.

9. 68 elderly men failed a medical test because of defects in at least one of these organs: the heart, lungs and kidneys. 30 heart disease, 30 lung disease and 33 kidney disease. 7 of them had both lung and heart diseases, 10 had lung and kidney diseases while 11 had kidney and heart diseases. Draw a Venn diagram to illustrate this information. Find the number of elderly men
   (i) who suffered from all the three diseases,
   (ii) had only lung disease.

10. A number of people were asked whether they liked drinks of orange, lemon or grape flavour. The replies showed:

85 liked orange	45 liked orange and lemon
65 liked grape	40 liked lemon and grape
90 liked lemon	15 liked lemon, orange and grape
30 liked orange and grape	25 liked none of the three

Find
   (i) the total number of people interviewed,
   (ii) the number who liked orange alone,
   (iii) the number who liked lemon alone,
   (iv) the number who liked grape alone.

11. 80 boys took the GCE examination in the three subjects: Geography, English and Mathematics, and none of them failed all three subjects. It was noted that 8 passed English only and 10 passed Mathematics only; 7 passed Mathematics and Geography but not English, 40 passed English and Mathematics and 21 passed English and Geography. Altogether, 54 passed English.
   (a) Draw a Venn diagram to represent this information.
   (b) From the Venn diagram, find the number of pupils who
      (i) passed Geography only,
      (ii) passed all three subjects.

12. It is given that $\xi = \{x : -20 \leqslant x \leqslant 25\}$, $A = \{x : -20 < x < 25\}$, $B = \{x : -5 < x \leqslant 7\}$ and $C = \{x : 7 \leqslant x < 15\}$.
Write down expressions for the following, illustrating each solution on a number line.
   (i) A'
   (ii) $B \cap C$
   (iii) $B \cup C$
   (iv) $C \cap A'$

13. $A = \{(x, y) : (x, y)$ lies on the line $y = \frac{2}{5}x - 7\}$ and $B = \{(x, y) : (x, y)$ lies on the line $y = ax - b\}$. If $n(A \cap B) = 0$, state the value of a and give a possible value for b.

14. If A and B are disjoint sets such that $n(A) = 84$, $n(B) = 16$, find
   (i) $n(A \cap B)$,
   (ii) $n(A \cup B)$.

15. If $A \subseteq B$, $n(A) = 42$, $n(B) = 75$, find
   (i) $n(A \cap B)$,
   (ii) $n(A \cup B)$.

16. If $n(A) = p$, $n(B) = q$ and $n(A \cup B) = r$, find $n(A \cap B)$ in terms of p, q and r.

17. Given that $n(A \cap B) = 5$, $n(A \cap B') = 28$ and $n(A' \cap B) = 23$, find
   (i) $n(A)$,
   (ii) $n(B)$,
   (iii) $n(A \cup B)$.

18. Given that $n(\xi) = 80$, $n(A) = 50$ and $n(B) = 32$, find the greatest possible value of $n(A \cup B)'$.

19. Given that $n(\xi) = 68$, $n(A) = 32$ and $n(B) = 54$, find the least possible value of $n(A \cap B)'$.

ADVANCED LEVEL

20. In a school, each pupil is required to take part in at least one of the following events: swimming, judo or tennis. A prefect reported that in his class of 45, 22 pupils participated in swimming, 14 in judo, and 32 in tennis; 5 in both swimming and judo, 8 in both judo and tennis, 7 in both swimming and tennis and 2 in all three events. Was the prefect conveying the correct information? Show your argument by means of a Venn diagram.

21. In a certain community centre, there are 90 youths. Of these, 43 take up dancing, 42 sewing and 48 swimming; 16 dancing and sewing, 17 sewing and swimming and 22 swimming and dancing. If each takes up at least one activity and x youths take up all three activities, express these facts in a Venn diagram. Show clearly the number in each separate region in terms of x, and form an equation satisfied by x. Hence, find the value of x.

22. In a class of 40 students, 15 students failed their Math paper and 19 students failed their Geography paper. Find
    (i) the greatest number of students that failed both papers,
    (ii) the least number of students that failed both papers,
    (iii) the maximum number of students who failed only one paper.

1. Venn diagrams are useful in solving problems involving classification and catalogue.
2. Given two disjoint sets A and B, $\mathbf{n}(A \cup B) = \mathbf{n}(A) + \mathbf{n}(B)$.
3. Given two intersecting sets A and B, $\mathbf{n}(A \cup B) = \mathbf{n}(A) + \mathbf{n}(B) - \mathbf{n}(A \cap B)$.
4. We can obtain
    - the greatest intersection when union is the smallest, i.e. when one set is a proper subset of another.
    - the smallest intersection when union is the greatest, i.e. when two sets are disjoint.

Review Exercise 2

1. Of the 44 houses in the neighbourhood of Jalan Chantek, 32 have mango trees and 28 have rambutan trees. How many houses have both mango and rambutan trees if 2 houses have neither?

2. Wellserve Trading Company has 43 employees and they drink either tea or coffee. If 28 of them drink tea and 10 of them drink tea but not coffee, how many employees drink coffee but not tea?

3. In a junior college, all pre-university one students must take part in athletics or swimming. If 170 take part in athletics and 192 take part in swimming, how many pre-university one students are there if
    (i) no pre-university one students can take part in both athletics and swimming?
    (ii) all pre-university one students can take part in both sports and 35 students do so?

4. A recent TV survey revealed the following: 59% of the TV viewers watched Channel 8, 48% watched Channel 5 and 46% watched Channel 12; 29% watched Channels 5 and 8; 20% watched Channels 5 and 12, 30% watched Channels 8 and 12 while 9% watched all the three channels. Find the percentage of viewers who
   (i) watched Channel 5 and 8 but not Channel 12,
   (ii) watched exactly two channels,
   (iii) did not watch any of the channels.

5. In an examination, Part A was attempted by 70 students, Part B by 50 and Part C by 42. 30 students attempted both parts A and B, 8 attempted both Parts B and C, 28 attempted both Parts A and C, and 3 attempted all three parts.
   (i) How many students attempted Part A but not Parts B and C?
   (ii) How many students attempted Part B but not Parts A and C?
   (iii) How many students attempted at least two parts?

6. The sets A and B are such that $n(A) = 28$, $n(A \cap B) = 12$ and $n(A \cup B) = 45$. Find the value of $n(B)$.

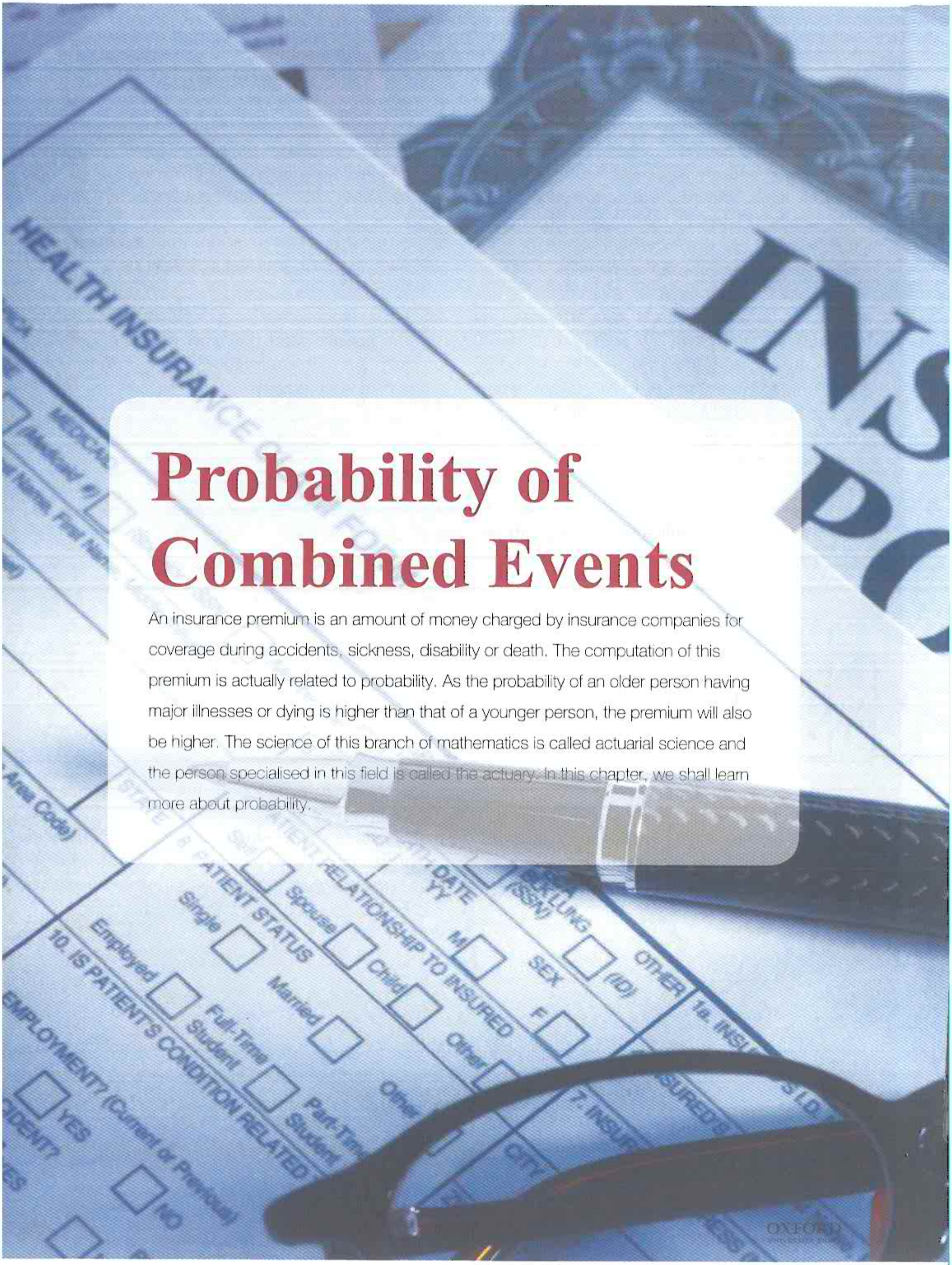
7. If $n(\xi) = 25$, $n(A) = 15$, $n(B) = 12$, find the greatest and the least values of the following.
   (i) $n(A \cap B)$ **(ii)** $n(A \cup B)$

8. Given that $n(\xi) = 20$, $n(A) = 13$ and $n(B) = 9$, find the greatest possible value of $n(A \cup B)'$.

9. Given that $n(\xi) = 80$, $n(A) = 60$ and $n(B) = 55$, find the greatest and the least possible values of $n(A \cap B)$. Illustrate your answer in both cases with a Venn diagram.

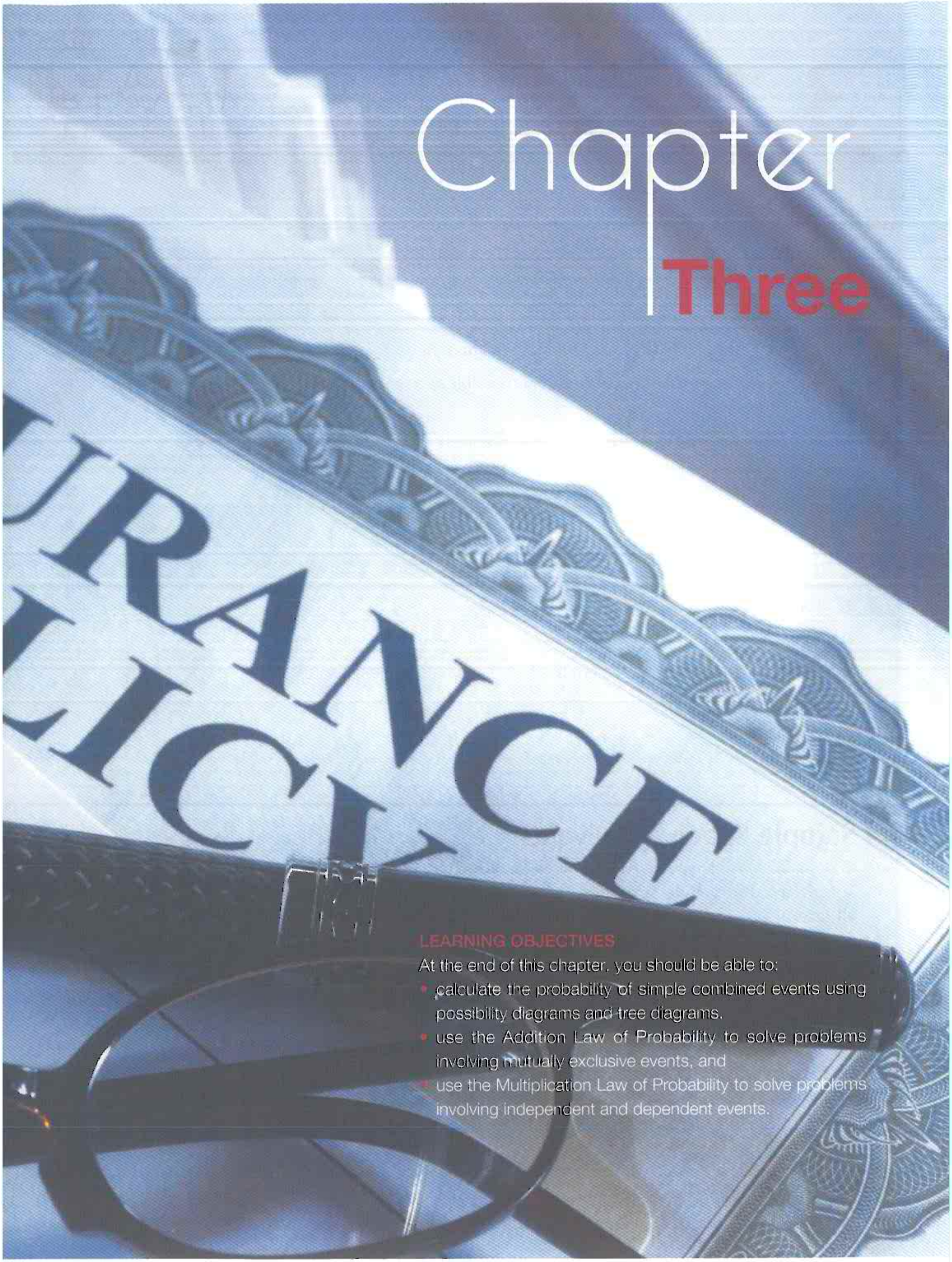
Challenge Yourself

A trade development board organised conventions abroad for its 150 members in the jewellery industry. The conventions fell under three categories: trading, manufacturing and design. 52 attended the trading category, 46 the manufacturing and 32 the design; 20 attended the trading and manufacturing category and 18 the trading and design; 8 attended the manufacturing and design but not the trading category and 14 attended the trading and manufacturing but not the design category. Find the percentage of members who attended none of the conventions.



Probability of Combined Events

An insurance premium is an amount of money charged by insurance companies for coverage during accidents, sickness, disability or death. The computation of this premium is actually related to probability. As the probability of an older person having major illnesses or dying is higher than that of a younger person, the premium will also be higher. The science of this branch of mathematics is called actuarial science and the person specialised in this field is called the actuary. In this chapter, we shall learn more about probability.



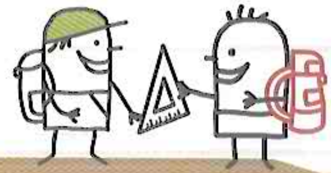
Chapter Three

LEARNING OBJECTIVES

At the end of this chapter, you should be able to:

- calculate the probability of simple combined events using possibility diagrams and tree diagrams.
- use the Addition Law of Probability to solve problems involving mutually exclusive events, and
- use the Multiplication Law of Probability to solve problems involving independent and dependent events.

3.1 Probability of Single Events



Recap

In this section, we will revise what you have learnt in Book 2 on finding the probability of *single events*, but we will use *set notations* learnt in Book 2 to describe the sample space and events.

The following is a summary of what we have learnt in Book 2.

1. A sample space is the collection of all the possible outcomes of a probability experiment.
2. In a probability experiment with m equally likely outcomes, if k of these outcomes favour the occurrence of an event E, then the probability, $P(E)$ of the event happening is given by:

$$\boxed{P(E) = \frac{\text{Number of favourable outcomes for event } E}{\text{Total number of possible outcomes}} = \frac{k}{m}}$$

3. For any event E, $0 \leqslant P(E) \leqslant 1$.
4. $P(E) = 0$ if and only if E is an **impossible** event, i.e. it will *never* occur.
5. $P(E) = 1$ if and only if E is a **certain** event, i.e. it will *definitely* occur.
6. For any event E, $P(\text{not } E) = 1 - P(E)$.

Sample Space and Events

In probability, we often make use of *set notations* to describe the sample space and events. The **sample space**, usually denoted by S, is the set containing **all** *possible outcomes* of a probability experiment while an **event** is a set of *favourable outcomes*. An event is a subset of the sample space.

Let us consider a probability experiment in which a card is chosen at random from a deck of 12 cards, numbered 1, 2, 3, ..., 12.

Since every number from 1 to 12 is a possible outcome of this experiment, the sample space can be written as:

$S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$
$\therefore n(S) = 12$

RECALL

The number of elements in a set E is denoted as $n(E)$.

Suppose A is the event that a prime number is chosen. Since the prime numbers in S are 2, 3, 5, 7 and 11, event A can be written as:

$$A = \{2, 3, 5, 7, 11\}$$
$$\therefore \text{n}(A) = 5$$

Fig. 3.1 shows the representation of S and A in a Venn diagram. We observe that in this case, the event A is a *proper subset* of S. The set A', the *complement* of the set A, is also represented in Fig. 3.1. For this experiment, $A' = \{1, 4, 6, 8, 9, 10, 12\}$.

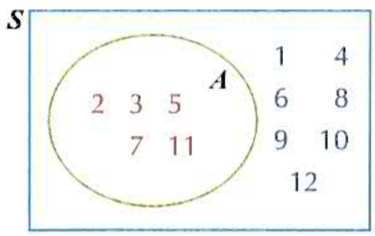


Fig. 3.1

RECALL

A' is the complement of the set A, and it is the set of elements in the **universal set**, which are not members of A.

ATTENTION

The sample space is the collection of all possible outcomes. Therefore, for a probability experiment, the sample space is the universal set.

Therefore, we can obtain the probability of event A occurring, i.e. P(A), by calculating $\text{P}(A) = \frac{\text{n}(A)}{\text{n}(S)} = \frac{5}{12}$, where n($A$) is the number of favourable outcomes for event A and n(S) is the total number of possible outcomes.

In general, in a sample space with a finite number of equally likely outcomes, the probability of an event E occurring, i.e. P(E) is

$$\boxed{\text{P}(E) = \frac{\text{n}(E)}{\text{n}(S)}.}$$

From the above experiment, we also observe that we can represent P(not A) as P(A'), which is the probability of all outcomes except those in A occurring,

$$\begin{aligned}\text{i.e. P(a prime number is not chosen)} &= \text{P}(A') \\ &= \frac{\text{n}(A')}{\text{n}(S)} \\ &= \frac{7}{12} \\ &= 1 - \text{P}(A)\end{aligned}$$

ATTENTION

For an event E, $\text{P}(E) + \text{P}(E') = 1$.

Therefore, in general, for an event E,

$$\boxed{\text{P}(E') = 1 - \text{P}(E).}$$

Thinking Time

For two events A and B, what do $\text{P}(A \cup B)$ and $\text{P}(A \cap B)$ mean? Use Venn diagram(s) to explain your answers.

In Worked Example 1, we will revise probability concepts taught in Book 2.

Worked Example 1

(Probability of Single Events)

The numbers 2, 3, 5 and 8 are written on four cards and these are placed on a table. Two of these cards are selected at random to form a two-digit number. List the sample space and hence find the probability that the number formed is

(i) odd,
(ii) divisible by 7,
(iii) prime,
(iv) not prime.

Solution:

Let S represent the sample space.
Let A be the event that the number formed is odd.
Let B be the event that the number formed is divisible by 7.
Let C be the event that the number formed is prime.

Then,

$S = \{23, 25, 28, 32, 35, 38, 52, 53, 58, 82, 83, 85\}$

$A = \{23, 25, 35, 53, 83, 85\}$

$B = \{28, 35\}$

$C = \{23, 53, 83\}$

$\therefore$ $\mathrm{n}(S) = 12$, $\mathrm{n}(A) = 6$, $\mathrm{n}(B) = 2$ and $\mathrm{n}(C) = 3$.

(i) $$\begin{aligned} \mathrm{P}(A) &= \frac{\mathrm{n}(A)}{\mathrm{n}(S)} \\ &= \frac{6}{12} \\ &= \frac{1}{2} \end{aligned}$$

(ii) $$\begin{aligned} \mathrm{P}(B) &= \frac{\mathrm{n}(B)}{\mathrm{n}(S)} \\ &= \frac{2}{12} \\ &= \frac{1}{6} \end{aligned}$$

(iii) $$\begin{aligned} \mathrm{P}(C) &= \frac{\mathrm{n}(C)}{\mathrm{n}(S)} \\ &= \frac{3}{12} \\ &= \frac{1}{4} \end{aligned}$$

(iv) $$\begin{aligned} \mathrm{P}(C') &= 1 - \mathrm{P}(C) \\ &= 1 - \frac{1}{4} \\ &= \frac{3}{4} \end{aligned}$$

Problem Solving Tip

To determine which numbers in the sample space are prime numbers, first eliminate the non-prime numbers — even numbers and numbers that end with a '5'. Then check the remaining numbers to see if they are prime numbers.

Consider the number 23.
$\sqrt{23} = 4.8$ (to 1 d.p.), so the largest prime less than or equal to $\sqrt{23}$ is 3. Since 23 is not divisible by either 2 or 3, then 23 is a prime number. Using this method, how do we determine that 53 and 83 are prime numbers?

ATTENTION

For **(iv)**, we can also obtain the same answer by calculating $\frac{\mathrm{n}(C')}{\mathrm{n}(S)}$.

However, since we have already obtained the answer for P(C) in **(iii)**, a quicker way will be to use P(C') = 1 – P(C).

PRACTISE NOW 1

SIMILAR QUESTIONS

Exercise 3A Questions 1, 9–10

A two-digit number is formed using the digits 2, 3 and 5. Repetition of digits is allowed.

(a) List the sample space,

(b) Find the probability that the two-digit number formed

(i) is prime,

(ii) contains the digit '2',

(iii) is divisible by 4,

(iv) is divisible by 13,

(v) is not divisible by 13.

Worked Example 2

(Probability of a Single Event)

There are 3 blue balls and 1 red ball in a bag. The balls are identical except for their colour. A ball is drawn at random from the bag. Find the probability that the ball drawn is blue.

Solution:

Let S represent the sample space and E the event that the ball drawn is blue.

Then $S = \{B_1, B_2, B_3, R\}$ and $E = \{B_1, B_2, B_3\}$.

$$\therefore \mathrm{P}(E) = \frac{\mathrm{n}(E)}{\mathrm{n}(S)} = \frac{3}{4}$$

Problem Solving Tip

Although the 3 blue balls are identical, they are still distinct. Hence, we need to distinguish between them by labelling them as B_1, B_2 and B_3. We cannot write $S = \{B, B, B, R\}$ because in set notation, it should be $\{B, R\}$; and we cannot write $S = \{B, R\}$ because $\mathrm{P}(B) = \frac{1}{2}$, which is wrong.

PRACTISE NOW 2

SIMILAR QUESTIONS

Exercise 3A Questions 2–3, 8, 19

A letter is chosen at random from the word 'CLEVER'. Find the probability that the letter chosen is

(i) an 'E',

(ii) a 'C' or a 'R',

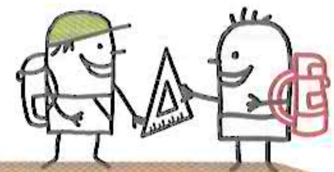
(iii) a 'K',

(iv) a consonant.

ATTENTION

In Sets, $S = \{x : x$ is a letter of the word 'CLEVER'$\} = \{$C, L, E, V, R$\}$ because we do not distinguish between the two 'E's as they are the same letter. However, in probability, we treat the two 'E's as distinct since there is a higher chance of choosing 'E' than each of the other letters.

3.2 Simple Combined Events, Possibility Diagrams and Tree Diagrams



Possibility Diagrams

In this section, we will learn how to list the sample space of an experiment involving two or more objects (e.g. rolling two dice), and calculate probabilities for simple combined events.

The possible outcomes for rolling a fair die are 1, 2, 3, 4, 5 and 6, and we write the sample space as {1, 2, 3, 4, 5, 6}.

How do we write the possible outcomes for rolling two fair dice? We can represent a possible outcome by using an **ordered pair**, e.g. (2, 3) means that the first die shows a '2' and the second die shows a '3'; which is different from (3, 2). So what does (3, 2) mean?

How can we write the sample space for rolling two fair dice?

Is {(1, 1), (1, 2), (1, 3), ..., (6, 6)} clear enough? Listing out all the outcomes would be very tedious and we may miss out some outcomes.

Therefore, there is a need to use a different method to represent the sample space. Fig. 3.2 shows one way of drawing a **possibility diagram** to represent the sample space for rolling two fair dice.

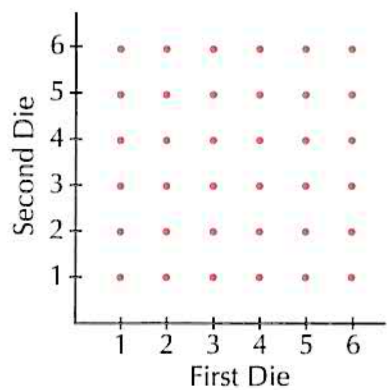


Fig. 3.2

A **possibility diagram** is used when each outcome of the sample space has two **components**. For example, in the above case, an outcome (represented by a red dot •) is determined by the values displayed by the first and second dice. From the above possibility diagram, we observe that the *total number of possible outcomes* is $6 \times 6 = 36$. We can also calculate the probability of certain events using a possibility diagram, as shown in Worked Example 3.

Worked Example 3

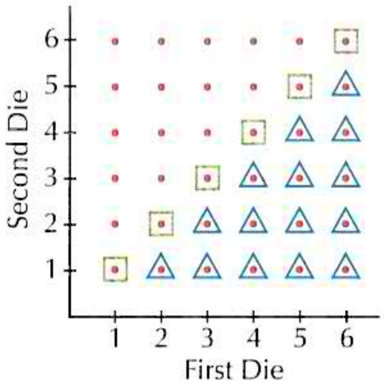
(Use of Possibility Diagram)

Two fair dice are rolled. What is the probability that

(i) both dice show the same number,

(ii) the number shown on the first die is greater than the number shown on the second die?

Solution:



Problem Solving Tip

Mark out the favourable outcomes on the possibility diagram.

Count the number of □ for **(i)** and the number of △ for **(ii)**.

(i) P(both dice show the same number) $= \frac{6}{36}$

$= \frac{1}{6}$

(ii) P(number shown on first die is greater than the number shown on second die)

$= \frac{15}{36} = \frac{5}{12}$

PRACTISE NOW 3

SIMILAR QUESTIONS

Exercise 3A Questions 4, 11, 20

1. A fair tetrahedral die (4-sided die) and a fair 6-sided die are rolled simultaneously. The numbers on the tetrahedral die are 1, 2, 5 and 6 while the numbers on the 6-sided die are 1, 2, 3, 4, 5 and 6.

(a) Display all the outcomes of the experiment using a possibility diagram.

(b) Using the possibility diagram or otherwise, find the probability that

(i) both dice show the same number,

(ii) the number shown on the tetrahedral die is greater than the number shown on the 6-sided die,

(iii) the numbers shown on both dice are prime numbers.

2. A bag contains five cards and the cards are numbered 1, 2, 3, 4 and 5. A card is drawn at random from the bag and its number is noted. The card is then replaced and a second card is drawn at random from the bag. Using a possibility diagram, find the probability that

(i) the number shown on the second card is greater than the number shown on the first card,

(ii) the sum of the two numbers shown is greater than 7,

(iii) the product of the two numbers shown is greater than 10.

There is another way to draw a possibility diagram to represent the sample space for rolling two fair dice, as shown in Worked Example 4.

Worked Example 4

(Use of Possibility Diagram)

Two fair dice are rolled. Find the probability that the sum of the numbers shown on the dice is

(i) equal to 5, **(ii)** even.

Solution:

First Die

Second Die

+	1	2	3	4	5	6
1	$\boxed{2}$	3	$\boxed{4}$	5	$\boxed{6}$	7
2	3	$\boxed{4}$	5	$\boxed{6}$	7	$\boxed{8}$
3	$\boxed{4}$	5	$\boxed{6}$	7	$\boxed{8}$	9
4	5	$\boxed{6}$	7	$\boxed{8}$	9	$\boxed{10}$
5	$\boxed{6}$	7	$\boxed{8}$	9	$\boxed{10}$	11
6	7	$\boxed{8}$	9	$\boxed{10}$	11	$\boxed{12}$

(i) $\text{P(sum is equal to 5)} = \frac{4}{36}$

$= \frac{1}{9}$

(ii) $\text{P(sum is even)} = \frac{18}{36}$

$= \frac{1}{2}$

ATTENTION

The sum of the numbers are shown in each cell. In your own possibility diagram, you can draw double lines to avoid accidentally counting the numbers in the first row and in the first column, when counting the number of favourable outcomes.

Problem Solving Tip

Mark out the favourable outcomes on the possibility diagram.

Count the number of ○ for **(i)** and the number of □ for **(ii)**.

PRACTISE NOW 4

SIMILAR QUESTIONS

Exercise 3A Questions 5–6, 12–15, 21

1. The numbers on a fair tetrahedral die are 1, 2, 5 and 6 while the numbers on a fair 6-sided die are 1, 2, 3, 4, 5 and 6. The two dice are rolled at the same time and the scores on both dice are recorded. The possibility diagrams below display separately some of the values of the sum and product of the two scores.

Tetrahedral die

6-sided die

+	1	2	5	6
1		3		
2				
3			8	
4				
5				
6	7			

Tetrahedral die

6-sided die

×	1	2	5	6
1				
2		4		
3				18
4				
5				
6			30	

(a) Copy and complete the possibility diagrams.

(b) Using the possibility diagrams, find the probability that the sum of the scores is

(i) even, (ii) divisible by 3,

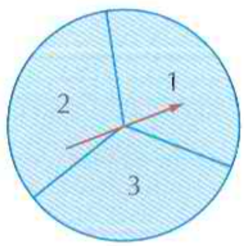
(iii) a perfect square, (iv) less than 2.

(c) Using the possibility diagrams, find the probability that the product of the scores is

(i) odd, (ii) larger than 12,

(iii) a prime number, (iv) less than 37.

2. A circular card is divided into 3 equal sectors with scores of 1, 2 and 3. The card has a pointer pivoted at its centre. The pointer is spun twice. Each time the pointer is spun, it is equally likely to stop at any of the sectors.



(a) With the help of a possibility diagram, find the probability that

(i) each score is a '1',

(ii) at least one of the scores is a '3'.

(b) In a game, a player spins the pointer twice. His final score is the larger of the two individual scores if they are different and their common value if they are the same. The possibility diagram below shows the player's final score.

	1	2	3
1	1		
2			
3		3	

(i) Copy and complete the possibility diagram.

(ii) Using the diagram, find the probability that his final score is even.

(iii) Using the same diagram, find the probability that his final score is a prime number.

Tree Diagrams

The sample space for tossing a fair coin is $\{H, T\}$.

The sample space for tossing two fair coins can be represented by a possibility diagram, as shown in Fig. 3.3.

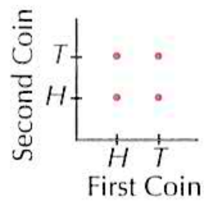


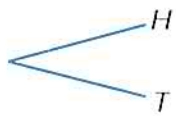
Fig. 3.3

How can we represent the sample space for tossing three fair coins?

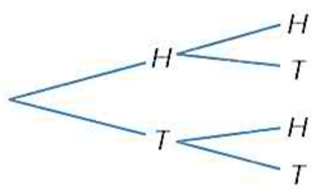
We use a different type of diagram called a **tree diagram** to represent the sample space, as shown in Fig. 3.4. The following steps show how the tree diagram is constructed.

In a family, there are two children and one of them is a boy. What is the probability that the other child is a girl?

1. When the first coin is tossed, there are two possible outcomes, head (H) or tail (T), so we start with a point and draw two branches H and T.



2. The second coin is then tossed. Regardless of the outcome of the first toss, the second coin would also yield either a H or a T, thus we draw two branches after the H and the T from the first toss as shown below. There are a total of $2 \times 2 = 4$ branches, i.e. there are 4 possible outcomes at this stage.



3. The third coin could also yield two outcomes when the first two outcomes are HH, HT, TH or TT. Thus we obtain the tree diagram as shown in Fig. 3.4.

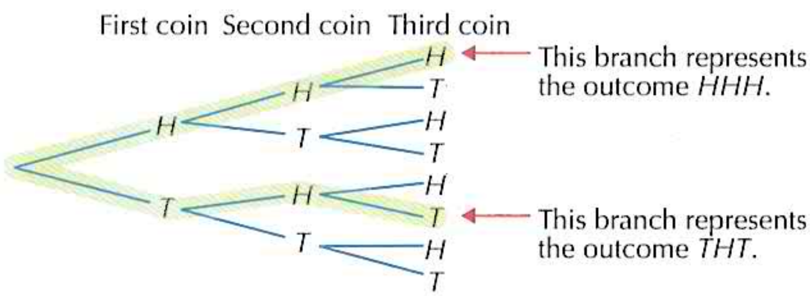


Fig. 3.4

From Fig. 3.4, we observe that there are a total of $2 \times 2 \times 2 = 8$ branches, i.e. the *total number of possible outcomes* is 8.

In summary,

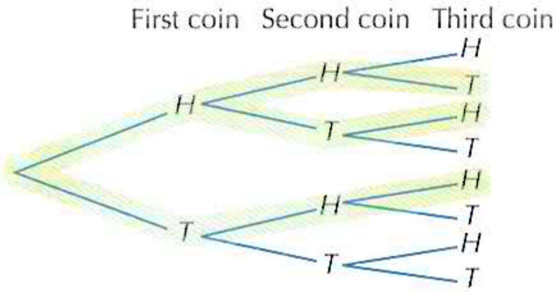
Example of Experiment	Components of Each Outcome	Representation of Sample Space
Tossing 1 coin	1	List of outcomes in a set
Tossing 2 coins	2	Possibility diagram or tree diagram
Tossing 3 coins	3	Tree diagram

Worked Example 5

(Use of Tree Diagram)

Three fair coins are tossed. Find the probability that
(i) there are two heads and one tail,
(ii) there is at least one tail.

Solution:



(i) P(two heads and one tail) $= \frac{3}{8}$ (see shaded regions)

(ii) P(at least one tail) $= 1 - \text{P(no tail)}$

$$= 1 - \text{P(three heads)}$$

$$= 1 - \frac{1}{8}$$

$$= \frac{7}{8}$$

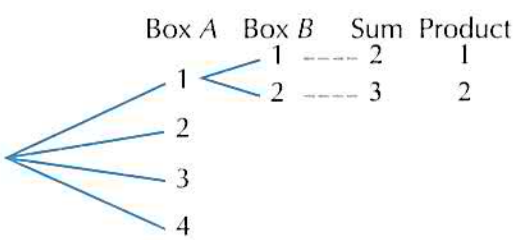
PRACTISE NOW 5

SIMILAR QUESTIONS

Exercise 3A Questions 7, 16–18

1. Michael is a darts player. There is an equal probability that he will hit or miss the bull's-eye. He aims for the bull's-eye and attempts 3 throws. Using a tree diagram, find the probability that
(i) he misses the bull's-eye once,
(ii) he hits the bull's-eye at least once.

2. Box A contains 4 pieces of paper numbered 1, 2, 3 and 4. Box B contains 2 pieces of paper numbered 1 and 2. One piece of paper is removed at random from each box.
(a) Copy and complete the following tree diagram.



(b) Find the probability that
(i) at least one '1' is obtained,
(ii) the sum of the two numbers is 3,
(iii) the product of the two numbers is at least 4,
(iv) the sum is equal to the product.

INFORMATION

Students may also use a possibility diagram to solve, if no diagram is provided.

Exercise 3A

BASIC LEVEL

1. A fair coin and a fair 6-sided die are tossed and rolled, respectively. Using set notations, list the sample space of the experiment.

2. A box contains 7 pens, 3 of which are faulty. A pen is drawn from the box at random. Using set notations, list the sample space of this probability experiment and find the probability that the pen drawn is not faulty.

   Suppose that the first pen drawn is not faulty and it is not replaced in the box. A second pen is now drawn from the box. Using set notations, find the probability that the second pen drawn is faulty.

3. Each letter of the word 'POSSIBILITY' is written on identical cards. One card is chosen at random. Using set notations, find the probability that the letter on the chosen card is
   (i) a 'S', **(ii)** a 'P' or an 'I',
   (iii) a vowel, **(iv)** a consonant.

4. A box contains three cards bearing the numbers 1, 2 and 3. A second box contains four cards bearing the numbers 2, 3, 4 and 5. A card is chosen at random from each box.
   (a) Display all the possible outcomes of the experiment using a possibility diagram.
   (b) With the help of the possibility diagram, calculate the probability that
   (i) the cards bear the same number,
   (ii) the numbers on the cards are different,
   (iii) the larger of the two numbers on the cards is 3.

5. Six cards numbered 0, 1, 2, 3, 4 and 5 are placed in a box and well-mixed. A card is drawn at random from the box and the number on the card is noted before it is replaced in the box. The cards in the box are thoroughly mixed again and a second card is drawn at random from the box. The sum of the two numbers is then obtained.
   (a) Copy and complete the possibility diagram below, giving all the possible sums of the two numbers. Some of the possible sums are shown.

First number

Second number

+	0	1	2	3	4	5
0						
1	1			4		
2						
3						
4		5				
5						

   (b) How many possible outcomes are there in the sample space of this experiment?
   (c) What is the probability that the sum of the two numbers
   (i) will be 7,
   (ii) will be a prime number,
   (iii) will not be a prime number,
   (iv) will be even,
   (v) will not be even?
   (d) Which sum is more likely to occur, the sum of 7 or the sum of 8?

6. It is given that $X = \{4, 5, 6\}$ and $Y = \{7, 8, 9\}$. An element x is selected at random from X and an element y is selected at random from Y. The possibility diagrams below display separately some of the values of $x + y$ and xy.

		x		
	+	4	5	6
	7	11		
y	8			
	9		14	

		x		
	×	4	5	6
	7			42
y	8		40	
	9			

(a) Copy and complete the possibility diagrams.

(b) Find the probability that the sum $x + y$ is
(i) prime, **(ii)** greater than 12,
(iii) at most 14.

(c) Find the probability that the product xy is
(i) odd, **(ii)** even,
(iii) at most 40.

7. A fair coin is tossed three times. Display all the possible outcomes of the experiment using a tree diagram.

From your tree diagram, find the probability of obtaining
(i) three heads, **(ii)** exactly two heads,
(iii) at least two heads.

INTERMEDIATE LEVEL

8. Bag P contains a red, a blue and a white marble while bag Q contains a blue and a red marble. The marbles are identical except for their colour. A marble is picked at random from both bag P and bag Q. List all the possible outcomes of the sample space.

Find the probability that the two marbles selected are
(i) of the same colour, **(ii)** blue and red,
(iii) of different colours.

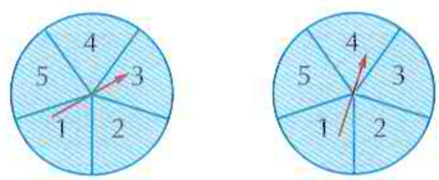
9. A two-digit number is formed using the digits 1, 2 and 3. Repetition of digits is allowed.

(a) List the sample space.

(b) Find the probability that the two-digit number formed is
(i) divisible by 3,
(ii) a perfect square,
(iii) a prime number,
(iv) a composite number.

10. The three daughters-in-law of Mrs Chan are happily awaiting the arrival of their bundles of joy within the year. List the sample space of the sexes of the three babies, given that the babies are equally likely to be either a boy or a girl.
Hence, find the probability that Mrs Chan will have
(i) three grandsons,
(ii) two grandsons and one granddaughter,
(iii) one grandson and two granddaughters.

11. In an experiment, two spinners are constructed with spinning pointers as shown in diagrams below. Both pointers are spun. Each time the pointer is spun, it is equally likely to stop at any sector.



First spinner Second spinner

(a) Find the probability that the pointers will point at
(i) numbers on the spinners whose sum is 6,
(ii) the same numbers on both spinners,
(iii) different numbers on the spinners,
(iv) two different prime numbers.

(b) What is the probability that the number on the first spinner will be less than the number on the second spinner?

12. In a game, the player throws a fair coin and a fair 6-sided die simultaneously. If the coin shows a head, the player's score is the score on the die. If the coin shows a tail, then the player's score is twice the score on the die. Some of the player's possible scores are shown in the possibility diagram below.

Die

Coin	1	2	3	4	5	6
H	1					
T			6			

(a) Copy and complete the possibility diagram.

(b) Using the diagram, find the probability that the player's score is

(i) odd,

(ii) even,

(iii) a prime number,

(iv) less than or equal to 8,

(v) a multiple of 3.

13. Two fair 6-sided dice were thrown together and the difference of the resulting numbers on their faces was calculated. Some of the differences are shown in the possibility diagram below.

First die

Second die −	1	2	3	4	5	6
1	0					
2			1			4
3						
4						
5						
6		4				0

(a) Copy and complete the possibility diagram.

(b) Using the diagram, find the probability that the difference of the two numbers is

(i) 1, **(ii)** non-zero,

(iii) odd, **(iv)** a prime number,

(v) more than 2.

14. A bag contains 5 identical balls which are numbered 1, 2, 4, 5 and 7. Two balls are drawn at random, one after another and without replacement.

Find the probability that the

(i) numbers obtained on both balls are prime,

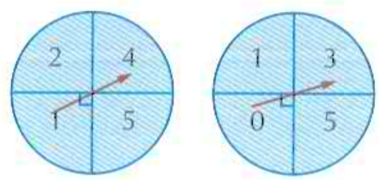
(ii) sum of the numbers obtained is odd,

(iii) product of the numbers obtained is greater than 20,

(iv) difference in the numbers obtained is less than 7,

(v) product of the numbers obtained is divisible by 9.

15. The diagrams below show two circular cards, each with a pointer pivoted at its centre. The first card is divided into 4 equal sectors with scores 1, 2, 4 and 5. The second card is divided into 4 equal sectors with scores 0, 1, 3 and 5. In a game, both pointers are spun. Each time the pointer is spun, it is equally likely to stop at any sector.



Find the probability that the

(i) scores on both cards are the same,

(ii) scores on both cards are prime,

(iii) sum of the scores is odd,

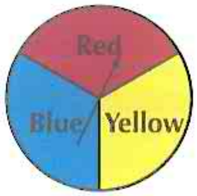
(iv) sum of the scores is divisible by 5,

(v) sum of the scores is 6 or less,

(vi) product of the scores is not 0,

(vii) product of the scores is greater than 11.

16. A spinner with three equal sectors (shown below) and a fair coin are used in a game. The spinner is spun once and the coin is tossed once. Each time the pointer is spun, it is equally likely to stop at any sector.



Calculate the probability of getting
(i) red on the spinner and tail on the coin,
(ii) blue or yellow on the spinner and head on the coin.

17. A bag contains 3 cards numbered 1, 3 and 5. A second bag contains 3 cards numbered 1, 2 and 7. One card is drawn at random from each bag. Calculate the probability that the two numbers obtained
(i) are both odd,
(ii) are both prime,
(iii) have a sum greater than 4,
(iv) have a sum that is even,
(v) have a product that is prime,
(vi) have a product that is greater than 20,
(vii) have a product that is divisible by 7.

18. A fair die is made from a tetrahedron such that each of its four faces is printed with one number. The numbers are 1, 2, 3 and 4.

(a) When the die is rolled, what is the probability that
(i) it will land with the face printed '4' down,
(ii) it will land such that the sum of the three upper faces is an odd number?

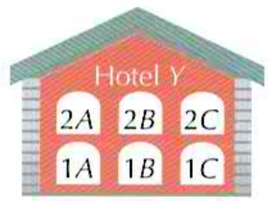
(b) If the same die is rolled and a fair coin is tossed at the same time, list all the possible outcomes using a tree diagram.

ADVANCED LEVEL

19. A box contains 7 electrical components. The box was dropped in transit and 1 of the components became defective, but not visibly. The components are taken out from the box at random and tested until the defective component is obtained.

What is the probability that the defective component is the first component tested?

20. Hotel Y is a two-storey hotel, with rooms (and their respective room numbers) arranged as shown in the diagram below. Rooms are allocated at random when guests arrive and each guest is allocated one room. Kate and Nora arrive at Hotel Y on a particular day. Upon their arrival, none of the rooms in Hotel Y are occupied.



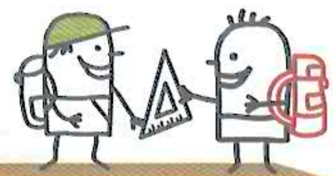
(a) With the help of the possibility diagram, find the probability that Nora and Kate
(i) stay next to each other,
(ii) stay on different storeys,
(iii) do not stay next to each other.

(b) Suppose the hotel accepts Kate's request that she only wants to be allocated rooms on the second floor, what will be the probability that she will be staying next to Nora?

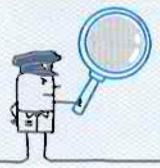
21. Two fair tetrahedral dice and a fair 6-sided die are rolled simultaneously. The numbers on the tetrahedral dice are 1, 2, 3 and 4 while the numbers on the 6-sided die are 1, 2, 3, 4, 5 and 6.

What is the probability that the score on the 6-sided die is greater than the sum of scores of the two tetrahedral dice?

3.3 Addition Law of Probability and Mutually Exclusive Events



In this section, we will learn the conditions for adding probabilities.



Investigation

Mutually Exclusive and Non-Mutually Exclusive Events

Eight cards numbered 1 to 8 are placed in a box.
A card is drawn at random.
Let A be the event of drawing a card with a prime number.
Let B be the event of drawing a card with a multiple of 4.
Let C be the event of drawing a card with an odd number.

1. List the sample space.

Part 1: Mutually Exclusive Events

2. List the favourable outcomes for event A and find the probability that A will occur, i.e. P(A).
3. List the favourable outcomes for event B and find the probability that B will occur, i.e. P(B).
4. Is there any **overlap** between the favourable outcomes for event A and the favourable outcomes for event B? That is, are there any outcomes that favour the occurrence of **both** event A and event B? These two events are said to be **mutually exclusive**.
5. List the favourable outcomes for event A or event B, and find the probability that the combined event A or B, or $A \cup B$ will occur, i.e. P(A or B) or P($A \cup B$).
6. Is $P(A \cup B) = P(A) + P(B)$ in this case? Can you explain why?

Part 2: Non-Mutually Exclusive Events

7. List the favourable outcomes for event C and find the probability that C will occur, i.e. P(C).
8. Is there any overlap between the favourable outcomes for event A and the favourable outcomes for event C? That is, are there any outcomes that favour the occurrence of **both** event A and event C? These two events are said to be **non-mutually exclusive.**
9. List the favourable outcomes for event A or event C, and find the probability that the combined event A or C, or $A \cup C$ will occur, i.e. P(A or C) or P($A \cup C$).
10. Is $P(A \cup C) = P(A) + P(C)$ in this case? Can you explain why?

SIMILAR QUESTIONS

Exercise 3B Questions 9–10

From the investigation, we can conclude that if two events A and B cannot occur at the same time (i.e. the events are **mutually exclusive**), then P(A or B) or $P(A \cup B) = P(A) + P(B)$.

On the other hand, if two events A and C can occur at the same time (i.e. the events are **non-mutually exclusive**), then P(A or C) or $P(A \cup C) \neq P(A) + P(C)$.

In general, the **Addition Law of Probability** states that

if A and B are ***mutually exclusive events***, P(A or B) or $P(A \cup B) = P(A) + P(B)$.

Worked Example 6

(Probability involving Mutually Exclusive Events)

A card is drawn at random from a standard pack of 52 playing cards. Find the probability that the card is

(i) an Ace or a King,

(ii) a heart or a diamond,

(iii) neither a King nor a Queen.

RECALL

There are four suits in a standard pack of 52 playing cards, i.e. club ♣, diamond ♦, heart ♥ and spade ♠.

Each suit has 13 cards, i.e. Ace, 2, 3, ..., 10, Jack, Queen and King.

All the clubs and spades are black in colour.

All the diamonds and hearts are red in colour.

All the Jack, Queen and King cards are picture cards.

Solution:

(i) P(Ace or King) = P(Ace) + P(King)

$$= \frac{4}{52} + \frac{4}{52}$$
$$= \frac{1}{13} + \frac{1}{13}$$
$$= \frac{2}{13}$$

Problem Solving Tip

Since only **one** card is drawn, the events of drawing an Ace and a King cannot occur at the same time, i.e. the events are ***mutually exclusive***.

By identifying that the two events are mutually exclusive, we apply the **Addition Law of Probability** to obtain the answer, i.e. P(Ace or King) = P(Ace) + P(King).

(ii) P(heart or diamond) = P(heart) + P(diamond)

$$= \frac{13}{52} + \frac{13}{52}$$
$$= \frac{1}{4} + \frac{1}{4}$$
$$= \frac{1}{2}$$

(iii) P(King or Queen) = P(King) + P(Queen)

$$= \frac{4}{52} + \frac{4}{52}$$
$$= \frac{1}{13} + \frac{1}{13}$$
$$= \frac{2}{13}$$

∴ P(neither King nor Queen) = 1 – P(King or Queen)

$$= 1 - \frac{2}{13}$$
$$= \frac{11}{13}$$

Problem Solving Tip

Alternative solution for **(iii)**:

Number of cards excluding King and Queen cards = 52 – 8

= 44

P(neither King nor Queen) = $\frac{44}{52}$

$= \frac{11}{13}$

PRACTISE NOW 6

SIMILAR QUESTIONS

Exercise 3B Questions 1–3, 5

A card is drawn at random from a standard pack of 52 playing cards.
Find the probability of drawing

(i) a picture card or an Ace,

(ii) an Ace or a card bearing a number which is divisible by 3,

(iii) a King or a Queen,

(iv) neither a Jack nor an Ace.

Worked Example 7

(Probability involving Mutually Exclusive Events)

The probabilities of three teams, L, M and N winning a football competition are $\frac{1}{4}$, $\frac{1}{8}$ and $\frac{1}{10}$, respectively.
Assuming only one team can win, calculate the probability that

(i) either L or M wins, **(ii)** neither L nor N wins.

Solution:

(i) $$\begin{aligned} P(L \text{ or } M \text{ wins}) &= P(L \text{ wins}) + P(M \text{ wins}) \\ &= \frac{1}{4}+\frac{1}{8} \\ &= \frac{3}{8} \end{aligned}$$

(ii) $$\begin{aligned} P(L \text{ or } N \text{ wins}) &= P(L \text{ wins}) + P(N \text{ wins}) \\ &= \frac{1}{4}+\frac{1}{10} \\ &= \frac{7}{20} \end{aligned}$$

$$\begin{aligned} P(\text{neither } L \text{ nor } N \text{ wins}) &= 1 - P(L \text{ or } N \text{ wins}) \\ &= 1-\frac{7}{20} \\ &= \frac{13}{20} \end{aligned}$$

ATTENTION

Since only one team can win, the events of each of the teams, L, M and N winning are ***mutually exclusive***.

How do we deduce that there are more than three teams in the competition?

From **(ii)**,
Is P(neither L nor N wins)
= P(L not winning)
+ P(N not winning)?
Explain.

PRACTISE NOW 7

SIMILAR QUESTIONS

Exercise 3B Questions 4, 6–8

The probabilities of four teams, P, Q, R and S winning the National Football Championship are $\frac{1}{5}$, $\frac{1}{6}$, $\frac{1}{7}$ and $\frac{1}{8}$ respectively. Assuming only one team can win the championship, find the probability that

(i) either P or Q wins the championship,

(ii) Q or R or S wins the championship,

(iii) none of these teams wins the championship.

Exercise 3B

BASIC LEVEL

1. Eleven cards numbered 11, 12, 13, 14,..., 21 are placed in a box. A card is removed at random from the box. Find the probability that the number on the card is
   (i) even,
   (ii) prime,
   (iii) either even or prime,
   (iv) divisible by 3,
   (v) neither even nor prime.

2. A bag contains 7 red, 5 green and 3 blue marbles. A marble is selected at random from the bag. Find the probability of selecting
   (i) a red marble,
   (ii) a green marble,
   (iii) either a red or a green marble,
   (iv) neither a red nor a green marble.

3. The letters of the word 'MUTUALLY' and the word 'EXCLUSIVE' are written on individual cards and the cards are put into a box. A card is picked at random. What is the probability of picking
   (i) the letter 'U',
   (ii) the letter 'E',
   (iii) the letter 'U' or 'E',
   (iv) a consonant,
   (v) the letter 'U' or a consonant,
   (vi) the letter 'U' or 'E' or 'L'?

4. The probability of a football team winning any match is $\frac{7}{10}$ and the probability of losing any match is $\frac{2}{15}$. What is the probability that
   (i) the team wins or loses a particular match,
   (ii) the team neither wins nor loses a match?

INTERMEDIATE LEVEL

5. A card is drawn at random from a standard pack of 52 playing cards. Find the probability of drawing
   (i) a King or a Jack,
   (ii) a Queen or a card bearing a prime number,
   (iii) a card bearing a number that is divisible by 3 or by 5,
   (iv) neither a King nor a Jack.

6. When a golfer plays any hole, the probabilities that he will take 4, 5 or 6 strokes are $\frac{1}{14}$, $\frac{2}{7}$ and $\frac{3}{7}$ respectively. He never takes less than 4 strokes. Calculate the probability that in playing a hole, he will take
   (i) 4 or 5 strokes,
   (ii) 4, 5 or 6 strokes,
   (iii) more than 6 strokes.

7. In a basketball tournament, three of the participating teams are Alpha, Beta and Gamma. The probabilities of each of these three teams winning the tournament are $\frac{4}{15}$, $\frac{1}{10}$ and $\frac{1}{5}$ respectively. Find the probability that
   (i) Alpha or Gamma will win the tournament,
   (ii) Alpha, Beta or Gamma will win the tournament,
   (iii) neither Alpha nor Gamma will win the tournament,
   (iv) none of these three teams will win the tournament.

8. Every year, only one student can win the Student of the Year Award. The probabilities of Priya, Rui Feng and Amirah winning the award are $\frac{1}{3}$, $\frac{1}{8}$ and $\frac{1}{20}$, respectively. What is the probability that
   (i) one of them will win the award,
   (ii) none of them will win the award,
   (iii) Priya and Rui Feng will not win the award?

9. In a probability experiment, three fair coins are tossed, one after another.

(a) Display all the possible outcomes of the experiment using a tree diagram.

(b) For the experiment, the events A, B, C and D are defined as follows:

A: All three coins show heads.

B: At least two coins show tails.

C: Exactly one coin shows a head.

D: The sides appear alternately.

For each part, identify if the following events are mutually exclusive.

(i) A, B (ii) C, D
(iii) B, C (iv) A, C
(v) B, D (vi) A, B, C

ADVANCED LEVEL

10. In a game, Jun Wei attempts to score a penalty kick against a goalkeeper who will try to save his shot. There is an equal chance that he will score or miss his penalty kick. Jun Wei has three chances to score, and the game ends once Jun Wei scores a penalty kick.

(a) Draw a tree diagram to show all the possible outcomes. What is the total number of outcomes?

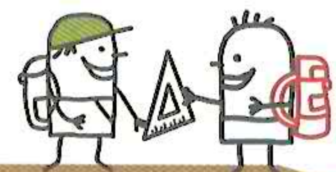
(b) Events A and B are defined as follows:

A: exactly two penalty kicks are attempted.

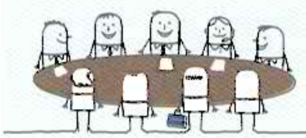
B: at most two penalty kicks are attempted.

Are A and B mutually exclusive events? Explain your answer.

3.4 Multiplication Law of Probability and Independent Events



In this section, we will learn another type of diagram to represent the sample space, and the conditions for multiplying probabilities.



Class Discussion

Choosing a Diagram to Represent the Sample Space

Discuss in pairs.

There are 3 blue balls and 2 red balls in a bag. The balls are identical except for their colour. A ball is drawn at random from the bag and is replaced. A second ball is then drawn at random from the bag.

1. Try representing the sample space for this probability experiment using
(a) a possibility diagram, and
(b) a tree diagram.

2. Is it easy or tedious to represent the sample space in each diagram?

From the class discussion, we observe that it is still possible to draw a possibility diagram as shown in Fig. 3.5**(a)**. But what happens if there is a third draw, or if there are 8 blue balls and 2 red balls? Then it is not possible to draw a possibility diagram for the former case, and it will be very tedious to draw a 10-by-10 tree diagram for the latter case. Similarly, it is very tedious to draw a tree diagram with $5 \times 5 = 25$ branches for the investigation.

Therefore, there is a need to simplify the tree diagram to represent the sample space. Fig 3.5**(b)** shows the use of a simplified **tree diagram** to represent the sample space for the above experiment, where B represents 'blue' and R represents 'red', and the probability on each branch represents the probability for the occurrence of the outcome at the end of the branch.

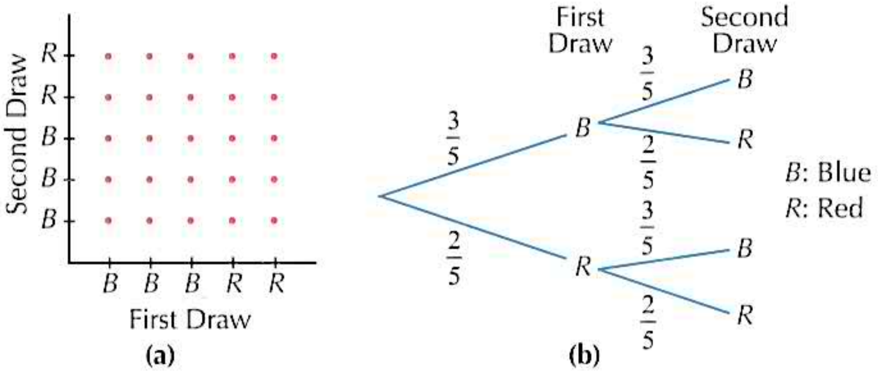
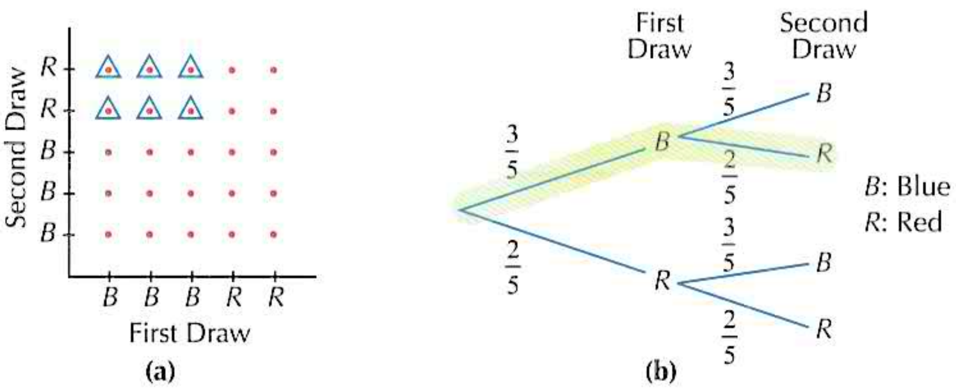


Fig. 3.5

Fig. 3.6 shows how we could use the possibility diagram in **(a)** and the tree diagram in **(b)** to find P(BR), the probability that the first ball drawn is blue and the second ball drawn is red. The answer is $\frac{6}{25}$ in both cases.



number of blue balls

number of red balls

$$P(BR) = \frac{3 \times 2}{5 \times 5} = \frac{6}{25}$$

total number of balls in each draw

probability of drawing first blue ball

probability of drawing red ball after first blue ball

$$P(BR) = \frac{3}{5} \times \frac{2}{5} = \frac{6}{25}$$

Fig. 3.6

In other words, we can ***multiply along the connected branches*** of a probability tree to find the above answer.

Worked Example 8

(Use of Tree Diagram)

There are 7 green marbles and 3 yellow marbles in a bag. The marbles are identical except for their colour. A marble is drawn at random from the bag, and is replaced. A second marble is then drawn at random from the bag. Find the probability that

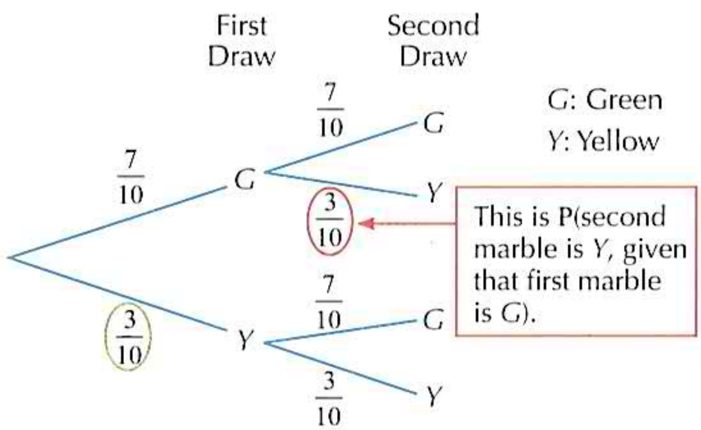
(i) the first marble drawn is yellow,

(ii) the second marble drawn is yellow given that the first marble drawn is green,

(iii) the first marble drawn is green and the second marble drawn is yellow,

(iv) the second marble drawn is yellow.

Solution:



(i) P(first marble is Y) $= \frac{3}{10}$ (see — $\frac{\text{Number of yellow marbles in bag for first draw}}{\text{Total number of marbles in bag for first draw}}$)

(ii) P(second marble is Y, given that first marble is G)

$= \frac{3}{10}$ (see — $\frac{\text{Number of yellow marbles in bag for second draw, given first marble is green}}{\text{Total number of marbles in bag for second draw}}$)

(iii) P(first marble is G and second marble is Y) or P(GY) $= \frac{7}{10} \times \frac{3}{10}$

$$= \frac{21}{100}$$

(iv) P(second marble is Y) = P(GY) + P(YY)

$$= \left(\frac{7}{10} \times \frac{3}{10}\right) + \left(\frac{3}{10} \times \frac{3}{10}\right)$$

$$= \frac{21}{100} + \frac{9}{100}$$

$$= \frac{30}{100}$$

$$= \frac{3}{10}$$

PRACTISE NOW 8

SIMILAR QUESTIONS

Exercise 3C Questions 1–3, 8–9

A box contains 5 blue pens and 7 red pens. The pens are identical except for their colour. A pen is selected at random from the box and its colour is noted. The pen is replaced back into the box. A second pen is then selected at random from the box. Find the probability that

(i) the first pen selected is red,

(ii) the second pen selected is blue, given that the first pen selected is blue,

(iii) the first pen selected is blue and the second pen selected is blue,

(iv) the second pen selected is blue,

(v) no blue pen was selected.

Independent Events

Two events are **independent events** if the chance of one of them occurring **does not** affect the chance of the other event occurring. From Worked Example 8, we observe that:

P(second marble is Y, given that first marble is G) = $\frac{3}{10}$, and

P(second marble is Y, given that first marble is Y) = $\frac{3}{10}$.

These two probabilities are equal, regardless of whether the first marble drawn is green or yellow, because the first marble is replaced in the bag before drawing the second marble. In other words, the first event of drawing a green or yellow marble **does not** affect the second event of drawing a yellow marble. We say that the second event is **independent** of the first event.

ATTENTION

Independent events are *not* the same as mutually exclusive events.

Let A be the event that the first marble drawn is green and B be the event that the second marble drawn is yellow.

From Worked Example 8**(iii)**, we observe that P(A and B) = $\frac{7}{10} \times \frac{3}{10}$, where $\frac{3}{10}$ is the probability that the second marble is yellow, given that the first marble is green. Since P(B) is also equal to $\frac{3}{10}$ from **(iv)**, we can write P(A and B) = P(A) × P(B) in this case.

In general, the **Multiplication Law of Probability** states that

> if A and B are **independent events**,
> P(A and B) or $P(A \cap B) = P(A) \times P(B)$.

Worked Example 9

(Probability involving Independent Events)

There are 25 boys and 15 girls in a class. 12 of the boys and 5 of the girls wear spectacles. A class monitor and a class monitress are selected at random from the 25 boys and the 15 girls, respectively. What is the probability that both the class monitor and monitress wear spectacles?

Solution:

$$\text{P(monitor wears spectacles)} = \frac{12}{25}$$

$$\text{P(monitress wears spectacles)} = \frac{5}{15} = \frac{1}{3}$$

Since the selections of the monitor and the monitress are independent,

$$\text{P(monitor and monitress wear spectacles)} = \frac{12}{25} \times \frac{1}{3} = \frac{4}{25}$$

PRACTISE NOW 9

SIMILAR QUESTIONS

Exercise 3C Questions 4, 10–11, 17

1. Workers from a company work in either the 'Administrative' Department or the 'Technical' Department. There are 18 men and 12 women in the company. 12 men and 4 women are from the 'Technical' Department. A chairman and a chairwoman are selected at random from the 18 men and the 12 women respectively. Find the probability that
   (i) both the chairman and chairwoman are from the 'Technical' Department,
   (ii) the chairman is from the 'Administrative' Department and the chairwoman is from the 'Technical' Department.

2. Michael has two laptops, Laptop X and Laptop Y. In any one year, the probability of Laptop X breaking down is 0.1 and the probability of Laptop Y breaking down is 0.35. In any one year, what is the probability that
   (i) both laptops break down,
   (ii) Laptop X breaks down but Laptop Y does not,
   (iii) exactly one of the laptops breaks down?

Dependent Events

Investigation

Dependent Events

There are 7 green marbles and 3 yellow marbles in a bag. The marbles are identical except for their colour. Two marbles are drawn at random from the bag (i.e. without any replacement).

1. Copy and complete the probabilities on the probability tree in Fig. 3.7.

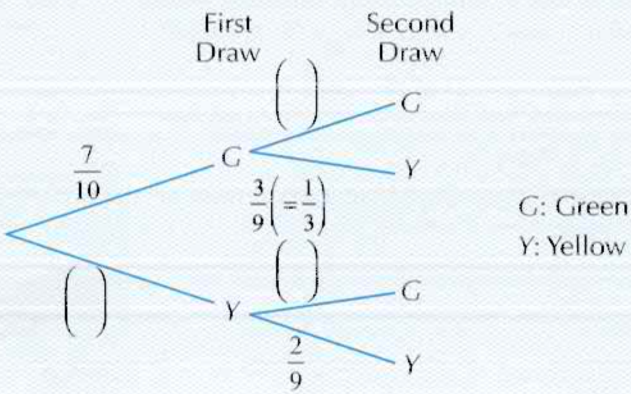


Fig. 3.7

2. Find the probability that
   (i) the second marble drawn is yellow, given that the first marble drawn is green,
   (ii) the second marble drawn is yellow, given that the first marble drawn is yellow.

3. Are the probabilities in Question 2**(i)** and **(ii)** equal? Does the probability of drawing a yellow marble in the second draw **depend** on the outcome in the first draw? Explain.

4. Find the probability that the second marble drawn is yellow. Is this probability equal to the probabilities in Question 2**(i)** and **(ii)**?

5. Let A be the event that the first marble drawn is green and B be the event that the second marble drawn is yellow.
   (i) Is event B independent or dependent on event A? Explain.
   (ii) Does the Multiplication Law of Probability, P(A and B) = P(A) × P(B) apply in this case?

From the investigation, we observe that the Multiplication Law of Probability, P(A and B) = P(A) × P(B), does not apply if A and B are **dependent** events.

However, when we are finding the probability for P(A and B) in Question 5**(ii)** in the investigation, we still multiply the probabilities across two connected branches, i.e. P(A and B) = P(GY) = $\frac{7}{10} \times \frac{3}{9} = \frac{7}{30}$, where $\frac{3}{9} \neq$ P(B). We can still multiply across two connected branches because the second probability on the branch is not the probability of the second event B.

Worked Example 10

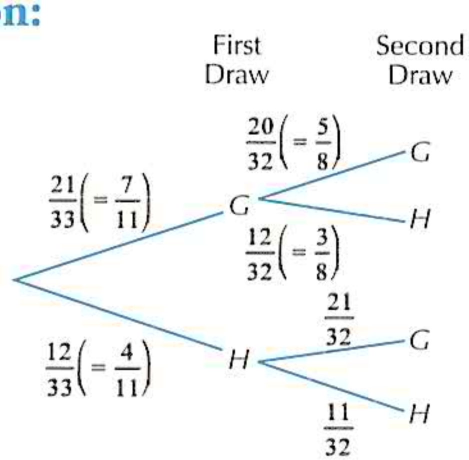
(Probability involving Dependent Events)

Out of 33 students in a class, 21 study Geography and 12 study History. No student studies both subjects. Two students are picked at random from the class. Find the probability that

(i) the first student studies History and the second student studies Geography,

(ii) one student studies History while the other student studies Geography.

Solution:



G: Geography
H: History

Problem Solving Tip

Although the question does not say that the first student is not replaced in the class before picking the second student, this is assumed.

This is because if the two students are picked together, it is the same as picking at random (same as the **no replacement case**).

(i) $\text{P}(HG) = \frac{4}{11} \times \frac{21}{32}$
$= \frac{21}{88}$

(ii) P(H and G) = P(HG) + P(GH)
$= \left(\frac{4}{11} \times \frac{21}{32}\right) + \left(\frac{7}{11} \times \frac{3}{8}\right)$
$= \frac{21}{88} + \frac{21}{88}$
$= \frac{21}{44}$

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PRACTISE NOW 10

SIMILAR QUESTIONS

Exercise 3C Questions 5–7, 12–16, 18–19

1. Mr Lim, a Science teacher, needs two students to assist him with a Science demonstration. Two students are picked at random from his class of 16 boys and 12 girls. Using a tree diagram, find the probability that
   (i) the first student is a boy and the second student is a girl,
   (ii) one student is a boy while the other student is a girl,
   (iii) at least one of the students is a girl.

2. A bag contains 8 red balls, 7 blue balls and 1 white ball. Two balls are drawn from the bag at random, one after another, without replacement. Find the probability that
   (i) the first ball is red and the second ball is blue,
   (ii) one ball is red while the other ball is blue,
   (iii) the two balls are of the same colour.

Performance Task

The mathematical constant pi, π is used to find the circumference or the area of a circle. A commonly used value of pi is 3.142. Since π is an irrational number, it has an infinite, non-repeating number of decimal digits, i.e. its value is 3.141 592....

We can approximate the value of π by **simulation modelling**, which involves the use of a computer program to generate a scenario based on a set of rules in order to study the outcomes of the interactions of the variables in the model.

For this question, we will use a spreadsheet to generate random points that fall inside a square of length 2 units, as shown in Fig. 3.8.

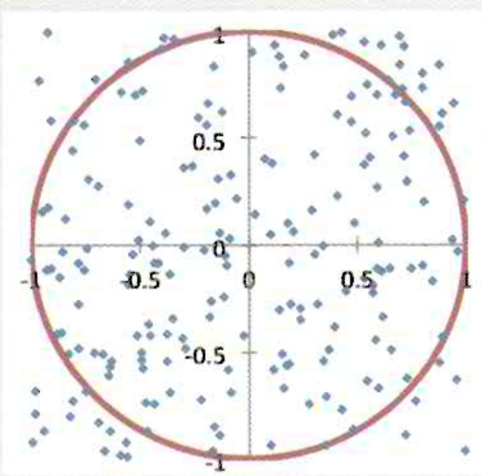


Fig. 3.8

We are interested to find out the number of points that fall inside the unit circle, i.e. a circle of radius 1 unit. This will help us to estimate the area of the unit circle.

(a) What is the area of the unit circle?

Set up the following in a spreadsheet, as shown in Fig. 3.9. The formula in each of the cells in Columns A and B, from A2 to A51 and from B2 to B51 is =2*RAND()-1. Enter the given formula in the cell A2 to generate a random number. Click on the small black square in the lower-right corner of the cell A2, hold and drag it down to the cell A51. This will generate 50 random numbers, from the cell A2 to A51. Repeat this process for Column B.

Hence, we will generate 50 points with coordinates (x, y), which fall within a square of length 2 units.

A2 | f_x =2*RAND()-1

	A	B	C	D	E	F
1	x	y	Point	Inside	Total	Area
2	-0.01729	-0.03454	1	37	50	2.96
3	-0.91536	0.919387	0			
4	-0.1197	-0.78173	1			
5	0.844383	-0.85694	0			
6	0.759291	0.059825	1			
7	0.967762	-0.25256	0			
8	-0.62087	0.620782	1			
9	0.22774	-0.53177	1			
10	-0.13354	-0.17948	1			
11	-0.52942	-0.29039	1			

Fig. 3.9

The formula for the cell C2 in Column C is =IF((A2^2+B2^2)<=1,1,0). This means that if $x^2 + y^2 \leqslant 1$ (i.e. if the point falls within the unit circle), the cell will take the value of 1. Otherwise, the cell will take the value of 0. Similarly, we click and hold the small black square in the lower-right corner of the cell C2, and drag it down to the cell C51.

The formula in D2 is =COUNTIF(C:C,"=1"). This will count the number of cells in Column C with the value of 1.

(b) What does the value in D2 tell you about the points?

The formula in E2 is =COUNTIF(C:C,">=0"). This will count all the points inside the square.

The cell F2 gives an estimate of the area of the unit circle, or an estimate of the value of π.

(c) What formula should you use in F2?

Select the cells from A2 to B51. Insert a **scatter plot with markers only**. It should look like the diagram in **(a)**, but with fewer points and without the circle.

Record the value in F2. For most spreadsheets, you can press the 'F9' button to re-generate another 50 random points. Record the new value in F2, and take the average of the two values to find the mean area of the unit circle based on 100 random points.

Continue this process until you get 1000 random points.

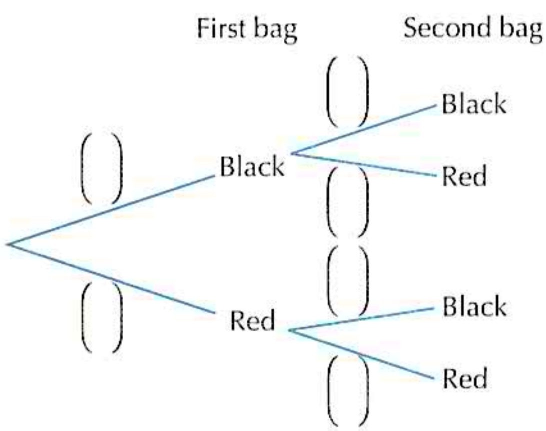
(d) What is the mean area of the unit circle based on 1000 random points? Is it close enough to the value of π?

Exercise 3C

BASIC LEVEL

1. Ethan has two bags each containing 5 black marbles and 4 red marbles. He takes one marble at random from each bag.

(a) Copy and complete the tree diagram shown below.



(b) Find the probability that Ethan draws

(i) a black marble from the first bag,

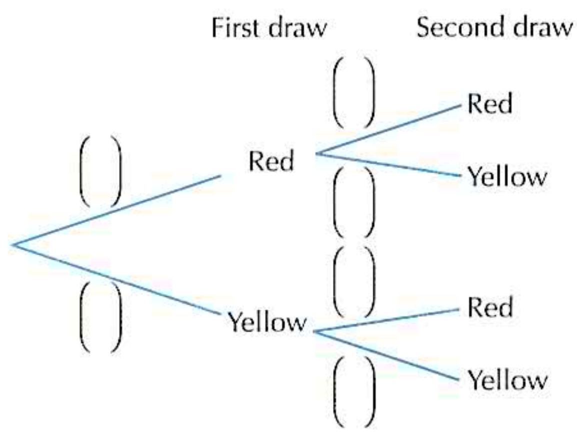
(ii) a red marble from the second bag, given that he draws a black marble from the first bag,

(iii) a black marble from the first bag and a red marble from the second bag,

(iv) a red marble from the second bag.

2. A bag contains 6 red balls and 4 yellow balls. A ball is chosen at random and then put back into the bag. The process is carried out twice.

(a) Copy and complete the tree diagram shown below.



(b) Find the probability of choosing

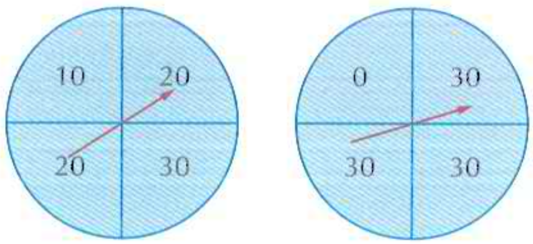
(i) two red balls,

(ii) one ball of each colour,

(iii) a yellow ball on the second draw.

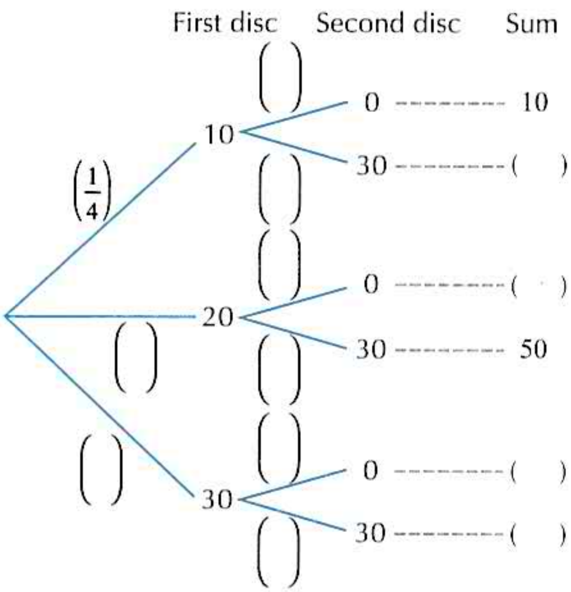
3. The diagram below shows two discs, each with four equal sectors. Each disc has a pointer which, when spun, is equally likely to come to rest in any of the four equal sectors.

In a game, the player spins each pointer once. His score is the sum of the numbers shown by the pointers.



First disc Second disc

(a) Copy and complete the tree diagram shown.



(b) With the help of the diagram, calculate the probability that

(i) the first number obtained is less than or equal to the second number obtained,

(ii) the second number obtained is zero.

(c) If the player's score is between 10 and 50 but excluding 10 and 50, he receives \$2. If his score is more than 40, he receives \$5. Otherwise, he receives nothing. What is the probability that he receives

(i) \$2, **(ii)** \$5,

(iii) \$2 or \$5, **(iv)** nothing?

4. Ethan takes either Bus A or Bus B to school every day. Bus A and B either arrive punctually or late. The probabilities of Bus A and B arriving punctually are $\frac{2}{3}$ and $\frac{7}{8}$, respectively. Find the probability that

(i) both buses are punctual,

(ii) Bus A is late while Bus B is punctual,

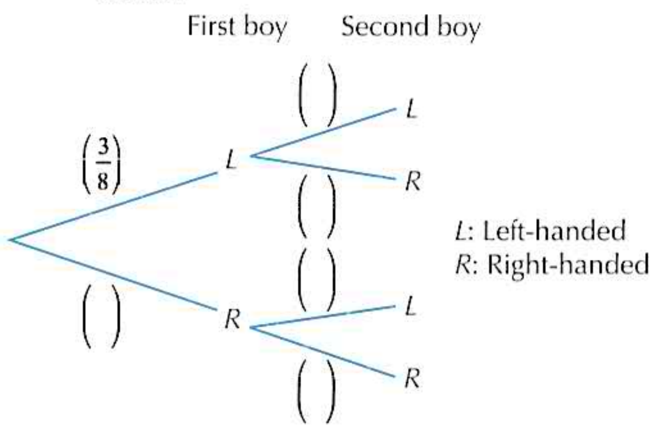
(iii) exactly one of the buses is late.

5. In a group of 8 boys, 3 are left-handed. The remaining 5 boys are right-handed. If a boy is chosen at random from the group, state the probability that the boy chosen is left-handed.

(a) A second boy is then chosen at random from the remaining 7 boys. What is the probability that the second boy chosen is also left-handed, given that the first boy chosen is left-handed?

On another occasion, 2 boys are chosen at random from the same group of 8 boys.

(b) Copy and complete the tree diagram shown below.



(c) From the tree diagram in **(b)**, find the probability that

(i) the first boy chosen is right-handed and the second boy chosen is left-handed,

(ii) both boys chosen are left-handed,

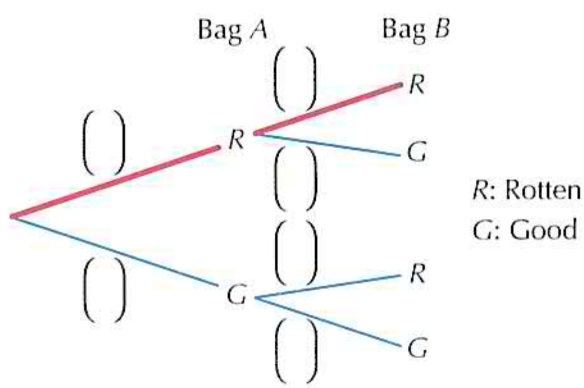
(iii) the second boy chosen is left-handed.

6. A class has 30 girls and 15 boys. Two representatives are to be selected at random from the class. Find the probability that
   (i) the first representative is a girl,
   (ii) the second representative is a girl, given that the first representative is a boy,
   (iii) the first representative is a boy and the second representative is a girl,
   (iv) a boy and a girl are selected as representatives.

7. A bag contains 6 green and 4 blue cards.
   (a) A card is drawn at random. Find the probability that it is green.
   (b) The card drawn is returned to the bag and after mixing the cards thoroughly, Shirley takes two cards at random from the bag, one after another. Using a tree diagram, calculate the probability that Shirley has taken out
      (i) two green cards,
      (ii) one card of each colour,
      (iii) at least one blue card.

INTERMEDIATE LEVEL

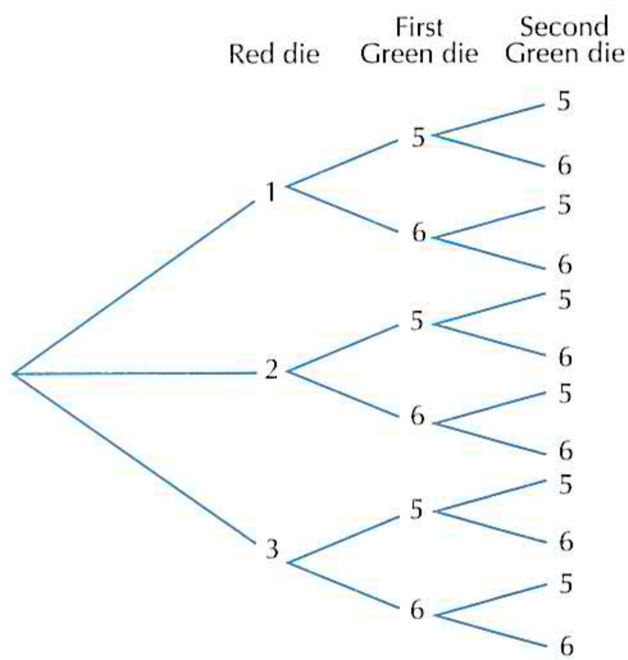
8. Bag A contains 20 potatoes, 4 of which are rotten. Bag B contains 12 potatoes, 3 of which are rotten. Jun Wei selects one potato at random from each bag.
   (a) Complete the tree diagram below to show the possible outcomes of Jun Wei's selections.



   (b) Jun Wei wants to find out the probability of selecting two rotten potatoes. He multiplies the probabilities along the 'RR' branch (highlighted in red) and he says that he is using the 'Multiplication Law of Probability'.

   Do you agree with what Jun Wei says? Explain your answer clearly.

9. A red die has the number 1 on one face, the number 2 on two faces and the number 3 on three faces. Two green dice each have the number 6 on one face and the number 5 on five faces. The three dice are rolled together.
   (a) Copy and complete the tree diagram shown below by writing the probabilities on the 'branches'.



   (b) Using the tree diagram, calculate the probability of obtaining
      (i) 2 on the red die, 5 on the first green die and 6 on the second green die,
      (ii) 3 on the red die and 6 on each of the two green dice,
      (iii) exactly two sixes,
      (iv) a sum of 12,
      (v) a sum which is divisible by 3.

10. A woman goes to the supermarket once a month for grocery shopping.

The probability that she buys a sack of rice is $\frac{4}{9}$.

Find the probability that

(i) she will not buy a sack of rice in a particular month,

(ii) she will not buy a sack of rice in two particular consecutive months,

(iii) she buys a sack of rice in just one of two particular months.

11. The table below shows the number of male and female employees working in the front office, middle office and back office of an investment bank.

Gender \ Department	Front Office	Middle Office	Back Office
Male	40	55	38
Female	36	35	52

Three representatives, one from each department, are selected at random to attend a seminar. What is the probability that

(i) all three representatives are females,

(ii) the representative from the front office is a male while the others are females,

(iii) exactly one of the representatives is a male?

12. In a wardrobe, there are 16 shirts, of which 8 are black, 6 are white and 2 are blue. The shirts are identical except for their colour.

(a) If two shirts are taken out of the wardrobe, find the probability that

(i) both are black,

(ii) one shirt is black and the other is white,

(iii) the two shirts are of the same colour.

(b) If a third shirt is taken out from the wardrobe, calculate the probability that all three shirts are black.

(c) Can we use the Multiplication Law of Probability to obtain the answer in **(b)**? Explain your answer.

13. Ten cards are marked with the letters P, R, O, P, O, R, T, I, O and N respectively. These cards are placed in a box. Two cards are drawn at random, without replacement. Calculate the probability that

(i) the first card bears the letter 'O',

(ii) the two cards bear the letters 'P' and 'O' in that order,

(iii) the two cards bear the letters 'P' and 'O' in any order,

(iv) the two cards bear the same letter.

14. Five balls numbered 1, 2, 5, 8 and 9 are put in a bag.

(a) One ball is selected at random from the bag. Write down the probability that it is numbered '8'.

(b) On another occasion, two balls are selected at random from the bag. Find the probability that

(i) the number on each ball is even,

(ii) the sum of the numbers on the balls is more than 10,

(iii) the number on each ball is not a prime number,

(iv) only one ball bears an odd number.

15. Box A contains 7 blue balls and 5 yellow balls. Box B contains 3 blue balls and 7 yellow balls. One ball is removed at random from Box A and placed into Box B. After thoroughly mixing the balls, a ball is drawn at random from Box B and placed back into Box A.

(a) Draw a tree diagram to illustrate this experiment.

(b) Find the probability that at the end of the experiment, Box A has

(i) more yellow balls than blue balls,

(ii) exactly 7 blue and 5 yellow balls,

(iii) twice as many blue balls as yellow balls.

16. Class A has 18 boys and 17 girls and Class B has 14 boys and 22 girls. A student from Class A is transferred to Class B. The teacher selects a student at random from the extended Class B. Find the probability that the student selected is

(i) the student who was initially from Class A,

(ii) a boy.

ADVANCED LEVEL

17. In any given year, the probabilities of a volcanic eruption in each of the countries A, B and C are 0.03, 0.12 and 0.3, respectively. For any given year, find the probability that

(i) volcanic eruptions will occur in all three countries,

(ii) no volcanic eruptions will occur,

(iii) there is at least one volcanic eruption,

(iv) there are exactly two volcanic eruptions.

18. A bag contains 10 red balls, 9 blue balls and 7 yellow balls. Three balls are drawn in succession without replacement. By drawing a tree diagram or otherwise, find the probability of obtaining

(i) a red and two blue balls in that order,

(ii) a red, a yellow and a blue ball in that order,

(iii) three balls of different colours.

19. A game is such that a fair die is rolled repeatedly until a '6' is obtained. Find the probability that

(a) **(i)** the game ends on the third roll,

(ii) the game ends on the fourth roll,

(iii) the game ends by the fourth roll.

(b) Suppose now that the game is such that the same die is rolled repeatedly until two '6's are obtained. Find the probability that

(i) the game ends on the third roll,

(ii) the game ends on the third roll and the sum of the scores is odd.

1.

Example of Experiment	Components of Each Outcome	Representation of Sample Space
Tossing 1 coin	1	List of outcomes in a set
Tossing 2 coins	2	Possibility diagram or tree diagram
Tossing 3 coins	3	Tree diagram

2. The **Addition Law of Probability** states that if A and B are mutually exclusive events, then P(A or B) or $P(A \cup B) = P(A) + P(B)$.

3. The **Multiplication Law of Probability** states that if A and B are independent events, then P(A and B) or $P(A \cap B) = P(A) \times P(B)$.

Review Exercise 3

1. A man throws a die and a coin. Find the probability that he will get
   (i) the number '3' followed by a head,
   (ii) an even number followed by a tail.

2. Two balanced dice are thrown together. Find the probability that they will show
   (i) the same number,
   (ii) two even numbers,
   (iii) two odd numbers,
   (iv) one odd and one even number.

3. 50 discs, numbered from 1 to 50, are placed in a bowl. One disc is picked at random. Find the probability that the number on the disc
   (i) is greater than 28,
   (ii) includes the digit '3',
   (iii) is prime,
   (iv) is divisible by 4.

4. Assuming that the birthdays of people are equally likely to occur in any month, find the probability that
   (i) two people selected at random are born in the same month,
   (ii) three people selected at random are not born in the same month,
   (iii) four people selected at random are born in the same month.

5. On any day, the probability that Huixian will miss her bus is $\frac{1}{7}$. Find the probability that
   (i) she will catch her bus on a particular day,
   (ii) she will miss her bus on two particular consecutive days,
   (iii) she will miss her bus on just one of two particular consecutive days,
   (iv) she will catch her bus on three particular consecutive days.

6. The probabilities of Rui Feng, Michael and Khairul winning the gold medal for the 100-metre freestyle swimming competition are $\frac{1}{2}$, $\frac{1}{6}$ and $\frac{1}{8}$, respectively.
   Find the probability that
   (i) one of them wins the gold medal,
   (ii) none of them wins the gold medal,
   (iii) Rui Feng fails to win the gold medal.

7. Six discs, with the numbers 1 to 6 written on each of them, are placed in a bag. Two discs are drawn at random from the bag and placed side by side to form a two-digit number. By drawing a possibility diagram, find the probability that the number formed is
   (i) divisible by 2,
   (ii) divisible by 5,
   (iii) a prime number,
   (iv) a perfect square.

8. The letters of the word 'FOLLOW' are written on six individual cards. The cards are placed face down on the table and their positions are rearranged randomly. The cards are turned over one at a time. For each of the following cases, find the probability that

   (i) the first two cards turned over will each have the letter 'O' written on them,

   (ii) the second card turned over will have the letter 'F' written on it,

   (iii) the word 'FOLLOW' is obtained, in that order.

9. A box contains x white chocolates and y dark chocolates. Khairul selects a chocolate from the box, followed by Priya who also selects a chocolate. Find, in terms of x and y, the probability that

   (i) Khairul selects a dark chocolate,

   (ii) Khairul selects a white chocolate while Priya selects a dark chocolate,

   (iii) the chocolates selected by them are of different types.

10. In a city, the probability that it will rain on any particular day is $\frac{1}{4}$. The probability of a traffic jam is $\frac{2}{5}$ when it rains and $\frac{1}{5}$ when it does not rain. What is the probability that there will be a traffic jam in the city on a particular day?

11. In a class of 30 students, 20 are boys and 10 are girls. Of the 10 girls, 6 travel to school by bus and 4 travel by car.

   (a) If two students are selected at random, calculate the probability that

      (i) one is a girl and one is a boy,

      (ii) no girls are selected.

   (b) If two of the 10 girls are selected at random, calculate the probability that

      (i) both travel to school by bus,

      (ii) both travel to school by different means of transportation,

      (iii) at least one travels to school by bus.

12. A weather forecast station describes the weather for the day as either fine or wet. If the weather is fine today, the probability that it will be fine the next day is 0.8. If the weather is wet today, the probability that it will be wet the next day is 0.6.

   Given that Monday is wet, find the probability that

   (i) the next two days will also be wet,

   (ii) Tuesday will be wet and Wednesday will be fine,

   (iii) there will be one fine day and one wet day for the next two days,

   (iv) two of the next three days will be wet.

13. A bowl of sweets contains 2 fruit gums, 3 mints and 5 toffees. Three sweets are to be chosen at random, without replacement, from the bowl. Calculate the probability that

   (i) the first two sweets chosen will be different,

   (ii) the three sweets chosen will be the same,

   (iii) of the three sweets chosen, the first two will be the same and the third will be a toffee.

14. Three airplanes are scheduled to land at either Terminal 1 or Terminal 2 of the airport. The probabilities of each airplane landing at Terminal 1 are $\frac{3}{4}$, $\frac{2}{3}$ and $\frac{5}{6}$ respectively. Find the probability that

   (i) all three airplanes land at Terminal 2,

   (ii) exactly two airplanes land at Terminal 1,

   (iii) exactly one airplane lands at Terminal 1.

15. A bag contains 5 red, 7 yellow and 1 white disc. Two discs are taken out in succession without replacement. By drawing a tree diagram or otherwise, find the probability of getting

(i) two red discs,

(ii) a red and a yellow disc in that order,

(iii) two white discs,

(iv) two discs of different colours.

16. Three national servicemen Raj, Rui Feng and Farhan took part in a rifle shooting competition. The probabilities that Raj, Rui Feng and Farhan will each hit the target are $\frac{2}{3}$, $\frac{3}{5}$ and $\frac{4}{7}$, respectively.

(a) Three of them fire one shot each simultaneously at the target. Find the probability that

(i) all three men hit the target,

(ii) all three men miss the target,

(iii) exactly two of them hit the target,

(iv) at least one man hits the target.

(b) In a game, Raj, Rui Feng and Farhan fire one shot each at the target, in that order. Once the target is hit, the game ends and the winner is the one who hits the target first. Find the probability that

(i) the game ends after two shots,

(ii) the game ends after three shots,

(iii) the game ends by the third shot.

Challenge Yourself

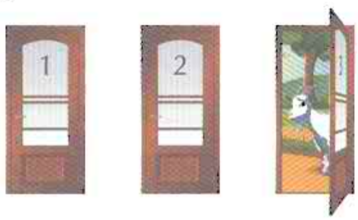
1. Ethan rolls two identical fair six-sided dice but does not know the result. He was told that one of the scores is a '3'. Given this information, what is the probability that both of the scores are '3's?

   Hint: Analyse the possible outcomes using a possibility diagram.

2. During a game show, the host picks you to take part in a contest and you are given the following scenario.

   There are three closed doors and you are invited to pick one of them. There is a brand new car behind one of the doors, and a goat behind each of the other two doors. *The host knows what is behind the doors.*

   Suppose that you pick Door 1 and the host opens Door 3 because he knows that the car is not behind it, as shown below.



   You are then given the option to switch to Door 2. Should you **switch doors** to increase your probability of winning the car?
   Hint: This is counter-intuitive. Search on the Internet for an explanation to this famous probability puzzle and compare it with your own reasoning.

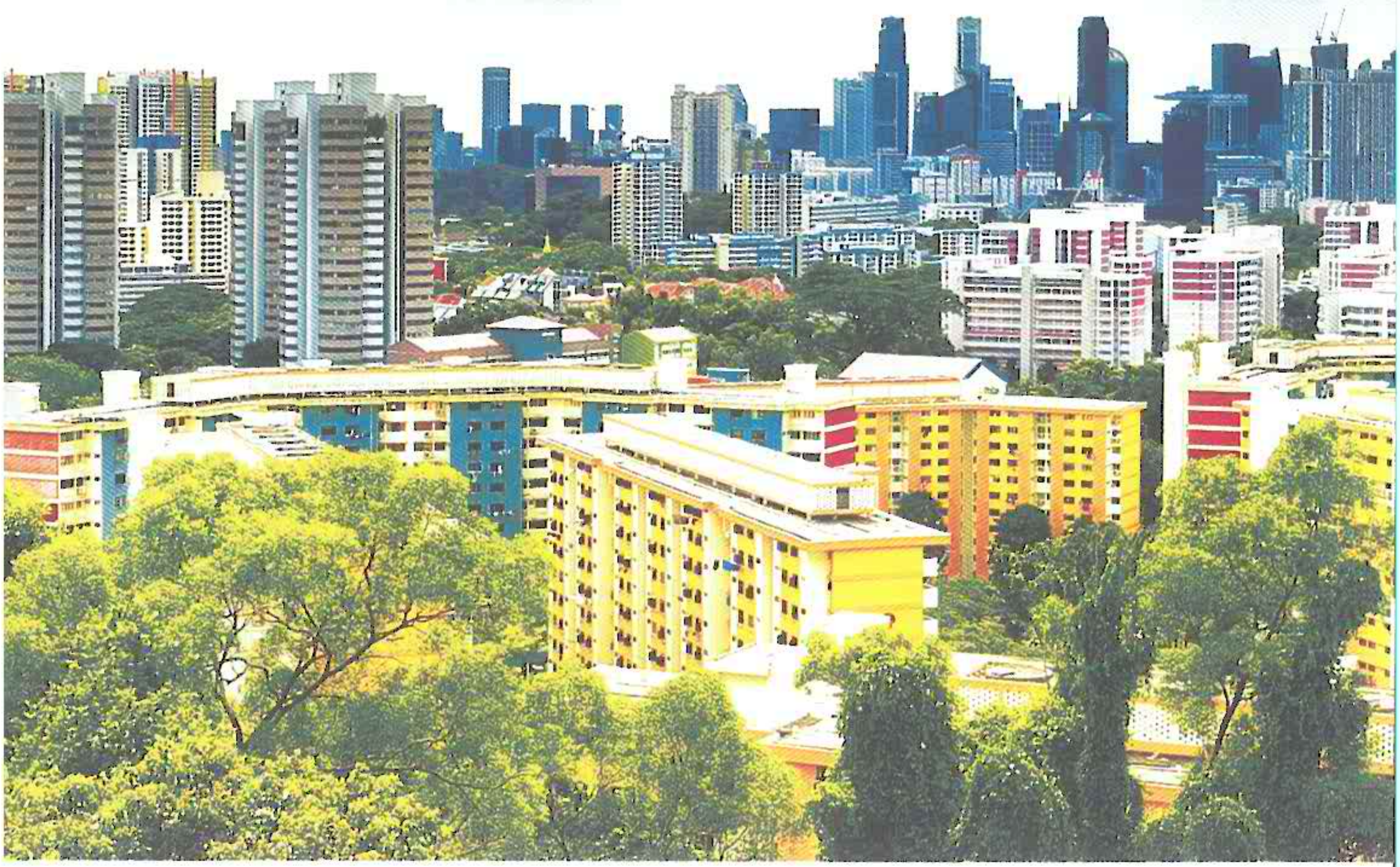
3. **(a)** There are 367 students in the school hall. What is the probability that at least two of the students have their birthday falling on the same day of the year (i.e. they do not have to be born in the same year)?

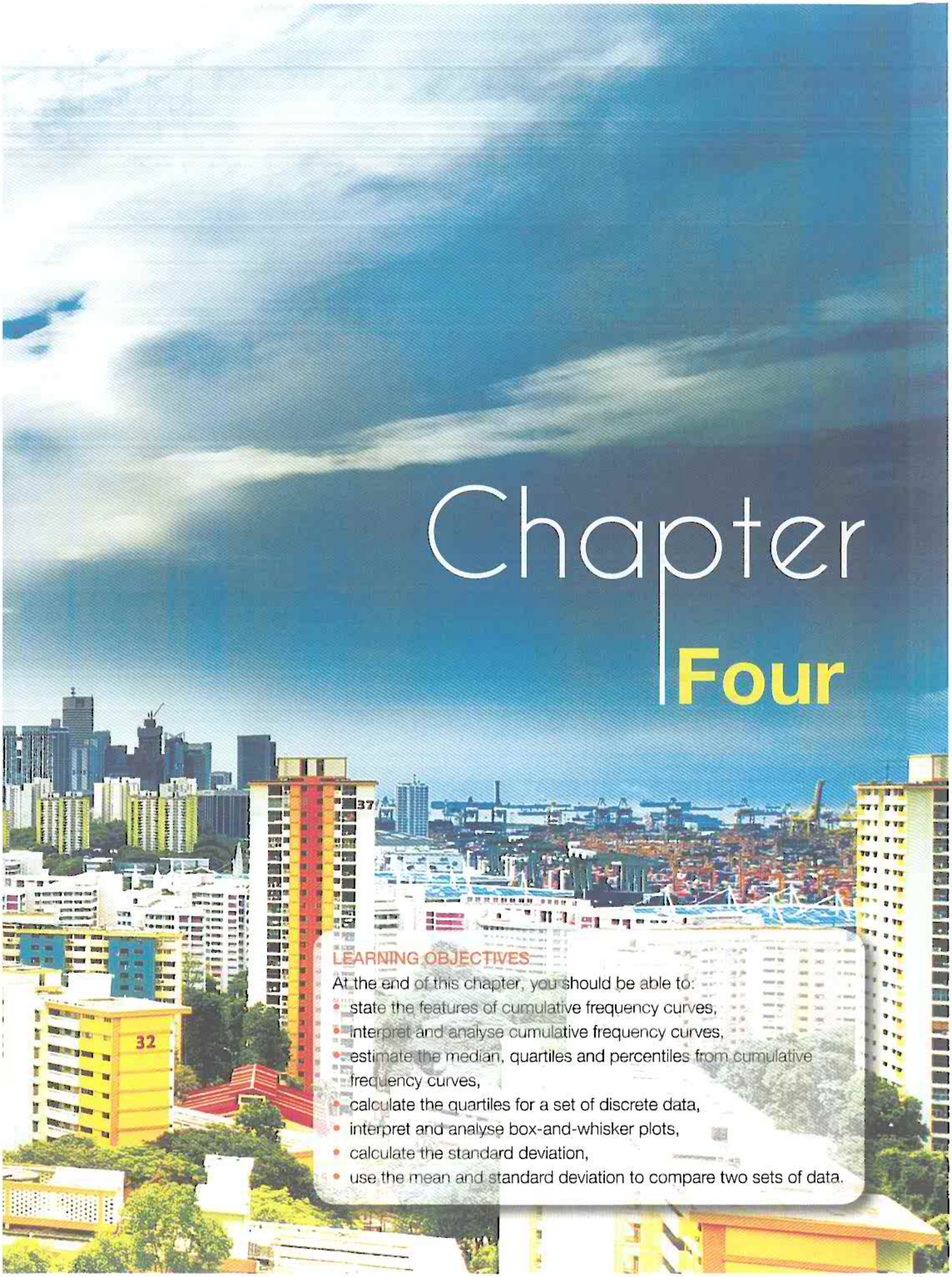
   (b) There are 40 students in the classroom. What is the probability that at least two of the students have their birthday falling on the same day of the year? Is the probability very high?

   (c) What is the least number of students in a classroom for the probability that at least two of them have their birthday falling on the same day of the year to be greater than 0.5?

Statistical Data Analysis

In Singapore, the daily mean maximum temperature was 31.2°C in 2012 and 31.3°C in 2013. Based on these statistics, can we conclude that the temperatures for both years were about the same? To have a better understanding of the temperatures for both years, we should also know how the temperatures were spread throughout the year. In this chapter, we will learn how to measure the spread of a set of data using statistics such as the interquartile range and standard deviation.





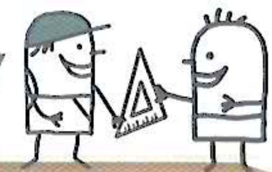
Chapter Four

LEARNING OBJECTIVES

At the end of this chapter, you should be able to:

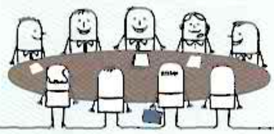
- state the features of cumulative frequency curves,
- interpret and analyse cumulative frequency curves,
- estimate the median, quartiles and percentiles from cumulative frequency curves,
- calculate the quartiles for a set of discrete data,
- interpret and analyse box-and-whisker plots,
- calculate the standard deviation,
- use the mean and standard deviation to compare two sets of data.

4.1 Cumulative Frequency Table and Curve



Cumulative Frequency Table

In this section, we will learn how to present a set of data by constructing a **table of cumulative frequencies**.



Class Discussion

Constructing a Table of Cumulative Frequencies

Discuss in pairs.

Table 4.1 **(a)** shows the frequency table for the number of hours spent surfing the Internet by 40 students on a particular day, while Table 4.1 **(b)** shows the corresponding **table of cumulative frequencies** (or cumulative frequency table). To find the cumulative frequency for a particular hour k, we must *add up the frequencies* which are less than or equal to k, i.e. $t \leqslant k$.

For example, the cumulative frequency for 4 hours, i.e. $t \leqslant 4$ is $3 + 5 = 8$.

Number of Hours Spent Surfing the Internet by 40 Students

Number of hours, t	Frequency
$0 \leqslant t \leqslant 2$	3
$2 < t \leqslant 4$	5
$4 < t \leqslant 6$	16
$6 < t \leqslant 8$	12
$8 < t \leqslant 10$	4

(a)

Number of hours, t	Cumulative Frequency
$t \leqslant 2$	3
$t \leqslant 4$	$3 + 5 = 8$

(b)

Table 4.1

1. Using the information from Table 4.1 **(a)**, copy and complete Table 4.1 **(b)**.
2. Using your answers in Table 4.1 **(b)**, find the number of students who surf the Internet for
   (i) 6 hours or less,
   (ii) more than 8 hours,
   (iii) more than 4 hours but not more than 10 hours.
3. What does the last entry under 'Cumulative Frequency' of Table 4.1 **(b)** represent? Explain your answer.

From the class discussion, we have learnt that the cumulative frequency for a particular value can be obtained by *adding up* the frequencies which are *less than or equal* to that value. In other words, the cumulative frequency is a 'running total' of frequencies. The cumulative frequency table allows us to gather information such as the number of students whose score is below a passing mark or the number of animal species shorter than or equal to a certain length, as shown below.

PRACTISE NOW

SIMILAR QUESTIONS

Exercise 4A Questions 1–2, 8

The lengths of 40 insects of a certain species were measured, to the nearest millimetre. The frequency distribution is given in the table below.

Length (x mm)	Frequency
$25 < x \leqslant 30$	1
$30 < x \leqslant 35$	3
$35 < x \leqslant 40$	6
$40 < x \leqslant 45$	12
$45 < x \leqslant 50$	10
$50 < x \leqslant 55$	6
$55 < x \leqslant 60$	2

(a) Using the table given, construct a cumulative frequency table.

(b) Using the cumulative frequency table which you have constructed, find the number of insects which are

(i) 50 mm or less in length,

(ii) more than 45 mm in length,

(iii) more than 35 mm but less than or equal to 50 mm in length.

Cumulative Frequency Curve

In this section, we will learn how to draw and interpret a **cumulative frequency curve.**

In Worked Example 1, we will make use of the *cumulative frequencies* from Table 4.1**(b)** to learn how to draw a cumulative frequency curve.

Worked Example 1

(Drawing and Interpretation of a Cumulative Frequency Curve)

The table below shows the cumulative frequencies for the number of hours (t) spent by 40 students surfing the Internet, on a particular day.

Number of hours, t	$t \leqslant 2$	$t \leqslant 4$	$t \leqslant 6$	$t \leqslant 8$	$t \leqslant 10$
Cumulative Frequency	3	8	24	36	40

(a) Using a scale of 1 cm to represent 1 hour on the horizontal axis and 1 cm to represent 5 students on the vertical axis, draw a cumulative frequency curve for the data given in the table.

(b) Using the cumulative frequency curve, estimate

(i) the number of students who surf the Internet for 5 hours or less,

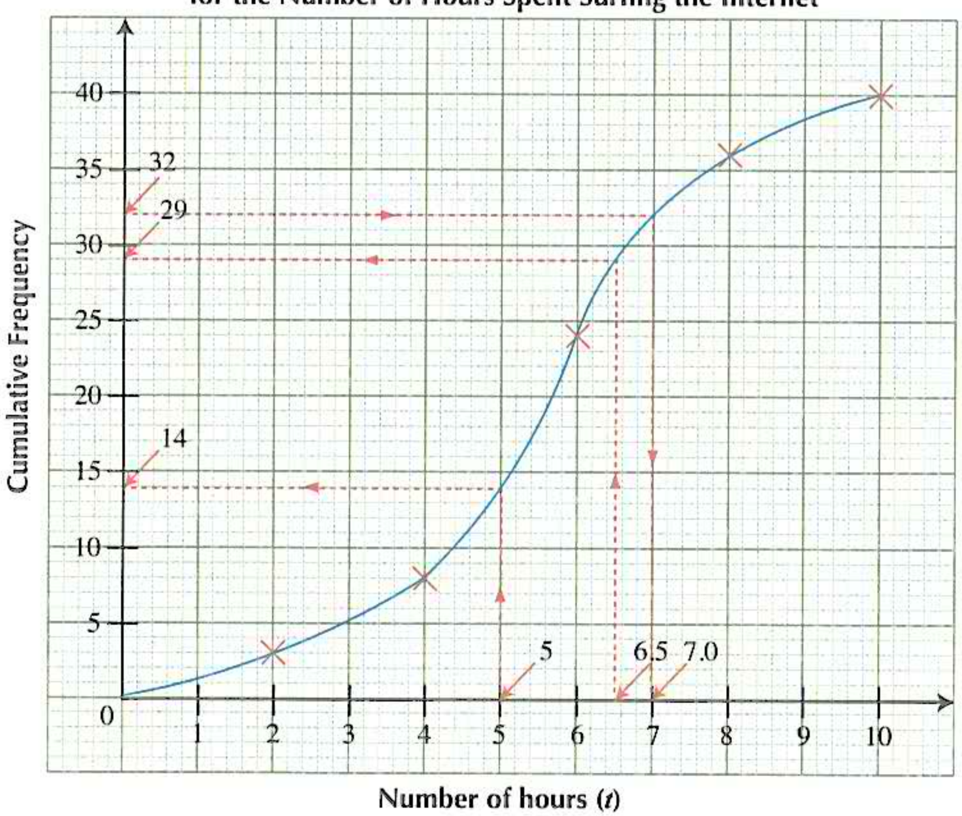
(ii) the percentage of students who surf the Internet for more than 6.5 hours,

(iii) the value of t, such that 80% of the students surf the Internet for t hours or less.

Solution:

(a)

Cumulative Frequency Curve for the Number of Hours Spent Surfing the Internet



Problem Solving Tip

To plot a cumulative frequency curve:

Step 1: Label the vertical axis, i.e. 'Cumulative Frequency'.

Step 2: Label the horizontal axis, i.e. 'Number of hours (t)'.

Step 3: Plot the points on the graph paper, i.e. (2, 3), (4, 8), (6, 24), (8, 36) and (10, 40).

Step 4: Join all the points with a smooth curve.

ATTENTION

The reading of '14' indicates that 14 students surf the Internet for **less than or equal** to 5 hours.

(b) (i) From the curve, the number of students who surf the Internet for 5 hours or less is 14.

(ii) From the curve, the number of students who surf the Internet for 6.5 hours or less is 29.

$\therefore$ $40 - 29 = 11$ students surf the Internet for more than 6.5 hours.

$\therefore$ The percentage of students who surf the Internet for more than 6.5 hours is $\frac{11}{40} \times 100\% = 27.5\%$.

(iii) 80% of the students means $\frac{80}{100} \times 40 = 32$, i.e. 32 students surf the Internet for t hours or less.

From the curve, $t = 7.0$.

ATTENTION

The answer can only be accurate up to half of a small square grid.

PRACTISE NOW 1

SIMILAR QUESTIONS

Exercise 4A Questions 3–5, 9

The table below shows the amount of milk (in litres) produced by each of the 70 cows of a dairy farm, on a particular day.

Amount of milk (x litres)	Number of cows
$0 < x \leqslant 4$	7
$4 < x \leqslant 6$	11
$6 < x \leqslant 8$	17
$8 < x \leqslant 10$	20
$10 < x \leqslant 12$	10
$12 < x \leqslant 14$	5

(a) Copy and complete the following cumulative distribution table for the data given.

Amount of milk (x litres)	Number of cows
$x \leqslant 4$	7
$x \leqslant 6$	18
$x \leqslant 8$	
$x \leqslant 10$	
$x \leqslant 12$	
$x \leqslant 14$	

(b) Using a scale of 1 cm to represent 1 litre on the horizontal axis and 1 cm to represent 5 cows on the vertical axis, draw a cumulative frequency curve for the data given.

(c) Using the curve in **(b)**, estimate

(i) the number of cows that produce less than or equal to 9.4 litres of milk,

(ii) the fraction of the 70 cows that produce more than 7.4 litres of milk,

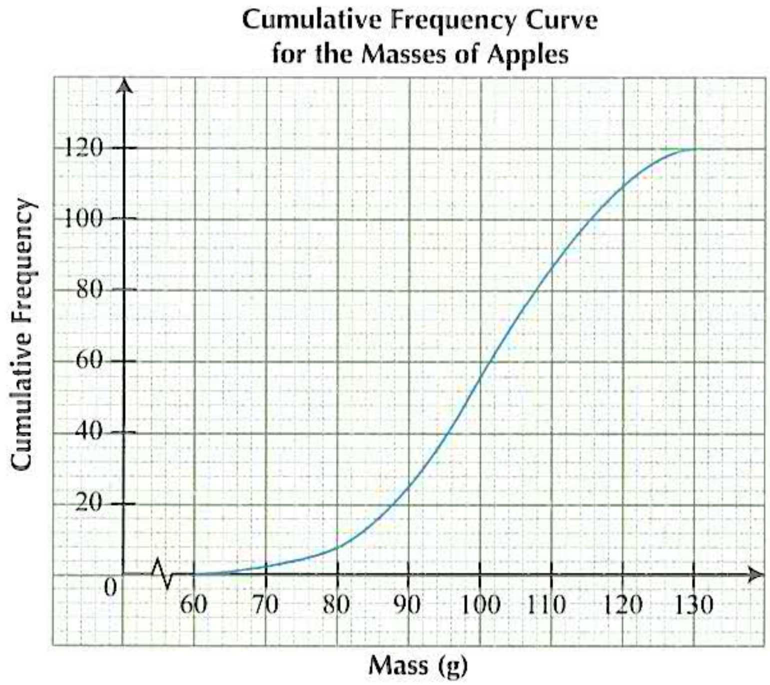
(iii) the value of x, if 70% of the cows produce more than x litres of milk.

In Worked Example 1, the upper-end points of the cumulative frequency classes are given as 'less than or equal to', i.e. $t \leqslant k$. The cumulative frequencies can also be computed by having the upper-end points as 'less than', i.e. $t < k$. In Worked Example 2, a *'less-than' cumulative frequency curve* is used.

Worked Example 2

(Interpretation of a 'Less-than' Cumulative Frequency Curve)

The 'less-than' cumulative frequency curve shows the distribution of the masses (g) of 120 apples.



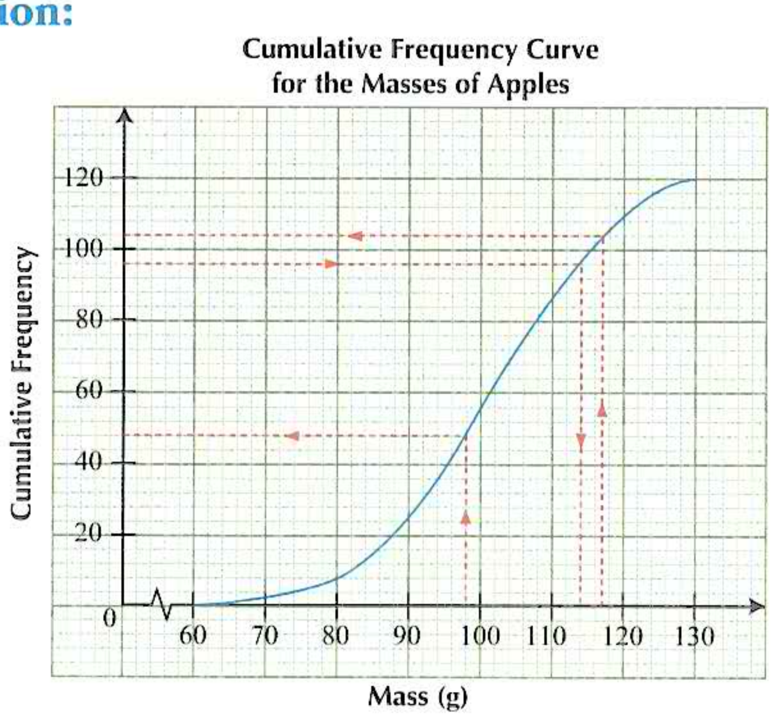
Estimate from the curve

(i) the number of apples having masses less than 98 g,

(ii) the fraction of the total number of apples having masses 117 g or more,

(iii) the value of k, given that 20% of the apples have masses k g or more.

Solution:



(i) From the curve, it is estimated that 48 apples have masses less than 98 g.

(ii) From the curve, 104 apples have masses less than 117 g.

$\therefore$ 120 – 104 = 16 apples have masses 117 g or more.

$\therefore$ The required fraction is $\frac{16}{120} = \frac{2}{15}$.

(iii) 20% of $120 = \frac{20}{100} \times 120$
$= 24$

$\therefore$ 24 apples have masses k g or more, i.e. 120 – 24 = 96 apples have masses less than k g.

From the curve, 96 apples have masses less than 114 g.

$\therefore k = 114$

ATTENTION

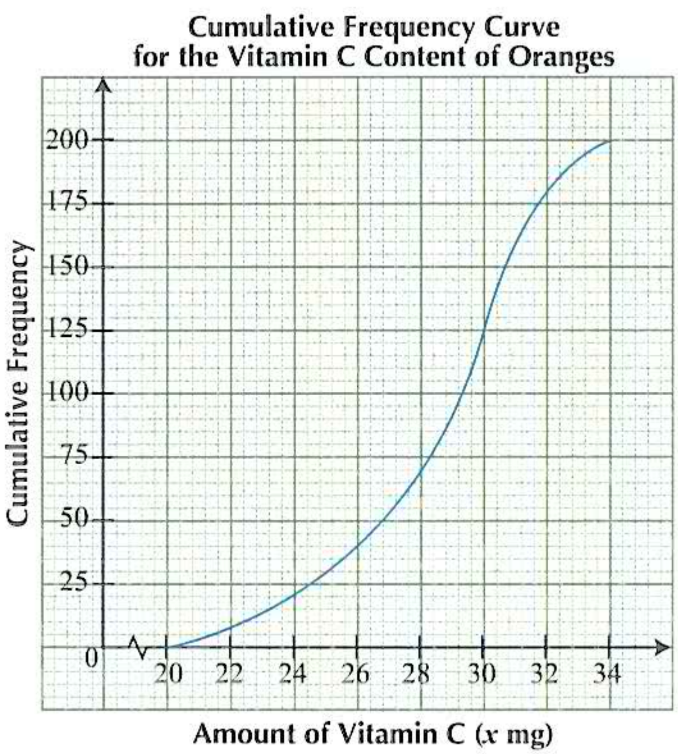
For **(i)**, the reading of 48 indicates that 48 apples have masses **less than** 98 g.

PRACTISE NOW 2

SIMILAR QUESTIONS

Exercise 4A Questions 6–7, 10

The Vitamin C content of 200 oranges is measured. The cumulative frequency curve below shows the Vitamin C content, x mg, and the number of oranges having Vitamin C content less than x mg.



Use the curve to estimate

(i) the number of oranges having Vitamin C content less than 32 mg,

(ii) the fraction of the total number of oranges having Vitamin C content of 26 mg or more,

(iii) the value of p, given that 40% of the oranges have Vitamin C content of p mg or more.

Exercise 4A

BASIC LEVEL

1. 120 students took a Mathematics examination and their results are shown in the table below.

Marks (m)	Number of students
$0 < m \leqslant 10$	3
$10 < m \leqslant 20$	12
$20 < m \leqslant 30$	9
$30 < m \leqslant 40$	11
$40 < m \leqslant 50$	17
$50 < m \leqslant 60$	19
$60 < m \leqslant 70$	20
$70 < m \leqslant 80$	14
$80 < m \leqslant 90$	10
$90 < m \leqslant 100$	5

(a) Construct a table of cumulative frequencies for the given data.

(b) Using the table in **(a)**, find the number of students who

(i) scored less than or equal to 30 marks,

(ii) scored more than 80 marks,

(iii) scored more than 40 marks but not more than 90 marks.

2. 230 students took part in a physical fitness test and were required to do pull-ups. The number of pull-ups done by the students is shown in the frequency table below.

Number of Pull-ups (x)	Frequency
$0 \leqslant x < 6$	69
$6 \leqslant x < 8$	63
$8 \leqslant x < 10$	28
$10 \leqslant x < 12$	24
$12 \leqslant x < 16$	19
$16 \leqslant x < 20$	14
$20 \leqslant x < 25$	13

(a) Copy and complete the following cumulative frequency table.

Number of Pull-ups (x)	Cumulative Frequency
$x < 6$	69
$x < 8$	
$x < 10$	
$x < 12$	
$x < 16$	
$x < 20$	
$x < 25$	

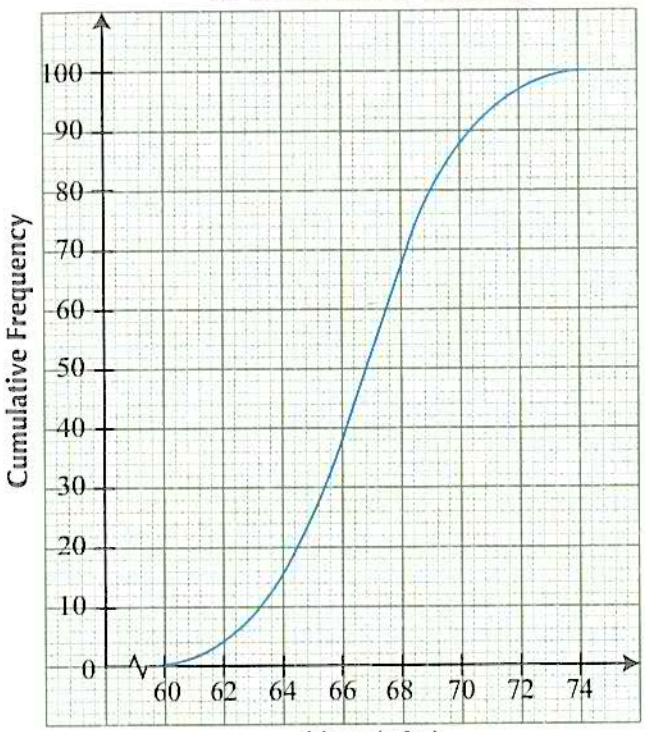
(b) Students have to do at least 12 pull-ups to qualify for the Gold Award, 8 pull-ups to qualify for the Silver Award and 6 pull-ups for the Bronze Award. Using the table in **(a)**, find the number of students who achieved for the

(i) Gold Award, **(ii)** Silver Award,

(iii) Bronze Award.

3. The masses, in kg, of 100 students are measured. The cumulative frequency curve shows the mass, x kg, and the number of students with masses less than or equal to x kg.

Cumulative Frequency Curve for the Masses of Students



Use the curve to estimate

(i) the number of students whose masses are less than or equal to 65 kg,

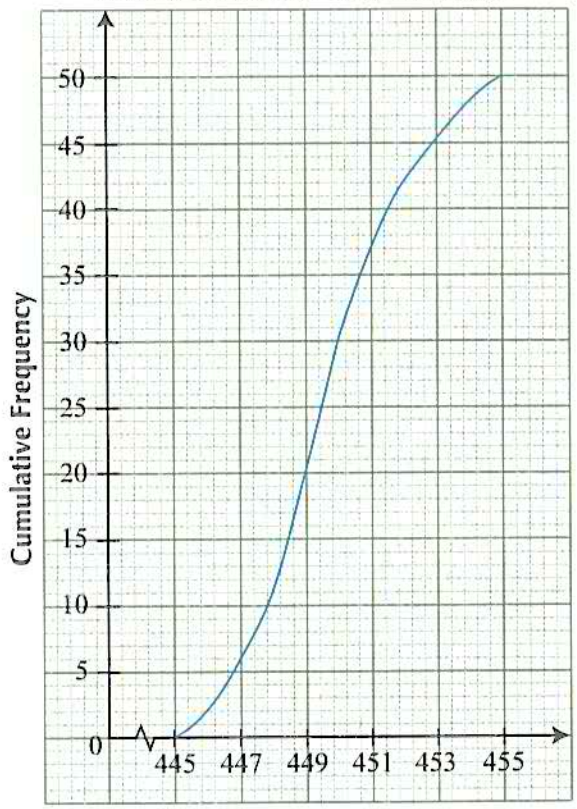
(ii) the number of students whose masses are more than 68.6 kg,

(iii) the percentage of the total number of students whose masses are more than 64.4 kg.

INTERMEDIATE LEVEL

4. The masses of 50 loaves of bread from a bakery are measured. The cumulative frequency curve below shows the mass y g, and the number of loaves of bread which are less than or equal to y g.

Cumulative Frequency Curve for the Masses of Loaves of Bread



From the graph, estimate

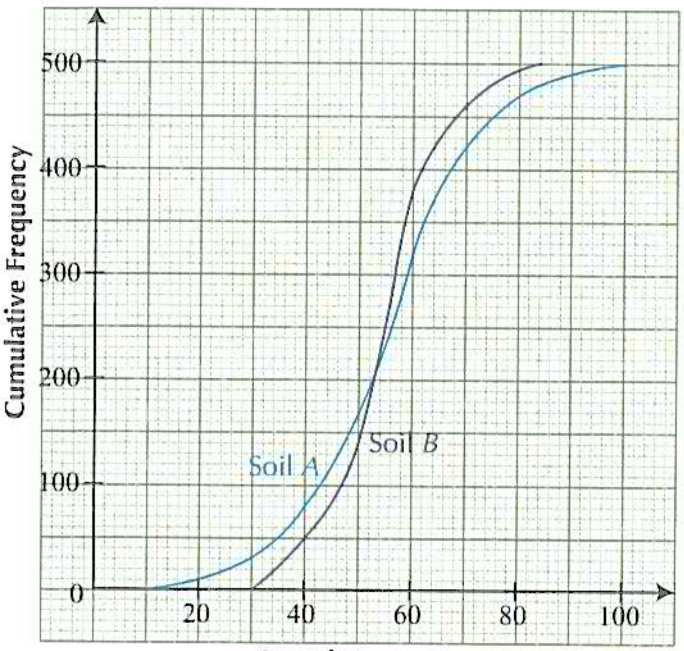
(i) the number of loaves of bread having masses less than or equal to 450.4 g,

(ii) the number of loaves of bread which are rejected because they are underweight or overweight, given that a loaf is underweight if its mass is 446.2 g or less, and overweight if its mass is more than 453.6 g,

(iii) the value of x, if $\frac{3}{10}$ of the loaves of bread have masses more than x g.

5. 500 earthworms were collected from a sample of Soil A and 500 earthworms from Soil B, and their lengths were measured. The cumulative frequency curve below shows the length x mm, and the number of earthworms which have lengths less than or equal to x mm.

Cumulative Frequency Curve for the Lengths of Earthworms



(a) For both Soil A and Soil B, use the graphs to estimate

(i) the number of earthworms having lengths less than or equal to 46 mm,

(ii) the percentage of earthworms having lengths greater than 76 mm,

(iii) the value of a, if 18% of the earthworms have lengths a mm or less.

(b) Which soil produced the longest earthworm among the 1000 earthworms?

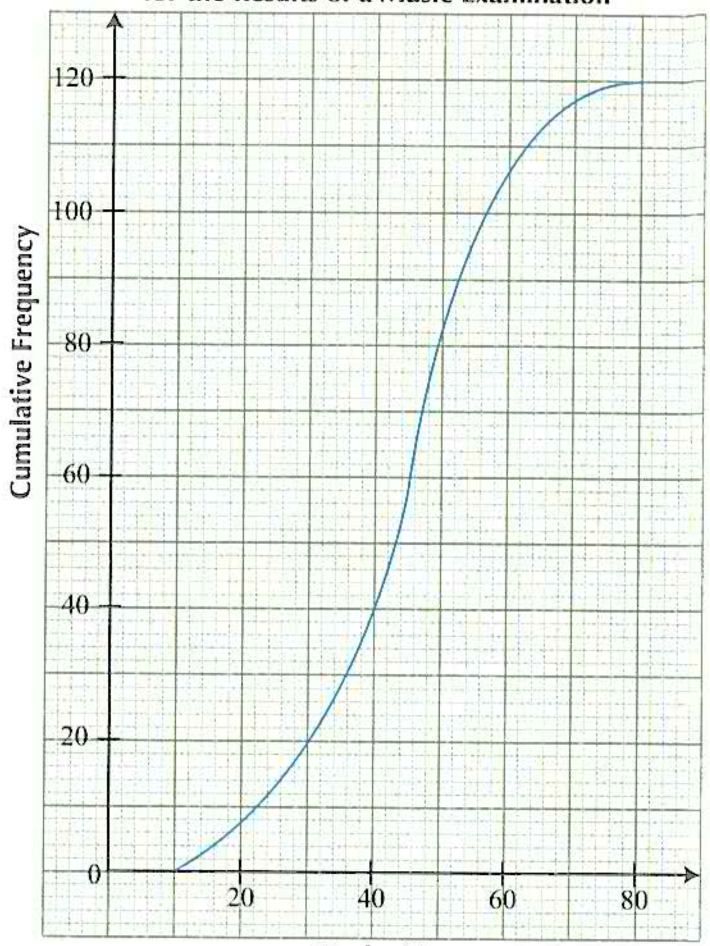
(c) Earthworms which grew more than 60 mm are said to be 'satisfactory'. From the graph, estimate the percentage of 'satisfactory' earthworms from

(i) Soil A,

(ii) Soil B.

6. 120 students took a music examination. The cumulative frequency curve below shows the results (x marks) and the number of students who obtained less than x marks. The highest possible mark is 80.

Cumulative Frequency Curve for the Results of a Music Examination



From the graph, estimate

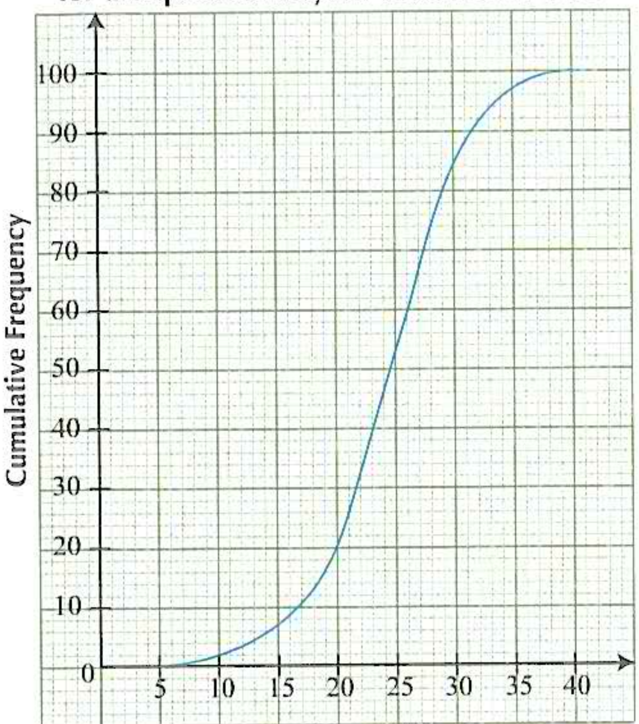
(i) the number of students who scored less than 45 marks,

(ii) the fraction of the total number of students who failed the music examination, given that 34 is the lowest mark to pass the examination,

(iii) the value of a, if 27.5% of the students obtained at least a marks in the music examination.

7. The speeds of 100 bicycles on a cycling lane at East Coast Park are recorded. The cumulative frequency curve below shows the speed u km/h, and the number of bicycles which travelled at a speed less than u km/h.

Cumulative Frequency Curve for the Speeds of Bicycles at East Coast Park



Use the curve to estimate

(i) the number of bicycles that travelled at a speed less than 18 km/h,

(ii) the fraction of the total number of bicycles that travelled at a speed greater than or equal to 29 km/h,

(iii) the value of v, if 40% of the bicycles have a speed less than v km/h.

8. The table below shows the cumulative frequencies for the annual income of 200 households in a certain district.

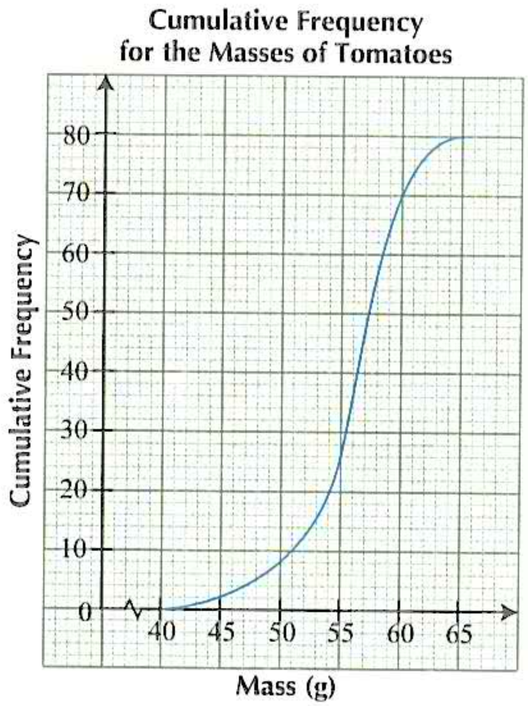
Annual Income ($\$x$, in thousands)	Cumulative Frequency
$x = 0$	0
$x < 20$	41
$x < 40$	78
$x < 60$	99
$x < 80$	118
$x < 100$	164
$x < 120$	200

(i) **On a sheet of graph paper**, draw a histogram to represent the frequency distribution.

(ii) If a household is selected at random from this district, what is the range of annual income the household is most likely to earn?

ADVANCED LEVEL

9. The cumulative frequency curve shows the distribution of the masses (in grams) of 80 tomatoes produced at a nursery.



(a) (i) Tomatoes with masses more than 56 g are rated as grade A tomatoes. Find the percentage of grade A tomatoes.

(ii) Estimate the value of y if 15% of the tomatoes are y g or less. These are rated as grade C tomatoes.

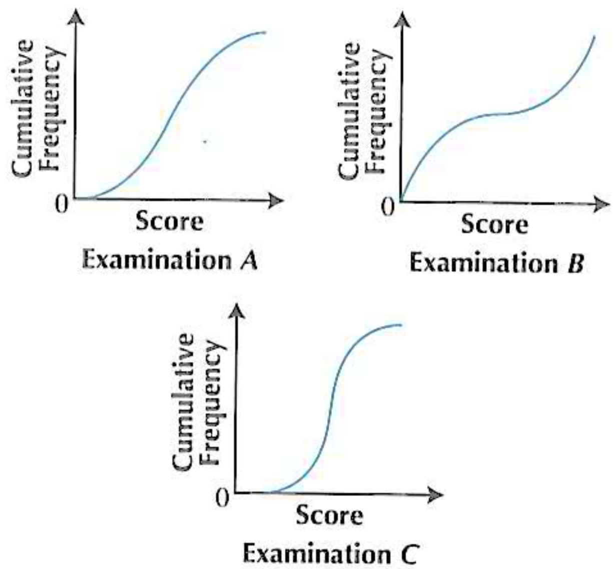
(iii) Find the number of grade B tomatoes which are between grades A and C.

(b) (i) From the curve, the data are transformed into the frequency distribution table below. Copy and complete the table.

Masses (x g)	Frequency
$40 < x \leq 45$	
$45 < x \leq 50$	
$50 < x \leq 55$	
$55 < x \leq 60$	
$60 < x \leq 65$	

(ii) Using the table, find an estimate of the mean mass of a tomato produced at the nursery.

10. The cumulative frequency curves for the results of three different Mathematics examinations are shown below. All three examinations are attempted by the same 1000 students.



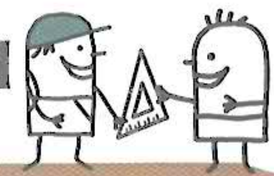
Out of the three Mathematics examinations, explain clearly which one is likely to be

(i) the most challenging,

(ii) the least challenging.

Explain your answers.

4.2 Median, Quartiles, Percentiles, Range and Interquartile Range



In Book 2, we have learnt how to find the **median** of a set of data. The median is a measure of the average and is the 'middle value' when the data are arranged in an ascending order. In this section, we will learn how to find the quartiles, range and interquartile range for both **discrete** and **continuous** data.

Discrete Data

Discrete data refers to a set of data which only takes on *distinct values*. For example, a data set showing the number of phone calls received in a day can only take on distinct values such as 1, 5, 12, etc., but not 1.5, $4\frac{2}{3}$, etc.

Consider the following set of distinct data arranged in **ascending order**:

Set A: 2, 5, 6, 7, 8, 12, 14, 16, 20, 21, 30

The total number of data values is 11, i.e. $n = 11$.

In Book 2, we have learnt that for a set of discrete data, the **median** is the value of the data in the middle position, i.e. the 6th position in the case for Fig. 4.1.

2 5 6 7 8 12 14 16 20 21 30

↑ **median** (12)

Fig. 4.1

From Fig. 4.1, we see that the median 12 divides the data in 2 equal halves, with 5 values on each side of the median.

In Fig. 4.2, we consider the 5 values on the left of the median. The middle value of these 5 values is 6 and it is called the **lower quartile** or the **first quartile Q_1**.

$\underbrace{2 \quad 5 \quad 6 \quad 7 \quad 8}_{\text{lower half}} \quad 12 \quad 14 \quad 16 \quad 20 \quad 21 \quad 30$

lower quartile Q_1 (6) **median** Q_2 (12)

Fig. 4.2

The first quartile can be considered as the 'first-quarter value'. 25% (or one quarter) of the data is less than or equal to this value.

Since the median is the middle value or 'second-quarter value', the **median** is also called the **second quartile Q_2**. 50% (or half) of the data is less than or equal to this value.

Similarly, in Fig. 4.3, we consider the 5 values on the right of the median. The middle value of the 5 values is 20, and it is called the **upper quartile** or the **third quartile Q_3**. 75% (or three quarters) of the data is less than or equal to this value.

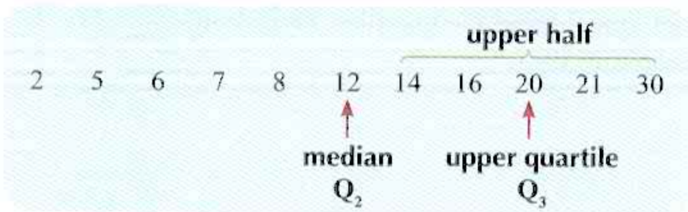


Fig. 4.3

From Fig. 4.2 and 4.3, we see that the **quartiles** obtained by the above method divide the data which is arranged in ascending order into 4 roughly equal parts.

Now that we have learnt how to find the median and quartiles for a given set of data, we shall learn how to measure the **spread** of the data by using the **range** and the **interquartile range**.

Fig. 4.4 shows the range and interquartile for the data values in Set A. The median, Q_1, Q_3, range and the interquartile range are indicated in the dot diagram as shown.

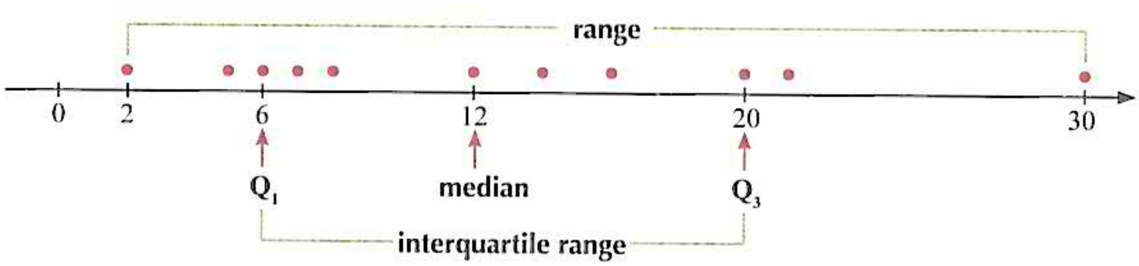


Fig. 4.4

These measures of spread show the **degree of variation** or how 'spread out' the data values are.

For Set A,

Range = Largest value – Smallest value
$= 30 - 2$
$= 28$

Interquartile range = $Q_3 - Q_1$
$= 20 - 6$
$= 14$

Internet Resources

There are other formulae and methods to find the lower quartile, Q_1 and the upper quartile, Q_3. Search on the Internet for more information. But we will use the method shown in the textbook.

ATTENTION

The interquartile range is the range of the **middle 50%** of the data.

The interquartile range is a better measure of the spread of the data than the range because it tells us how the **middle 50%** of the data are distributed. The range only consists the difference between the largest and the smallest values of the set of data.

The interquartile range is not affected by extreme values as it does not consider the behaviour of the lower 25% or upper 25% of the data.

A statistical measure that takes into account the behaviour of every value of a data set is introduced in Section 4.4.

Worked Example 3

(Finding and Interpreting the Range and Interquartile Range for a Set of Discrete Data with an Even Number of Data Values)

The data below shows the marks for a multiple choice quiz with 20 questions, taken by 8 students.

10, 12, 12, 13, 9, 17, 11, 14

(i) For the given set of data, find Q_1, Q_2 and Q_3.

(ii) Find the range.

(iii) Find the interquartile range.

Solution:

(i) Arranging the given data in ascending order:

$$\overbrace{9 \quad 10 \underset{\substack{\uparrow \\ Q_1}}{\;} 11 \quad 12}^{\textbf{lower half}} \underset{\substack{\uparrow \\ Q_2 \\ \textbf{median}}}{\;} \overbrace{12 \quad 13 \underset{\substack{\uparrow \\ Q_3}}{\;} 14 \quad 17}^{\textbf{upper half}}$$

For the given data, $n = 8$.

$\therefore Q_2 = \dfrac{12+12}{2} = 12$ (When n is even, the median is the average of the two middle values.)

$Q_1 = \dfrac{10+11}{2} = 10.5$ (When the number of data in the lower half is even, Q_1 is the average of the two middle values.)

$Q_3 = \dfrac{13+14}{2} = 13.5$ (When the number of data in the upper half is even, Q_3 is the average of the two middle values.)

(ii) Range $= 17 - 9$
$= 8$

(iii) Interquartile Range $= Q_3 - Q_1$
$= 13.5 - 10.5$
$= 3$

PRACTISE NOW 3

SIMILAR QUESTIONS

Exercise 4B Questions 1–3

1. The following set of data shows the number of sit-ups done in 1 minute by 10 students during a physical fitness test.

12, 22, 36, 10, 14, 45, 59, 44, 38, 25

(i) For the given set of data, find Q_1, Q_2 and Q_3.
(ii) Find the range.
(iii) Find the interquartile range.

2. Another physical fitness test is conducted one month later. However, only 9 of the students took the test as one of the students is sick. The following set of data shows the number of sit-ups done in 1 minute by these 9 students.

23, 54, 15, 32, 16, 26, 47, 9, 35

(i) For the given set of data, find Q_1, Q_2 and Q_3.
(ii) Find the range.
(iii) Find the interquartile range.

Continuous Data

Continuous data refers to data which can take on any value within a range of numbers. For example, a data set showing the height (cm) of 30 girls in a class can take on values such as 150.4, 169.34, 150, etc.

For continuous data, we can *estimate* the quartiles from the cumulative frequency curve.

Let us look at the cumulative frequency curve from Worked Example 2 on page 79, where $n = 120$.

Cumulative Frequency Curve for the Masses of Apples

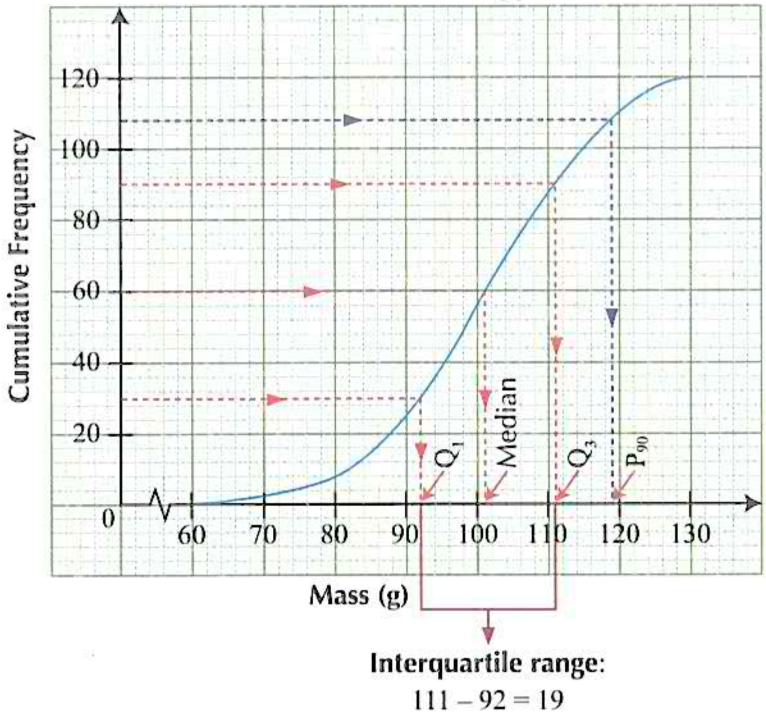


Fig. 4.5

For continuous data, we can obtain the quartiles by dividing the data set into 4 equal parts.

When the cumulative frequency is $\frac{n}{2} = \frac{120}{2} = 60$, from Fig. 4.5, the mass of the apples is 101.

Since the median is the middle value, then the median is 101 g.

Similarly, $\frac{n}{4} = \frac{120}{4} = 60$ and $\frac{3}{4}n = \frac{3}{4} \times 120 = 90$.

$\therefore$ From Fig. 4.5, $Q_1 = 92$ g and $Q_3 = 111$ g.

Thus, the **interquartile range** is $Q_3 - Q_1 = 111 - 92 = 19$ g (as shown in Fig 4.5).

The **range** is the difference between the largest end-point and smallest end-point, i.e. $130 - 60 = 70$ g.

Percentiles

For continuous data, we can also find a measure called **percentiles,** which are values that divide the data set into 100 equal parts.

For example, in Fig. 4.5, 90% of the distribution (i.e. $\frac{90}{100} \times 120 = 108$ apples) have masses less than 119 g.

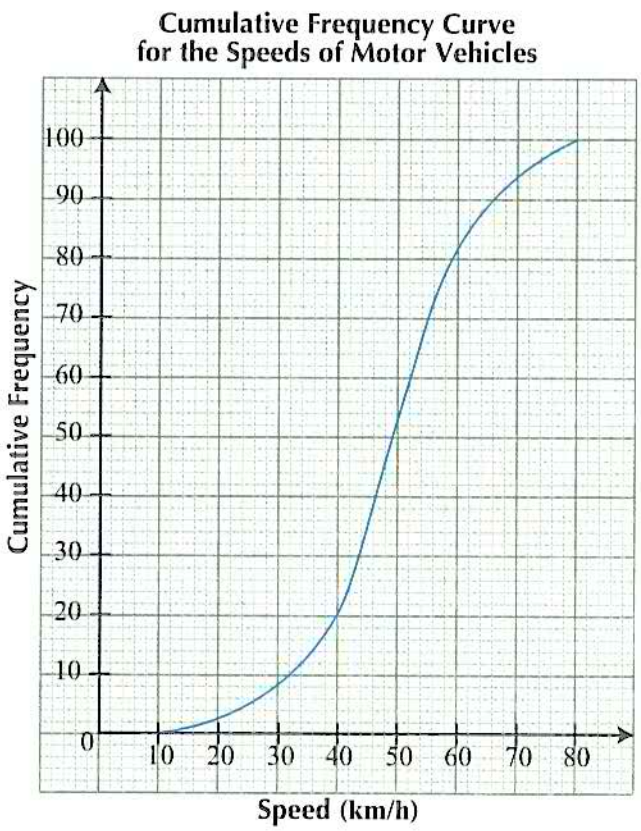
We say that the **90^{th} percentile, P_{90}** $= 119$ g.

Since the median value of 101 means that 50% of the distribution, i.e. 60 apples have masses less than 101 g, the **median** is also called the **50^{th} percentile, P_{50}**. Similarly, $Q_1 = P_{25}$ and $Q_3 = P_{75}$.

Worked Example 4

(Estimating the Quartiles, Interquartile Range and Percentiles from a Cumulative Frequency Curve)

The cumulative frequency curve represents the instantaneous speeds of 100 motor vehicles taken at a particular point on a street.



Estimate

(i) the median, the lower and the upper quartiles,

(ii) the interquartile range,

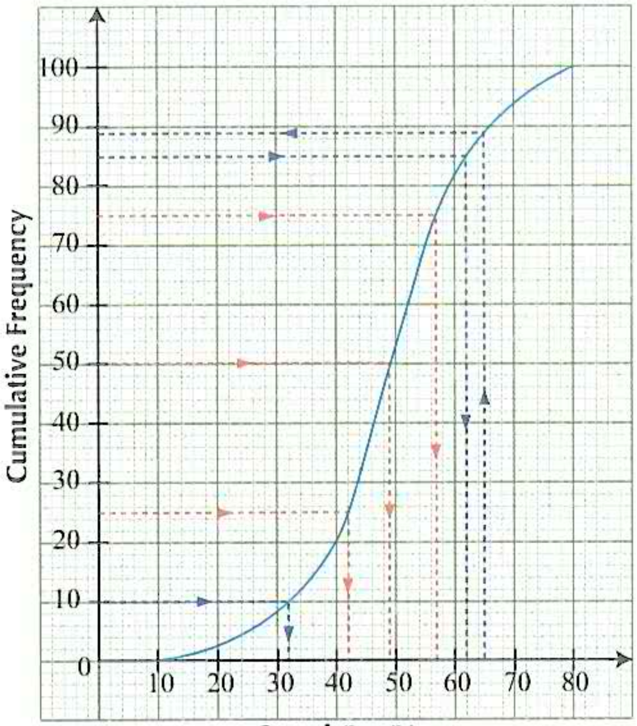
(iii) the range of the speed,

(iv) the 10^{th} percentile,

(v) the value of v, if 85% of motor vehicles have speeds less than or equal to v km/h.

Solution:

(a)



(i) For this set of data, $n = 100$.

$\therefore \frac{n}{2} = 50$, $\frac{n}{4} = 25$ and $\frac{3n}{4} = 75$

From the graph, median speed = 49 km/h,
lower quartile = 42 km/h,
upper quartile = 57 km/h.

(ii) Interquartile range = 57 – 42
= 15 km/h

(iii) Range = 80 – 10
= 70 km/h

(iv) 10% of the total frequency $= \frac{10}{100} \times 100$
$= 10$

From the graph, the 10[th] percentile = 32 km/h.

(v) 85% of drivers $= \frac{85}{100} \times 100 = 85$, i.e. 85 drivers have speeds less than or equal to v km/h.

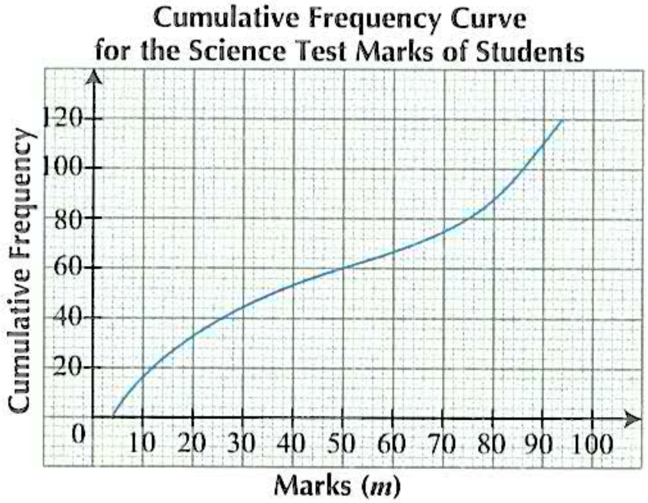
From the graph, $v = 62$.

PRACTISE NOW 4

SIMILAR QUESTIONS

Exercise 4B Questions 4–7

120 students take a Science test. The cumulative frequency curve shows the test marks (m) and the number of students scoring less than m marks.



From the graph, estimate

(i) the median, the lower quartile and the upper quartile,

(ii) the interquartile range,

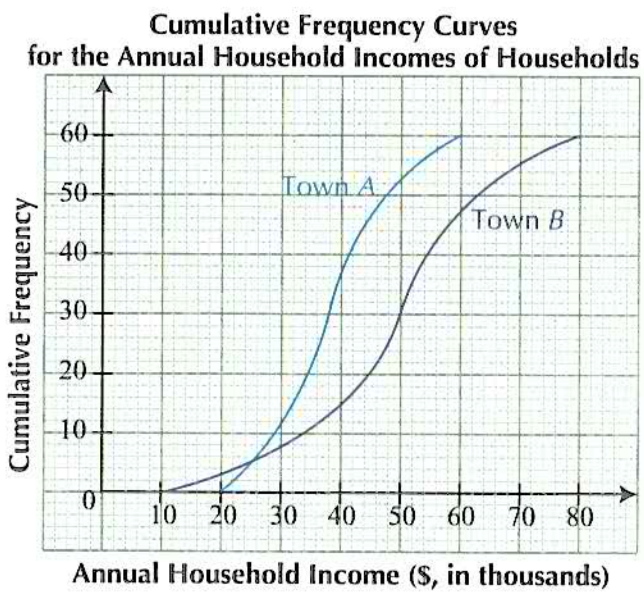
(iii) the 10th and 80th percentiles,

(iv) the passing mark if 60% of the students passed the test.

Worked Example 5

(Comparing and Analysing Two Cumulative Frequency Curves)

The diagram below shows the cumulative frequency curves for the annual incomes (in thousands of dollars) of 60 households in two towns, A and B.



(a) For Town A, find
   (i) the median income level,
   (ii) interquartile range.

(b) For Town B, find
   (i) the median income level,
   (ii) the interquartile range.

(c) 'Households in Town B generally have higher income levels than households in Town A'. Do you agree? Explain your answer.

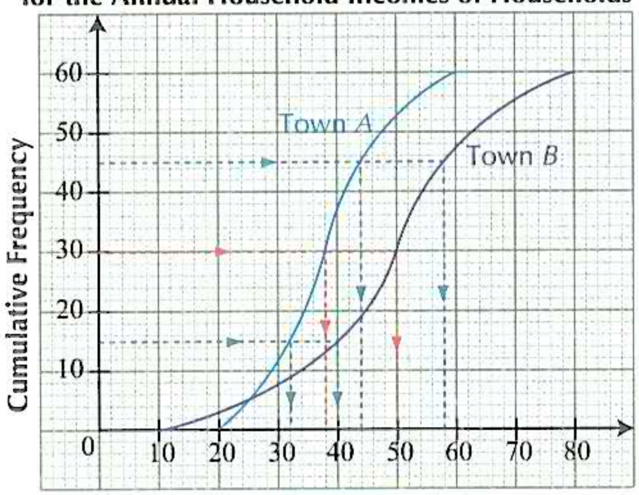
(d) Which town is more likely to have an 'income-gap' problem? Justify your answer.

Solution:

(a) For each set of data, $n = 60$.

$\therefore \frac{n}{2} = 30$, $\frac{n}{4} = 15$ and $\frac{3n}{4} = 45$

Cumulative Frequency Curve for the Annual Household Incomes of Households



(i) From the graph, median income level of Town A = \$38 000.

(ii) From the graph, lower quartile = \$32 000
upper quartile = \$44 000

$\therefore$ Interquartile range of Town A = \$44 000 – \$32 000
= \$12 000

(b) (i) From the graph, median income level of Town B = \$50 000.

(ii) From the graph, lower quartile = \$40 000
upper quartile = \$58 000

$\therefore$ Interquartile range of Town B = \$58 000 – \$40 000
= \$18 000

(c) Agree. The median annual income level of Town B is higher by \$50 000 – \$38 000 = \$12 000.

(d) Town B. The interquartile range of \$18 000 for Town B is much higher than the interquartile range of \$12 000 for Town A.

ATTENTION

For **(d)**, in Town B, the interquartile range of \$18 000 indicates that the difference (or gap) in income levels between the bottom 25% of households and top 25% of households is \$18 000.

Therefore, it is more likely to have an 'income-gap' problem than Town A, whose interquartile range of \$12 000 is much smaller.

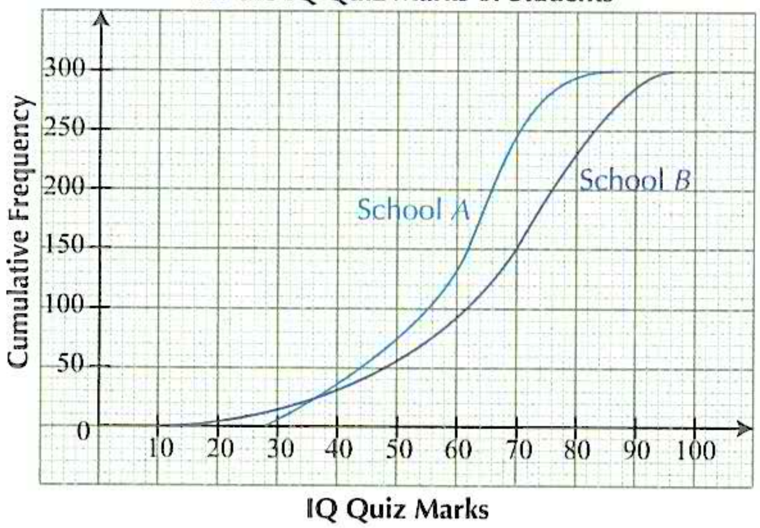
PRACTISE NOW 5

SIMILAR QUESTIONS

Exercise 4B Questions 8–12

300 students, each from School A and School B, participated in an IQ quiz. The maximum marks for the quiz is 100. The cumulative frequency curves below show the distribution of the marks scored by the students from each of the two schools.

Cumulative Frequency Curve for the IQ Quiz Marks of Students



(a) For School A, estimate

(i) the median, **(ii)** the interquartile range.

(b) For School B, estimate the

(i) the median, **(ii)** the interquartile range.

(c) State, with a reason, if School A or School B performed better overall.

(d) In which school were the quiz marks more consistent? Justify your answer.

Exercise 4B

BASIC LEVEL

1. Find the range, lower quartile, median, upper quartile and interquartile range for the following sets of data.

(a) 7, 6, 4, 8, 2, 5, 10

(b) 63, 80, 54, 70, 51, 72, 64, 66

(c) 14, 18, 22, 10, 27, 32, 40, 16, 9

(d) 138, 164, 250, 184, 102, 244, 168, 207, 98, 86

(e) 10.4, 8.5, 13.1, 11.8, 6.7, 22.4, 4.9, 2.7, 15.1

2. The following set of data shows the number of distinctions scored by 10 classes for a particular examination. Each class has 40 students.

0, 1, 6, 9, 24, 0, 27, 6, 9, 29

(i) For the given data, find the median, the lower and upper quartiles.

(ii) Find the range and the interquartile range.

3. The stem-and-leaf diagram below represents the Mathematics quiz marks of 20 students.

Stem	Leaf
0	9
1	2
2	1 2 8
3	0
4	0 1 1 2 8 9
6	0
7	2 3 9
8	7 7 8
9	5

Key: 0 | 9 means 9 marks

Find

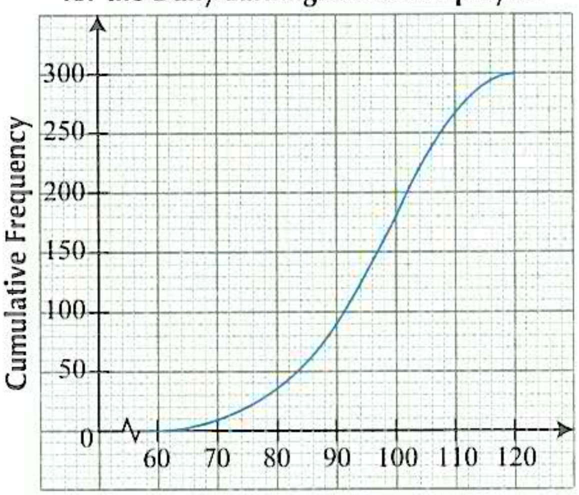
(i) the median mark,

(ii) the range,

(iii) the interquartile range.

4. The graph shows the cumulative frequency curve for the daily earnings of 300 employees in a company.

Cumulative Frequency Curve for the Daily Earnings of 300 Employees



Daily Earnings (dollars)

(a) Use the graph to estimate

(i) the median, the lower and upper quartiles,

(ii) the interquartile range.

(b) Find the

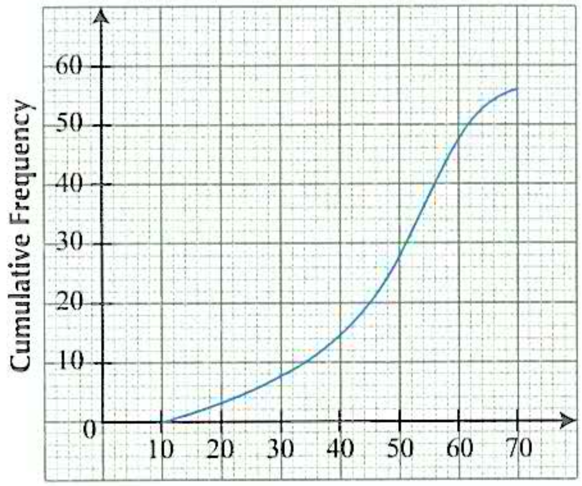
(i) 20^{th} percentile,

(ii) 90^{th} percentile,

of the daily earnings of the employees.

5. The following diagram shows the cumulative frequency curve for the heights, in cm, of 56 plants grown under experimental conditions.

Cumulative Frequency Curve for the Heights of Plants



Height (cm)

Use the curve to estimate

(i) the median height,

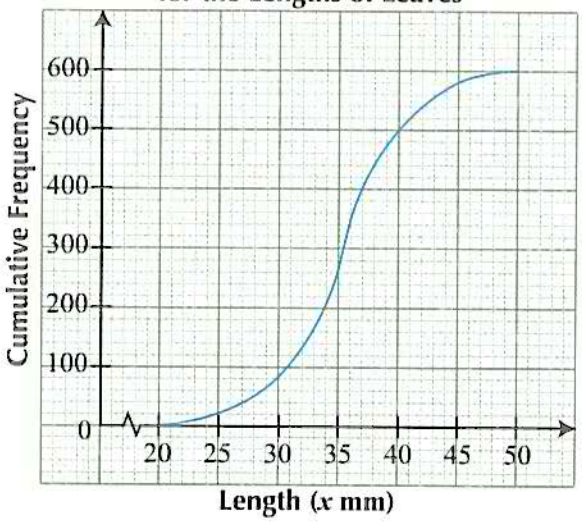
(ii) the upper quartile,

(iii) the lower quartile,

(iv) the number of plants having heights greater than 57 cm.

INTERMEDIATE LEVEL

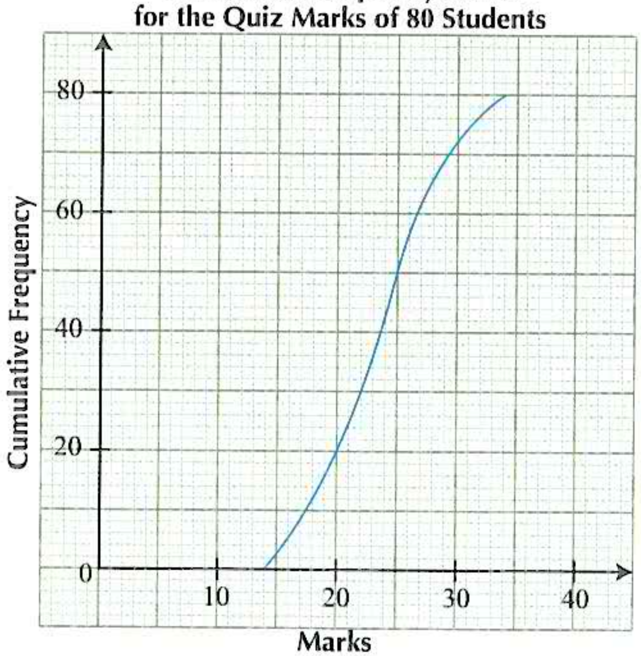
6. The following diagram shows the cumulative curve for the lengths of 600 leaves from a tree.



(a) Use the graph to find
  (i) the median length,
  (ii) the interquartile range.

(b) Given that 65% of the leaves are considered healthy if their length is longer than h mm, use the graph to find the value of h.

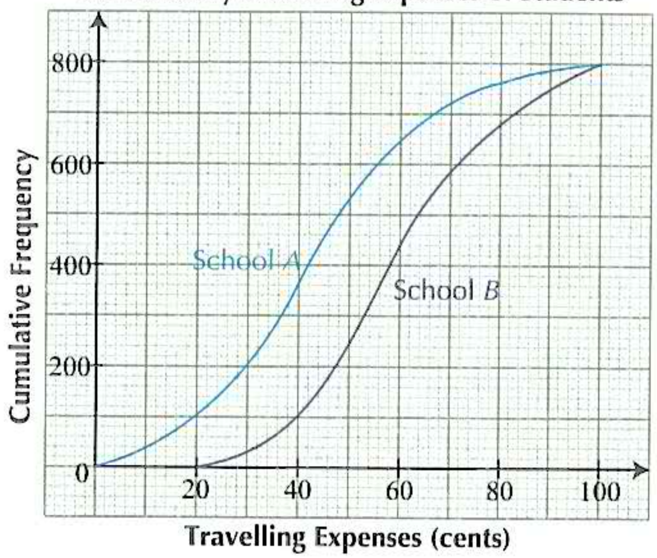
7. 80 students participated in a multiple choice quiz which consists of 40 questions. The cumulative curve shows the marks scored.



(a) Estimate from the graph
  (i) the median mark,
  (ii) the upper quartile,
  (iii) the interquartile range,
  (iv) the number of participants who scored more than or equal to 26 marks but less than 30 marks.

(b) Given that 37.5% of the students passed the quiz, use the graph to find the passing mark.

8. The graph shows the cumulative frequency curves of the daily travelling expenses of 800 students in two schools, A and B.



(a) Use the graph to estimate the median travelling expenses of the students from
  (i) School A, **(ii)** School B.

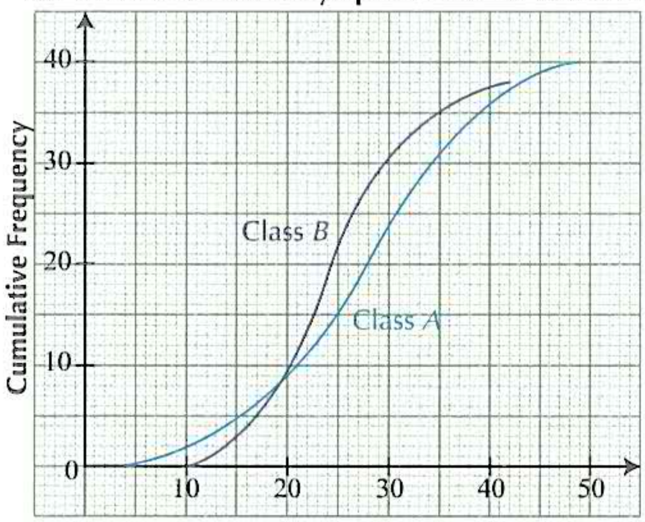
(b) Find the interquartile range of the travelling expenses of the students from
  (i) School A, **(ii)** School B.

(c) Find the 80^{th} percentile of the travelling expenses of the students from School B.

(d) State, with a reason, whether the students from School A or School B spend more on daily travelling expenses.

9. All the students from two classes, A and B, took the same Mathematics Olympiad examination paper. The cumulative frequency curves below show the scores for the two classes.

Cumulative Frequency Curves for the Mathematics Olympiad Scores of Students



(i) Estimate the lower quartile, median, and upper quartile in Class A.

(ii) How many students are there in Class B?

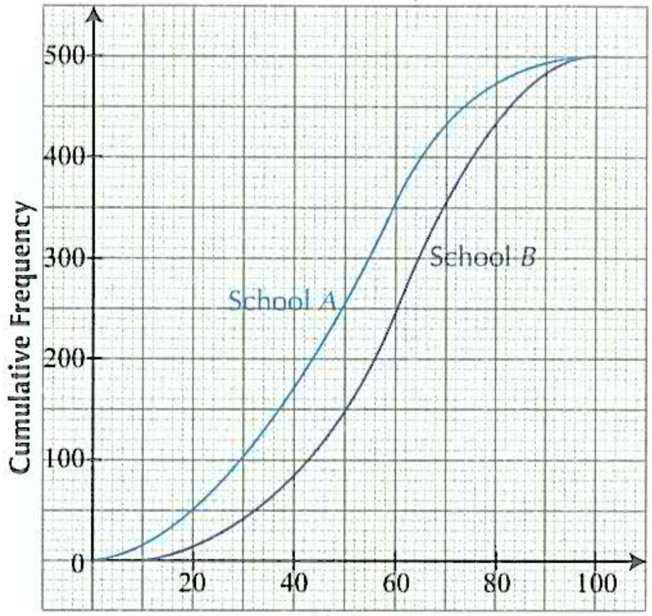
(iii) Find the interquartile range of Class B.

(iv) Estimate the percentage of the students from Class B who received a gold award, given that the qualifying mark for a gold award is more than 38.

(v) Do you agree with the statement that 'Class A generally performed better and their results are more consistent'? Justify your answer.

10. The cumulative frequency curves show the distribution of marks scored by 500 cadets in a physical fitness test from each of the two military schools, A and B.

Cumulative Frequency Curves for Marks Scored by Cadets



(a) For School A, estimate from the graph,

(i) the median mark,

(ii) the 70th percentile,

(iii) the interquartile range,

(iv) the number of cadets who scored less than 43 marks,

(v) the passing mark given that 60% of the cadets passed the physical fitness test.

(b) It is given that a distinction grade is equivalent to 70 marks and above. Find the percentage of cadets who scored distinctions in each school.

(c) 'Cadets from School B performed better in general, than School A'. Do you agree? Give two reasons to support your answer.

ADVANCED LEVEL

11. The table below shows the Pollutant Standards Index (PSI) of City X and City Y, measured in the same period of 10 days. A higher PSI reading indicates worse air quality, and vice versa.

City X	City Y
80 65 21 81 16	103 79 99 121 200
23 37 50 53 100	308 114 171 198 235

(a) For the PSI data given for City X, find

(i) the range,

(ii) the median,

(iii) interquartile range.

(b) For the PSI data given for City Y, find

(i) the range,

(ii) the median,

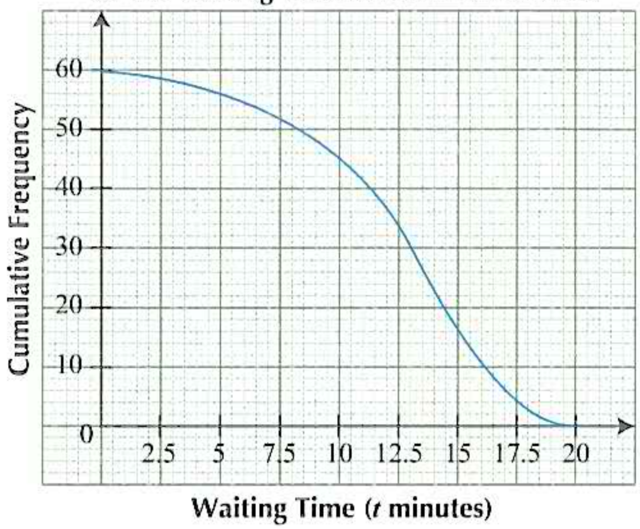
(iii) the interquartile range.

(c) Which city's data show a greater spread?

(d) Compare and comment on the air quality of the two cities. Give two reasons to support your answer.

12. The waiting times (in minutes) of 60 clients at a bank, on a particular day were measured. The cumulative frequency curve shows the waiting times (t), and the number of clients with waiting times **more than** t minutes.

Cumulative Frequency Curve for the Waiting Times of Clients in a Bank



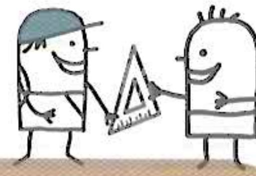
(a) **(i)** Estimate the lower quartile, median and upper quartile of the waiting times in the bank.

(ii) Find the interquartile range.

(b) Find the percentage of clients who waited for not more than 15 minutes at the bank.

(c) For the same 60 clients, a second cumulative frequency curve is plotted to show the waiting times (t), and the number of clients with waiting times **less than or equal** to t minutes. What does the intersection of the two cumulative curves represent? Explain your answer clearly.

4.3 Box-and-Whisker Plots



In this section, we will learn how to draw and interpret a **box-and-whisker** plot, which is another way to show the distribution of a set of data.

Let us look at Worked Example 4 on page 83 again, where estimates were obtained from the cumulative frequency curve for the speeds of 100 motor vehicles. In the example,

- the maximum speed is 80 km/h,
- the minimum speed is 10 km/h,
- the median speed is 49 km/h,
- the lower quartile (Q_1) is 42 km/h and
- the upper quartile (Q_3) is 57 km/h.

We can present this information on a **box-and-whisker plot**. To begin, we draw a horizontal **number line** using a suitable scale. The number line must be long enough to contain all the data points. On top of the number line, the positions of the **MIN** (minimum speed), the **MAX** (maximum speed) and the quartiles are indicated, as shown in Fig. 4.6.

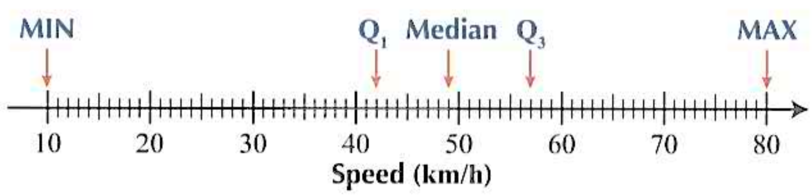


Fig. 4.6

As shown in Fig. 4.7, a rectangular box is drawn above the number line, with the left side at the lower quartile and the right side at the upper quartile. A vertical line is then drawn inside the box to indicate the median. This rectangular box represents the **box** of a box-and-whisker plot.

Above the number line, the MIN and the MAX are marked. Two line segments are then drawn to connect the MIN and MAX to the sides of the box. These two line segments represent the **whiskers** of a box-and-whisker plot.

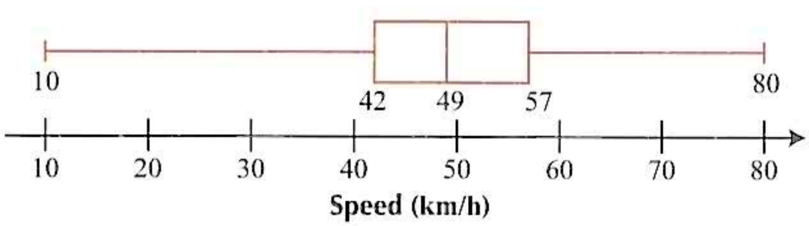


Fig. 4.7

ATTENTION

The number line must be drawn with an arrow pointing *to the right*.

ATTENTION

It is important to *label* the values of the MIN, MAX, the quartiles and the median on the box-and-whisker plot if there is no graph grid (e.g. Fig. 4.7).

With or without the grid, any units used for the number line, e.g. speed (km/h), must also be indicated.

Therefore, the final figure in Fig. 4.7 is called a **box-and-whisker plot**.

From the box-and-whisker plot,

Range = MAX – MIN
= 80 – 10
= 70 km/h,

and

Interquartile Range = $Q_3 - Q_1$
= 57 – 42
= 15 km/h.

A box-and-whisker plot is a way of summarising a set of data.

If we are interested in only the five values (i.e. min, max, Q_1, Q_2 and Q_3), then we use the box-and-whisker plot. But if we need to find the cumulative frequencies or percentiles, then we use the cumulative frequency curve.

When comparing two sets of data, it is easier to use the box-and-whisker plot than the cumulative frequency curve because we will usually compare only the medians and the interquartile ranges.

Worked Example 6

(Drawing a Box-and-Whisker Plot)
Draw a box-and-whisker plot for the given set of data.

10, 4, 3, 16, 14, 13, 4, 7, 11, 5, 17, 14

Solution:

Arranging the given data in ascending order:

lower half | **upper half**

3 4 4 5 7 10 ¦ 11 13 14 14 16 17

Q_1 **median** Q_3

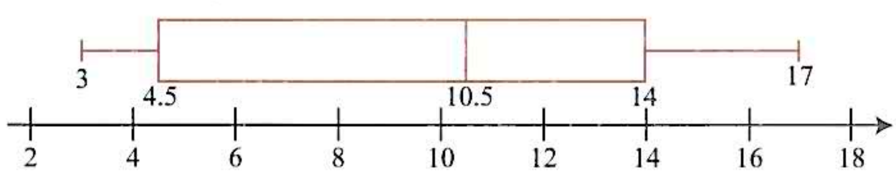
For the given data, $n = 12$, MIN = 3 and MAX = 17.

$\therefore$ Median $= \frac{10+11}{2} = 10.5$

$Q_1 = \frac{4+5}{2} = 4.5$ (When the number of data in the lower half is even, Q_1 is the average of the two middle values.)

$Q_3 = \frac{14+14}{2} = 14$ (When the number of data in the upper half is even, Q_3 is the average of the two middle values.)

The box-and-whisker plot is drawn below.



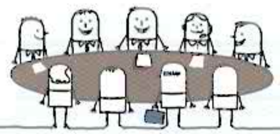
PRACTISE NOW 6

SIMILAR QUESTIONS

Exercise 4C Question 1(a)–(d)

Draw a box-and-whisker plot for the given set of data.

20, 14, 23, 9, 7, 13, 29, 9, 16



Class Discussion

Vertical Box-and-Whisker Plots

Box-and-whisker plots can also be drawn vertically.
Table 4.2 shows the summary statistics for two sets of data, *A* and *B*.

	Set *A*	Set *B*
MIN	20	10
MAX	120	110
Q_1	36	80
Median	50	90
Q_3	70	100

Table 4.2

Fig. 4.8 shows the box-and-whisker plot, which is drawn vertically for the data in Set *A*.

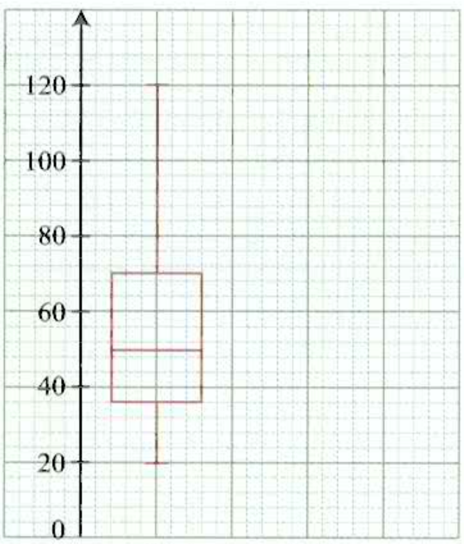


Fig. 4.8

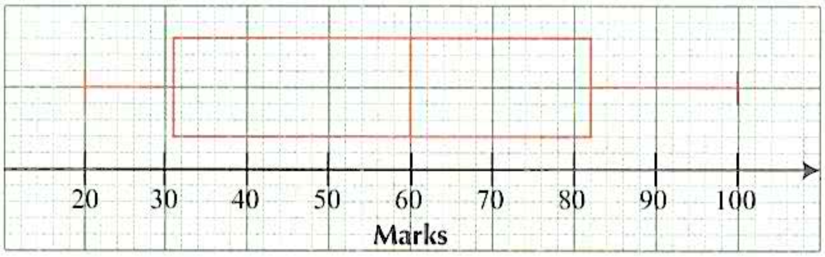
1. On the square grid and scale given in Fig. 4.8, draw a vertical box-and-whisker plot for the data in Set *B*.
2. What do the heights of the rectangular boxes represent? Compare the heights of the two rectangular boxes corresponding to the data in Set *A* and Set *B*.
3. From the height of the rectangular boxes, what can we infer about the **spread** of the data in Set *A* and Set *B*?

From the class discussion, we have learnt that box-and-whisker plots can also be drawn vertically. Box-and-whisker plots give us a visualisation of the **spread** of a set of data and also facilitate comparisons between two or more sets of data.

Worked Example 7

(Interpreting a Box-and-Whisker Plot)

A class of students took an English proficiency test. The results are represented by a box-and-whisker plot, as shown below.

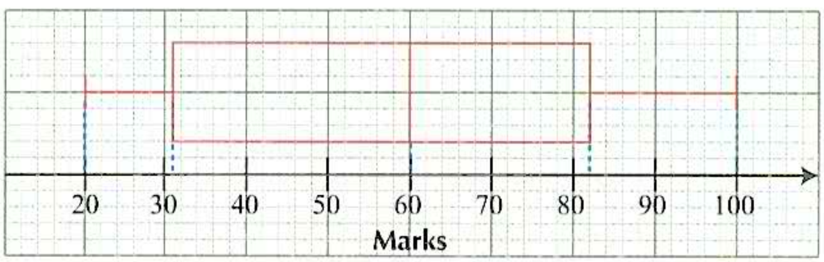


(i) State the median mark.

(ii) Find the range of the marks of the class.

(iii) Find the interquartile range of the mark.

Solution:



(i) From the box-and whisker plot, the median score is 60 marks.

(ii) Range = MAX – MIN

$= 100 - 20$

$= 80$

(iii) Interquartile range $= Q_3 - Q_1$

$= 82 - 31$

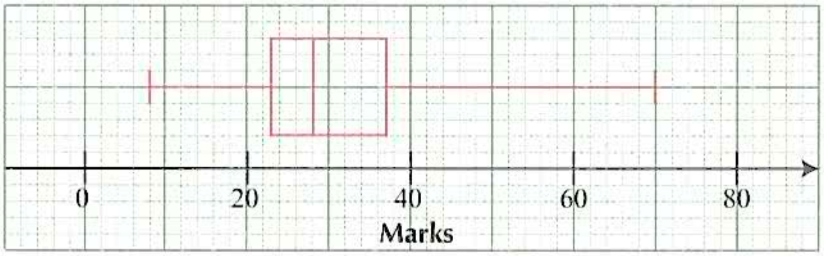
$= 51$

PRACTISE NOW 7

SIMILAR QUESTIONS

Exercise 4C Questions 2–5

A class of 50 students took a Geography test. The results are represented by a box-and-whisker plot, as shown below. The maximum mark of the test is 80.



(i) the median mark,

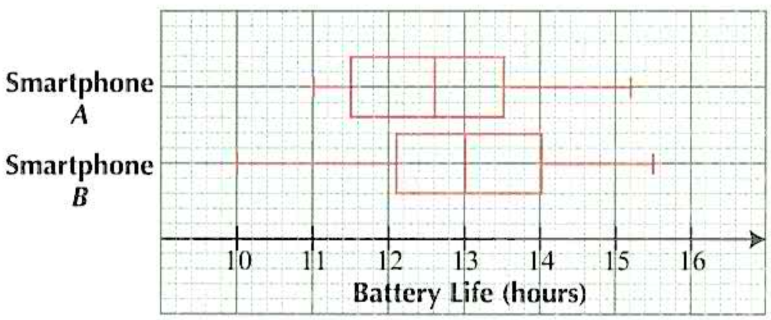
(ii) the range,

(iii) the interquartile range.

Worked Example 8

(Interpreting and Comparing Two Box-and-Whisker Plots)

The box-and-whisker plots show the distribution of the battery life (hours) of two brands of smartphones, Smartphone A and Smartphone B. 150 smartphones of each type were fully charged and tested for their battery lives.



(a) For Smartphone A, use the diagram to find,

(i) the range, **(ii)** the median,

(iii) the interquartile range.

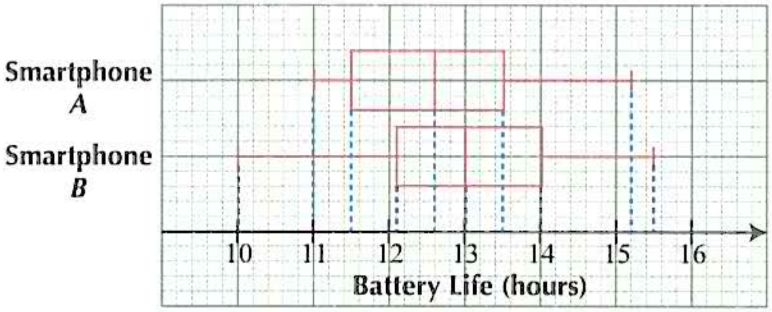
(b) For Smartphone B, use the diagram to find

(i) the range, **(ii)** the median,

(iii) the interquartile range.

(c) Which brand of smartphone has a longer battery life on average? State a reason.

Solution:



(a) For Smartphone A,

(i) range = MAX – MIN
= 15.2 – 11
= 4.2 hours

(ii) median = 12.6 hours

(iii) interquartile range = $Q_3 - Q_1$
= 13.5 – 11.5
= 2 hours

(b) For Smartphone B,

(i) range = MAX – MIN
= 15.5 – 10
= 5.5 hours

(ii) median = 13 hours

(iii) interquartile range = $Q_3 - Q_1$
= 14 – 12.1
= 1.9 hours

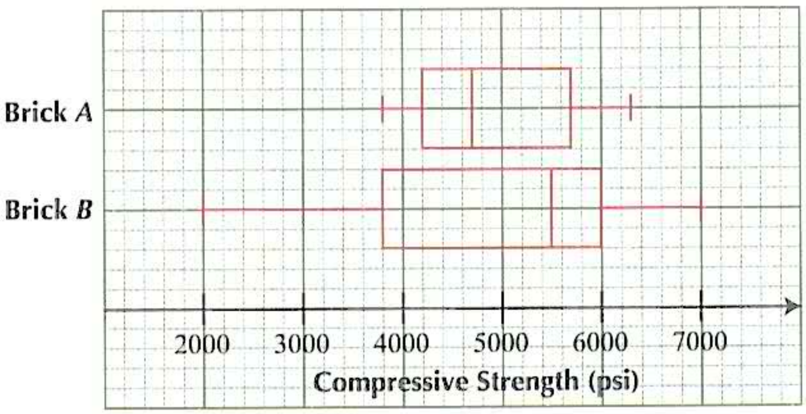
(c) Smartphone B. Smartphone B has a longer median battery life.

PRACTISE NOW 8

SIMILAR QUESTIONS

Exercise 4C Questions 6–12

A developer can choose between two different types of bricks for the construction of a new shopping complex. The box-and-whisker plots show the results of tests on the compressive strength of 200 bricks, measured in pounds per square inch (psi) of the two types of bricks. The higher the value of the psi, the stronger the brick.



(a) For Brick A, find

(i) the range,

(ii) the median,

(iii) the interquartile range.

(b) For Brick B, find

(i) the range,

(ii) the median,

(iii) the interquartile range.

(c) On average, which type of brick is stronger? State a reason to support your answer.

Internet Resources

In the study of Statistics, an **outlier** is an observation that is 'far away' from other observations in a data set.

Search on the Internet to find out more about outliers and how quartiles and the interquartile range are used to determine if a certain data point is an outlier.

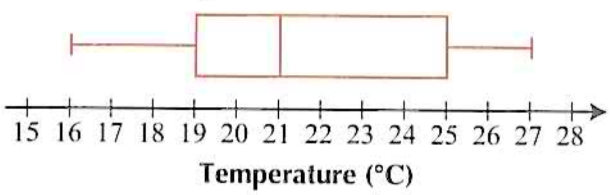
In addition, box-and-whisker plots with outliers are also drawn differently. Find out also how a box-and-whisker plot is drawn for a data set with outliers.

Exercise 4C

BASIC LEVEL

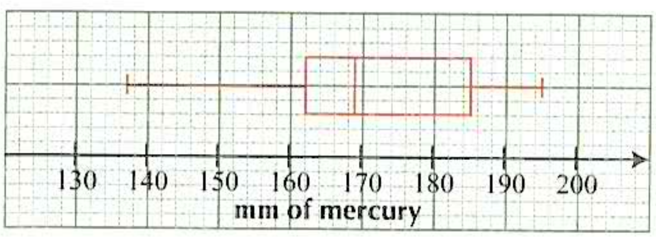
1. Draw a box-and-whisker plot for each of the following sets of data.
   (a) 1, 14, 9, 8, 20, 11, 5
   (b) 45, 51, 57, 43, 45, 60, 58, 54
   (c) 3, 6, 11, 2, 17, 22, 15, 8, 21, 3, 15, 12
   (d) 79, 87, 66, 96, 98, 87, 82, 77, 93

2. The following diagram shows the box-and-whisker plot for the daily temperature (°C) from 1st June to 30th June in a city.



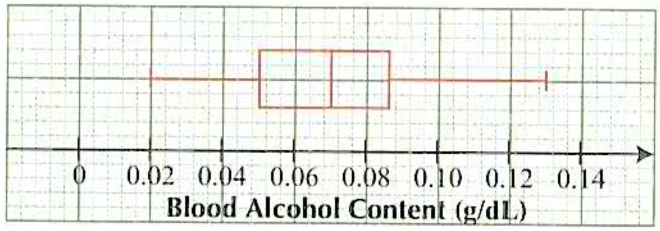
   (i) State the lower quartile, median and upper quartile of the temperature.
   (ii) Find the range of the temperature in June.

3. The box-and-whisker plot below shows the blood pressure level (in mm of mercury) of patients who have taken a certain prescription drug.



   (i) State the median blood pressure level of the patients.
   (ii) Find the interquartile range.

4. The following diagram shows the box-and whisker plot for the alcohol content (grams per decilitre of blood) in the blood of drivers who were given breathalyser tests.



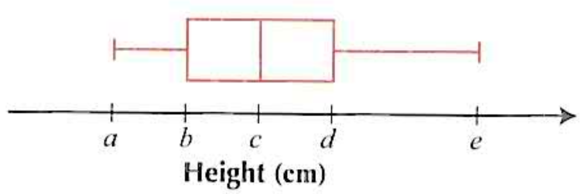
   (i) State the lower quartile, median and upper quartile of the alcohol content of the drivers.
   (ii) Compare the spread of the alcohol content between the highest 25% and the lowest 25% of the drivers.

INTERMEDIATE LEVEL

5. The heights of basketball players (cm) in a NBA team are given below.

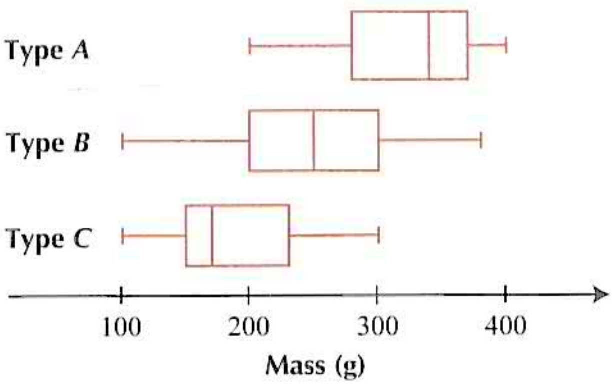
168,	180,	185,	192,	192,	195,
195,	196,	198,	200,	205,	213

   The data can be represented in the box-and-whisker plot below.



   (i) Find the values of a, b, c, d and e.
   (ii) Calculate $d - b$. What does it represent?
   (iii) Calculate $e - a$. What does it represent?

6. The following box-and-whisker plots show the masses (g) of three types of apples.

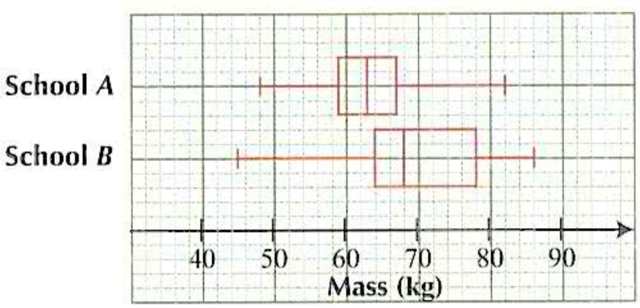


(a) Which type of apples has
  (i) the highest median mass,
  (ii) the lowest median mass?

(b) Which type of apples has masses which are more evenly distributed?

(c) Which type of apples has masses which have a greater spread?

7. The box-and-whisker plots show the masses (kg) of Secondary Four students from School *A* and School *B*.

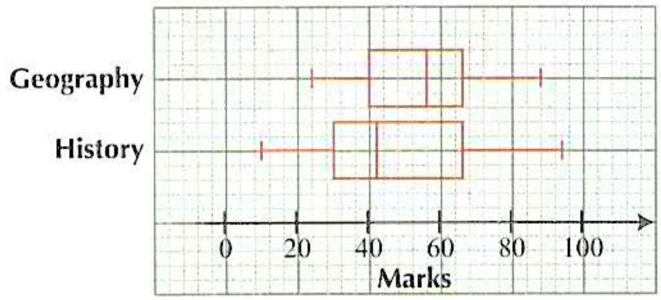


(a) For School *A*, find
  (i) the range,
  (ii) the median,
  (iii) the interquartile range.

(b) For School *B*, find
  (i) the range,
  (ii) the median,
  (iii) the interquartile range.

(c) 'Students from School *B* are generally heavier than students from School *A*.' Do you agree with this statement? Give a reason for your answer.

8. The box-and-whisker plots show the marks obtained by some students in the History and Geography examinations. The maximum mark for both examinations is 100.



(a) For the Geography examination, find
  (i) the range,
  (ii) the median,
  (iii) the interquartile range.

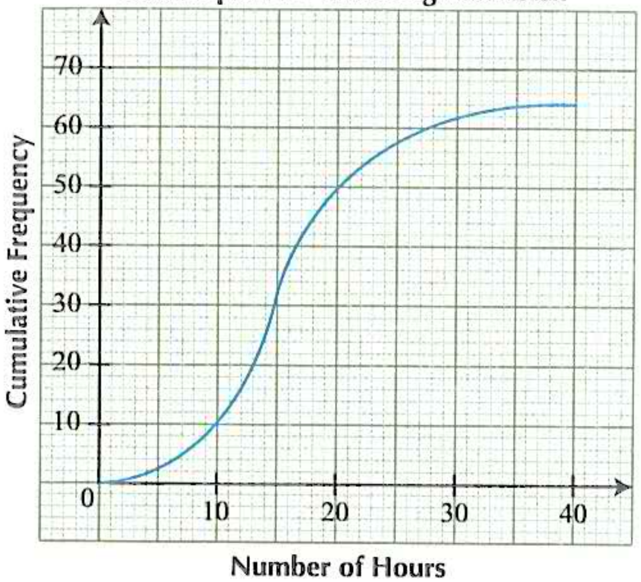
(b) For the History examination, find
  (i) the range,
  (ii) the median,
  (iii) the interquartile range.

(c) Nora said that the Geography examination is easier than the History examination. Do you agree with Nora? Give two reasons for your answer.

(d) Which examination has a wider spread of marks? Give a reason for your answer.

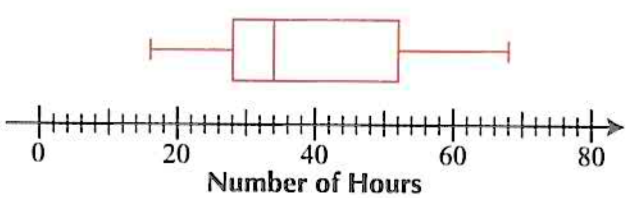
9. 64 adults were asked to indicate the weekly number of hours they spent watching television. The cumulative frequency curve below shows the information obtained.

Cumulative Frequency Curve for the Weekly Number of Hours Adults Spent on Watching Television



(a) Use the graph to estimate
  (i) the median,
  (ii) the interquartile range,
  (iii) the number of adults who spent more than 25 hours per week watching television.

The box-and-whisker plot below shows the number of hours that a group of 64 teenagers spent watching television.



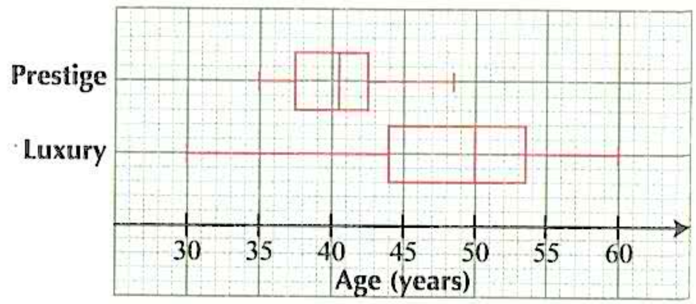
(b) Find
  (i) the median,
  (ii) the interquartile range.

(c) 'Teenagers spent more time watching television in general.' Do you agree? Give a reason to support your answer.

(d) Compare and comment on the spread of the time spent watching television of these two groups of people.

ADVANCED LEVEL

10. The box-and-whisker plots show the distribution of the ages (in years) of 60 members from Prestige Country Club and Luxury Country Club.



(a) For Prestige Country Club, find
  (i) the median age,
  (ii) the interquartile range.

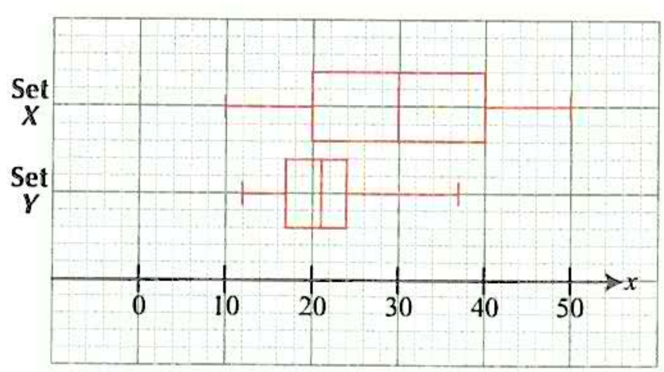
(b) For Luxury Country Club, find
  (i) median age,
  (ii) the interquartile range,

(c) For the box-and-whisker plot for Luxury Country Club, the left whisker is much longer than the right whisker. Explain what this means.

(d) Which country club shows a greater spread of ages?

(e) Comment briefly on the distribution of ages between the members in Prestige Country Club and Luxury Country Club.

11. The following diagrams show the box-and-whisker plots for two sets of data, X and Y.



(a) For each set of data, find
  (i) the median,
  (ii) the range,
  (iii) the interquartile range.

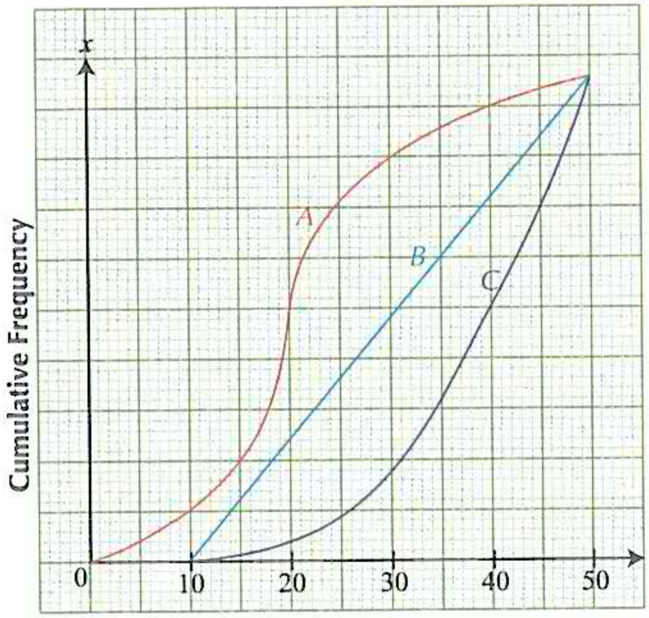
(b) Which set of data has a more balanced spread?

(c) Which set of data has a greater spread?

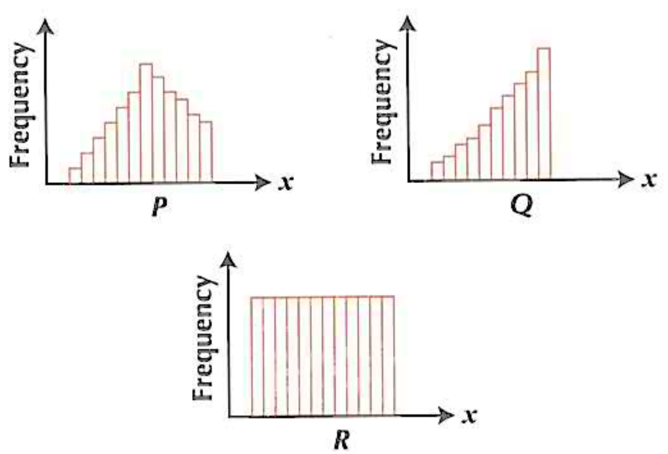
(d) Which set of data has a lower median?

(e) Which one of the cumulative curves (A, B or C) shown below best represents the set X?

Cumulative Frequency Curves for Data in Set X

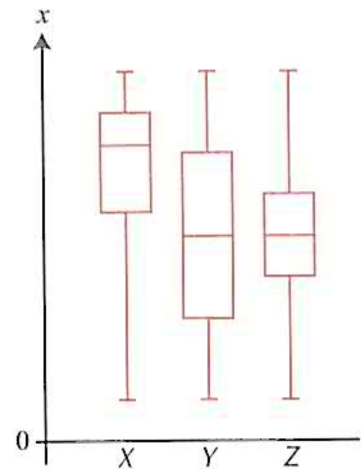


(f) The histograms below (P, Q and R) show the frequency distributions for the three cumulative frequency curves for data in Set X (A, B and C in part **(e)**).
Match each of the three curves to their respective histograms. Justify your answers.



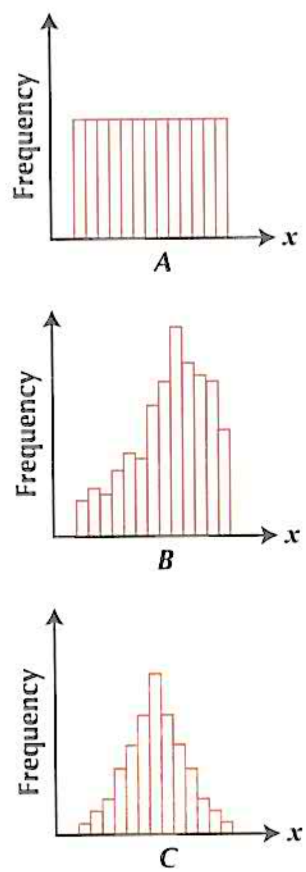
(g) Describe a context for each of the histograms P, Q and R.

12. The box-and-whisker diagrams for three sets of data, X, Y and Z, are shown below.

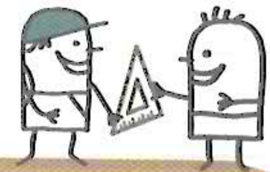


The histograms below (A, B and C) show the frequency distributions for the three sets of data.

Match each of the three data sets to their respective histograms. Justify your answers.



4.4 Standard Deviation



We will learn a new statistical measure, **standard deviation**, to describe the distribution of a set of data.

Investigation

Are Averages Adequate for Comparing Distributions?

Fig. 4.9**(a)** and **(b)** show the dot diagrams for two sets of data, Set A and Set B, both with size $n = 6$, mode = 3, median = 3 and mean = 3.

Although the three averages (mode, median and mean) are all equal to 3 for Set A and Set B, the two distributions are different.

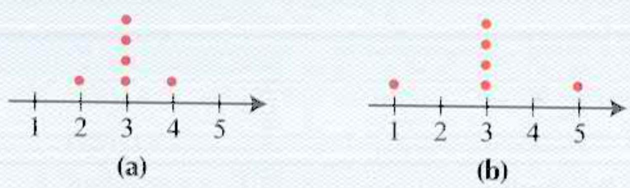


Fig. 4.9

1. Draw another two dot diagrams with distributions such that $n = 6$, and with the mode, median and mean all equal to 3.
2. Are the three averages (mode, median and mean) adequate for comparing two sets of data? Explain.

From the above investigation, we have learnt that two sets of data can have the same averages (mode, median and mean), but the distributions can still be different. Therefore, there is a need for another method to measure the **spread** of the data or distribution.

In Section 4.2, we have learnt how to find the interquartile range for both discrete and continuous data. The **interquartile range** is a measure of the spread of the data about the **median**. It tells us about the *range of the middle 50%* of the distribution. It is often used when the median is the appropriate measure of the average of the data, and we have learnt in Book 2 when we should use the median.

As mentioned in Section 4.2, we will learn in this section a new measure of spread which describes how the data are spread about the mean and which also takes into account all the values of the data set. It is often used when the mean is the appropriate measure of the average of the data.

In Book 2, we have learnt a formula for calculating the **mean**, $\bar{x}$ of a set of data,

$$\boxed{\bar{x} = \frac{\sum fx}{\sum f},}$$

where f is the frequency of each data value x.

If $f = 1$ for each data value x, then

$$\boxed{\bar{x} = \frac{\sum x}{n},}$$

where $n = \sum f$ is the size of the data.

Investigation

Obtaining a Formula for a New Measure of Spread

Table 4.3 shows the temperatures, in degree Celsius (°C) of two cities, City A and City B on a particular day, taken at 4-hour intervals.

Time	Temperature of City A (°C)	Temperature of City B (°C)
0000	25	21
0400	24	15
0800	26	23
1200	33	36
1600	31	41
2000	29	32

Table 4.3

Part 1: Mean Temperatures

1. Find the mean temperature of City A and of City B.
2. Are the mean temperatures of both cities equal?
3. By looking at Table 4.3 closely, what can you say about the spread of the temperatures of City A as compared to the spread of the temperatures of City B in relation to the respective mean temperatures?

Part 2: Spread of the Temperatures

4. In order to find a better measure of the spread of the temperatures, copy and complete Table 4.4 for City A. The first row has been done for you.

x	$x - \bar{x}$
25	$25 - 28 = -3$
24	
26	
33	
31	
29	
Sum	$\sum(x - \bar{x}) =$

Table 4.4

5. Fig. 4.10 shows the graphs of the temperatures of both cities. Compare the graphs and decide which set of data is more spread out.

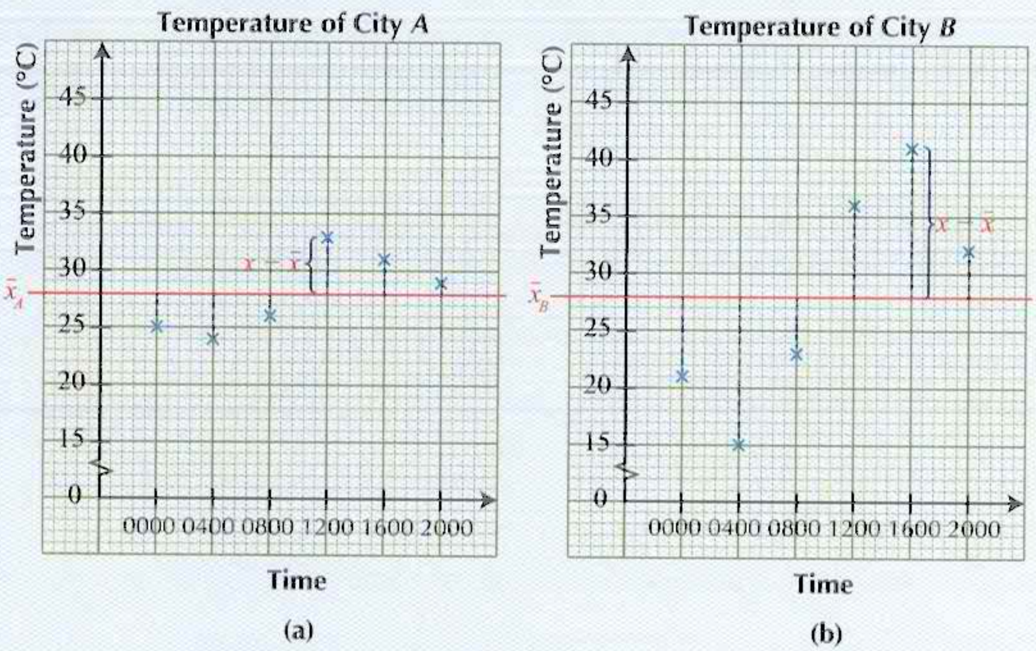


Fig. 4.10

6. Instead of a graph, we need to obtain a formula for measuring spread. Consider the value of $\sum(x - \bar{x})$. You have obtained this value for City A in Table 4.4. Use a similar method to obtain the value of $\sum(x - \bar{x})$ for City B. Compare the values obtained for City A and City B. Is this a good measure of spread? Why?

7. Now consider $\sum(x - \bar{x})^2$. Find the value of $\sum(x - \bar{x})^2$ for City A and City B and compare these values. Do you think it is a good measure of spread? Why?

8. What happens to $\sum(x-\bar{x})^2$ if the temperatures are taken at 2-hour intervals instead of 4-hour intervals, i.e. what happens to $\sum(x-\bar{x})^2$ if we have 12 data values instead of 6 data values? Does this mean that the spread will increase when there are more data values?

9. Find the value of $\frac{\sum(x-\bar{x})^2}{n}$ for City A and City B. Do you think it is a good measure of spread? Why?

10. The unit for temperature is °C. However, the unit for $\frac{\sum(x-\bar{x})^2}{n}$ is $(°C)^2$ because we have squared $(x-\bar{x})$. Hence we need to take the square root of $\frac{\sum(x-\bar{x})^2}{n}$ to make the unit consistent.

    Find the value of $\sqrt{\frac{\sum(x-\bar{x})^2}{n}}$ for City A.

    $\sqrt{\frac{\sum(x-\bar{x})^2}{n}}$ is called the **standard deviation**. It measures how the temperatures are spread about the mean $\bar{x}$.

11. Calculate the standard deviation $\sqrt{\frac{\sum(x-\bar{x})^2}{n}}$ for City B.

12. Compare the standard deviation for both cities. Which standard deviation is larger? What does it mean when the standard deviation is larger?

From the above investigation, we have learnt how the formula for the **standard deviation** comes about:

$$\textbf{Standard Deviation} = \sqrt{\frac{\sum(x-\bar{x})^2}{n}}$$

INFORMATION

Some students may have observed that, instead of squaring $x-\bar{x}$ as in the investigation, the absolute value of $x-\bar{x}$ i.e. $|x-\bar{x}|$ can be used in order to eliminate the negative signs.

There is actually such a measure of spread called the **mean absolute deviation (MAD)**, i.e. $\frac{\sum|x-\bar{x}|}{n}$, but statisticians have found that the standard deviation is more useful for higher-level Statistics.

Alternative Formula for Standard Deviation

There is an alternative formula for standard deviation:

$$\text{Standard Deviation} = \sqrt{\frac{\sum x^2}{n} - \bar{x}^2}$$

It is easier to compute the standard deviation using this formula than the first formula, i.e. $\sqrt{\frac{\sum(x-\bar{x})^2}{n}}$.

Internet Resources

Go to http://www.shinglee.com.sg/StudentResources/ and open the worksheet 'Standard Deviation' to find out how to obtain the alternative formula for standard deviation, i.e. $\sqrt{\frac{\sum x^2}{n} - \bar{x}^2}$, from $\sqrt{\frac{\sum(x-\bar{x})^2}{n}}$.

Worked Example 9

(Finding the Standard Deviation Using the Alternative Formula)

The data for the temperature of City A is shown below. Find the standard deviation **using the alternative formula** $\sqrt{\frac{\sum x^2}{n} - \bar{x}^2}$.

Time	Temperature of City A
0000	25
0400	24
0800	26
1200	33
1600	31
2000	29

Solution:

x	x^2
25	625
24	576
26	676
33	1089
31	961
29	841
$\sum x = 168$	$\sum x^2 = 4768$

ATTENTION

Compare the solution in this Worked Example with Table 4.4. Which formula is easier to use?

$$\text{Mean } \bar{x} = \frac{\sum x}{n} = \frac{168}{6} = 28$$

$$\text{Standard deviation} = \sqrt{\frac{\sum x^2}{n} - \bar{x}^2} = \sqrt{\frac{4768}{6} - 28^2} = 3.27 \text{ (to 3 s.f.)}$$

PRACTISE NOW 9

SIMILAR QUESTIONS

Exercise 4D Questions 1(a)–(c)

The table below shows the number of grammatical errors made by Shirley in eight English essays submitted this semester. Find the standard deviation of the number of errors made. Show your working clearly.

Essay Number	1	2	3	4	5	6	7	8
Number of Errors	6	9	15	26	10	14	21	3

Use of Calculator to find Standard Deviation for Ungrouped Data

In Worked Example 9, we can also make use of the statistical functions of scientific calculators to find the standard deviation directly. The data presented in Worked Example 9 is an example of **ungrouped data**.

We use the same data points (temperature of City A) as in Worked Example 9, i.e.

$\boxed{25, 24, 26, 31, 33, 29}$

Before we start, we must always remember to clear all the data currently stored in the calculator memory. To do so, press $\boxed{\text{SHIFT}}\boxed{9}$(CLR) $\boxed{2}\boxed{=}\boxed{\text{AC}}$.
Follow the steps below to obtain the standard deviation.

STEPS

1. $\boxed{\text{MODE}}$
2. $\boxed{2}$(STAT) (this changes the calculator to 'Statistics' mode)
3. $\boxed{1}$(1-VAR)
4. Enter the data one at a time, i.e.
   $\boxed{2}\boxed{5}\boxed{=}$ $\boxed{2}\boxed{4}\boxed{=}$ $\boxed{2}\boxed{6}\boxed{=}$
   $\boxed{3}\boxed{1}\boxed{=}$ $\boxed{3}\boxed{3}\boxed{=}$ $\boxed{2}\boxed{9}\boxed{=}$
5. $\boxed{\text{AC}}$
6. $\boxed{\text{SHIFT}}\boxed{1}$
7. $\boxed{4}$(VAR)
8. $\boxed{3}$($x\sigma n$) (the screen displays $x\sigma n$)
9. $\boxed{=}$ (the screen displays the value of the standard deviation, i.e. 3.265...)

ATTENTION

The buttons on calculators vary with different models. Refer to the instruction manual of your calculator.

ATTENTION

To obtain the mean after step 9, repeat step 5 and 7 and continue with

Step 8. $\boxed{2}$ ($\bar{x}$)

Step 9. $\boxed{=}$

We can also use the calculator to find the **mean**, i.e. $\bar{x}$, for **ungrouped data** first.

Steps 1 to 7 are the same.

Step 8. $\boxed{2}$ ($\bar{x}$)

Step 9. $\boxed{=}$

PRACTISE NOW

The ages of 7 people are 16, 21, 22, 18, 20, 12 and 24 years. **With the help of a calculator**, find the standard deviation of their ages.

SIMILAR QUESTIONS

Exercise 4D Questions 2(a)-(c), 8-10, 14-15

Standard Deviation for Grouped Data

For **grouped data**, the formula for finding standard deviation is essentially the same:

$$\text{Standard Deviation} = \sqrt{\frac{\sum f(x-\bar{x})^2}{\sum f}} \quad \text{or} \quad \sqrt{\frac{\sum fx^2}{\sum f} - \bar{x}^2},$$

where the mean, $\bar{x} = \dfrac{\sum fx}{\sum f}$.

Similarly, the second formula, i.e. $\sqrt{\dfrac{\sum fx^2}{\sum f} - \bar{x}^2}$, is easier to use for the computation of the standard deviation.

INFORMATION

For **grouped data**, both the mean and the standard deviation computed are estimates as the **mid-value** is used to represent the data in each group.

Worked Example 10

(Finding the Standard Deviation for Grouped Data)

100 Secondary Four students, each from School A and School B, were asked for the amount of time they spent watching television each week. The results are given in the table below.

School A	
Number of Hours	**Number of Students**
$10 < x \leqslant 15$	3
$15 < x \leqslant 20$	12
$20 < x \leqslant 25$	19
$25 < x \leqslant 30$	36
$30 < x \leqslant 35$	22
$35 < x \leqslant 40$	8

School B	
Mean	26.3 hours
Standard Deviation	5.12 hours

(i) Find an estimate of the mean and standard deviation of the number of hours spent watching television by the 100 students from School A, showing your working clearly.

(ii) Compare and comment briefly on the results of the two schools.

Solution:

(i)

Number of Hours	Frequency	Mid-value (x)	fx	fx^2
$10 < x \leq 15$	3	12.5	37.5	468.75
$15 < x \leq 20$	12	17.5	210	3675
$20 < x \leq 25$	19	22.5	427.5	9618.75
$25 < x \leq 30$	36	27.5	990	27 225
$30 < x \leq 35$	22	32.5	715	23 237.5
$35 < x \leq 40$	8	37.5	300	11 250
Sum	$\sum f = 100$		$\sum fx = 2680$	$\sum fx^2 = 75\,475$

$$\text{Mean } \bar{x} = \frac{\sum fx}{\sum f} = \frac{2680}{100} = 26.8 \text{ hours}$$

$$\text{Standard deviation} = \sqrt{\frac{\sum fx^2}{\sum f} - \bar{x}^2} = \sqrt{\frac{75\,475}{100} - 26.8^2} = 6.04 \text{ hours (to 3 s.f.)}$$

∴ For School A, the mean is 26.8 hours and the standard deviation is 6.04 hours.

(ii) The students in both schools spent approximately the same number of hours, on average, watching television. However, School A has a higher standard deviation, which indicates that there is a greater spread in the number of hours spent watching television, i.e. some students spent long hours while some spent very little time watching television.

RECALL

In order to obtain the mean for a set of **grouped data**, we need to calculate the **mid-value** of each class interval.

As the calculation of the standard deviation involves the mean, the mid-value of each class interval needs to be calculated.

The calculator can also be used to obtain the values of Σfx and Σfx^2.

ATTENTION

For Worked Example 10, use the first formula

i.e. $\left(\sqrt{\frac{\sum f(x-\bar{x})^2}{\sum f}}\right)$ to compute the standard deviation and compare which formula is easier to use.

PRACTISE NOW 10

SIMILAR QUESTIONS

Exercise 4D Questions 3–6, 11

30 students, each from Class A and Class B took the same examination. Information on the examination results is shown in the tables below.

Class A

Marks	$0 < x \leqslant 4$	$4 < x \leqslant 8$	$8 < x \leqslant 12$	$12 < x \leqslant 16$	$16 < x \leqslant 20$
Frequency	3	8	14	2	3

Class B

Mean	9.7
Standard Deviation	3.1

(a) For Class A, find an estimate of

(i) the mean mark,

(ii) the standard deviation,

showing your working clearly.

(b) Compare and comment briefly on the results for the two classes.

Use of Calculator to find Standard Deviation for Grouped Data

Similarly, we can use the calculator to find the standard deviation directly for grouped data. The data for School A in Worked Example 10 is an example of grouped data, as the data was grouped according to the given class intervals. The data for School A is shown again, in Table 4.5 below.

Hours	Mid-value (x)	Frequency
$10 < x \leqslant 15$	12.5	3
$15 < x \leqslant 20$	17.5	12
$20 < x \leqslant 25$	22.5	19
$25 < x \leqslant 30$	27.5	36
$30 < x \leqslant 35$	32.5	22
$35 < x \leqslant 40$	37.5	8

Table 4.5

Similarly, we press SHIFT 9 (CLR) 2 = AC to clear the calculator memory.

In addition, we need to switch on the 'FREQ' column in order to input the frequency values.

To do so, key in SHIFT MODE ↓ 4 (STAT) 1 (ON).

Follow the steps below to obtain the standard deviation.

STEPS

1. $\boxed{\text{MODE}}$
2. $\boxed{2}$ (STAT) (this changes the calculator to 'Statistics' mode)
3. $\boxed{1}$ (1-VAR) (we will see the column 'FREQ')
4. Enter the data (**mid-value**) and its corresponding **frequency**, one at a time. Use the arrow keys to move to the position where you want to input the values.

X	FREQ
$\boxed{1}\boxed{2}\boxed{.}\boxed{5}\boxed{=}$	$\boxed{3}\boxed{=}$
$\boxed{1}\boxed{7}\boxed{.}\boxed{5}\boxed{=}$	$\boxed{1}\boxed{2}\boxed{=}$
$\boxed{2}\boxed{2}\boxed{.}\boxed{5}\boxed{=}$	$\boxed{1}\boxed{9}\boxed{=}$
$\boxed{2}\boxed{7}\boxed{.}\boxed{5}\boxed{=}$	$\boxed{3}\boxed{6}\boxed{=}$
$\boxed{3}\boxed{2}\boxed{.}\boxed{5}\boxed{=}$	$\boxed{2}\boxed{2}\boxed{=}$
$\boxed{3}\boxed{7}\boxed{.}\boxed{5}\boxed{=}$	$\boxed{8}\boxed{=}$

5. $\boxed{\text{AC}}$
6. $\boxed{\text{SHIFT}}\boxed{1}$
7. $\boxed{4}$ (VAR)
8. $\boxed{3}$ ($x\sigma n$) (the screen displays $x\sigma n$)
9. $\boxed{=}$ (the screen displays the value of the standard deviation, i.e. 6.042...)

PRACTISE NOW

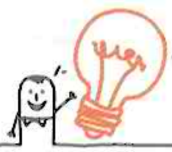
SIMILAR QUESTIONS

Exercise 4D Questions 7(a)–(b), 12–13, 16

The table shows the frequency distribution of the masses of 60 snails in grams.

Mass (g)	Number of Snails
$0 < x \leqslant 10$	1
$10 < x \leqslant 20$	2
$20 < x \leqslant 30$	10
$30 < x \leqslant 40$	18
$40 < x \leqslant 50$	20
$50 < x \leqslant 60$	6
$60 < x \leqslant 70$	3

With the help of a calculator, find an estimate of the standard deviation for the given data.



Thinking Time

A meteorologist has calculated the mean temperature and standard deviation of a particular day, with temperatures measured at hourly intervals. The calculation was done using statistical software.

Mean: 29.5°C Standard Deviation: 3.2°C

Due to a systematic error in the software, the hourly measurements taken were all overestimated by 1.5°C.

Explain clearly how the measured mean and standard deviation have been affected by this error.

Class Discussion

Matching Histograms with Data Sets

Discuss in groups of three.

Table 4.6 shows the mean, median and standard deviation of several data sets. Each data set has the same sample size.

Data set	Mean	Median	Standard Deviation
I	70	60	13
II	90	90	30
III	85	60	25
IV	110	120	23
V	83	100	17
VI	81	80	19

Table 4.6

Fig. 4.11 shows six histograms *A* to *F*.

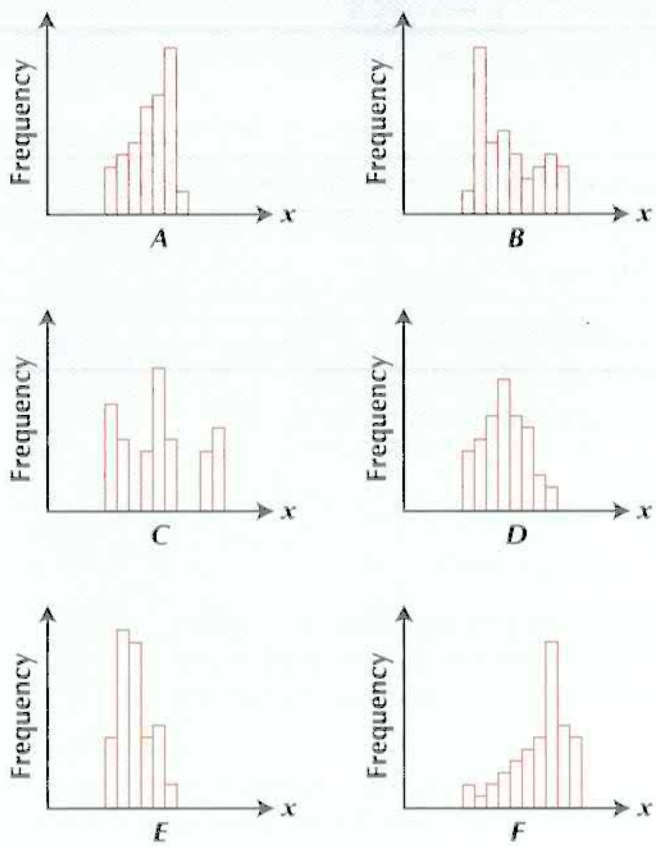
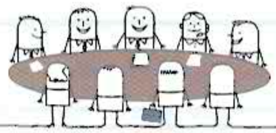


Fig. 4.11

Match the histograms to the data sets. Explain your answers.



Class Discussion

Can We Always Trust the Statistics We Read?

Discuss in groups of four.

Part 1: The Choice of Averages

Read the news report on an employment survey, as shown in Fig. 4.12.

NEWS REPORT

Fresh graduates of Statville National University (SNU) have the highest starting salary.

86% of the 2014 SNU cohort who graduated participated in an employment survey. The survey revealed that the average monthly starting salary of SNU fresh graduates in full-time employment is \$3580, the highest among all public universities in the country.

Fig. 4.12

Table 4.7 shows the actual summary statistics from the employment survey. However, this was not published in the news report.

Summary Statistics of Employment Survey

Mean Starting Salary	\$3580	Lower Quartile	\$2785
Median Starting Salary	\$2960	Upper Quartile	\$3692
Modal Starting Salary	\$2850	90^{th} Percentile	\$5120

Table 4.7

1. From Table 4.7, how do we interpret
   (i) the median starting salary, **(ii)** the 90^{th} percentile?
2. Find the numerical difference between the mean starting salary and median starting salary. What does the difference suggest about the distribution of starting salaries among SNU's fresh graduates, and what could have caused this difference?
3. Combining your analysis from Questions 1 and 2 and Table 4.7, do you think that the 'average monthly starting salary' used in the news report gives an accurate description of the starting salaries of fresh graduates from SNU? Explain your answer.
4. What are some points that you can learn about the **choice of averages** presented in everyday statistics?

Part 2: The Collection of Statistical Data

Fig. 4.13 shows a printed advertisement by a toothpaste company, Superclean. Study Fig. 4.13 and answer the questions.

Fig. 4.13

5. In Fig. 4.13, do you think that the statistic '90% of respondents in a recent survey recommend Superclean Toothpaste' is credible? Give two reasons to support your answer.

6. The Statville Advertising Standards Authority has banned this advertisement as it believes that it is '**misleading**' consumers. With reference to Fig. 4.14, explain why the advertisement in Fig. 4.13 is considered to be misleading.

Fig. 4.14

7. What are some points that you can learn about the **collection of statistical data**?

Part 3: The Display of Statistical Data

Statistical graphs and diagrams are often used to summarise or highlight data findings. Study each of the diagrams below.

An investment company displayed the bar graph in Fig. 4.15 to its prospective clients, to demonstrate the positive outlook in investing in one of its financial products.

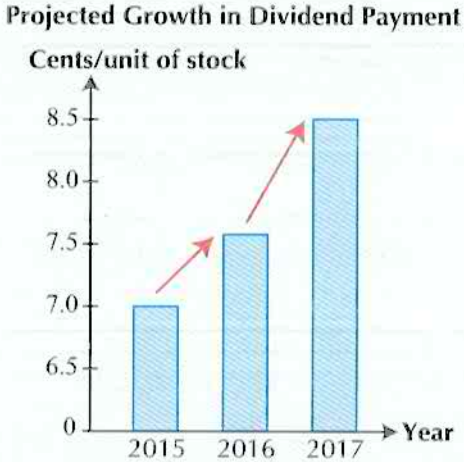


Fig. 4.15

8. Explain what is misleading about the bar graph.

The Chief Executive Officer of a big smartphone company used the following 3D pie chart to present his company's market share in the global smartphone market during a company presentation.

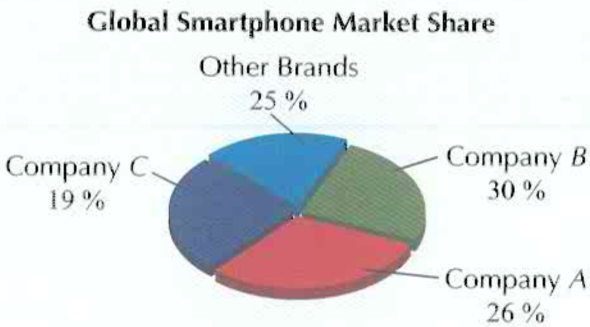


Fig. 4.16

9. **(i)** Which company do you think used the 3D pie chart in Fig. 4.16? Explain your answer.

(ii) Explain what is misleading about this 3D pie chart.

10. What are some points that you can learn about the **display of statistical data**?

Exercise 4D

BASIC LEVEL

1. Calculate the standard deviation of each set of data. Show your working clearly.

(a) 3, 4, 5, 7, 8, 10, 13
(b) 28, 25, 32, 20, 30, 19, 22, 24, 27, 23
(c) –5, –4, 0, 1, 4, –2

2. Use your calculator to find the standard deviation of each set of data.

(a) 128, 135, 156, 123, 144, 130
(b) 0, 1, 25, 14, 2, 16, 22, 4
(c) 39.6, 12, 13.5, 22.6, 31.3, 8.4, 5.5, 4.7

3. The distribution of marks scored by students for a class quiz is shown in the table below.

Marks	2	3	4	5	6	7	8
Number of Students	5	7	6	4	9	3	6

Calculate the standard deviation for the marks. Show your working clearly.

4. The number of goals scored per match by Spurs United during the soccer league season is shown in the frequency table below.

Number of Goals Scored per Match	0	1	2	3	4	5	6
Number of Matches	10	8	7	6	2	3	1

Calculate the standard deviation. Show your working clearly.

5. Find an estimate of the standard deviation for the following set of data. Show your working clearly.

x	Frequency
$0 < x \leqslant 5$	4
$5 < x \leqslant 10$	12
$10 < x \leqslant 15$	20
$15 < x \leqslant 20$	24
$20 < x \leqslant 25$	16
$25 < x \leqslant 30$	4

6. The weekly salaries, in dollars, of 60 workers in a company are shown in the table below.

Salary (\$)	Frequency
$200 < x \leqslant 220$	8
$220 < x \leqslant 240$	23
$240 < x \leqslant 260$	16
$260 < x \leqslant 280$	3
$280 < x \leqslant 300$	10

Find an estimate of the standard deviation of the weekly salary of the workers. Show your working clearly.

7. Use your calculator to find an estimate of the standard deviation of each of the following sets of data.

(a)

x	Frequency
$30 < x \leqslant 40$	16
$40 < x \leqslant 50$	25
$50 < x \leqslant 60$	35
$60 < x \leqslant 70$	14
$70 < x \leqslant 80$	10

(b)

y	Frequency
$70 \leqslant y < 75$	4
$75 \leqslant y < 80$	11
$80 \leqslant y < 85$	15
$85 \leqslant y < 90$	24
$90 \leqslant y < 95$	18
$95 \leqslant y < 100$	9
$100 \leqslant y < 105$	3

INTERMEDIATE LEVEL

8. The results for an IQ quiz taken by 8 students from Class A and Class B are shown below. The maximum score for the quiz is 25.

Class A: 4, 6, 6, 7, 8, 10, 11, 12
Class B: 0, 1, 1, 2, 3, 14, 17, 25

(i) Calculate the mean and standard deviation for Class A and Class B. Show your working clearly.

(ii) Compare and comment briefly on the results of Class A and Class B.

9. Priya scored x marks in a Mathematics quiz and her friends' scores were 5, 16, 6, 10 and 4. The mean mark of these six students is 10.

(i) Find the value of x.

(ii) Find the standard deviation of the marks of the six students.

(iii) How did Priya perform for the quiz, relative to her friends?

10. Mrs Tan reads bedtime stories to her daughter, Kate every night. The time taken for Kate to fall asleep on each night in a particular week is shown in the table below.

Day	Time Taken to Fall Asleep (Minutes)
Monday	23
Tuesday	15
Wednesday	8
Thursday	13
Friday	28
Saturday	6
Sunday	15

(i) Calculate the mean and standard deviation of the time taken for Kate to fall asleep.

Mrs Tan brought Kate on a holiday to New Zealand for a week. The time taken for Kate to fall asleep on each of the nights during the week is shown in the table below.

Day	Time Taken to Fall Asleep (Minutes)
Monday	20
Tuesday	12
Wednesday	5
Thursday	10
Friday	25
Saturday	3
Sunday	12

(ii) Calculate the mean and standard deviation of the time taken for Kate to fall asleep during the week in New Zealand.

(iii) Compare and comment on the answers in parts **(i)** and **(ii)**.

11. Two trains, A and B, are scheduled to arrive at a station at certain time. The times (in minutes) by which the trains arrived after the scheduled time were recorded in the table below.

Time (minutes)	Number of days for Train A	Number of days for Train B
2	3	4
3	2	3
4	5	9
5	12	9
6	10	7
7	6	5
8	1	3
9	1	0

(i) For each train, calculate the mean of the data and standard deviation of the data.

(ii) Which train is more consistently arriving late? Briefly explain your answer.

(iii) Which train is more punctual on the whole? Briefly explain your answer.

12. The waiting time, in minutes, for 60 patients at two hospitals are given in the tables below.

Stamford Hospital

Time (minutes)	Number of Patients
$20 < t \leqslant 22$	5
$22 < t \leqslant 24$	11
$24 < t \leqslant 26$	27
$26 < t \leqslant 28$	13
$28 < t \leqslant 30$	4

Hillview Hospital

Mean	Standard Deviation
25	3.2

(a) For Stamford Hospital, find an estimate of the

(i) mean waiting time,

(ii) standard deviation.

(b) Compare, briefly, the waiting time for the two hospitals.

13. The table shows the daily temperatures of two cities in Sahara Desert over a period of 50 days.

Temperature (°C)	Number of Days	
	City A	City B
$35 \leqslant x < 40$	1	2
$40 \leqslant x < 45$	4	14
$45 \leqslant x < 50$	12	16
$50 \leqslant x < 55$	23	10
$55 \leqslant x < 60$	7	5
$60 \leqslant x < 65$	3	3

(a) For each city, calculate an estimate of the

(i) mean temperature,

(ii) standard deviation.

(b) Which city is warmer on the whole? State a reason.

(c) Which city's daily temperature is more consistent? Explain your answer.

ADVANCED LEVEL

14. It is given that the six numbers, 10, 6, 18, x, 15 and y have a mean of 9 and a standard deviation of 6. Find the value of x and of y.

15. A set Q contains n numbers, which has a mean of 5 and a standard deviation of 1.8. Two additional numbers (from sets A, B or C) are to be added to the set Q.

A: {–2, 12}	B: {5, 16}	C: {4, 6}

(i) Which of the sets, when added to set Q, will result in a new mean which is unchanged from the original mean?

(ii) Using your answer in **(i)** or otherwise, state which of the sets, when added to set Q, will result in a new standard deviation which is closest to the original standard deviation.

16. The tables below show the masses of 100 students from Brighthill School and 100 students from Hogwarts School. (All masses are corrected to the nearest 5 kg.)

Brighthill School

Mass (x kg)	45	50	55	60	65	70
Number of Students	5	36	28	22	7	2

Hogwarts School

Mass (y kg)	40	45	50	55	60	65	70	75	80
Number of Students	7	21	24	6	3	26	8	1	4

The National Statistics Division requires the combined statistics (mean and standard deviation) of both schools.

(i) Can we use $\frac{\bar{x}+\bar{y}}{2}$ to find the combined mean? Explain your answer.

(ii) Can we add the standard deviations of the masses for both schools to find the combined standard deviation? Explain your answer.

(iii) Find an estimate of the combined mean and standard deviation of all 200 students.

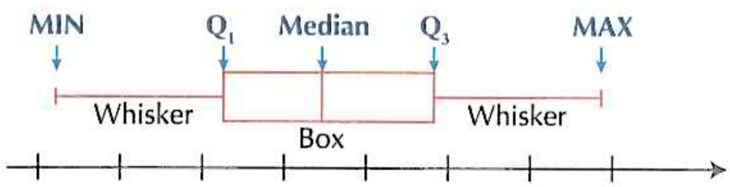
1. A **table of cumulative frequencies** is a way of presenting a set of data. It can be obtained from a frequency table. The cumulative distribution can be displayed graphically by a **cumulative frequency curve**.

2. The **range** of a set of data is the difference between the largest value and the smallest value.

3. The **interquartile range** is the difference between the **upper quartile** (Q_3) and the **lower quartile** (Q_1). It measures the spread of the **middle 50%** of the data.

4. A cumulative frequency curve can be used to *estimate* the **median**, **quartiles** and **percentiles** of a distribution. It can also be used to obtain estimates such as how many students scored less than a certain mark, etc.

5. A **box-and-whisker plot** illustrates the range, lower quartile (Q_1), median and upper quartile (Q_3) of a frequency distribution.



6. The **standard deviation** measures the spread of a set of data about the mean.

7. For **ungrouped data**, the standard deviation is

$$\boxed{\sqrt{\frac{\sum(x-\bar{x})^2}{n}} \quad \text{or} \quad \sqrt{\frac{\sum x^2}{n}-\bar{x}^2},}$$

where the mean, $\bar{x}=\dfrac{\sum x}{n}$.

8. For **grouped data**, the standard deviation is

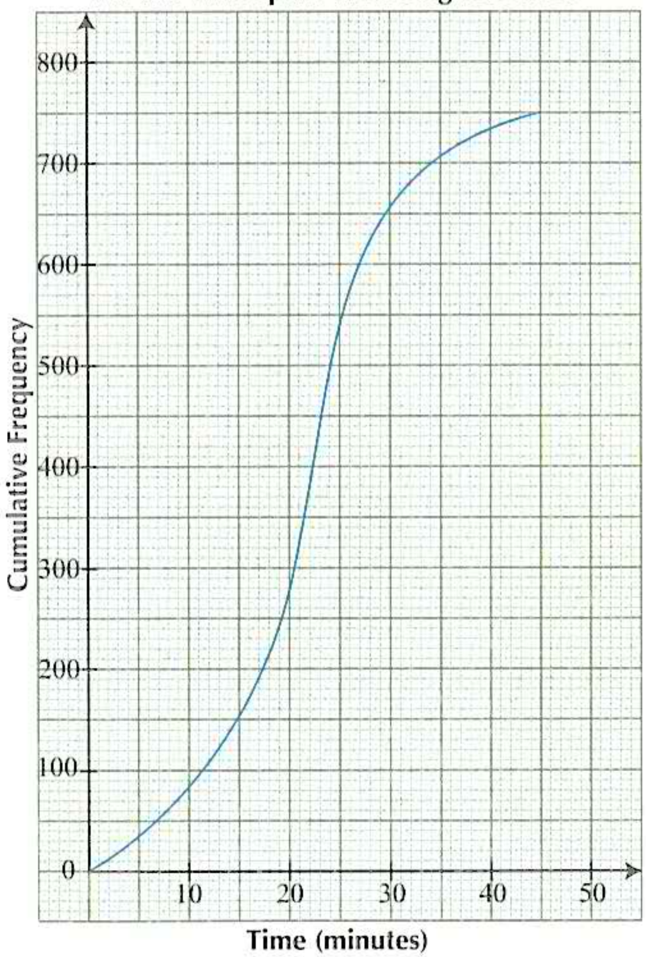
$$\boxed{\sqrt{\frac{\sum f(x-\bar{x})^2}{\sum f}} \quad \text{or} \quad \sqrt{\frac{\sum fx^2}{\sum f}-\bar{x}^2},}$$

where the mean, $\bar{x}=\dfrac{\sum fx}{\sum f}$.

Review Exercise 4

1. The amount of time spent by 750 students of Hillcrest School to travel from home to school on a particular morning is shown in the cumulative frequency curve.

Cumulative Frequency Curve for the Time Spent Travelling to School



(a) Use the graph to estimate
   (i) the number of students who take less than 17.5 minutes to travel to school,
   (ii) the fraction of the 750 students who take at least 27 minutes to travel to school,
   (iii) the value of x, given that 40% of the 750 students take at least x minutes to travel to school.

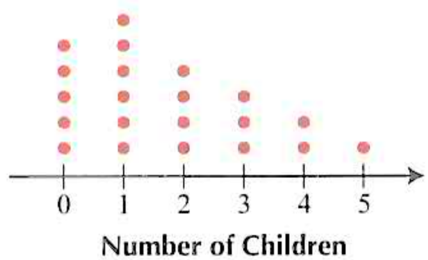
(b) Estimate the 90^{th} percentile.

2. The daily amounts of money (in dollars) spent by Kate over a period of 14 days are recorded as follows:

10, 120, 20, 5, 9, 12, 30
15, 13, 23, 19, 20, 84, 9

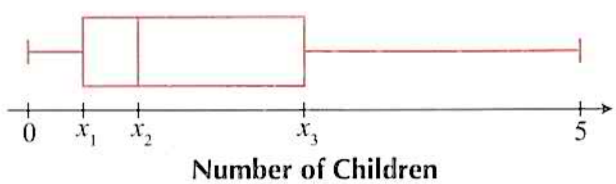
Find
   (i) the median of the data,
   (ii) the interquartile range,
   (iii) the mean and standard deviation of the data.

3. A survey was done to find out the number of children (aged 13 or below) in a family.



(a) Find the standard deviation of the number of children in a family.

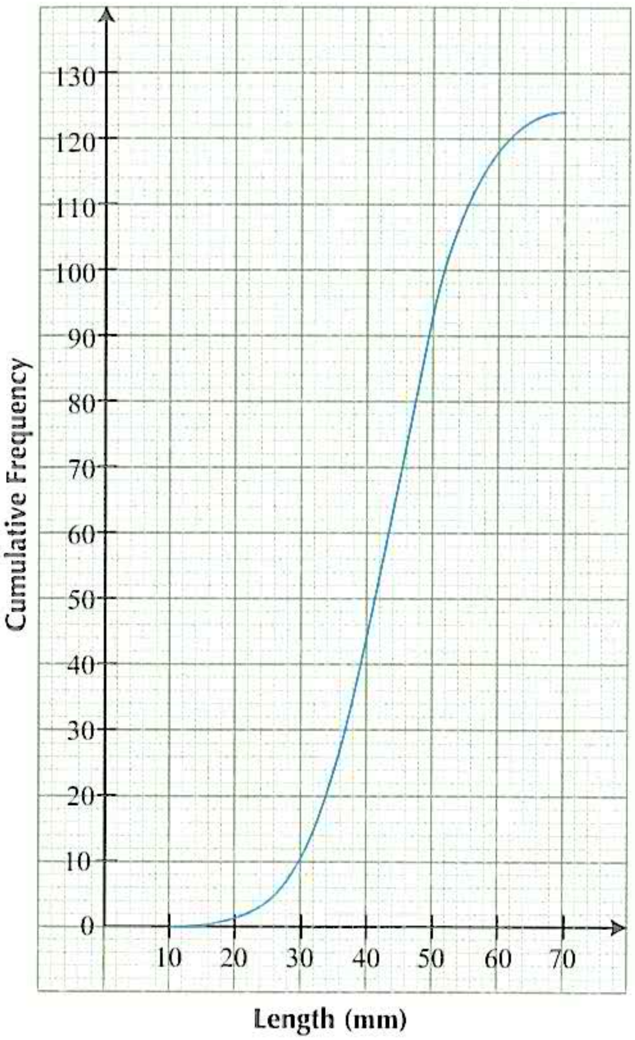
(b) A box-and-whisker plot is drawn to represent the data.



   (i) Find the values of x_1, x_2 and x_3.
   (ii) Find the interquartile range of the data.

4. In an agricultural experiment, the lengths (mm) of 124 ears of barley from Australia were measured. The cumulative frequency curve shows the length x mm, and the number of ears of barley with lengths less than or equal to x mm.

Cumulative Frequency Curve for the Lengths of Ears of Barley



(a) Using the graph, estimate
  (i) the median length,
  (ii) lower and upper quartiles.

(b) Find the interquartile range.

(c) Using the graph, find the number of ears of barley with lengths
  (i) greater than 55 mm,
  (ii) not greater than 25 mm or greater than 64 mm.

5. In a rifle range, Vishal and Jun Wei fired 6 shots each at a target. The table shows the distance, in millimetres, of each shot from the centre of the target.

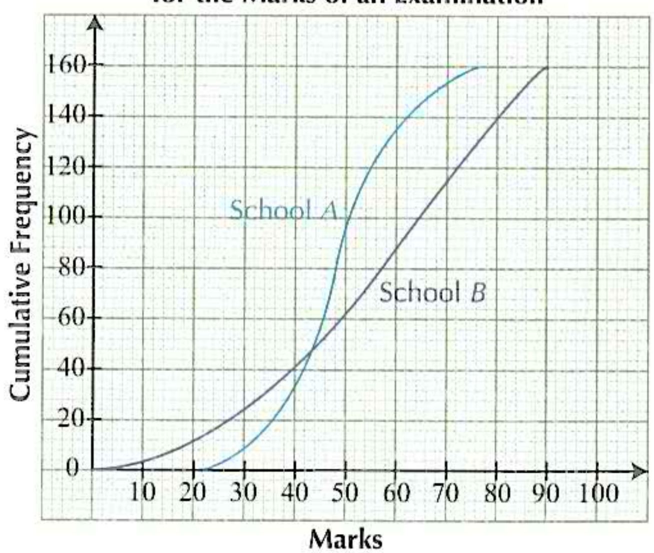
Shooter	Distance from centre of target (mm)					
Vishal's shots	47	16	32	1	19	35
Jun Wei's shots	20	9	16	43	13	4

(a) For each shooter, calculate
  (i) the mean distance from the centre of the target,
  (ii) the standard deviation.

(b) Make two comparisons between the shots fired by Vishal and Jun Wei.

6. There are 160 students taking the same examination paper in each of the two schools. The cumulative frequency curves show the marks scored by the students.

Cumulative Frequency Curve for the Marks of an Examination



(a) For the students in School A, use the graph to find
  (i) the median,
  (ii) interquartile range.

(b) For the students in School B, use the graph to find
  (i) the median,
  (ii) interquartile range.

(c) Estimate the percentage of students from School B who scored more than 80 marks.

(d) Make two comparisons between the scores of the students from School A and School B.

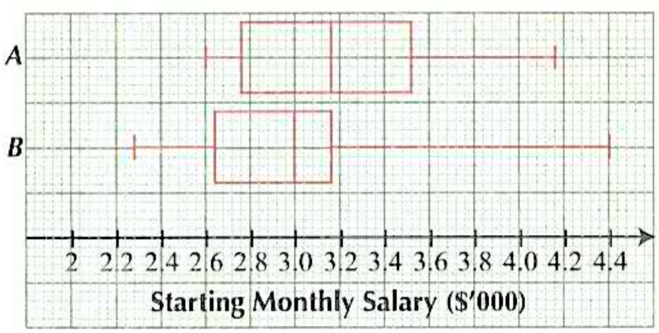
7. The following table shows the lifespans, to the nearest hour, for 100 light bulbs produced by two companies, Brightworks and Lumina.

Lifespans (hours)	Number of light bulbs	
	Brightworks	Lumina
$600 \leq t < 700$	2	8
$700 \leq t < 800$	9	10
$800 \leq t < 900$	16	12
$900 \leq t < 1000$	21	16
$1000 \leq t < 1100$	29	r
$1100 \leq t < 1200$	18	18
$1200 \leq t < 1300$	5	12
Mean	p	989.5
Standard Deviation	q	t

(i) Find the values of p, q, r and t.

(ii) Make two comparisons between the lifespans of the light bulbs produced by Brightworks and Lumina.

8. The box-and-whisker plots below show the distributions of the starting monthly salaries of fresh graduates from two universities, A and B.



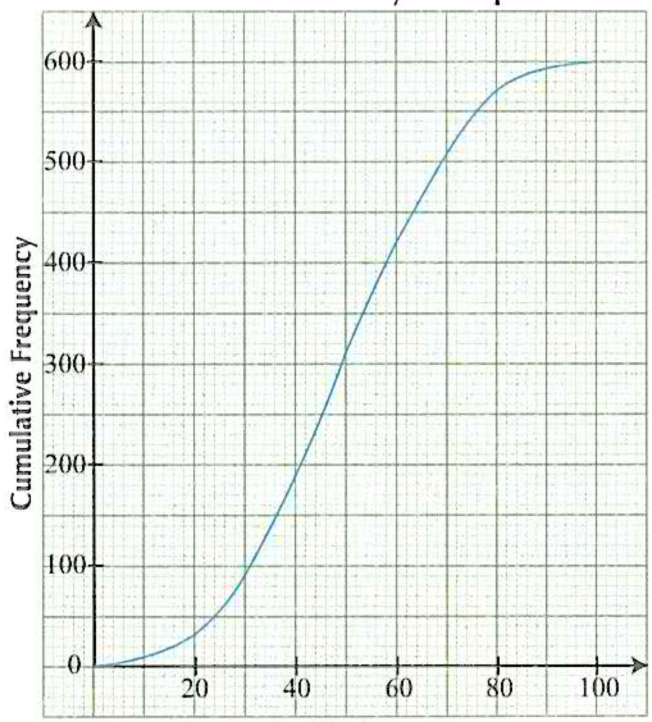
(i) Find the interquartile range for both universities.

(ii) 'On average, fresh graduates from university A get a higher starting salary than fresh graduates from university B'. Do you agree? Give a reason for your answer.

(iii) Which university has a bigger proportion of fresh graduates getting more than $3500 for their starting salary? Give a reason for your answer.

9. The cumulative frequency curve below shows the distribution of the marks scored by 600 students in a Mathematics examination in Euler High School.

Cumulative Frequency Curve for Marks Scored by 600 Pupils



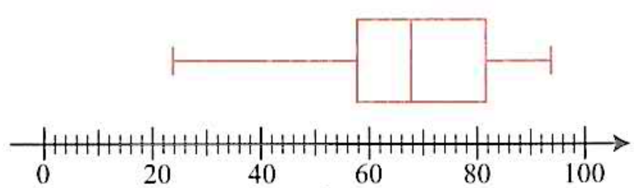
(a) Use the graph to estimate

(i) the median mark,

(ii) the passing mark such that 60% of the students will pass the examination.

(b) Indicate clearly the upper and lower quartiles on the graph and find the interquartile range.

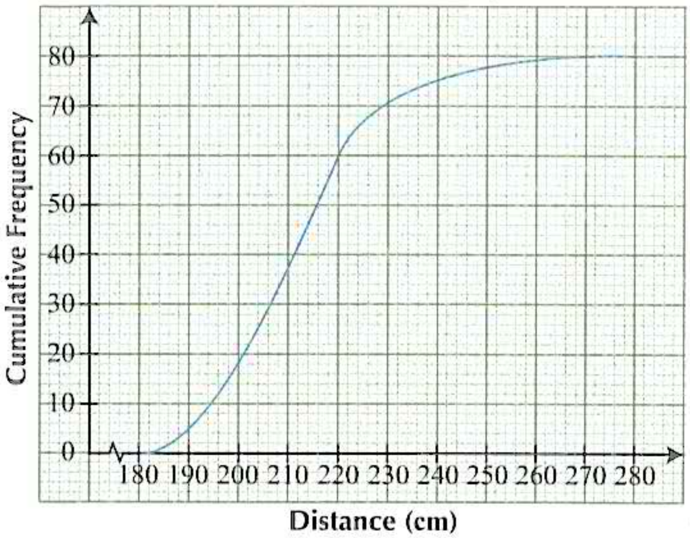
The box-and-whisker plot gives the information on the marks scored by 600 students in the same examination in Fermat High School.



(c) Find the median mark and the interquartile range. Hence, comment briefly on the performance of the students in the two schools.

10. The cumulative frequency curve below represents the standing broad jump distance (cm) of 80 Secondary Four boys.

Cumulative Frequency Curve for the Standing Broad Jump Distance of Secondary Four Boys



(a) Copy and complete the grouped frequency table of the standing broad jump distance of each boy.

Distance (cm)	Frequency
$180 \leqslant x < 200$	
$200 \leqslant x < 220$	
$220 \leqslant x < 240$	
$240 \leqslant x < 260$	
$260 \leqslant x < 280$	

(b) Using the grouped frequency table, find an estimate of

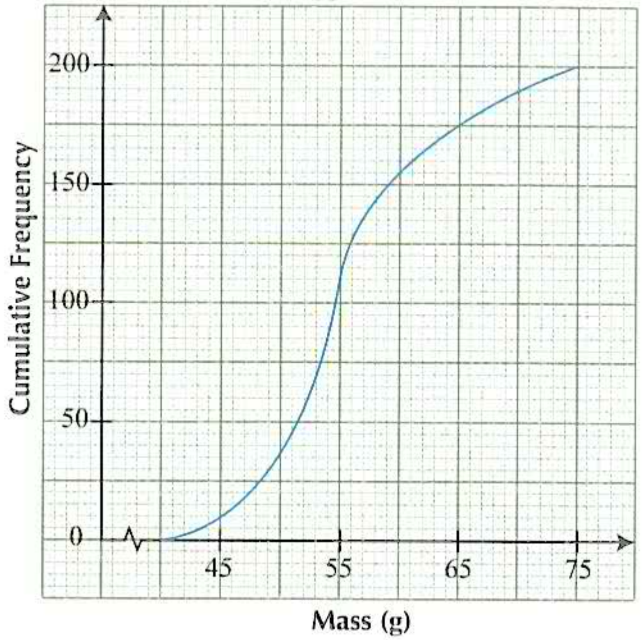
(i) the mean standing broad jump distance,

(ii) the standard deviation.

(c) Another batch of 80 students who have taken the standing broad jump test have the same median but a larger standard deviation. Describe how its cumulative frequency curve will differ from the given curve.

11. The masses of 200 eggs from Rainbow Farm were measured and the results are illustrated by the cumulative frequency curve below.

Cumulative Frequency Curve for the Masses of Eggs from Rainbow Farm



The eggs are graded according to their masses in grams:

Grade 1: $62 \text{ g} < m \leqslant 75 \text{ g}$
Grade 2: $51 \text{ g} < m \leqslant 62 \text{ g}$
Grade 3: $40 \text{ g} < m \leqslant 51 \text{ g}$

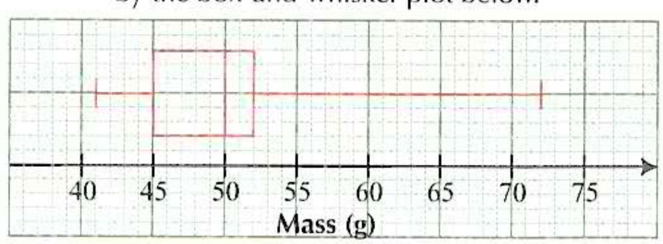
(a) Using the curve, estimate

(i) the median mass,

(ii) the interquartile range,

(iii) the percentage of eggs in each grade.

(b) The masses of 200 eggs from Skyhi Farm were also measured and the results are represented by the box-and-whisker plot below.



(i) State the median mass and the interquartile range.

(ii) 5% of the eggs produced by Skyhi Farm are Grade 1. Make two comparisons between the quality of eggs from the two farms.

Challenge Yourself

1. A set of data, Set A is given below.

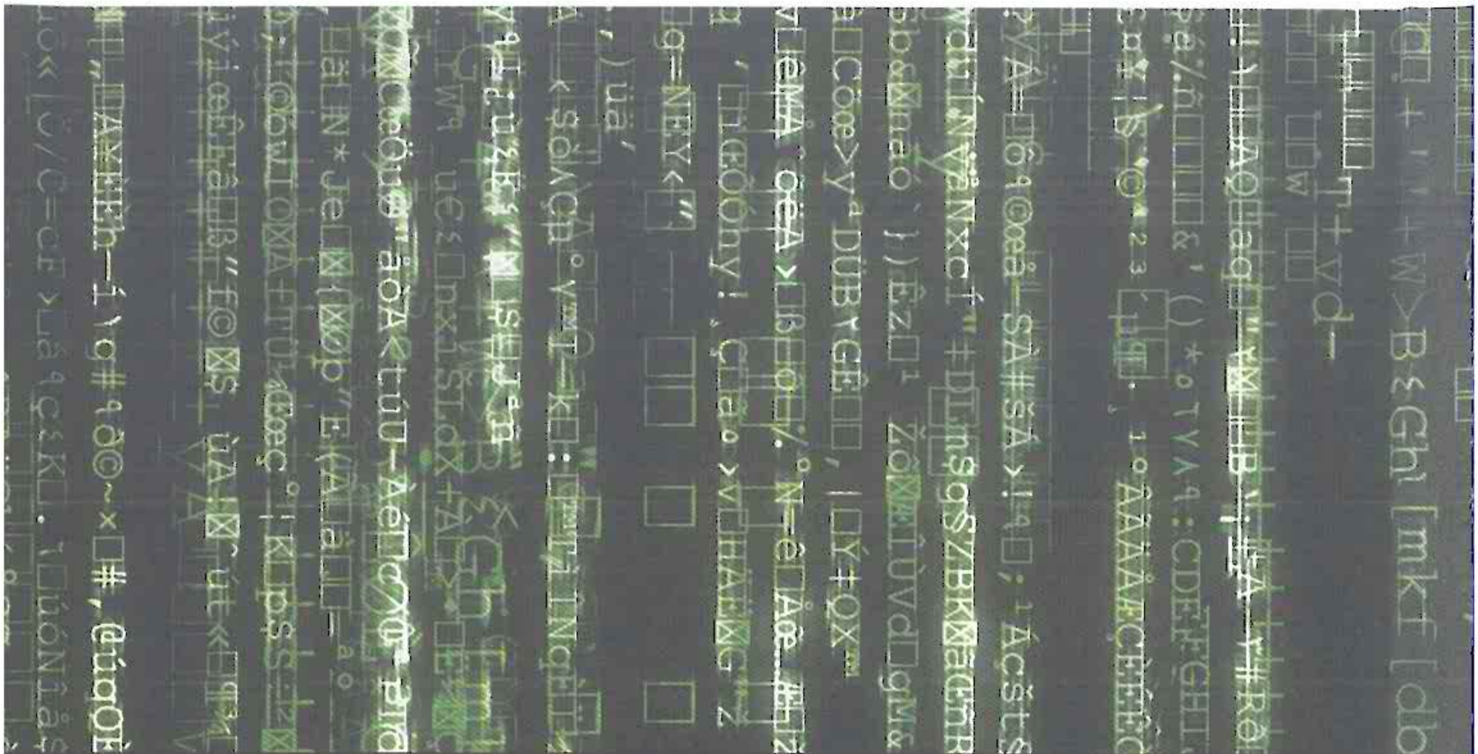
   Set A: 9, 11, 16, 24, 34

   Another four sets of data, Sets W, X, Y and Z are shown below.

   Set W: −9, −11, −16, −24, −34
   Set X: 14, 16, 21, 29, 39
   Set Y: 18, 22, 32, 48, 68
   Set Z: 16, 26, 34, 39, 41

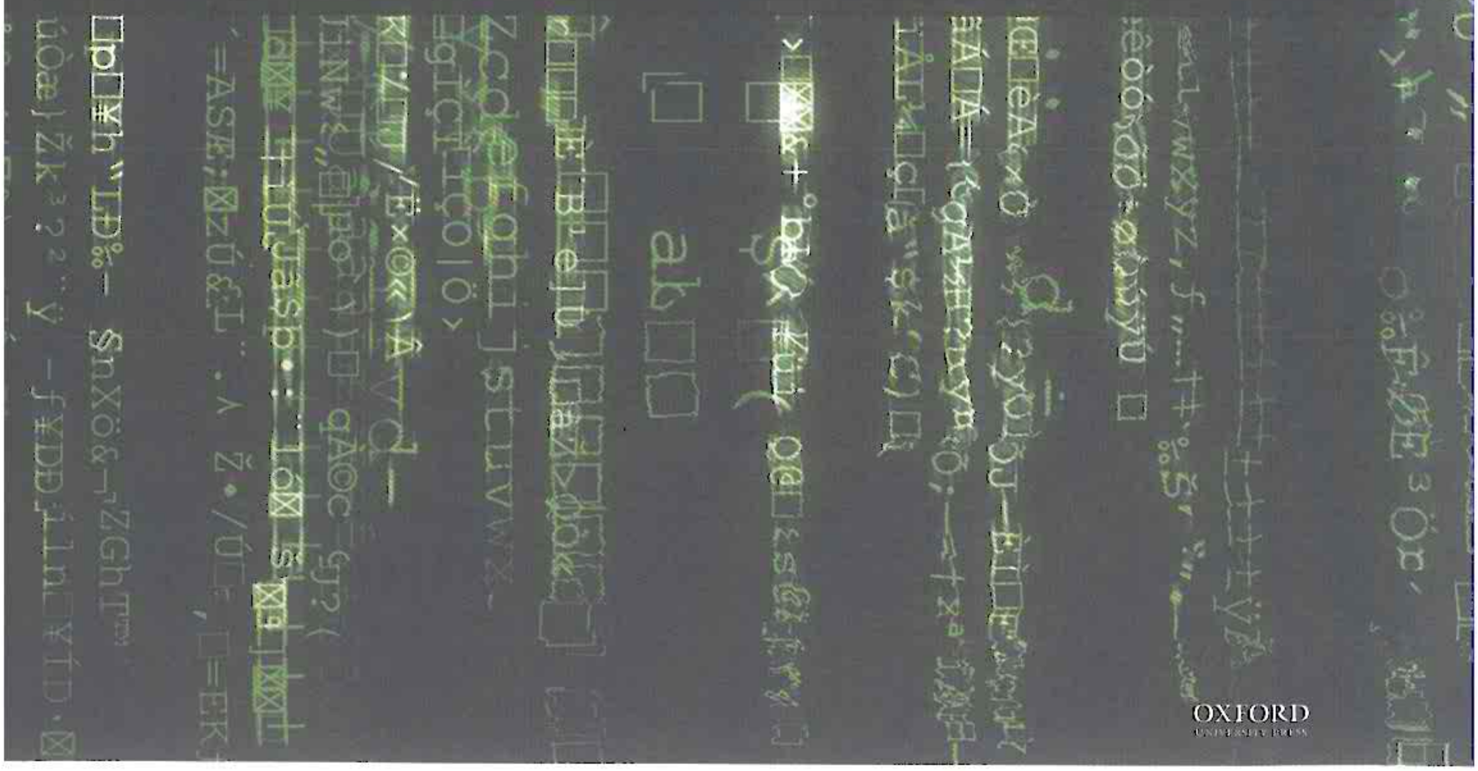
   Without calculating the standard deviation for any of the above sets, explain clearly which of the sets W, X, Y or Z has the same standard deviation as that of Set A.
   Hint: The standard deviation describes how the data are spread about the mean.

2. Two sets, M and N, have the same mean, standard deviation and data size, i.e. n. Give an example of such a pair of sets.



Matrices

In the movie The Matrix, the matrix code is a way of representing the activity of the virtual reality environment known as the Matrix. The downward-flowing green characters in the matrix code include mirror images of half-width kana characters, and Latin letters and numerals. In Mathematics, a matrix is a rectangular array of numbers which can be used to represent information in the real world. In this chapter, we will learn about matrices and some of their applications.



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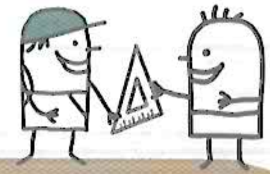
Chapter Five

LEARNING OBJECTIVES

At the end of this chapter, you should be able to:

- display information in the form of a matrix of any order,
- interpret the data in a given matrix,
- add and subtract two matrices of the same order,
- multiply a matrix by a scalar,
- multiply two matrices,
- find the inverse of a 2×2 matrix,
- solve problems involving addition, subtraction and multiplication of matrices.

5.1 Introduction to Matrices



Matrix Notations

Class Discussion

Defining a Matrix

Table 5.1 shows the number of pens of three different brands in two stationery shops owned by the same boss during a stock-take.

	Brand *A*	Brand *B*	Brand *C*
Shop 1	16	7	69
Shop 2	58	64	76

Table 5.1

These pieces of information can be represented using a **matrix**, which is a **rectangular array of numbers**.

$$\begin{array}{cc} & \text{3 columns} \\ & \downarrow \quad \downarrow \quad \downarrow \\ \text{2 rows} \begin{array}{c} \rightarrow \\ \rightarrow \end{array} & \begin{pmatrix} 16 & 7 & 69 \\ 58 & 64 & 76 \end{pmatrix} \end{array}$$

1. How many rows and columns does the above matrix have?

2. The numbers in the matrix are called the **elements** of the matrix. The elements in the first row of the above matrix represent the number of pens of each of the three brands in Shop 1. What do the elements in the second row represent?

3. The elements in the first column of the above matrix represent the number of pens of Brand *A* in each of the two shops. What do the elements in the second column, and in the third column represent?

Since this matrix has 2 **rows** and 3 **columns**, we say that the **order** of this matrix is 2 by 3 (also written as 2×3), or this is a 2×3 matrix.

4. Represent the information in Table 5.1 using a 3×2 matrix.

ATTENTION

'Matrix' is pronounced as 'may-trix', not 'mat-trix'. The plural of 'matrix' is 'matrices'.

ATTENTION

In a matrix, the rows are always horizontal and the columns vertical. In real life, we sometimes use the term 'row' differently, e.g. we ask people to queue up in two (vertical) rows, when in fact we mean 'columns'.

ATTENTION

When stating the order of a matrix, we always write the number of rows first.

PRACTISE NOW

SIMILAR QUESTIONS

Exercise 5A Questions 1, 3, 7

1. Write down the order of each of the following matrices.

(a) $\begin{pmatrix} 2 & 3 & 10 & -8 \\ -1 & 0 & 7\frac{1}{2} & 4 \end{pmatrix}$ (b) $\begin{pmatrix} -3 & 2 \\ 1.7 & 5 \end{pmatrix}$ (c) $\begin{pmatrix} 14 \\ 7 \\ 76 \end{pmatrix}$

(d) $\begin{pmatrix} -3 & 4 & 9 & 0 \end{pmatrix}$ (e) (7) (f) (0)

2. The table below shows the number of students in a class and the sports that they like best (i.e. they can only choose one sport).

	Soccer	Basketball	Swimming	Other Sports
Boys	14	3	5	2
Girls	1	8	3	4

(i) Represent the data in the table by using a matrix **M**.

(ii) How many boys like swimming best?

(iii) Find the sum of the elements in the first column of **M**. What does this sum represent?

(iv) How do you use the elements of **M** to find the number of girls in the class? What is the answer?

To summarise, an $m \times n$ **matrix**, or a matrix of **order** $m \times n$, refers to a matrix with m rows and n columns.

A matrix with one row, such as $\begin{pmatrix} -3 & 4 & 9 & 0 \end{pmatrix}$ is called a **row matrix.**

A matrix with one column, such as $\begin{pmatrix} 14 \\ 7 \\ 76 \end{pmatrix}$ is called a **column matrix.**

A $n \times n$ matrix refers to a matrix with the same number of rows and columns, e.g. $\begin{pmatrix} -3 & 2 \\ 1.7 & 5 \end{pmatrix}$. It is called a **square matrix**.

If every element in a matrix is 0, the matrix is called a **zero matrix** (or a **null matrix**), and is usually denoted by $\mathbf{0}$, e.g. (0), $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 & 0 \end{pmatrix}$.

Thinking Time

1. Write down a 3×3 square matrix.
2. Write down a 1×2 zero matrix and a 2×3 zero matrix.
3. Is (0) equal to 0? Explain.

Equal Matrices

Two matrices $\mathbf{A}$ and $\mathbf{B}$ are **equal** if and only if

(a) both matrices have the *same order*, and

(b) their *corresponding elements* are equal.

For example, if $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, then $\mathbf{A} = \mathbf{B}$.

ATTENTION

The corresponding elements of two matrices refer to the elements in the same position of both matrices.

Thinking Time

1. Are $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 2 & 0 \\ 3 & 0 \end{pmatrix}$ equal? Explain.
2. Are $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ equal? Explain.
3. Are $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$ equal? Explain.

Worked Example 1

(Problem involving Equal Matrices)

If $\mathbf{A} = \begin{pmatrix} 20 & b \\ c & 16 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 4a & a+3 \\ d-4 & d \end{pmatrix}$ and $\mathbf{A} = \mathbf{B}$, find the values of a, b, c and d.

Solution:

$\mathbf{A} = \mathbf{B}$, so $\begin{pmatrix} 20 & b \\ c & 16 \end{pmatrix} = \begin{pmatrix} 4a & a+3 \\ d-4 & d \end{pmatrix}$.

Equating the corresponding elements, we have

$20 = 4a,$ -------- (1)

$b = a + 3,$ -------- (2)

$c = d - 4,$ -------- (3) and

$16 = d.$ -------- (4)

From (1),

$\therefore\ a = \frac{20}{4}$

$= 5$

From (2),

$b = 5 + 3$

$= 8$

Substitute $d = 16$ (4) into $c = d - 4$ (3)

$\therefore\ c = 16 - 4$

$= 12$

$\therefore\ a = 5, b = 8, c = 12$ and $d = 16$

PRACTISE NOW 1

SIMILAR QUESTIONS

Exercise 5A Questions 2, 4–6

If $\mathbf{X} = \begin{pmatrix} 8a & 5 \\ c & -9 \end{pmatrix}$, $\mathbf{Y} = \begin{pmatrix} 16 & a+b \\ d+3 & 3d \end{pmatrix}$ and $\mathbf{X} = \mathbf{Y}$, find the values of a, b, c and d.

Exercise 5A

BASIC LEVEL

1. Write down the order of each of the following matrices.

(a) $\begin{pmatrix} -1 & 3 \\ 0 & 2 \\ 5 & 6 \end{pmatrix}$

(b) $\begin{pmatrix} 7 & 6\frac{1}{2} & -8 \end{pmatrix}$

(c) $\begin{pmatrix} -8 & -1 & 7 \\ 3 & 0 & 4.3 \\ 5 & -9 & 12 \end{pmatrix}$

(d) $\begin{pmatrix} 5 \\ 0 \end{pmatrix}$

(e) (-13)

(f) $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

2. Which of the following pairs of matrices are equal? If they are not equal, state the reason.

(a) $\mathbf{A} = \begin{pmatrix} -4 \\ 5 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} -4 \\ 5 \end{pmatrix}$

(b) $\mathbf{C} = \begin{pmatrix} 7 & 8 \\ -8 & 2 \end{pmatrix}$, $\mathbf{D} = \begin{pmatrix} 7 & -8 \\ 8 & 2 \end{pmatrix}$

(c) $\mathbf{P} = \begin{pmatrix} -2 \\ 7 \end{pmatrix}$, $\mathbf{Q} = \begin{pmatrix} -2 & 7 \end{pmatrix}$

(d) $\mathbf{X} = \begin{pmatrix} 0 & 6 \\ 0 & -3 \end{pmatrix}$, $\mathbf{Y} = \begin{pmatrix} 6 \\ -3 \end{pmatrix}$

INTERMEDIATE LEVEL

3. The table below shows the number of students in a class and the type of fruits that they like best among the fruits given (i.e. they can only choose one type of fruit among the fruits given).

	Apple	Orange	Banana	Durian
Boys	4	0	5	6
Girls	8	7	5	3

(i) Represent the data in the table by using a matrix **F**.

(ii) Which is the type of fruit that is liked best by an equal number of boys and girls?

(iii) Find the sum of the elements in the first row of **F**. What does this sum represent?

(iv) How do you use the elements of **F** to find the number of students who like durian best? What is the answer?

4. Which of the following pairs of matrices are equal? If they are not equal, state the reason.

(a) $\mathbf{P} = \begin{pmatrix} 6 \\ -3 \end{pmatrix}$, $\mathbf{Q} = (0)$

(b) $\mathbf{X} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $\mathbf{Y} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

5. State which of the following matrices are equal.

$A = \begin{pmatrix} 5 & 7 \end{pmatrix}$ $B = \begin{pmatrix} 5 \\ 7 \end{pmatrix}$

$C = \begin{pmatrix} 7 & 5 \end{pmatrix}$ $D = \begin{pmatrix} 7 \\ 5 \end{pmatrix}$

$E = \begin{pmatrix} 3 & 7 \\ 5 & -4 \end{pmatrix}$ $F = \begin{pmatrix} -4 & 5 \\ 7 & 3 \end{pmatrix}$

$G = \begin{pmatrix} 3 & 5 \\ 7 & -4 \end{pmatrix}$ $H = \begin{pmatrix} 3 & 7 \\ 5 & -4 \end{pmatrix}$

$I = \begin{pmatrix} 7 \\ 5 \end{pmatrix}$ $J = \begin{pmatrix} 3 & 2 & 7 \end{pmatrix}$

$K = \begin{pmatrix} 3 \\ 2 \\ 7 \end{pmatrix}$ $L = \begin{pmatrix} 3 & 5 \\ 7 & -4 \end{pmatrix}$

$M = \begin{pmatrix} 7 \\ 2 \\ 3 \end{pmatrix}$ $N = \begin{pmatrix} 3 & 2 & 7 \end{pmatrix}$

$O = \begin{pmatrix} 7 & 5 \end{pmatrix}$ $P = \begin{pmatrix} -4 & 5 \\ 7 & 3 \end{pmatrix}$

$Q = \begin{pmatrix} 5 \\ 7 \end{pmatrix}$

6. Find the values of the unknowns in each of the following.

(a) $\begin{pmatrix} 2 & 3 \\ 5 & k \end{pmatrix} = \begin{pmatrix} 2a & b \\ c & 7 \end{pmatrix}$

(b) $\begin{pmatrix} 3 & 5 & b \\ 7 & -3 & c \end{pmatrix} = \begin{pmatrix} a & 5 & 13 \\ d & -a & 6 \end{pmatrix}$

(c) $\begin{pmatrix} 2x & 18 \\ 3y & 36 \end{pmatrix} = \begin{pmatrix} 14 & 2k \\ 15 & 6h \end{pmatrix}$

(d) $\begin{pmatrix} 2x-3 & y+4 \end{pmatrix} = \begin{pmatrix} 7 & 6 \end{pmatrix}$

(e) $\begin{pmatrix} \frac{1}{2}x & x+4 \\ 5 & 3y \end{pmatrix} = \begin{pmatrix} 3 & h \\ k-9 & 27 \end{pmatrix}$

(f) $\begin{pmatrix} 2x-5 & y-4 \\ z+3 & 5k \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

ADVANCED LEVEL

7. The table below shows the total number of goals scored in each of the soccer matches between 4 teams in a tournament. Each of the 4 teams plays against one another only once.

	Team *A*	Team *B*	Team *C*	Team *D*
Team *A*	0	3	1	7
Team *B*	3	0	4	2
Team *C*	1	4	0	5
Team *D*	7	2	5	0

(i) Represent the data in the table using a matrix **S**.

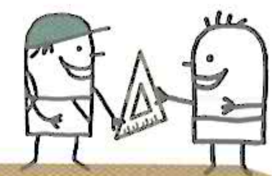
(ii) What is the total number of goals scored in the match between Team *C* and Team *D*?

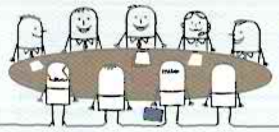
(iii) What do the zeros in **S** represent?

(iv) Find the sum of the elements in the second row of **S**. What does this sum represent?

(v) Can you explain why the elements in **S** are symmetrical about the diagonal of zeros?

5.2 Addition and Subtraction of Matrices





Class Discussion

Addition of Matrices

Let us return to the context of the matrix for Table 5.1 in the class discussion on page 137: $\mathbf{M} = \begin{pmatrix} 16 & 7 & 69 \\ 58 & 64 & 76 \end{pmatrix}$

Suppose the boss of the stationery shops receives a new stock of 100 pens of each brand. How should he distribute them? Should he give an equal number of pens of each brand to each of the two shops, or more pens to the shop with less stock?

Suppose he decides to distribute the pens according to this matrix:

$$\mathbf{N} = \begin{pmatrix} 70 & 80 & 50 \\ 30 & 20 & 50 \end{pmatrix}$$

1. How do you add the two matrices, **M** and **N**, to give the final stock of the pens of each brand in each shop?

$$\begin{aligned} \mathbf{M}+\mathbf{N} &= \begin{pmatrix} 16 & 7 & 69 \\ 58 & 64 & 76 \end{pmatrix} + \begin{pmatrix} 70 & 80 & 50 \\ 30 & 20 & 50 \end{pmatrix} \\ &= \begin{pmatrix} 16+70 & 7+80 & 69+\_\_\_ \\ 58+\_\_\_ & \_\_\_\_\_ & \_\_\_\_\_ \end{pmatrix} \\ &= \begin{pmatrix} 86 & \_\_\_ & \_\_\_ \\ \_\_\_ & \_\_\_ & \_\_\_ \end{pmatrix} \end{aligned}$$

Hence, when we add two matrices, we get a new matrix whose elements are the sum of the corresponding elements of the two matrices.

2. Can you add two matrices of different orders, e.g.

$$\begin{pmatrix} 16 & 7 & 69 \\ 58 & 64 & 76 \end{pmatrix} + \begin{pmatrix} 70 & 80 \\ 30 & 20 \end{pmatrix}?$$

Explain.

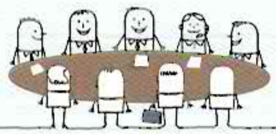
3. How about adding the following two matrices?

$$\begin{pmatrix} 16 & 7 & 69 \\ 58 & 64 & 76 \end{pmatrix} + \begin{pmatrix} 70 & 80 & 0 \\ 30 & 20 & 0 \end{pmatrix}$$

Explain.

In general, if two matrices $\mathbf{A}$ and $\mathbf{B}$ have the **same order**, then $\mathbf{A}+\mathbf{B}$ is obtained by *adding the corresponding elements* of $\mathbf{A}$ and $\mathbf{B}$, e.g.

$$\text{if } \mathbf{A}=\begin{pmatrix} p & q \\ r & s \end{pmatrix} \text{ and } \mathbf{B}=\begin{pmatrix} w & x \\ y & z \end{pmatrix}, \text{ then } \mathbf{A}+\mathbf{B}=\begin{pmatrix} p+w & q+x \\ r+y & s+z \end{pmatrix}.$$



Class Discussion

Subtraction of Matrices

The final stock for the two stationery shops for the previous class discussion is now represented by the matrix $\mathbf{X}=\begin{pmatrix} 86 & 87 & 119 \\ 88 & 84 & 126 \end{pmatrix}$.

After one month, the boss finds that the stock of pens left is given by $\mathbf{Y}=\begin{pmatrix} 30 & 24 & 98 \\ 61 & 67 & 117 \end{pmatrix}$.

1. How do you subtract matrix $\mathbf{Y}$ from matrix $\mathbf{X}$ to give the quantity of the pens of each brand that were sold?

$$\mathbf{X}-\mathbf{Y}=\begin{pmatrix} 86 & 87 & 119 \\ 88 & 84 & 126 \end{pmatrix}-\begin{pmatrix} 30 & 24 & 98 \\ 61 & 67 & 117 \end{pmatrix}$$

$$=\begin{pmatrix} 86-30 & 87-24 & 119-\_\_\_ \\ 88-\_\_\_ & \_\_\_\_\_ & \_\_\_\_\_ \end{pmatrix}$$

$$=\begin{pmatrix} 56 & \_\_\_ & \_\_\_ \\ \_\_\_ & \_\_\_ & \_\_\_ \end{pmatrix}$$

Hence, when we subtract one matrix from another, we get a new matrix whose elements are the difference of the corresponding elements of the two matrices.

2. Do you think you can find the difference of two matrices if their orders are different?

In general, if two matrices $\mathbf{A}$ and $\mathbf{B}$ have the **same order**, then $\mathbf{A}-\mathbf{B}$ is obtained by *subtracting the corresponding elements* of $\mathbf{B}$ from $\mathbf{A}$, e.g.

$$\text{if } \mathbf{A}=\begin{pmatrix} p & q \\ r & s \\ t & u \end{pmatrix} \text{ and } \mathbf{B}=\begin{pmatrix} w & x \\ y & z \\ a & b \end{pmatrix}, \text{ then } \mathbf{A}-\mathbf{B}=\begin{pmatrix} p-w & q-x \\ r-y & s-z \\ t-a & u-b \end{pmatrix}.$$

Worked Example 2

(Simple Addition and Subtraction of Matrices)

If $\mathbf{A} = \begin{pmatrix} 6 & 3 \\ 5 & 8 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 4 & -2 \\ 0 & 7 \end{pmatrix}$, $\mathbf{C} = \begin{pmatrix} -9 & 0 \\ 10 & -1 \end{pmatrix}$

and $\mathbf{D} = \begin{pmatrix} -1 & 3 \\ 6 & -2 \\ 0 & 8 \end{pmatrix}$, evaluate each of the following if possible. If it is not possible, explain why.

(a) $\mathbf{A} + \mathbf{B}$ **(b)** $\mathbf{A} - \mathbf{B} + \mathbf{C}$ **(c)** $\mathbf{C} - \mathbf{D}$

Solution:

(a)
$$\mathbf{A} + \mathbf{B} = \begin{pmatrix} 6 & 3 \\ 5 & 8 \end{pmatrix} + \begin{pmatrix} 4 & -2 \\ 0 & 7 \end{pmatrix} = \begin{pmatrix} 10 & 1 \\ 5 & 15 \end{pmatrix}$$

(b)
$$\mathbf{A} - \mathbf{B} + \mathbf{C} = \left[\begin{pmatrix} 6 & 3 \\ 5 & 8 \end{pmatrix} - \begin{pmatrix} 4 & -2 \\ 0 & 7 \end{pmatrix}\right] + \begin{pmatrix} -9 & 0 \\ 10 & -1 \end{pmatrix}$$
$$= \begin{pmatrix} 2 & 5 \\ 5 & 1 \end{pmatrix} + \begin{pmatrix} -9 & 0 \\ 10 & -1 \end{pmatrix}$$
$$= \begin{pmatrix} -7 & 5 \\ 15 & 0 \end{pmatrix}$$

(c) $\mathbf{C} - \mathbf{D}$ is not possible because $\mathbf{C}$ and $\mathbf{D}$ have different orders.

Problem Solving Tip

For **(b)**, we can also evaluate $\mathbf{A} - \mathbf{B} + \mathbf{C}$ straightaway, e.g. the first element is equal to $6 - 4 + (-9) = -7$.

PRACTISE NOW 2

SIMILAR QUESTIONS

Exercise 5B Questions 1–6

1. If $\mathbf{P} = \begin{pmatrix} 5 & 8 \\ 12 & 6 \end{pmatrix}$, $\mathbf{Q} = \begin{pmatrix} 0 & -3 \\ -4 & 7 \end{pmatrix}$, $\mathbf{R} = \begin{pmatrix} -7 & 0 \\ 6 & 0 \end{pmatrix}$ and $\mathbf{S} = \begin{pmatrix} -7 \\ 6 \end{pmatrix}$,
evaluate each of the following if possible. If it is not possible, explain why.
(a) $\mathbf{P} + \mathbf{Q}$ **(b)** $\mathbf{P} - \mathbf{Q} + \mathbf{R}$ **(c)** $\mathbf{R} - \mathbf{S}$

2. The marks of a Mathematics test and a Science test for three students are shown in matrix $\mathbf{P}$. The total possible score for each test is 50 marks.

$$\mathbf{P} = \begin{array}{c} \begin{matrix} \text{Raj} & \text{Ethan} & \text{Farhan} \end{matrix} \\ \begin{pmatrix} 49 & 28 & 39 \\ 47 & 45 & 21 \end{pmatrix} \end{array} \begin{array}{l} \\ \text{Mathematics} \\ \text{Science} \end{array}$$

In the second test for Mathematics and for Science, Raj scores 46 marks for Mathematics and 42 marks for Science, Ethan scores 40 marks for Mathematics and 38 marks for Science, and Farhan scores 31 marks for Mathematics and 35 marks for Science.
(i) Express the marks for the second test by using the matrix $\mathbf{Q}$ with the same order as $\mathbf{P}$.
(ii) Find the matrix $\mathbf{P} + \mathbf{Q}$.
(iii) Explain what the numbers in the matrix in **(ii)** represent.

Thinking Time

1. Is matrix addition commutative, i.e. $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$?
2. Is matrix addition associative, i.e. $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$?
3. Is matrix subtraction commutative, i.e. $\mathbf{A} - \mathbf{B} = \mathbf{B} - \mathbf{A}$?
4. Is matrix subtraction associative, i.e. $(\mathbf{A} - \mathbf{B}) - \mathbf{C} = \mathbf{A} - (\mathbf{B} - \mathbf{C})$?

Exercise 5B

BASIC LEVEL

1. Evaluate each of the following if possible. If it is not possible, explain why.

(a) $\begin{pmatrix} 3 & 4 \\ 8 & -5 \end{pmatrix} + \begin{pmatrix} 4 & 6 \\ 3 & 0 \end{pmatrix}$

(b) $\begin{pmatrix} 7 \\ -8 \end{pmatrix} + \begin{pmatrix} 5 \\ -9 \end{pmatrix}$

(c) $\begin{pmatrix} 2 & 8 & -3 \end{pmatrix} + \begin{pmatrix} -4 & 7 & 0 \end{pmatrix}$

(d) $\begin{pmatrix} 5 \\ -2 \end{pmatrix} + \begin{pmatrix} 5 & -2 \end{pmatrix}$

(e) $\begin{pmatrix} 2 & -3 & 8 \\ 10 & 5 & 4 \end{pmatrix} - \begin{pmatrix} 5 & 6 & 7 \\ -3 & 0 & 12 \end{pmatrix}$

(f) $\begin{pmatrix} 12 \\ -8.3 \\ 4 \end{pmatrix} - \begin{pmatrix} 8 \\ 1.7 \\ 0 \end{pmatrix}$

(g) $\begin{pmatrix} 8 & 9 \\ -7 & 6 \end{pmatrix} + \begin{pmatrix} 4 \\ 8 \end{pmatrix}$

(h) $\begin{pmatrix} 8 & 9 \\ -7 & 6 \end{pmatrix} + \begin{pmatrix} 4 & 0 \\ 8 & 0 \end{pmatrix}$

INTERMEDIATE LEVEL

2. Evaluate each of the following if possible. If it is not possible, explain why.

(a) $\begin{pmatrix} 3 \\ 4 \end{pmatrix} + \begin{pmatrix} -1 \\ 5 \end{pmatrix} - \begin{pmatrix} 6 \\ 7 \end{pmatrix}$

(b) $\begin{pmatrix} 4 & -1 \\ 3 & 2 \end{pmatrix} + \begin{pmatrix} 3 & 2 \\ -5 & 4 \end{pmatrix} - \begin{pmatrix} -6 & 4 \\ 2 & 1 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 3 \end{pmatrix} - \begin{pmatrix} 3 & 4 \end{pmatrix} + \begin{pmatrix} -2 & 6 \end{pmatrix}$

(d) $\begin{pmatrix} 3 & 1 & 5 \\ -7 & 8 & -2 \end{pmatrix} - \begin{pmatrix} 2 & -1 & 0 \\ 5 & -2 & 6 \end{pmatrix} + \begin{pmatrix} 7 & 5 & 8 \\ -2 & 4 & -9 \end{pmatrix}$

(e) $\begin{pmatrix} 2 \\ 3 \\ -5 \end{pmatrix} + \begin{pmatrix} 5 \\ 6 \\ 0 \end{pmatrix} - \begin{pmatrix} 2 \\ -8 \end{pmatrix}$

(f) $\begin{pmatrix} 4 & -3 \\ 2 & 5 \\ -8 & 9 \end{pmatrix} - \begin{pmatrix} -3 & 2 \\ 7 & -1 \\ 6 & -3 \end{pmatrix} + \begin{pmatrix} 4 & 5 \\ 0 & -6 \\ 2 & 8 \end{pmatrix}$

(g) $\begin{pmatrix} 2 & 5 \\ -3 & 6 \end{pmatrix} + \begin{pmatrix} -5 & 0 \\ 8 & 9 \end{pmatrix} - \begin{pmatrix} -8 & 6 & 0 \\ 2 & 8 & 0 \end{pmatrix}$

(h) $(5) - (-6) + (3)$

3. The marks of a Mathematics test and an English test for three students are shown in matrix **P**. The total possible score for each test is 50 marks.

$$\begin{array}{cc} & \begin{array}{ccc} \text{Nora} & \text{Shirley} & \text{Amirah} \end{array} \\ \mathbf{P} = & \begin{pmatrix} 41 & 38 & 29 \\ 39 & 33 & 36 \end{pmatrix} \begin{array}{l} \text{Mathematics} \\ \text{English} \end{array} \end{array}$$

In the second test for Mathematics and for English, Nora scores 42 marks for Mathematics and 33 marks for English, Shirley scores 35 marks for Mathematics and 40 marks for English, and Amirah scores 38 marks for Mathematics and 37 marks for English.

(i) Express the marks for the second test by using the matrix **Q** with the same order as **P**.

(ii) Find the matrix $\mathbf{P} + \mathbf{Q}$.

(iii) Explain what the numbers in the matrix in **(ii)** represent.

4. If $\mathbf{A} = \begin{pmatrix} 5 & -5 \\ -4 & 9 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 & 3 \\ -2 & 4 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 0 & 2 \\ -1 & 4 \end{pmatrix}$, find the value of each of the following.

(i) $\mathbf{A} + \mathbf{B}$ **(ii)** $\mathbf{B} + \mathbf{A}$

(iii) $\mathbf{B} + \mathbf{C}$ **(iv)** $\mathbf{C} + \mathbf{B}$

(v) $\mathbf{A} + (\mathbf{B} + \mathbf{C})$ **(vi)** $(\mathbf{A} + \mathbf{B}) + \mathbf{C}$

5. If $\mathbf{A} = \begin{pmatrix} 3 & 1 \\ 4 & -2 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 4 & -1 \\ 3 & -4 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, evaluate

(i) $\mathbf{A} - \mathbf{B}$, **(ii)** $\mathbf{B} - \mathbf{A}$,

(iii) $\mathbf{B} - \mathbf{C}$, **(iv)** $\mathbf{A} - (\mathbf{B} - \mathbf{C})$,

(v) $(\mathbf{A} - \mathbf{B}) - \mathbf{C}$.

ADVANCED LEVEL

6. The stocks for Chinese, Malay and Tamil textbooks for Secondary 1, 2, 3, 4 and 5 in a school bookshop on 1st December and 1st January are shown in matrix **A** and matrix **B** respectively.

$$\begin{array}{cc} & \begin{array}{ccccc} \text{Sec 1} & \text{Sec 2} & \text{Sec 3} & \text{Sec 4} & \text{Sec 5} \end{array} \\ \mathbf{A} = & \begin{pmatrix} 240 & 210 & 195 & 304 & 195 \\ 95 & 120 & 116 & 102 & 100 \\ 100 & 94 & 132 & 132 & 110 \end{pmatrix} \begin{array}{l} \text{Chinese} \\ \text{Malay} \\ \text{Tamil} \end{array} \end{array}$$

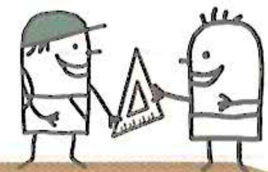
$$\begin{array}{cc} & \begin{array}{ccccc} \text{Sec 1} & \text{Sec 2} & \text{Sec 3} & \text{Sec 4} & \text{Sec 5} \end{array} \\ \mathbf{B} = & \begin{pmatrix} 24 & 13 & 5 & 11 & 27 \\ 12 & 18 & 9 & 17 & 13 \\ 10 & 14 & 12 & 21 & 8 \end{pmatrix} \begin{array}{l} \text{Chinese} \\ \text{Malay} \\ \text{Tamil} \end{array} \end{array}$$

Between 1st December and 1st January, no new stocks of these books arrived at the bookshop.

(i) Find the matrix $\mathbf{A} - \mathbf{B}$.

(ii) Explain what the numbers in the matrix in **(i)** represent.

5.3 Matrix Multiplication



Multiplication of a Matrix by a Scalar

Class Discussion

Multiplying a Matrix by a Scalar

In the previous class discussion on page 144, the number of pens sold after one month is represented by the matrix $\mathbf{P} = \mathbf{X} - \mathbf{Y} = \begin{pmatrix} 56 & 63 & 21 \\ 27 & 17 & 9 \end{pmatrix}$.

Suppose the boss decides to order two times the number of pens sold in the previous month.

1. How do you multiply matrix **P** by a constant number to give the quantity of the pens of each brand ordered by the boss for each shop?

$$\begin{aligned} 2\mathbf{P} &= 2\begin{pmatrix} 56 & 63 & 21 \\ 27 & 17 & 9 \end{pmatrix} \\ &= \begin{pmatrix} 2\times 56 & 2\times 63 & 2\times \underline{\quad} \\ 2\times \underline{\quad} & \underline{\qquad} & \underline{\qquad} \end{pmatrix} \\ &= \begin{pmatrix} 112 & \underline{\quad} & \underline{\quad} \\ \underline{\quad} & \underline{\quad} & \underline{\quad} \end{pmatrix} \end{aligned}$$

We call the constant number 2 a 'scalar' as compared to a matrix $\begin{pmatrix} 2 \end{pmatrix}$ or a vector $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$.

2. Does the order of a matrix matter if it is multiplied by a scalar?

In general, if a matrix **A** is multiplied by a scalar k, every element in **A** is multiplied by k, e.g.

$$\boxed{\text{if } \mathbf{A} = \begin{pmatrix} p & q \\ r & s \end{pmatrix}, \text{ then } k\mathbf{A} = \begin{pmatrix} kp & kq \\ kr & ks \end{pmatrix}.}$$

Worked Example 3

(Addition, Subtraction and Scalar Multiplication of Matrices)

(a) If $\mathbf{A} = \begin{pmatrix} 5 & -3 \\ 1 & 6 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} -2 & 7 \\ -4 & 0 \end{pmatrix}$, evaluate $2\mathbf{A} + 3\mathbf{B}$.

(b) Find the value of a and of b in the following:
$-2\begin{pmatrix} a & b \end{pmatrix} + \begin{pmatrix} -5 & 8 \end{pmatrix} = \begin{pmatrix} 7 & 3 \end{pmatrix}$.

Solution:

(a)
$$\begin{aligned} 2\mathbf{A} + 3\mathbf{B} &= 2\begin{pmatrix} 5 & -3 \\ 1 & 6 \end{pmatrix} + 3\begin{pmatrix} -2 & 7 \\ -4 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 10 & -6 \\ 2 & 12 \end{pmatrix} + \begin{pmatrix} -6 & 21 \\ -12 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 4 & 15 \\ -10 & 12 \end{pmatrix} \end{aligned}$$

(b)
$$\begin{aligned} -2\begin{pmatrix} a & b \end{pmatrix} + \begin{pmatrix} -5 & 8 \end{pmatrix} &= \begin{pmatrix} 7 & 3 \end{pmatrix} \\ \begin{pmatrix} -2a & -2b \end{pmatrix} + \begin{pmatrix} -5 & 8 \end{pmatrix} &= \begin{pmatrix} 7 & 3 \end{pmatrix} \\ \begin{pmatrix} -2a-5 & -2b+8 \end{pmatrix} &= \begin{pmatrix} 7 & 3 \end{pmatrix} \end{aligned}$$

Equating the corresponding elements, we have

$$\begin{aligned} -2a - 5 &= 7 \\ -2a &= 7 + 5 \\ &= 12 \\ a &= \frac{12}{-2} \\ &= -6 \end{aligned} \qquad \text{and} \qquad \begin{aligned} -2b + 8 &= 3 \\ -2b &= 3 - 8 \\ &= -5 \\ b &= \frac{-5}{-2} \\ &= 2\frac{1}{2} \end{aligned}$$

$\therefore\ a = -6,\ b = 2\frac{1}{2}$

PRACTISE NOW 3

SIMILAR QUESTIONS

Exercise 5C Questions 1–4, 10

1. If $\mathbf{A} = \begin{pmatrix} 2 & -1 \\ -3 & 6 \\ 5 & 8 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 4 & 7 \\ 0 & -9 \\ -10 & 11 \end{pmatrix}$, evaluate

(i) $3\mathbf{A} + 2\mathbf{B}$, **(ii)** $4\mathbf{B} - 3\mathbf{A}$.

2. Find the value of x and of y in the following.

(a) $-2\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 4 \\ 2 \end{pmatrix} = 3\begin{pmatrix} 6 \\ -2 \end{pmatrix}$

(b) $2\begin{pmatrix} 3 & x \\ 0 & y \end{pmatrix} + \begin{pmatrix} 0 & 4 \\ 6 & -2 \end{pmatrix} = 3\begin{pmatrix} 2 & 5 \\ 2 & 14 \end{pmatrix}$

Worked Example 4

(Problem involving Addition, Subtraction and Scalar Multiplication of Matrices)

The number of ships arriving at a harbour every weekday from Monday to Friday is given in the matrix **A**, and for Saturday and Sunday in the matrix **B**.

$$\begin{array}{cc} & \begin{array}{cc}\text{Passenger Ships} & \text{Cargo Ships}\end{array} \\ \mathbf{A} = & \begin{pmatrix} 8 & 3 \\ 7 & 5 \end{pmatrix} \end{array} \begin{array}{l} \\ \text{Dock 1} \\ \text{Dock 2} \end{array}$$

$$\begin{array}{cc} & \begin{array}{cc}\text{Passenger Ships} & \text{Cargo Ships}\end{array} \\ \mathbf{B} = & \begin{pmatrix} 11 & 2 \\ 12 & 1 \end{pmatrix} \end{array} \begin{array}{l} \\ \text{Dock 1} \\ \text{Dock 2} \end{array}$$

(i) Find the total number of ships of each type arriving at each of the docks from Monday to Friday, expressing your results in matrix form.

(ii) Evaluate the matrix $5\mathbf{A} + 2\mathbf{B}$.

(iii) Explain what the elements of the matrix in **(ii)** represent.

Solution:

(i) Total number of ships of each type arriving at each of the docks from Monday to Friday is given in the matrix $5\mathbf{A}$.

$$5\mathbf{A} = 5\begin{pmatrix} 8 & 3 \\ 7 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 40 & 15 \\ 35 & 25 \end{pmatrix}$$

(ii) $$5\mathbf{A} + 2\mathbf{B} = \begin{pmatrix} 40 & 15 \\ 35 & 25 \end{pmatrix} + 2\begin{pmatrix} 11 & 2 \\ 12 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 40 & 15 \\ 35 & 25 \end{pmatrix} + \begin{pmatrix} 22 & 4 \\ 24 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 62 & 19 \\ 59 & 27 \end{pmatrix}$$

(iii) The elements of $5\mathbf{A} + 2\mathbf{B}$ represent the number of each type of ships arriving at each of the docks from Monday to Sunday.

PRACTISE NOW 4

SIMILAR QUESTIONS

Exercise 5C Question 5

A chartered bus carries passengers daily from Monday to Friday. The number of adults and children it carries each morning and afternoon is given in the matrix $\mathbf{D}$.

$$\mathbf{D} = \begin{array}{c} \begin{array}{cc} \text{Adults} & \text{Children} \end{array} \\ \begin{pmatrix} 15 & 25 \\ 21 & 8 \end{pmatrix} \end{array} \begin{array}{l} \\ \text{Morning} \\ \text{Afternoon} \end{array}$$

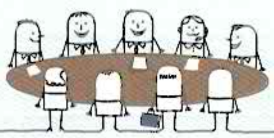
(i) Find the total number of adults and children carried by the bus from Monday to Friday in the mornings and afternoons, expressing your results in matrix form.

(ii) The bus carries a total of 14 adults and 10 children every Saturday morning, and 18 adults and 7 children every Saturday afternoon. Represent this information using the matrix $\mathbf{E}$.

(iii) Evaluate the matrix $5\mathbf{D} + \mathbf{E}$ and explain what this matrix represents.

Multiplication of a Matrix by another Matrix

Unlike matrix addition and subtraction, multiplication of a matrix by another matrix is not a direct extension of ordinary multiplication.



Class Discussion

Multiplying a Matrix with another Matrix

In the class discussion on page 144, the number of pens sold is represented by the matrix:

$$\mathbf{P} = \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{pmatrix} 56 & 63 & 21 \\ 27 & 17 & 9 \end{pmatrix} \end{array} \begin{array}{l} \\ \text{Shop 1} \\ \text{Shop 2} \end{array}$$

Suppose the selling price of each pen of brands A, B and C is \$1.50, \$2 and \$1.80 respectively, and this is represented by the matrix $\mathbf{Q} = \begin{pmatrix} 1.5 \\ 2 \\ 1.8 \end{pmatrix} \begin{array}{l} A \\ B \\ C \end{array}$.

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1. What is the total amount collected from the sales of pens from each shop? Show your working and represent your answer in the matrix form as follows:

$$\underbrace{\begin{array}{c} \\ \text{Shop 1} \\ \text{Shop 2} \end{array}\!\!\begin{array}{c} \begin{matrix} A & \;B & \;C \end{matrix} \\ \begin{pmatrix} 56 & 63 & 21 \\ 27 & 17 & 9 \end{pmatrix} \end{array}}_{\mathbf{P}} \underbrace{\begin{pmatrix} 1.5 \\ 2 \\ 1.8 \end{pmatrix}}_{\mathbf{Q}} \begin{array}{c} A \\ B \\ C \end{array} = \begin{pmatrix} (56\times 1.5)+(63\times 2)+(21\times \_\_) \\ (27\times \_\_)+(\_\_\times \_\_)+(\_\_\times \_\_) \end{pmatrix} \begin{array}{c} \text{Shop 1} \\ \text{Shop 2} \end{array}$$

$$= \underbrace{\begin{pmatrix} \_\_ \\ \_\_ \end{pmatrix}}_{\mathbf{R}} \begin{array}{c} \text{Shop 1} \\ \text{Shop 2} \end{array}$$

Order of matrix: 2×3 (**P**), 3×1 (**Q**), 2×1 (**R**)

2. Look at the orders of **P**, **Q** and the product **R**. Can you explain the relationship between the orders?

3. By looking at their orders, state the conditions for multiplication of two matrices to be possible.

In general, for any two matrices **A** and **B**, the product **A** × **B** (or simply **AB**) is only possible if

$$\boxed{\text{number of columns of } \mathbf{A} = \text{number of rows of } \mathbf{B}.}$$

In other words,

$$\underset{m\times n}{\mathbf{A}} \times \underset{n\times p}{\mathbf{B}} = \underset{m\times p}{\mathbf{C}}$$

must be equal

Thinking Time

Justify if the following two matrices can be multiplied together by checking their orders. If it is possible, write down the order of the product.

(a) $\begin{pmatrix} 2 & -3 & 5 \\ -7 & 0 & 8 \end{pmatrix}\begin{pmatrix} 4 & -9 \\ -5 & 10 \\ 21 & 6 \end{pmatrix}$

(b) $\begin{pmatrix} 4 & -9 \\ -5 & 10 \\ 21 & 6 \end{pmatrix}\begin{pmatrix} 2 & -3 & 5 \\ -7 & 0 & 8 \end{pmatrix}$

(c) $\begin{pmatrix} 2 & 3 \end{pmatrix}\begin{pmatrix} 7 & 8 \\ -9 & 4 \end{pmatrix}$

(d) $\begin{pmatrix} 7 & 8 \\ -9 & 4 \end{pmatrix}\begin{pmatrix} 2 & -3 \end{pmatrix}$

(e) $\begin{pmatrix} 7 \\ 8 \\ -5 \end{pmatrix}\begin{pmatrix} -7 & 2 & 3 \end{pmatrix}$

(f) $\begin{pmatrix} -7 & 2 & 3 \end{pmatrix}\begin{pmatrix} 7 \\ 8 \\ -5 \end{pmatrix}$

In the previous class discussion on page 143, suppose the boss decides to increase the selling price of each brand of pens by 40¢($0.40) as shown below:

$$\underbrace{\begin{array}{c} \\ \text{Shop 1} \\ \text{Shop 2} \end{array}\begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{pmatrix} 56 & 63 & 21 \\ 27 & 17 & 9 \end{pmatrix} \end{array}}_{\mathbf{P}} \underbrace{\begin{array}{c} \begin{array}{cc} \text{Old Price} & \text{New Price} \end{array} \\ \begin{pmatrix} 1.5 & 1.9 \\ 2 & 2.4 \\ 1.8 & 2.2 \end{pmatrix} \end{array}}_{\mathbf{Q}} = \underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{\mathbf{R}}$$

Order: 2×3 $\quad$ 3×2 $\quad$ 2×2

equal

Fig. 5.1

Since the orders of the two matrices **P** and **Q** are 2 by 3 and 3 by 2 respectively, the order of the product **R** must be 2 by 2, as shown in Fig. 5.1.

To obtain a (element in **first row**, **first column** of **R**) we multiply the corresponding elements in the **first row** of **P** and the **first column** of **Q** as shown in Fig. 5.2, before adding the three products to obtain 247.8.

$$\begin{pmatrix} 56 & 63 & 21 \\ 27 & 17 & 9 \end{pmatrix} \begin{pmatrix} 1.5 & 1.9 \\ 2 & 2.4 \\ 1.8 & 2.2 \end{pmatrix} = \begin{pmatrix} 56(1.5)+63(2)+21(1.8) & b \\ c & d \end{pmatrix}$$

$$= \begin{pmatrix} 247.8 & b \\ c & d \end{pmatrix}$$

Fig. 5.2

To obtain b (element in **first row**, **second column** of **R**), we multiply the corresponding elements in the **first row** of **P** and the **second column** of **Q** as shown in Fig. 5.3, before adding the three products to give 303.8.

$$\begin{pmatrix} 56 & 63 & 21 \\ 27 & 17 & 9 \end{pmatrix} \begin{pmatrix} 1.5 & 1.9 \\ 2 & 2.4 \\ 1.8 & 2.2 \end{pmatrix} = \begin{pmatrix} 247.8 & 56(1.9)+63(2.4)+21(2.2) \\ c & d \end{pmatrix}$$

$$= \begin{pmatrix} 247.8 & 303.8 \\ c & d \end{pmatrix}$$

Fig. 5.3

Since **Q** has no third column, then we move on to the **second row** of **P**.

In other words, we start with the first row of the first matrix **P** and we must finish all the multiplication along all the columns of the second matrix **Q**, before proceeding to the second row of the second matrix **P**.

To obtain c (element in **second row**, **first column** of **R**), we multiply the corresponding elements in the **second row** of **P** and the **first column** of **Q** as shown in Fig. 5.4, before adding the three products to obtain 90.7.

$$\begin{pmatrix} 56 & 63 & 21 \\ 27 & 17 & 9 \end{pmatrix}\begin{pmatrix} 1.5 & 1.9 \\ 2 & 2.4 \\ 1.8 & 2.2 \end{pmatrix} = \begin{pmatrix} 247.8 & 303.8 \\ (27)(1.5)+(17)(2)+(9)(1.8) & d \end{pmatrix}$$

$$= \begin{pmatrix} 247.8 & 303.8 \\ 90.7 & d \end{pmatrix}$$

Fig. 5.4

To obtain d (element in **second row**, **second column** of **R**), we multiply the corresponding elements in the **second row** of **P** and the **second column** of **Q** as shown in Fig. 5.5, before adding the three products to obtain 111.9.

$$\begin{pmatrix} 56 & 63 & 21 \\ 27 & 17 & 9 \end{pmatrix}\begin{pmatrix} 1.5 & 1.9 \\ 2 & 2.4 \\ 1.8 & 2.2 \end{pmatrix} = \begin{pmatrix} 247.8 & 303.8 \\ 90.7 & (27)(1.9)+(17)(2.4)+(9)(2.2) \end{pmatrix}$$

$$= \begin{pmatrix} 247.8 & 303.8 \\ 90.7 & 111.9 \end{pmatrix}$$

Fig. 5.5

Therefore, we obtain the product **R**, i.e. $\mathbf{PQ} = \mathbf{R} = \begin{pmatrix} 247.8 & 303.8 \\ 90.7 & 111.9 \end{pmatrix}$.

Worked Example 5

(Multiplication of Two Matrices)

Evaluate the following matrix products where possible. If not possible, explain why not.

(a) $\begin{pmatrix} -2 & 5 \\ 1 & -6 \end{pmatrix}\begin{pmatrix} 9 & 2 \\ 1 & -2 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & -4 & 2 \end{pmatrix}\begin{pmatrix} 2 \\ 6 \\ -\frac{1}{2} \end{pmatrix}$

(c) $\begin{pmatrix} -2 \\ 1 \end{pmatrix}\begin{pmatrix} 3 & -1 \\ 1 & -4 \end{pmatrix}$

Solution:

(a) $$\begin{pmatrix} -2 & 5 \\ 1 & -6 \end{pmatrix}\begin{pmatrix} 9 & 2 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} (-2)(9)+(5)(1) & (-2)(2)+(5)(-2) \\ (1)(9)+(-6)(1) & (1)(2)+(-6)(-2) \end{pmatrix}$$
$$= \begin{pmatrix} -13 & -14 \\ 3 & 14 \end{pmatrix}$$

(b) $$\begin{pmatrix} 1 & -4 & 2 \end{pmatrix}\begin{pmatrix} 2 \\ 6 \\ -\frac{1}{2} \end{pmatrix} = \left((1)(2)+(-4)(6)+(2)\left(-\frac{1}{2}\right)\right)$$
$$= (-23)$$

(c) The order of $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$ is 2×1.

Number of columns of $\begin{pmatrix} -2 \\ 1 \end{pmatrix} = 1$

Number of rows of $\begin{pmatrix} 3 & -1 \\ 1 & -4 \end{pmatrix} = 2$

Since number of columns of $\begin{pmatrix} 2 \\ 1 \end{pmatrix} \neq$ number of rows of $\begin{pmatrix} 3 & -1 \\ 1 & -4 \end{pmatrix}$, then matrix multiplication is not possible.

ATTENTION

(a) Product exists as number of columns of first matrix = number of rows of second matrix = 2

(b) Product exists as number of columns of first matrix = number of rows of second matrix = 3

PRACTISE NOW 5

Evaluate all the matrix multiplication in Thinking Time Questions 1**(a)** – **(f)** on page 152, if it is possible.

SIMILAR QUESTIONS

Exercise 5C Questions 6(a)–(h), 7–9, 11

Thinking Time

In general, is $\mathbf{AB} = \mathbf{BA}$ for any two matrices $\mathbf{A}$ and $\mathbf{B}$?

Exercise 5C

BASIC LEVEL

1. Simplify each of the following.

(a) $2\begin{pmatrix} 1 & -2 & 3 \end{pmatrix}$

(b) $4\begin{pmatrix} -2 \\ 1 \end{pmatrix}$

(c) $\frac{1}{2}\begin{pmatrix} 6 \\ 4 \\ -8 \end{pmatrix}$

(d) $\frac{1}{3}\begin{pmatrix} 6 & 15 \\ 21 & -24 \end{pmatrix}$

(e) $-2\begin{pmatrix} -1 & 0.5 & 3 \\ -0.8 & 2 & 1.2 \end{pmatrix}$

(f) $5\begin{pmatrix} 1 & 5 \\ -4 & 3 \\ -1 & 2 \end{pmatrix}$

(g) $3\begin{pmatrix} 6 & \frac{1}{2} & 1 \\ 0 & 2 & \frac{1}{3} \\ 5 & -4 & -2 \end{pmatrix}$

2. Simplify each of the following.

(a) $2\begin{pmatrix} -1 \\ 5 \end{pmatrix} + 3\begin{pmatrix} 4 \\ -3 \end{pmatrix}$

(b) $2\begin{pmatrix} 3 & 1 & 5 \end{pmatrix} - 4\begin{pmatrix} -1 & 3 & 2 \end{pmatrix}$

(c) $5\begin{pmatrix} 1 & 3 \\ -4 & 6 \end{pmatrix} - 2\begin{pmatrix} -3 & -1 \\ 4 & 2 \end{pmatrix}$

(d) $3\begin{pmatrix} 0 & 4 & 1 \\ 5 & 0 & -1 \end{pmatrix} - 4\begin{pmatrix} -1 & 3 & 0 \\ -2 & 1 & -1 \end{pmatrix}$

3. If $\mathbf{A} = \begin{pmatrix} 4 & 4 \\ 2 & 7 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 1 & 4 \\ 3 & -5 \end{pmatrix}$, find the following.

(i) $\mathbf{A} + \mathbf{B}$
(ii) $\mathbf{A} + 2\mathbf{B}$
(iii) $\mathbf{A} - \mathbf{B} - \mathbf{C}$
(iv) $2\mathbf{A} - 2\mathbf{C} + 3\mathbf{B}$

INTERMEDIATE LEVEL

4. Find the values of the unknowns in each of the following.

(a) $a\begin{pmatrix} 2 \\ 2 \end{pmatrix} + b\begin{pmatrix} 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 8 \end{pmatrix}$

(b) $3\begin{pmatrix} 2x \\ y \end{pmatrix} + 3\begin{pmatrix} x \\ 3y \end{pmatrix} = \begin{pmatrix} 18 \\ 36 \end{pmatrix}$

(c) $2\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} - \begin{pmatrix} 3 & 5 \\ c & 6 \end{pmatrix} = \begin{pmatrix} a & b \\ 7 & d \end{pmatrix}$

(d) $2\begin{pmatrix} 5 & 3 & 2 \\ 1 & 6 & 3 \end{pmatrix} + \begin{pmatrix} a & b & c \\ -2 & -4 & 5 \end{pmatrix} = \begin{pmatrix} 9 & 12 & 6 \\ d & e & f \end{pmatrix}$

5. The monthly fee, in dollars, charged by three childcare centres is given in the matrix $\mathbf{C}$.

$$\mathbf{C} = \begin{array}{c} \begin{pmatrix} 680 \\ 720 \\ 635 \end{pmatrix} \end{array} \begin{array}{l} \text{Childcare Centre } X \\ \text{Childcare Centre } Y \\ \text{Childcare Centre } Z \end{array}$$

(i) Find the annual fees, in dollars, charged by each of the 3 childcare centres, giving your answers in matrix form.

(ii) During the June and December holidays, all the 3 childcare centres offer some special programmes for an additional cost given in the matrices $\mathbf{J}$ and $\mathbf{D}$ respectively.

$$\mathbf{J} = \begin{pmatrix} 150 \\ 120 \\ 200 \end{pmatrix} \begin{array}{l} \text{Childcare Centre } X \\ \text{Childcare Centre } Y \\ \text{Childcare Centre } Z \end{array}$$

$$\mathbf{D} = \begin{pmatrix} 180 \\ 150 \\ 200 \end{pmatrix} \begin{array}{l} \text{Childcare Centre } X \\ \text{Childcare Centre } Y \\ \text{Childcare Centre } Z \end{array}$$

Evaluate the matrix $12\mathbf{C} + \mathbf{J} + \mathbf{D}$ and explain what this matrix represents.

(iii) Mrs Yeo wants to enroll her son in one of the childcare centres, and she is interested in the June special programmes only. Which childcare centre charges the lowest fees?

6. Evaluate the following matrix products if possible. If not possible, explain why not.

(a) $\begin{pmatrix} 4 & 3 \\ -1 & 5 \end{pmatrix}\begin{pmatrix} -2 \\ 6 \end{pmatrix}$

(b) $\begin{pmatrix} -3 & 1 \\ 0 & 8 \end{pmatrix}\begin{pmatrix} 6 & 5 \\ 8 & 7 \end{pmatrix}$

(c) $\begin{pmatrix} 6 \\ 7 \end{pmatrix}\begin{pmatrix} -1 & 3 \end{pmatrix}$

(d) $\begin{pmatrix} -1 & 3 \end{pmatrix}\begin{pmatrix} 6 \\ 7 \end{pmatrix}$

(e) $\begin{pmatrix} 1 \\ 8 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ -11 & 20 \end{pmatrix}$

(f) $\begin{pmatrix} -1 & 2 \\ 8 & 5 \\ 3 & -7 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ -1 & 4 \end{pmatrix}$

(g) $\begin{pmatrix} 3 & 8 & 0 & 5 \\ -1 & 0 & 7 & 6 \\ 4 & 9 & -2 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ -3 \\ 0 \\ 5 \end{pmatrix}$

(h) $\begin{pmatrix} \frac{1}{2} \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 \end{pmatrix}$

7. If $\begin{pmatrix} 1 & 5 \\ 3 & p \end{pmatrix}\begin{pmatrix} q \\ 7 \end{pmatrix} = \begin{pmatrix} 50 \\ 35 \end{pmatrix}$, find the value of p and of q.

8. If $\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 1 & 5 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 0 \\ 2 & k \end{pmatrix}$, find

(i) $\mathbf{AB}$, (ii) $\mathbf{BA}$,

(iii) the value of k if $\mathbf{AB} = \mathbf{BA}$.

9. If $\mathbf{A} = \begin{pmatrix} 8 & -3 \\ 7 & 5 \end{pmatrix}$ and $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, evaluate the following.

(i) $\mathbf{AI}$ (ii) $\mathbf{IA}$

Is $\mathbf{AI} = \mathbf{A} = \mathbf{IA}$? For information, $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is called an **identity matrix**.

ADVANCED LEVEL

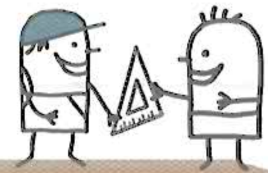
10. If $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\mathbf{C} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $\mathbf{D} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, find an expression in terms of $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$ for the matrix $\begin{pmatrix} 7 & 6 \\ 4 & 3 \end{pmatrix}$.

11. (i) Give an example of two 2×2 matrices $\mathbf{A}$ and $\mathbf{B}$ such that $\mathbf{AB} \neq \mathbf{BA}$.

(ii) Give an example of two 2×2 matrices $\mathbf{A}$ and $\mathbf{B}$ such that $\mathbf{AB} = \mathbf{BA}$.

(iii) For a 2×2 matrix $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, is $\mathbf{A}^2 = \begin{pmatrix} a^2 & b^2 \\ c^2 & d^2 \end{pmatrix}$, where $\mathbf{A}^2 = \mathbf{A} \times \mathbf{A}$? Why?

5.4 Determinant of a Matrix



Consider the matrix $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 4 & 5 \end{pmatrix}$.

The number obtained from subtracting the product of the elements on the *other diagonal* from the product of the elements on the *leading diagonal* i.e.
$(1 \times 5) - (2 \times 4) = -3$ is known as the **determinant** of the matrix $\mathbf{A}$. We write $|\mathbf{A}| = -3$ to denote that the determinant of $\mathbf{A}$ is -3.

The determinant can only be computed from square matrices. In this book, we will look at the determinant of only 2×2 matrices.

In general, given a matrix $\begin{pmatrix} p & q \\ r & s \end{pmatrix}$, the expression $ps - qr$ is known as the **determinant** of the matrix.

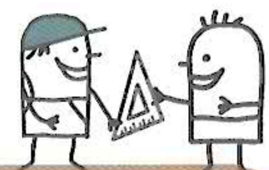
PRACTISE NOW

SIMILAR QUESTIONS

Exercise 5D Questions 1(a), 11

Given a matrix $\mathbf{Q} = \begin{pmatrix} 2 & -1 \\ -2 & 4 \end{pmatrix}$, find $|\mathbf{Q}|$.

5.5 Inverse of a Matrix



Definition of Inverse Matrices

In this section, we will study the **inverse** of a matrix.

Consider two matrices $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix}$.

$$\mathbf{AB} = \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}\begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I}$$

$$\mathbf{BA} = \begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I}$$

i.e. $\mathbf{AB} = \mathbf{BA} = \mathbf{I}$

In this case, $\mathbf{A}$ is said to be the inverse of $\mathbf{B}$, and $\mathbf{A}$ is denoted by $\mathbf{B}^{-1}$.
$\mathbf{B}$ is also the inverse of $\mathbf{A}$, and $\mathbf{B}$ is denoted by $\mathbf{A}^{-1}$.

Worked Example 6

(Determining if a Matrix is the Inverse of another Matrix)

If $\mathbf{A} = \begin{pmatrix} 4 & 1 \\ 11 & 3 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 3 & -1 \\ -11 & 4 \end{pmatrix}$, find **AB** and **BA**. Determine the relationship between **A** and **B**.

Solution:

$$\mathbf{AB} = \begin{pmatrix} 4 & 1 \\ 11 & 3 \end{pmatrix}\begin{pmatrix} 3 & -1 \\ -11 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I}$$

$$\mathbf{BA} = \begin{pmatrix} 3 & -1 \\ -11 & 4 \end{pmatrix}\begin{pmatrix} 4 & 1 \\ 11 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I}$$

Since $\mathbf{AB} = \mathbf{BA} = \mathbf{I}$, **A** is the inverse of **B** and **B** is the inverse of **A** i.e. $\mathbf{A} = \mathbf{B}^{-1}$ and $\mathbf{B} = \mathbf{A}^{-1}$.

PRACTISE NOW 6

If $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 13 & 7 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 7 & -1 \\ -13 & 2 \end{pmatrix}$, find **AB** and **BA**. Determine the relationship between **A** and **B**.

SIMILAR QUESTIONS

Exercise 5D Questions 2, 3

Finding the Inverse of a Matrix

Case 1: Determinant of given matrix is 1

Given $\mathbf{A} = \begin{pmatrix} 9 & 5 \\ 7 & 4 \end{pmatrix}$, find $\mathbf{A}^{-1}$.

Step 1: Find $|\mathbf{A}|$.

$$|\mathbf{A}| = (9 \times 4) - (5 \times 7)$$
$$= 1$$

Step 2: Interchange the elements on the leading diagonal of the given matrix.

$$\begin{pmatrix} 9 & 5 \\ 7 & 4 \end{pmatrix} \rightarrow \begin{pmatrix} 4 & 5 \\ 7 & 9 \end{pmatrix}$$

Step 3: Change the signs of the elements on the other diagonal.

$$\begin{pmatrix} 4 & 5 \\ 7 & 9 \end{pmatrix} \rightarrow \begin{pmatrix} 4 & -5 \\ -7 & 9 \end{pmatrix}$$

$$\therefore \mathbf{A}^{-1} = \begin{pmatrix} 4 & -5 \\ -7 & 9 \end{pmatrix}$$

Case 2: Determinant of given matrix is neither 1 nor 0

Given $\mathbf{A} = \begin{pmatrix} 6 & 4 \\ 10 & 7 \end{pmatrix}$, find $\mathbf{A}^{-1}$.

Step 1: Find $|\mathbf{A}|$.

$$|\mathbf{A}| = (6 \times 7) - (4 \times 10)$$
$$= 2$$

Step 2: Interchange the elements on the leading diagonal of the given matrix.

$$\begin{pmatrix} 6 & 4 \\ 10 & 7 \end{pmatrix} \rightarrow \begin{pmatrix} 7 & 4 \\ 10 & 6 \end{pmatrix}$$

Step 3: Change the signs of the elements on the other diagonal.

$$\begin{pmatrix} 7 & 4 \\ 10 & 6 \end{pmatrix} \rightarrow \begin{pmatrix} 7 & -4 \\ -10 & 6 \end{pmatrix}$$

Step 4: Multiply with a scalar equivalent to the reciprocal of the determinant.

$$\frac{1}{2}\begin{pmatrix} 7 & -4 \\ -10 & 6 \end{pmatrix} = \begin{pmatrix} \frac{7}{2} & -2 \\ -5 & 3 \end{pmatrix}$$

$$\therefore \mathbf{A}^{-1} = \begin{pmatrix} \frac{7}{2} & -2 \\ -5 & 3 \end{pmatrix}$$

In general,

> the matrix $\begin{pmatrix} p & q \\ r & s \end{pmatrix}$ has the inverse $\mathbf{A}^{-1} = \dfrac{1}{ps-qr}\begin{pmatrix} s & -q \\ -r & p \end{pmatrix}$ where $ps - qr \neq 0$.

Thinking Time

Why is it not possible to find the inverse of a matrix when its determinant is 0?

INFORMATION

When a matrix does not possess an inverse, it is known as a **singular matrix**.

Worked Example 7

(Determining the Inverse of Matrices)

Given $\mathbf{A} = \begin{pmatrix} 7 & 3 \\ 14 & 9 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 3 & -4 \\ -3 & -2 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 4 & 8 \\ 3 & 6 \end{pmatrix}$, find $\mathbf{A}^{-1}$, $\mathbf{B}^{-1}$ and $\mathbf{C}^{-1}$ where possible.

Solution:

$$|\mathbf{A}| = (7 \times 9) - (3 \times 14)$$
$$= 21$$
$$\neq 0$$

$\therefore$ $\mathbf{A}^{-1}$ exists.

$$\mathbf{A}^{-1} = \frac{1}{21}\begin{pmatrix} 9 & -3 \\ -14 & 7 \end{pmatrix}$$
$$= \begin{pmatrix} \frac{9}{21} & \frac{-3}{21} \\ \frac{-14}{21} & \frac{7}{21} \end{pmatrix}$$
$$= \begin{pmatrix} \frac{3}{7} & -\frac{1}{7} \\ -\frac{2}{3} & \frac{1}{3} \end{pmatrix}$$

$$|\mathbf{B}| = [3 \times (-2)] - [(-4) \times (-3)]$$
$$= -18$$
$$\neq 0$$

$\therefore$ $\mathbf{B}^{-1}$ exists.

$$\mathbf{B}^{-1} = \frac{1}{-18}\begin{pmatrix} -2 & 4 \\ 3 & 3 \end{pmatrix}$$
$$= \begin{pmatrix} \frac{-2}{-18} & \frac{-4}{-18} \\ \frac{3}{-18} & \frac{3}{-18} \end{pmatrix}$$
$$= \begin{pmatrix} \frac{1}{9} & -\frac{2}{9} \\ -\frac{1}{6} & -\frac{1}{6} \end{pmatrix}$$

$$|\mathbf{C}| = (4 \times 6) - (8 \times 3)$$
$$= 0$$

$\therefore$ $\mathbf{C}^{-1}$ does not exist. Hence, $\mathbf{C}$ is a singular matrix.

ATTENTION

$\mathbf{C}^{-1}$ is not defined when $|\mathbf{C}| = 0$ since $\frac{1}{0}$ is not defined.

PRACTISE NOW 7

Given $\mathbf{A} = \begin{pmatrix} 6 & 3 \\ 7 & 2 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 & 1 \\ -9 & -3 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 2 & 1 \\ 8 & 4 \end{pmatrix}$, find $\mathbf{A}^{-1}$, $\mathbf{B}^{-1}$ and $\mathbf{C}^{-1}$ where possible.

SIMILAR QUESTIONS

Exercise 5D Questions 1(b), 4–6

Worked Example 8

(Problems involving Inverse Matrices)

$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 7 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}$. Write down the inverse of **A**. Hence, find **P** and **Q** in the following equations.

(i) $\mathbf{AP} = \mathbf{B}$

(ii) $\mathbf{QA} = \mathbf{B}$

Solution:

$|\mathbf{A}| = (1 \times 7) - (2 \times 3)$

$= 1$

$\therefore \mathbf{A}^{-1}$ exists.

$\mathbf{A}^{-1} = \begin{pmatrix} 7 & -2 \\ -3 & 1 \end{pmatrix}$

(i)

$\mathbf{AP} = \mathbf{B}$

$\mathbf{A}^{-1}\mathbf{AP} = \mathbf{A}^{-1}\mathbf{B}$

$(\mathbf{A}^{-1}\mathbf{A})\mathbf{P} = \mathbf{A}^{-1}\mathbf{B}$ (Associative Law of Matrix Multiplication)

$\mathbf{IP} = \mathbf{A}^{-1}\mathbf{B}$ (where $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$)

$\mathbf{P} = \begin{pmatrix} 7 & -2 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}$

$= \begin{pmatrix} 26 & 8 \\ -11 & -3 \end{pmatrix}$

(ii)

$\mathbf{QA} = \mathbf{B}$

$\mathbf{QAA}^{-1} = \mathbf{BA}^{-1}$

$\mathbf{Q}(\mathbf{AA}^{-1}) = \mathbf{BA}^{-1}$ (Associative Law of Matrix Multiplication)

$\mathbf{QI} = \mathbf{BA}^{-1}$ (where $\mathbf{AA}^{-1} = \mathbf{I}$)

$\mathbf{Q} = \begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 7 & -2 \\ -3 & 1 \end{pmatrix}$

$= \begin{pmatrix} 22 & -6 \\ -2 & 1 \end{pmatrix}$

PRACTISE NOW 8

SIMILAR QUESTIONS

Exercise 5D Questions 7–10

$\mathbf{A} = \begin{pmatrix} 2 & -2 \\ 7 & -3 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 6 & 5 \\ 3 & 1 \end{pmatrix}$. Write down the inverse of **A**. Hence, find **P** and **Q** in the following equations.

(i) $\mathbf{AP} = \mathbf{B}$

(ii) $\mathbf{QA} = \mathbf{B}$

Exercise 5D

BASIC LEVEL

1. Find the **(a)** determinants and **(b)** inverses of the given matrices where possible.

(i) $\begin{pmatrix} 3 & 4 \\ 5 & 6 \end{pmatrix}$ **(ii)** $\begin{pmatrix} 2 & -1 \\ 3 & -\frac{3}{2} \end{pmatrix}$

(iii) $\begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}$ **(iv)** $\begin{pmatrix} 3 & 10 \\ 2 & 7 \end{pmatrix}$

(v) $\begin{pmatrix} -8 & -4 \\ -4 & -2 \end{pmatrix}$ **(vi)** $\begin{pmatrix} 1 & 3 \\ -2 & 6 \end{pmatrix}$

(vii) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ **(viii)** $\begin{pmatrix} -1 & 3 \\ 1 & -3 \end{pmatrix}$

2. If $\mathbf{A} = \begin{pmatrix} 2 & 3 \\ 5 & 8 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 8 & -3 \\ -5 & 2 \end{pmatrix}$, find **AB** and **BA**. Determine the relationship between **A** and **B**.

3. If $\mathbf{P} = \begin{pmatrix} 7 & 9 \\ 3 & 4 \end{pmatrix}$ and $\mathbf{Q} = \begin{pmatrix} 4 & -9 \\ -3 & 7 \end{pmatrix}$, find **PQ** and **QP**. Determine the relationship between **P** and **Q**.

4. Find the inverses of the following matrices where possible.

(a) $\begin{pmatrix} p & q \\ r & s \end{pmatrix}$ **(b)** $\begin{pmatrix} a & 0 \\ 0 & \frac{1}{a} \end{pmatrix}$

(c) $\begin{pmatrix} -a & b \\ a & b \end{pmatrix}$ **(d)** $\begin{pmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{pmatrix}$

(e) $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ **(f)** $\begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}$

(g) $\begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{3} \end{pmatrix}$

5. Given that $\mathbf{A} = \begin{pmatrix} 2 & 4 \\ 3 & 5 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 2 & 4 \\ 3 & 6 \end{pmatrix}$, calculate $|\mathbf{A}|$ and $|\mathbf{B}|$.

Hence, find the inverse of **A** and **B** where possible.

INTERMEDIATE LEVEL

6. Given that the determinant of the matrix $\begin{pmatrix} 2a & -4 \\ -1 & 5 \end{pmatrix}$ is 16, find the value of a. Hence, write down the inverse of the matrix.

7. Write down the inverse of $\begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix}$ and use it to solve the equation.

(i) $\mathbf{A}\begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & -3 \end{pmatrix}$

(ii) $\begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix}\mathbf{B} = \begin{pmatrix} 3 & 2 \\ -4 & -6 \end{pmatrix}$

8. $\mathbf{A} = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix}$. Write down the inverse matrices $\mathbf{A}^{-1}$ and $\mathbf{B}^{-1}$.

Hence use your results to find the matrices **P** and **Q** such that

(i) $\mathbf{AP} = \mathbf{B}$ **(ii)** $\mathbf{QA} = \mathbf{B}$

9. Given that $\mathbf{A} = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 2 & 5 \\ -1 & -3 \end{pmatrix}$, write down the inverse matrix of **A**. Use your result to find the matrices **P** and **Q** such that

(i) $\mathbf{AP} = \mathbf{B}$, **(ii)** $\mathbf{QA} = \mathbf{B}$

10. Given that $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 3 & 4 \\ -2 & 1 \end{pmatrix}$, write down the inverse matrix $\mathbf{A}^{-1}$ and use it to solve the following equations.

(i) $\mathbf{AX} = \mathbf{B} - \mathbf{A}$ **(ii)** $\mathbf{YA} = 3\mathbf{B} + 2\mathbf{A}$

ADVANCED LEVEL

11. $\mathbf{A} = \begin{pmatrix} 2 & 3 \\ 0 & 4 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} \frac{1}{2} & -k \\ 0 & 2a \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 6 & 2 \\ -3 & h \end{pmatrix}$.

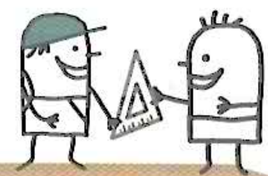
(i) Evaluate $\mathbf{A}^2$.

(ii) If $\mathbf{AB} = \mathbf{I}$, find the value of k and of a.

(iii) Find the value of h for which $|\mathbf{A}| = |\mathbf{C}|$.

(iv) If $|\mathbf{B}| = |\mathbf{C}|$, find the value of h when $a = 3$.

5.6 Applications of Matrices



Worked Example 9

(Application of Matrices in Calculating Costs)

A bakery produces 3 different types of bread: white bread (W), wholemeal bread (M) and multi-grain bread (G). Delivery is made to 2 distribution outlets in the following way:

> Outlet A receives 60 loaves of W, 50 loaves of M and 30 loaves of G.
>
> Outlet B receives 40 loaves of W, 70 loaves of M, and 20 loaves of G.

The costs of one loaf of W, M and G are \$2.10, \$2.70 and \$2.90 respectively.

It is given that $\mathbf{P} = \begin{pmatrix} 60 & 50 & 30 \\ 40 & 70 & 20 \end{pmatrix}$ and $\mathbf{Q} = \begin{pmatrix} 2.1 \\ 2.7 \\ 2.9 \end{pmatrix}$.

(a) **(i)** Evaluate $\mathbf{PQ}$.

(ii) Explain what the answer in **(i)** represents.

(b) In a particular month, Outlets A and B receive 27 and 25 such deliveries respectively. Form two matrices so that their product will give the total cost of the bread delivered to the 2 outlets. Find the product.

Solution:

(a) (i) $\mathbf{PQ} = \begin{pmatrix} 60 & 50 & 30 \\ 40 & 70 & 20 \end{pmatrix}\begin{pmatrix} 2.1 \\ 2.7 \\ 2.9 \end{pmatrix} = \begin{pmatrix} 348 \\ 331 \end{pmatrix}$

(ii) **PQ** gives the costs of the bread delivered to Outlets A and B respectively.

(b) $\begin{pmatrix} 27 & 25 \end{pmatrix}\begin{pmatrix} 348 \\ 331 \end{pmatrix} = \begin{pmatrix} 17\,671 \end{pmatrix}$

∴ The total cost of bread delivered to the 2 outlets is \$17 671.

ATTENTION

For **(b)**, an alternative method is

$\begin{pmatrix} 348 & 331 \end{pmatrix}\begin{pmatrix} 27 \\ 25 \end{pmatrix}$

$= \begin{pmatrix} 17\,671 \end{pmatrix}$.

PRACTISE NOW 9

SIMILAR QUESTIONS

Exercise 5E Questions 1–6

1. Huixian and Lixin take a multiple choice test. The matrices **X** and **Y** show the results of the test and the marks awarded respectively.

$$\mathbf{X} = \begin{array}{c} \begin{array}{ccc} \text{Correct} & \text{No attempt} & \text{Incorrect} \end{array} \\ \begin{pmatrix} 16 & 0 & 4 \\ 12 & 5 & 3 \end{pmatrix} \end{array} \begin{array}{l} \text{Huixian} \\ \text{Lixin} \end{array} \qquad \mathbf{Y} = \begin{array}{c} \text{Marks} \\ \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} \end{array} \begin{array}{l} \text{Correct} \\ \text{No attempt} \\ \text{Incorrect} \end{array}$$

(i) How many questions are there in the test?
(ii) Evaluate **XY**.
(iii) Explain what your answer to **(ii)** represents.

2. An otah factory produces four types of seafood otahs, namely fish (F), prawn (P), squid (S) and mixed seafood otah (M), for distribution to its five outlets across Singapore.

Tampines outlet receives	70 of F, 120 of P, 90 of S and 80 of M,
Bedok outlet receives	120 of F, 150 of S and 140 of M,
City Hall outlet receives	150 of P, 85 of S and 60 of M,
Toa Payoh outlet receives	200 of F, 140 of P and 70 of S,
Jurong East outlet receives	80 of F, 110 of P and 95 of M.

(i) The cost of each stick of F, P, S and M is 35¢, 40¢, 45¢ and 38¢ respectively. Write down two matrices such that the elements of their product under matrix multiplication give the total cost of otah delivered to each outlet. Evaluate this product.
(ii) During the month of December, Tampines outlet received 45 such deliveries, Bedok outlet 42 deliveries, City Hall outlet 38 deliveries, Toa Payoh outlet 55 deliveries and Jurong East outlet 52 deliveries.
Write down two matrices such that the elements of their product give the total number of sticks of each type of otah delivered by the factory. Obtain this product and hence, find the total number of otah supplied by the factory to these outlets.
(iii) With the information provided in **(i)** and **(ii)**, write down two matrices so that the product will give the total revenue derived by the otah factory in the month of December.

Worked Example 10

(Application of Matrices in Simultaneous Equations)

Solve the following sets of equations using the matrix method.

(a) $3x - 4y = 7$, $5x - 7y = 12$

(b) $9x + 6y = 14$, $6x + 4y = 30$

(c) $3x + 2y = 7$, $6x + 4y = 14$

Explain what your answers represent.

Solution:

(a) The simultaneous equations may be written in matrix form as

$$\begin{pmatrix} 3 & -4 \\ 5 & -7 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 \\ 12 \end{pmatrix}$$

$$\text{Determinant of } \begin{pmatrix} 3 & -4 \\ 5 & -7 \end{pmatrix} = -21 - (-20)$$
$$= -1$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & -4 \\ 5 & -7 \end{pmatrix}^{-1}\begin{pmatrix} 7 \\ 12 \end{pmatrix}$$
$$= \begin{pmatrix} 7 & -4 \\ 5 & -3 \end{pmatrix}\begin{pmatrix} 7 \\ 12 \end{pmatrix}$$
$$= \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$\therefore x = 1, y = -1$

The graphs of $3x - 4y = 7$ and $5x - 7y = 12$ intersect at the point $(1, -1)$.

RECALL

The solution of two simultaneous linear equations gives the coordinates of the point of intersection between two lines.

(b) The simultaneous equations may be written in matrix form as

$$\begin{pmatrix} 9 & 6 \\ 6 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 14 \\ 30 \end{pmatrix}$$

$$\text{Determinant of } \begin{pmatrix} 9 & 6 \\ 6 & 4 \end{pmatrix} = 36 - 36$$
$$= 0$$

Hence $\begin{pmatrix} 9 & 6 \\ 6 & 4 \end{pmatrix}$ is a singular matrix and its inverse matrix does not exist.

The graphs of $9x + 6y = 14$ and $6x + 4y = 30$ represent two *parallel lines*. There is *no solution* since there is no intersection between the two lines.

(c) The simultaneous equations may be written in matrix form as

$$\begin{pmatrix} 3 & 2 \\ 6 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 \\ 14 \end{pmatrix}$$

$$\text{Determinant of } \begin{pmatrix} 3 & 2 \\ 6 & 4 \end{pmatrix} = 12 - 12$$
$$= 0$$

Hence $\begin{pmatrix} 3 & 2 \\ 6 & 4 \end{pmatrix}$ is a singular matrix and its inverse matrix does not exist.

The second equation is obtained from the first by multiplying throughout by 2. Thus, the graphs of $3x + 2y = 7$ and $6x + 4y = 14$ represent the *same line*, i.e. the two lines coincide. There is an *infinite number of solutions*. Some solutions include $(1, 2)$ and $(3, -1)$.

PRACTISE NOW 10

SIMILAR QUESTIONS

Exercise 5E Questions 7–10

Solve the following sets of equations using the matrix method.

(a) $4x - y = 5$, $6x + 2y = 18$

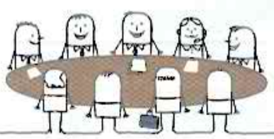
(b) $4x + y = 15$, $12x + 3y = 13$

(c) $8x + 2y = 14$, $4x + y = 7$

Explain what your answers represent.

In general,

> when a pair of simultaneous equations is expressed in the matrix form, the simultaneous equations will represent
> - two intersecting lines, if the determinant is not 0.
> - two parallel lines or the same straight line if the determinant is 0.



Class Discussion

Interesting Properties of Matrices

We will examine some of the properties of matrices.

1. In the real numbers system, given two real numbers a and b, if $ab = 0$ this would imply that either $a = 0$ or $b = 0$ or both a and b are zero.

   However, this may not hold true in Matrices.

   Example:

   Given $\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}$,

   $$\mathbf{AB} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}$$

   $$= \begin{pmatrix} \quad & \quad \end{pmatrix}$$

   However, $\mathbf{A}$ ______ 0 and $\mathbf{B}$ ______ 0.

2. In the real numbers system, given three real numbers p, q and r, if $pq = pr$, it implies that either $p = 0$ or $q = r$. However, this may not hold true in Matrices.
Example:

Given $\mathbf{P} = \begin{pmatrix} 2 & 0 \\ 3 & 0 \end{pmatrix}$, $\mathbf{Q} = \begin{pmatrix} 2 & 3 \\ 0 & 0 \end{pmatrix}$ and $\mathbf{R} = \begin{pmatrix} 2 & 3 \\ 0 & 3 \end{pmatrix}$,

$$\mathbf{PQ} = \begin{pmatrix} 2 & 0 \\ 3 & 0 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix}$$

$$\mathbf{PR} = \begin{pmatrix} 2 & 0 \\ 3 & 0 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 0 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix}$$

However, **P** _____ 0 and **Q** _____ **R**.

3. In the real numbers system, any number multiplied by or with 1 will give itself, i.e. $1 \times n = n \times 1 = n$, where n is any real number.
In Matrices, the identity matrix **I** plays a similar role i.e. given a matrix **A**, $\mathbf{IA} = \mathbf{AI} = \mathbf{A}$.
The identity matrix is a square matrix with the number 1 on its leading diagonal and 0 anywhere else.
Discuss a proof for the property.

4. In the real numbers system, we obtain the number 1 when we multiply a number by its reciprocal (or inverse) i.e. $n \times \frac{1}{n} = 1$, where n is any real number.
In Matrices, we obtain the identity matrix when we multiply a matrix by its inverse, i.e. given a matrix **A**, $\mathbf{A} \times \mathbf{A}^{-1} = \mathbf{I}$.
Discuss a proof for the property.

Investigation

Encoding and Decoding Messages

Matrices can be used to encode messages.

Suppose you are a secret agent working on a top-secret mission. You need to report to your boss about a secret room that you have found, but you are afraid your enemies will interpret your message. So you have to send your message in code. The message is:

SECRET ROOM IN BASEMENT

Part A:
A simple method of encoding a message is to use the encoding key shown in Fig. 5.6.

A ➔ D
B ➔ E
C ➔ F
⋮
W ➔ Z
X ➔ A
Y ➔ B
Z ➔ C

Fig. 5.6

Usually, we write the encoded message in blocks of 4 letters.
The first 3 blocks are: VHFU HWUR RPLQ

1. Find the remaining parts of the encoding message.

If your enemies intercept this encoded message, they can break it easily by using **frequency analysis**. In English, the 3 most frequent letters used are E, followed by T and then A.

2. Which letter occurs the most often in the above encoded message? Does it correspond to E, T or A in the original message?

Your enemies will try H → E (i.e. I → F, J → G, etc.) and decode the entire message. If it does not make sense, they will try H → T (i.e. I → U, J → V, etc.), and so forth, until the decoded message makes sense. Therefore there is a need for a more secure coding system.

Part B:

First, represent each letter by a number, as shown in Fig. 5.7:

A → 1

B → 2

C → 3

⋮

Z → 26

Fig. 5.7

So the first 2 letters of the message is: S → 19 and E → 5.

The matrix encoding key is $\begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}$.

To encode the first 2 letters, $\begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}\begin{pmatrix} 19 \\ 5 \end{pmatrix} = \begin{pmatrix} 62 \\ 105 \end{pmatrix}$.

62, when divided by 26, leaves a remainder of 10, which represents J. Another way is to subtract 26 continuously from 62 until you reach a number between 1 and 26 inclusive, e.g. $62 - 26 = 36$; $36 - 26 = 10$.

3. What letter does 105 represent?

You can encode all the letters at one go. The following shows the encoding of the first 6 letters. You must fill the second matrix *column by column*, not row by row.

$$\begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}\begin{pmatrix} 19 & 3 & 5 \\ 5 & 18 & 20 \end{pmatrix} = \begin{pmatrix} 62 & 27 & 35 \\ 105 & 51 & 65 \end{pmatrix} \rightarrow \begin{pmatrix} 10 & 1 & 9 \\ 1 & 25 & 13 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} \text{J} & \text{A} & \text{I} \\ \text{A} & \text{Y} & \text{M} \end{pmatrix}$$

So, SECRET is encoded as JAAYIM.

4. Can you use frequency analysis to break the code? For example, what letter(s) in the original message does the letter A in the encoded message represent?

5. Encode the rest of the message using the above matrix encoding key. Then write down the entire encoded message in blocks of 4 letters.

INFORMATION

Although it is harder to decode matrix encoding, we can still analyse the frequency of blocks of 2 letters using a computer software in order to try to break the code.

Part C:

Suppose your boss replies using the same coding system:

JMGF QDOU QPFW

The matrix decoding key for the above encoding is $\begin{pmatrix} 2 & -1 \\ -5 & 3 \end{pmatrix}$.

Notice that $\begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ -5 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I}$, the 2 by 2 identity matrix (see Exercise 5C Question 9).

$\begin{pmatrix} 2 & -1 \\ -5 & 3 \end{pmatrix}$ is called the **inverse matrix** of $\begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}$.

6. Decode the above message.

Hint: If you get a negative number, e.g. –11, you add 26 continuously until you reach a number between 1 and 26 inclusive, e.g. $-11 + 26 = 15 \rightarrow$ O.

Internet Resources

In the field of **cryptography**, the method of encoding and decoding a message is called a *cipher*. The cipher used in part B and C of this investigation is called the **'Hill cipher'**. Search on the Internet for more examples of different types of ciphers, e.g. RSA cipher and examine how they are used, their limitations and their real-world applications.

Journal Writing

We can use matrices to rotate a point on the Cartesian plane.

For example, Fig. 5.8 shows a point $P(2, 3)$. We want to rotate P 90° anti-clockwise about the origin to give the image P'.

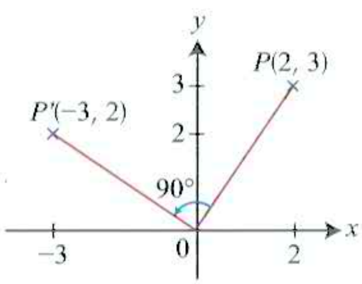


Fig. 5.8

A 90° anti-clockwise rotation about the origin O can be represented by the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

By matrix multiplication, $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$.

$\therefore$ The coordinates of the image of P under the rotation is $(-3, 2)$.

1. Find the image of $Q(5, 4)$ if Q is rotated 90° anti-clockwise about O.
2. Using the point $P(2, 3)$, try to find the matrices for the following transformations.
   (i) 90° clockwise rotation about O,
   (ii) 180° rotation about O,
   (iii) reflection in the x-axis,
   (iv) reflection in the y-axis.
   Hint: For reflection in the x-axis and the y-axis, will one of the coordinates remain unchanged after transformation?
3. When plotted on a Cartesian plane, a point can be rotated, reflected or translated. This is known as **transformation** of a point. Search on the Internet to find
   (i) a matrix to rotate a point through any angle about the origin,
   (ii) a matrix to reflect a point in a line that does not pass through the origin.
   In animations, a point in a three-dimensional (3D) field may need to be translated to another point. Search on the Internet to find
   (iii) a matrix to transform a point in 3D graphics for movie-making.

Then write a short article to explain how we can use transformation matrices for movie-making.

Exercise 5E

BASIC LEVEL

1. In a soccer tournament, 4 teams play against one another twice. The matrices $\mathbf{P}$ and $\mathbf{Q}$ show the results of the matches and the points awarded respectively.

$$\mathbf{P} = \begin{array}{c} \\ \end{array}\begin{array}{ccc} \text{Win} & \text{Draw} & \text{Lose} \\ \end{array}$$
$$\mathbf{P} = \begin{pmatrix} 5 & 1 & 6 \\ 8 & 4 & 0 \\ 2 & 3 & 7 \\ 3 & 4 & 5 \end{pmatrix} \begin{array}{l} \text{Team } A \\ \text{Team } B \\ \text{Team } C \\ \text{Team } D \end{array}$$

$$\mathbf{Q} = \begin{array}{c} \text{Points} \\ \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} \end{array} \begin{array}{l} \\ \text{Win} \\ \text{Draw} \\ \text{Lose} \end{array}$$

   (i) How many matches does each team play during the tournament?
   (ii) Evaluate $\mathbf{PQ}$.
   (iii) Explain what your answer to **(ii)** represents.

2. Solve the following simultaneous equations by the matrix method where possible. Where there is no solution, explain why this is so.
   (a) $x - y = 7$, $x + y = 11$
   (b) $x + 3y = 6$, $2x + y = 4$
   (c) $3x + 2y = 7$, $5x - y = 3$
   (d) $3x + 6y = 25$, $3x + 4y = 17$
   (e) $6x + 7y = 4$, $5x + 6y = 3$
   (f) $2x - 5y = 1$, $3x - 7y = 2$
   (g) $2x - 4y = 8$, $x - 2y = 6$
   (h) $4x + 2y = 10$, $2x + y = 5$

INTERMEDIATE LEVEL

3. The price of tickets to a musical at Marina Bay Sands Theatre is as follows:

VIP Reserve: \$130   A Reserve: \$115
B Reserve: \$90   C Reserve: \$75

The number of tickets sold for three nightly performances are as follows:

	VIP Reserve	A Reserve	B Reserve	C Reserve
Friday	220	430	555	355
Saturday	245	485	520	310
Sunday	280	430	515	375

Write down two matrices only such that the elements of their product will give the total amount of ticket sales for the three nightly performances. Hence, calculate the total amount collected for the three nightly performances.

4. A pie company operates three outlets selling sardine, mushroom, chicken, vegetable and apple pies. The table below shows the number of pies sold in a day in each of the three outlets. The prices for each type of pie are also included.

	Outlet *A*	Outlet *B*	Outlet *C*	Prices
Sardine	85	65	38	\$2.80
Mushroom	74	84	42	\$2.40
Chicken	80	70	56	\$2.60
Vegetable	60	52	40	\$3.00
Apple	82	94	56	\$2.50

(i) Write down two matrices only such that the product will give the total takings of each outlet and hence, calculate the takings for the day, for each of the outlets.

(ii) Hence, calculate the total takings for the pie company.

5. The table below shows the number of cups of tea, tea with milk, coffee, and coffee with milk, sold during breakfast hours by 3 drinks stalls in a neighbourbood.

	Tea	Tea with milk	Coffee	Coffee with milk
Albert Drink Stall	22	32	42	28
Best Drink Stall	18	26	36	32
Chandra Drink Stall	27	24	52	25

(i) If the price is \$0.90 for a cup of tea, \$1.00 for a cup of tea with milk, \$1.10 for a cup of coffee and \$1.20 for a cup of coffee with milk, form two matrices only such that the product will give the total amount of money collected by the three different stalls from the sales of these drinks during the breakfast hours.

(ii) During the month of January, Albert Drink Stall operates on 26 days, Best Drink Stall operates on 29 days and Chandra Drink Stall operates on 30 days. Assuming that each of the stalls sells the same number of cups of drinks during each of the days, use matrix multiplication to find the total amount collected by all three stalls during the month of January.

6. During the Family Day for a multi-national company, the organiser ordered T-shirts of various sizes for its employees. The table below gives the orders of the T-shirts of the various sizes.

Size	Extra -large	Large	Medium	Small
Men	220	240	180	85
Women	50	60	210	135
Children	10	40	200	250

The cost of an extra-large, large, medium and small T-shirt is \$15, \$13.50, \$12 and \$10 respectively. Evaluate the product.

(i) Write down two matrices only such that the elements of their product under matrix multiplication give the total cost of the T-shirts ordered for the men, women and children respectively. Evaluate the product.

(ii) Evaluate the matrix product

$$\begin{pmatrix} 220 & 240 & 180 & 85 \\ 50 & 60 & 210 & 135 \\ 10 & 40 & 200 & 250 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

and explain what the elements in the product represent.

(iii) Evaluate the matrix product

$$\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 220 & 240 & 180 & 85 \\ 50 & 60 & 210 & 135 \\ 10 & 40 & 200 & 250 \end{pmatrix}$$

and explain what the elements in the product represent.

(iv) Write down two matrices such that the elements of their product under matrix multiplication will give the **total cost** of the T-shirts ordered. Hence, find the total cost.

7. Find the inverse of the matrix $\begin{pmatrix} 3 & 4 \\ 2 & 6 \end{pmatrix}$ and use it to solve the simultaneous equations $3x + 4y = 18$ and $2x + 6y = 22$.

8. Find the inverse of the matrix $\begin{pmatrix} 5 & 7 \\ 2 & 3 \end{pmatrix}$. Use this inverse to find the coordinates of the point of intersection of the lines $5x + 7y = 19$ and $2x + 3y = 8$. Explain why the method fails when applied to the lines $3x + 4y = 7$ and $6x + 8y = 9$.

9. Find the inverse of the matrix $\begin{pmatrix} 7 & -11 \\ 2 & -3 \end{pmatrix}$ and use it to solve the simultaneous equations $7x - 11y = 10$ and $2x - 3y = 3$. Explain why the method fails when applied to the lines $3x + 2y = 7$ and $6x + 4y = 14$.

10. Express the simultaneous equations $2x + ky = 7$ and $4x - 9y = 4$ in matrix form. Given that the above equations have no solution, find the value of k.

ADVANCED LEVEL

11. A hamper company packs four different gift hampers. The table below gives the contents of each type of hamper.

Type	Abalone (Cans)	Groundnuts (Packets)	Chocolate (Boxes)	Candy (Boxes)	Biscuits (Packets)
Happiness	2	6	5	4	5
Prosperity	3	8	2	3	2
Bumper Harvest	4	9	3	6	3
Good Fortune	3	5	6	3	4

The cost price of each item is as follows.

A can of abalone	\$30
A packet of groundnuts	\$1.80
A box of chocolates	\$4.80
A box of candy	\$3.50
A packet of biscuits	\$2.40

(i) Write down two matrices only such that the product of the elements under matrix multiplication will give the cost price of each type of hamper. Evaluate this product.

(ii) The company receives an order for 85 Happiness hampers, 90 Prosperity hampers, 80 Bumper Harvest hampers and 120 Good Fortune hampers. Form two matrices so that the elements of the product of these matrices will give the total cost price of the hampers ordered. Evaluate this product.

(iii) The company intends to make a profit of 30% on each Happiness hamper, 25% on each Prosperity hamper, 20% on each Bumper Harvest hamper and 15% on each Good Fortune hamper. Form two matrices such that the elements of their product will give the selling price of each of the hampers. Evaluate this product.

12. Four components P, Q, R and S are produced by a toy company. Each company undergoes three different manufacturing processes: cutting, grinding and polishing. The number of minutes required for each manufacturing process for each component, the cost (in cents per minute) of each manufacturing process, and the number of components needed to meet an order are given in the following table.

Manufacturing Process	Cutting	Grinding	Polishing	Number of components
Number of minutes for P	4	5	6	60
Number of minutes for Q	3	6	7	80
Number of minutes for R	5	8	6	90
Number of minutes for S	6	4	5	80
Cost in cents per minute	12	15	24	–

(i) Write down two matrices such that the elements of their product give the costs for components P, Q, R and S, respectively. Calculate their product.

(ii) Using the result in **(i)**, find the total cost of the order.

13. A small catering firm provides three types of economy buffet lunch. The table below shows the ingredients for one set of each type of buffet lunch. Each set of buffet lunch is for 20 people.

Type of buffet lunch	Amount of ingredients needed for one set (kg)				
	Mutton	Chicken	Fish	Vegetable	Rice
Mutton and fish	1.2	0	1.4	2.6	5.2
Chicken and fish	0	1.6	1.6	2.8	4.7
Mutton and chicken	1.4	1.8	0	3	4.4

(i) Using matrix multiplication, find a matrix whose elements give the total amount of mutton, chicken, fish, vegetable and rice needed for providing 280 sets of 'mutton and fish' lunch, 320 sets of 'chicken and fish' lunch, and 360 sets of 'mutton and chicken' lunch.

(ii) The costs of 1 kg of mutton, chicken, fish, vegetable and rice are \$12.50, \$5.20, \$7.80, \$1.40 and \$1.10 respectively. Using the result obtained in **(i)**, find the total cost incurred in **(i)** by matrix multiplication.

1. Two matrices **A** and **B** are **equal** if and only if
   (a) both matrices have the **same order**, and
   (b) their *corresponding elements* are equal.

2. If two matrices **A** and **B** have the same order, then **A** + **B** and **A** – **B** are obtained by adding the corresponding elements of **A** and **B**, or subtracting the corresponding elements of **B** from **A**, respectively, e.g.

$$\text{if } \mathbf{A} = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} w & x \\ y & z \end{pmatrix}, \text{ then}$$
$$\mathbf{A}+\mathbf{B} = \begin{pmatrix} p+w & q+x \\ r+y & s+z \end{pmatrix} \text{ and } \mathbf{A}-\mathbf{B} = \begin{pmatrix} p-w & q-x \\ r-y & s-z \end{pmatrix}.$$

3. If a matrix **A** is multiplied by a scalar k, every element in **A** is multiplied by k, e.g.

$$\text{if } \mathbf{A} = \begin{pmatrix} p & q & r \\ s & t & u \end{pmatrix}, \text{ then } k\mathbf{A} = \begin{pmatrix} kp & kq & kr \\ ks & kt & ku \end{pmatrix}.$$

4. For any two matrices **A** and **B**, the product **AB** is only possible if

$$\boxed{\text{number of columns of } \mathbf{A} = \text{number of rows of } \mathbf{B}.}$$

In other words,

$$\underset{m \times n}{\mathbf{A}} \times \underset{n \times p}{\mathbf{B}} = \underset{m \times p}{\mathbf{C}}$$

must be equal

For example, if $\mathbf{A} = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} w & x \\ y & z \end{pmatrix}$, then

$$\mathbf{AB} = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} w & x \\ y & z \end{pmatrix} = \begin{pmatrix} pw+qy & px+qz \\ rw+sy & rx+sz \end{pmatrix}.$$

5. If $\mathbf{A} = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$, det $\mathbf{A} = ps - qr$ and $\mathbf{A}^{-1} = \dfrac{1}{ps-qr}\begin{pmatrix} s & -q \\ -r & p \end{pmatrix}$, provided det $\mathbf{A} \neq 0$. $\mathbf{AA}^{-1} = \mathbf{I}$

Review Exercise 5

1. Evaluate each of the following.

(a) $\begin{pmatrix} 3 & 2 \\ 4 & -1 \end{pmatrix}+\begin{pmatrix} 5 & 4 \\ 3 & 2 \end{pmatrix}-\begin{pmatrix} 6 & 3 \\ 1 & -2 \end{pmatrix}$

(b) $\begin{pmatrix} 2 & 3 & -4 \\ 6 & -1 & 3 \end{pmatrix}+\begin{pmatrix} 3 & 1 & 5 \\ -3 & 2 & 7 \end{pmatrix}+\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$

(c) $\begin{pmatrix} 2 & 3 \\ 4 & -7 \\ 5 & -3 \end{pmatrix}-\begin{pmatrix} 4 & 5 \\ -2 & 7 \\ 6 & -1 \end{pmatrix}+\begin{pmatrix} -3 & 4 \\ -1 & 7 \\ -6 & 2 \end{pmatrix}$

(d) $\begin{pmatrix} 3 \\ 5 \end{pmatrix}-\begin{pmatrix} 4 \\ 7 \end{pmatrix}+\begin{pmatrix} 5 \\ -3 \end{pmatrix}$

(e) $\begin{pmatrix} 1 & 3 \end{pmatrix}-\begin{pmatrix} 3 & 2 \end{pmatrix}+\begin{pmatrix} 6 & 5 \end{pmatrix}$

(f) $\begin{pmatrix} 1 & 0 & 7 \end{pmatrix}+\begin{pmatrix} 3 & -2 & 4 \end{pmatrix}-\begin{pmatrix} 7 & 3 & -5 \end{pmatrix}$

2. Given that $\mathbf{A}=\begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}$, $\mathbf{B}=\begin{pmatrix} 5 & a \\ c & 4 \end{pmatrix}$ and $\mathbf{C}=\begin{pmatrix} b & 6 \\ 4 & d \end{pmatrix}$, find the values of a, b, c and d when

(i) $2\mathbf{A}+\mathbf{B}=\mathbf{C}$, **(ii)** $3\mathbf{A}-2\mathbf{B}=4\mathbf{C}$.

3. Evaluate each of the following matrix products if it exists.

(a) $\begin{pmatrix} 1 \\ 3 \end{pmatrix}\begin{pmatrix} 3 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & 3 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$ (d) $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}\begin{pmatrix} 3 & 0 & 1 \end{pmatrix}$

(e) $\begin{pmatrix} 7 & 9 \\ 3 & -5 \end{pmatrix}\begin{pmatrix} 2 & 6 \end{pmatrix}$

(f) $\begin{pmatrix} -2 & 3 \\ -1 & -2 \end{pmatrix}\begin{pmatrix} \frac{1}{2} \\ 1\frac{1}{2} \end{pmatrix}$

(g) $\begin{pmatrix} 0 & 2 \\ 3 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

(h) $\begin{pmatrix} 0 & -2 \\ -1 & 1 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 0 & -4 \end{pmatrix}$

(i) $\begin{pmatrix} 2 & 1 & 3 \\ -1 & -1 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$

(j) $\begin{pmatrix} 1 & 3 & 2 \end{pmatrix}\begin{pmatrix} 3 & -2 \\ 1 & 4 \\ -1 & 2 \end{pmatrix}$

4. Find the values of the unknowns in each of the following.

(a) $\begin{pmatrix} 1 & 3 & 2 \\ 0 & 1 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ a \\ 2 \end{pmatrix}=\begin{pmatrix} 5 \\ b \end{pmatrix}$

(b) $\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 2 & 1 \\ 3 & 0 \end{pmatrix}\begin{pmatrix} 2 \\ 5 \end{pmatrix}+\begin{pmatrix} 3 \\ 2 \end{pmatrix}$

(c) $\begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix}\begin{pmatrix} a & -4 \\ b & 0 \end{pmatrix}=\begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}+\begin{pmatrix} 0 & -3 \\ 6 & 2c \end{pmatrix}$

5. Given that the determinant of the matrix $\begin{pmatrix} 4 & 2 \\ 7 & -b \end{pmatrix}$ is 22, find the value of b. Hence, write down the inverse of the matrix.

6. Find the inverse of the following matrices where possible.

(a) $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ (b) $\begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}$

(c) $\begin{pmatrix} 2 & 8 \\ 1 & 4 \end{pmatrix}$ (d) $\begin{pmatrix} 3 & 5 \\ 4 & 8 \end{pmatrix}$

(e) $\begin{pmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix}$

7. Solve the following simultaneous equations by the matrix method where possible. Where there is no solution, explain why this is so.

(a) $2x + y = 90$
$x + 3y = 90$

(b) $x + y = 2$
$y - x = 2$

(c) $12x - 6y = 24$
$2x - y = 4$

(d) $25x + 5y = 9$
$5x + y = 2$

8. Three Roti Prata stalls serve two different types of Roti Prata and three different types of curry. The table shows the number of each type of food served during a busy morning and prices of each type of food.

	Plain Prata	Egg Prata	Mutton Curry	Chicken Curry	Fish Curry
Stall *A*	450	240	120	80	60
Stall *B*	250	140	80	60	20
Stall *C*	280	120	50	30	24
Cost of each item	\$1.00	\$1.50	\$6.50	\$5.50	\$4.80

Write down two matrices **P** and **Q** such that **PQ** will give the total amount collected from the three different stalls. Evaluate this product.

9. A drinks factory delivers Coke, Sprite, Root Beer and Pepsi to three different coffee shops.

Shop *A* receives 12 cartons of Coke, 8 cartons of Sprite, 12 cartons of Root Beer and 15 cartons of Pepsi.

Shop *B* receives 15 cartons of Coke, 16 cartons of Root Beer and 14 cartons of Pepsi.

Shop *C* receives 20 cartons of Sprite, 25 cartons of Root Beer and 16 cartons of Pepsi.

(i) The cost per carton is \$8.40 for Coke, \$7.80 for Sprite, \$8.80 for Root Beer and \$8.20 for Pepsi. Write down two matrices only such that the elements of their product under matrix multiplication give the total cost of drinks delivered to each shop. Evaluate this product.

(ii) In the first quarter of the year, shop *A* received 22 deliveries, shop *B* received 18 deliveries and shop *C* received 25 deliveries. Use matrix multiplication to find the total amount of money the factory collected from the three shops during this period.

10. The table below shows the number of coins collected by four drinks machines in a big shopping centre.

	10 cents	20 cents	50 cents	\$1
Machine *A*	480	460	620	430
Machine *B*	350	450	385	540
Machine *C*	420	520	420	620
Machine *D*	380	452	250	486

Using matrix multiplication twice, find the total amount collected by the four machines.

11. Six S-League teams took part in a competition and the results are shown in the table below.

	Played	Won	Drawn	Lost
Lions	18	11	2	5
Balestier	20	7	2	11
Clementi	19	4	5	10
Rovers	18	7	4	7
Geylang	22	12	1	9
Wellington	19	9	2	8

A win gains 3 points, a draw 1 point, and a loss 0 point.

(i) Write down two matrices such that the elements of their product will display the total number of points gained by each team and hence, calculate the total number of points gained by each team.

(ii) The organiser of the competition has an award system for all the teams taking part in the competition. A game played is awarded \$300, a win \$500, a draw \$200, and a loss will result in a deduction of \$300. Set up two matrices such that the elements of their product will give the total amount awarded to each of the six teams. Hence, calculate the total amount awarded to each team.

Challenge Yourself

1. In each of the following cases, find the matrix **A** which satisfies the given relationship.

   (a) $\mathbf{A} + \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} -4 \\ 3 \end{pmatrix}$

   (b) $2\mathbf{A} - \begin{pmatrix} 0 & 6 \\ 9 & -7 \end{pmatrix} = 3\begin{pmatrix} 4 & 8 \\ -3 & 5 \end{pmatrix}$

2. **(i)** Find a 2×2 matrix **X** such that
   $\begin{pmatrix} 5 & 9 \\ 1 & 2 \end{pmatrix}\mathbf{X} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

   (ii) Find a 2×2 matrix **Y** such that
   $\mathbf{Y}\begin{pmatrix} 5 & 9 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

   (iii) Is $\mathbf{X} = \mathbf{Y}$? What is so special about the matrices **X** and **Y**?

3. **(a)** In algebra, $a \times b = 0$ implies $a = 0$ or $b = 0$.
   In matrices, $\begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 3 & 0 \end{pmatrix} = \mathbf{0}$, where
   $\mathbf{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is the 2×2 zero matrix, but
   $\begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix} \neq \mathbf{0}$ and $\begin{pmatrix} 0 & 0 \\ 3 & 0 \end{pmatrix} \neq \mathbf{0}$.

   In other words, for two matrices **A** and **B**, $\mathbf{AB} = \mathbf{0}$ does not imply $\mathbf{A} = \mathbf{0}$ or $\mathbf{B} = \mathbf{0}$. Give another example of two 2×2 matrices **X** and **Y**, where $\mathbf{XY} = \mathbf{0}$, but $\mathbf{X} \neq \mathbf{0}$ and $\mathbf{Y} \neq \mathbf{0}$.

   (b) In algebra, $ab = ac$ implies that

   $$ab - ac = 0$$
   $$a(b - c) = 0$$
   $$a = 0 \quad \text{or} \quad b = c.$$

   In matrices, if **A**, **B** and **C** are 2×2 matrices such that $\mathbf{AB} = \mathbf{AC}$, investigate whether this will imply that either $\mathbf{A} = \mathbf{0}$ or $\mathbf{B} = \mathbf{C}$.

4. A square matrix **X** is an **idempotent matrix** if $\mathbf{XX} = \mathbf{X}$.
   For example, the 2 by 2 matrix $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ is an idempotent matrix because
   $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.
   Can you come up with another example of a 2×2 idempotent matrix? What about an example of a 3×3 idempotent matrix?

5. In the chapter opener, the 'matrix code' is described as a representation of the virtual reality known as the Matrix. In the movie itself, the creator of the Matrix is known as the 'Architect'.

   Suppose now that you are the 'Architect' of a simplified virtual reality with 10 people, defined by 3 different traits – height (cm), mass (kg) and intelligence quotient (IQ). As the 'Architect', you have programmed your virtual reality using a 10×3 matrix **X**, shown below.

   $$\mathbf{X} = \begin{array}{c} \begin{array}{ccc} \text{Height} & \text{Mass} & \text{IQ} \end{array} \\ \begin{pmatrix} 160 & 60 & 120 \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ 172 & 79 & 100 \end{pmatrix} \end{array} \begin{array}{c} \text{Person 1} \\ \vdots \\ \vdots \\ \vdots \\ \text{Person 10} \end{array}$$

   Now, you wish to change the quantities of some of these traits to form a new virtual reality, defined by matrix **Y**. Explain clearly how you can make the following changes by using the matrix operations which you have learnt, i.e. by adding, subtracting, or multiplying another matrix (or a scalar) to **X**.

   (i) You wish to concurrently make all the 10 people in the virtual reality taller by 5 cm and lighter by 1.2 kg. Other traits remain constant.

   (ii) You wish to increase the IQ of all 10 people in the virtual reality by 5%. Other traits remain constant.

Further Geometrical Transformations

The picture shows a microscope which is an important instrument commonly found in science laboratories. It is used to produce enlarged images of specimens that are being studied. By adjusting the nose piece of a microscope, the viewer can study the specimen at different levels of magnification. This illustrates the idea of enlargement. In Book 2, we learnt three isometric transformations: reflection, rotation and translation. Now, we shall study a non-isometric transformation – enlargement and how we can represent transformations using matrices.

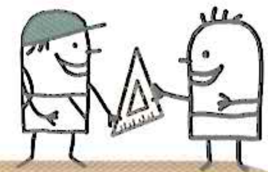
Chapter Six

LEARNING OBJECTIVES

At the end of this chapter, you should be able to:

- enlarge a figure with a whole negative and disminishing scale factor,
- find the centre and scale factor of enlargement given the original figure and its enlarged image,
- link transformations with Matrices,
- find the image figure of an object under a combination of transformations.

6.1 Enlargement



Recap

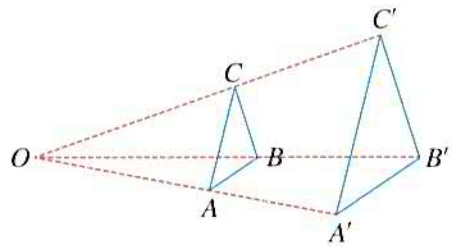


Fig. 6.1

Fig. 6.1 shows two similar triangles ABC and $A'B'C'$. $\triangle A'B'C'$ is an enlargement to $\triangle ABC$. We say that $\triangle ABC$ is transformed onto $\triangle A'B'C'$ by an enlargement with centre O and scale factor $\dfrac{OA'}{OA}$.

Construction Steps to Enlarge a Figure

In Book 2, we have learnt to construct scale drawings. In this section, we will learn to construct figures under enlargement. Fig. 6.2 shows a triangle ABC being enlarged 3 times (scale factor 3) to triangle $A_1B_1C_1$ with origin, O, as the *centre of enlargement* or *point of enlargement*.

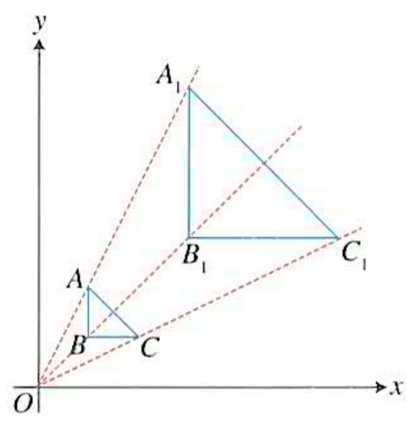


Fig. 6.2

Construction Steps:

1. Join the point of enlargement O to A and produce OA.
2. From O, mark off a distance equal to 3 times the length of OA on OA produced to get the point A_1.
3. Repeat the above procedure for points B and C to get B_1 and C_1.
4. Join A_1B_1, B_1C_1 and A_1C_1 to get the enlarged figure $A_1B_1C_1$.

How many times larger is the image as compared to the original triangle in Fig. 6.2?

In reflection, rotation and translation, there is no change in the shape and size of the image. They are called **isometric transformations**. In enlargement, there is no change in the shape, angle and orientation of the figure but the size of the image changes. Enlargement is a non-isometric transformation. We have learnt in Book 2 that the images formed under enlargement yield similar figures.

Fig. 6.3 shows a quadrilateral $ABCD$ being enlarged to quadrilateral $A'B'C'D'$ with E as the centre of enlargement and scale factor 2.

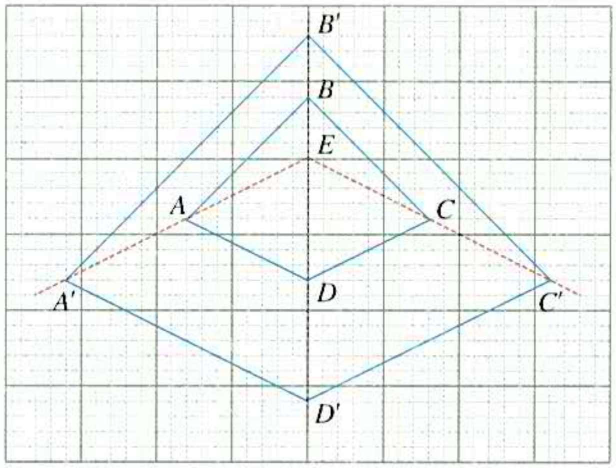


Fig. 6.3

We can also say that quadrilateral $A'B'C'D'$ is being "enlarged" to quadrilateral $ABCD$ with E as the centre of enlargement and scale factor $\frac{1}{2}$, even though the image $ABCD$ is smaller than the original figure $A'B'C'D'$.

The term enlargement in mathematics may thus refer to the enlarging or diminishing of a figure depending on the scale factor involved.

Fig. 6.4 shows $\triangle ABC$ being enlarged with a scale factor of -2 and E as the centre of enlargement.

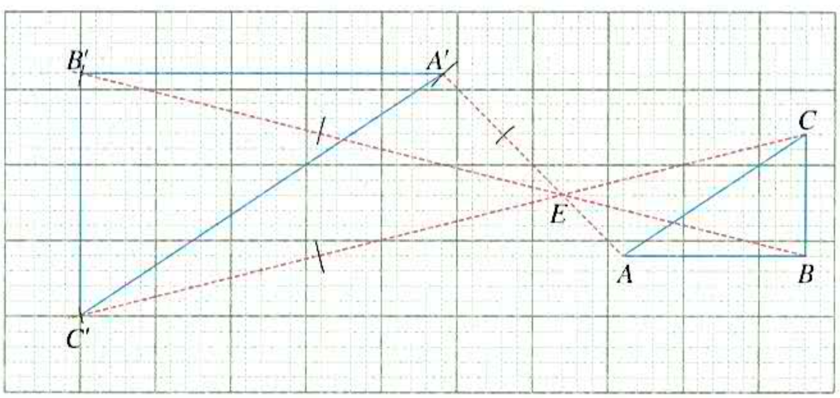


Fig. 6.4

Construction Steps:

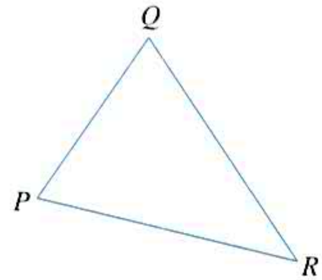
1. Join A to E and produce in the direction of AE.
2. With E as the centre and radius equal to $2AE$, mark off the image A' on AE produced.
3. Repeat Step 1 and Step 2 for points B and C to get the images B' and C'.

We have learnt that if the scale factor is greater than 1, the image is enlarged; if it is between 0 and 1, the image is diminished. Compare the points of A and A'. For a negative scale factor, what can you say about the corresponding points of the image and the original figure?

Worked Example 1

(Construction of Enlarged Figures with Scale Factor Greater than 1)

Enlarge $\triangle PQR$ with P as the point of enlargement and scale factor 2.

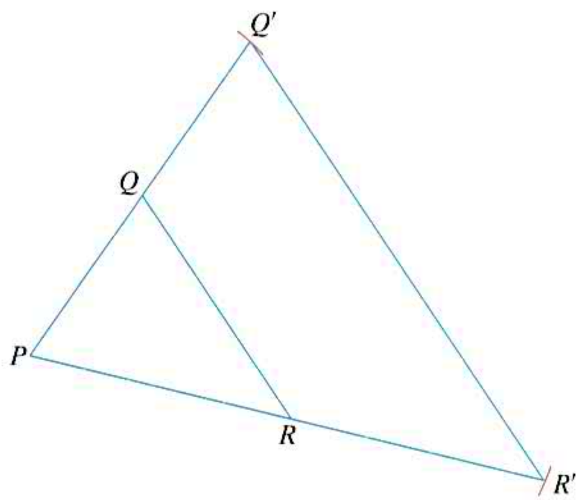


Solution:

Construction Steps:

1. Produce PQ.
2. With P as centre, use your compasses to mark the point Q' on PQ produced such that $PQ' = 2PQ$.
3. Repeat the procedure for the point R to get R'.

The diagram below shows the enlarged triangle $PQ'R'$.



ATTENTION

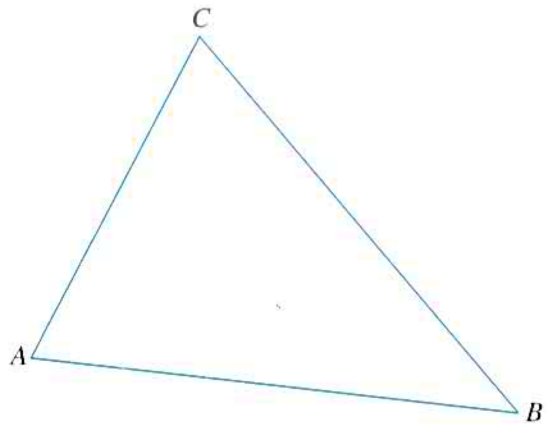
Notice that all the points in the triangle have been transformed except point P, which is the only invariant point.

PRACTISE NOW 1

SIMILAR QUESTIONS

Exercise 6A Question1(a)–(e), 2–5, 13(a)

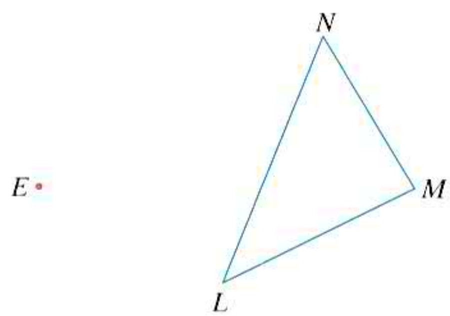
Enlarge $\triangle ABC$ with A as the centre of enlargement and scale factor 2.5.



Worked Example 2

(Construction of Enlarged Figure with Scale Factor Between 0 and 1)

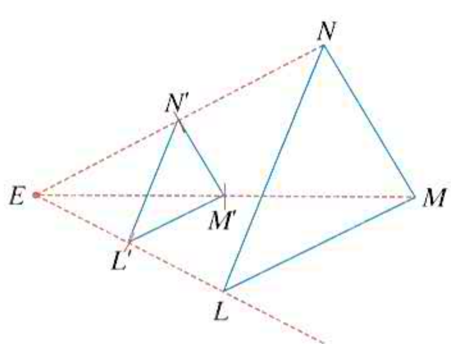
Enlarge $\triangle LMN$ with E as the point of enlargement and scale factor $\frac{1}{2}$.



Solution:

Construction Steps:

1. Join E to L.
2. With E as centre, use your compasses to mark off the point L' on EL such that $EL' = \frac{1}{2} EL$.
3. Repeat the above procedure for points M and N to get points M' and N'.
4. Join $L'M'$, $M'N'$ and $L'N'$ to obtain the image triangle $L'M'N'$.

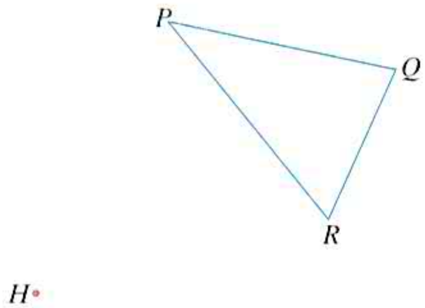


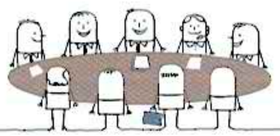
PRACTISE NOW 2

SIMILAR QUESTIONS

Exercise 6A Questions 1(f), 6, 7, 13(b)

Enlarge $\triangle PQR$ with H as the point of enlargement and scale factor $\frac{1}{2}$.





Class Discussion

Enlargement in our surroundings

Explore your surroundings to find examples of enlargement being put into use. Discuss your observations in class.

Centre of Enlargement and Scale Factor

Fig. 6.5 shows how to find the centre of enlargement and the scale factor, with $ABCD$ as the original figure and $A_1B_1C_1D_1$ as the image.

Internet Resources

Use the open tool, Geometer's Sketchpad, to explore enlargement where you can alter the centre and scale factor of enlargement.

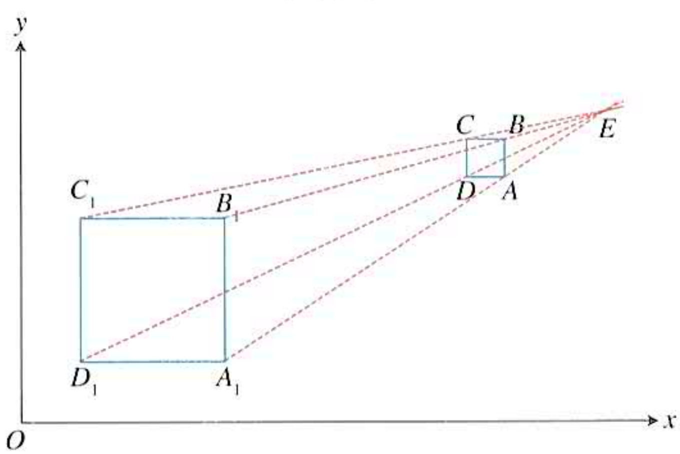


Fig. 6.5

Construction Steps:

1. Join any two corresponding points from the original figure and the image (e.g. A_1A and B_1B).
2. Extend A_1A and B_1B.
3. The point of intersection of these lines yields the centre of enlargement E.
4. The scale factor, $k = \dfrac{A_1B_1}{AB} = \dfrac{B_1C_1}{BC} = \dfrac{C_1D_1}{CD} = \dfrac{D_1A_1}{DA}$.

   Alternatively, $k = \dfrac{A_1E}{AE} = \dfrac{B_1E}{BE} = \dfrac{C_1E}{CE} = \dfrac{D_1E}{DE}$.

Worked Example 3

(Finding Centre and Scale Factor of Enlargement)

$\triangle PQR$ has vertices $P(1, 2)$, $Q(5, 1)$ and $R(4, 4)$. An enlargement maps $\triangle PQR$ onto $\triangle PAB$. Given that the coordinates of A are $(m, 0)$, plot $\triangle PQR$ on a sheet of graph paper and use construction to find

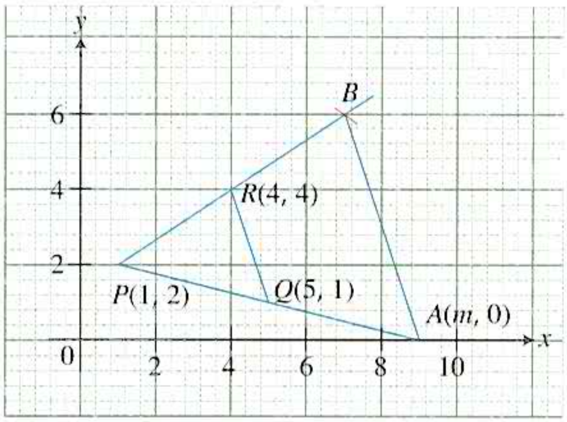
(a) the centre of enlargement,

(b) the value of m,

(c) the scale factor of the enlargement,

(d) the coordinates of the point B.

Solution:



(a) From the graph plotted, we see that P is the invariant point and hence, the centre of enlargement is (1, 2).

(b) PQ is produced to meet A on the x-axis. A is the point (9, 0).

$\therefore\ m = 9$

(c) $PQ = QA$.

$\therefore$ scale factor $= \dfrac{PA}{PQ} = \dfrac{2}{1} = 2$

(d) Produce PR. Draw an arc to cut PR produced at B such that $PB = 2PR$.

The coordinates of B are (7, 6).

PRACTISE NOW 3

SIMILAR QUESTIONS

Exercise 6A Questions 8–11, 14–17

The vertices of $\triangle ABC$ are $A(1, 1)$, $B(1, -1)$ and $C(2, 2)$. Under an enlargement, $\triangle ABC$ is mapped onto $\triangle PQR$ whose vertices have coordinates $P(3, 2)$, $Q(3, -2)$ and $R(5, 4)$. Plot these two triangles on a sheet of graph paper and find

(a) the coordinates of the centre of enlargement,

(b) the scale factor.

Area of Enlarged Figures

In Book 3, we have learnt that the ratio of the areas of two similar figures is the square of the ratio of their corresponding lengths. Fig. 6.6 shows $\triangle ABC$ being enlarged to $\triangle APQ$ with the centre of enlargement at A and scale factor k.

Let $AB = x$ cm and $AC = y$ cm, then $AP = kx$ cm and $AQ = ky$ cm.

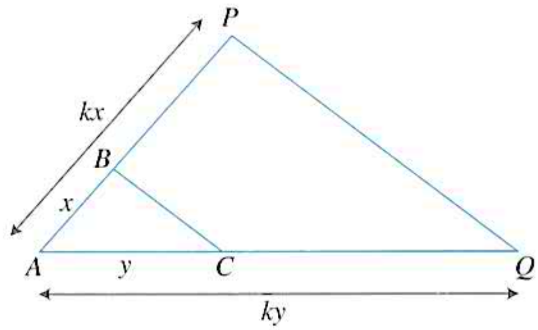


Fig. 6.5

Now,

area of $\triangle ABC = \frac{1}{2} \times x \times y \times \sin A$ and

area of $\triangle APQ = \frac{1}{2} \times kx \times ky \times \sin A$

$$\therefore \frac{\text{area of } APQ}{\text{area of } ABC} = \frac{\frac{1}{2}(kx)(ky)\sin A}{\frac{1}{2}xy\sin A} = k^2$$

In general,

$$\boxed{\textbf{area of image} = k^2 \times \textbf{(area of the original figure)}}$$

Worked Example 4

(Finding Area of Enlarged Figure using Scale Factor)

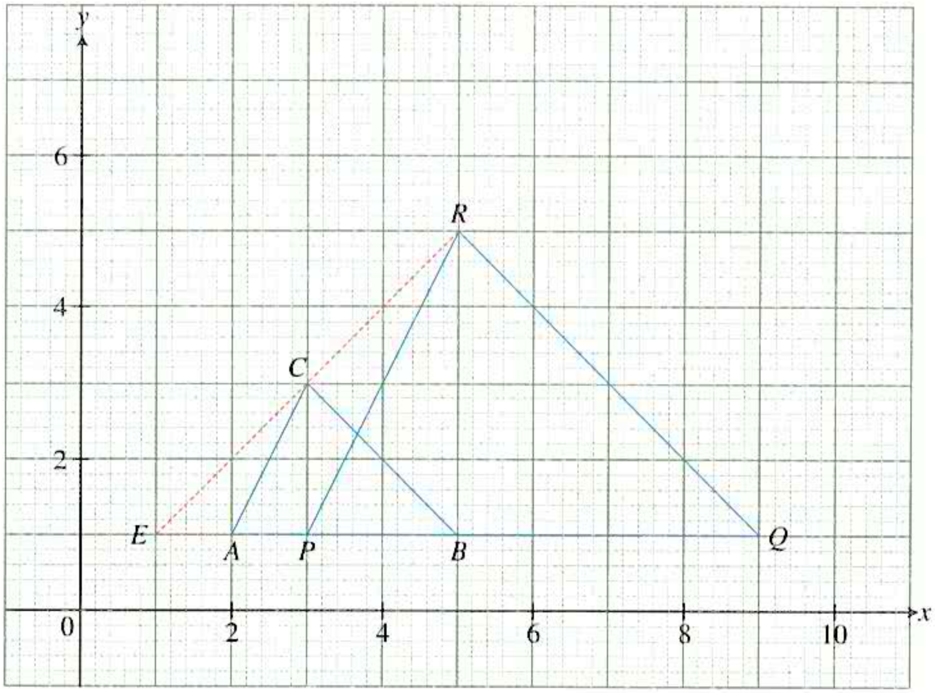
The image of $\triangle ABC$ under an enlargement is $\triangle PQR$. Given that the coordinates of the triangles are $A(2, 1)$, $B(5, 1)$, $C(3, 3)$, $P(3, 1)$, $Q(9, 1)$ and $R(5, 5)$, draw the two triangles on a sheet of graph paper and find

(a) the coordinates of the centre of enlargement,

(b) the scale factor,

(c) the area of $\triangle ABC$ and of $\triangle PQR$.

Solution:



(a) Join RC and produce; join PA and produce. RC and PA produced meet at point $E(1, 1)$. Hence, the coordinates of the centre of enlargement are (1, 1).

(b) Scale factor $= \dfrac{PQ}{AB} = \dfrac{6}{3} = 2$

(c) Length of $AB = 3$ units

Height of $\triangle ABC = 2$ units

$\therefore$ Area of $\triangle ABC = \dfrac{1}{2}(3)(2)$

$= 3 \text{ units}^2$

Area of $\triangle PQR = 2^2 \times (3 \text{ units}^2)$

$= 12 \text{ units}^2$

PRACTISE NOW 4

SIMILAR QUESTIONS

Exercise 6A Questions 12, 18–20

The coordinates of $\triangle ABC$ are $A(1, 1)$, $B(7, 1)$ and $C(5, 6)$. $\triangle ABC$ is mapped onto $\triangle APR$ by an enlargement of scale factor 3. Draw the two triangles on a sheet of graph paper and find

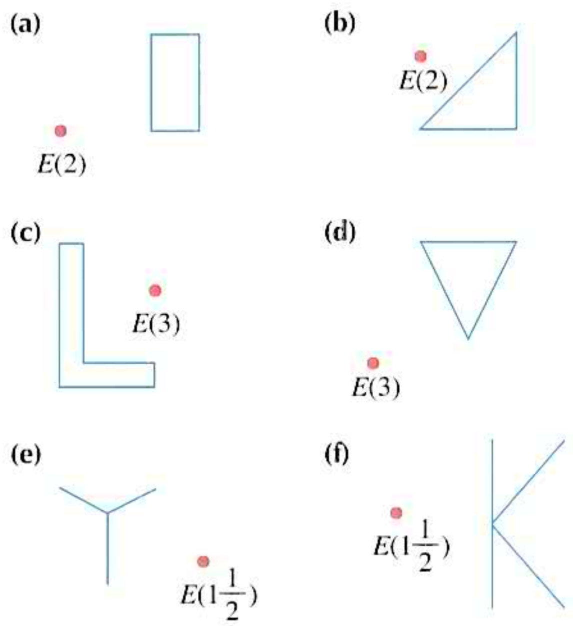
(a) the coordinates of the centre of enlargement,

(b) the area of $\triangle APR$.

Exercise 6A

BASIC LEVEL

1. On a sheet of paper, enlarge the following figures with centre of enlargement E and scale factors each indicated in brackets.



2. Draw on graph paper the triangle ABC with vertices $A(2, 1)$, $B(2, 5)$ and $C(4, 2)$. Enlarge $\triangle ABC$ with the origin as the point of enlargement and scale factor 2.

3. Draw on graph paper the triangle PQR with vertices $P(2, 2)$, $Q(5, 3)$ and $R(3, 5)$. Enlarge $\triangle PQR$ with Q as the point of enlargement and scale factor 3.

4. Draw on graph paper the quadrilateral $PQRS$ with vertices $P(2, 2)$, $Q(7, 2)$, $R(6, 6)$ and $S(4, 6)$. Enlarge $PQRS$ with $E(4, 4)$ as the centre of enlargement and scale factor $1\frac{1}{2}$.

5. The vertices of $\triangle ABC$ are $A(1, 1)$, $B(3, -1)$ and $C(0, 0)$. $\triangle ABC$ is enlarged to $\triangle PQR$ with $E(4, 4)$ as the centre of enlargement and scale factor $\frac{1}{2}$. Find the coordinates of P, Q and R.

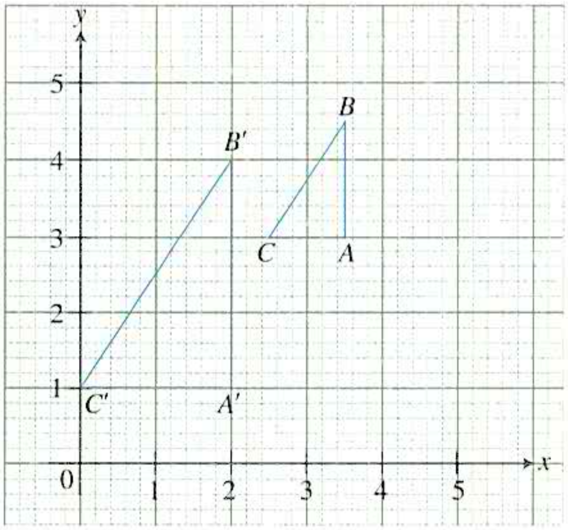
6. The coordinates of $\triangle ABC$ are $A(2, 1)$, $B(7, 1)$ and $C(4, 4)$. $\triangle ABC$ is mapped onto $\triangle APQ$ by an enlargement scale factor 2.
   - **(a)** State the centre of enlargement.
   - **(b)** Find the coordinates of P and Q.

7. The vertices of $\triangle PQR$ are $P(1, 2)$, $Q(2, 6)$ and $R(8, 1)$. $\triangle PQR$ is mapped onto $\triangle LMN$ by an enlargement centre $E(2, 4)$ and scale factor $\frac{1}{2}$. Find the coordinates of L, M and N by construction.

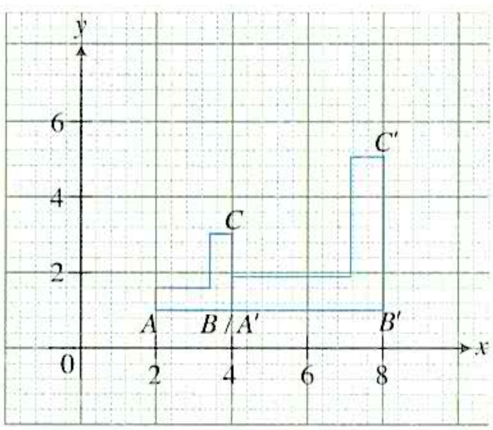
8. The coordinates of the quadrilateral $ABCD$ are $A(2, 3)$, $B(6, 2)$, $C(10, 5)$ and $D(8, 8)$. Find the image of the point
   - **(a)** A under an enlargement centre at $(0, 2)$ and scale factor 2,
   - **(b)** B under an enlargement centre at $(4, 0)$ and scale factor 3,
   - **(c)** C under an enlargement centre at $(8, 4)$ and scale factor -2,
   - **(d)** D under an enlargement centre at $(1, 2)$ and scale factor $\frac{1}{2}$.

9. Find the centre of enlargement and scale factor for each of the following enlargements which map ABC onto $A'B'C'$.

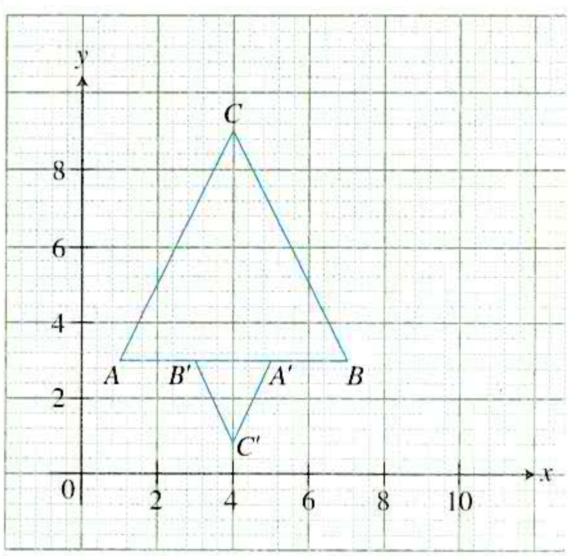
(a)



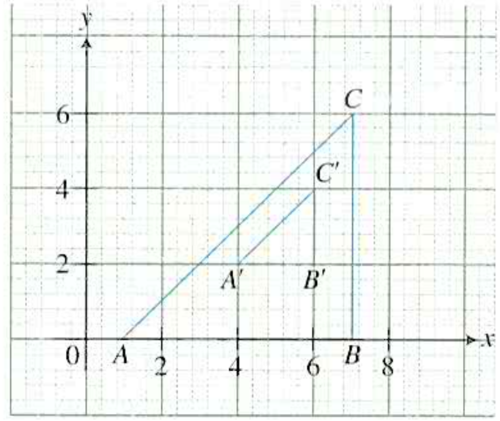
(b)



(c)



(d)



10. $\triangle ABC$ with vertices $A(2, 1)$, $B(4, 3)$ and $C(3, 6)$ is transformed into $\triangle A'B'C'$ under an enlargement, with centre $(1, 1)$ and scale factor 3. Illustrate these points on a clearly-labelled diagram, marking the positions of $\triangle ABC$ and $\triangle A'B'C'$.

11. Enlarge the following triangles with the centre of enlargement E and scale factor k as given.

(a) $A(1, 3)$, $B(2, 5)$ and $C(6, 1)$; $E(0, 0)$; $k = 2$

(b) $P(1, 4)$, $Q(4, 1)$ and $R(5, 6)$; $E(1, 2)$; $k = -2$

(c) $X(1, 1)$, $Y(2, 3)$ and $Z(4, 2)$; $E(1, 1)$; $k = 3$

(d) $J(4, 4)$, $H(6, 7)$ and $K(3, 9)$; $E(8, 4)$; $k = \dfrac{1}{2}$

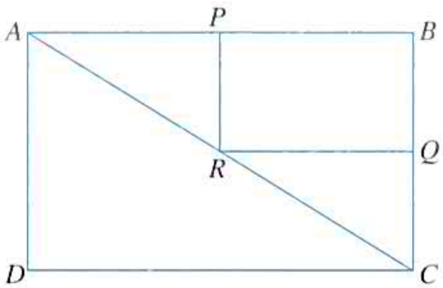
(e) $L(4, 1)$, $M(4, 3)$ and $N(1, 3)$; $E(1, 0)$; $k = -3$

12. In each of the following parts, $\triangle ABC$ is mapped onto $\triangle PQR$ by an enlargement. Plot the object and image and find the centre of enlargement and the scale factor in each case:

	Object	Image
(a)	$A(4, 4)$, $B(7, 4)$, $C(4, 6)$	$P(1, 2)$, $Q(7, 2)$, $R(1, 6)$
(b)	$A(3, 3)$, $B(5, 1)$, $C(2, 1)$	$P(5, 7)$, $Q(11, 1)$, $R(2, 1)$
(c)	$A(10, 4)$, $B(10, 8)$, $C(6, 8)$	$P(13, 3)$, $Q(13, 9)$, $R(7, 9)$
(d)	$A(7, 3)$, $B(13, 3)$, $C(13, 0)$	$P(3, 7)$, $Q(1, 7)$, $R(1, 8)$
(e)	$A(2, 4)$, $B(5, 3)$, $C(4, 6)$	$P(1, 6)$, $Q(7, 4)$, $R(5, 10)$
(f)	$A(3, 1)$, $B(6, 3)$, $C(1, 4)$	$P(6, 13)$, $Q(0, 9)$, $R(10, 7)$

INTERMEDIATE LEVEL

13.



In the figure, $ABCD$ is a rectangle and P and Q are the midpoints of AB and BC, respectively.

(a) $\triangle APR$ is mapped by an enlargement centre A and scale factor 2. Name the image figure.

(b) $ABCD$ is mapped by an enlargement centre B and scale factor $\frac{1}{2}$. Name the image figure.

14. Under an enlargement with centre (4, 3) and scale factor –2, the line AB with coordinates $A(9, 2)$ and $B(6, 6)$ is mapped onto the line PQ. Find

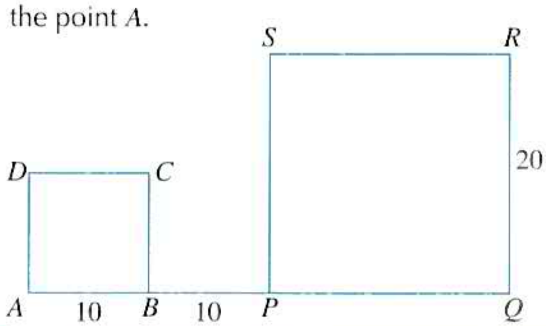
(a) the coordinates of P and Q,

(b) the length of PQ.

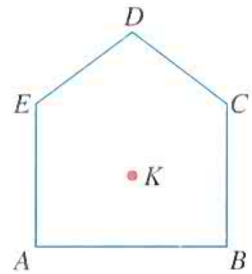
15. $\triangle ABC$ is enlarged onto $\triangle A'B'C'$ with the origin as centre of enlargement and scale factor –2. If the coordinates of $\triangle A'B'C'$ are $A'(-2, -2)$, $B'(-10, -4)$ and $C'(-4, -6)$, find the coordinates of $\triangle ABC$.

16. $\triangle ABC$ is enlarged onto $\triangle A'B'C'$ with B as the centre of enlargement and scale factor –2. If the coordinates of $\triangle A'B'C'$ are $A'(7, 7)$, $B'(3, 3)$ and $C'(9, 3)$, and B is the point (3, 3), find the coordinates of A and C.

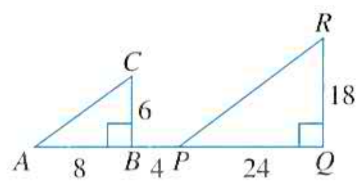
17. In the figure, $ABPQ$ is a straight line. The square $PQRS$ is the image of $ABCD$ under an enlargement. Given that $AB = 10$ cm, $BP = 10$ cm and $RQ = 20$ cm, find the distance of the centre of enlargement from the point A.



18. The figure shows the front view of a hut. Copy this diagram and using K as the centre of enlargement, construct a similar view of the hut four times the original area.



19. In the figure, $\triangle ABC$ is enlarged to $\triangle PQR$ with scale factor 3 and centre of enlargement E. Given that $BP = 4$ cm, $AB = 8$ cm, $BC = 6$ cm, $PQ = 24$ cm and $QR = 18$ cm, copy the diagram (not drawn to scale), using a scale of 1 cm to represent 4 cm, and locate E. Measure the length of EA.



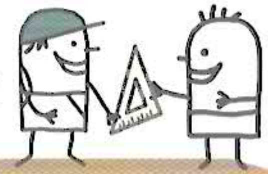
ADVANCED LEVEL

20. Draw accurately the parallelogram $ABCD$ in which $AD = 5$ cm, $DC = 9$ cm, and $ADC = 60°$. Mark the point L on AB such that $AL = 3$ cm and draw the lines LD and AC to meet at O.

(a) State the scale factor and the centre of the enlargement which maps $\triangle ALO$ onto $\triangle CDO$.

(b) Draw a square which has one corner on OD, one corner on OC and two of its sides, both on the same side of O, parallel to DC. Hence draw the square which has one corner on OD, one corner on OC and two corners on DC.

6.2 Transformation Matrices for Reflection and Rotation



Geometrical transformations can be represented by matrices. Earlier in book 2, we have seen how column matrices are used in representing translations.

A point (x, y) can be written as the column matrix $\begin{pmatrix} x \\ y \end{pmatrix}$. If $\begin{pmatrix} x \\ y \end{pmatrix}$ is pre-multiplied by a 2×2 matrix, the result will be another column matrix describing a different point.

For example

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}$$

We say that $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ transforms or maps the point (x, y) onto the point $(-y, x)$. Thus, any 2×2 matrix can be regarded as representing a geometrical transformation.

Worked Example 5

(Finding Image Points with Transformation Matrix)

$A(3, 1)$, $B(6, 3)$ and $C(4, 8)$ are the vertices of a triangle. Plot these points on a diagram. Pre-multiply the points by the matrix $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ to obtain the images A', B' and C'.

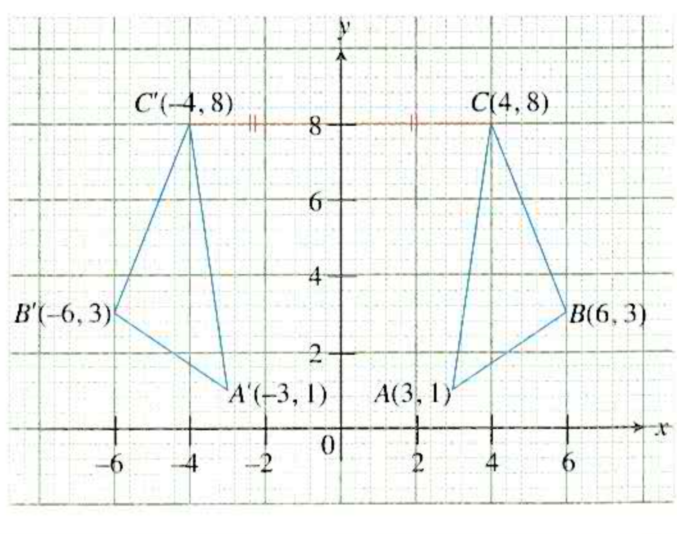
Plot these images on the same diagram and deduce the transformation represented by this matrix.

Solution:

The results below give the images of the points.

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 6 \\ 3 \end{pmatrix} = \begin{pmatrix} -6 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 4 \\ 8 \end{pmatrix} = \begin{pmatrix} -4 \\ 8 \end{pmatrix}$$

An examination of triangles ABC and $A'B'C'$ shows that $\triangle A'B'C'$ is the reflection of $\triangle ABC$ in the y-axis. Hence the matrix $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ represents reflection in the y-axis.



PRACTISE NOW 5

SIMILAR QUESTIONS

Exercise 6B Questions 1, 2, 4

$\triangle ABC$ has vertices $A(1, 1)$, $B(4, 2)$ and $C(2, 4)$. Represent these points on graph paper. Pre-multiply the coordinates by the matrix $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ and plot the points on the same graph. Deduce the transformation represented by this matrix.

Worked Example 6

(Finding Image Points with Transformation Matrix)

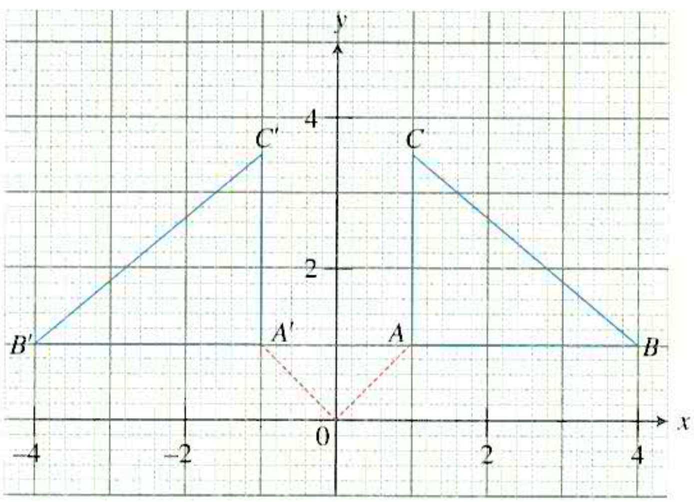
Pre-multiply the coordinates of the triangle $A(1, 1)$, $B(1, 4)$ and $C(3, 1)$ by the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ to get A', B', and C'. Plot these points on a diagram and deduce what this matrix represents.

Solution:

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 4 \end{pmatrix} = \begin{pmatrix} -4 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

The above working can be simplified as shown below:

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 & 3 \\ 1 & 4 & 1 \end{pmatrix} = \begin{pmatrix} -1 & -4 & -1 \\ 1 & 1 & 3 \end{pmatrix}$$



The figure shows $\triangle ABC$ and its image, $\triangle A'B'C'$ under a rotation about the origin O. By joining OA and OA', it is evident that A is rotated through 90° anticlockwise about the origin. Thus $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is a matrix representing a 90° rotation anticlockwise about the origin.

PRACTISE NOW 6

SIMILAR QUESTIONS

Exercise 6B Question 3

Pre-multiply the coordinates of the triangle $A(2, 2)$, $B(4, 2)$ and $C(4, 5)$ by the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ to obtain A', B' and C'. Plot points A, B, C, A', B' and C' on graph paper and deduce the transformation represented by this matrix.

Determination of Simple Transformation Matrices

Fig. 6.6 shows the vertices of a quadrilateral $OABC$, where O is the origin, $A(1, 0)$, $B(1, 1)$ and $C(0, 1)$. Under a reflection in the x-axis, the points O and A are invariant. The point B will be mapped onto the point $B'(1, -1)$, while the point C onto $C'(0, -1)$.

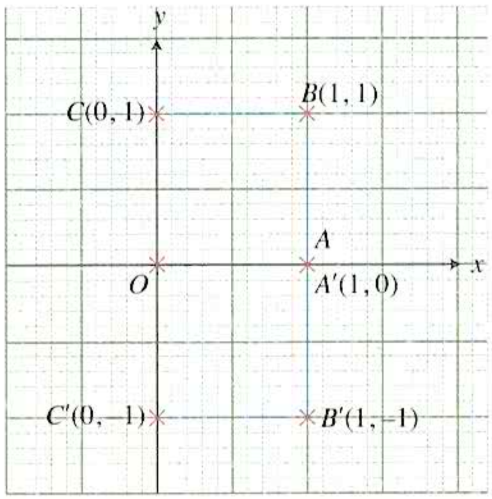


Fig 6.6

To find the 2×2 matrix that represents a reflection in the x-axis, let the transformation matrix be $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Now, let $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ transform the point A onto A' and the point C onto C',

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Since $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is the identity matrix, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

To find the 2×2 matrix that represents a $180°$ rotation about the origin,

let the transformation matrix representing a $180°$ rotation be $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

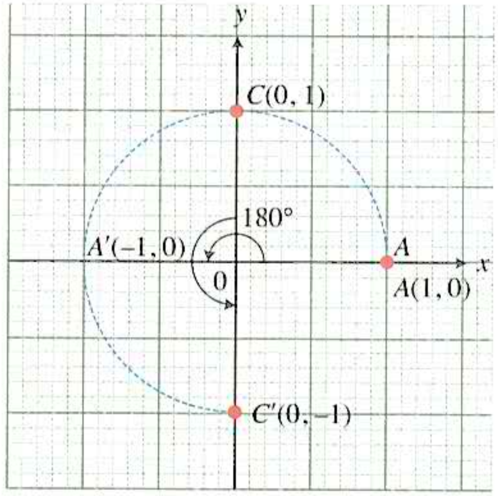


Fig. 6.7

Fig. 6.7 shows the points $A(1, 0)$ and $C(0, 1)$ and their images A' and C' under a 180° rotation about the origin O.

Now, $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

Hence, the matrix representing any particular transformation can be obtained simply by finding the images of the points (1, 0) and (0, 1).

We choose points (1, 0) and (0, 1) whenever we can. They form the identity matrix which is most convenient to apply when finding a transformation matrix.

Worked Example 7

(Determination of Transformation Matrix)

Find the 2×2 transformation matrix which maps the point $(3, -2)$ and the point $(-1, 4)$ onto $(-1, 4)$ and $(7, 2)$, respectively.

Solution:

Let the transformation matrix be $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Method 1:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 3 & -1 \\ -2 & 4 \end{pmatrix} = \begin{pmatrix} -1 & 7 \\ 4 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 3a - 2b & -a + 4b \\ 3c - 2d & -c + 4d \end{pmatrix} = \begin{pmatrix} -1 & 7 \\ 4 & 2 \end{pmatrix}$$

Thus we have $3a - 2b = -1$ ....(1)

$-a + 4b = 7$ ....(2)

$3c - 2d = 4$ ....(3)

$-c + 4d = 2$ ....(4)

Solving (1) and (2) simultaneously, we get $a = 1$, $b = 2$.
Solving (3) and (4) simultaneously, we get $c = 2$, $d = 1$.

$\therefore$ The required matrix is $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$.

Method 2:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 3 & -1 \\ -2 & 4 \end{pmatrix} = \begin{pmatrix} -1 & 7 \\ 4 & 2 \end{pmatrix}$$

Post-multiplying both sides by the inverse of $\begin{pmatrix} 3 & -1 \\ -2 & 4 \end{pmatrix}$, we have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 3 & -1 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} 3 & -1 \\ -2 & 4 \end{pmatrix}^{-1} = \begin{pmatrix} -1 & 7 \\ 4 & 2 \end{pmatrix}\frac{1}{10}\begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}$$

$$\begin{aligned} \begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \frac{1}{10}\begin{pmatrix} -1 & 7 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix} \\ &= \frac{1}{10}\begin{pmatrix} 10 & 20 \\ 20 & 10 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \end{aligned}$$

$$\therefore \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

PRACTISE NOW 7

SIMILAR QUESTIONS

Exercise 6B Questions 5–12

Find the matrix of the transformation which maps the point (1, 0) onto (3, 1) and the point (0, 1) onto (4, 3).

Exercise 6B

BASIC LEVEL

1. Represent the coordinates of the vertices of a triangle $A(1, 1)$, $B(3, 2)$ and $C(2, 4)$ on graph paper. Pre-multiply the coordinates by the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and represent them on the same diagram. Deduce the transformation represented by this matrix.

2. The square with vertices (0, 0), (1, 0), (0, 1) and (1, 1) is transformed by the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Plot these points and their images on graph paper. Is there any point in the square not affected by this transformation? What do you call such a point?

3. Pre-multiply the coordinates of the triangle $A(2, 2)$, $B(1, 5)$ and $C(4, 3)$ by the matrix $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. Plot the points of A, B and C on graph paper and also plot the images obtained on the same diagram. Deduce the transformation represented by the matrix.

4. A triangle whose vertices are $A(2, 3)$, $B(4, 3)$ and $C(4, 6)$ is transformed onto $\triangle A'B'C'$ by the transformation matrix $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. Find the coordinates of the points A', B' and C'. Plot the points A, B, C, A', B' and C' on a piece of graph paper and hence describe the transformation. Write down the matrix which will map $\triangle A'B'C'$ back onto $\triangle ABC$.

5. Find the images of the points $A(2, 1)$, $B(3, 2)$, $C(-2, 5)$, $D(-3, -4)$ and $E(-7, -3)$ under the transformation whose matrix is $\begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$.

6. Find the images of the points $A(0, 3)$, $B(4, 0)$, $C(-3, 4)$, $D(7, -4)$ and $E(-2, -5)$ under the transformation whose matrix is $\begin{pmatrix} -2 & 0 \\ 1 & -1 \end{pmatrix}$.

INTERMEDIATE LEVEL

7. A triangle whose vertices are $A(1, 2)$, $B(5, 2)$ and $C(4, 6)$ is transformed by $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ under T to the triangle $A'B'C'$. Describe the transformation T. Evaluate the matrices T^2 and T^{-1} and describe the transformation they represent. [T^2 is the matrix representing 2 successive transformations represented by T.]

8. Find the matrix of the transformation which maps the point (1, 2) onto (4, 4) and the point (3, –2) onto (4, –4).

9. Determine the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ which maps (1, 2) onto (7, –2) and (–1, 1) onto (–1, –1).

10. If T is a transformation represented by the matrix $\begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$, find the image of (2, 3), (4, 3) and (3, 5) under T. Evaluate T^2 and T^{-1}.

ADVANCED LEVEL

11. M is a reflection in the line $y = x$.
    (i) Find the coordinates of the image of (1, 3) under M.
    (ii) Find the coordinates of the image of (2, 5) under M^{-1}.
    (iii) Find the coordinates of the image of (3, 7) under M^5.
    (iv) Write down the matrix which represents M^5 and M^{-1}.

    [M^5 represents 5 successive reflections in the line $y = x$.]

12. R is an anticlockwise rotation of 90° about the origin.
    (i) Find the coordinates of the image of
    (a) (3, 5),
    (b) (7, –4)
    under R.
    (ii) Find the coordinates of the image of
    (a) (3, 4),
    (b) (–2, –3)
    under R^{-1}.
    (iii) Find the coordinates of the image of (2, 5) under
    (a) R,
    (b) R^8.
    (iv) Write down the matrix which represents
    (a) R,
    (b) R^4,
    (c) R^9.

    [R^8 represents 8 successive anticlockwise rotations of 90° about the origin.]

6.3 Transformation Matrix for Enlargement

Plot the points $A(2, 2)$, $B(4, 3)$ and $C(5, 1)$ on graph paper. Pre-multiply the coordinates of A, B and C by the matrix $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ to obtain the points A_1, B_1 and C_1.

Plot these points on the same graph and deduce the transformation represented by the matrix $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$.

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 2 & 4 & 5 \\ 2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 8 & 10 \\ 4 & 6 & 2 \end{pmatrix}$$

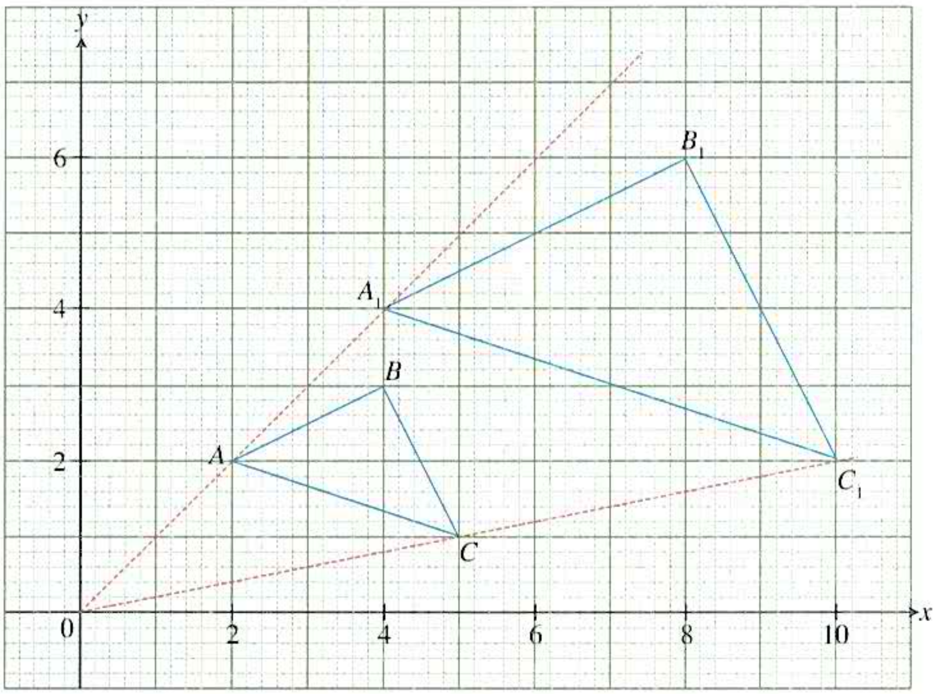


Fig. 6.8

Fig. 6.8 shows the $\triangle ABC$ and its image $\triangle A_1B_1C_1$. Both triangles are similar and $A_1B_1 = 2AB$, $B_1C_1 = 2BC$ and $A_1C_1 = 2AC$. Therefore the given matrix denotes an enlargement of scale factor 2 and centre of enlargement at the origin. Find the area of $\triangle A_1B_1C_1$. How many times is it larger than the area of $\triangle ABC$?

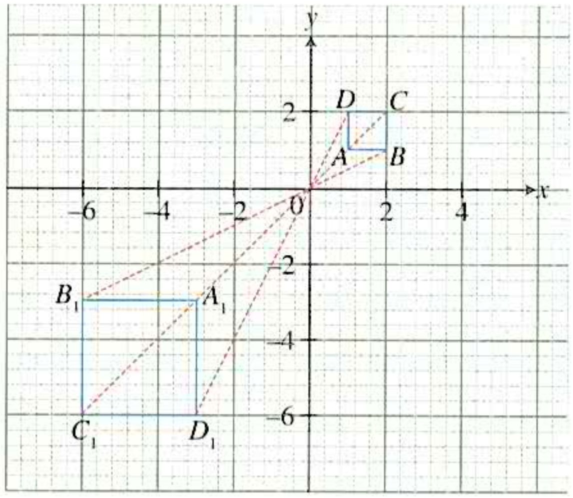
Worked Example 8

(Finding Image Points with Transformation Matrix)

Pre-multiply the points $A(1, 1)$, $B(2, 1)$, $C(2, 2)$, $D(1, 2)$ by the matrix $\begin{pmatrix} -3 & 0 \\ 0 & -3 \end{pmatrix}$ to get $A_1B_1C_1D_1$. Plot the square $ABCD$ and its image on a diagram. Deduce the transformation represented by this matrix.

Solution:

$$\begin{pmatrix} -3 & 0 \\ 0 & -3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 2 & 1 \\ 1 & 1 & 2 & 2 \end{pmatrix} = \begin{pmatrix} -3 & -6 & -6 & -3 \\ -3 & -3 & -6 & -6 \end{pmatrix}$$



The figure shows the square $ABCD$ and its image $A_1B_1C_1D_1$. Notice that $A_1B_1 = 3AB$, $A_1D_1 = 3AD$, etc.

Thus $\begin{pmatrix} -3 & 0 \\ 0 & -3 \end{pmatrix}$ represents an enlargement centre at origin and scale factor −3 because the image is on the opposite side of the centre of enlargement to the original figure.

PRACTISE NOW 8

SIMILAR QUESTIONS

Exercise 6C Questions 1–5

$\triangle ABC$ with vertices $A(1, 1)$, $B(5, 4)$ and $C(3, 6)$ is transformed into $\triangle A'B'C'$ by the matrix $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$. Find the coordinates of A', B' and C'.

In general,

> $\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$ represents an enlargement matrix of *scale factor k*, with the *origin* as *centre of enlargement*.

An enlargement with centre at points other than the origin is not represented by a general 2×2 matrix.

Exercise 6C

BASIC LEVEL

1. $\triangle ABC$ with vertices $A(1, 2)$, $B(1, 4)$ and $C(5, 2)$ is mapped onto $\triangle A'B'C'$ by the transformation matrix $\begin{pmatrix} 2\frac{1}{2} & 0 \\ 0 & 2\frac{1}{2} \end{pmatrix}$. Find

   (i) the coordinates of A', B' and C',

   (ii) the area of $\triangle ABC$,

   (iii) the area of $\triangle A'B'C'$.

INTERMEDIATE LEVEL

2. Draw a quadrilateral with points $A(1, 1)$, $B(3, 1)$, $C(4, 3)$ and $D(2, 4)$. Find the coordinates of the image of $ABCD$ and the value of $\left(\dfrac{\text{area of image of } ABCD}{\text{area of } ABCD}\right)$ under the transformation represented by the matrix

   (i) $\begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$, (ii) $\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$.

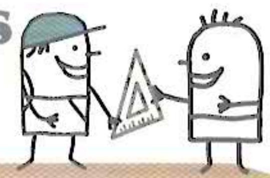
3. $\triangle ABC$ with vertices $A(6, 1)$, $B(12, 8)$ and $C(2, 4)$ is transformed into $\triangle A'B'C'$ with vertices $A'\left(9, 1\frac{1}{2}\right)$, $B'(18, 12)$ and $C'(3, 6)$. Plot the points A, B, C and A', B', C', and find the transformation matrix that maps $\triangle ABC$ onto $\triangle A'B'C'$.

4. $\triangle ABC$ with vertices $A(1, 1)$, $B(4, 2)$ and $C(4, 3)$ is transformed into $\triangle A_1B_1C_1$ by the matrix $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$. $\triangle A_1B_1C_1$ is transformed into $\triangle A_2B_2C_2$ by the matrix $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$. Find the scale factor for both the transformations and a single transformation matrix that will map $\triangle ABC$ onto $\triangle A_2B_2C_2$.

ADVANCED LEVEL

5. $\triangle ABC$ with vertices $A(3, 6)$, $B(12, 6)$ and $C(6, 9)$ is transformed into $\triangle A_1B_1C_1$ by the matrix $\begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix}$. $\triangle A_1B_1C_1$ is then transformed into $\triangle A_2B_2C_2$ by the matrix $\begin{pmatrix} -6 & 0 \\ 0 & -6 \end{pmatrix}$. Find the coordinates of $\triangle A_1B_1C_1$ and $\triangle A_2B_2C_2$ and determine a single matrix that will map $\triangle ABC$ onto $\triangle A_2B_2C_2$.

6.4 Inverse Transformations and Combined Transformations



Inverse Transformations

When $\triangle ABC$ is transformed to $\triangle PQR$ by a transformation, its inverse transformation will be when $\triangle PQR$ can be transformed back to $\triangle ABC$.

Inverse transformation by matrices is simply the inverse of a given matrix.

Inverse Reflection

Inverse reflection is reflecting a figure twice in the same line of reflection. Whenever a reflection takes it back on itself, it is always an inverse.

E.g. Reflecting $\triangle ABC$ about the y-axis will give us $\triangle PQR$. Its inverse will be when $\triangle PQR$ is reflected back onto $\triangle ABC$ by reflecting it about the y-axis again.

Inverse Rotation

Inverse rotation is a rotation in the opposite direction of equal magnitude about the same centre.

E.g. With the origin as centre, if $\triangle ABC$ is rotated clockwise 90° to $\triangle PQR$, its inverse will be when $\triangle PQR$ is rotated 90° anticlockwise about the same centre.

Inverse Translation

The inverse of a certain translation is movement of an equal magnitude in the opposite direction.

E.g. The translation vector $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$ is the inverse of the translation vector $\begin{pmatrix} -4 \\ -3 \end{pmatrix}$.

Inverse Enlargement

The inverse of an enlargement is an enlargement about the same centre having a reciprocal scale factor.

E.g. If $\triangle ABC$ is enlarged to $\triangle PQR$ about the origin by a scale factor of 3, its inverse will be when $\triangle PQR$ is enlarged to $\triangle ABC$ by a scale factor of $\frac{1}{3}$ about the same centre of enlargement.

The table below shows the list of matrix equations for some of the basic transformations.

	Transformation	Matrix Equation
1	Translation $\begin{pmatrix} p \\ q \end{pmatrix}$	$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} p \\ q \end{pmatrix}$
2	Reflection in the x-axis	$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$
3	Reflection in the y-axis	$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$
4	Reflection in the line $y = x$	$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$
5	Reflection in the line $y = -x$	$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$
6	90° clockwise rotation about the origin	$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$
7	90° anticlockwise rotation about the origin	$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$
8	180° rotation about the origin	$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$
9	Enlargement with scale factor k with origin as the centre of enlargement	$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

Combined Transformations

Now, let us look at how transformations can be combined.

If M represents a reflection in the y-axis and R represents a 90° anticlockwise rotation about the origin, then MR represents a 90° anticlockwise rotation about the origin followed by a reflection in the y-axis while RM denotes a reflection in the y-axis followed by a 90° anticlockwise rotation about the origin.

Consider the point $K(2, 3)$. Under MR, K will be mapped onto $(-3, 2)$ under R and then onto $(3, 2)$ under M, i.e. MR$(2, 3) = (3, 2)$ [see Fig. 6.9**(a)**].

Under RM, $K(2, 3)$ will be mapped onto $(-2, 3)$ under M and then onto $(-3, -2)$ under R, i.e. RM$(2, 3) = (-3, -2)$ [see Fig. 6.9**(b)**].

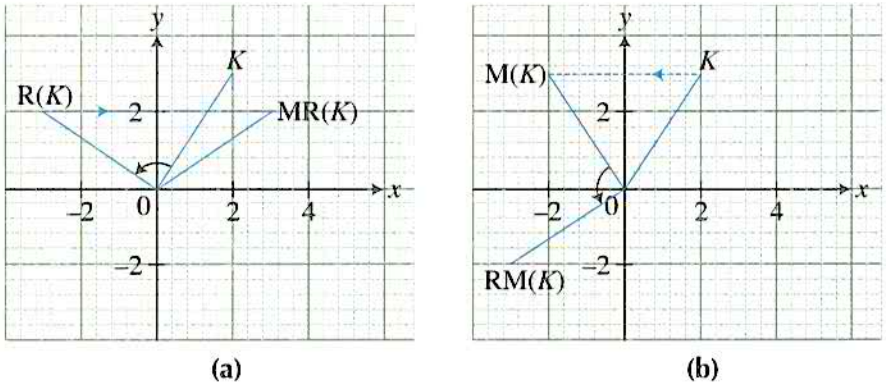


Fig. 6.9

From Fig. 6.9, we observe that MR ≠ RM.

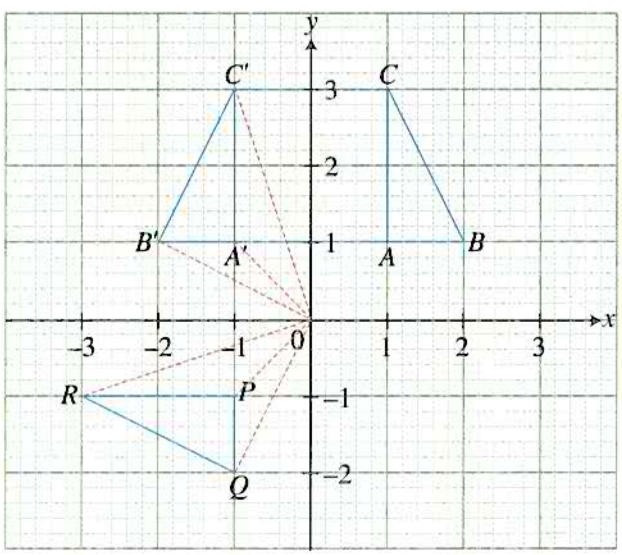
In general, the combination of two transformations is non-commutative.

Worked Example 9

(Finding Image Points under Combined Transformations)

Plot $\triangle ABC$ whose vertices are $A(1, 1)$, $B(2, 1)$ and $C(1, 3)$ on graph paper. $\triangle ABC$ is reflected in the y-axis followed by a rotation of 90° anticlockwise about the origin to obtain $\triangle PQR$. Plot $\triangle PQR$ on the same graph paper.

Solution:



We observe that $\triangle ABC$ is reflected in the y-axis to $\triangle A'B'C'$. $\triangle A'B'C'$ is then rotated through 90° anticlockwise about the origin to obtain $\triangle PQR$ whose coordinates are $P(-1, -1)$, $Q(-1, -2)$ and $R(-3, -1)$.

Thinking Time

Would you obtain the same result if $\triangle ABC$ is first rotated through 90° anticlockwise about O and then reflected in the y-axis? Check your answer by doing the transformations on graph paper.

PRACTISE NOW 9

SIMILAR QUESTIONS

Exercise 6D Questions 1–5

Using a scale of 1 cm to represent 1 unit on each axis, draw x- and y-axes for $-5 \leqslant x \leqslant 5$ and $-5 \leqslant y \leqslant 5$.

(a) The vertices of $\triangle ABC$ are $A(1, 2)$, $B(5, 1)$ and $C(3, 4)$. Draw and label $\triangle ABC$.

(b) $\triangle ABC$ undergoes a double transformation: a reflection in the x-axis (M), followed by a translation (T) of 5 units in the negative x-direction and 5 units in the positive y-direction. Plot the image of $\triangle ABC$ under TM.

Worked Example 10

(Finding Image Points under Combined Transformations)

The transformation T is a translation of 2 units upwards, parallel to the y-axis, and the transformation M is a reflection in the line $y = x$.

Given that P is the point (1, 2), find the coordinates of the image of P under the following transformations.

(a) T^2 **(b)** M^2 **(c)** TM **(d)** MT

Solution:

(a) The translation T is represented by $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$.

$$\therefore T^2\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 6 \end{pmatrix}$$

(b) Under M, $(a, b) \rightarrow (b, a)$.

$$M^2\begin{pmatrix} a \\ b \end{pmatrix} = M\begin{pmatrix} b \\ a \end{pmatrix}$$

$$= \begin{pmatrix} a \\ b \end{pmatrix}$$

$$M^2\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

ATTENTION

Notice that M^2 maps any point onto itself.

(c) $\text{TM}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \text{T}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$$= \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

(d) $\text{MT}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \text{M}\left[\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \end{pmatrix}\right]$

$$= \text{M}\begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

$$= \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

PRACTISE NOW 10

SIMILAR QUESTIONS

Exercise 6D Questions 6, 9

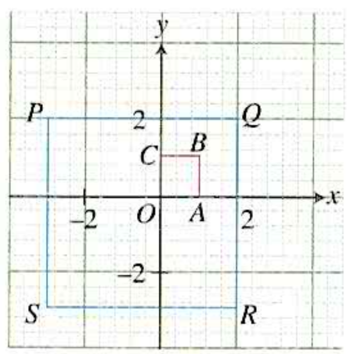
If M denotes reflection in the x-axis and R denotes rotation about the origin through 90° anticlockwise, find

(a) MR(2, 3), **(b)** (x, y), if RM$(x, y) = (3, 1)$.

Worked Example 11

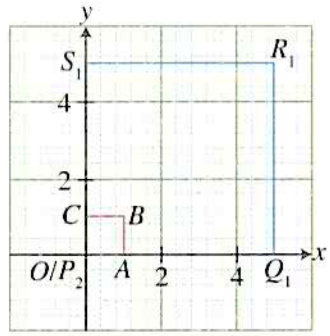
(Finding Image Points under Successive Transformations)

Sketch and describe three successive transformations under which $OABC$ will be transformed into $PQRS$.

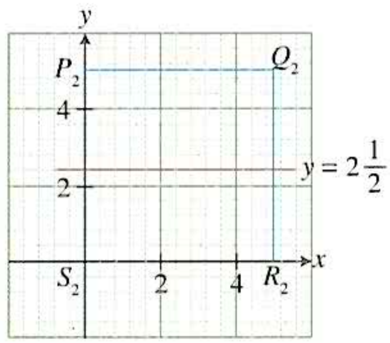


Solution:

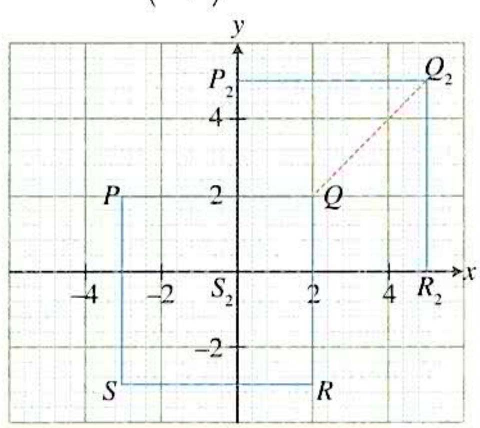
Step 1: $OABC$ is transformed into $P_1Q_1R_1S_1$ under an enlargement with centre O, scale factor 5.



Step 2: $P_1Q_1R_1S_1$ is reflected in the line $y = 2\frac{1}{2}$ to $P_2Q_2R_2S_2$.



Step 3: $P_2Q_2R_2S_2$ is then translated to $PQRS$ by $\begin{pmatrix} -3 \\ -3 \end{pmatrix}$.



PRACTISE NOW 11

SIMILAR QUESTIONS

Exercise 6D Questions 7, 8

The coordinates of $\triangle ABC$ are $A(7, 4)$, $B(7, 0)$ and $C(5, 4)$. The coordinates of $\triangle PQR$ are $P(10, 4)$, $Q(10, 10)$ and $R(7, 4)$. Draw $\triangle ABC$ and $\triangle PQR$ on a sheet of graph paper and describe two successive transformations that will map $\triangle ABC$ onto $\triangle PQR$.

Worked Example 12

(Finding Image Points under Successive Transformations)

Using a scale of 1 cm to 1 unit on each axis, draw the x- and y-axes for $-3 \leqslant x \leqslant 6$ and $-5 \leqslant y \leqslant 5$.

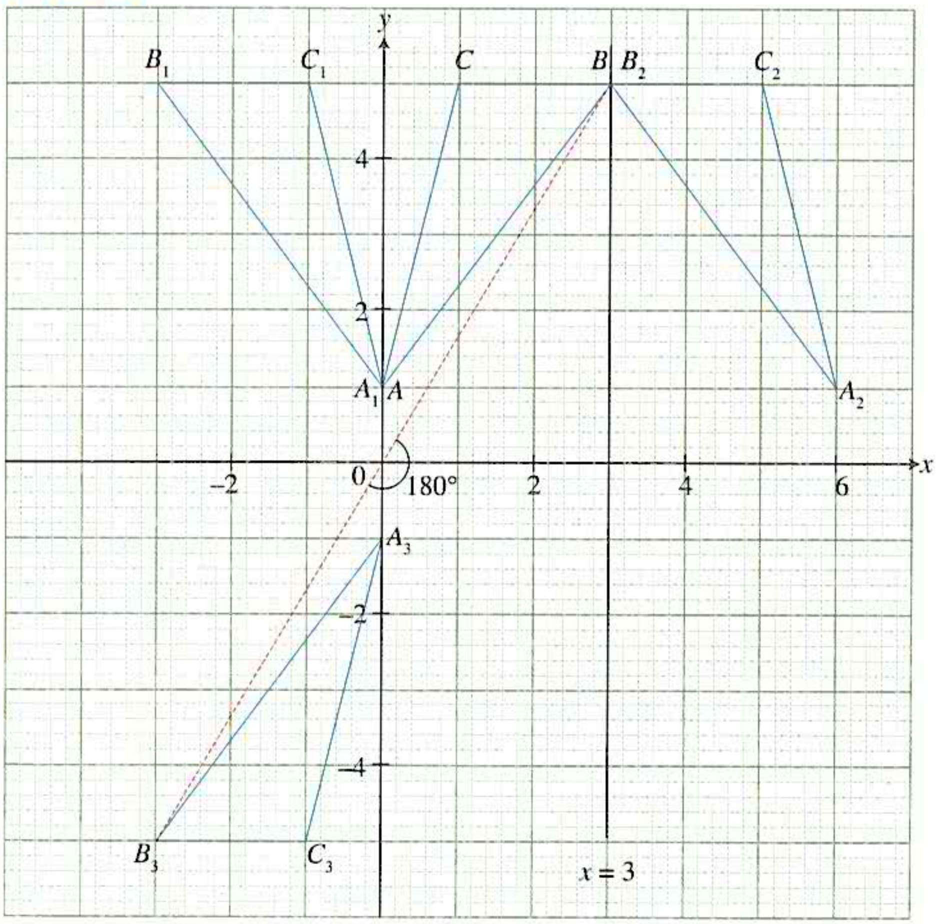
(a) The vertices of $\triangle ABC$ are $A(0, 1)$, $B(3, 5)$ and $C(1, 5)$. Draw and label $\triangle ABC$.

(b) $\triangle ABC$ is mapped onto $\triangle A_1B_1C_1$ by a reflection in the y-axis. $\triangle A_1B_1C_1$ is then translated to $\triangle A_2B_2C_2$ by a translation T represented by $\begin{pmatrix} 6 \\ 0 \end{pmatrix}$. Draw $\triangle A_1B_1C_1$ and $\triangle A_2B_2C_2$ on your graph. State the single transformation that will map $\triangle ABC$ directly onto $\triangle A_2B_2C_2$.

(c) $\triangle ABC$ is mapped onto $\triangle A_3B_3C_3$ by an enlargement scale factor -1 and centre at origin. Draw $\triangle A_3B_3C_3$ in your graph. Give another geometrical transformation description of this enlargement.

Solution:

(a)



(b) The transformation that will map $\triangle ABC$ onto $\triangle A_2B_2C_2$ is a reflection in the line $x = 3$.

(c) The transformation represents a rotation of 180° about the origin.

PRACTISE NOW 12

SIMILAR QUESTIONS

Exercise 6D Questions 10–18

Using a scale of 1 cm to 1 unit on each axis, draw the x- and y-axes for $-5 \leqslant x \leqslant 5$ and $-5 \leqslant y \leqslant 5$.

(a) The vertices of $\triangle ABC$ are $A(1, 2)$, $B(4, 1)$ and $C(3, 4)$. Draw and label $\triangle ABC$.

(b) $\triangle ABC$ undergoes a double transformation: a reflection in the y-axis (M) followed by an anticlockwise rotation of 90° about the origin (R). Plot the image of $\triangle ABC$ under

(i) M, **(ii)** RM.

(c) Describe a single transformation that will map $\triangle ABC$ onto $\triangle A''B''C''$ where $\triangle A''B''C''$ is the image of $\triangle ABC$ under RM.

Exercise 6D

BASIC LEVEL

1. If R represents a reflection in the y-axis and T a translation of 2 units in the positive x-direction and 4 units in the positive y-direction, find the image of the point (1, 3) under the combined transformation represented by

(a) RT, **(b)** TR.

2. Find the coordinates of the image of the point (5, 2) under a reflection in the x-axis followed by a clockwise rotation of 90° about the origin.

3. Points are reflected in the y-axis and their images are rotated through 90° anticlockwise about O. Find the coordinates of the final image of the point

(a) (2, −3), **(b)** (−4, −1).

4. Find the coordinates of the image of (7, −2) under a translation of 4 units in the negative x-direction and 3 units in the positive y-direction followed by a 180° rotation about the origin.

5. E is an enlargement with centre (0, 0) with scale factor 2 and T is a translation represented by 2 units in the positive x-direction and 1 unit in the positive y-direction. Find the final image of the point (2, 1) under

(a) ET, **(b)** T^2.

INTERMEDIATE LEVEL

6. If M is a reflection in the y-axis and T is a translation represented by $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$, find

(a) MT(2, 3),

(b) (x, y) if TM$(x, y) = (4, 3)$.

7. Under a reflection in the line $y = 3$, the point $A(5, 1)$ is mapped onto A_1. Find the coordinates of A_1. A reflection in the line $y = 8$ will map the point A_1 onto the point A_2. Find the coordinates of A_2. Given that A_2 is the reflection of A in the line $y = k$, find the value of k.

8. A reflection in the line $y = 0$ will map the point P onto P_1 and a reflection in the line $x = 0$ will map the point P_1 onto P_2. Describe a single transformation that will map P directly onto P_2.

9. The transformation T is the translation $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and the transformation M is a reflection in the line $y + x = 0$. Given that P is the point (2, 5), write down the coordinates of the image of P under the following transformations.

   (a) T^2 **(b)** M^2

   (c) M^8 **(d)** MT

   (e) TM

10. The coordinates of $\triangle ABC$ are $A(2, 2)$, $B(5, 2)$ and $C(3, 4)$. $\triangle ABC$ is mapped onto $\triangle PQR$ by means of the following successive transformations.

    (a) A clockwise rotation of 90° about (0, 0).

    (b) A reflection in the line $x = 0$.

    Draw $\triangle ABC$ and $\triangle PQR$ on a sheet of graph paper and describe a single transformation that will map $\triangle ABC$ onto $\triangle PQR$.

11. The coordinates of $\triangle ABC$ are $A(4, 0)$, $B(5, 0)$ and $C(5, 2)$. $\triangle ABC$ is mapped onto $\triangle PQR$ by means of the following successive transformations.

    (a) An enlargement scale factor 2, centre at (4, 0).

    (b) A reflection in the line $x = 4$.

    (c) A 90° clockwise rotation about the point (1, 1).

    Draw $\triangle ABC$ and $\triangle PQR$ on a sheet of graph paper and label the vertices clearly.

12. The transformation R is a 90° clockwise rotation about (0, 2) and the translation T is given by the column vector $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$. If A is the point (4, 5), find the coordinates of R(A) and TR(A).

13. A is the point (5, 1) and R is a transformation which gives a 90° anticlockwise rotation about the origin. T is a translation represented by the column vector $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$. If RT($A$) = B, TR(A) = C and $R^2(A) = D$, find the coordinates of B, C and D.

14. The coordinates of $\triangle ABC$ are $A(1, -1)$, $B(1, 0)$ and $C(3, -1)$. $\triangle ABC$ is transformed into $\triangle A_1B_1C_1$ by the following successive transformations.

    (a) A reflection in the x-axis.

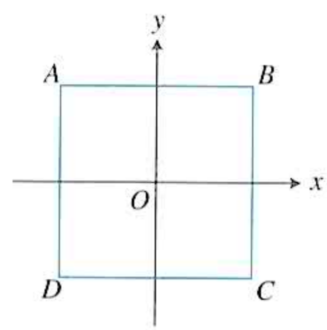
    (b) An enlargement, centre origin, scale factor 3.

    Draw $\triangle ABC$ on a sheet of graph paper and construct $\triangle A_1B_1C_1$ on the same graph. State the coordinates of A_1.

15. R represents a 90° anticlockwise rotation about O and M represents a reflection in the x-axis.

    (a) Sketch on a copy of the diagram, the image of the square $ABCD$, correctly lettered, under the transformation MR (i.e. R followed by M).

    (b) State a single transformation which is equivalent to MR.



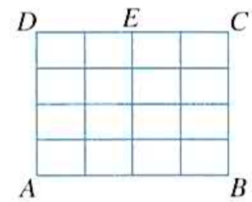
16. $\triangle ABC$ with vertices $A(1, 1)$, $B(2, 3)$ and $C(-1, 4)$ is first reflected in the x-axis and then rotated through 90° anticlockwise about the origin. Calculate the new coordinates of A, B and C. A square $PQRS$ is transformed by the above transformations into a square whose vertices are at the points with coordinates (0, 5), (3, 5), (3, 8) and (0, 8) respectively. Calculate the coordinates of P, Q, R and S.

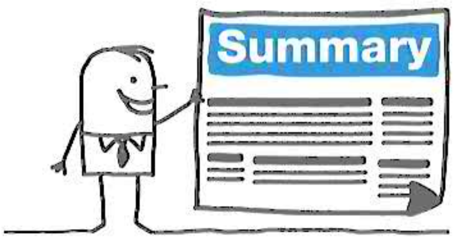
17. A square $ABCD$ with vertices $A(2, 4)$, $B(4, 4)$, $C(4, 6)$ and $D(2, 6)$ is enlarged with centre $P(0, 5)$ and scale factor 3, to $A_1B_1C_1D_1$. Construct the images formed and state the coordinates of C_1. Taking A as the centre of enlargement and scale factor -3, construct the images of $A_2B_2C_2D_2$ formed under this transformation and state the coordinates of C_2.

ADVANCED LEVEL

18. The rectangle $ABCD$ is divided into 16 equal rectangles. The point P is such that the area of $\triangle APB$ is equal to one quarter of the area of rectangle $ABCD$, and the point Q, lying on AB, is such that $\triangle AEB$ is an enlargement of $\triangle APQ$.

Mark P and Q clearly on a copy of the diagram.





1. An **enlargement** is defined by its *centre* and *scale factor*. The scale factor affects the size and position of the image. The centre of enlargement is the only possible invariant point.
2. Under an enlargement with the scale factor k, **image area = k^2 × area of original figure.**

Review Exercise 6

1. P is the point $(2, -1)$. Find the coordinates of the image of P under

(a) a reflection in the y-axis,

(b) a 90° anticlockwise rotation about $(0, 0)$,

(c) a translation represented by 1 unit in the positive x-direction and 5 units in the positive y-direction.

2. K is the point $(3, -1)$. Find the coordinates of the image of K under

(a) a reflection in the x-axis followed by a reflection in the y-axis,

(b) a reflection in the y-axis followed by a reflection in the x-axis,

(c) a 90° anticlockwise rotation about $(0, 0)$, followed by a reflection in the y-axis,

(d) a 180° rotation about $(2, 0)$ followed by a reflection in the line $y = 3$.

3. A translation maps the point $(-2, 3)$ onto the point $(2, 5)$ and the point $(5, -2)$ onto the point (x, y). Find the values of x and y.

4. Using a scale of 1 cm to 1 unit on both axes, draw on graph paper the flag formed by the points (1, 1), (1, 2), (1, 3) and (2, 2). Draw the image of the flag under

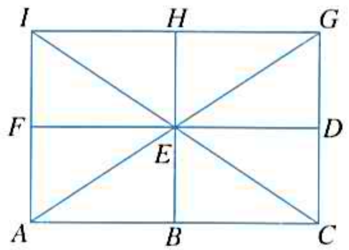
(a) a 90° anticlockwise rotation about (0, 0),

(b) a reflection in the x-axis,

(c) an enlargement of scale factor 2 and centre of enlargement at (0, 0),

(d) a 180° rotation about (0, 0), followed by a reflection in the y-axis.

5.



The diagram shows a rectangle $ACGI$ divided into four smaller equal rectangles.

(a) Describe a single transformation that maps $ABEF$ onto $ACGI$.

(b) Describe a single transformation that maps $\triangle ABE$ onto $\triangle EDG$.

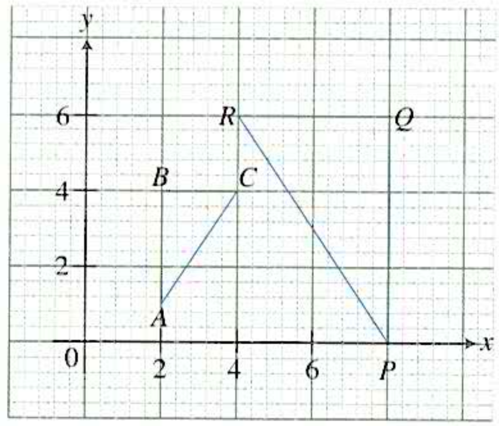
(c) Describe a single transformation that maps $\triangle ACI$ onto $\triangle GIC$.

(d) Describe two successive transformations that maps $\triangle ABE$ onto $\triangle IHE$.

(e) Describe two successive transformations that map $\triangle ACG$ onto $\triangle CAI$.

6. Under an enlargement, centre (0, 3), the line PQ with coordinates $P(2, 5)$ and $Q(3, 1)$ are mapped onto the points $(4, p)$ and (m, n). Find the values of p, m and n.

7. A transformation H followed by another transformation K will map $\triangle ABC$ onto $\triangle PQR$. Describe H and K completely.



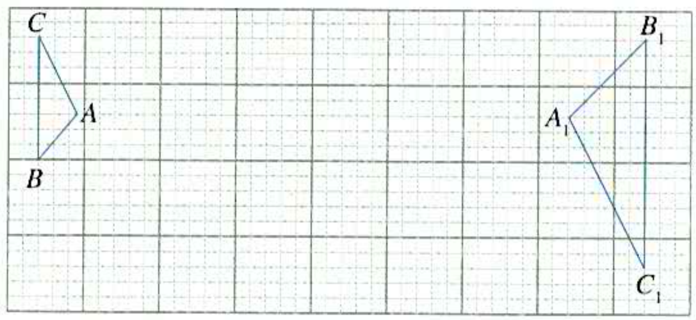
8. $\triangle ABC$ is mapped onto $\triangle A'B'C'$ by means of an enlargement centre A and scale factor 4. Write down the values of the ratios of

(a) the side $B'C'$ to the side BC,

(b) the size of $AB'C'$ to the size of ABC,

(c) the area of $\triangle AB'C'$ to the area of $\triangle ABC$.

9.



In the figure, $\triangle A_1B_1C_1$ is an enlargement of $\triangle ABC$. If O is the centre of enlargement, copy the diagram and find O by construction. State the value of the ratio

(a) $AB : A_1B_1$,

(b) area of $\triangle A_1B_1C_1$: area of $\triangle ABC$.

10. Under an enlargement E, the point $A(1, 3)$, is mapped onto $A'(1, 7)$ and $B(3, 2)$ is mapped onto $B'(7, 4)$. Plot the points A, B, A' and B' on a sheet of graph paper and find

(a) the coordinates of the centre of enlargement,

(b) the scale factor of the enlargement,

(c) the coordinates of the image of the point (2, 2) under E,

(d) the coordinates of the point whose image is (4, 7)

11. The coordinates of the vertices of $\triangle ABC$ are $A(2, 0)$, $B(2, 2)$ and $C(6, 2)$. Under an enlargement centre at origin and scale factor –2, $\triangle ABC$ is mapped onto $\triangle A_1B_1C_1$. Find

(a) the coordinates of A_1, B_1 and C_1,

(b) the ratio of A_1B_1 to AB,

(c) the ratio of area of $\triangle ABC$ to the area of $\triangle A_1B_1C_1$.

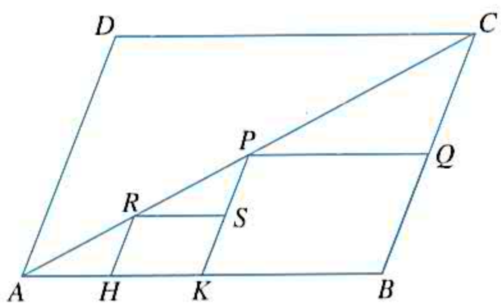
12. The transformation G is a reflection in the y-axis.

(a) The point $A(3, 4)$ is mapped onto the point B under G. Find the coordinates of B and write down the equation of the invariant line.

(b) The point B is now mapped onto the point C by a transformation P which is a 90° anticlockwise rotation about the origin. Find the coordinates of C.

(c) Describe the single transformation which is equivalent to G followed by P.

13.



In the figure, $ABCD$ is a parallelogram. P is the midpoint of AC and R is the midpoint of AP. The parallelogram $PQBK$ and $RSKH$ are drawn. Describe fully the transformations which will map the following.

(a) $\triangle ARH$ onto $\triangle ACB$

(b) $\triangle APK$ onto $\triangle PCQ$

(c) $\triangle ABC$ onto $\triangle CDA$

(d) $RSKH$ onto $PQBK$

(e) $HKSR$ onto $ABCD$

(f) $PQBK$ onto $DCBA$

Challenge Yourself

1. On graph paper, draw $\triangle ABC$ at $A(3, 1)$, $B(5, 1)$ and $C(5, 2)$. Draw the line $y = x$. $\triangle ABC$ is transformed into $\triangle A_1B_1C_1$ by a reflection in the line $y = x$. $\triangle A_1B_1C_1$ is then transformed into $\triangle A_2B_2C_2$ by a reflection in the x-axis. Draw $\triangle A_1B_1C_1$ and $\triangle A_2B_2C_2$. Describe a single transformation that will map $\triangle ABC$ onto $\triangle A_2B_2C_2$.

2. The transformation P is a reflection in the line $y = 0$ and the transformation Q is a reflection in the line $x = 0$. Describe a single transformation equivalent to

(a) PQ,

(b) QP.

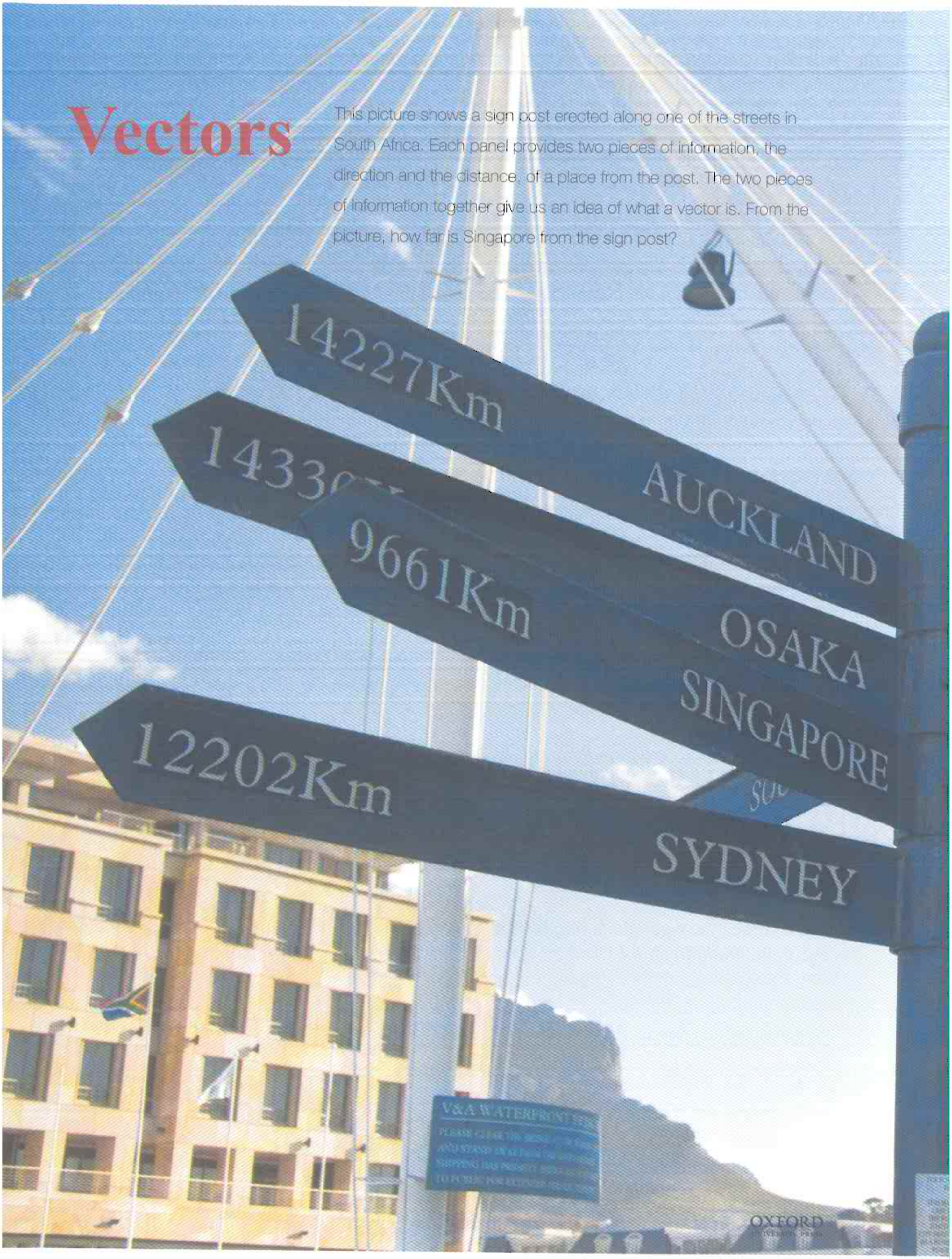
3. The transformation P is a 90° clockwise rotation about the origin and the transformation Q is a reflection in the x-axis. Describe a single transformation equivalent to

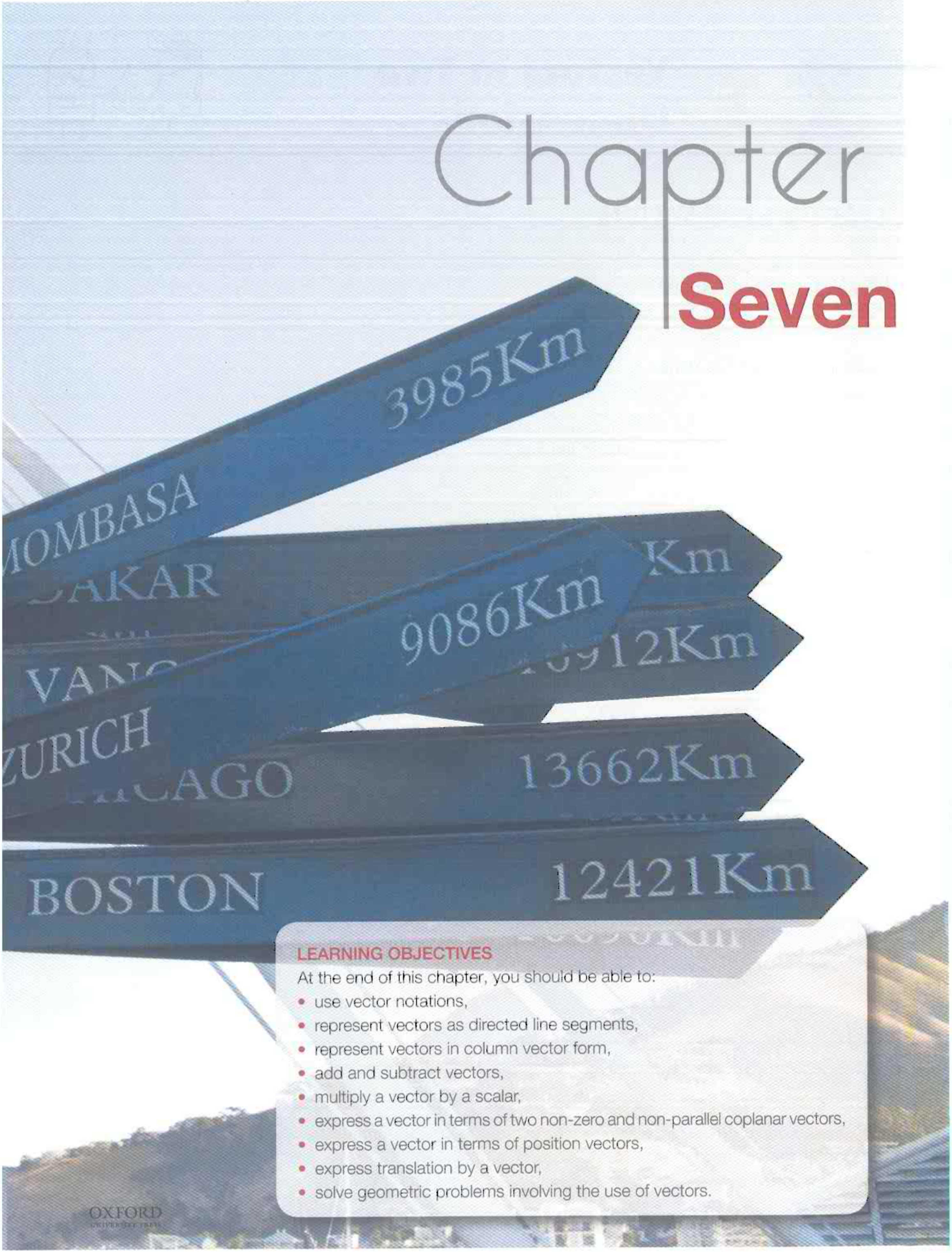
(a) PQ,

(b) QP.

Vectors

This picture shows a sign post erected along one of the streets in South Africa. Each panel provides two pieces of information, the direction and the distance, of a place from the post. The two pieces of information together give us an idea of what a vector is. From the picture, how far is Singapore from the sign post?





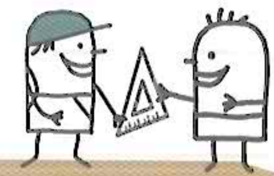
Chapter Seven

LEARNING OBJECTIVES

At the end of this chapter, you should be able to:

- use vector notations,
- represent vectors as directed line segments,
- represent vectors in column vector form,
- add and subtract vectors,
- multiply a vector by a scalar,
- express a vector in terms of two non-zero and non-parallel coplanar vectors,
- express a vector in terms of position vectors,
- express translation by a vector,
- solve geometric problems involving the use of vectors.

7.1 Vectors in Two Dimensions



Scalars and Vectors

Class Discussion

Scalar and Vector Quantities

Discuss in pairs.

Amirah walks 100 metres due North from point P, as shown in Fig. 7.1.
Nora also walks 100 metres from point P, but due East.

Fig. 7.1

1. On Fig. 7.1, draw the route taken by Nora.
2. Although both Amirah and Nora walk the same distance of 100 metres each, do they end up at the same point? Why or why not?
3. In the real world, distance is not enough to describe motion. What else do you need?

From the class discussion, we realise that there are two types of quantities.

The first type of quantity is called a **scalar**, which is a quantity with a **magnitude** only. For example, distance is a scalar. In the class discussion above, the distance covered by both Amirah and Nora is 100 metres (magnitude).

The second type of quantity is called a **vector**, which is a quantity with both a **magnitude** and a **direction**. For example, displacement is a vector. In the class discussion above, the displacement of Amirah from P is 100 metres (magnitude) in the North direction.

Thinking Time

Another real world example of a scalar is speed (e.g. 50 km/h), while another example of a vector is velocity (e.g. 50 km/h southwards).

Can you think of other examples of scalars and vectors?

Representation of Vectors

A vector can be represented by a **directed line segment**, where the *direction* of the line segment is that of the vector, and the *length* of the line segment represents the magnitude of the vector.
Fig. 7.2 shows some examples of vectors.

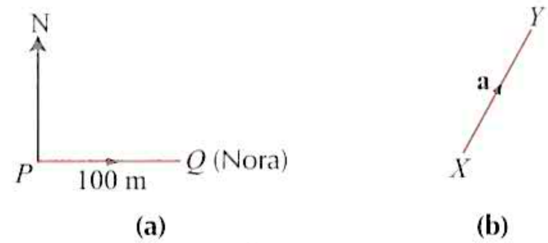


Fig. 7.2

In Fig. 7.2**(a)**, the displacement vector is denoted by $\overrightarrow{PQ}$, where P is the **starting or initial point**, and Q is the **ending or terminal point**. The magnitude of PQ is denoted by $|\overrightarrow{PQ}|$. In this case, $|\overrightarrow{PQ}| = 100$ m.

In Fig. 7.2**(b)**, another way to denote the vector $\overrightarrow{XY}$ is $\mathbf{a}$, and its magnitude is $|\mathbf{a}|$. When we write, we cannot bold the letter '**a**', so we write it as $\underset{\sim}{a}$ and $|\underset{\sim}{a}|$.

Vectors on Cartesian Plane

Fig. 7.3 shows two vectors lying on a Cartesian plane.

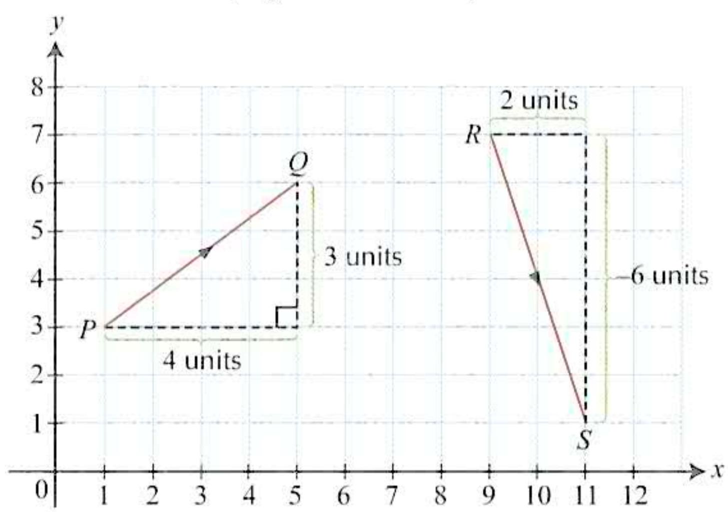


Fig. 7.3

We can move P to Q by moving 4 units in the positive x-direction and 3 units in the positive y-direction, so another way to describe the vector $\overrightarrow{PQ}$ is to use a **column vector**, i.e.

$$\overrightarrow{PQ} = \begin{pmatrix} 4 \\ 3 \end{pmatrix},$$

where the first entry 4 in the column matrix represents the number of units in the x-direction and the second entry 3 represents the number of units in the y-direction. 4 and 3 are called the **components** of the vector $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$, where 4 is the x-component and 3 is the y-component.

Similarly, $\overrightarrow{RS} = \begin{pmatrix} 2 \\ -6 \end{pmatrix}$. Is $\begin{pmatrix} 2 \\ -6 \end{pmatrix}$ equal to $\begin{pmatrix} 1 \\ -3 \end{pmatrix}$? Explain.

To find the magnitude of $\overrightarrow{PQ}$, we can use Pythagoras' Theorem to find the length of the line segment PQ, i.e.

$$|\overrightarrow{PQ}| = \sqrt{4^2 + 3^2} = 5 \text{ units}.$$

What is the magnitude of $\overrightarrow{RS}$?

In general, the magnitude of a column vector $\mathbf{a} = \begin{pmatrix} x \\ y \end{pmatrix}$ is given by

$$\boxed{|\mathbf{a}| = \sqrt{x^2 + y^2}\,.}$$

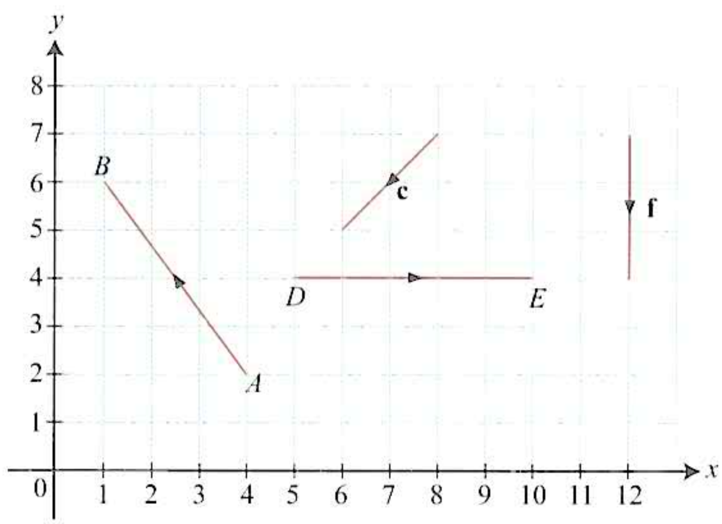
In particular, the magnitude of a *horizontal* vector $\mathbf{b} = \begin{pmatrix} x \\ 0 \end{pmatrix}$ is $|\mathbf{b}| = x$ and the magnitude of a *vertical* vector $\mathbf{c} = \begin{pmatrix} 0 \\ y \end{pmatrix}$ is $|\mathbf{c}| = y$.

PRACTISE NOW

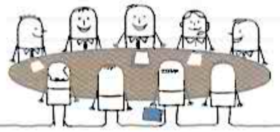
SIMILAR QUESTIONS

Exercise 7A Questions 1(a)–(e), 5, 10

Express each of the vectors in the diagram as a column vector and find its magnitude.



Equal Vectors



Class Discussion

Equal Vectors

Discuss in pairs. You may need to measure the length of the vectors.
Fig. 7.4 shows 5 vectors **a**, **b**, **c**, **d** and **e**.

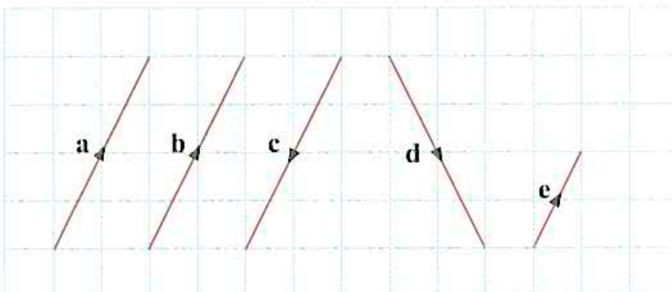


Fig. 7.4

1. Compare vector **a** with each of the vectors **b**, **c**, **d** and **e** respectively.
   What is similar and different between vector **a** and each of the other vectors?

2. Compare the x and y components of the vectors **a** and **b**. What do you observe?

In summary, **a**, **b**, **c** and **e** have the same or opposite direction. We say that the vectors are parallel. In particular,

- **a** and **b** have the same magnitude and direction. We say that the two vectors are **equal** and we write **a** = **b**.
- **a** and **c** have the same magnitude but opposite in direction. **c** is called the **negative** of vector **a** and we write **a** = −**c**.
- **a** and **e** have different magnitudes but the same direction. We will learn more about these vectors in Section 7.4.

ATTENTION

'Opposite direction' is not the same as 'different directions'. In the class discussion above, **a** and **c** are in **opposite direction** but **a** and **d** have **different directions**.

Furthermore, the x and y components of two equal vectors **a** and **b** are equal.

∴ For equal vectors represented in column vector form,

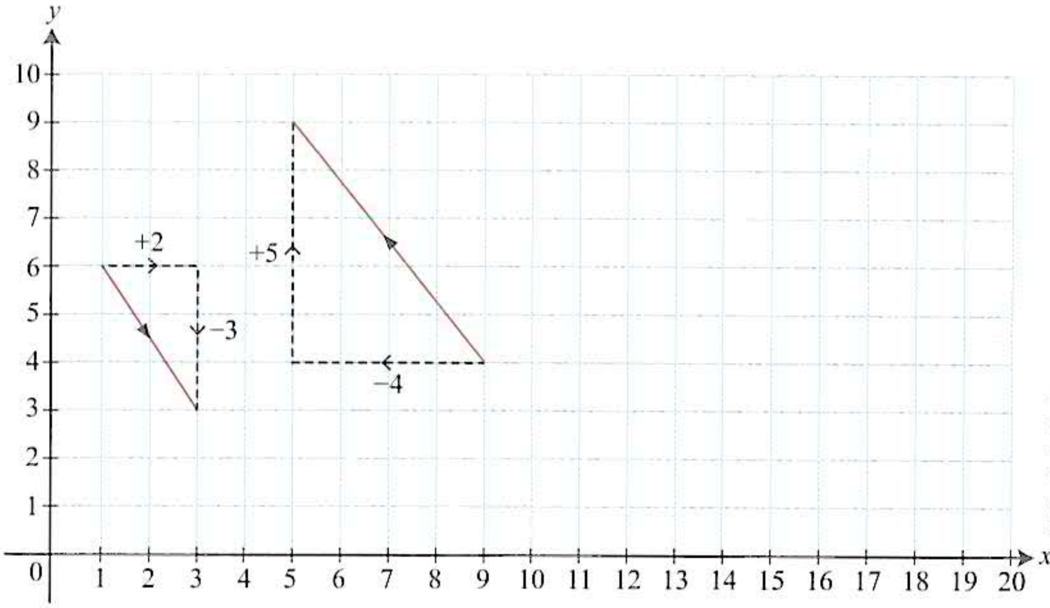
$$\boxed{\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} r \\ s \end{pmatrix} \text{ if and only if } p = r \text{ and } q = s.}$$

Worked Example 1

(Drawing a Vector on a Cartesian Plane)

On the 1 unit by 1 unit square grid below, draw the column vector $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} -4 \\ 5 \end{pmatrix}$.

Solution:



Problem Solving Tip

For $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$, choose any starting point. Move 2 units in the positive x-direction, and 3 units in the negative y-direction to the ending point. Join the starting and ending points with a directed line segment. Since the y-component of the column vector is negative, the starting point should be higher.

INFORMATION

Note that different students will draw the column vectors using different starting points but the vectors are equal if the x and y components are equal in magnitude and direction.

PRACTISE NOW 1

Draw the following column vectors on the square grid provided in Worked Example 1.

(i) $\begin{pmatrix} 4 \\ -6 \end{pmatrix}$ **(ii)** $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$ **(iii)** $\begin{pmatrix} 4.5 \\ 0 \end{pmatrix}$ **(iv)** Negative of $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

SIMILAR QUESTIONS

Exercise 7A Questions 2(a)–(e), 6(a)–(f), 11

Thinking Time

(a) Is the vector $\begin{pmatrix} 4 \\ -6 \end{pmatrix}$ in **(i)** equal to the vector $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$ in Worked Example 1?

Are there any similarities and differences? Compare their magnitude and direction and describe the relationship between the two vectors.

(b) Is the vector $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$ in **(ii)** equal to the vector $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$ in Worked Example 1?

Are there any similarities and differences? Compare their magnitude and direction and describe the relationship between the two vectors.

Worked Example 2

(Magnitude and Direction of Two Vectors)

Two column vectors **a** and **b** are such that $\mathbf{a} = \begin{pmatrix} x-2 \\ 3-y \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 4-x \\ y-6 \end{pmatrix}$.

(a) If $\mathbf{a} = \mathbf{b}$,

(i) find the value of x and of y,

(ii) write down the negative of **a** as a column vector,

(iii) show that $|\mathbf{a}| = |\mathbf{b}| = \sqrt{\frac{13}{4}}$.

(b) If $|\mathbf{a}| = |\mathbf{b}|$,

(i) express y in terms of x,

(ii) explain why **a** may not be equal to **b**.

Solution:

(a) (i) Since $\begin{pmatrix} x-2 \\ 3-y \end{pmatrix} = \begin{pmatrix} 4-x \\ y-6 \end{pmatrix}$, then

$x - 2 = 4 - x$ and $3 - y = y - 6$

$2x = 6$ $\qquad$ $2y = 9$

$x = 3$ $\qquad$ $y = 4\frac{1}{2}$

$\therefore x = 3$ and $y = 4\frac{1}{2}$

(ii)

$$\mathbf{a} = \begin{pmatrix} x-2 \\ 3-y \end{pmatrix} = \begin{pmatrix} 3-2 \\ 3-4\frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 \\ -1\frac{1}{2} \end{pmatrix}$$

Negative of $\mathbf{a} = -\mathbf{a}$

$$= -\begin{pmatrix} 1 \\ -1\frac{1}{2} \end{pmatrix} = \begin{pmatrix} -1 \\ 1\frac{1}{2} \end{pmatrix}$$

(iii) $|\mathbf{a}| = \sqrt{1^2 + \left(-1\frac{1}{2}\right)^2}$

$= \sqrt{1^2 + \left(-\frac{3}{2}\right)^2}$

$= \sqrt{1 + \frac{9}{4}}$

$= \sqrt{\frac{13}{4}}$

$\mathbf{b} = \begin{pmatrix} 4-x \\ y-6 \end{pmatrix} = \begin{pmatrix} 1 \\ -1\frac{1}{2} \end{pmatrix}$, so $|\mathbf{b}| = \sqrt{1^2 + \left(-1\frac{1}{2}\right)^2}$

$= \sqrt{\frac{13}{4}}$

$\therefore |\mathbf{a}| = |\mathbf{b}| = \sqrt{\frac{13}{4}}$ (shown)

(b) **(i)**

$$|\mathbf{a}| = |\mathbf{b}|$$
$$\sqrt{(x-2)^2 + (3-y)^2} = \sqrt{(4-x)^2 + (y-6)^2}$$
$$x^2 - 4x + 4 + 9 - 6y + y^2 = 16 - 8x + x^2 + y^2 - 12y + 36 \quad \text{(squaring both sides)}$$
$$\cancel{x^2} - 4x + 4 + 9 - 6y + \cancel{y^2} = 16 - 8x + \cancel{x^2} + \cancel{y^2} - 12y + 36$$
$$-4x + 13 - 6y = -8x - 12y + 52$$
$$6y = -4x + 39$$
$$y = \frac{39 - 4x}{6}$$

ATTENTION

For **(b)(i)**, if $x = 3$, then $y = \frac{39-4(3)}{6} = 4\frac{1}{2}$, which is the value of x and of y in **(a)(i)**. For this special case, $\mathbf{a} = \mathbf{b}$. For other values of x and y, $\mathbf{a} \neq \mathbf{b}$.

(ii) $\mathbf{a}$ may not be equal to $\mathbf{b}$ because only their magnitudes are equal but they may have different directions.

PRACTISE NOW 2

SIMILAR QUESTIONS

Exercise 7A Questions 3, 7, 8, 12

Two column vectors $\mathbf{a}$ and $\mathbf{b}$ are such that $\mathbf{a} = \begin{pmatrix} x+2 \\ 4-y \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 10-x \\ y-5 \end{pmatrix}$.

(a) If $\mathbf{a} = \mathbf{b}$,

(i) find the value of x and of y,

(ii) write down the negative of $\mathbf{a}$ as a column vector,

(iii) show that $|\mathbf{a}| = |\mathbf{b}| = \sqrt{\frac{145}{4}}$.

(b) If $|\mathbf{a}| = |\mathbf{b}|$,

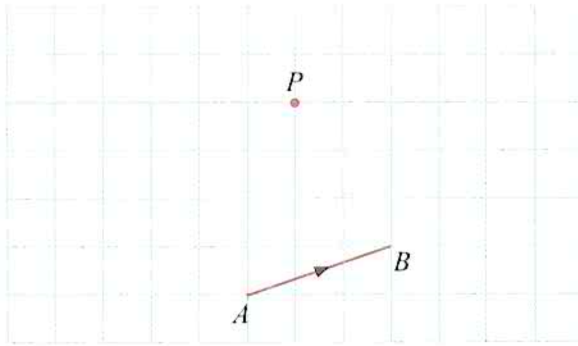
(i) express y in terms of x,

(ii) explain why $\mathbf{a}$ may not be equal to $\mathbf{b}$.

Worked Example 3

(Vectors in a Parallelogram)

The figure below shows the positions of the points P, A and B where $\overrightarrow{AB} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$.



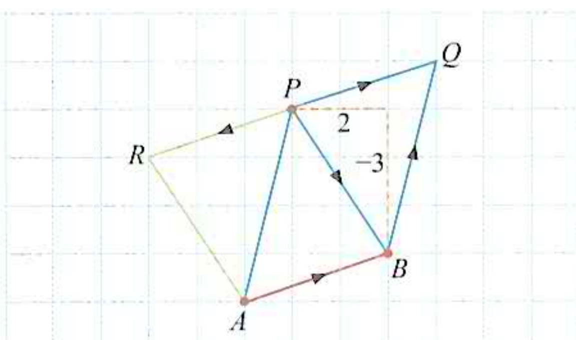
(i) Express $\overrightarrow{PB}$ as a column vector.

(ii) Q is a point such that $ABQP$ is a parallelogram. Express $\overrightarrow{BQ}$ as a column vector.

(iii) R is a point such that $ABPR$ is a parallelogram. Express $\overrightarrow{PR}$ as a column vector.

(iv) Do the two vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ have the same magnitude? Is $\overrightarrow{PQ} = \overrightarrow{PR}$? Why or why not?

Solution:



From the above diagram,

(i) $\overrightarrow{PB} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$ **(ii)** $\overrightarrow{BQ} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$ **(iii)** $\overrightarrow{PR} = \begin{pmatrix} -3 \\ -1 \end{pmatrix}$

(iv) $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ have the same magnitude but $\overrightarrow{PQ} \neq \overrightarrow{PR}$ because they do not have the same direction.

Problem Solving Tip

In **(ii)**, $ABQP$ is a parallelogram means that the vertices must be in this order:

$A \to B \to Q \to P$.

To draw the parallelogram $ABQP$, we note that $PQ = AB$ and $PQ \parallel AB$, i.e. $\overrightarrow{PQ} = \overrightarrow{AB}$.

Problem Solving Tip

Alternatively, for **(ii)**,

$\overrightarrow{BQ} = \overrightarrow{AP} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$.

For **(iii)**, $\overrightarrow{PR} = -\overrightarrow{AB} = -\begin{pmatrix} 3 \\ 1 \end{pmatrix}$

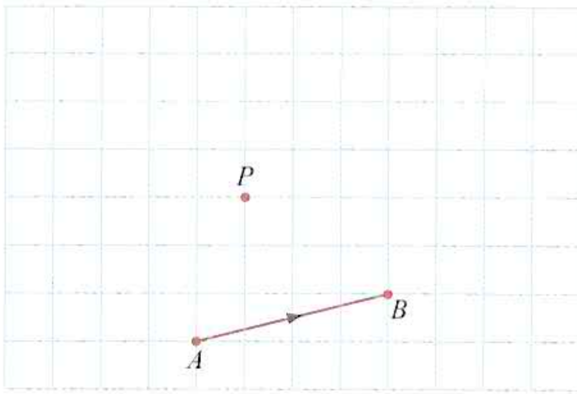
$= \begin{pmatrix} -3 \\ -1 \end{pmatrix}$.

PRACTISE NOW 3

SIMILAR QUESTIONS

Exercise 7A Questions 4, 9, 13

The figure below shows the positions of the points P, A and B where $\overrightarrow{AB} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$.



(i) Express $\overrightarrow{PB}$ as a column vector.

(ii) Q is a point such that $ABQP$ is a parallelogram. Express $\overrightarrow{BQ}$ as a column vector.

(iii) R is a point such that $ABPR$ is a parallelogram. Express $\overrightarrow{PR}$ as a column vector.

(iv) Do the two vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ have the same magnitude? Is $\overrightarrow{PQ} = \overrightarrow{PR}$? Why or why not?

Exercise 7A

BASIC LEVEL

1. Find the magnitude of each of the following vectors.

(a) $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ **(b)** $\begin{pmatrix} -5 \\ 12 \end{pmatrix}$

(c) $\begin{pmatrix} -7 \\ -2 \end{pmatrix}$ **(d)** $\begin{pmatrix} 0 \\ -6\frac{1}{2} \end{pmatrix}$

(e) $\begin{pmatrix} 8 \\ 0 \end{pmatrix}$

2. Write down the negative of each of the following vectors.

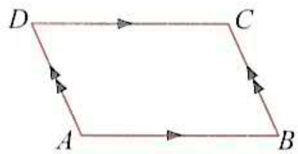
(a) $\begin{pmatrix} 12 \\ -7 \end{pmatrix}$ **(b)** $\begin{pmatrix} -2 \\ 0 \end{pmatrix}$

(c) $\begin{pmatrix} 4 \\ 8 \end{pmatrix}$ **(d)** $\begin{pmatrix} -3 \\ -1.2 \end{pmatrix}$

(e) $\begin{pmatrix} 0 \\ 3\frac{1}{4} \end{pmatrix}$

3. If $\mathbf{p} = \begin{pmatrix} a \\ 3 \end{pmatrix}$, $\mathbf{q} = \begin{pmatrix} -2 \\ a+2b \end{pmatrix}$ and $\mathbf{p} = \mathbf{q}$, find the value of a and of b.

4. $ABCD$ is a parallelogram. It is given that $\overrightarrow{AB} = \begin{pmatrix} 7 \\ 0 \end{pmatrix}$ and $\overrightarrow{BC} = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$.



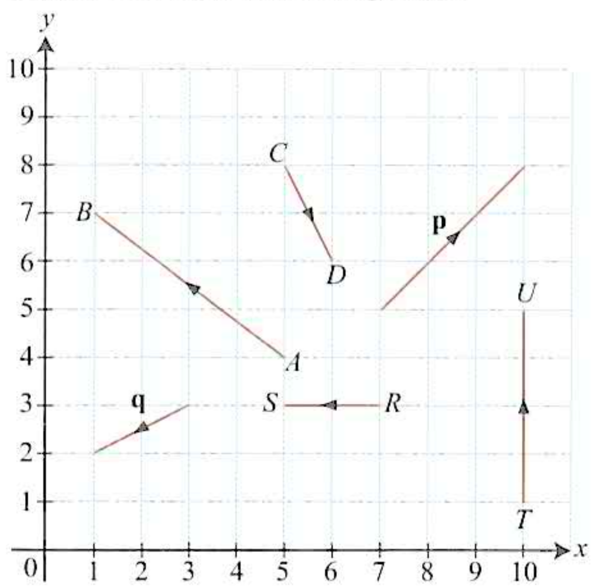
(a) Find the value of $|\overrightarrow{AB}|$.

(b) Express each of the following as a column vector:

(i) $\overrightarrow{DC}$ **(ii)** $\overrightarrow{DA}$

INTERMEDIATE LEVEL

5. Express each of the vectors in the diagram as a column vector and find its magnitude.



6. On a sheet of squared paper or graph paper, draw the following column vectors. You need to draw the x-axis and y-axis, and indicate the scale on the squared paper or graph paper.

(a) $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ **(b)** $\begin{pmatrix} -4.5 \\ 8 \end{pmatrix}$

(c) $\begin{pmatrix} -5 \\ 7 \end{pmatrix}$ **(d)** $\begin{pmatrix} 0 \\ -2\frac{1}{2} \end{pmatrix}$

(e) Negative of $\begin{pmatrix} 6 \\ -1 \end{pmatrix}$ **(f)** Negative of $\begin{pmatrix} -3 \\ 0 \end{pmatrix}$

7. Two column vectors $\mathbf{a}$ and $\mathbf{b}$ are such that $\mathbf{a} = \begin{pmatrix} x-3 \\ 2-y \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 5-x \\ y-9 \end{pmatrix}$.

(a) If $\mathbf{a} = \mathbf{b}$,

(i) find the value of x and of y,

(ii) write down the negative of $\mathbf{a}$ as a column vector,

(iii) show that $|\mathbf{a}| = |\mathbf{b}| = \sqrt{\frac{53}{4}}$.

(b) If $|\mathbf{a}| = |\mathbf{b}|$,

(i) express y in terms of x,

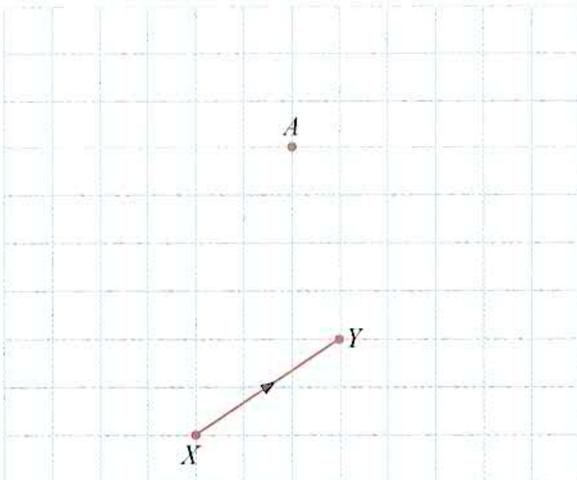
(ii) explain why $\mathbf{a}$ may not be equal to $\mathbf{b}$.

8. If $\overrightarrow{AB} = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$ and $\overrightarrow{CD} = \begin{pmatrix} 0 \\ 5 \end{pmatrix}$,

(i) show that $|\overrightarrow{AB}| = |\overrightarrow{CD}|$,

(ii) explain why $\overrightarrow{AB} \neq \overrightarrow{CD}$ even though $|\overrightarrow{AB}| = |\overrightarrow{CD}|$.

9. The figure below shows the positions of the points A, X and Y where $\overrightarrow{XY} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$.



(i) Express $\overrightarrow{AY}$ as a column vector.
(ii) B is a point such that $XYBA$ is a parallelogram. Express $\overrightarrow{YB}$ as a column vector.
(iii) C is a point such that $XYAC$ is a parallelogram. Express $\overrightarrow{AC}$ as a column vector.
(iv) Do the two vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ have the same magnitude? Is $\overrightarrow{AB} = \overrightarrow{AC}$? Why or why not?

ADVANCED LEVEL

10. If $\mathbf{a} = \begin{pmatrix} n \\ -3 \end{pmatrix}$, find the possible values of n such that $|\mathbf{a}| = 7$, leaving your answer in square root form if necessary.

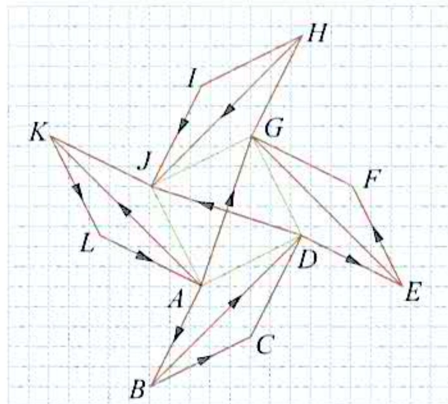
11. On a sheet of squared paper or graph paper, draw the following column vectors. You need to draw the x-axis and y-axis, and indicate the scale on the squared paper or graph paper.

(a) Two times of $\begin{pmatrix} 5 \\ -3 \end{pmatrix}$

(b) Three times of the negative of $\begin{pmatrix} -4 \\ 3 \end{pmatrix}$

12. If $\mathbf{u} = \begin{pmatrix} 13s \\ 4t \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} 6t+20 \\ 18-7s \end{pmatrix}$ and $\mathbf{u} = \mathbf{v}$, find the value of s and of t.

13. The figure below consists of a square $ADGJ$ and four identical rhombuses $AJKL$, $GHIJ$, $DEFG$ and $ABCD$.



(a) (i) Explain why $\overrightarrow{AB} = \overrightarrow{IJ}$.
(ii) Name two other vectors that are equal to $\overrightarrow{AB}$.

(b) Name all the vectors that are equal to
(i) $\overrightarrow{KL}$, (ii) $\overrightarrow{DE}$,
(iii) $\overrightarrow{BC}$, (iv) $\overrightarrow{AK}$.

(c) Give a reason why $\overrightarrow{AG} \neq \overrightarrow{DJ}$.

(d) The line segments BD and HJ have the same length and are parallel. Explain why $\overrightarrow{BD} \neq \overrightarrow{HJ}$.

(e) Give a vector that has the same magnitude but opposite direction to
(i) $\overrightarrow{BC}$, (ii) $\overrightarrow{EF}$,
(iii) $\overrightarrow{LA}$.

7.2 Addition of Vectors

A boat left Changi Jetty (P) for Pulau Ubin (Q) 2.1 km away on a bearing of 298° (see Fig. 7.5). Then it sailed 1.9 km away from Q on a bearing of 081°. Another boat left Changi Jetty and travelled 1.3 km north. Did they arrive at the same destination?

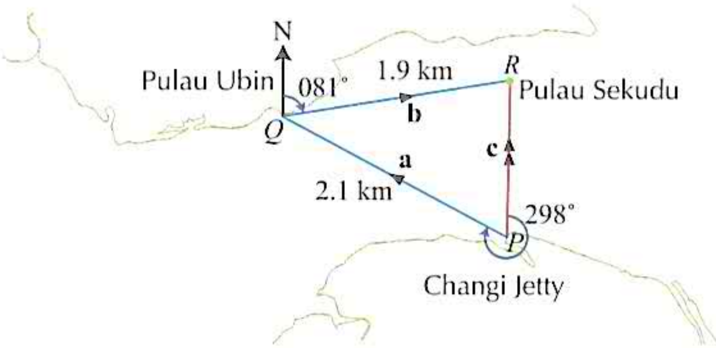
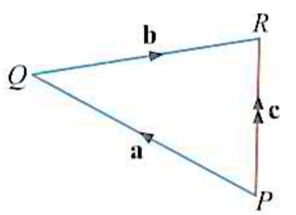


Fig. 7.5

Yes, they arrived at Pulau Sekudu (R).

This is the concept behind the *addition of vectors*. We can think of vector $\overrightarrow{PQ}$ as a **translation** (i.e. movement) from P to Q. Moving from P to Q and subsequently from Q to R is the same as moving from P to R.



We define the addition of two vectors $\overrightarrow{PQ}$ and $\overrightarrow{QR}$ as

$$\overrightarrow{PQ} + \overrightarrow{QR} = \overrightarrow{PR} \qquad \text{or} \qquad \mathbf{a} + \mathbf{b} = \mathbf{c}$$

start — end — start — end

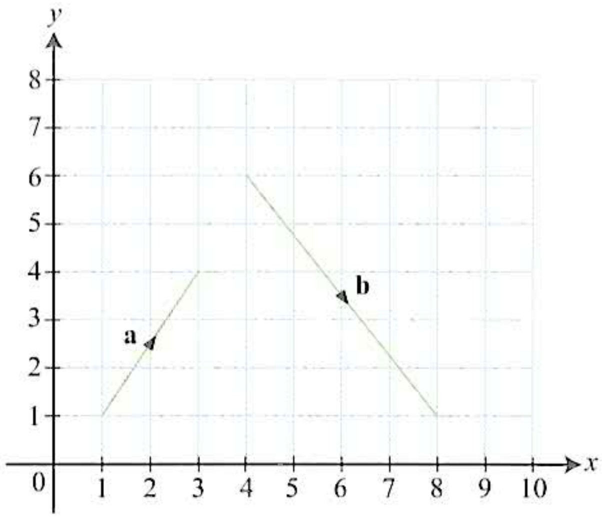
must be the same point for vector addition

$\overrightarrow{PR}$ is called the **vector sum** or **resultant vector** and we use a *double arrow* to indicate it.

Worked Example 4

(Vector Addition of Two Vectors)

The diagram shows two vectors **a** and **b**.



(i) Draw the sum of the two vectors **a** and **b**.

(ii) Express each of **a**, **b**, and **a** + **b** as a column vector.

(iii) How do you obtain **a** + **b** from **a** and **b** using column vectors directly?

(iv) Find the value of |**a**|, of |**b**| and of |**a** + **b**|.

(v) Is |**a** + **b**| = |**a**| + |**b**|? Explain why or why not, using the diagram that you have drawn.

Internet Resources

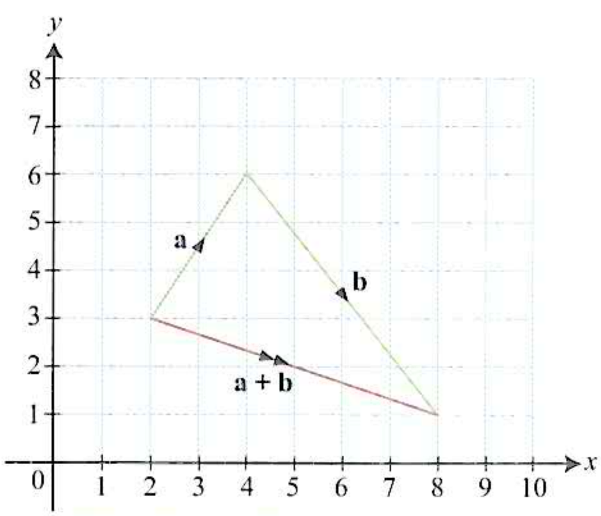
There are several mini-applications (also known as applets) on the Internet to help students practise vector addition. Search on the Internet to find out more.

Solution:

(i) **Method 1** (Triangle Law of Vector Addition)

Copy the vector **a** on a sheet of squared paper by using its column vector $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ as a guide. From the **ending point of a**, start drawing the vector $\mathbf{b} = \begin{pmatrix} 4 \\ -5 \end{pmatrix}$.

Then draw a directed line segment from the starting point of **a** to the ending point of **b**. This is the resultant vector **a** + **b**.



This method is the **Triangle Law of Vector Addition**.

INFORMATION

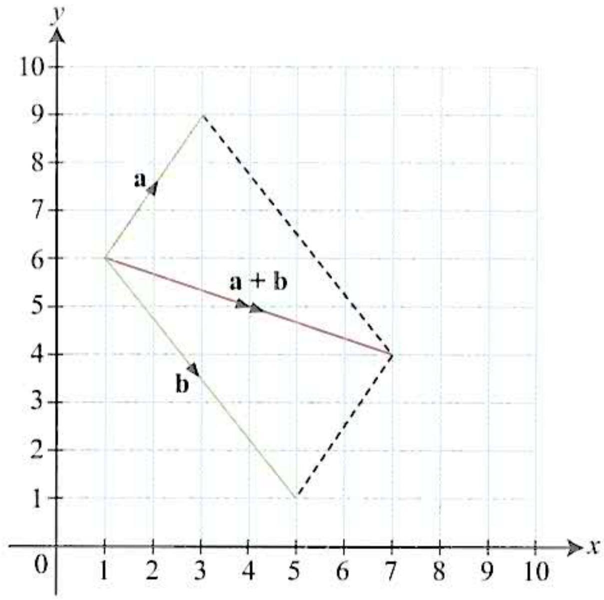
Can you draw **b** first, followed by **a**?

The resultant vector is **b** + **a**.

Note: **a** + **b** = **b** + **a**

Method 2 (Parallelogram Law of Vector Addition)

Draw **a** and **b** from the **same starting point**. Then complete the parallelogram. The resultant vector **a** + **b** also has the same starting point as **a** and as **b**.



(ii) From the diagram,

$$\mathbf{a} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 4 \\ -5 \end{pmatrix} \text{ and } \mathbf{a} + \mathbf{b} = \begin{pmatrix} 6 \\ -2 \end{pmatrix}.$$

(iii) We observe from **(ii)** that

$$\begin{aligned} \mathbf{a} + \mathbf{b} &= \begin{pmatrix} 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 4 \\ -5 \end{pmatrix} \\ &= \begin{pmatrix} 2+4 \\ 3+(-5) \end{pmatrix} \\ &= \begin{pmatrix} 2+4 \\ 3-5 \end{pmatrix} \\ &= \begin{pmatrix} 6 \\ -2 \end{pmatrix} \end{aligned}$$

(iv) $|\mathbf{a}| = \sqrt{2^2 + 3^2}$
$= \sqrt{13}$

$|\mathbf{b}| = \sqrt{4^2 + (-5)^2}$
$= \sqrt{41}$

$|\mathbf{a} + \mathbf{b}| = \sqrt{6^2 + (-2)^2}$
$= \sqrt{40}$

(v) $|\mathbf{a}| + |\mathbf{b}| = \sqrt{13} + \sqrt{41}$
$= 10.0$ (to 3 s.f.)

$|\mathbf{a} + \mathbf{b}| = \sqrt{40}$
$= 6.32$ (to 3 s.f.)

$\therefore |\mathbf{a} + \mathbf{b}| \neq |\mathbf{a}| + |\mathbf{b}|$

From the diagram, the 3 vectors **a**, **b**, and **a** + **b** form the sides of a triangle. Since the sum of the lengths of any two sides of a triangle is larger than the length of the third side, $|\mathbf{a} + \mathbf{b}| \neq |\mathbf{a}| + |\mathbf{b}|$.

ATTENTION

In this case, $|\mathbf{b}|$ is even larger than $|\mathbf{a} + \mathbf{b}|$.

INFORMATION

This method is the **Parallelogram Law of Vector Addition**.

From Worked Example 4, we have learnt two methods to draw the resultant vector for the addition of two vectors:

> **(i) Triangle Law of Vector Addition:**
> ending point of first vector **a** = starting point of second vector **b**.
>
> **(ii) Parallelogram Law of Vector Addition:**
> both vectors **a** and **b**, and the resultant **a** + **b**, all start from the same point.

In practice, it is usually easier to use the Triangle Law of Vector Addition. However, if a parallelogram has already been drawn for a question, it will be easier to use the Parallelogram Law of Vector Addition.

We have also learnt that for two *non-zero* vectors and *non-parallel* vectors, **a** and **b**,

$$\boxed{|\mathbf{a}+\mathbf{b}| \neq |\mathbf{a}| + |\mathbf{b}|.}$$

For column vectors,

$$\boxed{\begin{pmatrix} p \\ q \end{pmatrix} + \begin{pmatrix} r \\ s \end{pmatrix} = \begin{pmatrix} p+r \\ q+s \end{pmatrix}.}$$

Thinking Time

When do you think $|\mathbf{a}+\mathbf{b}|$ will be equal to $|\mathbf{a}| + |\mathbf{b}|$? Explain.

PRACTISE NOW 4

SIMILAR QUESTIONS

Exercise 7B Questions 1(a)–(c), 2, 9(a)–(d), 15(a)–(d), 16

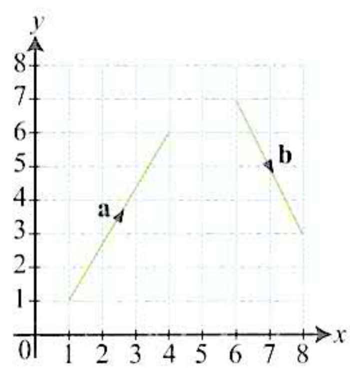
1. **(i)** Draw the sum of the two vectors **a** and **b**, using both the Triangle Law of Vector Addition and the Parallelogram Law of Vector Addition.

(ii) Express each of **a**, **b**, and **a** + **b** as a column vector.

(iii) How do you obtain **a** + **b** from **a** and **b** using column vectors directly?

(iv) Find the value of $|\mathbf{a}|$, of $|\mathbf{b}|$ and of $|\mathbf{a}+\mathbf{b}|$.

(v) Is $|\mathbf{a}+\mathbf{b}| = |\mathbf{a}| + |\mathbf{b}|$? Explain why or why not, using the diagram that you have drawn.



2. Simplify

(i) $\begin{pmatrix} 6 \\ 9 \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \end{pmatrix}$,

(ii) $\begin{pmatrix} 8 \\ -3 \end{pmatrix} + \begin{pmatrix} -10 \\ -5 \end{pmatrix} + \begin{pmatrix} 4 \\ 8 \end{pmatrix}$.

The principle of adding two vectors can be extended to any number of vectors. In Fig. 7.6 below,

$$\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} = \overrightarrow{AD}, \quad \text{or} \quad \begin{pmatrix} 0 \\ 3 \end{pmatrix} + \begin{pmatrix} 5 \\ 1 \end{pmatrix} + \begin{pmatrix} -2 \\ -5 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix} = \overrightarrow{AD},$$

i.e. $\overrightarrow{AD}$ is the result of the addition of all three vectors.

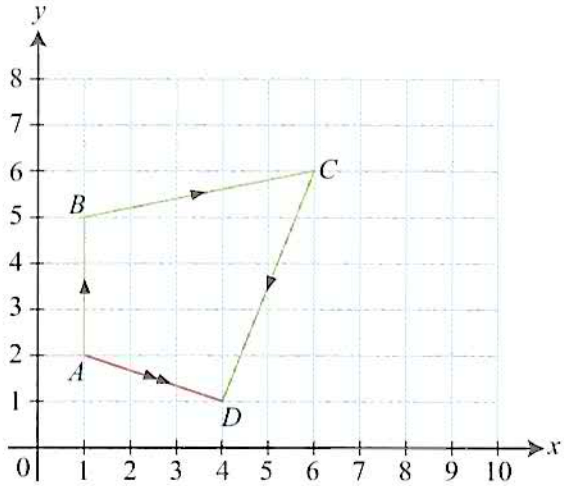


Fig. 7.6

Worked Example 5

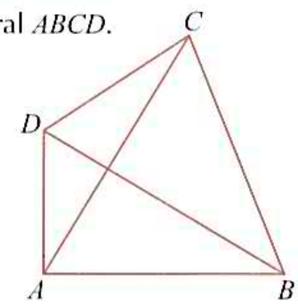
(Vector Addition of Two and More Vectors)

The diagram shows a quadrilateral $ABCD$.
Simplify

(i) $\overrightarrow{AB} + \overrightarrow{BC}$,

(ii) $\overrightarrow{DB} + \overrightarrow{AD}$,

(iii) $\overrightarrow{AC} + \overrightarrow{CB} + \overrightarrow{BD}$.



Solution:

(i) $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$

start, end start, end

check these are the same

(Triangle Law of Vector Addition)

(ii) $\overrightarrow{DB} + \overrightarrow{AD} = \overrightarrow{AD} + \overrightarrow{DB}$

$= \overrightarrow{AB}$ (Triangle Law of Vector Addition)

(iii) $\overrightarrow{AC} + \overrightarrow{CB} + \overrightarrow{BD} = (\overrightarrow{AC} + \overrightarrow{CB}) + \overrightarrow{BD}$

$= \overrightarrow{AB} + \overrightarrow{BD}$ (Triangle Law of Vector Addition)

$= \overrightarrow{AD}$ (Triangle Law of Vector Addition)

Problem Solving Tip

In fact, you can simplify all these vector additions without even looking at the quadrilateral. Just match the vertices:

$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$

start, end start, end

must be same vertex

PRACTISE NOW 5

SIMILAR QUESTIONS

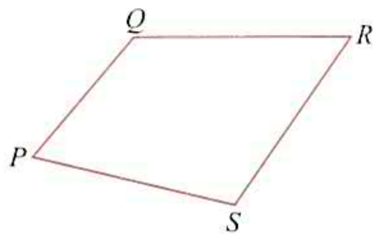
Exercise 7B Questions 3, 10

The diagram shows a quadrilateral $PQRS$.
Simplify

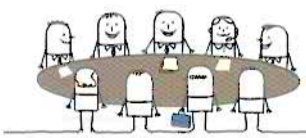
(i) $\overrightarrow{PQ} + \overrightarrow{QR}$,

(ii) $\overrightarrow{SR} + \overrightarrow{PS}$,

(iii) $\overrightarrow{PR} + \overrightarrow{RS} + \overrightarrow{SQ}$.



Zero Vector



Class Discussion

The Zero Vector

In Fig. 7.5, on page 228, one of the boats travelled from Changi Jetty (P) to Pulau Sekudu (R). Its journey is represented by the vector $\overrightarrow{PR}$.

Suppose the boat travelled back from Pulau Seduku to Changi Jetty; its journey will be represented by the vector $\overrightarrow{RP}$.

1. What do you think is the meaning of $\overrightarrow{PR} + \overrightarrow{RP}$?

2. How should you simplify $\overrightarrow{PR} + \overrightarrow{RP}$?

For the above class discussion, the boat went from Changi Jetty (P) to Pulau Seduku (R) and then back to Changi Jetty, i.e. the result of the whole journey is a zero displacement of the boat from Changi Jetty (P). In other words,

$$\boxed{\overrightarrow{PR} + \overrightarrow{RP} = \mathbf{0}}$$

$\mathbf{0}$ is called the **zero vector**. It is *not* a scalar.

INFORMATION

The zero vector $\mathbf{0}$ has a magnitude of 0, but it has no direction. However, it is still called a vector, unlike the scalar 0. The zero vector $\mathbf{0}$ is necessary to make vector addition 'closed' as the addition of two or more vectors will always be a vector.

Worked Example 6

(Problem involving the Zero Vector)

(a) Simplify $\begin{pmatrix} 3 \\ -4 \end{pmatrix} + \begin{pmatrix} -3 \\ 4 \end{pmatrix}$.

(b) Copy and complete the following vector equation:

$$\begin{pmatrix} 5 \\ \end{pmatrix} + \begin{pmatrix} \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Solution:

(a) $\begin{pmatrix} 3 \\ -4 \end{pmatrix} + \begin{pmatrix} -3 \\ 4 \end{pmatrix} = \begin{pmatrix} 3-3 \\ -4+4 \end{pmatrix}$

$= \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

(b) $\begin{pmatrix} 5 \\ 2 \end{pmatrix} + \begin{pmatrix} -5 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

From Worked Example 6, $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is the column vector form of the zero vector $\mathbf{0}$, i.e. $\mathbf{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$; $\begin{pmatrix} 3 \\ -4 \end{pmatrix}$ and $\begin{pmatrix} -3 \\ 4 \end{pmatrix}$ are the negatives of each other.

In general,

$$\boxed{\mathbf{a} + (-\mathbf{a}) = \mathbf{0} = (-\mathbf{a}) + (\mathbf{a}).}$$

PRACTISE NOW 6

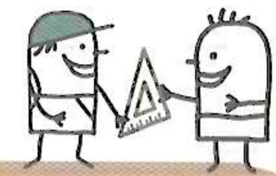
SIMILAR QUESTIONS

Exercise 7B Questions 4–6

(a) Simplify $\begin{pmatrix} 8 \\ -1 \end{pmatrix} + \begin{pmatrix} -8 \\ 1 \end{pmatrix}$.

(b) Copy and complete the following vector equation: $\begin{pmatrix} -6 \\ \end{pmatrix} + \begin{pmatrix} \\ 7 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

7.3 Vector Subtraction

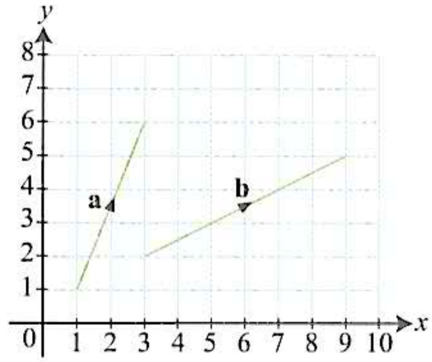


Difference of Two Vectors

Worked Example 7

(Vector Subtraction)

The diagram shows two vectors **a** and **b**.



(i) Draw the vector **a** – **b**.

(ii) By looking at the diagram that you have drawn, express each of **a**, **b** and **a** – **b** as a column vector.

(iii) How do you obtain **a** – **b** from **a** and **b** using column vectors directly?

(iv) Find the value of |**a**|, of |**b**| and of |**a** – **b**|.

(v) Is |**a** – **b**| = |**a**| – |**b**|?

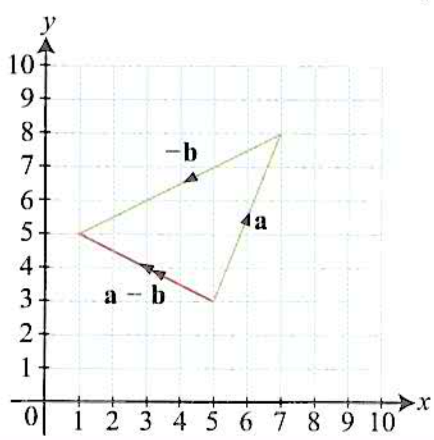
ATTENTION

It is not clear if we just say 'the difference of two vectors **a** and **b**'. We need to specify whether we mean **a** – **b** or **b** – **a**.

Solution:

(i) Method 1 (Addition of Negative Vector)

Copy the vector **a** on a sheet of squared paper by using its column vector $\begin{pmatrix} 2 \\ 5 \end{pmatrix}$ as a guide. From the ending point of **a**, start drawing the **negative of vector b**, i.e. –**b** = $\begin{pmatrix} -6 \\ -3 \end{pmatrix}$. Then draw a directed line segment from the starting point of **a** to the ending point of –**b**. This is the resultant vector **a** + (–**b**) = **a** – **b**.



INFORMATION

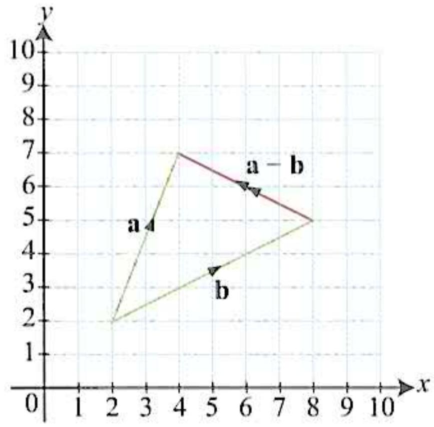
Can you draw –**b** first, followed by **a**? What happens if you draw **b** first, followed by –**a**?

Method 2 (Triangle Law of Vector Subtraction)

Draw **a** and **b** from the **same starting point**. Then draw a directed line segment from the ending point of **b** to the ending point of **a**. This is the resultant vector $\mathbf{a} - \mathbf{b}$ since $\mathbf{b} + (\mathbf{a} - \mathbf{b}) = \mathbf{a}$.

INFORMATION

What happens if you draw the resultant vector in the opposite direction?



(ii) From the diagram,

$$\mathbf{a} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 6 \\ 3 \end{pmatrix} \text{ and } \mathbf{a} - \mathbf{b} = \begin{pmatrix} -4 \\ 2 \end{pmatrix}.$$

(iii) We observe from **(ii)** that

$$\begin{aligned} \mathbf{a} - \mathbf{b} &= \begin{pmatrix} 2 \\ 5 \end{pmatrix} - \begin{pmatrix} 6 \\ 3 \end{pmatrix} \\ &= \begin{pmatrix} 2-6 \\ 5-3 \end{pmatrix} \\ &= \begin{pmatrix} -4 \\ 2 \end{pmatrix} \end{aligned}$$

(iv)

$$\begin{aligned} |\mathbf{a}| &= \sqrt{2^2 + 5^2} \\ &= \sqrt{29} \\ &= 5.39 \text{ (to 3 s.f.)} \end{aligned} \qquad \begin{aligned} |\mathbf{b}| &= \sqrt{6^2 + 3^2} \\ &= \sqrt{45} \\ &= 6.71 \text{ (to 3 s.f.)} \end{aligned}$$

$$\begin{aligned} |\mathbf{a} - \mathbf{b}| &= \sqrt{(-4)^2 + 2^2} \\ &= \sqrt{20} \\ &= 4.47 \text{ (to 3 s.f.)} \end{aligned}$$

(v) $|\mathbf{a} - \mathbf{b}| = \sqrt{20} = 4.47$

$|\mathbf{a}| - |\mathbf{b}| = \sqrt{29} - \sqrt{45} = 1.32$ (to 3 s.f.)

$\therefore |\mathbf{a} - \mathbf{b}| \neq |\mathbf{a}| - |\mathbf{b}|$

From Worked Example 7, we have learnt two methods to draw the resultant vector for vector subtraction.

In most vector problems with a diagram, the diagram will look like the triangle in Method 2 in Worked Example 7. Hence, we need to learn this method to obtain $\mathbf{a} - \mathbf{b}$ first.

In fact, we can also obtain $\mathbf{b} - \mathbf{a}$ from the triangle in Method 2 by drawing the resultant vector in the opposite direction.

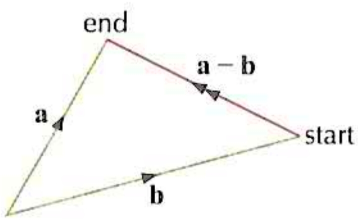
In general,

> **Triangle Law of Vector Subtraction:**
> both vectors **a** and **b** must *start from the same point.*

To determine the direction of **a** – **b** or **b** – **a**, just remember 'end minus start', as shown in Fig. 7.7.

(i) **a** – **b**
end start

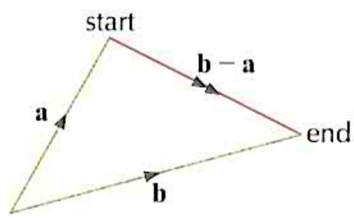
The arrow starts from the ending point of **b** and ends at the ending point of **a**:



(a)

(ii) **b** – **a**
end start

The arrow starts from the ending point of **a** and ends at the ending point of **b**.



(b)

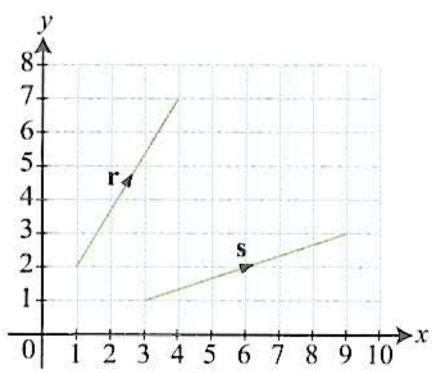
Fig. 7.7

PRACTISE NOW 7

SIMILAR QUESTIONS

Exercise 7B Questions 11(a)–(d)

The diagram shows two vectors **r** and **s**.



(i) Draw the vector **r** – **s**.

(ii) By looking at the diagram that you have drawn, express each of **r**, **s** and **r** – **s** as a column vector.

(iii) Find the value of |**r**|, of |**s**| and of |**r** – **s**|.

(iv) Is |**r** – **s**| = |**r**| – |**s**|?

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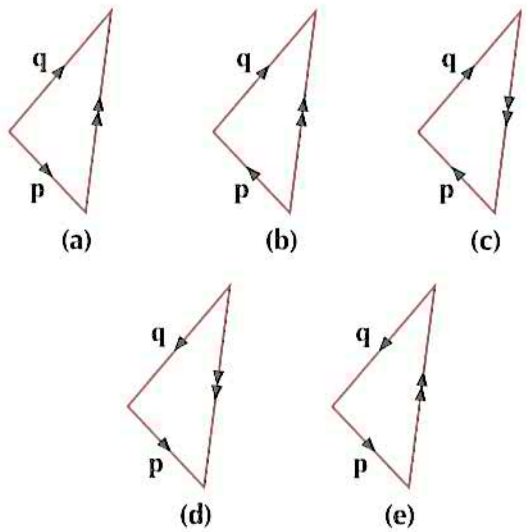
Thinking Time

When do you think $|\mathbf{a} - \mathbf{b}|$ will be equal to $|\mathbf{a}| - |\mathbf{b}|$? Explain.

Worked Example 8

(Vector Addition and Subtraction)

Find the resultant vector represented by the double arrow in each of the diagrams below.



Solution:

(a) The resultant vector is $\mathbf{q} - \mathbf{p}$.

(b) The resultant vector is $\mathbf{p} + \mathbf{q}$.

(c) The resultant vector is $-(\mathbf{p} + \mathbf{q}) = -\mathbf{p} - \mathbf{q}$.

(d) The resultant vector is $\mathbf{p} + \mathbf{q}$.

(e) The resultant vector is $-(\mathbf{p} + \mathbf{q}) = -\mathbf{p} - \mathbf{q}$.

Problem Solving Tip

For **(a)**, **p** and **q** start from the same point. Applying the **Triangle Law of Vector Subtraction**, the resultant vector is:

$$\underset{\text{end}}{\mathbf{q}} - \underset{\text{start}}{\mathbf{p}}$$

For **(b)**, **p** and **q** do not start from the same point, but **q** starts where **p** ends. Applying the **Triangle Law of Vector Addition**, the resultant vector is $\mathbf{p} + \mathbf{q}$.

For **(c)**, the resultant vector is the **negative** of the resultant vector in **(b)**.

In fact, the arrows 'go in a round trip', i.e. $\mathbf{p} + \mathbf{q} + (-\mathbf{p} - \mathbf{q}) = \mathbf{0}$.

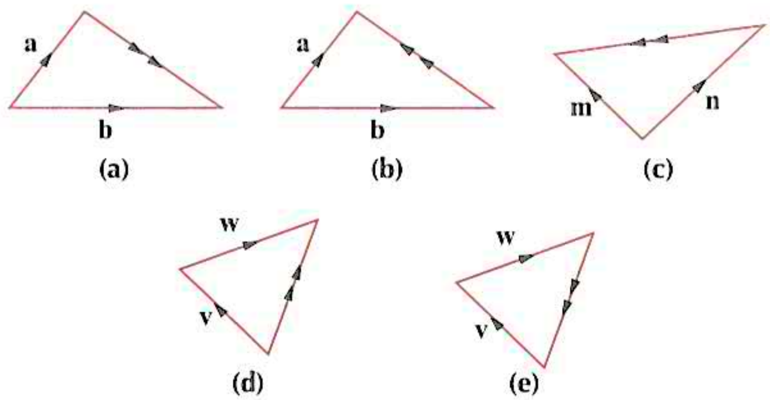
If **p** and **q** are in the same direction, what is the resultant vector?

PRACTISE NOW 8

SIMILAR QUESTIONS

Exercise 7B Questions 7(a)–(h)

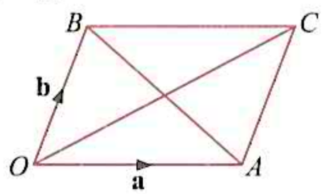
Find the resultant vector represented by the double arrow in each of the diagrams below.



Worked Example 9

(Vector Addition and Subtraction)

The diagram below shows a parallelogram $OACB$, where $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$.



Express the following vectors in terms of $\mathbf{a}$ and $\mathbf{b}$.

(i) $\overrightarrow{BC}$ **(ii)** $\overrightarrow{CA}$ **(iii)** $\overrightarrow{OC}$

(iv) $\overrightarrow{AB}$ **(v)** $\overrightarrow{BA}$

ATTENTION

Notice that $\overrightarrow{OC}$ and $\overrightarrow{AB}$ are the diagonals of the parallelogram. One diagonal represents vector addition, $\mathbf{a} + \mathbf{b}$, and the other diagonal represents vector subtraction, $\mathbf{a} - \mathbf{b}$ or $\mathbf{b} - \mathbf{a}$, depending on which direction.

Solution:

(i) $\overrightarrow{BC} = \overrightarrow{OA} = \mathbf{a}$

(ii) $\overrightarrow{CA} = \overrightarrow{BO} = -\overrightarrow{OB} = -\mathbf{b}$

(iii) $\overrightarrow{OC} = \mathbf{a} + \mathbf{b}$ (Parallelogram Law of Vector Addition)

or $\overrightarrow{OC} = \overrightarrow{OB} + \overrightarrow{BC} = \mathbf{a} + \mathbf{b}$ (Triangle Law of Vector Addition)

(iv) $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$ (Triangle Law of Vector Subtraction)

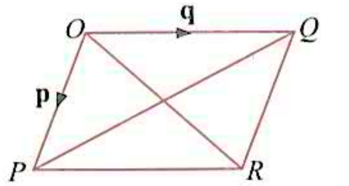
(v) $\overrightarrow{BA} = \mathbf{a} - \mathbf{b}$ (Triangle Law of Vector Subtraction)

PRACTISE NOW 9

SIMILAR QUESTIONS

Exercise 7B Question 12

The diagram below shows a parallelogram $OPRQ$, where $\overrightarrow{OP} = \mathbf{p}$ and $\overrightarrow{OQ} = \mathbf{q}$.



Express the following vectors in terms of **p** and **q**.

(a) $\overrightarrow{PR}$ **(b)** $\overrightarrow{RQ}$ **(c)** $\overrightarrow{OR}$ **(d)** $\overrightarrow{PQ}$ **(e)** $\overrightarrow{QP}$

For addition of vectors, we have seen at the start of Section 7.2 on page 228 that:

$$\overrightarrow{PQ} + \overrightarrow{QR} = \overrightarrow{PR}$$

start end start end

must be the same

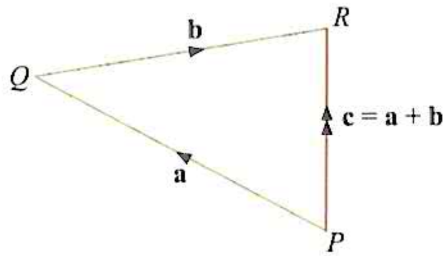
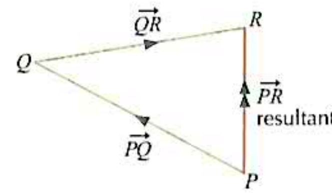


Fig. 7.8(a)

Problem Solving Tip



For subtraction of vectors, in terms of the vertices of the triangle in Fig. 7.8(**b**):

$$\overrightarrow{OA} - \overrightarrow{OB} = \overrightarrow{BA}$$

end start start end

must be the same

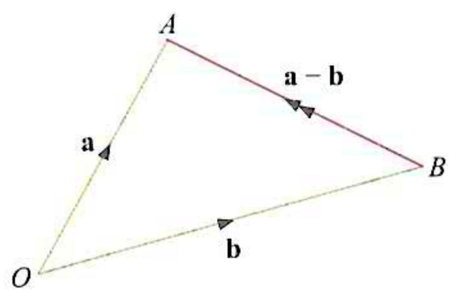


Fig. 7.8(b)

Problem Solving Tip

$$\begin{aligned}\overrightarrow{OA} - \overrightarrow{OB} &= \overrightarrow{OA} + (-\overrightarrow{OB}) \\ &= \overrightarrow{OA} + \overrightarrow{BO} \\ &= \overrightarrow{BO} + \overrightarrow{OA} \\ &= \overrightarrow{BA}\end{aligned}$$

Notice it still has the same idea of 'end minus start'.

Worked Example 10

(Vector Addition and Subtraction)

Simplify the following if possible.

(a) $\overrightarrow{PR} + \overrightarrow{RQ}$ (b) $\overrightarrow{PQ} - \overrightarrow{PR}$

(c) $\overrightarrow{PQ} - \overrightarrow{QR}$

Problem Solving Tip

You can draw a diagram to help you understand the relationship between the three vectors.

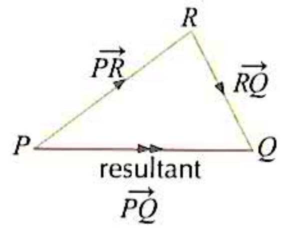
Solution:

We observe the following patterns in the labelling of the vectors.

(a) $\overrightarrow{PR} + \overrightarrow{RQ} = \overrightarrow{PQ}$

start end start end

check these are the same



(b) $\overrightarrow{PQ} - \overrightarrow{PR} = \overrightarrow{RQ}$ (start at R)

end end

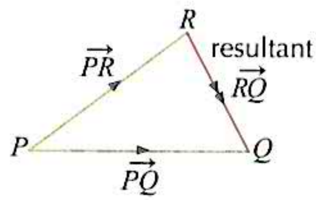
start start

check these are the same

or $\overrightarrow{PQ} - \overrightarrow{PR} = \overrightarrow{PQ} + \overrightarrow{RP}$

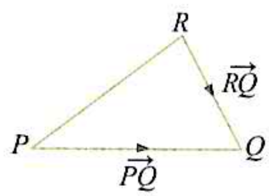
$= \overrightarrow{RP} + \overrightarrow{PQ}$

$= \overrightarrow{RQ}$



(c) $\overrightarrow{PQ} - \overrightarrow{QR}$

These are the same. However, this is not vector addition, so we cannot simplify this further using P, Q and R.



PRACTISE NOW 10

SIMILAR QUESTIONS

Exercise 7B Questions 13(a)–(f), 17, 18(a)–(f)

Simplify the following if possible.

(a) $\overrightarrow{AB} + \overrightarrow{BC}$ (b) $\overrightarrow{AB} - \overrightarrow{AC}$ (c) $\overrightarrow{AB} - \overrightarrow{BC}$

(d) $\overrightarrow{PQ} - \overrightarrow{PR}$ (e) $\overrightarrow{PQ} - \overrightarrow{RQ}$ (f) $\overrightarrow{PQ} + \overrightarrow{RP} - \overrightarrow{RS}$

From Worked Example 7, we have also learnt that for two non-zero and non-parallel vectors **a** and **b**,

$$\boxed{|\mathbf{a}-\mathbf{b}| \neq |\mathbf{a}|-|\mathbf{b}|.}$$

For column vectors,

$$\boxed{\begin{pmatrix} p \\ q \end{pmatrix} - \begin{pmatrix} r \\ s \end{pmatrix} = \begin{pmatrix} p-r \\ q-s \end{pmatrix}.}$$

Worked Example 11

(Addition and Subtraction of Column Vectors)

(a) Simplify $\begin{pmatrix} 3 \\ 4 \end{pmatrix} - \begin{pmatrix} 5 \\ -2 \end{pmatrix}$.

(b) Find the values of x and y in each of the following equations:

(i) $\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 10 \\ -7 \end{pmatrix} = \begin{pmatrix} -6 \\ 8 \end{pmatrix}$,

(ii) $\begin{pmatrix} 5 \\ 4x \end{pmatrix} - \begin{pmatrix} 3 \\ 8 \end{pmatrix} = \begin{pmatrix} x+2y \\ -2 \end{pmatrix}$.

Solution:

(a)
$$\begin{aligned}\begin{pmatrix} 3 \\ 4 \end{pmatrix} - \begin{pmatrix} 5 \\ -2 \end{pmatrix} &= \begin{pmatrix} 3-5 \\ 4-(-2) \end{pmatrix} \\ &= \begin{pmatrix} 3-5 \\ 4+2 \end{pmatrix} \\ &= \begin{pmatrix} -2 \\ 6 \end{pmatrix}\end{aligned}$$

(b) (i)
$$\begin{aligned}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 10 \\ -7 \end{pmatrix} &= \begin{pmatrix} -6 \\ 8 \end{pmatrix} \\ \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} -6 \\ 8 \end{pmatrix} - \begin{pmatrix} 10 \\ -7 \end{pmatrix} \\ &= \begin{pmatrix} -16 \\ 15 \end{pmatrix}\end{aligned}$$

$\therefore x = -16$ and $y = 15$

(ii)
$$\begin{aligned}\begin{pmatrix} 5 \\ 4x \end{pmatrix} - \begin{pmatrix} 3 \\ 8 \end{pmatrix} &= \begin{pmatrix} x+2y \\ -2 \end{pmatrix} \\ \begin{pmatrix} 2 \\ 4x-8 \end{pmatrix} &= \begin{pmatrix} x+2y \\ -2 \end{pmatrix}\end{aligned}$$

$2 = x + 2y$ ------ (1)

and $4x - 8 = -2$ ------ (2)

From (2), $4x = 6$

$x = 1\frac{1}{2}$

Substitute $x = 1\frac{1}{2}$ into (1),

$$\begin{aligned}1\frac{1}{2} + 2y &= 2 \\ 2y &= \frac{1}{2} \\ y &= \frac{1}{4}\end{aligned}$$

$\therefore x = 1\frac{1}{2}$ and $y = \frac{1}{4}$

PRACTISE NOW 11

SIMILAR QUESTIONS

Exercise 7B Questions 8(a)–(d), 14(a)–(d)

(a) Simplify the following:

(i) $\begin{pmatrix} 1 \\ 3 \end{pmatrix} - \begin{pmatrix} 8 \\ -6 \end{pmatrix}$ (ii) $\begin{pmatrix} -2 \\ -3 \end{pmatrix} - \begin{pmatrix} -5 \\ 4 \end{pmatrix} + \begin{pmatrix} -6 \\ 7 \end{pmatrix}$

(b) Find the values of x and y in each of the following equations:

(i) $\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -5 \\ 7 \end{pmatrix} = \begin{pmatrix} 8 \\ -3 \end{pmatrix}$ (ii) $\begin{pmatrix} 3 \\ y \end{pmatrix} - \begin{pmatrix} x \\ -9 \end{pmatrix} = \begin{pmatrix} 4 \\ x \end{pmatrix}$

Exercise 7B

BASIC LEVEL

1. Simplify the following:

(a) $\begin{pmatrix} 2 \\ 4 \end{pmatrix} + \begin{pmatrix} 7 \\ 5 \end{pmatrix}$ (b) $\begin{pmatrix} 3 \\ -5 \end{pmatrix} + \begin{pmatrix} -4 \\ 1 \end{pmatrix}$

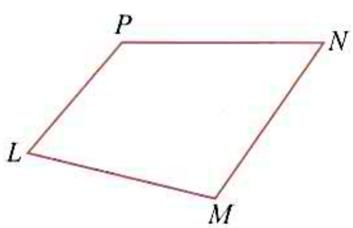
(c) $\begin{pmatrix} -9 \\ -2 \end{pmatrix} + \begin{pmatrix} 0 \\ -8 \end{pmatrix} + \begin{pmatrix} -3 \\ 7 \end{pmatrix}$

2. If $\mathbf{a} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} -7 \\ 2 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} 1 \\ -5 \end{pmatrix}$,

(a) determine whether $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$.

(b) determine whether $(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$.

3. The diagram shows a quadrilateral $PLMN$.



Simplify

(a) $\overrightarrow{LM} + \overrightarrow{MN}$, (b) $\overrightarrow{PN} + \overrightarrow{LP}$,

(c) $\overrightarrow{LN} + \overrightarrow{NM} + \overrightarrow{MP}$.

4. Simplify the following:

(a) $\begin{pmatrix} 12 \\ -6 \end{pmatrix} + \begin{pmatrix} -12 \\ 6 \end{pmatrix}$ (b) $\begin{pmatrix} 5 \\ 7 \end{pmatrix} + \begin{pmatrix} -5 \\ -7 \end{pmatrix}$

(c) $\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -x \\ -y \end{pmatrix}$

5. Copy and complete the following:

(a) $\begin{pmatrix} 9 \\ \end{pmatrix} + \begin{pmatrix} \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

(b) $\begin{pmatrix} -3 \\ \end{pmatrix} + \begin{pmatrix} \\ 7 \end{pmatrix} = \mathbf{0}$

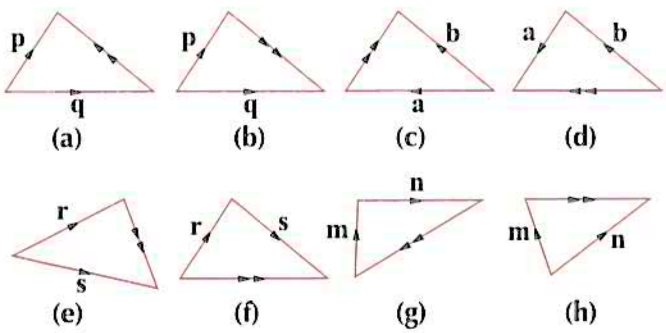
(c) $\begin{pmatrix} \\ p \end{pmatrix} + \begin{pmatrix} -q \\ \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

6. Simplify the following:

(a) $\overrightarrow{AB} + \overrightarrow{BA}$ (b) $\overrightarrow{PQ} + \overrightarrow{QR} + \overrightarrow{RP}$

(c) $\overrightarrow{MN} + \overrightarrow{LM} + \overrightarrow{NL}$

7. Find the vector that is marked with double arrows in each of the diagrams below.



8. Simplify the following:

(a) $\begin{pmatrix}5\\4\end{pmatrix}-\begin{pmatrix}3\\2\end{pmatrix}$

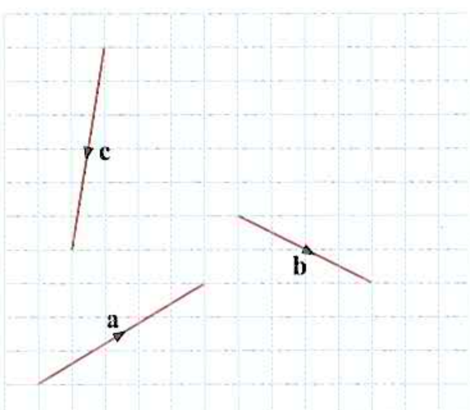
(b) $\begin{pmatrix}-1\\3\end{pmatrix}-\begin{pmatrix}-3\\-4\end{pmatrix}$

(c) $\begin{pmatrix}2\\3\end{pmatrix}+\begin{pmatrix}5\\-2\end{pmatrix}-\begin{pmatrix}7\\-3\end{pmatrix}$

(d) $\begin{pmatrix}4\\7\end{pmatrix}-\begin{pmatrix}-2\\5\end{pmatrix}-\begin{pmatrix}3\\-6\end{pmatrix}$

INTERMEDIATE LEVEL

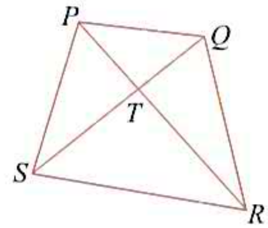
9. The diagram shows three vectors **a**, **b** and **c**.



On a sheet of squared paper or graph paper, draw appropriate triangles to illustrate the following vector additions:

(a) **a** + **b** (b) **b** + **a**
(c) **a** + **c** (d) **b** + **c**

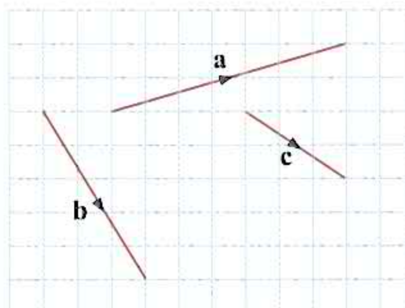
10. *PQRS* is a quadrilateral.



Simplify the following:

(a) $\overrightarrow{PT}+\overrightarrow{TR}$ (b) $\overrightarrow{SQ}+\overrightarrow{QR}$
(c) $\overrightarrow{TR}+\overrightarrow{ST}$ (d) $\overrightarrow{SQ}+\overrightarrow{QT}$
(e) $\overrightarrow{SQ}+\overrightarrow{QR}+\overrightarrow{PS}$ (f) $\overrightarrow{RQ}+\overrightarrow{QT}+\overrightarrow{TP}+\overrightarrow{PS}$

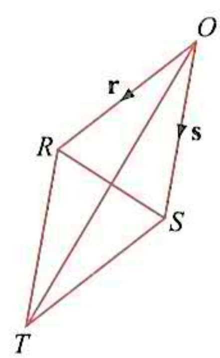
11. The diagram shows three vectors **a**, **b** and **c**.



On a sheet of squared paper or graph paper, use the Triangle Law of Vector Subtraction to illustrate the vector subtractions.

(a) **a** – **b** (b) **b** – **a**
(c) **a** – **c** (d) **c** – **b**

12. The diagram shows a parallelogram *ORTS* where $\overrightarrow{OR}$ = **r** and $\overrightarrow{OS}$ = **s**.



Express the following vectors in terms of **r** and/or **s**.

(a) $\overrightarrow{RT}$ (b) $\overrightarrow{TS}$ (c) $\overrightarrow{OT}$
(d) $\overrightarrow{RS}$ (e) $\overrightarrow{SR}$

13. Simplify the following if possible.

(a) $\overrightarrow{RS} + \overrightarrow{ST}$ (b) $\overrightarrow{RS} - \overrightarrow{RT}$

(c) $\overrightarrow{RT} - \overrightarrow{RS}$ (d) $\overrightarrow{RS} - \overrightarrow{ST}$

(e) $\overrightarrow{RS} - \overrightarrow{TS}$ (f) $\overrightarrow{RS} + \overrightarrow{TR} - \overrightarrow{TU}$

14. Find the value of x and of y in the following equations.

(a) $\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -3 \\ 2 \end{pmatrix} = \begin{pmatrix} 7 \\ -5 \end{pmatrix}$

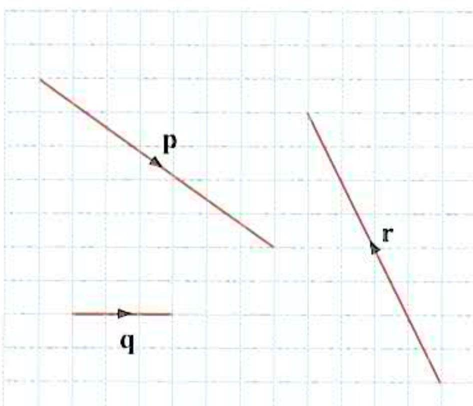
(b) $\begin{pmatrix} 3 \\ y \end{pmatrix} - \begin{pmatrix} x \\ -8 \end{pmatrix} = \begin{pmatrix} -6 \\ 9 \end{pmatrix}$

(c) $\begin{pmatrix} y \\ 3 \end{pmatrix} + \begin{pmatrix} -4 \\ 2x \end{pmatrix} = \begin{pmatrix} 6 \\ x \end{pmatrix}$

(d) $\begin{pmatrix} 2x \\ 5 \end{pmatrix} - \begin{pmatrix} y-3 \\ -10 \end{pmatrix} = \begin{pmatrix} 4 \\ 3y \end{pmatrix}$

ADVANCED LEVEL

15. The diagram shows three vectors **p**, **q** and **r**.



On a sheet of squared paper or graph paper, draw appropriate parallelograms to illustrate the following vector additions:

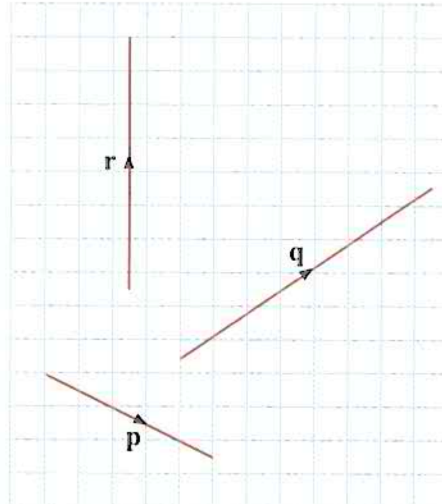
(a) **p** + **q**

(b) **q** + **p**

(c) **p** + **r**

(d) **q** + **r**

16. (a) Illustrate graphically the following vector sums using the vectors given in the diagram.



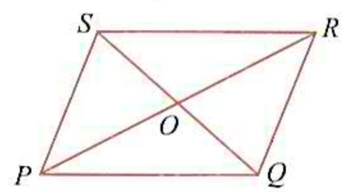
(i) **p** + **q** (ii) **q** + **p**

(iii) (**p** + **q**) + **r** (iv) **p** + (**q** + **r**)

(b) Is **p** + **q** = **q** + **p**? Explain.

(c) Is (**p** + **q**) + **r** = **p** + (**q** + **r**)? Explain.

17. $PQRS$ is a parallelogram. O is the point of intersection of its diagonals.



(a) Simplify

(i) $\overrightarrow{PQ} + \overrightarrow{PS}$,

(ii) $\overrightarrow{RO} - \overrightarrow{QO}$,

(iii) $\overrightarrow{PR} - \overrightarrow{SR} + \overrightarrow{SQ}$.

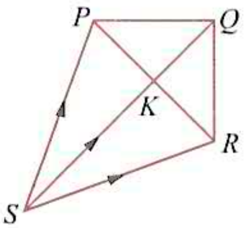
(b) If $\overrightarrow{PQ} = \mathbf{a}$ and $\overrightarrow{PS} = \mathbf{b}$, find in terms of **a** and/or **b**:

(i) $\overrightarrow{SR}$,

(ii) $\overrightarrow{PR}$,

(iii) $\overrightarrow{SQ}$.

18. In the figure below, the diagonals of $PQRS$ intersect at K. Find, for each of the following equations, a vector which can replace $\mathbf{u}$.



(a) $\overrightarrow{SK} + \mathbf{u} = \mathbf{0}$

(b) $\overrightarrow{SP} + \overrightarrow{PQ} + \mathbf{u} = \mathbf{0}$

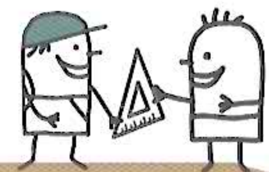
(c) $\overrightarrow{PS} + \overrightarrow{SK} + \overrightarrow{KR} = \mathbf{u}$

(d) $\overrightarrow{PK} + (-\overrightarrow{SK}) = \mathbf{u}$

(e) $\overrightarrow{PS} + (-\overrightarrow{RS}) = \mathbf{u}$

(f) $\overrightarrow{PQ} + \overrightarrow{QR} + (-\overrightarrow{PR}) = \mathbf{u}$

7.4 Scalar Multiples of a Vector



In Section 7.1, we have learnt that two vectors are **parallel** if they have the same or opposite direction, but they can have the same or different magnitudes.

Fig. 7.9 shows three parallel vectors where the length of $\mathbf{a}$ is twice the length of $\mathbf{b}$, and thrice the length of $\mathbf{c}$.

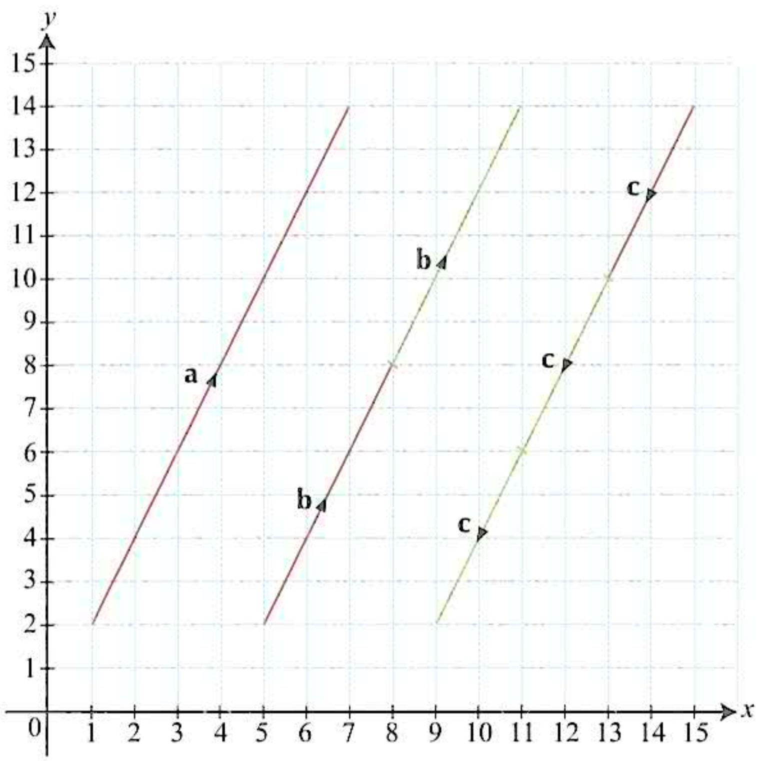


Fig. 7.9

We observe that $\mathbf{a} = \mathbf{b} + \mathbf{b} = 2\mathbf{b}$ and $\mathbf{a} = (-\mathbf{c}) + (-\mathbf{c}) + (-\mathbf{c}) = -3\mathbf{c}$.
$2\mathbf{b}$ and $3\mathbf{c}$ are called the **scalar multiple** of $\mathbf{b}$ and $\mathbf{c}$, respectively.

In general,

> if $\mathbf{a}$ and $\mathbf{b}$ are two **non-zero** and **parallel** vectors,
> then $\mathbf{a} = k\mathbf{b}$ for some scalar or real number $k \neq 0$.

In other words, if $\mathbf{a}$ and $\mathbf{b}$ are any two vectors and $\mathbf{a} = k\mathbf{b}$ for some real number k, then there are 3 possibilities. Either

(1) $\mathbf{a}$ and $\mathbf{b}$ are parallel,
(2) $\mathbf{a} = \mathbf{b} = \mathbf{0}$,
(3) $k = 0$.

Thinking Time

If $\mathbf{a} = k\mathbf{b}$, $\mathbf{a} \neq \mathbf{0}$, $\mathbf{b} \neq \mathbf{0}$ and $k \neq 0$, what does it mean if k is positive or negative?

ATTENTION

$2\begin{pmatrix} 3 \\ 6 \end{pmatrix}$ means

$2 \times \begin{pmatrix} 3 \\ 6 \end{pmatrix} = \begin{pmatrix} 2\times 3 \\ 2\times 6 \end{pmatrix} = \begin{pmatrix} 6 \\ 12 \end{pmatrix}$.

In Fig. 7.9, in terms of column vectors, $\mathbf{a} = \begin{pmatrix} 6 \\ 12 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} -2 \\ -4 \end{pmatrix}$.

$$\mathbf{a} = \begin{pmatrix} 6 \\ 12 \end{pmatrix} = 2\begin{pmatrix} 3 \\ 6 \end{pmatrix} = 2\mathbf{b} \text{ and } \mathbf{a} = \begin{pmatrix} 6 \\ 12 \end{pmatrix} = -3\begin{pmatrix} -2 \\ -4 \end{pmatrix} = -3\mathbf{c}.$$

Moreover, $|\mathbf{a}| = |2\mathbf{b}| = 2|\mathbf{b}|$ and $|\mathbf{a}| = |-3\mathbf{c}| = |-3||\mathbf{c}| = 3|\mathbf{c}|$.

ATTENTION

In Book 1, we have learnt that the absolute value of a number, e.g. –5, is 5. We write $|-5| = 5$. For a positive number, e.g. 5, $|5| = 5$.

There is a difference between $|k|$ and $|\mathbf{a}|$: $|k|$ is the absolute value of the scalar or real number k, while $|\mathbf{a}|$ is the magnitude of the vector $\mathbf{a}$.

In general,

> if $\mathbf{a} = \begin{pmatrix} x \\ y \end{pmatrix}$, then $k\mathbf{a} = \begin{pmatrix} kx \\ ky \end{pmatrix}$ and $|k\mathbf{a}| = |k||\mathbf{a}|$, for any real number k.

Worked Example 12

(Parallel Vectors and Opposite Vectors)

(a) State which of the following pairs of vectors are parallel.

(i) $\begin{pmatrix} 6 \\ 8 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ **(ii)** $\begin{pmatrix} 8 \\ -4 \end{pmatrix}, \begin{pmatrix} -6 \\ 3 \end{pmatrix}$

(iii) $\begin{pmatrix} 15 \\ -6 \end{pmatrix}, \begin{pmatrix} -5 \\ 3 \end{pmatrix}$

(b) Write down two vectors that are parallel to $\begin{pmatrix} -5 \\ 2 \end{pmatrix}$, one in the same direction, and one in the opposite direction.

Solution:

(a) (i) Since $\begin{pmatrix} 6 \\ 8 \end{pmatrix} = 2\begin{pmatrix} 3 \\ 4 \end{pmatrix}$, then $\begin{pmatrix} 6 \\ 8 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ are parallel.

(ii) Since $\begin{pmatrix} 8 \\ -4 \end{pmatrix} = -\frac{4}{3}\begin{pmatrix} -6 \\ 3 \end{pmatrix}$, then $\begin{pmatrix} 8 \\ -4 \end{pmatrix}$ and $\begin{pmatrix} -6 \\ 3 \end{pmatrix}$ are parallel.

(iii) If $\begin{pmatrix} 15 \\ -6 \end{pmatrix}$ and $\begin{pmatrix} -5 \\ 3 \end{pmatrix}$ are parallel, then there must be a value of k that satisfies

$\begin{pmatrix} 15 \\ -6 \end{pmatrix} = k\begin{pmatrix} -5 \\ 3 \end{pmatrix}$, but $15 = k(-5)$, i.e. $k = -3$ and $-6 = k(3)$, i.e. $k = -2 \neq -3$.

$\therefore \begin{pmatrix} 15 \\ -6 \end{pmatrix}$ and $\begin{pmatrix} -5 \\ 3 \end{pmatrix}$ are not parallel.

(b) A vector in the same direction as $\begin{pmatrix} -5 \\ 2 \end{pmatrix}$ is $2 \times \begin{pmatrix} -5 \\ 2 \end{pmatrix} = \begin{pmatrix} -10 \\ 4 \end{pmatrix}$.

A vector in the opposite direction as $\begin{pmatrix} -5 \\ 2 \end{pmatrix}$ is $-\begin{pmatrix} -5 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \end{pmatrix}$.

Problem Solving Tip

Observe:

$$\begin{pmatrix} -6 \\ 3 \end{pmatrix} \xrightarrow{\times k\,(=\,?)} \begin{pmatrix} 8 \\ -4 \end{pmatrix}$$

To obtain k, ask yourself:
$-6 \times ? = 8$, $3 \times ? = -4$
So, $k = \frac{8}{-6} = -\frac{4}{3}$.

If both values of k are different, then the two vectors are *not parallel*.

PRACTISE NOW 12

SIMILAR QUESTIONS

Exercise 7C Questions 1(a)–(c), 2(a)–(c), 6(a)–(c), 7(a)–(b), 13–14

1. (a) State which of the following pairs of vectors are parallel.

(i) $\begin{pmatrix} 6 \\ -9 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \end{pmatrix}$ **(ii)** $\begin{pmatrix} 14 \\ 18 \end{pmatrix}, \begin{pmatrix} -7 \\ 9 \end{pmatrix}$

(iii) $\begin{pmatrix} -3 \\ 6 \end{pmatrix}, \begin{pmatrix} 4 \\ -8 \end{pmatrix}$

(b) Write down two vectors that are parallel to $\begin{pmatrix} 4 \\ -3 \end{pmatrix}$, one in the same direction, and one in the opposite direction.

2. Given that $\begin{pmatrix} 4 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} 12 \\ p \end{pmatrix}$ are parallel vectors, find the value of p.

Worked Example 13

(Addition, Subtraction and Scalar Multiplication of Column Vectors)

(a) If $\mathbf{a} = \begin{pmatrix} 7 \\ -5 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$, express $2\mathbf{a} + 3\mathbf{b}$ as a column vector.

(b) If $\mathbf{u} = \begin{pmatrix} x \\ 4 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} -5 \\ y \end{pmatrix}$ and $\mathbf{u} - 2\mathbf{v} = \begin{pmatrix} 7 \\ 8 \end{pmatrix}$, find the value of x and of y.

Solution:

(a)

$$\begin{aligned} 2\mathbf{a} + 3\mathbf{b} &= 2\begin{pmatrix} 7 \\ -5 \end{pmatrix} + 3\begin{pmatrix} -2 \\ 4 \end{pmatrix} \\ &= \begin{pmatrix} 14 \\ -10 \end{pmatrix} + \begin{pmatrix} -6 \\ 12 \end{pmatrix} \\ &= \begin{pmatrix} 8 \\ 2 \end{pmatrix} \end{aligned}$$

(b)

$$\begin{aligned} \mathbf{u} - 2\mathbf{v} &= \begin{pmatrix} 7 \\ 8 \end{pmatrix} \\ \begin{pmatrix} x \\ 4 \end{pmatrix} - 2\begin{pmatrix} -5 \\ y \end{pmatrix} &= \begin{pmatrix} 7 \\ 8 \end{pmatrix} \\ \begin{pmatrix} x+10 \\ 4-2y \end{pmatrix} &= \begin{pmatrix} 7 \\ 8 \end{pmatrix} \end{aligned}$$

$\therefore \quad x + 10 = 7 \quad$ and $\quad 4 - 2y = 8$

i.e. $\quad x = -3 \qquad 2y = -4$

$\qquad\qquad\qquad\qquad y = -2$

$\therefore x = -3$ and $y = -2$

PRACTISE NOW 13

SIMILAR QUESTIONS

Exercise 7C Questions 3(a)–(c), 8(a)–(c), 15

1. If $\mathbf{u} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$, find a single column vector to represent the following:

(i) $\mathbf{u} + 3\mathbf{v}$ **(ii)** $3\mathbf{u} - 2\mathbf{v} - \mathbf{w}$

2. If $\mathbf{a} = \begin{pmatrix} x \\ y \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$ and $2\mathbf{a} + \mathbf{b} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$, find the value of x and of y.

Class Discussion

Graphical Representation of Vectors

Discuss in pairs.

Given that $\mathbf{a} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$, illustrate each of the following on a sheet of squared or graph paper.

(a) $2\mathbf{a} + 3\mathbf{b}$ **(b)** $2\mathbf{a} - 3\mathbf{b}$

Do you prefer to use the Triangle Law of Vector Addition or the Parallelogram Law of Vector Addition for **(a)**?

Do you prefer to use the Addition of Negative Vector or the Triangle Law of Vector Subtraction for **(b)**?

SIMILAR QUESTIONS

Exercise 7C Question 9

7.5 Expression of a Vector in Terms of Two Other Vectors

We have learnt that the sum or difference of two vectors is also a vector. Can we do the reverse? That is, can we express a vector as the sum or difference of two other vectors?

Fig. 7.10**(a)** shows 2 non-zero and non-parallel vectors $\mathbf{u}$ and $\mathbf{v}$, and the vector $\overrightarrow{AB}$.

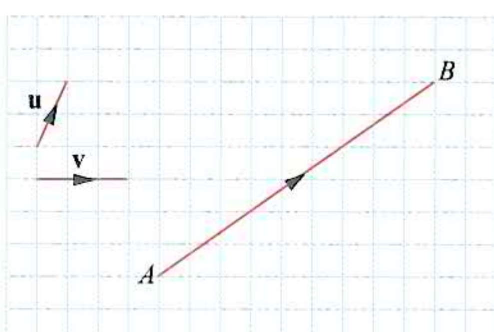


Fig. 7.10(a)

To express $\overrightarrow{AB}$ in terms of $\mathbf{u}$ and $\mathbf{v}$, we start from the point A and draw a line parallel to $\mathbf{u}$ (see Fig. 7.10(**b**)). Then we draw a line from B parallel to $\mathbf{v}$ (this line must be in the opposite direction as $\mathbf{v}$ in order to intersect the first line). Name the point of intersection of the two lines C.

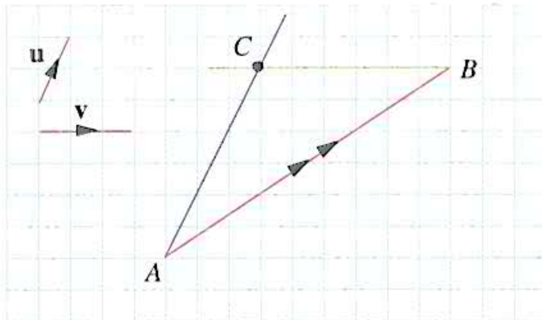


Fig. 7.10(b)

From the diagram, $\overrightarrow{AC} = 3\mathbf{u}$ and $\overrightarrow{CB} = 2\mathbf{v}$ (see Fig. 7.10(**c**)).

$\therefore \overrightarrow{AB} = 3\mathbf{u} + 2\mathbf{v}$

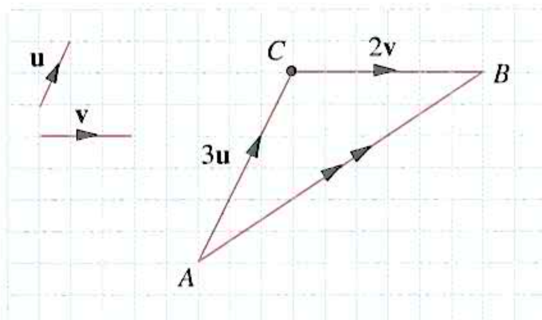


Fig. 7.10(c)

Alternatively, we can start from the point A and draw a line parallel to $\mathbf{v}$ first (see Fig. 7.10(**d**)). Then draw a line from B parallel to $\mathbf{u}$ to intersect the first line at C.

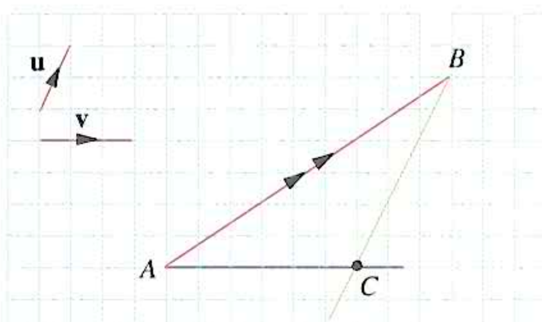


Fig. 7.10(d)

From the diagram, $\overrightarrow{AC} = 2\mathbf{v}$ and $\overrightarrow{CB} = 3\mathbf{u}$.

$\therefore \overrightarrow{AB} = 2\mathbf{v} + 3\mathbf{u} = 3\mathbf{u} + 2\mathbf{v}$

We observe that there is only one way to write the vector $\overrightarrow{AB}$ in terms of $\mathbf{u}$ and $\mathbf{v}$, i.e. $\overrightarrow{AB} = 3\mathbf{u} + 2\mathbf{v}$. We can also write $\overrightarrow{AB} = 2\mathbf{v} + 3\mathbf{u}$, as the order is not important.

ATTENTION

$3\mathbf{u} + 2\mathbf{v} = 2\mathbf{v} + 3\mathbf{u}$ as vector addition is commutative.

Class Discussion

Expressing a Vector in Terms of Two Other Vectors

Work in pairs.

Fig. 7.11 shows two non-zero and non-parallel vectors $\mathbf{u}$ and $\mathbf{v}$ and the vector $\overrightarrow{PQ}$. Express $\overrightarrow{PQ}$ in terms of $\mathbf{u}$ and $\mathbf{v}$.

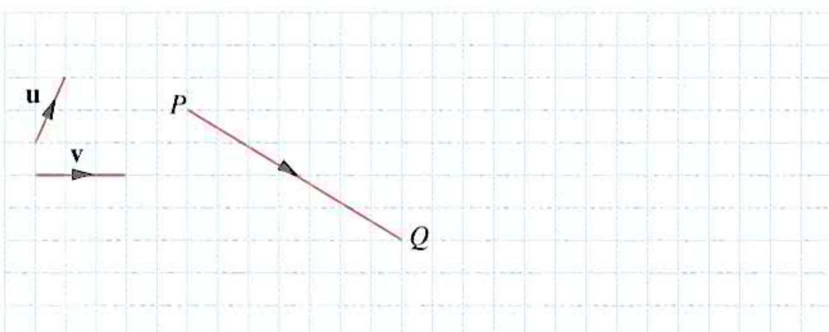


Fig. 7.11

SIMILAR QUESTIONS

Exercise 7C Question 10

Did your classmate and you get the same answer?

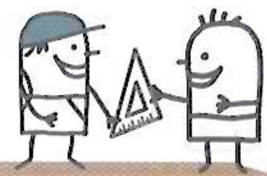
ATTENTION

'Coplanar vectors' means $\mathbf{u}$ and $\mathbf{v}$ must lie on the same plane.

In general, any vector $\overrightarrow{AB}$ can be expressed **uniquely** in terms of two other **non-zero** and **non-parallel** coplanar vectors $\mathbf{u}$ and $\mathbf{v}$, i.e.

$$\boxed{\overrightarrow{AB} = m\mathbf{u} + n\mathbf{v}.}$$

7.6 Position Vectors



In Section 7.1, we have learnt that we can express a vector lying on a Cartesian plane as a **column vector**.

For example, in Fig. 7.3 on page 218, $\overrightarrow{PQ} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$.

In fact, we can draw another vector $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$ on the Cartesian plane with a different starting point.

For example, in Fig. 7.4 on page 220, $\mathbf{a} = \mathbf{b} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$, but $\mathbf{a}$ and $\mathbf{b}$ have different starting points.

However, the **position vector** of a point P must have a fixed starting point. On a Cartesian plane, this starting point or reference point is usually the origin O.

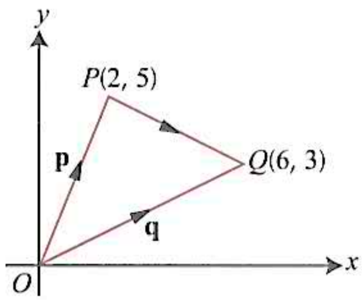


Fig. 7.12

Fig. 7.12 shows a point $P(2, 5)$. The position vector of P relative to O (or with respect to O) is $\overrightarrow{OP} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$. What is the position vector of Q relative to O?

Fig. 7.12 shows another vector $\overrightarrow{PQ}$, which does not start from O. However, we can express $\overrightarrow{PQ}$ in terms of the position vectors $\overrightarrow{OP}$ and $\overrightarrow{OQ}$. Using the Triangle Law of Vector Subtraction,

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \mathbf{q} - \mathbf{p}$$

start end end start

must be the same

Since the coordinates of P and Q are (2, 5) and (6, 3) respectively, then

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \begin{pmatrix} 6 \\ 3 \end{pmatrix} - \begin{pmatrix} 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \end{pmatrix}.$$

ATTENTION

We can also use the law of addition.

$$\begin{aligned} \overrightarrow{PQ} &= \overrightarrow{PO} + \overrightarrow{OQ} \\ &= -\overrightarrow{OP} + \overrightarrow{OQ} \\ &= -\mathbf{p} + \mathbf{q} \\ &= \mathbf{q} - \mathbf{p} \end{aligned}$$

In general,

$$\text{the position vector of } P(x, y) \text{ is } \overrightarrow{OP} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

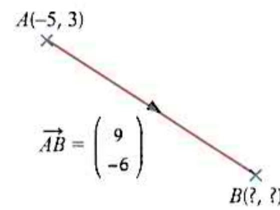
A vector $\overrightarrow{PQ}$ on the Cartesian plane can be expressed in terms of position vectors as follows:

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \mathbf{q} - \mathbf{p}$$

As mentioned in Section 7.2, vector $\overrightarrow{PQ}$ can be regarded as movement from P to Q and we call this a **translation** from P to Q. $\overrightarrow{PQ}$ becomes a **translation vector** which describes this movement. Translation vectors are expressed as column vectors, as shown in Worked Example 14 below.

ATTENTION

A translation can be represented by a **translation vector** $\overrightarrow{AB}$.



Worked Example 14

(Position Vectors)

(a) Write down the position vector of $P(2, -3)$ and of $Q(4, 1)$. Then express $\overrightarrow{PQ}$ as a column vector.

(b) A point $A(-5, 3)$ is translated by the **translation vector** $\overrightarrow{AB} = \begin{pmatrix} 9 \\ -6 \end{pmatrix}$ to the point B. Find the coordinates of B.

Solution:

(a) The position vector of P is $\overrightarrow{OP} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$.

The position vector of Q is $\overrightarrow{OQ} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$.

$$\therefore \overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \begin{pmatrix} 4 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

start end end start

must be the same

(b)

$$\overrightarrow{AB} = \begin{pmatrix} 9 \\ -6 \end{pmatrix}$$

$$\overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} 9 \\ -6 \end{pmatrix}$$

$$\overrightarrow{OB} - \begin{pmatrix} -5 \\ 3 \end{pmatrix} = \begin{pmatrix} 9 \\ -6 \end{pmatrix}$$

$$\overrightarrow{OB} = \begin{pmatrix} 9 \\ -6 \end{pmatrix} + \begin{pmatrix} -5 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} 4 \\ -3 \end{pmatrix}$$

$\therefore$ Coordinates of B are $(4, -3)$

ATTENTION

Alternative solution for **(a)**:

$$\overrightarrow{PQ} = \overrightarrow{PO} + \overrightarrow{OQ}$$
$$= -\overrightarrow{OP} + \overrightarrow{OQ}$$
$$= -\begin{pmatrix} 2 \\ -3 \end{pmatrix} + \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$
$$= \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

Alternative solution for **(b)**:
Point A is translated by vector $\overrightarrow{AB}$ to B.

i.e. $\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}$

$$\overrightarrow{OB} = \begin{pmatrix} -5 \\ 3 \end{pmatrix} + \begin{pmatrix} 9 \\ -6 \end{pmatrix}$$
$$= \begin{pmatrix} 4 \\ -3 \end{pmatrix}$$

PRACTISE NOW 14

SIMILAR QUESTIONS

Exercise 7C Questions 4(a)–(d), 5, 11–12, 16–17

(a) Write down the position vector of $P(8, -2)$ and of $Q(-1, 7)$. Then express $\overrightarrow{PQ}$ as a column vector.

(b) A point $A(6, -7)$ is translated by the translation vector $\overrightarrow{AB} = \begin{pmatrix} -4 \\ 5 \end{pmatrix}$ to the point B. Find the coordinates of B.

Exercise 7C

BASIC LEVEL

1. State which of the following pairs of vectors are parallel.

(a) $\begin{pmatrix} -2 \\ 1 \end{pmatrix}, \begin{pmatrix} -8 \\ 4 \end{pmatrix}$ **(b)** $\begin{pmatrix} 9 \\ 7 \end{pmatrix}, \begin{pmatrix} 18 \\ 21 \end{pmatrix}$

(c) $\begin{pmatrix} 6 \\ -8 \end{pmatrix}, \begin{pmatrix} -3 \\ 4 \end{pmatrix}$

2. Write down two vectors that are parallel to each of the following vectors, one in the same direction, and one in the opposite direction.

(a) $\begin{pmatrix} 8 \\ -7 \end{pmatrix}$ **(b)** $\begin{pmatrix} 3 \\ 9 \end{pmatrix}$

(c) $\begin{pmatrix} -6 \\ -2 \end{pmatrix}$

3. If $\mathbf{p} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$, $\mathbf{q} = \begin{pmatrix} 6 \\ -3 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} -3 \\ -4 \end{pmatrix}$, find a single column vector to represent the following:

(a) $\mathbf{p} + 2\mathbf{q}$ **(b)** $3\mathbf{p} - \frac{1}{2}\mathbf{q}$

(c) $4\mathbf{p} - 3\mathbf{q} + \mathbf{r}$

4. Write down the position vectors of the following points as column vectors.

(a) $A(4, 7)$ **(b)** $B(-2, 5)$

(c) $C(6, -1)$ **(d)** $D(-4, -9)$

5. If P, Q and R are the points $(3, -2)$, $(2, -4)$ and $(2, 3)$ respectively, express the following as column vectors.

(i) $\overrightarrow{PQ}$ **(ii)** $\overrightarrow{QR}$

(iii) $\overrightarrow{RP}$ **(iv)** $\overrightarrow{PR}$

INTERMEDIATE LEVEL

6. State which of the following pairs of vectors are parallel.

(a) $\begin{pmatrix} 6 \\ -3 \end{pmatrix}, \begin{pmatrix} -4 \\ 2 \end{pmatrix}$ **(b)** $\begin{pmatrix} -5 \\ 15 \end{pmatrix}, \begin{pmatrix} -3 \\ 9 \end{pmatrix}$

(c) $\begin{pmatrix} 7 \\ -8 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \end{pmatrix}$

7. **(a)** Given that $\begin{pmatrix} 5 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 20 \\ p \end{pmatrix}$ are parallel vectors, find the value of p.

(b) Given that $\begin{pmatrix} h \\ 12 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ -9 \end{pmatrix}$ are parallel vectors, find the value of h.

8. For each of the following, find the value of x and of y.

(a) $\mathbf{a} = \begin{pmatrix} x \\ y \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ and $\mathbf{a} + 2\mathbf{b} = \begin{pmatrix} 8 \\ 9 \end{pmatrix}$.

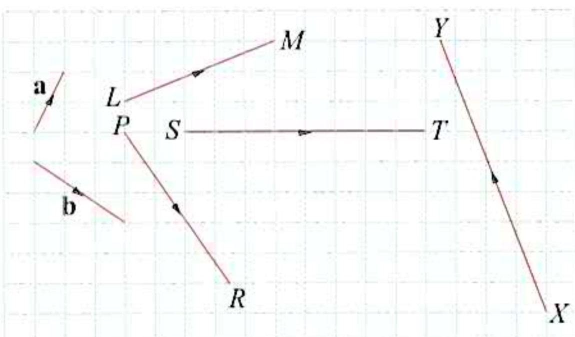
(b) $\mathbf{u} = \begin{pmatrix} 2 \\ y \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} x \\ 2 \end{pmatrix}$ and $4\mathbf{u} + \mathbf{v} = 2\begin{pmatrix} \frac{1}{2} \\ 9 \end{pmatrix}$.

(c) $\mathbf{p} = \begin{pmatrix} x \\ 5 \end{pmatrix}$, $\mathbf{q} = \begin{pmatrix} 6 \\ y \end{pmatrix}$ and $5\mathbf{p} - 2\mathbf{q} = \begin{pmatrix} 3 \\ 23 \end{pmatrix}$.

9. Given that $\mathbf{a} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, illustrate each of the following on a sheet of squared paper or graph paper.

(a) $2\mathbf{a} + \mathbf{b}$
(b) $3\mathbf{a} + 2\mathbf{b}$
(c) $\mathbf{a} - 2\mathbf{b}$
(d) $2\mathbf{a} - 3\mathbf{b}$
(e) $4\mathbf{a} + 3\mathbf{b}$
(f) $-3\mathbf{a} + 4\mathbf{b}$

10. The diagram below shows 2 non-parallel vectors $\mathbf{a}$ and $\mathbf{b}$. Using the squared grid below, express $\overrightarrow{LM}$, $\overrightarrow{PR}$, $\overrightarrow{ST}$ and $\overrightarrow{XY}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.



11. A point $A(-3, 8)$ is translated by the translation vector $\overrightarrow{AB} = \begin{pmatrix} -2 \\ -4 \end{pmatrix}$ to the point B. Find the coordinates of B.

12. $\overrightarrow{AB} = \begin{pmatrix} 9 \\ -15 \end{pmatrix}$ and $\overrightarrow{CD} = \frac{2}{3}\overrightarrow{AB}$.

(i) Express $\overrightarrow{CD}$ as a column vector.
(ii) Given that A is the point $(-2, 7)$, find the coordinates of the point B.
(iii) Given that D is the point $(8, -5)$, find the coordinates of the point C.

ADVANCED LEVEL

13. If $\begin{pmatrix} a \\ b \end{pmatrix}$ and $\begin{pmatrix} c \\ d \end{pmatrix}$ are two parallel vectors, explain why $\frac{a}{c} = \frac{b}{d}$.

14. It is given that $\mathbf{u} = \begin{pmatrix} -15 \\ 8 \end{pmatrix}$. If $\mathbf{u} = k\mathbf{v}$ where k is a positive constant and $|\mathbf{v}| = 51$, find the value of k. Hence find $\mathbf{v}$.

15. Given that $\overrightarrow{AB} = \begin{pmatrix} -3 \\ 5 \end{pmatrix}$, $\overrightarrow{CD} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$, $\overrightarrow{EF} = \begin{pmatrix} k \\ 7.5 \end{pmatrix}$ and $\overrightarrow{PQ} = \begin{pmatrix} \frac{1}{4} \\ 1 \end{pmatrix}$,

(i) express $2\overrightarrow{AB} + 5\overrightarrow{CD}$ as a column vector,
(ii) find the value of k if $\overrightarrow{EF}$ is parallel to $\overrightarrow{AB}$,
(iii) explain why $\overrightarrow{PQ}$ is parallel to $\overrightarrow{CD}$.

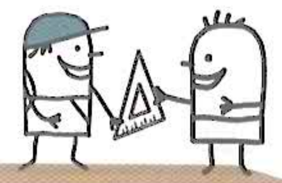
16. L is the point $(-3, 2)$ and M is the point $(t, 6)$.

(i) Express $\overrightarrow{LM}$ as a column vector.
(ii) If $\overrightarrow{LM}$ is parallel to $\mathbf{p} = \begin{pmatrix} 8 \\ 1 \end{pmatrix}$, find the value of t.
(iii) If instead, $|\overrightarrow{LM}| = |\mathbf{p}|$, find the two possible values of t.

17. P is the point $(2, -3)$ and $\overrightarrow{PQ} = \begin{pmatrix} 8 \\ -2 \end{pmatrix}$.

(i) Find the coordinates of Q.
(ii) Find the gradient of PQ.
(iii) If $\overrightarrow{PQ} = \begin{pmatrix} x \\ y \end{pmatrix}$, express the gradient of PQ in terms of x and of y.
(iv) If the gradient of PQ is $\frac{y}{x}$, express $\overrightarrow{PQ}$ in terms of x and of y.

7.7 Applications of Vectors



Vectors in Real-World Contexts

In the chapter opener, we observe that both magnitude and direction are necessary to describe the position of a place from another place. Search for the video 'The Waggle Dance of the Honeybee' on the Internet. It tells how honeybees, after discovering a new source of food, communicate the direction and distance of the food source to other honeybees when they return to their hive.

Internet Resources

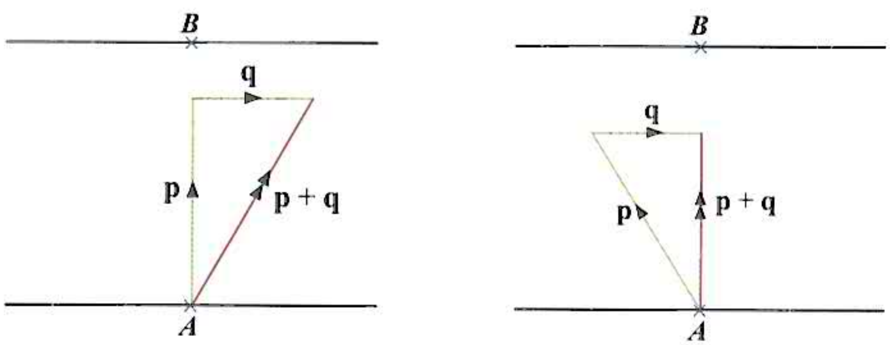
An application of vectors is in GPS (Global Positioning System) which makes use of complex vectors and geometric trilateration to determine the positions of objects. Search on the Internet for more information.

This is done by doing a waggle dance in the direction of the food source with reference to the direction of the Sun. What is amazing is that when the Sun changes its position in the sky as time passes, the honeybees will adjust the angle between the direction of the Sun and the direction of the food source accordingly. The distance of the food source from the hive is communicated by the duration of the dance. In general, every second of the dance indicates one kilometre from the food source. Therefore, we see the importance of vectors in real life.

In the Thinking Time in Section 7.1 on page 217, we have thought of some real-life examples of vectors. But what about real-life examples of the resultant of two vectors?

For example, Fig. 7.13 shows a boat crossing a river from A to B.

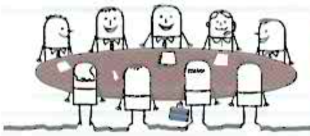
In Fig. 7.13**(a)**, the boat tries to travel in the direction of B from A, as indicated by $\mathbf{p}$. However, the water current, as indicated by $\mathbf{q}$, causes the boat to travel in the direction indicated by the resultant vector $\mathbf{p} + \mathbf{q}$. So the boat will not reach B.



(a) **(b)**

Fig. 7.13

To reach B from A, the boat must travel in the direction indicated by $\mathbf{p}$ in Fig 7.13**(b)**. Then the boat will end up travelling in the direction indicated by the resultant vector $\mathbf{p} + \mathbf{q}$.



Class Discussion

Real-Life Examples of Resultant Vectors

Discuss in pairs.

Think of other real life examples to illustrate the resultant of two vectors. It can be vector addition or subtraction.

Solving Geometric Problems involving Vectors

Vectors can be used to solve some geometric problems.

Worked Example 15

(Geometric Problems involving Vectors)

The coordinates of A, B and D are (1, 2), (6, 3) and (2, 8) respectively. Find the coordinates of C if $ABCD$ is a parallelogram.

Solution:

Since $ABCD$ is a parallelogram, then

$$\overrightarrow{DC} = \overrightarrow{AB}$$

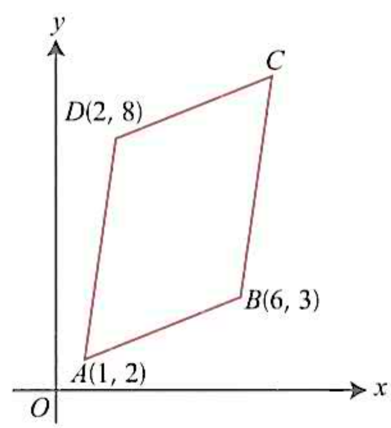
$$\overrightarrow{OC} - \overrightarrow{OD} = \overrightarrow{OB} - \overrightarrow{OA}$$

$$\overrightarrow{OC} - \begin{pmatrix} 2 \\ 8 \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\overrightarrow{OC} = \begin{pmatrix} 6 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 2 \\ 8 \end{pmatrix}$$

$$= \begin{pmatrix} 7 \\ 9 \end{pmatrix}$$

$\therefore$ The coordinates of C are (7, 9).



Problem Solving Tip

1. Obtain a vector equation relating the position vectors of the known points and C.
2. We use the properties of parallelograms here, i.e. $\overrightarrow{DC} = \overrightarrow{AB}$.
3. We can also use $\overrightarrow{AD} = \overrightarrow{BC}$ to obtain $\overrightarrow{OC}$.

$$\overrightarrow{OD} - \overrightarrow{OA} = \overrightarrow{OD} - \overrightarrow{OB}$$

$$\begin{pmatrix} 2 \\ 8 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \overrightarrow{OC} - \begin{pmatrix} 6 \\ 3 \end{pmatrix}$$

$$\overrightarrow{OC} = \begin{pmatrix} 2 \\ 8 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 6 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} 7 \\ 9 \end{pmatrix}$$

PRACTISE NOW 15

The coordinates of A, B and D are (3, 7), (–1, 2) and (5, –4) respectively. Find the coordinates of C if $ABCD$ is a parallelogram.

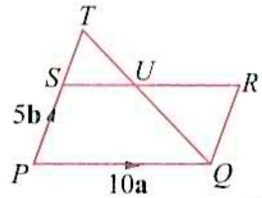
SIMILAR QUESTIONS

Exercise 7D Questions 1, 6, 13

Worked Example 16

(Geometric Problems involving Vectors)

In the diagram, $SPQR$ is a parallelogram where $\overrightarrow{PQ} = 10\mathbf{a}$ and $\overrightarrow{PS} = 5\mathbf{b}$.



The point U on SR is such that $SU = \frac{2}{5}SR$. The lines PS and QU, when produced, meet at T.

(a) Express the following in terms of **a** and/or **b**.

(i) $\overrightarrow{PR}$ **(ii)** $\overrightarrow{SU}$ **(iii)** $\overrightarrow{UR}$ **(iv)** $\overrightarrow{TU}$

(b) Calculate the value of

(i) $\dfrac{\text{area of } \triangle TSU}{\text{area of } \triangle QRU}$, **(ii)** $\dfrac{\text{area of } \triangle TSU}{\text{area of } \triangle PSU}$.

Solution:

(a) (i) $\overrightarrow{PR} = \overrightarrow{PQ} + \overrightarrow{QR} = \overrightarrow{PQ} + \overrightarrow{PS}$

$= 10\mathbf{a} + 5\mathbf{b}$

or $\overrightarrow{PR} = \overrightarrow{PQ} + \overrightarrow{PS}$ (Parallelogram Law of Vector Addition)

$= 10\mathbf{a} + 5\mathbf{b}$

(ii) $SU = \frac{2}{5}SR$

$\overrightarrow{SU} = \frac{2}{5}\overrightarrow{SR}$

$= \frac{2}{5}\overrightarrow{PQ}$

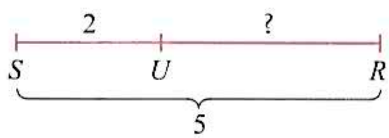
$= \frac{2}{5}(10\mathbf{a})$

$= 4\mathbf{a}$

(iii) $SU = \frac{2}{5}SR$, i.e. $\frac{SU}{SR} = \frac{2}{5}$.

From the diagram, $UR = 3$ parts.

$\therefore \frac{UR}{SR} = \frac{3}{5}$, i.e. $UR = \frac{3}{5}SR$.



$\overrightarrow{UR} = \frac{3}{5}\overrightarrow{SR} = \frac{3}{5}\overrightarrow{PQ} = \frac{3}{5}(10\mathbf{a}) = 6\mathbf{a}$

(iv) $\triangle TSU$ and $\triangle QRU$ are similar. (corresponding angles equal)

$\therefore \frac{TU}{QU} = \frac{SU}{RU}$, i.e. $\frac{TU}{UQ} = \frac{SU}{UR} = \frac{2}{3}$, from the diagram in **(iii)**.

$\therefore TU = \frac{2}{3}UQ$

$\overrightarrow{TU} = \frac{2}{3}\overrightarrow{UQ}$

$= \frac{2}{3}(\overrightarrow{UR} + \overrightarrow{RQ})$

$= \frac{2}{3}(\overrightarrow{UR} - \overrightarrow{PS})$

$= \frac{2}{3}(6\mathbf{a} - 5\mathbf{b})$

Problem Solving Tip

For **(a)(ii)**, to decide whether $SU = \frac{2}{5}SR$ implies $\overrightarrow{SU} = \frac{2}{5}\overrightarrow{SR}$ or $\overrightarrow{SU} = \frac{2}{5}\overrightarrow{RS}$, we need to check the direction of $\overrightarrow{SU}$ and $\overrightarrow{SR}$ or $\overrightarrow{RS}$ in the diagram.

Similarly for **(a)(iii)**, to decide whether $UR = \frac{3}{5}SR$ implies $\overrightarrow{UR} = \frac{3}{5}\overrightarrow{SR}$ or $\overrightarrow{UR} = \frac{3}{5}\overrightarrow{RS}$, we need to check the direction of $\overrightarrow{UR}$ and $\overrightarrow{SR}$ or $\overrightarrow{RS}$ in the diagram.

(b) (i) Since $\triangle TSU$ and $\triangle QRU$ are similar, then

$$\frac{\text{area of } \triangle TSU}{\text{area of } \triangle QRU} = \left(\frac{TU}{QU}\right)^2 \text{ (where } \frac{TU}{QU} = \frac{2}{3} \text{ in (a)(iii))}$$

$$= \left(\frac{2}{3}\right)^2$$

$$= \frac{4}{9}$$

Problem Solving Tip

For **(b)(i)**, since the two triangles are similar, we can use the formula:

$$\frac{A_2}{A_2} = \left(\frac{l_2}{l_1}\right)^2$$

(ii) $\overrightarrow{ST} = \overrightarrow{SU} + \overrightarrow{UT}$

$= \overrightarrow{SU} - \overrightarrow{TU}$

$= 4\mathbf{a} - \frac{2}{3}(6\mathbf{a} - 5\mathbf{b})$

$= 4\mathbf{a} - 4\mathbf{a} + \frac{10}{3}\mathbf{b}$

$= \frac{10}{3}\mathbf{b}$

For **(b)(ii)**, since the two triangles have the same height, then

$$\frac{\text{area of } \triangle TSU}{\text{area of } \triangle PSU} = \frac{\frac{1}{2} \times ST \times h}{\frac{1}{2} \times PS \times h} = \frac{ST}{PS}.$$

Hence we need to find $\overrightarrow{ST}$.

$$\frac{\text{Area of } \triangle TSU}{\text{Area of } \triangle PSU} = \frac{\frac{1}{2} \times ST \times h}{\frac{1}{2} \times PS \times h},$$ where h is the common height of $\triangle TSU$ and $\triangle PSU$.

$$= \frac{ST}{PS}$$

$$= \frac{\left|\frac{10}{3}\mathbf{b}\right|}{|5\mathbf{b}|}$$

$$= \frac{\frac{10}{3}|\mathbf{b}|}{5|\mathbf{b}|}$$

$$= \frac{10}{3} \div 5$$

$$= \frac{10}{3} \times \frac{1}{5}$$

$$= \frac{2}{3}$$

An alternative (and shorter) solution is to use the idea of similar triangles.

$$\frac{ST}{PS} = \frac{ST}{RQ}$$

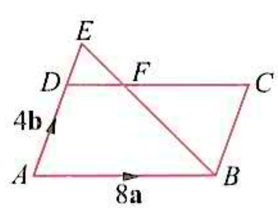
$$= \frac{2}{3} \text{ } (\triangle TSU \text{ is similar to } \triangle QRU)$$

PRACTISE NOW 16

SIMILAR QUESTIONS

Exercise 7D Questions 4, 5

In the diagram, $DABC$ is a parallelogram where $\overrightarrow{AB} = 8\mathbf{a}$ and $\overrightarrow{AD} = 4\mathbf{b}$. The point F on DC is such that $DF = \frac{1}{4}DC$. The lines AD and BF, when produced, meet at E.



(a) Express the following in terms of **a** and/or **b**.

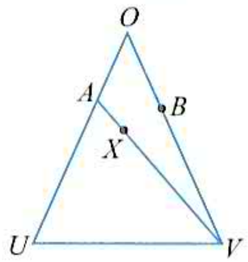
(i) $\overrightarrow{AC}$ **(ii)** $\overrightarrow{DF}$ **(iii)** $\overrightarrow{FC}$ **(iv)** $\overrightarrow{EF}$

(b) Calculate the value of

(i) $\frac{\text{area of } \triangle EDF}{\text{area of } \triangle BCF}$, **(ii)** $\frac{\text{area of } \triangle EDF}{\text{area of } \triangle ADF}$.

Worked Example 17

(Geometric Problems involving Vectors)



In the diagram, $\overrightarrow{OU} = 15\mathbf{u}$, $\overrightarrow{OV} = 9\mathbf{v}$, $\overrightarrow{OA} = \frac{1}{5}\overrightarrow{OU}$ and $\overrightarrow{OB} = \frac{1}{3}\overrightarrow{OV}$.

(i) Find the vectors $\overrightarrow{UV}$ and $\overrightarrow{AB}$ in terms of $\mathbf{u}$ and $\mathbf{v}$.

(ii) Given that $\overrightarrow{AX} = \frac{1}{5}\overrightarrow{AV}$, express the vector $\overrightarrow{XB}$ in terms of $\mathbf{u}$ and $\mathbf{v}$.

Solution:

(i) $\overrightarrow{UV} = \overrightarrow{OV} - \overrightarrow{OU} = 9\mathbf{v} - 15\mathbf{u}$ or $3(3\mathbf{v} - 5\mathbf{u})$

$\overrightarrow{OA} = \frac{1}{5}\overrightarrow{OU} = \frac{1}{5}(15\mathbf{u}) = 3\mathbf{u}$

$\overrightarrow{OB} = \frac{1}{3}\overrightarrow{OV} = \frac{1}{3}(9\mathbf{v}) = 3\mathbf{v}$

$\therefore \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = 3\mathbf{v} - 3\mathbf{u}$ or $3(\mathbf{v} - \mathbf{u})$

(ii) Since $\overrightarrow{AX} = \frac{1}{5}\overrightarrow{AV}$, then $\overrightarrow{OX} - \overrightarrow{OA} = \frac{1}{5}(\overrightarrow{OV} - \overrightarrow{OA})$.

$$\overrightarrow{OX} - 3\mathbf{u} = \frac{1}{5}(9\mathbf{v} - 3\mathbf{u})$$

$$\begin{aligned}\overrightarrow{OX} &= \frac{9}{5}\mathbf{v} - \frac{3}{5}\mathbf{u} + \frac{15}{5}\mathbf{u}\\ &= \frac{9}{5}\mathbf{v} + \frac{12}{5}\mathbf{u}\\ &= \frac{3}{5}(4\mathbf{u} + 3\mathbf{v})\end{aligned}$$

$$\begin{aligned}\therefore \overrightarrow{XB} = \overrightarrow{OB} - \overrightarrow{OX} &= 3\mathbf{v} - \frac{3}{5}(4\mathbf{u} + 3\mathbf{v})\\ &= \frac{15}{5}\mathbf{v} - \frac{12}{5}\mathbf{u} - \frac{9}{5}\mathbf{v}\\ &= \frac{6}{5}\mathbf{v} - \frac{12}{5}\mathbf{u}\\ &= \frac{6}{5}(\mathbf{v} - 2\mathbf{u})\end{aligned}$$

Problem Solving Tip

A common approach is to express a given vector equation in terms of position vectors.

For example, in **(ii)**, to obtain $\overrightarrow{XB}$ in terms of $\mathbf{u}$ and $\mathbf{v}$, we need to obtain $\overrightarrow{OX}$. This can be done by breaking down the given equation $\overrightarrow{AX} = \frac{1}{5}\overrightarrow{AV}$ in terms of position vectors.

Alternatively, we can use the Triangle Law of Vector Addition to express $\overrightarrow{XB}$ in terms of other known vectors.

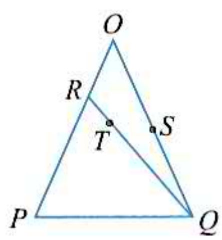
$$\begin{aligned}\overrightarrow{XB} &= \overrightarrow{XV} + \overrightarrow{VB}\\ &= \frac{4}{5}\overrightarrow{AV} + \frac{2}{3}\overrightarrow{VO}\\ &= \frac{4}{5}(\overrightarrow{AO} + \overrightarrow{OV}) + \frac{2}{3}\overrightarrow{VO}\\ &= \frac{4}{5}\overrightarrow{AO} + \frac{4}{5}\overrightarrow{OV} - \frac{2}{3}\overrightarrow{OV}\\ &= \frac{4}{5}\overrightarrow{AO} + \frac{2}{15}\overrightarrow{OV}\\ &= \frac{4}{5}(-3\mathbf{u}) + \frac{2}{15}(9\mathbf{v})\\ &= -\frac{12}{5}\mathbf{u} + \frac{6}{5}\mathbf{v}\\ &= \frac{6}{5}(\mathbf{v} - 2\mathbf{u})\end{aligned}$$

PRACTISE NOW 17

SIMILAR QUESTIONS

Exercise 7D Questions 2, 3, 7–12, 14–17

In the diagram $\overrightarrow{OP} = 9\mathbf{p}$, $\overrightarrow{OQ} = 3\mathbf{q}$, $\overrightarrow{OR} = \frac{1}{3}\overrightarrow{OP}$ and $\overrightarrow{OS} = \frac{1}{2}\overrightarrow{OQ}$.



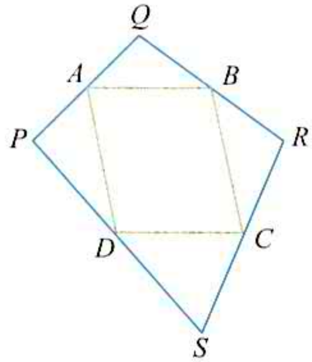
(i) Find the vectors $\overrightarrow{PQ}$ and $\overrightarrow{RS}$ in terms of $\mathbf{p}$ and $\mathbf{q}$.

(ii) Given that $\overrightarrow{RT} = \frac{1}{4}\overrightarrow{RQ}$, express the vector $\overrightarrow{TS}$ in terms of $\mathbf{p}$ and $\mathbf{q}$.

Worked Example 18

(Geometric Problems involving Vectors)

In the diagram, $PQRS$ is a quadrilateral, and A, B, C and D are the midpoints of PQ, QR, RS and SP respectively.



Show that

(i) PR is parallel to AB and $PR = 2AB$,

(ii) $ABCD$ is a parallelogram.

Solution:

(i) Let $\overrightarrow{QA} = \mathbf{a}$ and $\overrightarrow{QB} = \mathbf{b}$.

Then, $\overrightarrow{AB} = \overrightarrow{QB} - \overrightarrow{QA} = \mathbf{b} - \mathbf{a}$ (Triangle Law of Vector Subtraction)

Since $\overrightarrow{QP} = 2\overrightarrow{QA} = 2\mathbf{a}$ and $\overrightarrow{QR} = 2\overrightarrow{QB} = 2\mathbf{b}$,

Then, $\overrightarrow{PR} = \overrightarrow{QR} - \overrightarrow{QP} = 2\mathbf{b} - 2\mathbf{a}$

$= 2(\mathbf{b} - \mathbf{a})$

$= 2\overrightarrow{AB}$

Since $\overrightarrow{PR} = 2\overrightarrow{AB}$, then PR is parallel to AB and $PR = 2AB$.

(ii) Using the same reasoning in **(i)** for ΔSPR, we can show that PR is parallel to DC and $PR = 2DC$.

$\therefore$ AB is parallel to DC and $AB = DC$, i.e. $ABCD$ is a parallelogram.

Problem Solving Tip

(i) Showing that $PR \parallel AB$ and $PR = 2AB$ is the same as showing that $\overrightarrow{PR} = 2\overrightarrow{AB}$ (where $\overrightarrow{PR}$ and $\overrightarrow{AB}$ are in the same direction as shown in the diagram).

(ii) Showing that $ABCD$ is a parallelogram is the same as showing that $\overrightarrow{AB} = \overrightarrow{DC}$ (or AB is parallel to DC and $AB = DC$).

PRACTISE NOW 18

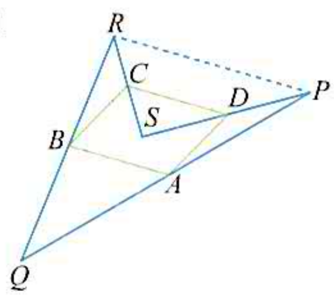
SIMILAR QUESTIONS

Exercise 7D Question 18

In the diagram, $PQRS$ is a quadrilateral, and A, B, C and D are the midpoints of PQ, QR, RS and SP respectively. Show that

(i) PR is parallel to AB and $PR = 2AB$,

(ii) $ABCD$ is a parallelogram.

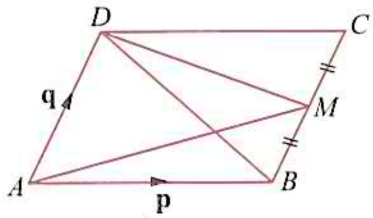


Exercise 7D

BASIC LEVEL

1. The coordinates of A, B and D are (2, 3), (7, 5) and (4, 9) respectively. Find the coordinates of C if $ABCD$ is a parallelogram.

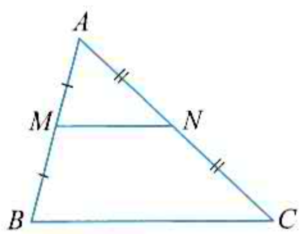
2. $ABCD$ is a parallelogram with M as the midpoint of BC.



If $\overrightarrow{AB} = \mathbf{p}$ and $\overrightarrow{AD} = \mathbf{q}$, express in terms of $\mathbf{p}$ and/or $\mathbf{q}$,

(i) $\overrightarrow{CM}$, **(ii)** $\overrightarrow{DB}$, **(iii)** $\overrightarrow{AM}$, **(iv)** $\overrightarrow{MD}$.

3.

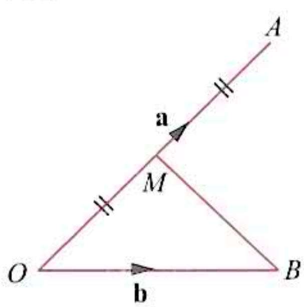


In the diagram, if $\overrightarrow{AB} = \mathbf{u}$, $\overrightarrow{AC} = \mathbf{v}$, and M and N are the midpoints of AB and AC respectively, express in terms of $\mathbf{u}$ and/or $\mathbf{v}$,

(i) $\overrightarrow{BC}$, **(ii)** $\overrightarrow{AM}$, **(iii)** $\overrightarrow{AN}$, **(iv)** $\overrightarrow{MN}$.

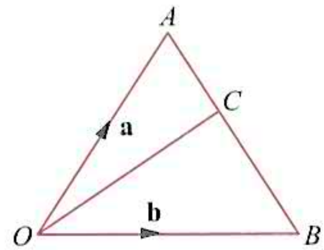
What can you say about $\overrightarrow{BC}$ and $\overrightarrow{MN}$?

4. In the diagram, $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$ and M is the midpoint of OA.



Find $\overrightarrow{BM}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

5. Given that $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$ and $\overrightarrow{AC} = \frac{2}{3}\overrightarrow{CB}$, find in terms of $\mathbf{a}$ and $\mathbf{b}$,

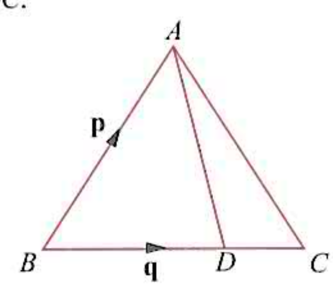


(i) $\overrightarrow{AB}$, **(ii)** $\overrightarrow{AC}$, **(iii)** $\overrightarrow{OC}$.

INTERMEDIATE LEVEL

6. The coordinates of P, Q and R are (1, 0), (4, 2) and (5, 4) respectively. Use a vector method to determine the coordinates of S if
   (i) $PQRS$ is a parallelogram,
   (ii) $PRQS$ is a parallelogram.

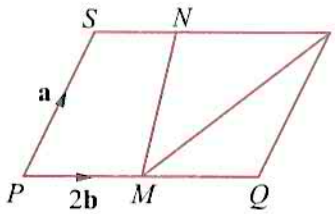
7. In the diagram, D is a point on BC such that $BD = 3DC$.



Given that $\overrightarrow{BA} = \mathbf{p}$ and $\overrightarrow{BD} = \mathbf{q}$, express in terms of $\mathbf{p}$ and/or $\mathbf{q}$,

(i) $\overrightarrow{BC}$, **(ii)** $\overrightarrow{AD}$, **(iii)** $\overrightarrow{CA}$.

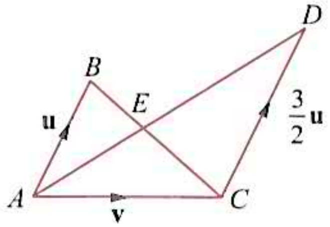
8. In the diagram, $PQRS$ is a parallelogram. M is the midpoint of PQ and N is on SR such that $SR = 3SN$.



Given that $\overrightarrow{PS} = \mathbf{a}$ and $\overrightarrow{PM} = 2\mathbf{b}$, express in terms of $\mathbf{a}$ and/or $\mathbf{b}$,

(i) $\overrightarrow{MR}$, **(ii)** $\overrightarrow{RN}$, **(iii)** $\overrightarrow{NM}$.

9. In the diagram, $\overrightarrow{AB} = \mathbf{u}$, $\overrightarrow{AC} = \mathbf{v}$, $\overrightarrow{CD} = \frac{3}{2}\mathbf{u}$, and $\overrightarrow{BE} = \frac{2}{5}\overrightarrow{BC}$.

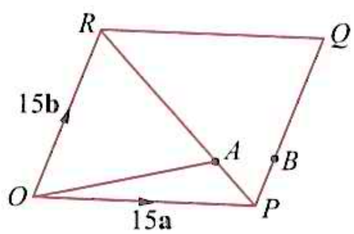


Express in terms of $\mathbf{u}$ and $\mathbf{v}$,

(i) $\overrightarrow{BC}$, **(ii)** $\overrightarrow{BE}$, **(iii)** $\overrightarrow{AD}$,

(iv) $\overrightarrow{AE}$, **(v)** $\overrightarrow{BD}$.

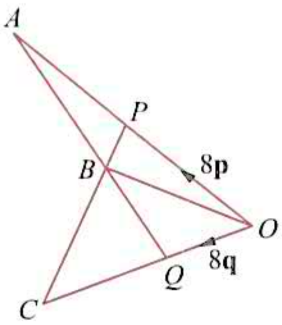
10.



$OPQR$ is a parallelogram. The point A on PR is such that $\overrightarrow{AR} = \frac{3}{4}\overrightarrow{PR}$. The point B on PQ is such that $\overrightarrow{PB} = \frac{1}{3}\overrightarrow{PQ}$. Given that $\overrightarrow{OP} = 15\mathbf{a}$ and $\overrightarrow{OR} = 15\mathbf{b}$, express the following vectors in terms of $\mathbf{a}$ and $\mathbf{b}$.

(i) $\overrightarrow{PR}$ **(ii)** $\overrightarrow{PA}$ **(iii)** $\overrightarrow{OA}$ **(iv)** $\overrightarrow{OB}$

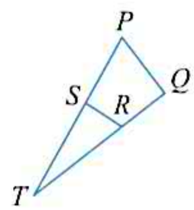
11.



OPA and OQC are straight lines, and PC intersects QA at B. Given that $\overrightarrow{OQ} = \frac{2}{3}\overrightarrow{QC}$, $\frac{PB}{BC} = \frac{1}{3}$, $\overrightarrow{OP} = 8\mathbf{p}$ and $\overrightarrow{OQ} = 8\mathbf{q}$, express the following vectors as simply as possible in terms of $\mathbf{p}$ and $\mathbf{q}$.

(i) $\overrightarrow{PC}$ **(ii)** $\overrightarrow{PB}$ **(iii)** $\overrightarrow{OB}$ **(iv)** $\overrightarrow{QB}$

12. Relative to the origin O which is not shown in the diagram, P is the point (1, 11), Q is the point (2, 8), R is the point (–1, 7), S is the point (–2, 8) and T is the point (–4, 6).



(a) Express the following as column vectors.

(i) $\overrightarrow{PQ}$ **(ii)** $\overrightarrow{SR}$

(iii) $\overrightarrow{RQ}$ **(iv)** $\overrightarrow{TQ}$

(b) Find the numerical value of the ratio $\frac{RQ}{TQ}$.

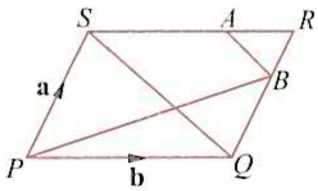
ADVANCED LEVEL

13. Given that A is the point (1, 2), $\overrightarrow{AB} = \begin{pmatrix} 4 \\ -5 \end{pmatrix}$, $\overrightarrow{AC} = \begin{pmatrix} 6 \\ 3 \end{pmatrix}$ and that M is the midpoint of BC, find

(i) $\overrightarrow{BC}$, **(ii)** $\overrightarrow{AM}$,

(iii) the coordinates of the point D such that $ABCD$ is a parallelogram.

14. $PQRS$ is a parallelogram. $\overrightarrow{BQ} = 2\overrightarrow{RB}$, $\overrightarrow{AR} = \frac{1}{3}\overrightarrow{SR}$, $\overrightarrow{PS} = \mathbf{a}$ and $\overrightarrow{PQ} = \mathbf{b}$.



(a) Express in terms of $\mathbf{a}$ and/or $\mathbf{b}$,

(i) $\overrightarrow{SA}$, **(ii)** $\overrightarrow{QB}$, **(iii)** $\overrightarrow{PB}$,

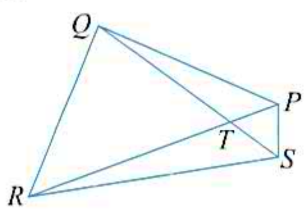
(iv) $\overrightarrow{QS}$, **(v)** $\overrightarrow{BA}$.

(b) Calculate the value of

(i) $\dfrac{BA}{QS}$, **(ii)** $\dfrac{\text{area of}\triangle ABR}{\text{area of}\triangle SQR}$,

(iii) $\dfrac{\text{area of}\triangle ABR}{\text{area of } PQRS}$.

15. In the diagram, T is the point of intersection of the diagonals of the quadrilateral $PQRS$. $\overrightarrow{PR} = 3\overrightarrow{PT}$, $\overrightarrow{PS} = 5\mathbf{b}$, $\overrightarrow{PQ} = 4\mathbf{a} + \mathbf{b}$ and $\overrightarrow{PR} = 3\mathbf{a} + 12\mathbf{b}$.



(a) Express, as simply as possible, in terms of $\mathbf{a}$ and $\mathbf{b}$,

(i) $\overrightarrow{RS}$, **(ii)** $\overrightarrow{RT}$, **(iii)** $\overrightarrow{RQ}$.

(b) Show that $\overrightarrow{QT} = 3(\mathbf{b} - \mathbf{a})$.

(c) Express $\overrightarrow{QS}$ as simply as possible, in terms of $\mathbf{a}$ and $\mathbf{b}$.

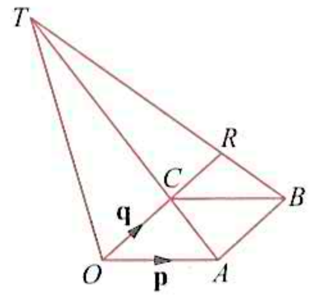
(d) Calculate the value of

(i) $\dfrac{QT}{QS}$, **(ii)** $\dfrac{\text{area of}\triangle PQT}{\text{area of}\triangle PQS}$,

(iii) $\dfrac{\text{area of}\triangle PQT}{\text{area of}\triangle RQT}$.

16. $OABC$ is a parallelogram and ACT is a straight line. OC is produced to meet BT at R. $BT = 4BR$, $\overrightarrow{OA} = \mathbf{p}$, $\overrightarrow{OC} = \mathbf{q}$ and $\overrightarrow{TC} = 3(\mathbf{p} - \mathbf{q})$.

(a) Express, as simply as possible, in terms of $\mathbf{p}$ and $\mathbf{q}$,



(i) $\overrightarrow{OT}$, **(ii)** $\overrightarrow{AT}$, **(iii)** $\overrightarrow{OB}$,

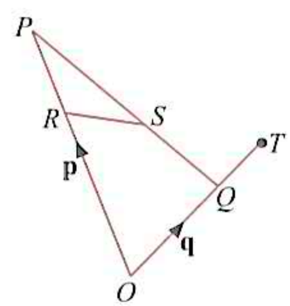
(iv) $\overrightarrow{BT}$, **(v)** $\overrightarrow{TR}$.

(b) Show that $\overrightarrow{CR} = \frac{3}{4}\mathbf{q}$.

(c) Find the value of

(i) $\dfrac{CR}{OC}$, **(ii)** $\dfrac{\text{area of}\triangle TCR}{\text{area of}\triangle TAB}$.

17. In the diagram, $\overrightarrow{OP} = \mathbf{p}$, $\overrightarrow{OQ} = \mathbf{q}$, $PS : SQ = 3 : 2$, $OQ : QT = 2 : 1$ and $OR : RP = 2 : 1$.



(a) Express, as simply as possible, in terms of $\mathbf{p}$ and/or $\mathbf{q}$,

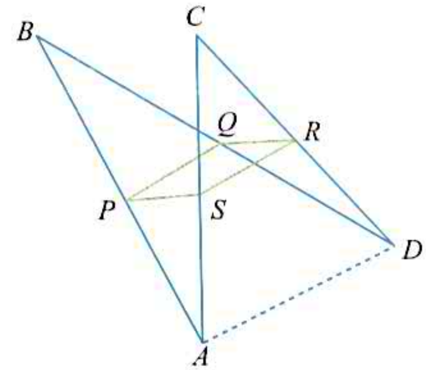
(i) $\overrightarrow{QP}$, **(ii)** $\overrightarrow{QS}$,

(iii) $\overrightarrow{OS}$, **(iv)** $\overrightarrow{ST}$.

(b) **(i)** Show that $\overrightarrow{RS} = k\overrightarrow{ST}$, where k is a constant.

(ii) Write down two facts about the points R, S and T.

18.

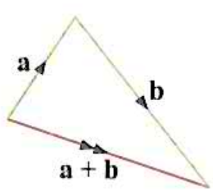


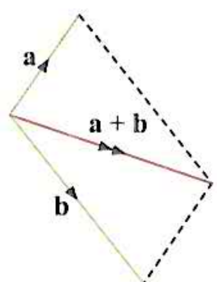
In the diagram, P, Q, R and S are the midpoints of AB, BD, CD and AC respectively. Show that

(i) PQ is parallel to AD and $PQ = \frac{1}{2}AD$,

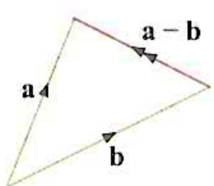
(ii) $PQRS$ is a parallelogram.

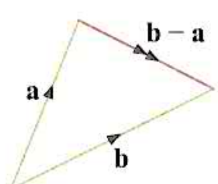
1. A scalar has magnitude only while a vector has both magnitude and direction.
2. Two vectors are equal if they have the same magnitude and direction.
3. The magnitude of a column vector $\mathbf{a} = \begin{pmatrix} x \\ y \end{pmatrix}$ is $|\mathbf{a}| = \sqrt{x^2 + y^2}$.
4. The **Triangle Law** or the **Parallelogram Law of Vector Addition** can be used to find the sum of two vectors:





5. The **Triangle Law of Vector Subtraction** can be used to find the difference of two vectors:





6. The addition of a vector $\mathbf{a}$ and its negative $-\mathbf{a}$ will give the zero vector $\mathbf{0}$, i.e. $(\mathbf{a}) + (-\mathbf{a}) = \mathbf{0}$.

7. If $\mathbf{a}$ and $\mathbf{b}$ are two **parallel vectors**, then $\mathbf{a} = k\mathbf{b}$ for some scalar or real number $k \neq 0$.

8. For column vectors, $\begin{pmatrix} p \\ q \end{pmatrix} + \begin{pmatrix} r \\ s \end{pmatrix} = \begin{pmatrix} p+r \\ q+s \end{pmatrix}$, $\begin{pmatrix} p \\ q \end{pmatrix} - \begin{pmatrix} r \\ s \end{pmatrix} = \begin{pmatrix} p-r \\ q-s \end{pmatrix}$
and $k\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} kp \\ kq \end{pmatrix}$.

9. Any vector $\overrightarrow{AB}$ can be expressed **uniquely** in terms of two other **non-zero** and **non-parallel** coplanar vectors $\mathbf{u}$ and $\mathbf{v}$, i.e. $\overrightarrow{AB} = m\mathbf{u} + n\mathbf{v}$.

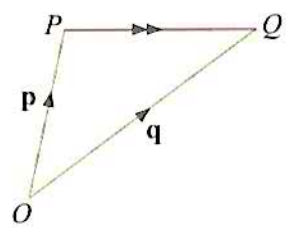
10. The position vector of a point $P(x, y)$ is $\overrightarrow{OP} = \begin{pmatrix} x \\ y \end{pmatrix}$.

11. A vector $\overrightarrow{PQ}$ on the Cartesian plane can be expressed in terms of position vectors as follows:

$$\underbrace{\overrightarrow{PQ}}_{\text{start, end}} = \overrightarrow{OQ} - \overrightarrow{OP} = \mathbf{q} - \mathbf{p}$$

start — end, end — start

must be the same



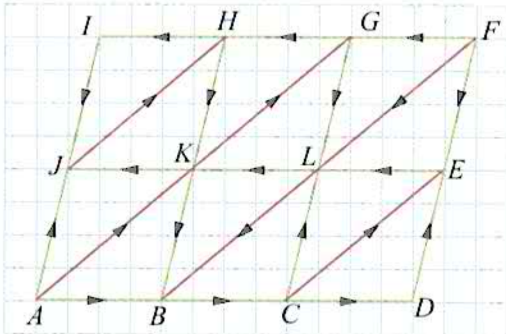
Review Exercise 7

1. Find the magnitude of each of the following vectors.
   (a) $\begin{pmatrix} 5 \\ -12 \end{pmatrix}$ (b) $\begin{pmatrix} -6 \\ 8 \end{pmatrix}$ (c) $\begin{pmatrix} 5 \\ 2 \end{pmatrix}$
   (d) $\begin{pmatrix} -7 \\ -1 \end{pmatrix}$ (e) $\begin{pmatrix} 0 \\ -3 \end{pmatrix}$

2. Given that $\overrightarrow{XY} = \begin{pmatrix} p \\ -2 \end{pmatrix}$, find the possible values of p such that $|\overrightarrow{XY}| = 5$ units.

3. Given that $\overrightarrow{AB} = \begin{pmatrix} -4 \\ 2 \end{pmatrix}$ and $\overrightarrow{CD} = \begin{pmatrix} p \\ -12 \end{pmatrix}$, find
   (i) $|\overrightarrow{AB}|$,
   (ii) the positive value of p if $|\overrightarrow{CD}| = 3|\overrightarrow{AB}|$.

4. If $\mathbf{a} = \begin{pmatrix} p+q \\ p \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 3 \\ q+1 \end{pmatrix}$ and $\mathbf{a} = \mathbf{b}$, find the value of p and of q.

5. 

(a) Without using negative vectors, name two vectors shown in the diagram that are equal to the following vectors:

(i) $\overrightarrow{IJ}$ (ii) $\overrightarrow{AJ}$ (iii) $\overrightarrow{HI}$

(iv) $\overrightarrow{BC}$ (v) $\overrightarrow{AK}$

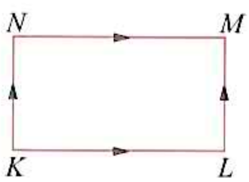
(b) Name a negative vector of

(i) $\overrightarrow{JH}$, (ii) $\overrightarrow{AB}$, (iii) $\overrightarrow{AJ}$.

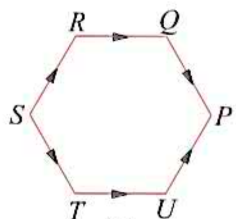
(c) Explain why

(i) $\overrightarrow{AB} \neq \overrightarrow{DE}$, (ii) $\overrightarrow{AK} \neq \overrightarrow{AB}$.

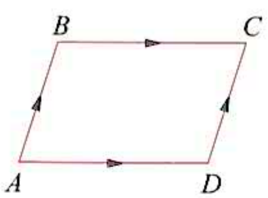
6. Copy and complete the equalities below in each of the diagrams **(a)** – **(d)**. The first equalities in **(a)** and **(b)** have been done for you.



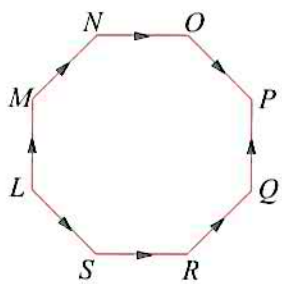
(a)



(b)



(c)



(d)

(a) $\overrightarrow{KN} = \overrightarrow{LM}$

$\overrightarrow{NM} =$ ____

(b) $\overrightarrow{SR} = \overrightarrow{UP}$

$\overrightarrow{RQ} =$ ____

$\overrightarrow{QP} =$ ____

(c) $\overrightarrow{AB} =$ ____

$\overrightarrow{BC} =$ ____

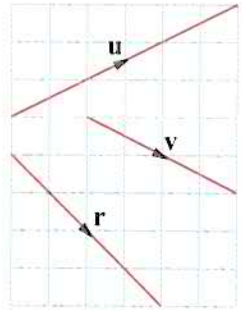
(d) $\overrightarrow{LM} =$ ____

$\overrightarrow{MN} =$ ____

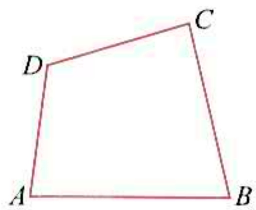
$\overrightarrow{NO} =$ ____

$\overrightarrow{OP} =$ ____

7. The figure below shows the vectors **u**, **v** and **r**. On a sheet of squared paper or graph paper, draw appropriate triangles to illustrate the following vector additions:



(i) $\mathbf{u} + \mathbf{v}$ (ii) $\mathbf{u} + \mathbf{r}$ (iii) $\mathbf{v} + \mathbf{r}$

8. 

ABCD is a quadrilateral. Simplify the following.

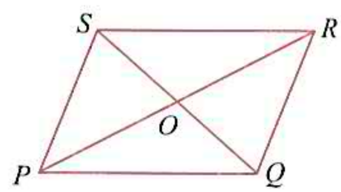
(i) $\overrightarrow{AD} + \overrightarrow{DC}$

(ii) $\overrightarrow{AB} + \overrightarrow{BD}$

(iii) $\overrightarrow{AC} + \overrightarrow{CB} + \overrightarrow{BD}$

(iv) $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA}$

9. From the given diagram, find a vector which can replace **x** in each of the following equations.



(i) $\overrightarrow{SO} + \mathbf{x} = \overrightarrow{SP}$ (ii) $\overrightarrow{PO} + \mathbf{x} = \overrightarrow{PR}$

(iii) $\mathbf{x} + \overrightarrow{SQ} = \overrightarrow{RQ}$ (iv) $\overrightarrow{PR} + \mathbf{x} = \mathbf{0}$

(v) $\overrightarrow{PQ} + \mathbf{x} + \overrightarrow{RS} = \overrightarrow{PS}$ (vi) $\overrightarrow{QR} + \overrightarrow{RS} + \mathbf{x} = \overrightarrow{PS}$

10. If $\mathbf{a} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$, express as column vectors:

(i) $\mathbf{a} - \mathbf{b}$ (ii) $\mathbf{b} - \mathbf{c}$

(iii) $\mathbf{a} - (\mathbf{b} + \mathbf{c})$ (iv) $\mathbf{a} - (\mathbf{b} - \mathbf{c})$

11. It is given that $\mathbf{u} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$ and $|\mathbf{v}| = 20$. If $\mathbf{u} = k\mathbf{v}$ where k is a positive constant, find the value of k. Hence, find $\mathbf{v}$.

12. $\mathbf{p} = \begin{pmatrix} 5 \\ -12 \end{pmatrix}$, $\mathbf{q} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 20 \\ m \end{pmatrix}$.

(a) Express $2\mathbf{p} + 3\mathbf{q}$ as a column vector.

(b) Find

(i) $|\mathbf{p}|$,

(ii) $|-\mathbf{p} + 2\mathbf{q}|$,

giving your answers correct to the nearest whole number.

(c) Given that $\mathbf{r}$ is parallel to $\mathbf{p}$, write down the value of m.

13. A, B, C and D are four points such that A is $(-5, 3)$, C is $(7, 4)$, $\overrightarrow{AB} = \begin{pmatrix} 6 \\ 1 \end{pmatrix}$ and $\overrightarrow{AD} = \begin{pmatrix} 8 \\ -9 \end{pmatrix}$.
Find

(i) the coordinates of B and D,

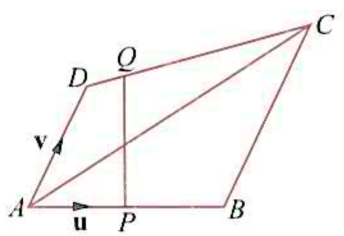
(ii) the vectors $\overrightarrow{BC}$ and $\overrightarrow{CD}$.

14. Two points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively, relative to the origin O. Given that A is the point $(7, 4)$ and $\overrightarrow{AB} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$, find

(i) $\mathbf{b}$,

(ii) the coordinates of the point C, such that $\overrightarrow{OC} = \overrightarrow{BA}$.

15. In the diagram, $ABCD$ is a trapezium with $AD \,//\, BC$ and $AD = \frac{2}{3}BC$. P and Q are points on AB and DC, respectively such that P is the midpoint of AB and $DQ = \frac{1}{4}DC$.



Given that $\overrightarrow{AB} = \mathbf{u}$ and $\overrightarrow{AD} = \mathbf{v}$, express in terms of $\mathbf{u}$ and $\mathbf{v}$,

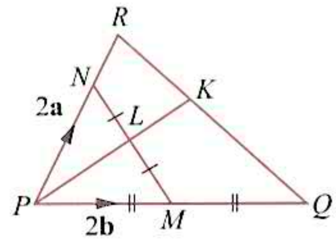
(i) $\overrightarrow{AC}$,

(ii) $\overrightarrow{DC}$,

(iii) $\overrightarrow{AQ}$,

(iv) $\overrightarrow{PQ}$.

16. In triangle PQR, the point N on PR is such that $PN = \frac{2}{3}PR$. M is the midpoint of PQ, L is the midpoint of MN, and PL produced meets RQ at K. $\overrightarrow{PL} = \frac{7}{12}\overrightarrow{PK}$, $\overrightarrow{PN} = 2\mathbf{a}$ and $\overrightarrow{PM} = 2\mathbf{b}$.



(a) Express, as simply as possible, in terms of $\mathbf{a}$ and/or $\mathbf{b}$,

(i) $\overrightarrow{NM}$,

(ii) $\overrightarrow{NL}$,

(iii) $\overrightarrow{PK}$,

(iv) $\overrightarrow{PR}$,

(v) $\overrightarrow{PQ}$.

(b) Express $\overrightarrow{RQ}$ as simply as possible, in terms of $\mathbf{a}$ and $\mathbf{b}$.

(c) Show that $\overrightarrow{KR} = \frac{3}{7}(3\mathbf{a} - 4\mathbf{b})$.

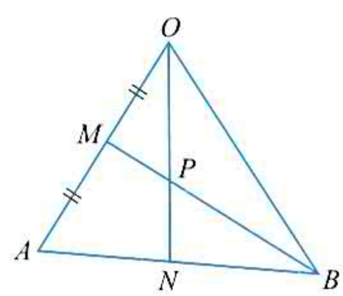
(d) Calculate the value of $\frac{KR}{QR}$.

(e) Calculate the value of

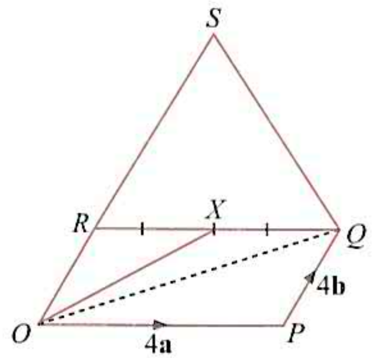
(i) $\dfrac{\text{area of } \triangle PKR}{\text{area of } \triangle PQR}$,

(ii) $\dfrac{\text{area of } \triangle PKN}{\text{area of } \triangle PQR}$.

17. In the diagram below, M is the midpoint of OA and $BP = 3PM$. Given that the position vectors of A and B relative to O are $\mathbf{a}$ and $\mathbf{b}$ respectively, find the position vector of P relative to O.



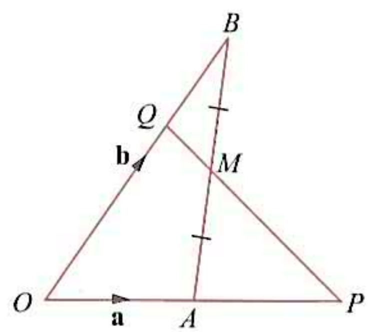
18.



$OPQR$ is a parallelogram and X is the midpoint of QR. OR is produced to S so that $\overrightarrow{OR} = \frac{1}{2}\overrightarrow{RS}$. Given that $\overrightarrow{OP} = 4\mathbf{a}$ and $\overrightarrow{PQ} = 4\mathbf{b}$, express the following vectors in terms of $\mathbf{a}$ and $\mathbf{b}$, giving your answers in the simplest form.

(i) $\overrightarrow{OQ}$ **(ii)** $\overrightarrow{OX}$ **(iii)** $\overrightarrow{QS}$

19. In the diagram, $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$, $\overrightarrow{OP} = 2\mathbf{a}$, and $OQ : QB = 2 : 1$, and M is the midpoint of AB.



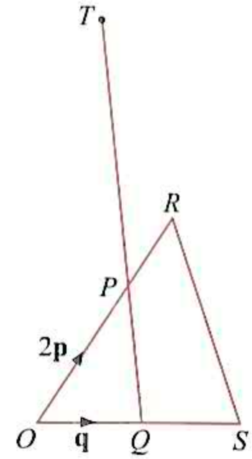
(a) Express, as simply as possible, in terms of $\mathbf{a}$ and/or $\mathbf{b}$,

(i) $\overrightarrow{OQ}$, **(ii)** $\overrightarrow{PQ}$,

(iii) $\overrightarrow{OM}$, **(iv)** $\overrightarrow{QM}$.

(b) Find the value of $\frac{PM}{MQ}$.

20. In the diagram, $\overrightarrow{OP} = 2\mathbf{p}$ and $\overrightarrow{OQ} = \mathbf{q}$. $\overrightarrow{OP} = 2\overrightarrow{PR}$ and $\overrightarrow{OQ} = \overrightarrow{QS}$. T is the point on QP produced where $TQ = 3PQ$.



(a) Express the following vectors, as simply as possible, in terms of $\mathbf{p}$ and/or $\mathbf{q}$.

(i) $\overrightarrow{QP}$ **(ii)** $\overrightarrow{OR}$

(iii) $\overrightarrow{SR}$ **(iv)** $\overrightarrow{ST}$

(b) Write down two facts about the points S, R and T.

(c) Find the value of $\frac{\text{area of}\triangle OPQ}{\text{area of}\triangle SPT}$.

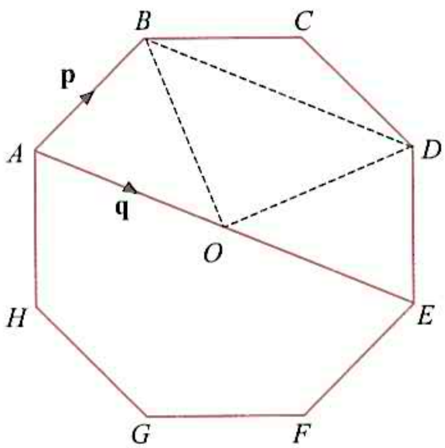
Challenge Yourself

1. By drawing a suitable diagram, show that for any two vectors $\mathbf{a}$ and $\mathbf{b}$,

   (i) $|\mathbf{a} + \mathbf{b}| \leqslant |\mathbf{a}| + |\mathbf{b}|$,

   (ii) $|\mathbf{a} - \mathbf{b}| \leqslant |\mathbf{a}| + |\mathbf{b}|$.

2. In the diagram, $ABCDEFGH$ is a regular octagon where O is its centre, $\overrightarrow{AB} = \mathbf{p}$ and $\overrightarrow{AE} = \mathbf{q}$.



   (a) Express, as simply as possible, in terms of $\mathbf{p}$ and/or $\mathbf{q}$,

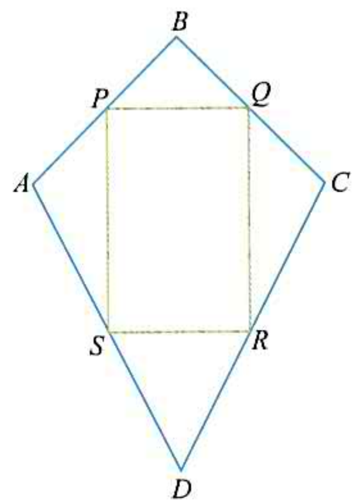
   (i) $\overrightarrow{EF}$, **(ii)** $\overrightarrow{BE}$.

   (b) Given that $|\overrightarrow{AE}| = 2$ units, find the exact value of $|\overrightarrow{BD}|$.

   (c) Hence, express $\overrightarrow{BD}$, as simply as possible, in terms of $\mathbf{p}$ and/or $\mathbf{q}$.

3. In the diagram, $ABCD$ is a kite, and P, Q, R and S are the midpoints of AB, BC, CD and DA respectively.

   Show that $PQRS$ is a rectangle.



Loci

The point on the end of each hand is a fixed distance away from the centre of the clock. Looking at the picture, can you describe the path traced by the point on the end of the minute hand in 60 minutes? In this chapter, we will learn how to describe paths traced by points as they travel under specific conditions.

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Chapter Eight

LEARNING OBJECTIVES

At the end of this chapter, you should be able to:

- construct simple loci of points in two dimensions,
- solve problems involving intersection of loci.

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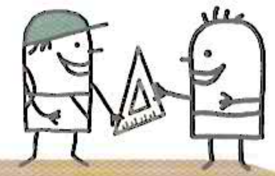
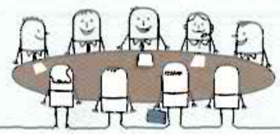
8.1 Introduction to Loci





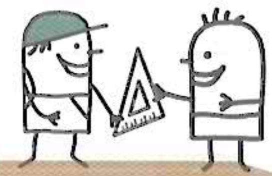
Class Discussion

Introduction to Loci

1. Work in pairs.
   (a) When a lift in a tall building moves up or down, what path will it follow? Describe and make a sketch of the path.
   (b) Discuss the path of a swinging pendulum bob. Describe and make a sketch of the path.
   (c) Describe and make a sketch of the path followed by the tip of the second hand of a wall clock.
   (d) Discuss the movement of the Earth around the Sun. Describe and make a sketch of the path taken by the Earth.
   (e) Discuss the movement of the tip of a man's nose when he is walking along a straight line on level ground. What assumptions do you have to make before you can simplify the description of the path? Make a sketch of the path.
2. Do you find that the movements of the objects discussed above follow regular paths? Are these paths regulated by certain conditions? Discuss the condition that regulates each path.
3. Do movements of objects always follow regular paths? If not, can you give some examples of movements of objects which we do not follow regular paths?
4. In your sketches of paths of the objects, are the paths represented by sets of points which show the different positions of the objects?

In Mathematics, these sets of points are known as **loci**. The singular of 'loci' is **locus**. From the class discussion, we can conclude that the locus of a variable point can be defined as *a set of points satisfying certain given conditions*. It can also be defined as *the path traced out by an object moving in a specific manner*. The loci which we will encounter often in Mathematics are straight lines. In this book, we will study loci in two dimensions.

8.2 Locus Theorems



Recap

In Book 1, we have learnt the properties and construction of perpendicular bisectors and angle bisectors.

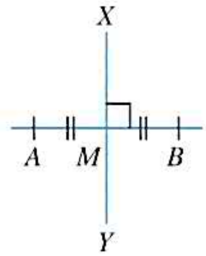


Fig. 8.1

Given that XY divides AB into two equal parts and is also perpendicular to AB, XY is known as the perpendicular bisector of AB. Any point on the perpendicular bisector of a line segment is equidistant from the two end points of the line segment.

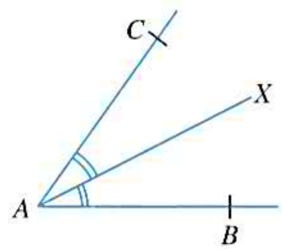


Fig. 8.2

Given that the ray AX divides $B\hat{A}C$ into two equal angles, i.e. $B\hat{A}X$ and $C\hat{A}X$, AX is known as the angle bisector of $B\hat{A}C$. Any point on the angle bisector of an angle is equidistant from the two sides of the angle.

In this section, we will learn how to construct loci using what we have learnt in basic geometrical construction.

Locus of a Point from another Point

Investigation

Locus Theorem 1

Part 1

1. In your sketch of the path followed by the tip of the second hand of a wall clock, what do you notice about the points representing different positions of the tip?
2. Describe the path.

Part 2

3. Mark a point O on the centre of a piece of paper.
4. Mark a number of points, each of which is exactly 2 cm from O. Imagine that you have marked a hundred or a thousand or even a million of such points. What can you conclude about the points, each of which is exactly 2 cm from O?

From the investigation, **Locus Theorem 1** states that:

> The locus of a point which is at a given distance d from a given point O is a circle with centre O and radius d.

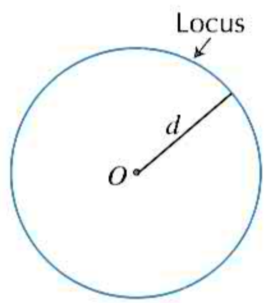


Fig. 8.3

PRACTISE NOW

Describe the locus of a point P at a constant distance of 5 cm from a fixed point Q.

SIMILAR QUESTIONS

Exercise 8A Questions 1, 2, 6(i), 11

Locus of a Point from a Straight Line

Investigation

Locus Theorem 2

Part 1

1. Discuss with a partner. How will you walk so that you will always be 2 m from a straight wall?
2. Try by walking a short distance, keeping the same distance away from one wall of the classroom.
3. What can you conclude from part 1 of this investigation?

Part 2

4. On a piece of paper, draw a line XY down the middle of the page.
5. Mark a number of points 3 cm from XY with some of the points on the right and some on the left of XY. Imagine that you have marked a million of such points. What can you say about the points, each of which is exactly 3 cm from XY?

From the investigation, **Locus Theorem 2** states that:

> The loci of a point at a given distance d from a given straight line XY are two straight lines, l_1 and l_2, parallel to XY and at a distance d from XY.

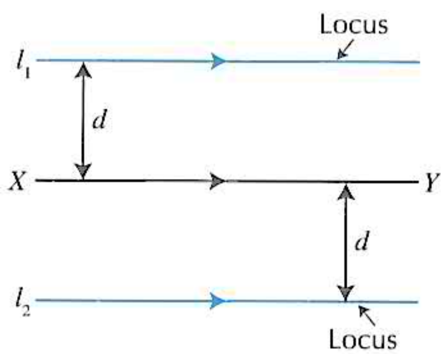


Fig. 8.4

Conversely, the locus of points equidistant from two parallel lines is a line that lies between the two lines, parallel and equidistant to both of them.

PRACTISE NOW

SIMILAR QUESTIONS

Describe the locus of a point P at a constant distance of 10 cm from a straight line RS.

Exercise 8A Questions 3, 4, 6(ii)

Locus of a Point Equidistant from Two Points

Investigation

Locus Theorem 3

Part 1

1. Discuss with a partner. How will you walk so that you will always be the same distance from two desks in the classroom?
2. Try walking a short distance such that you are equidistant from the two desks.
3. What can you conclude from part 1 of this investigation?

Part 2

4. On a piece of paper, draw a line XY 4 cm long across the middle of the page.
5. Mark a number of points, each of which is equidistant from X and Y. Have some of the points above and some below the line XY.
6. Imagine that you have marked a large number of such points. What can you say about the points, each of which is equidistant from X and Y?

From the investigation, **Locus Theorem 3** states that:

> The locus of a point which is equidistant from two given points X and Y is the perpendicular bisector of the line XY.

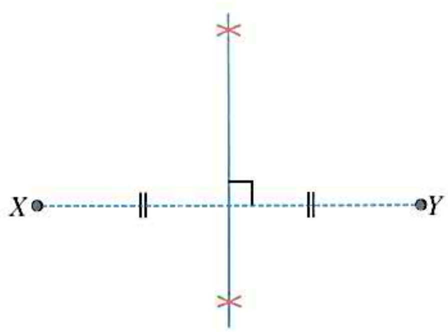


Fig. 8.5

PRACTISE NOW

Describe the locus of a point C equidistant from two points A and B that are 12 cm apart.

SIMILAR QUESTIONS

Exercise 8A Questions 5, 6(iii), 9

Locus of a Point Equidistant from Two Intersecting Lines

Investigation

Locus Theorem 4

Part 1

1. Discuss with a partner. How will you walk so that you will always be at the same distance from two adjacent walls in your classroom?
2. Try walking a short distance such that you are equidistant from the two walls.
3. What can you conclude from part 1 of this investigation?

Part 2

4. On a piece of paper draw two lines AB and XY to intersect at a point O.
5. Mark a number of points which are equidistant from AB and XY.
6. Imagine that you have marked a large number of such points. What can you say about the points, each of which is equidistant from AB and XY?

From the investigation, **Locus Theorem 4** states that:

> The locus of a point which is equidistant from two given intersecting straight lines, AB and XY, is a pair of straight lines, l_1 and l_2, which bisect the angles between the two given lines.

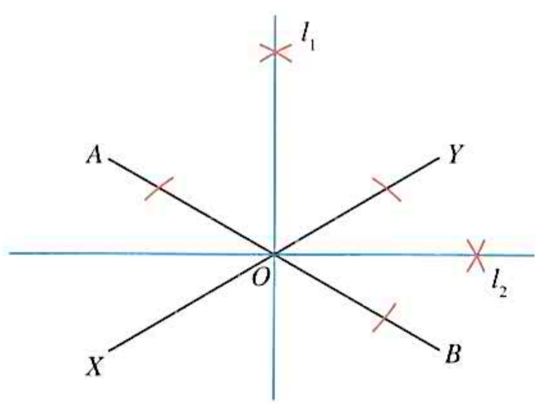


Fig. 8.6

PRACTISE NOW

Describe the locus of a point C equidistant from the diagonals of a square.

SIMILAR QUESTIONS

Exercise 8A Questions 6(iv), 7, 8, 10

Exercise 8A

BASIC LEVEL

1. Describe the locus of a point P which moves in a plane so that it is always 4 cm from a fixed point O in the plane.

2. X is a fixed point in a given plane. Draw the locus of a point P which is always 3.5 cm from X.

3. Draw a line AB, 7 cm long. A point P moves such that it is always 3 cm from AB. Draw the locus of P.

4. Describe the locus of a point Q which moves in a plane so that it is always 5 cm from a given straight line l.

5. Two points A and B are 7.5 cm apart. Draw the locus of a point P equidistant from A and B.

INTERMEDIATE LEVEL

6. Two straight lines AB and CD intersect at right angles at the point O. Draw each of the following using separate diagrams.
   (i) The locus of a point 2.5 cm from O.
   (ii) The locus of a point 3 cm from CD.
   (iii) The locus of a point equidistant from C and O.
   (iv) The locus of a point equidistant from OB and OD.

7. Draw two intersecting lines l and m. Draw the locus of a point P which moves such that it is equidistant from l and m.

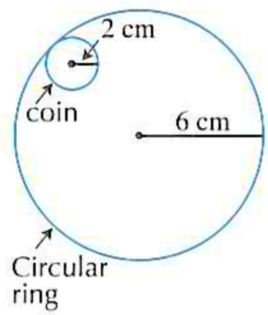
8. Construct an angle $X\hat{Y}Z$ equal to 60°. Draw the locus of a point P which moves such that it is equidistant from XY and YZ.

9. Construct $\triangle ABC$ in which $AB = 6$ cm, $BC = 7$ cm and $CA = 8$ cm. Draw the locus of P such that P is equidistant from A and C.

10. Construct $\triangle PQR$ in which $QR = 8$ cm, $P\hat{Q}R = 70°$ and $PR = 9$ cm. Construct the locus which represents points equidistant from PQ and QR.

ADVANCED LEVEL

11. A coin of radius 2 cm is placed flat on the floor and made to move inside a circular of radius 6 cm, also lying flat on the floor. The coin is always in contact with the ring as shown in the figure below. Describe and show diagrammatically the locus of the centre of the coin.



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8.3 Intersection of Loci

If two or more loci intersect at a point P, then P satisfies the conditions of the loci simultaneously.

Worked Example 1

(Problem involving Intersection of Loci)

(i) Using ruler and compasses, construct $\triangle ABC$ in which $AB = 8.8$ cm, $BC = 7$ cm and $CA = 5.6$ cm.

(ii) On the same diagram, draw

(a) the locus of a point which is 4.6 cm from A,

(b) the locus of a point which is equidistant from BA and BC.

(iii) Find the distance between two points which are 4.6 cm from A and equidistant from BA and BC. Give your answer in centimetres and correct to 1 decimal place.

Solution:

(i) Constructions Steps:

1. Using a ruler, draw $AB = 8.8$ cm.
2. Since C is 7 cm away from B, with B as centre and 7 cm as radius, draw arc 1.
3. Since C is 5.6 cm away from A, with A as centre and 5.6 cm as radius, draw arc 2 to cut arc 1 at C.
4. Join AC and BC.

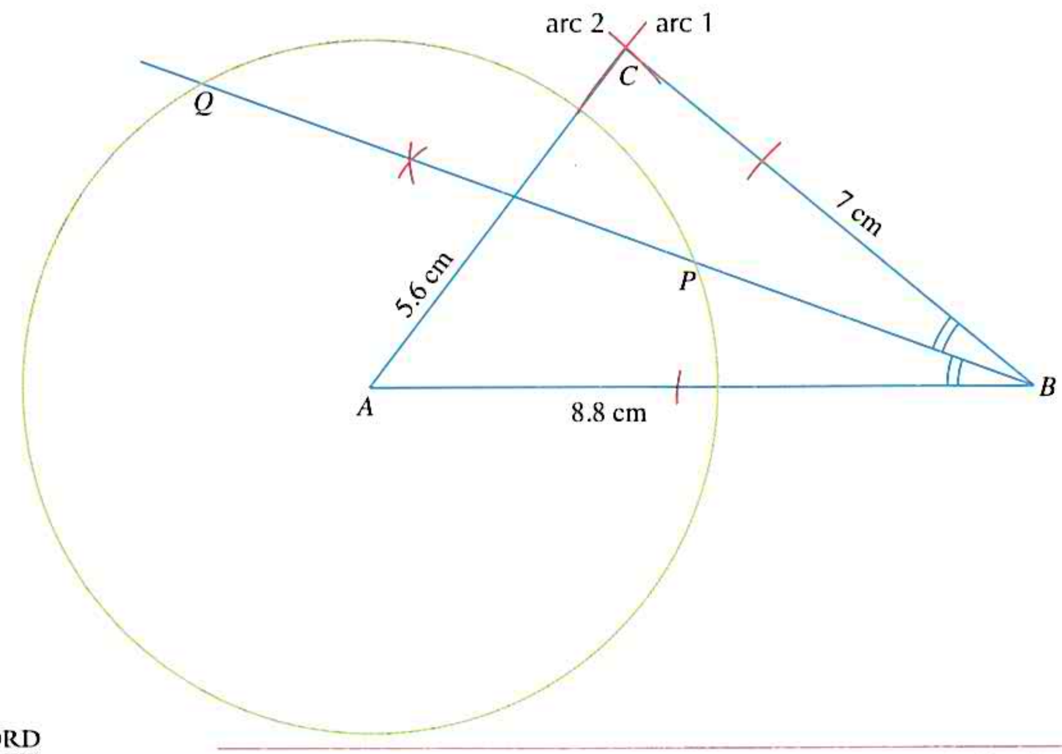
(ii) Constructions Steps:

(a) With A as centre and radius 4.6 cm, draw a circle. The locus of a point which is 4.6 cm from A is the circle drawn.

(b) The locus of a point which is equidistant from BA and BC is the angle bisector of $A\hat{B}C$.

(iii) Two points which are 4.6 cm from A and equidistant from BA and BC are P and Q, the intersection points of the two loci in **(ii)(a)** and **(ii)(b)**.

Length of $PQ = 7.0$ cm



PRACTISE NOW 1

SIMILAR QUESTIONS

Exercise 8B Questions 1-2

(i) Using ruler and compasses, construct $\triangle ABC$ in which $AB = 10$ cm, $BC = 6.1$ cm and $CA = 7.8$ cm.

(ii) On the same diagram, draw

(a) the locus of a point which is 5.5 cm from A.

(b) the locus of a point which is equidistant from BA and BC.

(iii) Find the distance between two points which are 5.5 cm from A and equidistant from BA and BC. Give your answer in centimetres and correct to 1 decimal place.

Worked Example 2

(Problem involving Intersection of Loci)

Construct and label $\triangle XYZ$ in which $XY = 8$ cm, $Y\hat{X}Z = 60°$ and $X\hat{Y}Z = 45°$.

(a) On your diagram,

(i) measure and write down the length of YZ,

(ii) draw the locus of a point which is equidistant from X and Z,

(iii) draw the locus of a point which is equidistant from ZX and ZY,

(iv) draw the locus of a point which is 3 cm from XY and on the same side of XY as Z.

(b) On your diagram,

(i) label the point P which is equidistant from the points X and Z and from the lines ZX and ZY,

(ii) label the point Q which is on the same side of XY as Z, is equidistant from X and Z and is 3 cm from the line XY,

(iii) measure and write down the length of PQ.

Solution:

Construction Steps:

1. Using a ruler, draw $XY = 8$ cm.
2. Since $\angle X = 60°$, using a protractor at X, mark off an angle of 60° and draw a line XH such that $Y\hat{X}H = 60°$.
3. Since $\angle Y = 45°$, using a protractor at Y, mark off an angle of 45° and draw a line YK such that $X\hat{Y}K = 45°$.
4. Label the intersection of XH and YK as Z.

(a) **(i)** Length of $YZ = 7.2$ cm

(ii) Draw the perpendicular bisector (line l) of XZ to obtain the locusof a point equidistant from X and Z.

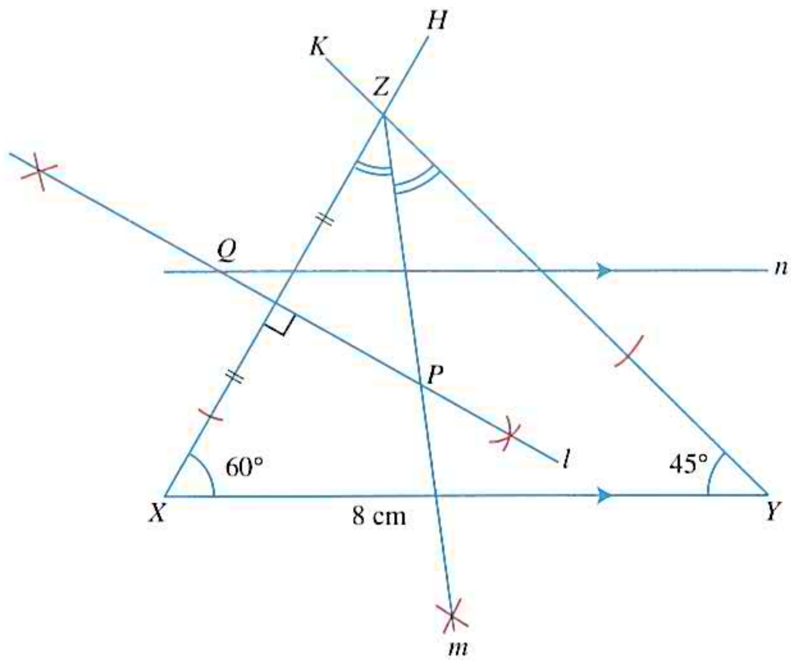
(iii) Draw the angle bisector (line m) of $X\hat{Z}Y$ to obtain the locus of a point equidistant from ZX and ZY.

(iv) Draw a line n parallel to XY and 3 cm from XY to obtain the locus of a point which is 3 cm from XY.

(b) **(i)** P is the point of intersection of l and m.

(ii) Q is the point of intersection of l and n.

(iii) Length of $PQ = 3.2$ cm.



PRACTISE NOW 2

SIMILAR QUESTIONS

Exercise 8B Questions 3-6

Construct and label $\triangle GHI$ in which $GH = 15$ cm, $GI = 10.7$ cm and $HI = 19.5$ cm.

(a) On your diagram,

(i) measure and write down the size of the angle facing the longest side.

(ii) draw the locus of a point which is equidistant from H and I.

(iii) draw the locus of a point which is equidistant from GI and HI.

(iv) draw the locus of a point which is 2.3 cm from HI and on the same side of HI as G.

(b) On your diagram,

(i) label the point P which is equidistant from the points H and I and from the lines GI and HI.

(ii) label the point Q which is on the same side of HI as G, is equidistant from H and I and is 2.3 cm from the line HI.

(iii) measure and write down the length of PQ.

Worked Example 3

(Problem involving Intersection of Loci)

In a single diagram, construct

(i) $\triangle ABC$ in which $AB = 7$ cm, $BC = 9$ cm and $CA = 6$ cm,

(ii) the locus of a point with distance 3 cm from A,

(iii) the point P which lies outside $\triangle ABC$, 3 cm from A and such that $PB = PC$,

(iv) the point Q which lies outside $\triangle ABC$, 3 cm from A and such that $Q\hat{A}C = A\hat{C}B$.
Measure the length of PQ.

Solution:

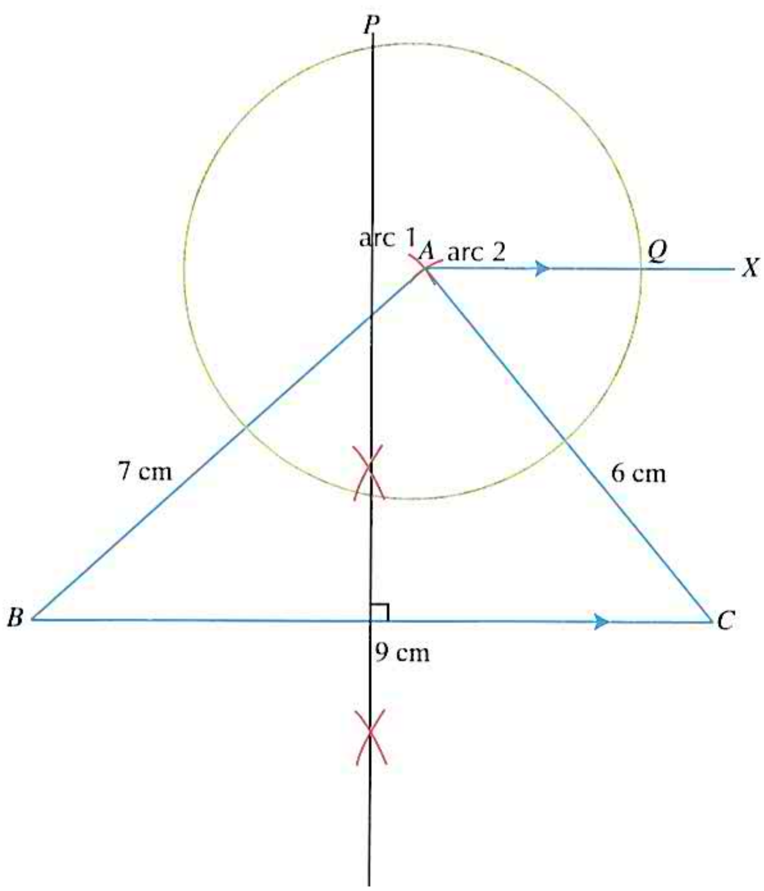
(i) **Constructions Steps:**

1. Using a ruler, draw $BC = 9$ cm.
2. Since A is 7 cm away from B, with B as centre and 7 cm as radius, draw arc 1.
3. Since A is 6 cm away from A, with A as centre and 6 cm as radius, draw arc 2 to cut arc 1 at C.
4. Join AC and BC.

(ii) Taking A as centre, draw a circle of radius 3 cm. The circle is the required locus.

(iii) Construct the perpendicular bisector of BC. The perpendicular bisector intersects the circle at P. P lies outside $\triangle ABC$, is the required point.

(iv) From A, construct a line AX parallel to BC. The point Q, the intersection point of AX and the circle, is the required point.
Length of $PQ = 4.7$ cm



PRACTISE NOW 3

SIMILAR QUESTIONS

Exercise 8B Question 7

In a single diagram, construct
(i) $\triangle ABC$ in which $AB = 6$ cm, $BC = 8.3$ cm and $CA = 8$ cm,
(ii) the locus of a point with distance 3 cm from A,
(iii) the point Q which lies inside $\triangle ABC$, 3 cm from A and such that $QB = QC$,
(iv) the point R which lies outside $\triangle ABC$, 3 cm from A and such that $R\hat{A}C = A\hat{C}B$.
Measure the length of RQ.

Exercise 8B

BASIC LEVEL

1. (i) Construct and label $\triangle XYZ$ in which $XY = 10$ cm, $YZ = 7.5$ cm and $XYZ = 60°$. Measure and write down the length of XZ.
(ii) On your diagram, construct the locus of a point
(a) 6 cm from Y,
(b) equidistant from X and Z.
(iii) The point P, inside $\triangle XYZ$, is 6 cm from Y and equidistant from the points X and Z.
(a) Label clearly, on your diagram, the point P.
(b) Measure and write down the length PX.

2. (i) Using ruler and compasses only, construct $\triangle PQR$ in which $PQ = 11$ cm, $PR = 8$ cm and $QR = 6$ cm.
(ii) On your diagram, construct the locus of a point
(a) 3.8 cm from R,
(b) equidistant from PQ and PR.
(iii) The point X, inside $\triangle PQR$ is 3.8 cm from R and equidistant from PQ and PR.
(a) Label clearly, on your diagram, the point X.
(b) Measure and write down the length XQ.

INTERMEDIATE LEVEL

3. A playground is in the shape of a triangle ABC in which $B\hat{A}C = 50°$, $AB = 70$ m and $BC = 80$ m.
(i) Using a scale of 1 cm to represent 10 m, make an accurate scale drawing of the playgound.
A seesaw S in the playground is 55 m from A and equidistant from B and C.
(ii) On the same diagram, draw the locus which represents all the points inside the triangle which are
(a) 55 m from A,
(b) equidistant from B and C.
(iii) Mark clearly on your diagram, the position of the seesaw S.
(iv) Measure the length BS and find the distance of the seesaw S from B.

4. A factory occupies a quadrilateral site $ABCD$ in which $AB = 110$ m, $B\hat{A}D = 65°$, $AD = 90$ m, $A\hat{D}C = 110°$ and $DC = 60$ m.
   (i) Using a scale of 1 cm to represent 10 m, construct a plan of the quadrilateral $ABCD$. Measure the angle ABC. Two fuel storage tanks, T_1 and T_2 are located 30 m from C and 15 m from BD.
   (ii) On the same diagram, draw the locus which represents all the points inside the quadrilateral which are
   (a) 30 m from C,
   (b) 15 m from BD.
   (iii) Mark clearly on your diagram, the position of the tanks T_1 and T_2.
   (iv) By measurement, find the distance between T_1 and T_2.

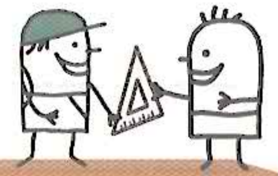
5. A metal sheet is in the form of a $\triangle XYZ$, where $XY = 8.8$ m, $X\hat{Y}Z = 64°$ and $Y\hat{X}Z = 41°$.
   (i) Using a scale of 1 cm to represent 1 m, construct an accurate scale drawing of the metal sheet.
   A hole is to be drilled on the metal sheet at the point O which is equidistant from X, Y and Z.
   (ii) On the same diagram, draw the locus which represents points inside the triangle which are equidistant from
   (a) X and Y,
   (b) Y and Z.
   (iii) Mark clearly on your diagram, the position of the point O.
   (iv) What is the distance of the hole from the corners X, Y and Z of the metal sheet?

6. A collar badge is in the shape of a triangle PQR in which $PQ = 14$ mm, $QPR = 60°$ and $PR = 12$ mm.
   (i) Using a scale of 1 cm to 1 mm, construct an accurate scale drawing of the collar badge.
   A pin is soldered to the back of the badge at a point X which is equidistant from PQ, QR and RP.
   (ii) On the same diagram, draw the locus which represents all the points inside the triangle which are equidistant from
   (a) PQ and QR,
   (b) QR and RP.
   (iii) Mark clearly on your diagram, the position of the point X.

ADVANCED LEVEL

7. The quadrilateral $ABCD$ represents a farmer's field in which $AB = 100$ m, $AD = 80$ m, $A\hat{D}C = 105°$, $B\hat{A}D = 90°$ and $A\hat{B}C = 75°$.
   (i) Using a scale of 1 cm to represent 10 m, construct an accurate scale drawing of the field.
   (ii) There is a mango tree at B and one at C. The farmer wishes to plant a third mango tree equidistant from B and C and 40 m from his house at A. On your diagram, draw the locus which represents points which are
   (a) equidistant from B and C,
   (b) 40 m from A.
   Label, with the letter M, the point where the farmer may plant the third mango tree. Find how far this mango tree is from the other two mango trees.
   (iii) A water storage tank in the field is 50 m from AD and is also equidistant from CB and CD. Find the position of the water storage tank on your diagram by using the intersection of two loci, labelling it clearly with the letter T.

8.4 Further Loci



In the previous section, we learnt about the fundamental theorems of loci. In this section, we will look at more loci theorems.

Loci involving Triangles with Constant Area

The locus of a point P which moves such that the area of $\triangle ABP$ remains constant is a set of points of the two lines a and b parallel to and equidistant from AB as shown in Fig. 8.7.

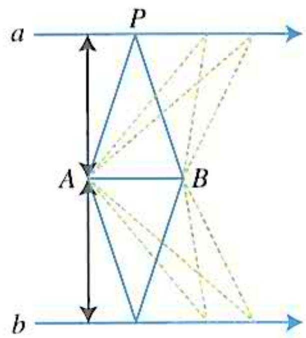


Fig. 8.7

If $AB = 4$ cm and the area of $\triangle ABP = 12\text{ cm}^2$, then the locus of P consists of two lines parallel to AB and 6 cm from AB as shown in Fig. 8.8.

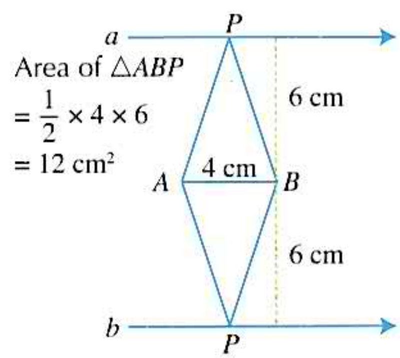


Fig. 8.8

Loci involving Circle Angle Properties

We have learnt in Book 3 that an angle in a semicircle is always equal to 90°. Hence, the locus of points such that $X\hat{P}Y = 90°$, where O is the centre of the circle and X, Y and P are points on the circumference, is the set of points excluding X and Y, with XY as diameter as shown in Fig. 8.9.

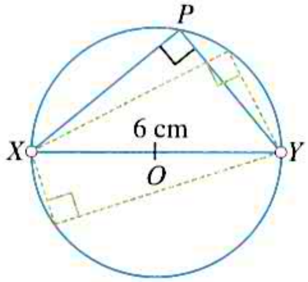


Fig. 8.9

ATTENTION

A circle O is used to indicate that a particular point is being excluded in the locus.

We have also learnt in Book 3 that an angle in the centre of a circle is twice that of any angle at the circumference subtended by the same arc. Hence, the locus of points such that $\angle XOY = 2\ \angle XPY$, where O is the centre of the circle and X, Y and P are points on the circumference, is the set of points on the arc XPY and its reflection in XY, excluding X and Y, as shown in Fig. 8.10.

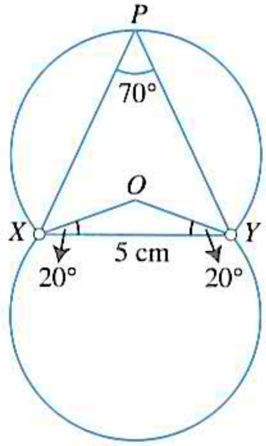
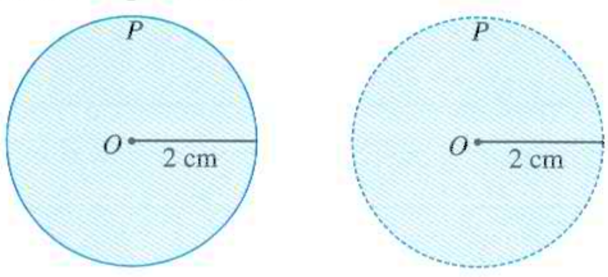


Fig. 8.10

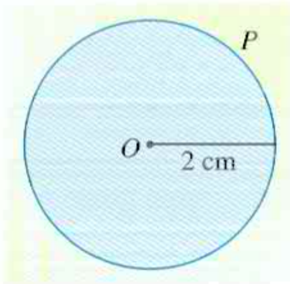
Fig. 8.11 **(a)** shows the locus of a point P whose distance from a fixed point O is $OP \leqslant 2$ cm. The locus is represented by the points inside and on the circumference of the circle with centre O and radius 2 cm. If $OP < 2$ cm, the locus of P will not include the points on the circumference. The circumference will be represented by a broken line as in Fig. 8.11 **(b)**.



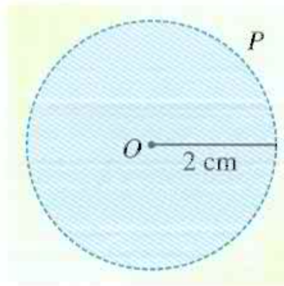
(a) **(b)**

Fig. 8.11

If $OP \geqslant 2$ cm, then the locus of P is the set of all the points outside the circle, including the points on the circumference as shown in Fig. 8.12**(a)**. If $OP > 2$ cm, then the locus of P is the set of all the points outside the circle as shown in Fig. 8.12**(b)**.



(a)



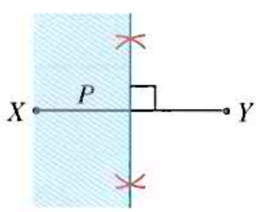
(b)

Fig. 8.12

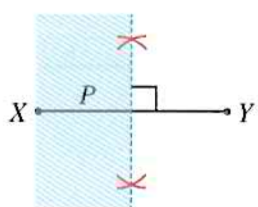
Extension of Locus of a Point Equidistant from Two Points

Earlier, we learnt that if X and Y are two fixed points, and if a point P moves in a plane such that $PX = PY$, then the locus of P is the perpendicular bisector of the line XY.

If P moves such that $PX \leqslant PY$, the locus of P is the set of points shown in the shaded region including all the points on the perpendicular bisector which is represented by a solid line as shown in Fig. 8.13**(a)**. If P moves such that $PX < PY$, the locus of P is the set of points shown in the shaded region, excluding all the points on the perpendicular bisector, which is represented by a broken line as shown in Fig. 8.13**(b)**.



(a)



(b)

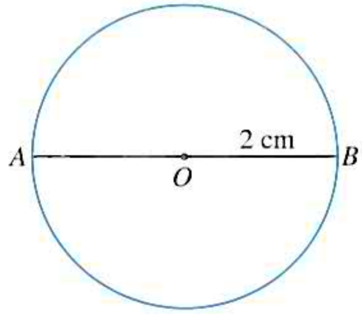
Fig. 8.13

In the cases where $PX \geqslant PY$ and $PX > PY$, we shade the other side of the perpendicular bisector since the point P is closer is closer to Y than it is to X.

Worked Example 4

(Further Loci)

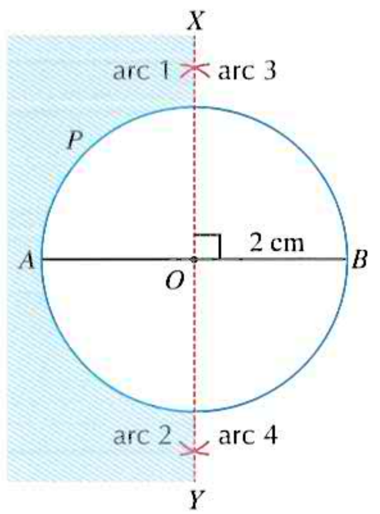
The figure below shows a circle with centre O and diameter AB, 4 cm long. Indicate, by shading, the locus of a point P which moves such that $OP \geqslant 2$ cm and $PA < PB$.



Solution:

Construction Steps:

1. Adjust the arms of the compasses so that the distance between the ends is more than half the length of AB, i.e. more than 2 cm. With A as centre, draw arc 1 above AB and draw arc 2 below AB.
2. Using the same radius in Step 1, with B as centre, draw arc 3 to cut arc 1 at X draw arc 4 to cut arc 2 at Y.
3. Join XY to obtain the perpendicular bisector of AB.
4. Since $OP \geqslant 2$ cm and $PA < PB$, shade the same side of the perpendicular bisector as A and outside the circle.

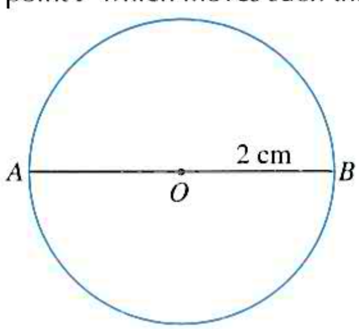


PRACTISE NOW 4

SIMILAR QUESTIONS

Exercise 8C Questions 1–5, 8, 9, 13

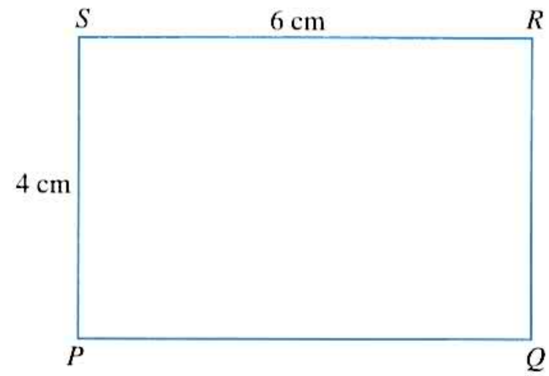
The figure below shows a circle with centre O and diameter AB, 4 cm long. Indicate, by shading, the locus of a point P which moves such that $OP \leqslant 2$ cm and $PA > PB$.



Worked Example 5

(Further Loci)

The figure below shows a rectangle $PQRS$ of length 6 cm and width 4 cm. A variable point X moves inside the rectangle such that $XP \leqslant 4$ cm, $XP \geqslant XQ$ and the area of $\triangle PQX \geqslant 3$ cm². Construct and shade the region in which X must lie.

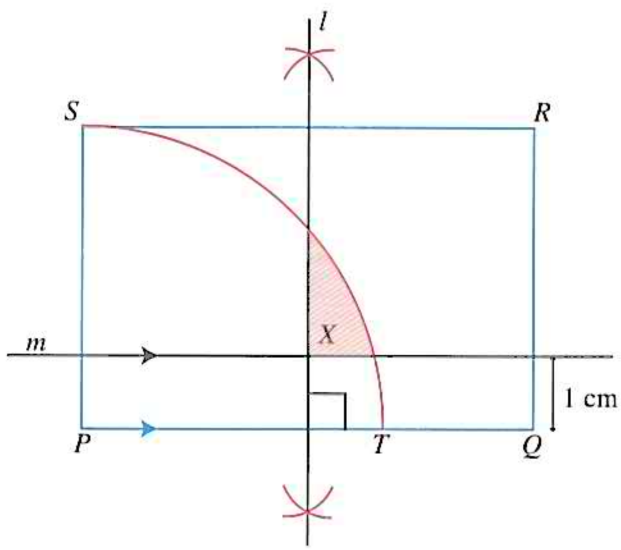


Solution:

Construction Steps:

1. With centre P and radius $PS = 4$ cm, draw an arc to cut PQ at T. Any point inside the quadrant SPT including the borders will satisfy the condition that $XP \leqslant 4$ cm.
2. Construct the perpendicular bisector l, of the line PQ. Any point inside the rectangle to the right of l including l will satisfy the condition $XP \geqslant XQ$.
3. Draw a line m parallel to and 1 cm away from PQ. Any point inside the rectangle above and including m will satisfy the condition that area of $\triangle PQX \geqslant 3$ cm².

   The shaded region is the region in which X must lie.

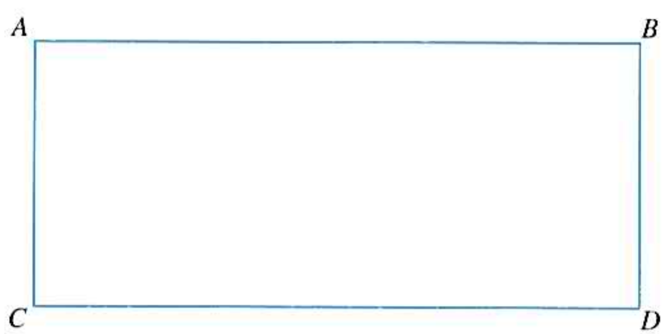


PRACTISE NOW 5

SIMILAR QUESTIONS

Exercise 8C Questions 6, 10

The figure below shows a rectangle $ABCD$ of length 8 cm and width 3.5 cm. A variable point Y moves inside the rectangle such that $YD \leqslant 4.5$ cm, $YD \geqslant YC$ and the area of $\triangle CYD \geqslant 4$ cm². Construct and shade the region in which Y must lie.



Worked Example 6

(Further Loci)

Construct $\triangle PQR$ in which $PQ = 9.5$ cm, $Q\hat{P}R = 100°$ and $PR = 7.2$ cm.

(i) On the same diagram,

(a) draw the locus of a point equidistant from P and R,

(b) draw the locus of a point equidistant from Q and R,

(c) draw the circle through P, Q and R.

(ii) Measure and write down the radius of the circle.

(iii) A is the point on the same side of QR as P such that $\triangle AQR$ is isosceles, with $QA = RA$ and $Q\hat{A}R = 100°$. Mark the point A clearly on your diagram.

Solution:

Construction Steps:

1. Using a ruler, draw $PQ = 9.5$ cm.
2. Since $\angle P = 100°$, using a protractor at P, mark off an angle of $100°$ and draw a line PK such that $Q\hat{P}K = 100°$.
3. Since R is 7.2 cm away from P, with P as centre and 7.2 cm as radius, draw an arc to cut PK at R.
4. Join QR.

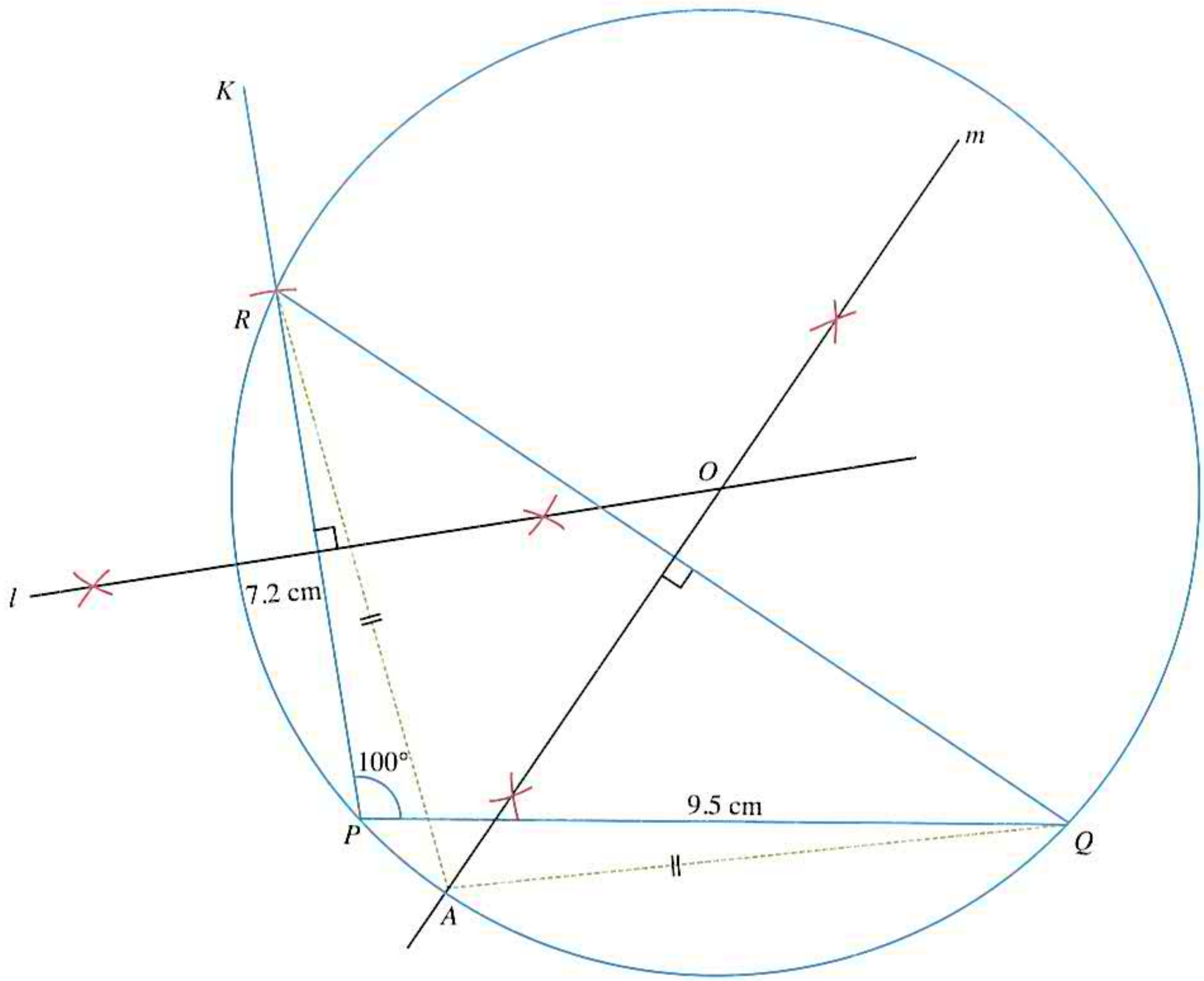
(i) **(a)** Construct the perpendicular bisector, line l, of PR.

(b) Construct the perpendicular bisector, line m, of QR.

(ii) l and m intersect at O which is equidistant from P, Q and R. With O as the centre and radius OP, draw the circle through P, Q and R.

(iii) By measurement, $OP \approx 6.5$ cm.

(iv) m cuts the circle at A so that $QA = RA$. $Q\hat{A}R = Q\hat{P}R = 100°$ ($\angle$s in the same segment)



PRACTISE NOW 6

SIMILAR QUESTIONS

Exercise 8C Questions 7, 11, 12, 14

Construct $\triangle ABC$ in which $AB = 6.8$ cm, $BC = 10.1$ cm and $AC = 6$ cm.

(i) On the same diagram,

(a) draw the locus of a point equidistant from A and B,

(b) draw the locus of a point equidistant from B and C,

(c) draw the circle through A, B and C.

(ii) Measure and write down the radius of the circle.

(iii) X is the point on the same side of BC as A such that $\triangle BXC$ is isosceles, with $BX = CX$ and $B\hat{X}C = 104°$. Mark the point X clearly on your diagram.

Exercise 8C

BASIC LEVEL

1. Construct, in a single diagram,

(i) $\triangle ABC$ in which $AB = 5$ cm, $B\hat{A}C = 45°$ and $A\hat{B}C = 60°$,

(ii) the perpendicular bisector of AB,

(iii) a circle with centre C of radius 4 cm.

If a point P moves inside ABC so that $PC < 4$ cm and $PA < PB$, shade the region in which P must lie.

2. Construct, in a single diagram,

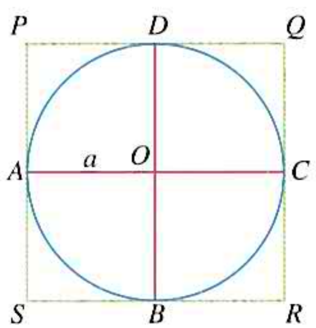
(i) $\triangle PQR$ with side $QR = 11$ cm, $PQ = 7$ cm and $PR = 9$ cm,

(ii) the locus of a point which is 5 cm from P,

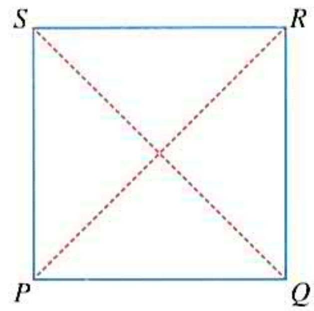
(iii) the locus of a point which is equidistant from PQ and PR.

If a point X moves inside $\triangle PQR$ so that $PX < 5$ cm and X is nearer to PQ than to PR, Indicate by suitable shading on your diagram the region in which X must lie.

3. A circle $ABCD$ of radius a is inscribed in a square $PQRS$. Indicate clearly the region in which the point X must lie, within the square $PQRS$, if $CX \geqslant AX$, $DX \geqslant BX$ and $OX \geqslant a$.



4. X is a point which moves inside the square $PQRS$ so that $XS > XQ$ and $XR > XP$. Make a copy of the diagram and indicate clearly, the region in which X must lie.



5. **(i)** Construct $\triangle ABC$ where the base $BC = 10$ cm, $A\hat{B}C = 40°$ and $AC = 8.5$ cm. Measure and write down the length of AB.

(ii) On your diagram, construct the locus of a point

(a) 4 cm from A,

(b) equidistant from BA and BC.

(iii) The point P, inside $\triangle ABC$, is more than 4 cm from A and nearer to BA than to BC. Indicate clearly by shading, the region of your diagram in which P must lie.

6. Draw a fixed straight line XY, 8 cm long. Draw the locus of P if $\triangle XYP$ has an area of 8 cm². Find the position of P which makes $X\hat{P}Y$ a right angle.

7. Construct, in a single diagram,

(i) $\triangle ABC$, with base $AB = 8.0$ cm, $AC = 9.5$ cm and $BC = 7.5$ cm,

(ii) the point X, such that $A\hat{B}X = 90°$ and X is equidistant from AB and AC.

Measure XB and state this length correct to the nearest centimetre.

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INTERMEDIATE LEVEL

8. Construct, in a single diagram,

(i) $\triangle LMN$ with sides $LM = 7$ cm, $LN = 8$ cm and $MN = 6$ cm,

(ii) the locus of a point which is 5 cm from L,

(iii) the locus of a point which is equidistant from MN and LN.

The position of a point P which lies outside $\triangle LMN$ is such that $LP < 5$ cm and P is nearer to MN than to LN. Indicate clearly by shading, the region in which the point P must lie.

9. **(i)** Using ruler and compasses only, construct $\triangle XYZ$ where the base $XY = 8$ cm, $YZ = 6$ cm and $XZ = 11$ cm. Measure and write down the size of $X\hat{Y}Z$.

(ii) On your diagram construct the locus of a point

(a) 3.8 cm from Y,

(b) equidistant from X and Z.

(c) equidistant from ZX and ZY.

(iii) The point M, inside $\triangle XYZ$, is less than 3.8 cm from Y, nearer to Z than to X and nearer to ZX than to ZY. Indicate clearly by shading, the region of your diagram in which M must lie.

10. $ABCD$ is a square of side 4 cm. A variable point P moves inside the square so that $PA \leqslant 4$ cm, $PC \leqslant PA$, and the area of $\triangle ABP \leqslant 6$ cm^2. Construct $ABCD$ accurately and shade the region in which P must lie.

11. Construct, in a single diagram,

(i) a circle, radius 5 cm, with diameter AC,

(ii) a point B on the circumference of the circle such that $AB = BC$,

(iii) the point D on the circumference of the circle, but on the side of AC opposite to B, such that $C\hat{A}D = 60°$,

(iv) the locus of a point equidistant from AB and AC.

Given that this locus cuts the circle again at P, measure PD.

12. Construct $\triangle ABC$ in which $AB = 8$ cm, $BC = 7.5$ cm, and $AC = 6$ cm. On the same diagram, construct

(i) the locus of point P on the same side of AB as the point C and such that area of $\triangle APB$ = area of $\triangle ACB$,

(ii) **(a)** the locus of a point equidistant from A and B,

(b) the locus of a point equidistant from A and C,

(c) the circle through A, B and C.

ADVANCED LEVEL

13. Construct $\triangle PQR$ in which $PQ = 10$ cm, $QR = 9$ cm and $RP = 7$ cm.

On the same diagram, draw

(i) **(a)** the locus of points equidistant from PQ and PR,

(b) the locus of points equidistant from QP and QR,

(c) the circle touching the sides of $\triangle PQR$.

(ii) Construct the locus of a point which is equidistant from

(a) PQ and PR,

(b) P and Q.

The point X moves inside the inscribed circle so that it is nearer to PQ than to PR and $PX \leqslant QX$. Indicate clearly by shading, the region in which X must lie.

14. Construct the parallelogram $ABCD$ in which $AB = 10$ cm, $AD = 6$ cm and $B\hat{A}D = 50°$. Measure and write down the length of BD. On the same diagram, construct

(i) the locus of a point P which moves so that it is equidistant from A and C,

(ii) the locus of a point Q which moves so that $B\hat{Q}D = 90°$.

The position of point X, which lies inside the parallelogram, is such that $AX \leqslant XC$ and $B\hat{X}D \leqslant 90°$. Indicate clearly by shading, the region in which the point X must lie.

Summary

1. The **locus** of points is the set of points satisfying certain conditions. It can also be defined as the path traced out by a moving object. The locus may be a straight line, a curve, a region in a plane or a region in a space.
2. **Locus Theorem 1:** The locus of a point which is at a given distance d from a given point O is a circle with centre O and radius d.
3. **Locus Theorem 2:** The loci of a point at a given distance d from a given straight line XY are two straight lines, l_1 and l_2, parallel to XY and at a distance d from XY.
4. **Locus Theorem 3:** The locus of a point which is equidistant from two given points X and Y is the perpendicular bisector of the line XY.
5. **Locus Theorem 4:** The locus of a point which is equidistant from two given intersecting straight lines, AB and XY, is a pair of straight lines, l_1 and l_2, which bisect the angles between the two given lines.
6. If two or more loci intersect at a point P, then P satisfies the conditions of the loci simultaneously.

Review Exercise 8

1. Construct, on separate diagrams,
   (i) the locus of a point that moves so that it is constantly 3 cm from a fixed point A, and
   (ii) the locus of points less than 3 cm from the fixed point A.

2. A wooden panel is in the shape of $\triangle ABC$, where $AB = 4.5$ m, $BC = 6.5$ m and $CA = 4$ m.
   (i) Using a scale of 2 cm to 1 m, construct an accurate drawing of $\triangle ABC$. Measure the largest angle.

   Peter wishes to cut the largest possible circle out from this wooden panel.

   (ii) On the same diagram, draw the locus of a point that is equidistant from
   (a) AB and AC, **(b)** AB and BC.
   (iii) Mark clearly with the letter O, the centre of the circle that touches the three sides of $\triangle ABC$.
   (iv) Draw the circle.
   (v) By measurement, find the radius of the largest circle that Peter can cut out from the wooden panel.

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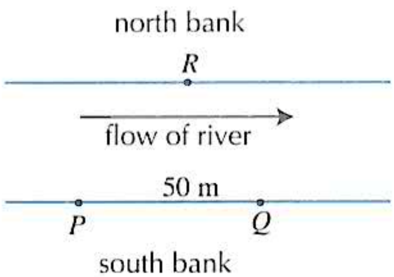
3. The figure below shows the sketch of a river which flows due east between parallel banks.
Two points P and Q are 50 m apart on the south bank. R is a point on the north bank such that the bearing of R from P is 042° and the bearing of Q from R is 140°.



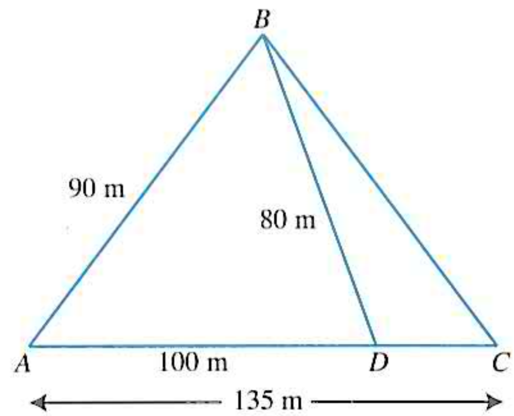
(i) Using a scale of 2 cm to 10 m, construct $\triangle PQR$. Measure PR and find, correct to the nearest metre, the width of the river.

(ii) A boat B_1 is 45 m from P and is equidistant from Q and R. On your diagram, draw the locus which represents points which are
 (a) 45 m from P,
 (b) equidistant from Q and R.
 Mark clearly the position of B_1.

(iii) A second boat B_2 is 15 m from QR and is equidistant from PR and QR. On your diagram, draw the locus which represents points which are
 (a) 15 m from QR,
 (b) equidistant from PR and QR.
 Mark clearly, the position of B_2.

(iv) By measurement, find the distance between the two boats correct to the nearest metre.

4. In the figure below, A represents a house which lies 135 m due west of a church C. Roads run from A to B and from B to C and a path leads from B to the point D on AC. $AB = 90$ m, $BD = 80$ m and $AD = 100$ m.



(i) Using a scale of 1 cm to 10 m, construct an accurate drawing of the diagram. Measure and write down the bearing of B from A.

(ii) A school S is equidistant from A and D. It is also equidistant from AB and AD. On your diagram, draw the locus which represents points which are equidistant from
 (a) A and D, (b) AB and AD.
 Mark clearly the position of the point S.

(iii) A swimming pool P is 80 m from the church C, 40 m from the road AB and nearer to A than C. On your diagram, draw the locus which represents points which are
 (a) 80 m from C, (b) 40 m from AB.
 Mark clearly the position of the point P.

(iv) By measurement, find the distance between the house A and the swimming pool P, correct to the nearest metre.

5. Draw $\triangle XYZ$ such that $XY = 9.1$ cm, $XZ = 6.8$ cm and $Y\hat{X}Z = 80°$.

(i) Measure and write down the length of YZ.

(ii) On your diagram,
 (a) draw the locus of a point equidistant from X and Y,
 (b) by making a further construction, mark clearly with the letter O the centre of the circle that passes through the three vertices of the triangle,
 (c) draw the circle.

6. (i) Construct $\triangle ABC$ in which $AB = 11$ cm, $A\hat{B}C = 102°$ and $BC = 8$ cm.
   (a) Measure and write down the length of AC.
   (b) Measure and write down the size of $B\hat{A}C$.
   (ii) On your diagram,
   (a) draw the locus of a point equidistant from A and B,
   (b) by making a further construction, mark clearly with the letter X the point which is equidistant from A, B and C,
   (c) measure and write down the length of AX.

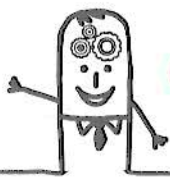
7. (i) Draw $\triangle PQR$ such that $PQ = 12.6$ cm, $QR = 10$ cm and $RP = 5$ cm. Measure and write down the size of $Q\hat{R}P$.
   (ii) Draw the locus of a point equidistant from Q and R.
   (iii) Draw the locus of a point A, on the same side of QR as P, such that $\triangle AQR$ and $\triangle PQR$ have the same area.
   (iv) Hence, mark and clearly label the point X, which is such that $XQ = XR$ and $\triangle XQR$ and $\triangle PQR$ have the same area.

8. (i) Construct $\triangle XYZ$ in which $XY = 10.2$ cm, $XZ = 11$ cm and $ZXY = 62°$. Measure and write down the length of YZ.
   (ii) Draw the locus of a point which is
   (a) equidistant from XZ and XY,
   (b) 6 cm from Z.
   (iii) Measure and write down the length of PQ, given that these two loci intersect at P and Q.
   (iv) Construct, on the same diagram, the rectangle $XYRS$ which is equal in area to $\triangle XYZ$ with R and S on the opposite side of XY to Z.

9. (i) Construct a quadrilateral $PQRS$ in which $PQ = 6$ cm, $PS = 6$ cm, $SR = 7$ cm, $QPR = 45°$ and $P\hat{Q}R = 60°$.
   (ii) On your diagram,
   (a) draw the locus of a point which is 2 cm from S,
   (b) draw the locus of a point X which is such that $P\hat{X}S = 90°$,
   (c) mark clearly with the letter A, the point outside the quadrilateral which is 2 cm from S and is such that $P\hat{A}S = 90°$.
   (iii) Measure and write down the length of PA.
   (iv) Locate on your diagram and mark clearly with the letter B, the point on SP, such that $\triangle PQR$ and $\triangle PQB$ have the same area.
   (v) Measure and write down the length of PB.

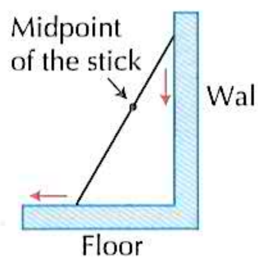
10. A farmer's field $ABCD$ which lies on horizontal ground is bounded on two sides by straight roads which meet at right angles at the point A. The point B lies on one of the roads and the point D lies on the other. Along the other two straight sides, BC and CD, there are fences.
   (i) Given that $AB = 120$ m, $AD = 80$ m, $A\hat{D}C = 110°$ and $A\hat{B}C = 56°$, use a scale of 1 cm to represent 10 m to construct an accurate scale drawing of the field.
   (ii) A tree in the field is 60 m from A and is equidistant from the fences BC and CD. On your diagram, draw
   (a) the locus which represents points which are 60 m from A,
   (b) the locus which represents points which are equidistant from BC and CD. Label, with the letter T, the point representing the position of the tree.
   (iii) A well in the field is 10 m from the road AB and equidistant from the points C and D. Represent the position of the well on your diagram by using the intersection of two loci, labeling it clearly with the letter W.
   (iv) When the angle of elevation of the sun is 15°, the shadow of the tree just reaches the corner B. By further drawing or otherwise, find, correct to the nearest metre, the height of the tree.

11. A playground is in the shape of a quadrilateral $ABCD$. It is given that $AB = 90$ m, $AD = 70$ m, $B\hat{A}D = 115°$, $A\hat{B}C = 100°$ and $A\hat{D}C = 85°$.
   (i) Using a scale of 1 cm to represent 10 m, make an accurate scale drawing of the playground.
   (ii) On your diagram, draw
      (a) the locus which represents points 80 m from B,
      (b) the locus which represents points equidistant from C and D.
   Label the point of intersection of these loci with the letter M.
   (iii) A point N on the playground is equidistant from BC and CD and is such that $A\hat{N}B = 90°$. Represent this point N on your diagram by again using the intersection of two loci.
   (iv) Measure and write down the length MN.
   (v) A square region on the playground, if represented on the scale drawing, would have an area of 5 cm². Calculate the length, in metres, of a side of the actual square.



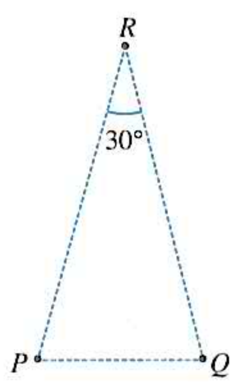
Challenge Yourself

1. A long stick leans vertically against a wall. The stick then slides in such a way that its upper end describes a vertical straight line down the wall, while the lower end crosses the floor in a straight line at right angles to the wall. Construct a number of positions of the midpoint of the stick. Hence, draw its locus.



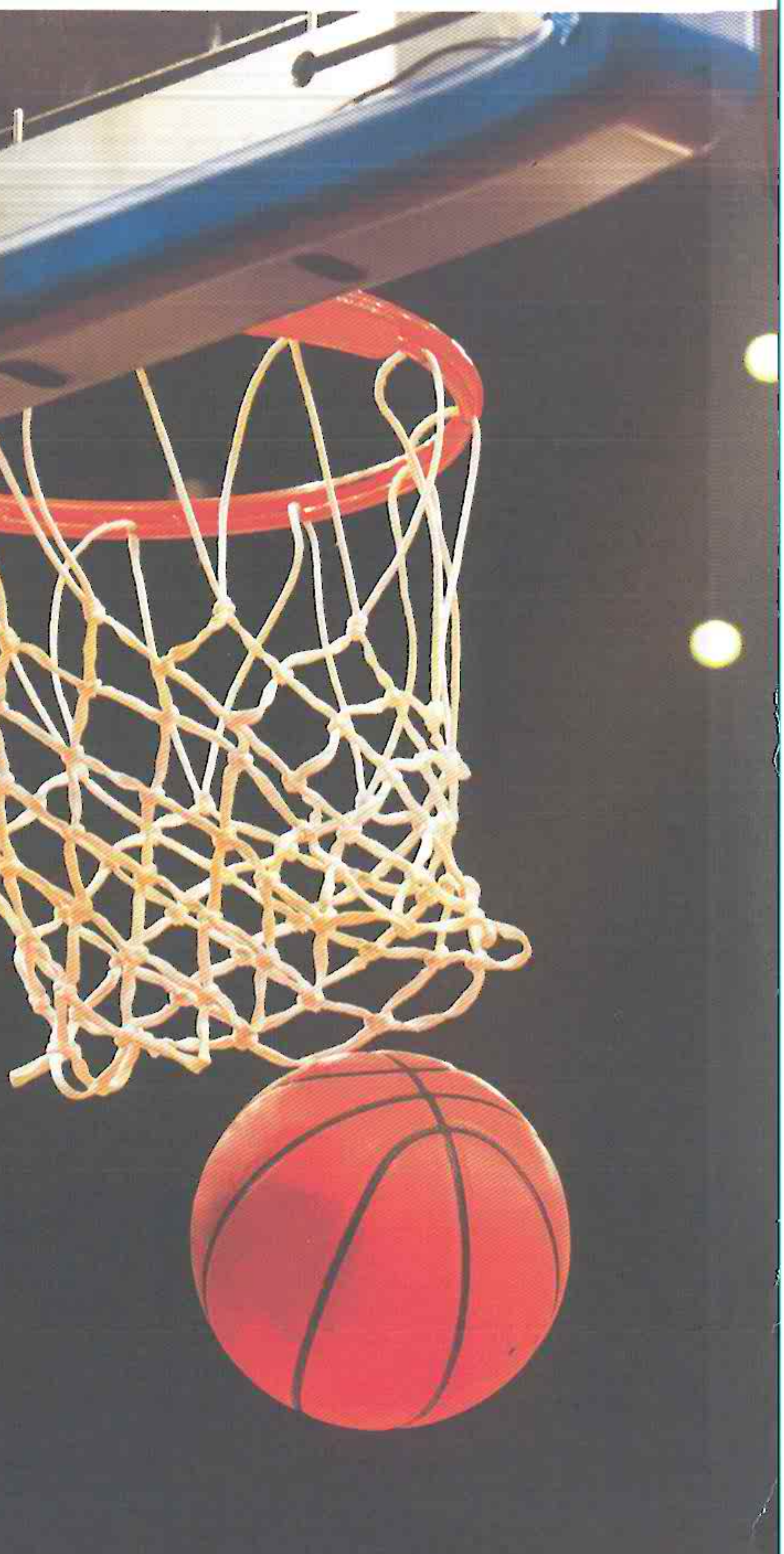
2. On the circumference of a circle of radius 5 cm, mark a fixed point A. If R is a variable point on the circumference and Q is the midpoint of AR, plot a number of positions of Q and hence, find its locus.

3. P and Q are fixed points, and R is a variable point such that $P\hat{R}Q = 30°$. Using a 30° set-square, plot a number of positions of R. Hence, draw its locus.



Revision: Numbers and Algebra

Most events in real life can be modelled by numbers and algebra. In basketball, the trajectory refers to the path taken by the ball after it has been shot by a player. Professional basketball players put in hours of practice on the court to master the perfect trajectory of a shot. Which type of mathematical equation will you use to model the trajectory of a basketball?



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Chapter Nine

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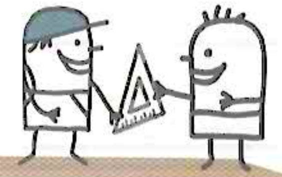
9.1 Numbers and Percentages



Worked Example 1

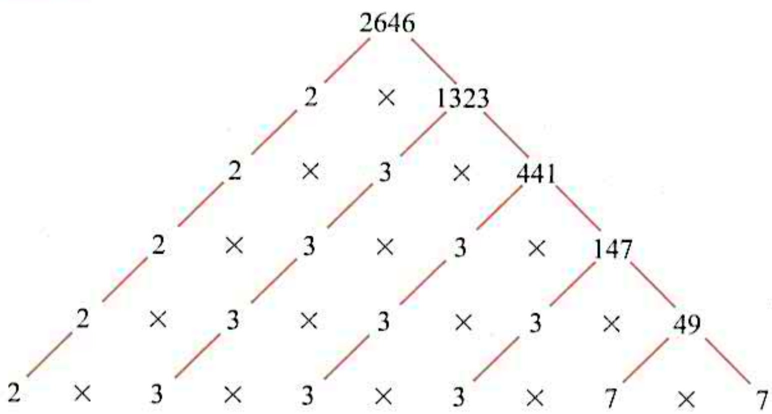
(Prime Factorisation)

(i) Written as a product of its prime factors,
$2646 = 2^x \times 3^y \times 7^z$.
Find the values of x, y and z.

(ii) Hence, find the greatest whole number that will divide both 2646 and 420 exactly.

Solution:

(i)



$2646 = 2 \times 3^3 \times 7^2$
$\therefore x = 1, y = 3, z = 2$

(ii) Divide 420 by the *smallest prime factor* and continue the process until we obtain 1.

start with the smallest *prime factor* →	2	420	
	2	210	← divide 420 by 2 to get 210
	3	105	
	5	35	
	7	7	
		1	← divide until we obtain 1

$420 = 2^2 \times 3 \times 5 \times 7$
$\therefore$ Greatest whole number that will divide both 2646 and 420 exactly
$= 2 \times 3 \times 7$
$= 42$

Problem Solving Tip

In **(i)**, we use a factor tree to express 2646 as a product of its prime factors.
In **(ii)**, the greatest whole number that will divide both 2646 and 420 exactly is the highest common factor (HCF) of the numbers.

Worked Example 2

(Laws of Indices)

Given that $\left(\frac{1}{4}\right)^x = 25^{-\frac{1}{2}} \times \sqrt[3]{125^2} - 5^0$, find the value of x.

Solution:

$$\begin{aligned}
\left(\frac{1}{4}\right)^x &= 25^{-\frac{1}{2}} \times \sqrt[3]{125^2} - 5^0 \\
4^{-x} &= \frac{1}{25^{\frac{1}{2}}} \times \left(\sqrt[3]{125}\right)^2 - 1 \\
&= \frac{1}{\sqrt{25}} \times 5^2 - 1 \\
&= \frac{1}{5} \times 5^2 - 1 \\
&= 5 - 1 \\
&= 4 \\
-x &= 1 \\
x &= -1
\end{aligned}$$

RECALL

- $a^{-n} = \frac{1}{a^n}$, if $a \neq 0$
- $a^0 = 1$, if $a \neq 0$
- $\sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$, if $a > 0$
- $a^{\frac{1}{n}} = \sqrt[n]{a}$, if $a > 0$

Worked Example 3

(Applications of Standard Form)

	Singapore	China	Germany
Population	5.39 million	0.133×10^{10}	81.7×10^6
Land area (km^2)	716	9.60 million	0.357 million

(a) 0.133×10^{10} can be written as k billion. Find the value of k. (1 billion = 10^9)

(b) Using information from the table above,

(i) find the number of times that the population in China is as large as that in Germany,

(ii) calculate the average number of people per square kilometre living in Singapore.

Solution:

(a) $0.133 \times 10^{10} = 1.33 \times 10^9$

$\therefore k = 1.33$

(b) **(i)** Number of times $= \frac{1.33 \times 10^9}{81.7 \times 10^6}$

$= 16.3$ (to 3 s.f.)

(ii) Average number of people per km^2 $= \frac{5.39 \times 10^6}{716}$

$= 7530$ (to 3 s.f.)

Worked Example 4

(Percentage in Practical Situations)

In January 2014, the price of a smartphone was \$990. In January 2015, the price of the same smartphone was reduced by 15%.

(i) Find its price in January 2015.

(ii) Given that the price of \$990 was a decrease of 25% over the price in January 2013, find the price of the smartphone in January 2013.

Problem Solving Tip

In percentage problems, we need to identify the base that represents 100%.

In **(i)**, the price in January 2014 represents the base (100%).

In **(ii)**, since the price decrease is based on that in January 2013, we use the price in January 2013 as a *base*, i.e. the price in January 2013 represents 100%.

Solution:

(i) Price in January 2015 $= \frac{85}{100} \times \$990$

$= \$841.50$

(ii) Price in January 2013 $= \frac{100}{75} \times \$990$

$= \$1320$

Revision 9A

1. **(i)** Calculate $\frac{68.1}{37 - 4.59^2}$, showing all the figures on your calculator display.

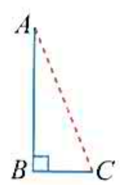
(ii) Give your answer in **(i)** correct to 1 decimal place.

2. A piece of metal has a mass of 121 grams, correct to the nearest gram.

(a) Write down the least possible mass of the piece of metal.

(b) The volume of the piece of metal is 14 cm^3, correct to the nearest cubic centimetre. Find the greatest possible mass of 1 cubic centimetre of the piece of metal.

3. As part of a Mathematics project during an exchange programme, Huixian is required to estimate the height of a building. She paces 20 m from B, the foot of the building, to a point C on level ground, and uses a clinometer to measure the angle of elevation ACB to be 79°. Find AB, the height of the building, giving your answer to a reasonable degree of accuracy.



4. The lowest temperatures recorded at the South Pole and in Singapore are −89.2 °C and 19.4 °C, respectively. Find

(i) the difference in the temperatures recorded,

(ii) the temperature that is mid-way between the two temperatures.

5. Lixin spent $\frac{2}{3}$ of her weekly allowance on food and $\frac{1}{4}$ of the remainder on transport. She had \$27 left. How much is her weekly allowance?

6. **(a)** Express 792 as a product of its prime factors.

(b) A number p has exactly 12 factors. Two of the factors are 4 and 15. Find the value of p.

7. **(a)** Express 525 as the product of its prime factors.

(b) Given that the LCM of 15, x and 35 is 525, find two possible values of x between 15 and 100.

(c) If $525k$ is a perfect square, find the smallest possible integer value of k.

8. When written as the product of their prime factors,

a is $2^4 \times 3^6$,
b is $2^2 \times 3 \times 5$,
c is $2 \times 3^3 \times 5 \times 7$.

Write down, as a product of its prime factors,

(a) the value of the square root of a,

(b) the LCM of a, b and c,

(c) the greatest number that will divide a, b and c exactly.

9. A patient has to take three types of medication – pill A once every 3 hours, pill B once every 4 hours and pill C once every 6 hours. He takes all three pills at 0900. At what time will he take all three pills together again?

10. **(a)** At a school's graduation dinner, there are 402 students and each student pays \$70.35. By approximating both the number of students and the amount paid by each student to 2 significant figures, estimate the total cost of the graduation dinner.

(b) The school photographer took a total of 680 photographs, each with an average file size of 2.5 megabytes. Calculate the total file size, in gigabytes, of the 680 photographs.
Hint: 1 megabyte = 10^6 bytes and
1 gigabyte = 10^9 bytes

11. At a warehouse sale, all prices are reduced by 22.5%. The price of a set of waterproof headphones during the sale is \$139.50.

(i) Find its original price.

(ii) The original price in **(i)** is inclusive of 7% Goods and Services Tax (GST). Find the amount of GST payable.

12. In 2012, the total trade in Singapore was \$985 billion.

(i) Express 985 billion in standard form.
(1 billion = 10^3 million)

In 2013, the total trade decreased by 0.5%.

(ii) Calculate the total trade in 2013 in billion, giving your answer correct to 2 significant figures.

13. A hydrogen atom has an atomic radius of h picometres (pm), where $h = 50$ and 1 pm = 10^{-12} m. Express this radius in metres. Give your answer in standard form.

14. Given that $P = 7.8 \times 10^5$ and $Q = 3.9 \times 10^3$, express each of the following in standard form.

(i) $2P - Q$ **(ii)** $0.3(P + 4Q)$

(iii) $\frac{P}{Q}$ **(iv)** $(2PQ)^{-\frac{1}{3}}$

15. Expressing your answer as a power of 11, find

(i) $11^7 \div 11^{-1}$, **(ii)** $\frac{1}{121^3}$,

(iii) $\sqrt[5]{11}$.

16. **(a)** Given that $36^c \times 2 = 12$, find the value of c.

(b) Simplify $1 \div 2x^{-5}$.

(c) Given that $8^2 \div 16^{\frac{3}{4}} = \frac{1}{2^m}$, find the value of m.

(d) Given that $10^n = \frac{10^5 \times 10}{(10^2)^3}$, find the value of n.

17. The water storage capacity of MacRitchie Reservoir is approximately 4 200 000 m^3.

(i) Convert 4 200 000 m^3 into cm^3, giving your answer in standard form.

(ii) Given that the water storage capacity of Bedok Reservoir is approximately 12.8 million cubic metres, express the storage capacity of Bedok Reservoir as a percentage of that of MacRitchie Reservoir.

18.

	Vietnam	Indonesia	Canada
Population	90.5 million	2.38×10^8	3.45×10^7
Number of Mobile Phones	7.23×10^7	237 million	26.5 million

(a) 2.38×10^8 can be written as k million. Find the value of k.

(b) Using information from the table above,

(i) find the number of times that the population in Vietnam is as large as that in Canada,

(ii) calculate the average number of mobile phones per person in Indonesia.

19. **(a)** The price of rice has increased by 25%. Calculate the percentage of rice consumption to be decreased so that there would be no increase in the expenditure of a household.

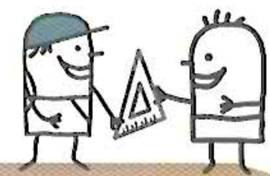
(b) The length of a rectangle is increased by 20% and its width is decreased by 20%. Find the percentage change, if any, in its area.

20. Two property agencies charge the following commissions to sell a piece of property.

	Chan and Partners	Lim and Partners
First $200 000	5%	3.5%
Remaining Selling Price	1.5%	2.5%

Mr Koh wants to sell a condominium for $2.8 million. Determine which property agency he should engage to sell the condominium. Explain your answer.

9.2 Proportion, Ratio, Rate and Speed



Worked Example 5

(Direct Proportion)

Given that y is directly proportional to x^3 and that $y = 24$ when $x = 2$,

(i) express y in terms of x,

(ii) find the value of x when $y = 10\frac{1}{8}$,

(iii) find the percentage increase in y when x is doubled.

Solution:

(i)
$$y = kx^3$$
When $x = 2$, $y = 24$,
$$\begin{aligned} 24 &= k(2)^3 \\ &= 8k \\ k &= 3 \end{aligned}$$
$\therefore y = 3x^3$

(ii) When $y = 10\frac{1}{8}$,
$$\begin{aligned} 10\frac{1}{8} &= 3x^3 \\ x^3 &= \frac{81}{24} \\ &= \frac{27}{8} \\ x &= \frac{3}{2} \\ &= 1\frac{1}{2} \end{aligned}$$

RECALL

If y is directly proportional to x, then $\frac{y}{x} = k$ or $y = kx$, where k is a constant and $k \neq 0$.

(iii) When $x = a$, $y = 3a^3$

When $x = 2a$, $y = 3(2a)^3 = 24a^3$

$$\begin{aligned} \text{Percentage increase in } y &= \frac{24a^3 - 3a^3}{3a^3} \times 100\% \\ &= \frac{21a^3}{3a^3} \times 100\% \\ &= 700\% \end{aligned}$$

Alternatively, if x is doubled, since y is directly proportional to x^3, y is increased to $2^3 = 8$ times of its original value, i.e. when $x_1 \to 2x_1$, then $y_1 \to 8y_1$.

Percentage increase in $y = \frac{8y_1 - y_1}{y_1} \times 100\% = 700\%$

Worked Example 6

(Problem involving Average Speed)

An aeroplane flies a distance of 4650 km from Singapore to Seoul at an average speed of 560 km/h.

(i) Convert 560 km/h into m/s.

(ii) Calculate the flight time, in hours and minutes, correct to the nearest minute.

Solution:

(i) $560 \text{ km/h} = \dfrac{560 \text{ km}}{1 \text{ h}}$

$= \dfrac{560\ 000 \text{ m}}{3600 \text{ s}}$ (convert 560 km into m and 1 h into s)

$= 155\frac{5}{9} \text{ m/s}$

(ii) Flight time $= \dfrac{4650}{560}$

$= 8.304$ h (to 4 s.f.)

$= 8$ h 18 min (to the nearest minute)

RECALL

Time taken $= \dfrac{\text{Distance travelled}}{\text{Speed}}$

Worked Example 7

(Problem involving Map Scale)

A map is drawn to a scale of 1 : 20 000.

(a) Calculate the actual distance, in kilometres, between two towns which are represented on the map by two points 15.5 cm apart.

(b) A lake has an area of 6 km^2. Find, in square centimetres, the area represented by the lake on the map.

Solution:

(a)

Map		**Actual**
1 cm	represents	20 000 cm
15.5 cm	represent	(15.5 × 20 000) cm
		= 310 000 cm
		= 3.1 km

∴ The actual distance between the towns is 3.1 km.

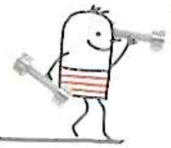
Problem Solving Tip

We should always write what we want to find on the right-hand side.

(b)

Actual		**Map**
20 000 cm	is represented by	1 cm
0.2 km	is represented by	1 cm
0.04 km^2	is represented by	1 cm^2
6 km^2	is represented by	$\frac{6}{0.04} = 150$ cm^2

∴ The area represented by the lake on the map is 150 cm^2.



Revision 9B

1. A farmer has enough food to feed 30 cows for 20 days. If 10 cows are added to the farm, find the number of days the same amount of food will feed all the cows.

2. In a factory, 4 workers can manufacture 80 chairs in 12 days. How long will 24 workers take to manufacture 300 chairs?

3. Nora, Michael and Ethan share 180 postcards in such a way that Nora has 2.5 times as many postcards as Michael, and Michael has 4 times as many postcards as Ethan. How many postcards does each of them have?

4. A sum of money is divided among Jun Wei, Amirah and Shirley in the ratio 3 : 4 : 5. After Shirley gives Jun Wei $30, the ratio becomes 9 : 10 : 11. Find the amount of money Jun Wei has now.

5. Nora deposited $800 in a bank at the end of 2013 and another $700 in the same bank at the end of 2014. The bank offers simple interest at a rate of 1.2% per annum. Find the total amount she has in the bank at the end of 2015.

6. One week before his trip to New Zealand, Rui Feng exchanged Singapore dollars (S$) for New Zealand dollars (NZ$) at a rate of NZ$0.96 = S$1.

   (i) Calculate, in NZ$, the amount received for S$1800.

   Upon his return, he exchanged his remaining NZ$350 into S$ at a rate of NZ$0.98 = S$1.

   (ii) Calculate, in S$, the amount received for NZ$350.

7. In 2014, Mr Chua earned a gross annual income of $220 000. Of this $220 000, the amount that will not be subjected to income tax is shown in the following table:

Personal relief	$3000
Child relief	$4000 per child
Parent relief	$5000 per parent
CPF contributions	$28 500

   Given that Mr Chua lives with 3 children and 2 parents, and that for the remaining income that will be taxed, the gross tax payable for the first $160 000 is $13 950 and the tax rate for the rest is 17%, find his income tax payable.

8. It is given that y is proportional to x^n. Write down the value of n when

   (i) y cm^3 is the volume of a solid sphere of radius x cm,

   (ii) an object travels a fixed distance at a speed of y m/s for x s.

9. Given that y varies inversely as x, copy and complete the table and express y in terms of x.

x	8	10	
y	2.5		0.8

10. A carton contains solid cones with equal heights. The volume, V cm^3, of each cone is directly proportional to the square of the radius, r cm. The cone with radius 5 cm has a volume of 240 cm^3.

    (a) **(i)** Find an equation connecting V and r.

    (ii) Find the volume of the cone with a radius of 8 cm, giving your answer correct to the nearest whole number.

    (b) One of the cones, P, has a radius 25% greater than that of another cone Q. Write down the ratio of the volume of cone P to that of cone Q.

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11. Given that y is directly proportional to x^2 and that the difference between the values of y when $x = 2$ and $x = 5$ is 32, express y in terms of x and hence, find the value of y when $x = 3$.

12. The force of attraction, F Newtons, between two magnets, is inversely proportional to the square of the distance, d cm, between them. When the magnets are 8 cm apart, the force of attraction is 10 Newtons.

    (a) Find a formula for F in terms of d.

    (b) Given that the force of attraction between two magnets is 25 Newtons, find the distance between the magnets.

    (c) When the magnets are at a certain distance apart, the force of attraction is 12 Newtons. Write down the force when the distance is doubled.

13. A map is drawn to a scale of 1 : 20 000.

    (a) This scale can be expressed as 1 cm represents n km. Find the value of n.

    (b) Given that the distance between two police stations on the map is 18 cm, find the actual distance, in kilometres, between the two police stations.

    (c) Given that a nature reserve has an actual area of 3.2 km^2, find the area, in square centimetres, of the nature reserve on the map.

14. On a map whose scale is 1 cm to 4 km, the distance between a university and a polytechnic is 6 cm and the area of the university is 40 cm^2.

    (a) Calculate the actual distance, in km, between the university and the polytechnic.

    (b) On another map whose scale is 1 : n, the area of the university is 0.4 cm^2. Find the value of n.

15. A model lorry is made on a scale of 1 : 20.

    (a) If the model lorry is 24 cm long, find the length of the actual lorry.

    (b) Given that the area of the load platform of the actual lorry is 10 m^2, find the area of the load platform of the model lorry in cm^2.

    (c) If the fuel tank of the model lorry is 30 cm^3, find the volume of the fuel that the actual lorry can hold, giving your answer in litres.

16. An aeroplane flies a distance of 3240 km from Singapore to Taipei City at an average speed of 700 km/h.

    (i) Convert 700 km/h into m/s.

    (ii) Calculate the flight time, in hours and minutes, correct to the nearest minute.

17. The diameter of a wheel of a mountain bicycle is 58.4 cm. If it travels at an average speed of 25 km/h, find the number of revolutions made by the wheel per minute, giving your answer correct to the nearest whole number. (Take π to be 3.142.)

18. A van driver has to cover a journey of 117 km in 6.5 hours. After $2\frac{3}{4}$ h, he finds that he has travelled 57 km. Calculate how much the van driver has to reduce his average speed in order to arrive at his destination on time.

19. A cyclist rode from Burnie to Devonport for 2 hours at a uniform speed of x km/h and then for another 1.5 hours at a uniform speed of $(x - 3)$ km/h.

    (i) Write down, in terms of x, an expression for the distance between Burnie and Devonport.

    After he had rested for half an hour at Devonport, he cycled back to Burnie at a uniform speed of $(x + 1)$ km/h for 3 hours.

    (ii) Write down, in terms of x, another expression for the distance between Burnie and Devonport.

    (iii) Form an equation in x and find the distance between Burnie and Devonport.

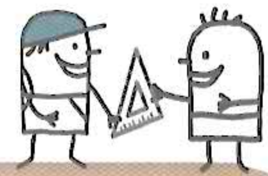
    (iv) Find the average speed for the entire journey.

20. A radar station transmits a signal which travels at a speed of 298 000 km/h. This signal, when reflected from an aircraft, returns to the transmitter at the same speed.

    (i) Convert 298 000 km/h into m/s, giving your answer in standard form.

    (ii) Find the difference in the times between the signals received by reflection from two aircrafts if one is 372.5 m further away from the radar station than the other. Give your answer in standard form.

9.3 Algebraic Manipulation and Formulae



Worked Example 8

(Algebraic Simplification and Expansion)

Simplify each of the following expressions.

(a) $4(3x - 2) - 3(2x - 7)$ **(b)** $6 - [(3x - 7) - (7x - 3)]$

(c) $(2x - 3)^2 - 4x(x - 5)$

Solution:

(a) $$\begin{aligned} 4(3x - 2) - 3(2x - 7) &= 12x - 8 - 6x + 21 \\ &= 6x + 13 \end{aligned}$$

(b) $$\begin{aligned} 6 - [(3x - 7) - (7x - 3)] &= 6 - [3x - 7 - 7x + 3] \\ &= 6 - [-4x - 4] \\ &= 6 + 4x + 4 \\ &= 4x + 10 \end{aligned}$$

(c) $$\begin{aligned} (2x - 3)^2 - 4x(x - 5) &= (2x)^2 - 2(2x)(3) + (3)^2 - 4x^2 + 20x \\ &= 4x^2 - 12x + 9 - 4x^2 + 20x \\ &= 8x + 9 \end{aligned}$$

RECALL

$(a - b)^2 = a^2 - 2ab + b^2$

Worked Example 9

(Factorisation, Algebraic Manipulation and Algebraic Fractions)

(a) Factorise $3xy + 2y - 12x - 8$ completely.

(b) Factorise each of the following expressions completely.

(i) $x^2 - 4y^2$ **(ii)** $x^2 + xy - 2y^2$

Hence simplify $\dfrac{x - y}{x^2 + xy - 2y^2} + \dfrac{2(x - 2y)}{x^2 - 4y^2}$ as a single fraction in its simplest form.

Solution:

(a) $$\begin{aligned} 3xy + 2y - 12x - 8 &= y(3x + 2) - 4(3x + 2) \\ &= (3x + 2)(y - 4) \end{aligned}$$ (arrange the terms into two groups)

(b) (i) $x^2 - 4y^2 = (x + 2y)(x - 2y)$

(ii) $x^2 + xy - 2y^2 = (x + 2y)(x - y)$

$$\begin{aligned} &\frac{x - y}{x^2 + xy - 2y^2} + \frac{2(x - 2y)}{x^2 - 4y^2} \\ &= \frac{x - y}{(x + 2y)(x - y)} + \frac{2(x - 2y)}{(x + 2y)(x - 2y)} \\ &= \frac{1}{x + 2y} + \frac{2}{x + 2y} \\ &= \frac{3}{x + 2y} \end{aligned}$$

(factorise the denominators using the answers in (i) and (ii))

RECALL

$a^2 - b^2 = (a + b)(a - b)$

Worked Example 10

(Changing the Subject of a Formula)

The formula for the volume, V cm³ of an object is given by $V = \left(\pi r^2 h + \frac{1}{3}\pi r^2 a\right)$ cm³.

(i) Make h the subject of the formula.

(ii) Given that $V = 54$, $r = 2.5$ and $a = 1$, find the value of h.

Solution:

(i)
$$V = \pi r^2 h + \frac{1}{3}\pi r^2 a$$
$$\pi r^2 h = V - \frac{1}{3}\pi r^2 a \quad \text{(bring the term containing } h \text{ to one side of the equation)}$$
$$h = \frac{V - \frac{1}{3}\pi r^2 a}{\pi r^2} \quad \text{(divide by } \pi r^2 \text{ on both sides)}$$
$$= \frac{V}{\pi r^2} - \frac{\frac{1}{3}\pi r^2 a}{\pi r^2}$$
$$= \frac{V}{\pi r^2} - \frac{a}{3}$$

(ii) Given that $V = 54$, $r = 2.5$ and $a = 1$,
$$h = \frac{V}{\pi r^2} - \frac{a}{3}$$
$$= \frac{54}{\pi(2.5)^2} - \frac{1}{3}$$
$$= 2.42 \text{ (to 3 s.f.)}$$

Worked Example 11

(Problem involving Algebraic Manipulation)

In a ring shop, there are n rings and x of them are bronze. There are 3 times as many silver rings as bronze rings, while the remaining rings are gold. A bronze ring can be sold for \$50, a silver ring for \$65 and a gold ring for \$105. Find, expressing your answer in terms of n and/or x,

(i) the number of gold rings,

(ii) the total value of all the rings if there are 3 times as many bronze rings as gold rings,

(iii) the total value of all the rings if there are 4 silver rings for every 3 gold rings.

Solution:

(i) Number of silver rings = $3x$

Number of gold rings $= n - 3x - x$
$$= n - 4x$$

(ii) Since the number of bronze rings is x, then the number of gold rings is $\frac{x}{3}$.

Total value of all the rings $= 50x + (65)(3x) + 105\left(\frac{x}{3}\right)$
$$= 50x + 195x + 35x$$
$$= 280x$$

(iii) Ratio of silver rings to gold rings = 4 : 3.

Since the number of silver rings is $3x$, the number of gold rings is $\frac{3x}{4} \times 3 = \frac{9x}{4}$.

Total value of all the rings $= 50x + (65)(3x) + 105\left(\frac{9x}{4}\right)$
$$= 50x + 195x + \frac{945}{4}x$$
$$= \frac{1925}{4}x$$

Worked Example 12

(Number Sequence and Problem Solving)

Study the pattern of a certain series shown in the table below.

Series	S Sum of Series	N Base of Last Term of Series	$N+1$	M $N(N+1)$
$1^3 + 2^3$	9	2	3	6
$1^3 + 2^3 + 3^3$	36	3	4	
$1^3 + 2^3 + 3^3 + 4^3$		4	5	20
$1^3 + 2^3 + 3^3 + 4^3 + 5^3$	225			
$1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3$				42

(a) Copy and complete the table.

(b) Write down a formula connecting S and M.

(c) **(i)** Using your answer in **(b)**, find the value of S when $N = 8$.

(ii) Verify your answer in **(c)(i)** by evaluating $1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3 + 7^3 + 8^3$.

(d) Using your answer in **(b)**, evaluate

(i) $1^3 + 2^3 + 3^3 + \ldots + 3375$,

(ii) $1^3 + 2^3 + 3^3 + \ldots + 24^3$.

(e) Write down a formula, in terms of n, for the sum of the series $1^3 + 2^3 + 3^3 + \ldots + n^3$.

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Solution:

(a)

Series	S Sum of Series	N Base of Last Term of Series	$N + 1$	M $N(N + 1)$
$1^3 + 2^3$	9	2	3	6
$1^3 + 2^3 + 3^3$	36	3	4	12
$1^3 + 2^3 + 3^3 + 4^3$	100	4	5	20
$1^3 + 2^3 + 3^3 + 4^3 + 5^3$	225	5	6	30
$1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3$	441	6	7	42

(b) $S = \left(\frac{M}{2}\right)^2 = \frac{M^2}{4}$

(c) (i) When $N = 8$,

$$M = N(N + 1) = 8 \times 9 = 72$$

$$\therefore S = \frac{72^2}{4} = 1296$$

(ii) $1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3 + 7^3 + 8^3 = 1 + 8 + 27 + 64 + 125 + 216 + 343 + 512$
$= 1296$

(d) (i) $3375 = 15^3$

$\therefore N = 15$ and $M = 15 \times 16 = 240$

$$S = \left(\frac{240}{2}\right)^2 = 14\ 400$$

$\therefore 1^3 + 2^3 + 3^3 + \ldots + 3375 = 14\ 400$

(ii) $N = 24$ and $M = 24 \times 25 = 600$

$$S = \left(\frac{600}{2}\right)^2 = 90\ 000$$

$\therefore 1^3 + 2^3 + 3^3 + \ldots + 24^3 = 90\ 000$

(e)
$$1^3 + 2^3 + 3^3 + \ldots + n^3 = \left[\frac{n(n+1)}{2}\right]^2 = \frac{1}{4}n^2(n+1)^2$$

Revision 9C

1. Simplify each of the following expressions.
   (a) $3(2x-1)-4(x-7)$
   (b) $14-3(5-4x)+6x$
   (c) $7(2y+3)-4(3-y)$
   (d) $9(5p-6)+4(7-13p)$
   (e) $5-3(q+r)-6(3r-2q)$
   (f) $(a+2b)^2-(a-2b)^2$

2. Factorise each of the following expressions completely.
   (a) $5x^2-20x^2y$
   (b) $x^2-4xy+4y^2$
   (c) $(3x+4y)^2-9z^2$
   (d) $6x^2-31x+35$
   (e) $5p^2+11p+2$

3. Simplify each of the following expressions.
   (a) $2[3a-2(3a-1)+4(a+1)]$
   (b) $8(x-y)-[x-y-3(y-z-x)]$
   (c) $2b(c-a)-[3c(a-b)-3a(b+c)]$
   (d) $3(a-c)-\{5(2a-3b)-[5a-7(a-b)]\}$

4. Simplify each of the following algebraic fractions.
   (a) $\dfrac{25a^2}{b^2c}\times\dfrac{bc^3}{100a^3}\div\dfrac{15}{c^2}$
   (b) $\dfrac{8a^5b^2c}{(-2ab)^2}$
   (c) $\dfrac{2a^2b^3}{3b}\div\dfrac{(2a)^2}{15ab^2}$
   (d) $\dfrac{a-1}{a-b}\div\dfrac{1-a}{a^2-b^2}$
   (e) $\dfrac{2x^2+11x+15}{x^2-9}$

5. Factorise each of the following expressions completely.
   (a) $x^2+3y+xy+3x$
   (b) $ab-bc-ac+c^2$
   (c) $ax-kx-ah+kh$
   (d) $20ac-4ad-15kc+3kd$
   (e) $6a^2+3ab-8ka-4kb$

6. Make the letter in the brackets the subject of each of the formulae below.
   (a) $ax^2+bx+c=0$ [b]
   (b) $\dfrac{1}{a}+\dfrac{b}{2}+\dfrac{3}{c}=k$ [c]
   (c) $\sqrt{4x^2-5k}=2x+3$ [x]
   (d) $v^2=u^2+2as$ [u]
   (e) $x=\sqrt[3]{\dfrac{a}{b-a}}$ [a]

7. Express each of the following as a single fraction in its simplest form.
   (a) $\dfrac{3}{4}+\dfrac{x-3}{2x}$
   (b) $\dfrac{2y+3}{9y^2-1}-\dfrac{5}{3y-1}$
   (c) $\dfrac{3a}{a-3}+\dfrac{2}{a+4}$
   (d) $\dfrac{2}{p^2+4p-5}-\dfrac{1}{p-1}$
   (e) $\dfrac{5x}{2x-y}+\dfrac{y}{3x-y}$

8. Express each of the following as a single fraction in its simplest form.
   (a) $\dfrac{1}{1-x}+\dfrac{2}{1+x}+\dfrac{2x}{x^2-1}$
   (b) $\dfrac{3}{x+2}-\dfrac{x-5}{x^2-4}+\dfrac{1}{x-2}$
   (c) $\dfrac{1}{2x-3}-\dfrac{2}{x+2}-\dfrac{2x-x^2}{2x^2+x-6}$
   (d) $\dfrac{5}{x-2}-\dfrac{3x+x^2}{x^2-x-2}+\dfrac{x}{x+1}$

9. Given that $\dfrac{3}{a}=\dfrac{2}{b}+\dfrac{1}{c}$,
   (i) make b the subject of the formula,
   (ii) find the value of b when $a=2\frac{1}{5}$ and $c=-3$.

10. The volume, V m^3, of a certain object is given by the formula $V = \frac{b^2}{2}(a + 3h)$, where h is the height of the object in metres and a and b are constants.

(a) Make h the subject of the formula.

(b) Find

(i) the volume of the object, V, when $h = 1.5$ m, $a = 0.2$ and $b = 1.2$,

(ii) the height of the object, h, when its volume is 12 m^3, $a = 2.5$ and $b = 0.4$.

(iii) the value(s) of b, when the volume and height of the object are 15 m^3 and 2.4 m respectively, and $a = 0.25$.

11. The equation of a hyperbola is given by $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, where (x, y) represent the coordinates of any point on the hyperbola and a and b are constants.

(i) Make y the subject of the equation.

(ii) Find the value(s) of y when $x = -5$, $a = 3$ and $b = 2$.

12. It takes x hours for Pipe A to fill a swimming pool with water. It takes Pipe B 1.5 hours longer than Pipe A to fill up the same swimming pool with water. Write an expression, in terms of x, for the fraction of the pool that

(i) Pipe A can fill in one hour,

(ii) Pipe B can fill in two hours,

(iii) **both** pipes can fill in one hour.

13. **(a)** Write down the next two terms in the sequence

$$\frac{1}{6}, \frac{5}{12}, \frac{2}{3}, \frac{11}{12}, \ldots$$

(b) Write down an expression, in terms of n, for the n^{th} term of the sequence

$$0, -1, -8, -27, -64, \ldots$$

14. The first four terms in a sequence of numbers, $u_1, u_2, u_3, u_4, \ldots$, are given below.

$$u_1 = 1^2 - 1 = 0$$
$$u_2 = 2^2 - 2 = 2$$
$$u_3 = 3^2 - 3 = 6$$
$$u_4 = 4^2 - 4 = 12$$

(i) Find the values of the next two terms of the sequence.

(ii) Find an expression, in terms of n, for the n^{th} term, u_n, of the sequence.

(iii) Find the term of the sequence which has value 110.

15. **(a)** In a class with $(x + 1)$ boys and $(y + 2)$ girls, the average age is q years old. If the average age of the boys is p years old, find the average age of the girls.

(b) In 2 years' time, Michael's age will be twice Ethan's age. If the sum of their ages in 5 years' time is x, find Michael's present age in terms of x.

16. Squares are placed to enclose numbers in the number array as shown in the diagram below.

1	2	3	4	5	6	. . .
2	4	6	8	10	12	. . .
3	6	9	12	15	18	. . .
4	8	12	16	20	24	. . .
5	10	15	20	25	30	. . .

The sum of the numbers in the first square, $S_1 = 1 = 1^2$.

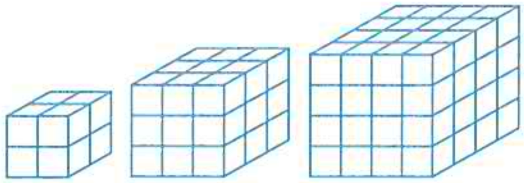
The sum of the numbers in the second square, $S_2 = 1 + 2 + 2 + 4 = 9 = 3^2$.

(i) Find S_3, S_4 and S_5.

(ii) Find a formula for S_n in terms of n.

(iii) The sum of the numbers in the k^{th} square is 44 100. Find the value of k.

17. Ethan arranged small cubes of 1 cm to form cubes with sides 2 cm, 3 cm, 4 cm, etc.



Ethan then painted the outer surfaces of each cube. He observed that for cubes with sides 2 cm ($n = 2$), there are 8 cubes with 3 faces painted. For cubes with sides 3 cm ($n = 3$), there are 8 cubes with 3 faces painted, 12 cubes with 2 faces painted, 6 cubes with only 1 face painted and 1 cube not painted at all.

He tabulated his findings as shown in the table below.

Size of Cubes	Number of Faces Painted			
	3	2	1	0
$n = 2$	8	0	0	0
$n = 3$	8	12	6	1
$n = 4$	a	b	c	d
$n = 5$				
⋮				

(a) Find the values of a, b, c and d.

(b) When $n = 10$, find the number of cubes with 3 faces painted.

(c) When $n = 5$, find the number of cubes that will not be painted.

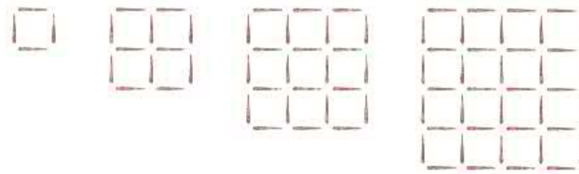
(d) When a cube has size n cm, find in terms of n, the number of cubes with

(i) 2 faces painted,

(ii) 1 face painted,

(iii) none of the faces painted.

18. Raj used toothpicks to construct a series of squares. The first four squares constructed are shown below.



For each square, let T represent the number of toothpicks used, S the total number of small squares formed and P the number of points at which 2 or more toothpicks meet. The values of T, S and P are tabulated as shown in the table below.

Number of Toothpicks Used, T	Number of Small Squares Formed, S	Number of Points at which 2 or More Toothpicks Meet, P
4	1	4
12	4	9
24	9	16
40	16	25
l	m	n

Study the number patterns in the table.

(i) Find the values of l, m and n.

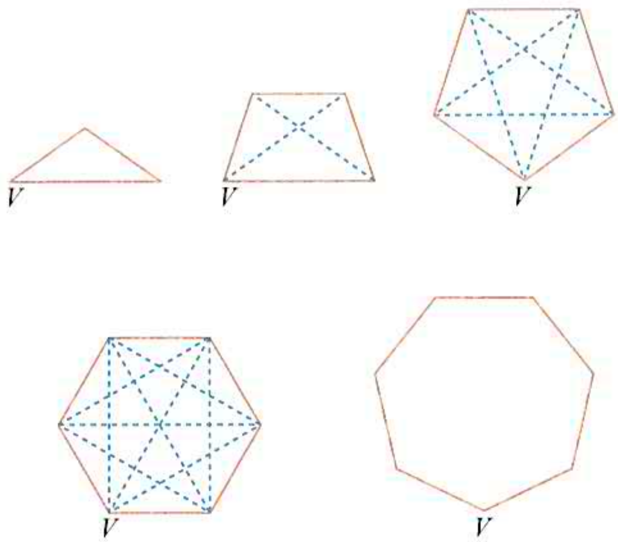
(ii) Write down a formula connecting T, S and P.

(iii) Using your answer in **(ii)**, find the value of P when $T = 364$ and $S = 169$.

(iv) Explain why the number 112 cannot appear in both the S and P columns.

(v) Explain why the number 4442 cannot appear in the T column.

19. The diagram and the table below show the total number of diagonals, d that can be drawn in an n-sided polygon and the number of diagonals, v that can be drawn from a vertex V.



Number of Sides (n)	3	4	5	6	7	8
Number of Diagonals drawn from V (v)	0	1	2	3	p	q
Total Number of Diagonals (d)	0	2	5	9	r	s

(a) Without drawing all the possible diagonals or considering the number patterns, find the value of p and of q in the table. Explain how you find them.

(b) By drawing all the possible diagonals or by considering the number patterns, find the value of r and of s.

(c) Write down a formula connecting n and v.

(d) By studying the three rows of numbers in the table,

(i) find an equation that connects n, v and d,

(ii) hence, express d in terms of n using the result in **(c)**,

(iii) find the total number of diagonals in a 30-sided polygon.

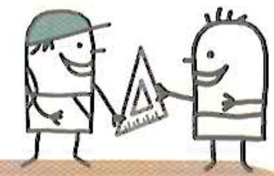
20. The first five terms of a sequence are 1024, 512, 256, p and 64.

(i) State the value of p.

(ii) Find, in terms of n, a formula for the n^{th} term of the sequence.

(iii) Given that the k^{th} term of the sequence is $\frac{1}{4}$, find the value of k.

9.4 Equations and Inequalities



Worked Example 13

(Solving a Linear Equation)

Solve the equation $3(x-1)+2(x+5)=27$.

Solution:

$$\begin{aligned}3(x-1)+2(x+5)&=27\\3x-3+2x+10&=27\\5x+7&=27\\5x&=20\\x&=4\end{aligned}$$

Worked Example 14

(Solving Simultaneous Linear Equations)

Solve the simultaneous equations

$$4x-3y=17,$$
$$2x+5y=15.$$

Solution:

Method 1: by substitution

$4x-3y=17$ ---------- (1)

$2x+5y=15$ ---------- (2)

From (2),

$2x=15-5y$

$x=\dfrac{15-5y}{2}$ ---------- (3)

Substitute (3) into (1):

$$\begin{aligned}4\left(\frac{15-5y}{2}\right)-3y&=17\\2(15-5y)-3y&=17\\30-10y-3y&=17\\13y&=13\\y&=1\\x&=\frac{15-5(1)}{2}\\&=5\end{aligned}$$

$\therefore x=5,\ y=1$

Method 2: by elimination

$4x-3y=17$ ---------- (1)

$2x+5y=15$ ---------- (2)

$2\times$ (2): $4x+10y=30$ ---------- (3)

(3) – (1): $(4x+10y)-(4x-3y)=30-17$

$$\begin{aligned}10y+3y&=13\\13y&=13\\y&=1\end{aligned}$$

Substitute $y=1$ into (1): $4x-3(1)=17$

$$\begin{aligned}4x-3&=17\\4x&=20\\x&=5\end{aligned}$$

$\therefore x=5,\ y=1$

Problem Solving Tip

You can check your solution by substituting the answers which you have obtained into the original equations.

Worked Example 15

(Solving a Fractional Equation)

Solve the equation $\frac{2}{x+3}+\frac{5}{2x+1}=4$.

Solution:

$$
\begin{aligned}
\frac{2}{x+3}+\frac{5}{2x+1} &= 4 \\
\frac{2(2x+1)+5(x+3)}{(x+3)(2x+1)} &= 4 \qquad \left(\frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd}\right) \\
\frac{4x+2+5x+15}{(x+3)(2x+1)} &= 4 \\
\frac{9x+17}{(x+3)(2x+1)} &= 4 \\
9x+17 &= 4(x+3)(2x+1) \qquad \text{(Multiply both sides by } (x+3)(2x+1)\text{)} \\
9x+17 &= 4(2x^2+x+6x+3) \\
9x+17 &= 4(2x^2+7x+3) \\
9x+17 &= 8x^2+28x+12 \\
8x^2+19x-5 &= 0
\end{aligned}
$$

ATTENTION

The LCM of the denominators of $\frac{2}{x+3}$ and $\frac{5}{2x+1}$ is $(x+3)(2x+1)$.

Comparing $8x^2+19x-5=0$ with $ax^2+bx+c=0$, we have $a=8$, $b=19$ and $c=-5$.

$$
\begin{aligned}
x &= \frac{-19\pm\sqrt{19^2-4(8)(-5)}}{2(8)} \qquad \left(x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}\right) \\
&= \frac{-19\pm\sqrt{521}}{16} \\
&= \frac{-19+\sqrt{521}}{16} \quad \text{or} \quad \frac{-19-\sqrt{521}}{16} \\
&= 0.239 \text{ (to 3 s.f.)} \quad \text{or} \quad -2.61 \text{ (to 3 s.f.)}
\end{aligned}
$$

Worked Example 16

(Solving a Linear Inequality)

Find the range of values of x for which $x-3 \leqslant 7$ and $4x-5>22$, and represent the solution set on a number line.

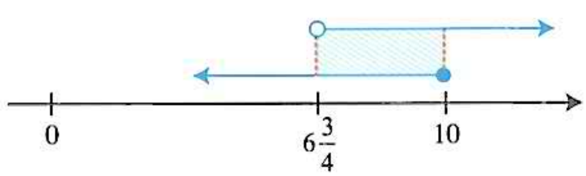
Solution:

$x-3 \leqslant 7$ and $4x-5>22$

$x \leqslant 10$ $\qquad$ $4x>27$

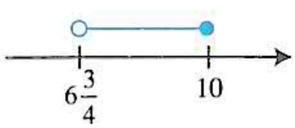
$x>\frac{27}{4}$

$x>6\frac{3}{4}$



$\therefore\ 6\frac{3}{4}<x \leqslant 10$

The solution set is



Worked Example 17

(Greatest and Least Possible Values)

Given that x and y are integers such that $-8 \leqslant x \leqslant 4$ and $-2 \leqslant y \leqslant 3$, find

(i) the greatest value of $x - y$,

(ii) the least value of $2x + y^2$,

(iii) the greatest value of xy,

(iv) the least value of $2x - 3y$.

Solution:

(i) Greatest value of $x - y = 4 - (-2)$ (Greatest value of $x - y$

$= 6$ = greatest value of x – smallest value of y)

(ii) Least value of $2x + y^2 = 2(-8) + 0^2$ (Least value of $2x + y^2$

$= -16$ = least value of $2x$ + least value of y^2)

(iii) Greatest value of $xy = (-8)(-2)$

$= 16$

(iv) Least value of $2x - 3y = 2(-8) - 3(3)$ (Least value of $2x - 3y$

$= -25$ = least value of $2x$ – greatest value of $3y$)

Problem Solving Tip

In **(iii)**, greatest value of $xy \neq$ greatest value of $x \times$ greatest value of y, which is $4 \times 3 = 12$.

Worked Example 18

(Linear Inequalities in Two Variables)

Show, unshaded, the regions satisfied by the following inequalities:

$$3x + y \geqslant 12,\ x + 2y \geqslant 8.$$

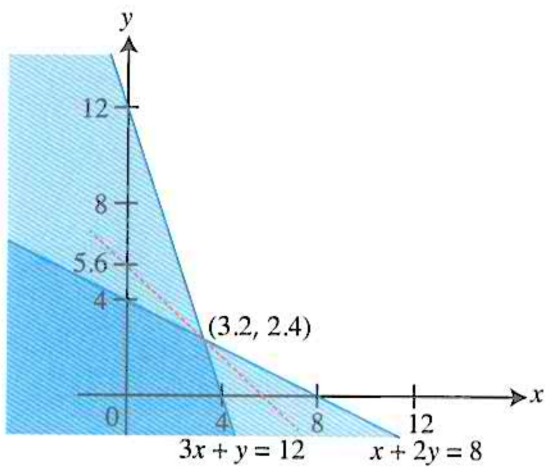
Hence, find the least value of $x + y$ satisfying the region.

Solution:

Draw the lines $3x + y = 12$ and $x + 2y = 8$.

Shade the regions not required by the inequalities: $3x + y \geqslant 12$ and $x + 2y \geqslant 8$

(i) Below $3x + y = 12$ **(ii)** Below $x + 2y = 8$

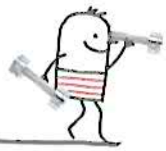


$x + y$ must be satisfied by the unshaded region.

If $x = 3.2$, $y = 2.4$, we obtain the least value of $x + y$.

Least value of $x + y = 3.2 + 2.4$

$= 5.6$



Revision 9D

1. Given that $-2 \leqslant x \leqslant 7$, find the value of
   (i) the largest integer value of x,
   (ii) the smallest integer value of x,
   (iii) the largest prime number of x.

2. Solve each of the following inequalities and illustrate each solution set on a number line.
   (a) $9x - 7 \leqslant 12$
   (b) $7 - 2x > 2$
   (c) $3 + 5x \geqslant 32$
   (d) $3x - 4 \geqslant \frac{1}{3}x - 2$
   (e) $12 < 3x - 1 < 27$

3. Solve each of the following pairs of simultaneous equations.
   (a) $x + 2y = 8,\ 3x + 2y = 12$
   (b) $x + y = \frac{5}{6},\ x - y = \frac{1}{6}$
   (c) $3a - 2b = 1,\ 5a + 3b = -11$
   (d) $3p - 4q - 24 = 0,\ 5p - 6q - 38 = 0$
   (e) $\frac{1}{4}x + \frac{3}{5}y = -4,\ \frac{1}{5}x + \frac{1}{4}y = -\frac{9}{10}$

4. Solve each of the following equations, giving your answers correct to 3 significant figures where necessary.
   (a) $(x - 5)(x + 3) = 7$
   (b) $(3x - 2)(2x + 7) = 0$
   (c) $2x^2 + 5x - 12 = 0$
   (d) $4x^2 - 3x = 7$
   (e) $7x^2 + 15x - 3 = 0$

5. Solve each of the following equations.
   (a) $3x(x + 4) + 28(x + 2) + 21 = 0$
   (b) $2(4x^2 + 23x) = 105$
   (c) $(x - 1)^2 - 16 = 0$
   (d) $\frac{5}{x-1} = 3 + \frac{4}{x}$
   (e) $\frac{x}{1+x} + \frac{x+1}{1-3x} = \frac{1}{4}$
   (f) $\frac{2}{3x} = \frac{x-2}{4(x+3)}$
   (g) $\frac{5}{x+2} - \frac{5}{x^2-4} = 0$

6. One solution of $2x^2 + qx - 2 = 0$ is $x = -2$. Find
   (i) the value of q,
   (ii) the other solution of the equation.

7. The sides of a rectangle are given as 9 cm and 6 cm, correct to the nearest cm. Find
   (a) the smallest possible value of the perimeter of the rectangle,
   (b) the largest possible value of the area of the rectangle.

8. (a) Find the integer value of a for which $-5 < 12 - 3a < -1$.
   (b) Find the odd integer values of b for which $b - 5 \leqslant 7$ and $3b - 2 \geqslant 11$.
   (c) Solve the inequality $\frac{2}{7} < 2x - 1 < \frac{9+2x}{12}$, illustrating the solution on a number line.

9. Given that $-5 \leqslant 4x + 1 \leqslant 2x + 9$ and $-6 \leqslant 2y - 2 \leqslant 8$, find
   (a) the greatest value of $x - y$,
   (b) the smallest value of $(x + y)(x - y)$.

10. A ball is thrown down from the top of a building. The distance, d m, of the ball above the ground can be modelled by the equation $d = 35 + 7t - 2t^2$, where t is the time in seconds after the ball is thrown. Find
    (i) the time(s) when the ball is 40 m above the ground,
    (ii) the time taken for the ball to reach the ground.

11. Given that $1 \leqslant x \leqslant 5$ and $-1 \leqslant y \leqslant 7$, find

(i) the greatest possible value of $2x - y$,

(ii) the least possible value of $2xy$,

(iii) the greatest possible value of $\frac{y}{x}$.

12. In a multiple-choice quiz of 25 questions, 2 points are awarded for a correct answer and 1 point is deducted for an incorrect answer. No points are deducted for an unanswered question. Raj answered only 21 questions. If Raj's total quiz score is not more than 33, find the maximum number of questions he answered correctly.

13. The diagonal of a rectangle exceeds its length by 4 cm and its width by 6.4 cm. Find the length of the diagonal.

14. The length of a page of a book is 3.4 cm more than the width. The area of a page is 125 cm². Given that the width is x cm,

(i) form an equation in x,

(ii) hence, find the length of a page, giving your answer correct to 2 decimal places.

15. A 2.4-km nature trail at MacRitchie Reservoir is divided into two sections, A and B. A cross-country runner took 9 minutes and 10 seconds to complete the trail. His average speed for Section A of the trail is 4 m/s, and his average speed for Section B of the trail is 5 m/s. Find the distance for each of the two sections of the trail, expressing your answers in kilometres.

16. A 2-digit number is such that the sum of its digits is 11. When the digits are reversed, it is 27 less than the original number. Find the original number.

17. A night train leaves Singapore for Segamat and returns to Singapore. The distance between Singapore and Segamat is 200 km.

(i) If the train travels from Singapore to Segamat at an average speed of x km/h, write down an expression for the time taken, in hours, for the journey.

(ii) On the return journey from Segamat to Singapore, the train increases its speed by 5 km/h. Write down an expression for the time taken, in hours, for the return journey.

(iii) If the difference in the time taken for the two journeys is 1 hour 15 minutes, form an equation in x and show that it reduces to $x^2 + 5x - 800 = 0$.

(iv) Solve the equation $x^2 + 5x - 800 = 0$ and hence, find the speed of the train from Segamat to Singapore.

18. The numerator of a fraction is 2 less than the denominator. When both the numerator and the denominator increase by 3, the fraction increases by $\frac{3}{20}$. Find the original fraction.

19. A fruitseller bought x apples for \$32 and sold them at a profit of 5 cents per apple.

(a) Write down, in terms of x, expressions for

(i) the cost price of one apple,

(ii) the selling price of one apple.

When he sold all except 20 of the apples, he found that he had received \$35.

(b) Form an equation in x and show that it reduces to $x^2 - 80x - 12\,800 = 0$.

(c) Solve the equation and hence find the number of apples the fruitseller bought.

20. A marine aquarium at SeaWorld can be filled by two pipes, A and B in 48 minutes if they are both operating at the same time. When operating by itself, Pipe B takes 40 minutes longer than Pipe A to fill the aquarium.

(a) Given that Pipe A takes x minutes to fill the aquarium, form an equation in x and show that it reduces to $x^2 - 56x - 1920 = 0$.

(b) **(i)** Solve the equation $x^2 - 56x - 1920 = 0$.

(ii) Briefly explain why one of the answers cannot be accepted.

(c) Find the time taken by Pipe B to fill the aquarium by itself, giving your answer in hours.

21. The masses of two packages, measured to the nearest 0.5 kg, are 3.5 kg and 5.5 kg. Calculate the upper and lower bounds of their combined mass.

22. The mass of a toy, in the shape of a cube, of sides 5 cm is given as 250 g. The length of the side is measured to the nearest millimetre and the mass is measured to the nearest gram. Given that density = $\frac{\text{mass}}{\text{volume}}$, calculate the maximum and minimum possible values for the density of the material.

23. Find the upper and lower bounds of $\frac{56 \times 2}{4}$ if each number is given to the nearest whole number. Give your answer to 3 significant figures.

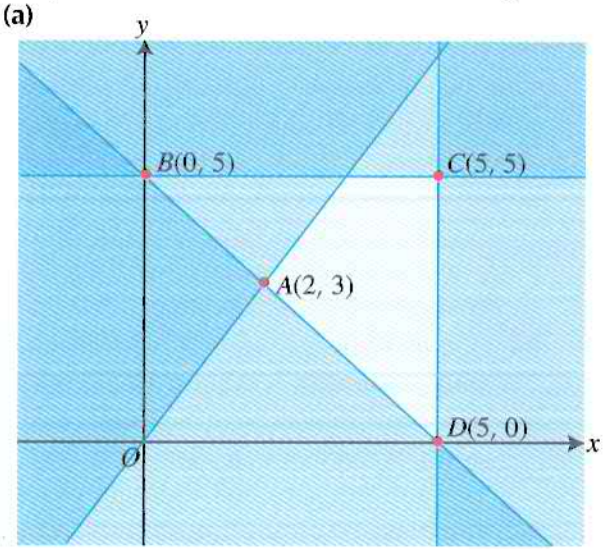
24. Show, unshaded, the regions satisfied by the following inequalities:

(a) $x \leqslant 1$, $y > 0$, $y < 2x + 3$

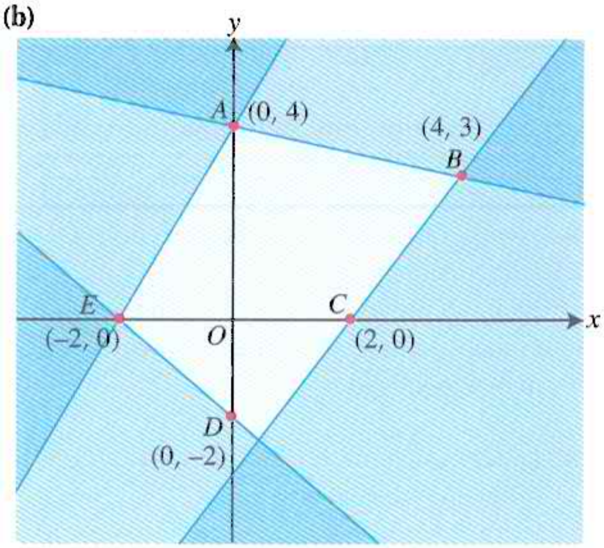
(b) $x < 0$, $2y \leqslant x + 8$, $y + 2x > 0$

25. In each of the following cases, write down the inequalities which define the unshaded region.

(a)



(b)



26. The set of points with coordinates (x, y) satisfies the three inequalities:

$y + x \leqslant 8$, $4y + 3x \geqslant 24$ and $3y \leqslant 2x$.

(i) Using 2 cm to represent 1 unit on each axis, construct accurately on graph paper, and clearly indicate by shading the unwanted regions, the region in which the set of points (x, y) must lie.

(ii) Using your graph, find the greatest value of $(x - y)$ for points in the region.

27. A children's book is to be published in both hardback and paperback editions. A bookstore is ordering more than 20 hardback copies, 40 or more paperback copies, at least 70 but not more than 80 copies altogether.

(a) Using x to represent the number of hardback copies and y to represent the number of paperback copies, write down the inequalities to represent these conditions.

(b) The point (x, y) represents x hardback copies and y paperback copies. Using a scale of 1 cm to represent 10 books on each axis, construct and indicate clearly, by shading the unwanted regions, the region in which (x, y) must lie.

(c) Given that the profit of each hardback copy is \$30 and the profit of each paperback copy is \$20, calculate the number of each edition that the bookstore must buy to give the maximum profit.

28. Leave unshaded the region defined by the following inequalities:

$x \geqslant 0$, $y \geqslant 0$, $x \leqslant 4$, $y \leqslant 6$ and $y \leqslant 8 - x$.

Find the largest value of $2x + 4y$ which satisfies the above inequalities.

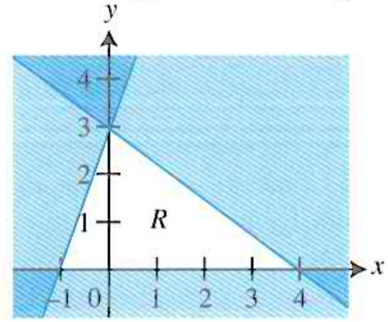
29. Leave unshaded the region defined by the following inequalities:

$x > 0$, $y > 0$, $x + 2y \leqslant 10$ and $y + 3x \leqslant 10$.

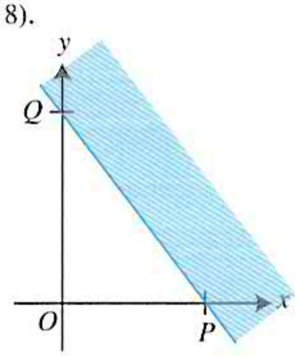
Find the largest value of $3x + 5y$ which satisfies the above inequalities.

30. In a graph, leave unshaded the region that simultaneously satisfies the following inequalities:
$x \geqslant 0$, $y \geqslant 0$, $x + y \geqslant 2$, $x + y \leqslant 8$ and $2y \leqslant x + 10$.
Find the maximum and minimum values of $2x - y$ which satisfy the above inequalities.

31. Write down the three inequalities needed to define the unshaded region R in the diagram.



32. In the diagram, P is the point $(6, 0)$ and Q is the point $(0, 8)$.

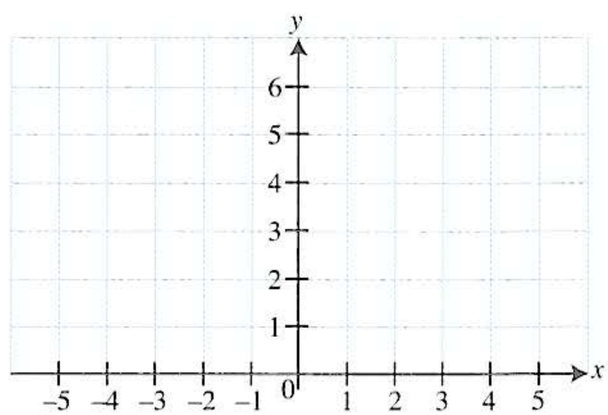


(i) Find the equation of the line PQ.

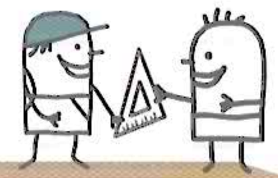
(ii) Write down the inequality needed to define the unshaded region.

33. Leave unshaded the region R defined by the following inequalities:

$x < 3$, $y < 5$, $x + 4y \geqslant 4$ and $4y \leqslant 5x + 20$.



9.5 Functions and Graphs

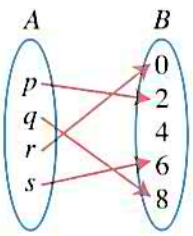


Worked Example 19

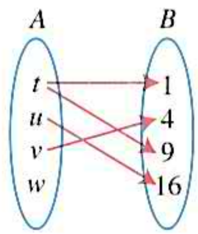
(Verifying if a Relation is a Function)

State, with reason, whether each of the following arrow diagrams defines a function.

(a)



(b)



Solution:

(a) The relation is a function since every element in the domain A has a unique image in the codomain B.

(b) The relation is not a function since the element t in the domain A has two images, 1 and 9, and the element w does not have an image in the codomain B.

Worked Example 20

(Inverse Functions)

A function f is defined as $f : x \mapsto \dfrac{9x-2}{x-2}$ where $x \neq 2$. Find the inverse function $f^{-1}(x)$ and state the value of x for which f^{-1} is not defined. Hence, evaluate $f^{-1}\left(\dfrac{1}{4}\right)$.

Solution:

$f(x) = \dfrac{9x-2}{x-2}$

Let $y = \dfrac{9x-2}{x-2}$

$\therefore$ $f(x) = y$ and $f^{-1}(y) = x$

$$y(x-2) = 9x-2$$
$$yx - 2y = 9x - 2$$
$$yx - 9x = 2y - 2$$
$$x(y-9) = 2(y-1)$$
$$x = \frac{2(y-1)}{y-9}$$
$$\therefore f^{-1}(y) = \frac{2(y-1)}{y-9}$$

Hence, $f^{-1}(x) = \dfrac{2(x-1)}{x-9}$ or $f^{-1} : x \mapsto \dfrac{2(x-1)}{x-9}$

f^{-1} is not defined when $x - 9 = 0$, i.e. $x = 9$.

$$f^{-1}\left(\frac{1}{4}\right) = \frac{2\left(\frac{1}{4}-1\right)}{\frac{1}{4}-9}$$
$$= \frac{6}{35}$$

Worked Example 21

(Sketching the Graph of $y = (x - p)^2 + q$)

Given the quadratic function $y = (x - 3)^2 - 1$,

(i) find the coordinates of the x- and y-intercepts,

(ii) write down the coordinates of the minimum point of the graph,

(iii) sketch the graph,

(iv) state the equation of the line of symmetry of the graph.

Solution:

(i) When $y = 0$,

$$(x - 3)^2 - 1 = 0$$
$$(x - 3)^2 = 1$$
$$x - 3 = 1 \quad \text{or} \quad x - 3 = -1$$
$$x = 4 \qquad\qquad x = 2$$

$\therefore$ The graph cuts the x-axis at (2, 0) and (4, 0).

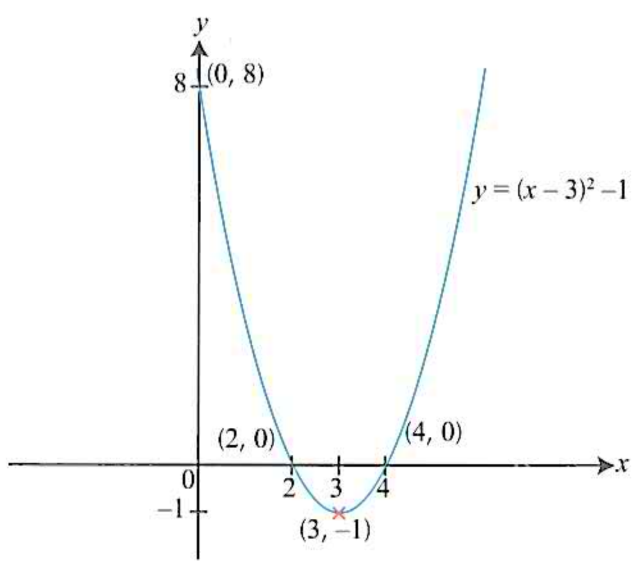
When $x = 0$,

$$y = (0 - 3)^2 - 1$$
$$= 8$$

$\therefore$ The graph cuts the y-axis at (0, 8).

(ii) The coordinates of the minimum point are (3, –1).

(iii) Since the coefficient of x^2 is 1, the graph opens upwards.



(iv) The equation of the line of symmetry is $x = 3$.

RECALL

For graphs of a quadratic function of the form $y = (x - p)^2 + q$, the coordinates of the minimum point are (p, q).

ATTENTION

A quadratic function is symmetrical about the line $x = a$, where a is the x-coordinate of its maximum or minimum point.

Worked Example 22

(Graphical Solutions)

The variables x and y are connected by the equation $y = x^2 + \frac{10}{x} - 8$. Some corresponding values of x and y are given in the following table.

x	0.5	1	1.5	2	3	4
y	12.3	3	p	1	4.3	10.5

(a) Find the value of p.

(b) Using a scale of 2 cm to represent 1 unit on the x-axis and 1 cm to represent 1 unit on the y-axis, draw the graph of $y = x^2 + \frac{10}{x} - 8$ for $0.5 \leqslant x \leqslant 4$.

(c) Use your graph to find the values of x in the interval $0.5 \leqslant x \leqslant 4$ for which $x^2 + \frac{10}{x} = 14$.

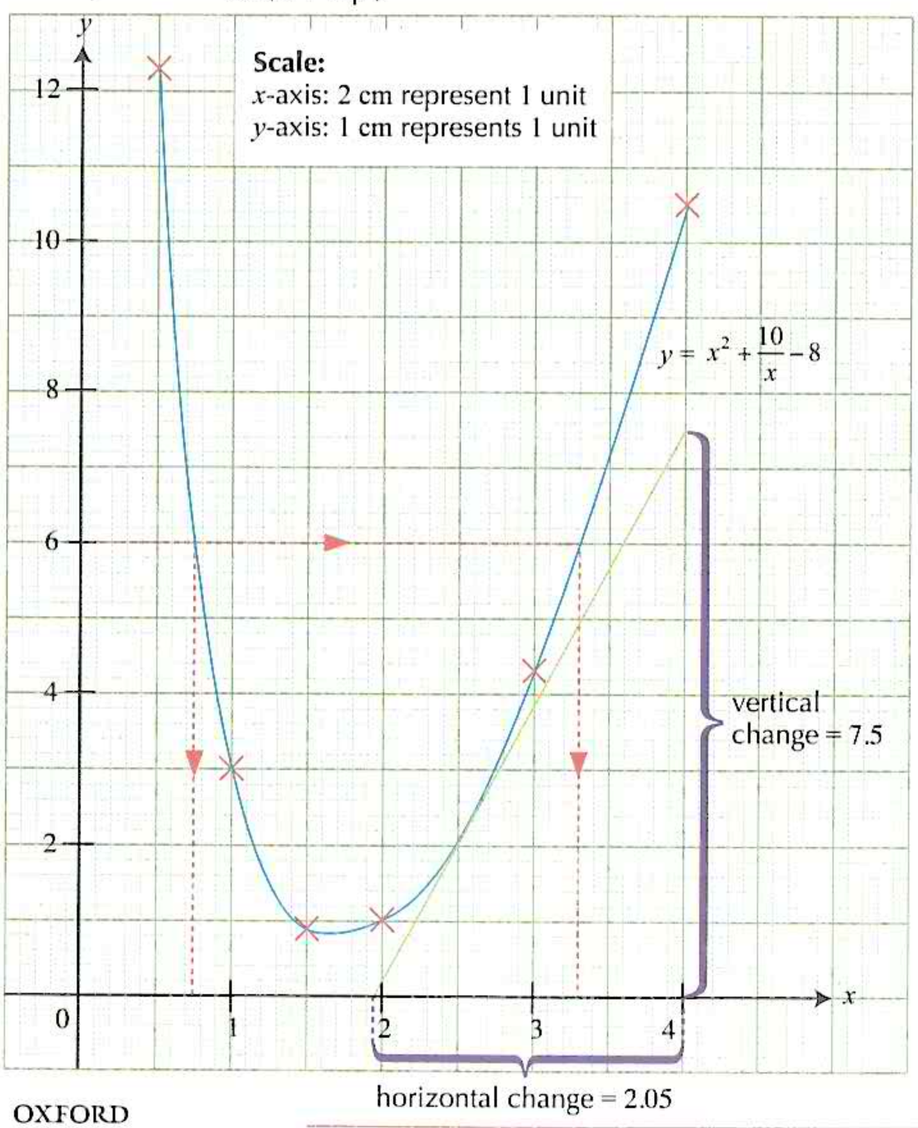
(d) By drawing a tangent, find the gradient of the curve at the point when $x = 2.5$.

Solution:

(a) When $x = 1.5$, $p = 1.5^2 + \frac{10}{1.5} - 8$

$= 0.9$ (to 1 d.p.)

(b)



(c) $x^2 + \dfrac{10}{x} = 14$

$x^2 + \dfrac{10}{x} - 8 = 6$

From the graph, when $y = 6$, $x = 0.75$ or $x = 3.3$.

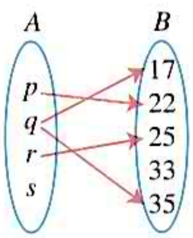
(d) From the graph,

$$\text{Gradient} = \frac{\text{vertical change}}{\text{horizontal change}}$$
$$= \frac{7.5 - 0}{4 - 1.95}$$
$$= \frac{7.5}{2.05}$$
$$= 3.66 \text{ (to 3 s.f.)}$$

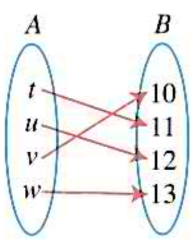
Revision 9E

1. State, with reason, whether each of the following arrow diagrams defines a function.

(a)



(b)



2. If $f(x) = 7x - \dfrac{1}{2}$ and $F(x) = 8x + 1$, express

(i) $f(a)$, **(ii)** $F(a+1)$,

(iii) $f\left(\dfrac{a}{7}\right) + F(a-1)$,

in terms of a.

3. A function f is defined by $f : x \mapsto 3x^2 - \dfrac{1}{4}$ for all real values of x. What are the images of -5, $\dfrac{1}{8}$ and 7 under f?

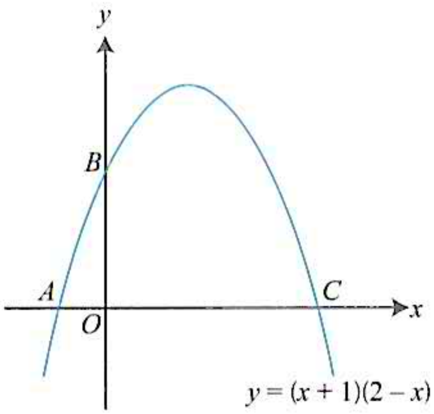
4. A function f is defined by $f : x \mapsto 2x + 11$. Find the inverse function $f^{-1}(x)$.

5. A function g is defined by $g : x \mapsto 2x - 5$ for all real values of x. What are the elements in the domain, whose images are -8, 0, 3 and $\dfrac{1}{3}$?

6. A function f is defined by $f : x \mapsto ax + b$. Given that $f(4) = 6$ and $f^{-1}(8) = 5$, find the value of a and of b. Hence find $f^{-1}(x)$.

7. A function g is defined by $g : x \mapsto \dfrac{5x-7}{x-4}$ where $x \neq 4$. Find $g^{-1}(x)$ and state the value of x for which g^{-1} is not defined.

8. In the figure, the curve $y = (x + 1)(2 - x)$ cuts the x-axis at the points A and C and the y-axis at B.



Find

(i) the coordinates of A, B and C,

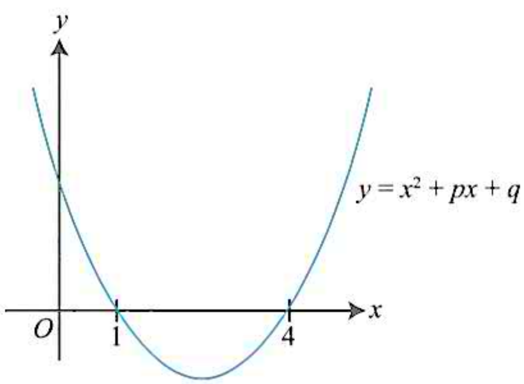
(ii) the equation of the line of symmetry of the curve.

9. The curve $y = x^2 + hx - 5$ cuts the y-axis at A and passes through the point $(1, -2)$. Find

(i) the coordinates of A,

(ii) the value of h.

10. The figure shows part of the graph of $y = x^2 + px + q$. The graph cuts the x-axis at $x = 1$ and $x = 4$.



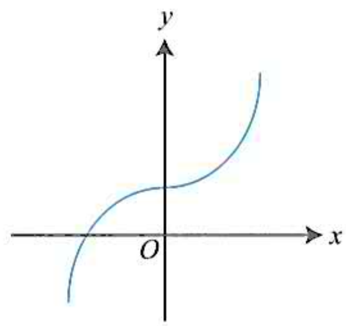
Find the value of p and of q.

11.

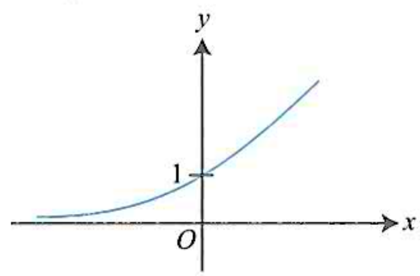
$y = x^2 + 4$	$y = 4^{-x}$	$y = 4 - x^3$	$y = \dfrac{4}{x}$
$y = \dfrac{4}{x^2}$	$y = 4^x$	$y = x^3 + 4$	$y = 4 - x^2$

By selecting one of the equations from the box above, write down a possible equation for each of the following sketch graphs.

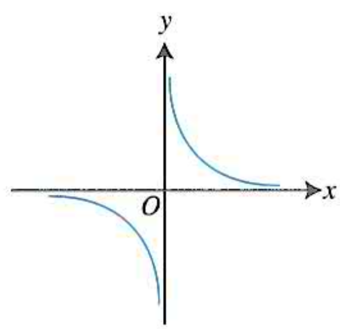
(a)



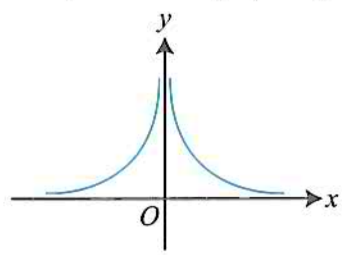
(b)



(c)

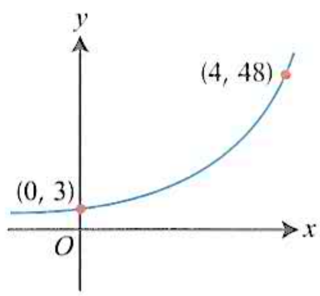


12. The sketch represents the graph of $y = x^n$.



Write down a possible value of n.

13. The sketch shows the graph of $y = ka^x$. The points (0, 3) and (4, 48) lie on the graph.



Find the value of k and of a.

14. On separate diagrams, sketch the graph of each of the following functions.

(a) $y = x^2 + 2$ **(b)** $x + y = 2$

15. The graph of $y = 9 - 4x^2$ intersects the y-axis at P and the x-axis at Q and R. Write down the coordinates of P, Q and R.

16. **(a)** Sketch the graph of $y = (x + 1)(x - 5)$.

(b) Write down the equation of the line of symmetry of the graph.

(c) Find the coordinates of the minimum point.

17. Given the quadratic function $y = -(x + 3)^2 + 1$,

(i) find the coordinates of the x- and y-intercepts,

(ii) write down the coordinates of the maximum point of the graph,

(iii) sketch the graph,

(iv) state the equation of the line of symmetry of the graph.

18. **(i)** Express $x^2 - 6x + 4$ in the form $(x - p)^2 + q$.

(ii) Hence, write down the coordinates of the minimum point of the graph $y = x^2 - 6x + 4$.

(iii) Sketch the graph of $y = x^2 - 6x + 4$.

(iv) State the equation of the line of symmetry of the graph.

19. **(i)** Express $x^2 + x + 3$ in the form $(x + p)^2 + q$.

(ii) Hence, write down the coordinates of the minimum point of the graph $y = x^2 + x + 3$.

(iii) Sketch the graph of $y = x^2 + x + 3$.

(iv) State the equation of the line of symmetry of the graph.

20. Consider the equation $-2x + y = -2$.

(a) Copy and complete the table.

x	–1	0	2
y			

(b) Using a scale of 4 cm to represent 1 unit on the x-axis and 2 cm to represent 1 unit on the y-axis, draw the graph of $-2x + y = -2$ for $-1 \leqslant x \leqslant 2$.

(c) **(i)** On the same axes in **(b)**, draw the graph of $y = 2$.

(ii) Find the area of the trapezium bounded by the lines $-2x + y = -2$, $y = 2$, and the x- and y-axes.

21. **(a)** Using a scale of 4 cm to represent 1 unit on the x-axis and 2 cm to represent 1 unit on the y-axis, draw the graph of $y = 2^x - 5$ for $-1.5 \leqslant x \leqslant 2.5$.

(b) Use your graph to find the value of x when $y = -1.5$.

(c) Explain why the graph of $y = 2^x - 5$ will not lie below the line $y = -5$ for all real values of x.

22. The variables x and y are connected by the equation $y = 2(x + 1)(x - 3)$. Some corresponding values of x and y are given in the following table.

x	-2	-1	0	1	2	3	4
y		0	-6	-8		0	10

(a) Using a scale of 2 cm to represent 1 unit on the x-axis and 1 cm to represent 2 units on the y-axis, draw the graph of $y = 2(x + 1)(x - 3)$ for $-2 \leqslant x \leqslant 4$.

(b) By drawing a tangent, find the gradient of the curve at the point where $x = 2$.

(c) **(i)** On the same axes, draw the graph of $y = 2x - 4$ for $-2 \leqslant x \leqslant 4$.

(ii) Hence, solve the equation $2(x + 1)(x - 3) = 2x - 4$.

(iii) State the range of values of x for which $2(x + 1)(x - 3) \leqslant 2x - 4$.

23. **(a)** Using a suitable scale, draw the graph of $y = \frac{5}{x} + 2x - 3$ for $0.5 \leqslant x \leqslant 7$.

(b) Find the gradient of the curve when $x = 3$.

(c) Use your graph to estimate the solutions to the equations

(i) $\frac{5}{x} + 2x - 8 = 0$, **(ii)** $\frac{5}{x} + x - 6 = 0$.

24. The variables x and y are connected by the equation $y = x^2(x - 2)$. The table below shows some values of x and the corresponding values of y.

x	-2	-1	0	1	2	3	4
y	-16		0		0		

(a) Copy and complete the table.

(b) Using a scale of 2 cm to represent 1 unit on the x-axis and 1 cm to represent 2 units on the y-axis, draw the graph of $y = x^2(x - 2)$ for $-2 \leqslant x \leqslant 4$.

(c) By drawing a tangent, find the gradient of the curve at the point where $x = 1.5$.

(d) Use your graph to solve the equation $x^2(x - 2) = -1$.

(e) On the same axes, draw the graph of $y = x - 2$ for $-2 \leqslant x \leqslant 4$.

(f) **(i)** Write down the x-coordinates of the points where the two graphs intersect.

(ii) These values of x are the solutions of the equation $x^3 + Ax^2 + Bx + C = 0$. Find the value of A, of B and of C.

25. The variables x and y are connected by the equation $y = x^2 + 3x - 3$. Some corresponding values of x and y are given in the following table.

x	-5	-4	-3	-2	-1	0	1	2
y	7	1	-3	-5	-5	-3	1	7

(a) Using a scale of 2 cm to represent 1 unit on the x-axis and 1 cm to represent 1 unit on the y-axis, draw the graph of $y = x^2 + 3x - 3$ for $-5 \leqslant x \leqslant 2$.

(b) Use your graph to find

(i) the values of x for which $x^2 + 3x = 8$,

(ii) the values of x for which $x^2 + 4x = 2$,

(iii) the range of values of x for which $x^2 + 3x - 3 \leqslant x$.

(c) By drawing a suitable tangent to your curve, find the coordinates of the point at which the gradient of the tangent is equal to 1.

26. The volume of an open rectangular box, made with metal of negligible thickness, is 35 m^3. The base of the box is a square of side x m.

(a) Find, in terms of x, an expression for the height of the box.

(b) The total external surface area of the base and the four sides is given as A m^2. Show that $A = x^2 + \frac{140}{x}$.

(c) The table below shows some values of x and the corresponding values of A correct to 1 decimal place.

x	2	2.5	3	3.5	4	4.5	5	5.5	6
A	74			52.3	51	51.4		55.7	

Copy and complete the table.

(d) Using a scale of 4 cm to represent 1 m, draw a horizontal x-axis for $2 \leqslant x \leqslant 6$.

Using a scale of 4 cm to represent 5 m^2, draw a vertical A-axis for $50 \leqslant A \leqslant 75$.

On your axes, plot the points given in the table and join them with a smooth curve.

(e) Use your graph to find

(i) two possible values for the length of the base which will give a total external surface area of 55 m^2,

(ii) the minimum value of A,

(iii) the height of the box for which the least amount of metal is used.

27. The table below shows the speed of a van over a period of 8 seconds.

Time (t s)	0	1	2	3	4	5	6	7	8
Speed (v m/s)	0	4.5	7.8	10.1	11.6	12.4	13	13.4	14

After the 8th second, the van moves with a constant speed of 14.0 m/s. Using a scale of 2 cm to represent 1 second on the horizontal t-axis and 1 cm to represent 1 m/s on the vertical v-axis, plot the graph of the motion of the van for $0 \leqslant t \leqslant 8$.

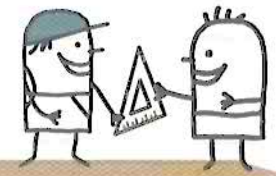
A sports car starts from rest at $t = 2$ and moves with an acceleration of 3 m/s^2. Plot the graph of the motion of the sports car for $2 \leqslant t \leqslant 8$.

Hence, find

(a) the time at which the two vehicles have the same speed,

(b) the acceleration of the van when $t = 5$.

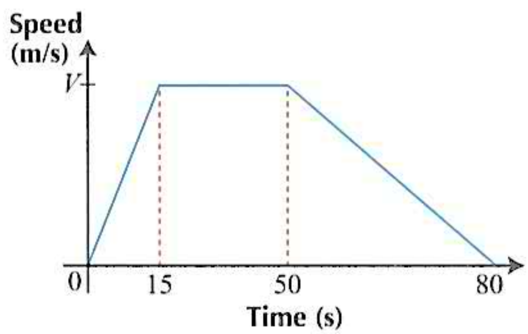
9.6 Graphs in Practical Situations



Worked Example 23

(Speed-Time Graph)

The figure shows the speed-time graph of a train over a period of 80 seconds. The train travelled a total distance of 850 m in the first 50 seconds.



Given that the distance travelled is given by the area under the speed-time graph, find

(i) the value of V,

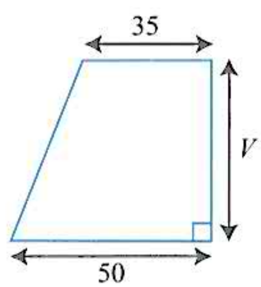
(ii) the acceleration of the train during the first 15 seconds,

(iii) the distance the train further travelled before it came to a stop.

Solution:

(i) Total distance during the first 50 s = 850 m

$$\begin{aligned}\text{Area of trapezium} &= 850\\ \frac{1}{2}\times(35+50)\times V &= 850\\ 42.5V &= 850\\ V &= 20\end{aligned}$$



RECALL

Area of trapezium
$= \frac{1}{2} \times$ sum of parallel sides $\times$ height

(ii) Acceleration for the first 15 s $= \dfrac{20-0}{15}$

$= 1\frac{1}{3}$ m/s^2

(iii) Distance travelled = Area under graph between t = 50 s and t = 80 s

= Area of triangle

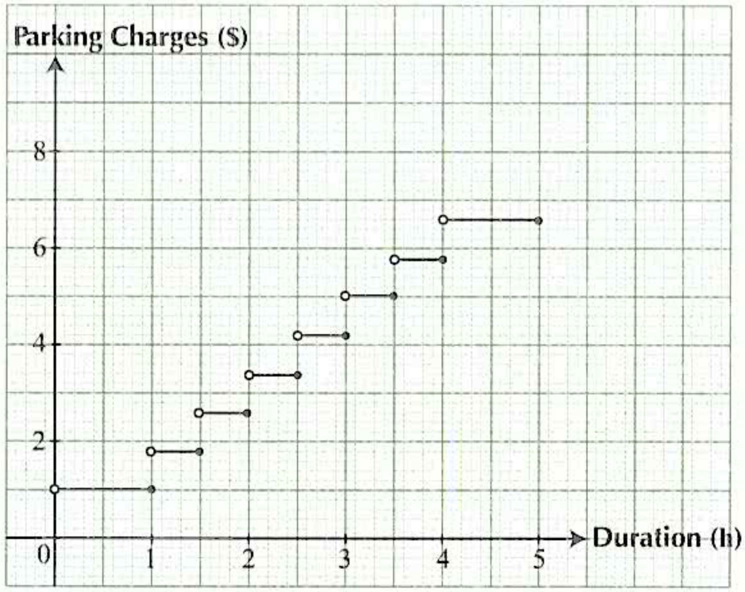
$= \dfrac{1}{2} \times 20 \times 30$

= 300 m

Worked Example 24

(Graph involving Rates)

The step-function graph below shows the parking charges for the first 5 hours at car park A.



Carpark B has the following charges:

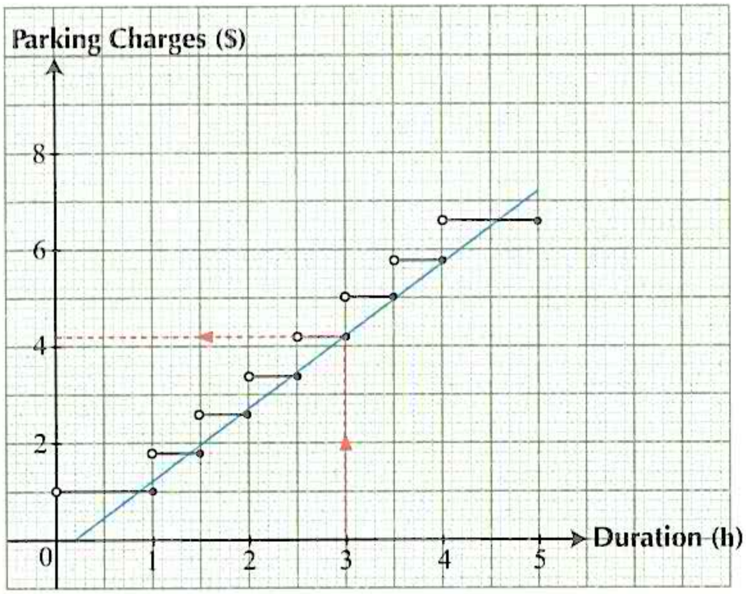
Free for the first 12 minutes 2.5 cents per minute thereafter.

(i) In the graph given above, draw a graph representing the parking charges of car park B.

(ii) Given that Mr Lee needs to park at either car park A or B from 1145 to 1445, determine which car park offers lower parking charges.

Solution:

(i)



RECALL

An empty node o indicates that the point is **excluded** while a shaded node • indicates that the point is **included**.

Problem Solving Tip

For a parking duration of 1 hour, the parking charge in car park B is $2.5 \times (60 - 12) = 120$ cents $= \$1.20$.

For a parking duration of 2 hours, the parking charge in car park B is $120 + 2.5 \times 60 = 270$ cents $= \$2.70$.

Alternatively, for every hour of parking after the first 12 minutes, the parking charge in car park B is $2.5 \times 60 = 150$ cents $= \$1.50$.

(ii) Time duration between 1145 and 1445 = 3 hours

From the graph, for a parking duration of 3 hours, the parking charge in car park A is \$4.20, and the parking charge in car park B = \$4.20.

∴ For 3 hours, both car parks have the same parking charge of \$4.20.

Revision 9F

1. The exchange rate for Hong Kong dollars (HK\$) to Singapore dollars (S\$) is HK\$100 to S\$16.

(a) Find

(i) HK\$55 in S\$, **(ii)** S\$20 in HK\$.

(b) Draw a graph to represent the relationship between the two currencies. By letting x and y be the amounts of Singapore dollars and Hong Kong dollars respectively, write down an equation for the graph.

2. The relationship between degree Celsius (°C) and degree Fahrenheit (°F) is given by the formula $F = \frac{9}{5}C + 32$.

(a) Find

(i) the value of C when $F = 98$,

(ii) the value of F when $C = 50$.

(b) Plot the graph of F against C for $0 \leqslant C \leqslant 100$.

(c) Find the increase in C when F increases from 70 °F to 120 °F.

3. The table shows the force, F in Newtons (N) applied to a pulley to raise a load L kg.

Mass of load (L kg)	20	40	80	120	160	200
Force (F N)	70	90	130	170	210	250

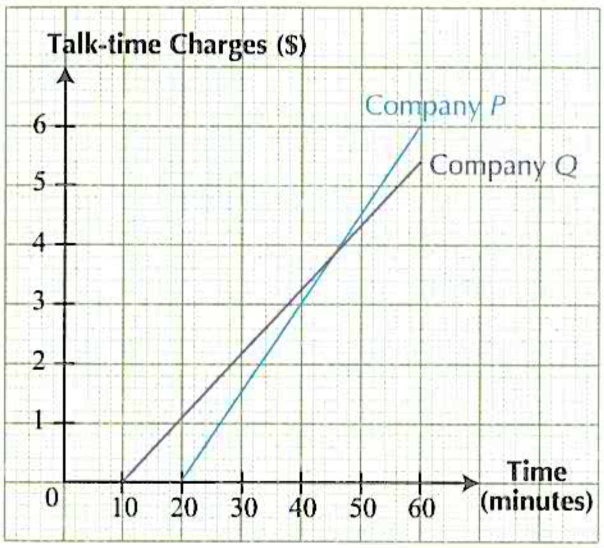
(a) Using a suitable scale, draw the graph of F against L.

(b) Using your graph, find the force required to raise a load of

(i) 56 kg, (ii) 190 kg.

(c) Using your graph, find the initial force required to operate the pulley.

4. The graph shows the talk-time charges offered by two mobile service providers, Company P and Company Q.



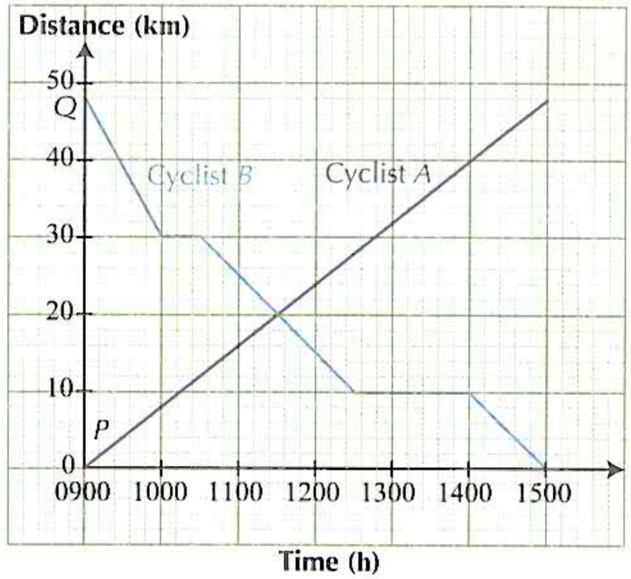
(a) (i) How much does Company P charge for 30 minutes of talk-time?

(ii) How much does Company Q charge for 55 minutes of talk-time?

(iii) Michael uses less than 45 minutes of talk-time per month. Which company offers him a better deal?

(b) Which company offers a lower rate (dollar per minute) of talk-time charges?

5. The distance-time graphs show the journeys of cyclist A and cyclist B. Cyclist A travelled from Town P to Town Q while cyclist B travelled from Town Q to Town P.



From the graph, find

(i) the distance between Town P and Town Q,

(ii) the duration cyclist B took to rest during the journey,

(iii) the distance of both cyclists from Town Q when they met,

(iv) the period of time for which cyclist B travelled the fastest,

(v) the average speed of cyclist B for the whole journey.

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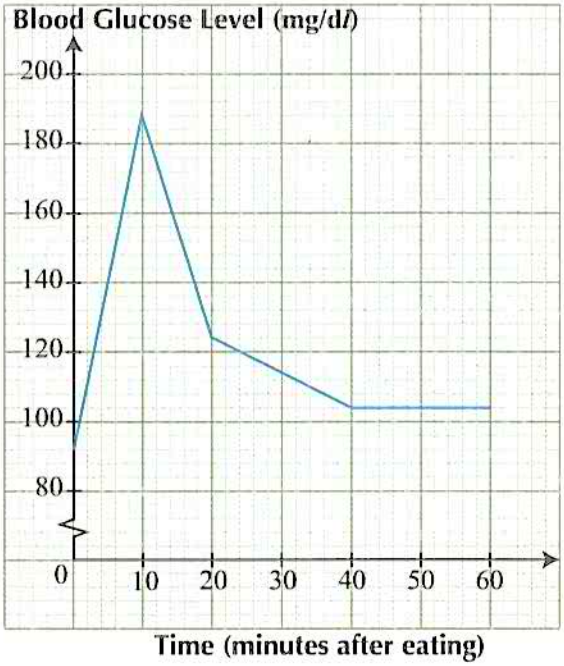
6. The graph below shows Huixian's blood glucose level (milligrams per decilitre) after she consumed a chocolate bar.

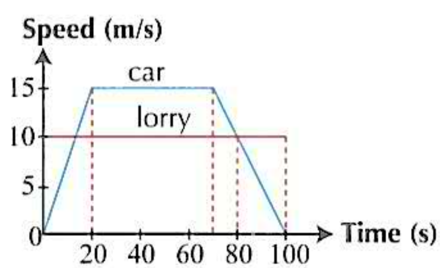


(i) Write down the duration for which Huixian's blood glucose level remained constant.

(ii) Find the rate of increase in her glucose level in the first 10 minutes.

(iii) Find the rate of decrease in her glucose level in the next 10 minutes.

7. The diagram shows the speed-time graphs of a car and a lorry travelling on the road for a period of 100 seconds.



Given that the distance travelled is given by the area under the speed-time graph, calculate

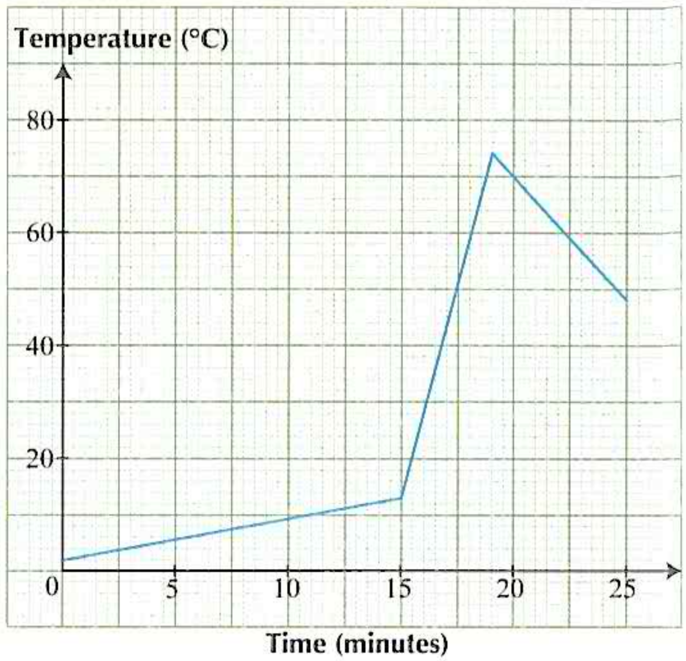
(i) the acceleration of the car in the first 20 seconds,

(ii) the distance travelled by the car during the 100 seconds,

(iii) the distance travelled by the lorry during the 100 seconds,

(iv) the time when the car overtakes the lorry.

8. Nora took a piece of pizza out of the refrigerator and allowed it to defrost for 15 minutes before heating it up in a microwave oven. When the pizza was heated, Nora found that it was too hot for her to consume and she waited for 5 minutes before eating the pizza. The graph shows the temperature of the frozen pizza after it was taken out from the refrigerator.



Find

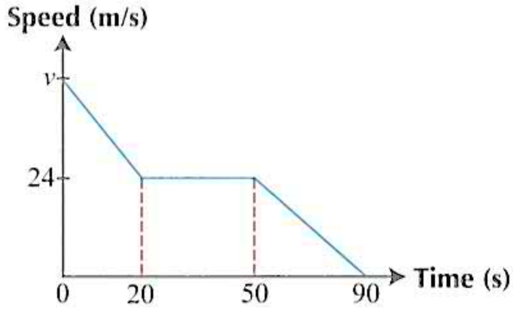
(i) the duration in which the pizza was being heated in the microwave oven,

(ii) the rate of increase in the temperature of the pizza in the first 15 minutes,

(iii) the temperature of the pizza just after it was heated up in the microwave oven,

(iv) the rate of increase in the temperature of the pizza when it was being heated up in the microwave oven.

9. The diagram shows the speed-time graph of a particle. The total distance travelled by the particle over a period of 90 seconds is 1.84 km.



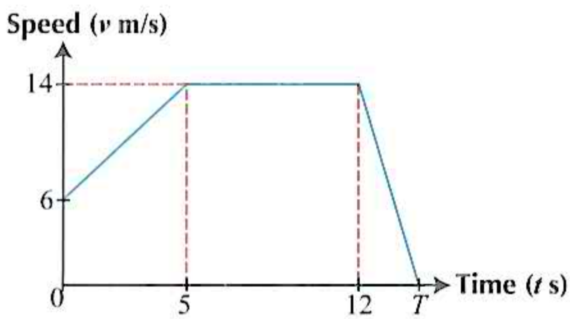
Given that the distance travelled is given by the area under the speed-time graph, find

(i) the value of v,

(ii) the acceleration during the first 10 seconds,

(iii) the distance travelled during the first 10 seconds of its motion.

10. The diagram shows the speed-time graph of a moving object.

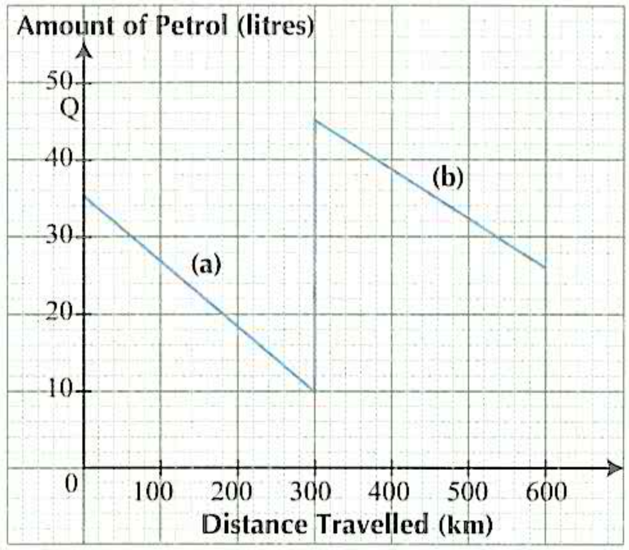


(i) Find the speed of the object when $t = 3$, given that the acceleration in the first 5 seconds is constant.

(ii) Given that the distance travelled is given by the area under the speed-time graph, find the average speed during the first 12 seconds of the journey.

(iii) Given that the deceleration of the object between 12 seconds and T seconds is 3.5 m/s^2, find the value of T.

11. Mr Tan is in China and he drove his car out to a mountain resort. The graph shows the amount of petrol in the fuel tank of his car during the journey. The currency used in China is the Renminbi (RMB).

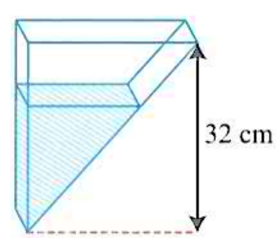


(i) Find the gradient of parts **(a)** and **(b)** of the graph and state what these gradients represent.

(ii) Give a reason for the difference in the gradient between the two parts of the graph.

(iii) Mr Tan stopped once at the top of the mountain to fill up the petrol tank. He paid RMB6.75 per litre. How much did he pay for the petrol in terms of Singapore dollars if the exchange rate is S\$1 = RMB4.88?

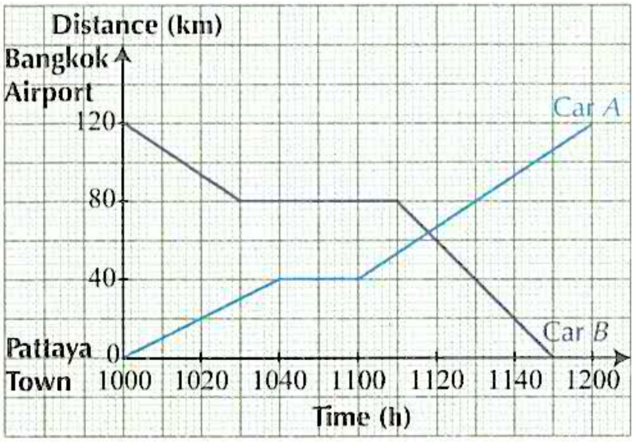
12. The diagram shows a container which is a prism with a triangular cross-section. Water is being poured into the container at a constant rate.



(i) Given that the height of the container is 32 cm and it takes 12 seconds to fill the container, find the time taken to fill the container to a depth of 24 cm.

(ii) Sketch a graph showing how the depth of the water varies during the 12 seconds.

13. The graphs below show the travel graph of car A travelling from Pattaya Town to Bangkok Airport, and the travel graph of car B travelling from Bangkok Airport to Pattaya Town.



(i) Find the average speed of car A and of car B.

(ii) How far away is car B from Bangkok Airport when it stopped? For how long did it stop?

(iii) How far away from Pattaya Town are the two cars when they met? What time did the two cars meet?

(iv) How far apart are the cars at 1130?

(v) Find the speed of car A at the time when it met car B.

14. A car starts from rest and accelerates at a uniform rate for 10 seconds to reach a speed of 20 m/s. The car then continues at a constant speed for 20 seconds, before decelerating at a uniform rate of 1.8 m/s² and eventually coming to rest.

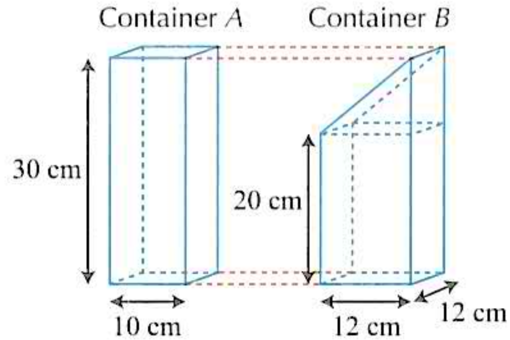
(a) Sketch the speed-time graph for the entire motion of the car.

(b) Find

(i) the acceleration for the first 10 seconds,

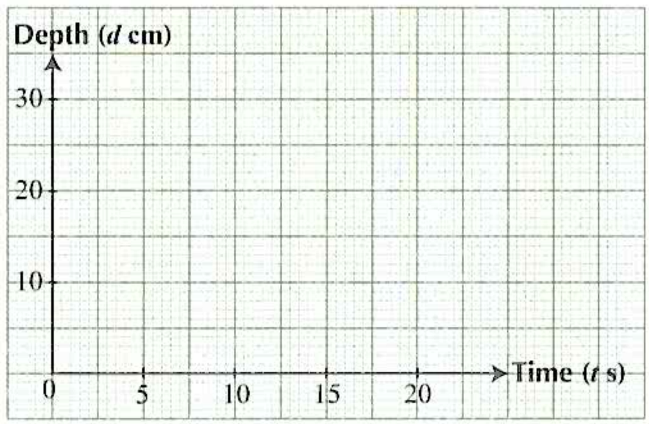
(ii) the total time taken for the entire motion of the car.

15. The figure shows two containers, each with a height of 30 cm and a width of 12 cm. The other dimensions are as shown. The containers were initially empty and it takes 20 seconds to fill each container at a constant rate.



(a) On the grid below, sketch the graph of the height of the water level against time for

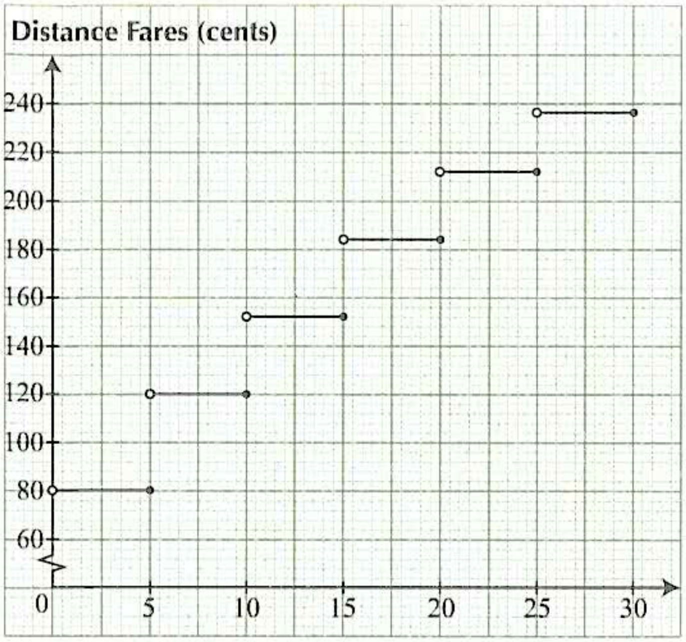
(i) container A, (ii) container B.



(b) From the graphs, find the depth of the water level at $t = 10$ for

(i) container A, (ii) container B.

16. The step-function graph shows the distance fares charged by Optimus Bus Services for the first 30 km of travel.



(a) Write down the fare for a distance of 18 km.

First Bus Services offers the following distance fares:

\$1.00 for the first 5 km and 6 cents for each subsequent kilometre.

(b) Insert the graph corresponding to the rates of the distance fare charged by First Bus Service.

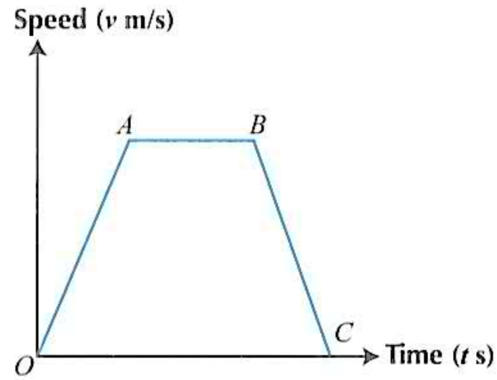
(c) Using your graph, for a distance of 25 km,

(i) determine which bus service charges a lower distance fare,

(ii) find the difference in the distance fares between the two bus services.

17. The diagram shows the speed-time graph of a sports car. The sections OA, AB and BC are represented by the following equations.

OA: $v = 8\frac{1}{3}t$ $\quad AB$: $v = 40$ $\quad BC$: $v = -12.5t + 175$



Find

(a) **(i)** the acceleration of the sports car,

(ii) the coordinates (t, v), for A, B and C,

(iii) the deceleration of the sports car.

(b) Given that the distance travelled is given by the area under a speed-time graph, find the distance travelled by the sports car.

18. A particle moves along a straight line from X to Y such that, t seconds after leaving X, its speed, v m/s, is given by $v = 2t^2 - 10t + 15$.

The table shows some values of t and the corresponding values of v.

t (s)	0	2	4	6	8	10	12
v (m/s)	15	3	a	27	63	b	183

(a) Find the value of a and of b.

(b) On a sheet of graph paper, using a scale of 1 cm to represent 1 second on the horizontal axis and 1 cm to represent 10 m/s on the vertical axis, draw the graph of $v = 2t^2 - 10t + 15$ for $0 \leqslant t \leqslant 12$.

(c) Using your graph to estimate

(i) the value of t when the speed is 45 m/s,

(ii) the gradient at $t = 5$, and explain what this value represents.

19. A load is lifted from ground level to a height of 50 metres in 25 seconds, stops for 10 seconds and then descends to the ground in 15 seconds. The table shows the height, h m, of the load on the upward and downward journeys, t seconds after leaving ground level.

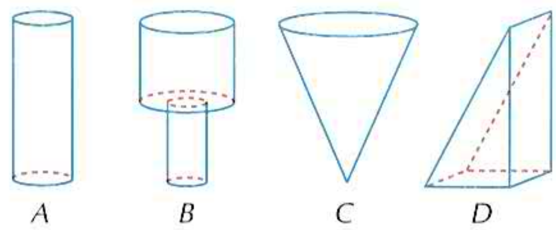
t (s)	0	5	10	15	20	25
h (m)	0	4	12	30	43	50

t (s)	35	40	45	50	55	60
h (m)	50	43	30	12	4	0

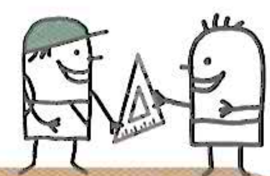
(i) Using a scale of 1 cm to represent 5 seconds on the horizontal axis and 1 cm to represent 5 metres on the vertical axis, plot the points given in the table and join them with a smooth curve.

(ii) Find the gradient of the graph at $t = 16$ and explain briefly what this gradient represents.

20. The diagrams below show four containers (not drawn to scale), i.e. A, B, C and D, each with a height of h cm. The containers are initially empty. It takes t seconds to fill each container with water at a **constant rate**.



On separate diagrams, sketch the graph of the depth of the water against time for each of the four containers.



9.7 Sets

Worked Example 25

(Use of Set Notations and Listing the Elements in a Set)

It is given that

$\xi = \{x : x \text{ is an integer such that } 12 \leqslant x \leqslant 25\}$,

$A = \{x : x \text{ is a prime number}\}$

and $B = \{x : x \text{ is an odd number}\}$.

List the elements in the following sets.

(i) $A \cap B$ **(ii)** $A' \cap B$

(iii) $(A \cup B)'$

Solution:

(i) $A = \{13, 17, 19, 23\}$

$B = \{13, 15, 17, 19, 21, 23, 25\}$

$A \cap B = \{13, 17, 19, 23\}$

(ii) $A' = \{12, 14, 15, 16, 18, 20, 21, 22, 24, 25\}$

$\therefore A' \cap B = \{15, 21, 25\}$

(iii) $A \cup B = \{13, 15, 17, 19, 21, 23, 25\}$

$\therefore (A \cup B)' = \{12, 14, 16, 18, 20, 22, 24\}$

Worked Example 26

(Drawing a Venn Diagram and Listing the Elements in a Set)

$\xi = \{x : x \text{ is an integer such that } 1 \leqslant x < 17\}$
$A = \{x : x \text{ is a factor of } 32\}$
$B = \{x : x \text{ is a perfect square}\}$

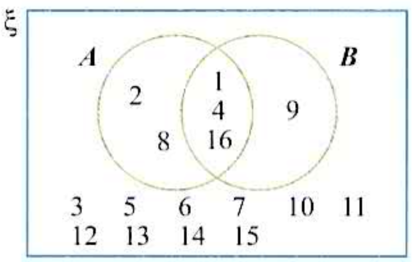
(i) Draw a Venn diagram to illustrate the information given.

(ii) Write down $A \cap B$.

(iii) List the elements in the set $A \cap B'$.

Solution:

(i)



Problem Solving Tip

It is easier to identify $A \cap B$ first, before drawing the Venn diagram.

(ii) $A \cap B = \{1, 4, 16\}$

(iii) From the diagram, $A \cap B' = \{2, 8\}$.

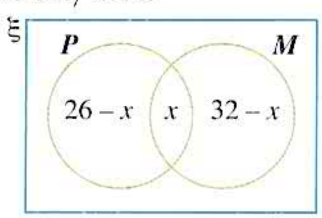
Worked Example 27

(Problem Solving using Venn Diagrams)

In a class of 45 students, 26 enjoy studying Physics and 32 enjoy studying Mathematics. Assuming that every student enjoys studying at least one subject, how many students enjoy studying both Mathematics and Physics?

Solution:

Let P = {students who enjoy studying Physics} and M = {students who enjoy studying Mathematics}.
Let x be the number of pupils who enjoy studying both subjects.
Since there are 26 in P altogether, there must be $(26 - x)$ who are only in P. Similarly, there must be $(32 - x)$ who are only in M.



Since all 45 students enjoy studying at least one of the two subjects, $P \cup M = \xi$.

$$(26 - x) + x + (32 - x) = 45$$
$$x = 13$$

$\therefore$ There are 13 students who enjoy studying both Mathematics and Physics.

Worked Example 28

(Application of Formula to find Extreme Values)

Given that $n(A) = 27$ and $n(B) = 14$, find the

(i) greatest value of $n(A \cup B)$,

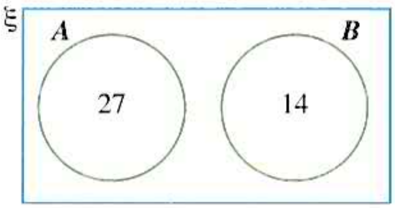
(ii) least value of $n(A \cup B)$,

illustrating the cases with Venn diagrams.

Solution:

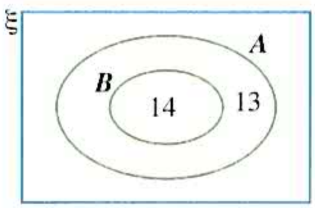
(i) $n(A \cup B)$ will have the greatest value when $(A \cap B) = \varnothing$.

$$\begin{aligned} n(A \cup B) &= n(A) + n(B) \\ &= 27 + 14 \\ &= 41 \end{aligned}$$



(ii) $n(A \cup B)$ will have the least value when $B \subseteq A$.

$n(A \cup B) = n(A) = 27$



Revision 9G

1. Let the universal set

$\xi = \{2, 3, 4, 5, 6, 7, 8, 9\}$,

$A = \{x : x \text{ is a prime number}\}$

and $B = \{x : x \text{ is an odd number}\}$.

List the elements in the following sets.

(i) $A \cup B$ **(ii)** $A \cap B'$

2. Suppose that

$\xi = \{a, b, c, d, e, f\}$,

A is a set of vowels

and $B = \{a, c, f\}$.

List the elements in the following sets.

(i) $A \cap B$ **(ii)** $A \cup B$

(iii) $A \cap B'$

3. It is given that

$\xi = \{x : x \text{ is an integer and } 1 \leqslant x \leqslant 10\}$,

$A = \{x : x \text{ is a prime number}\}$,

$B = \{x : x \text{ is an even number}\}$

and $C = \{x : x \text{ is a multiple of } 3\}$.

List the elements of the following sets.

(i) $A \cap B$ **(ii)** $A \cup B'$

(iii) $A \cap C'$

4. The universal set is the set of positive integers greater than 10 but less than 34. The sets A, B and C are defined as follows.

$A = \{x : x \text{ is a positive integer such that } 15 \leqslant x \leqslant 32\}$

$B = \{x : x \text{ is an odd number}\}$

$C = \{x : x \text{ is a prime number}\}$

List the elements in the following sets.

(i) A **(ii)** B

(iii) C **(iv)** $A \cap B$

(v) $B \cap C$ **(vi)** $B \cap C'$

5. A and B are two distinct sets such that $A \subset B$. Simplify

(i) $A \cap B$, **(ii)** $A \cup B$.

6. Given that

$A = \{x : x \text{ is an integer and } 50 \leqslant x \leqslant 100\}$,

$B = \{x : x \text{ is a positive integer and } x^2 \in A\}$

and $C = \{x : \sqrt{x} \text{ is a positive integer}\}$.

(i) List the elements in $A \cap C$.

(ii) List the elements in B.

7. It is given that

$\xi = \{x : x \text{ in an integer, } 1 \leqslant x \leqslant 12\}$,

$A = \{x : x \text{ is a factor of } 12\}$

and $B = \{x : x \text{ is an odd integer}\}$.

List the elements in the following sets.

(i) A **(ii)** $A' \cap B$

(iii) $A \cup B$

8. $\xi = \{x : x \text{ is an integer such that } 1 \leqslant x \leqslant 20\}$

$A = \{x : x \text{ is a multiple of } 2\}$

$B = \{x : x \text{ is a factor of } 28\}$

(a) Draw a Venn diagram to illustrate the information given.

(b) List the elements in the following sets.

(i) $A \cap B$ **(ii)** $A' \cup B$

(iii) $A' \cap B'$

9. $\xi = \{x : x \text{ is an integer such that } 1 \leqslant x \leqslant 24\}$

$A = \{x : x \text{ is a prime number}\}$

$B = \{x : x \text{ is a multiple of } 3\}$

$C = \{x : x \text{ is a factor of } 24\}$

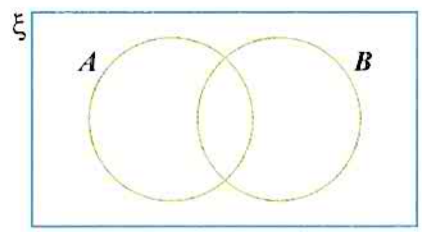
Given the information, list the elements in the following sets.

(i) C **(ii)** $A \cap C$

(iii) $B' \cap C$ **(iv)** $A \cap B'$

10. The Venn diagram represents the subsets A and B of the universal set ξ. Copy and shade on separate Venn diagrams, the region representing the set

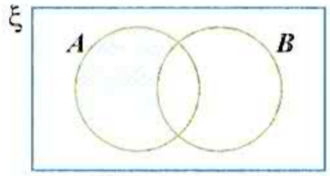
(i) $A \cap B'$, **(ii)** $A' \cap B'$.

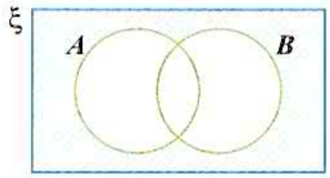


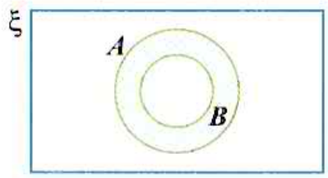
11. P and Q are two non-empty subsets of the universal set ξ. If $P \cap Q \neq \varnothing$, draw separate Venn diagrams and illustrate by shading the following sets.

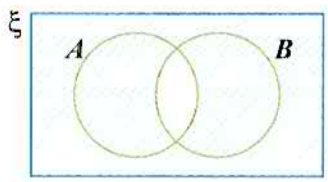
(i) Q' **(ii)** $P \cup Q'$

12. Write the set notation represented by the sets shaded in each of the following Venn diagrams.

(a)


(b)


(c)


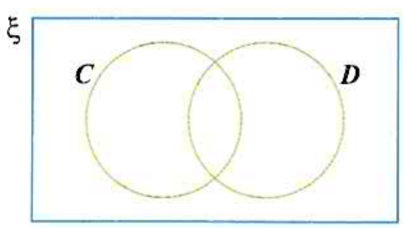
(d)


13. $\xi = \{x : x \text{ is a quadrilateral}\}$

$C = \{x : x \text{ is a quadrilateral with at least one right angle}\}$

$D = \{x : x \text{ is a quadrilateral with at least two equal sides}\}$

Q is a rectangle.
R is a rhombus with one angle equal to 40°.
S is a kite with angles 120°, 30°, 120° and 90°.
T is a quadrilateral with three angles 120°, 120° and 110°, and all unequal sides. On the Venn diagram below, write Q, R, S and T in the appropriate subsets.



14. $\xi = \{x : x \text{ is a quadrilateral}\}$
$A = \{x : x \text{ is a rectangle}\}$
$B = \{x : x \text{ is a parallelogram}\}$

(i) Draw a clearly labelled Venn diagram to show the relationship between the sets ξ, A and B.

(ii) Using set notation, describe the relationship between A and B.

(iii) K is a kite, R is a rhombus and S is a square.

On the Venn diagram in **(i)**, write K, R and S in the appropriate subsets.

15. Given that
$\xi = \{x : x \text{ is an integer such that } -10 \leqslant x \leqslant 15\}$,
$A = \{x : x \text{ is an integer such that } 14 - x > 3\}$
and $B = \{x : x \text{ is an integer such that } 1 - 2x < 10\}$,
list the elements of

(i) $(A \cup B)'$,

(ii) $A \cap B$,

(iii) $A \cap B'$.

16. $\xi = \{x : x \text{ is a real number}\}$
$A = \{x : x \text{ is a rational number}\}$
$B = \{x : x \text{ is an integer}\}$

(a) **(i)** Is $A = B$? Explain your answer.

(ii) Is $A \cup B = \xi$? Explain your answer.

(b) Draw a clearly labelled Venn diagram to illustrate the relationship between the sets ξ, A and B.

(c) Write down an element of the following sets.

(i) A' **(ii)** $A \cap B'$

17. There were 26 people queuing at the cashier of a departmental store. 14 of them were about to purchase footwear and 12 were about to purchase furniture. Find

(i) the greatest possible number of people buying both footwear and furniture,

(ii) the least possible number of people buying both footwear and furniture,

(iii) the greatest possible number of people buying only one type of item (either footwear or furniture and not both).

18. If $n(A) = 12$, $n(B) = 6$ and $n(A \cap B) = 4$, state the value of $n(A \cup B)$.

19. In a group of 50 people, 33 can dance and 26 can sing. How many can do both, dance and sing, if each of them has at least one of the two skills?

20. If $n(A) = 12$, $n(B) = 9$, $n(\xi) = 20$, find the greatest and the least values of the following.

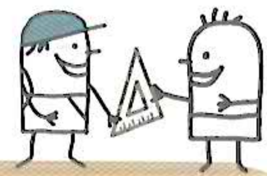
(i) $n(A \cap B)$ (ii) $n(A \cup B)$

21. If A and B are disjoint sets such that $n(A) = 11$, $n(B) = 6$, find

(i) $n(A \cap B)$, (ii) $n(A \cup B)$.

22. In a group of 60 people, 30 can cook Italian food, 40 can cook Chinese food and 24 can cook Indian food. 5 of them can cook both Italian and Chinese food, 2 of them can cook both Italian and Indian food and 4 of them can cook both Chinese and Indian food. There are 13 people who can cook all three – Italian, Chinese and Indian food. Draw a Venn diagram to illustrate this information and find the number of people who cannot cook.

9.8 Matrices



Worked Example 29

(Addition, Subtraction and Scalar Multiplication of Matrices)

(a) If $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ -3 & 5 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} -2 & 1 \\ -4 & 0 \end{pmatrix}$,

evaluate $2\mathbf{A} + 3\mathbf{B}$.

(b) Find the value of x and y in the equation

$3\begin{pmatrix} x & y \end{pmatrix} + \begin{pmatrix} -1 & 2 \end{pmatrix} = \begin{pmatrix} 8 & -2 \end{pmatrix}$.

Solution:

(a)
$$2\mathbf{A} + 3\mathbf{B} = 2\begin{pmatrix} 1 & 2 \\ -3 & 5 \end{pmatrix} + 3\begin{pmatrix} -2 & 1 \\ -4 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 2 & 4 \\ -6 & 10 \end{pmatrix} + \begin{pmatrix} -6 & 3 \\ -12 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} -4 & 7 \\ -18 & 10 \end{pmatrix}$$

(b)
$$3\begin{pmatrix} x & y \end{pmatrix} + \begin{pmatrix} -1 & 2 \end{pmatrix} = \begin{pmatrix} 8 & -2 \end{pmatrix}$$
$$\begin{pmatrix} 3x & 3y \end{pmatrix} + \begin{pmatrix} -1 & 2 \end{pmatrix} = \begin{pmatrix} 8 & -2 \end{pmatrix}$$
$$\begin{pmatrix} 3x-1 & 3y+2 \end{pmatrix} = \begin{pmatrix} 8 & -2 \end{pmatrix}$$

$3x - 1 = 8$ ---------- (1) (equating each element)

$3y + 2 = -2$ ---------- (2)

From (1),

$$3x = 8 + 1$$
$$= 9$$
$$x = 3$$

From (2),

$$3y = -2 - 2$$
$$= -4$$
$$y = -\frac{4}{3}$$
$$= -1\frac{1}{3}$$

$\therefore x = 3,\ y = -1\frac{1}{3}$

Worked Example 30

(Multiplication of Two Matrices)

Evaluate the following matrix products, if possible. If not possible, explain why.

(a) $\begin{pmatrix} 3 \\ 1 \end{pmatrix}\begin{pmatrix} 1 & 3 \end{pmatrix}$

(b) $\begin{pmatrix} 2 & 1 \\ 3 & 5 \end{pmatrix}\begin{pmatrix} -2 & 3 \\ 1 & 0 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 4 \\ 2 & 5 \end{pmatrix}\begin{pmatrix} 2 & 1 \end{pmatrix}$

(d) $\begin{pmatrix} 2 & 1 & 3 \\ 0 & -1 & 4 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$

Solution:

(a)
$$\begin{pmatrix} 3 \\ 1 \end{pmatrix}\begin{pmatrix} 1 & 3 \end{pmatrix} = \begin{pmatrix} 3\times1 & 3\times3 \\ 1\times1 & 1\times3 \end{pmatrix}$$
$$= \begin{pmatrix} 3 & 9 \\ 1 & 3 \end{pmatrix}$$

RECALL

The matrix product **AB** is only possible if the number of columns of **A** is equal to the number of rows of **B**.

(b)
$$\begin{pmatrix} 2 & 1 \\ 3 & 5 \end{pmatrix}\begin{pmatrix} -2 & 3 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2(-2)+1(1) & 2(3)+1(0) \\ 3(-2)+5(1) & 3(3)+5(0) \end{pmatrix}$$
$$= \begin{pmatrix} -3 & 6 \\ -1 & 9 \end{pmatrix}$$

(c) Matrix multiplication is not possible. The number of columns of the first matrix (i.e. 2) is not equal to the number of rows of the second matrix (i.e. 1).

(d)
$$\begin{pmatrix} 2 & 1 & 3 \\ 0 & -1 & 4 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2\times2+1\times(-1)+3\times1 \\ 0\times2+(-1)\times(-1)+4\times1 \end{pmatrix}$$
$$= \begin{pmatrix} 6 \\ 5 \end{pmatrix}$$

Worked Example 31

(Applications of Matrices)

Three bus companies operate bus services from Singapore to three destinations – Cameron Highlands, Penang and Kuala Lumpur. The table below shows the number of passengers who travelled by the three bus companies on a particular day.

	Cameron Highlands	Penang	Kuala Lumpur
Swift Bus	30	18	25
ABC Travel	34	23	35
Fast Tours	45	45	16

The prices (per passenger) to each of the three destinations are shown below.

Cameron Highlands:	\$38
Penang:	\$55
Kuala Lumpur:	\$30

(a) **(i)** Write down two matrices such that under matrix multiplication, the product indicates the total revenue earned by each company on that particular week. Evaluate this product.

(ii) Using your answer in **(i)**, write down two matrices such that under matrix multiplication, the product indicates the total amount earned by all three bus companies on that particular day.

(b) **(i)** Find $\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 30 & 18 & 25 \\ 34 & 23 & 35 \\ 45 & 45 & 16 \end{pmatrix}$.

(ii) Explain what the answer in **(b)(i)** represents.

Solution:

(a) **(i)** $\begin{pmatrix} 30 & 18 & 25 \\ 34 & 23 & 35 \\ 45 & 45 & 16 \end{pmatrix}\begin{pmatrix} 38 \\ 55 \\ 30 \end{pmatrix} = \begin{pmatrix} 2880 \\ 3607 \\ 4665 \end{pmatrix}$,

i.e. Swift Bus earned \$2880, *ABC* Travel earned \$3607 and Fast Tours earned \$4665.

(ii) $\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 2880 \\ 3607 \\ 4665 \end{pmatrix} = \begin{pmatrix} 11\,152 \end{pmatrix}$,

i.e. the total amount earned by all three bus companies is \$11 152.

(b) **(i)** $\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 30 & 18 & 25 \\ 34 & 23 & 35 \\ 45 & 45 & 16 \end{pmatrix} = \begin{pmatrix} 109 & 86 & 76 \end{pmatrix}$

(ii) Each element of the matrix product in **(b)(i)** represents the total number of bus rides made to each of the three destinations on that particular day.

Worked Example 32

(Matrices of Transformation)

$\triangle ABC$ with vertices $A(2, 4)$, $B(4, 6)$ and $C(6, 2)$ is transformed into $\triangle A_1B_1C_1$ by the matrix $\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$. $\triangle A_1B_1C_1$ is then transformed into $\triangle A_2B_2C_2$ by the matrix $\begin{pmatrix} -4 & 0 \\ 0 & -4 \end{pmatrix}$. Find the coordinates of the vertices of $\triangle A_1B_1C_1$ and $\triangle A_2B_2C_2$ and determine a single matrix that will map $\triangle ABC$ onto $\triangle A_2B_2C_2$. Deduce the transformation represented by this matrix.

Solution:

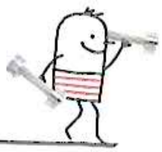
$$\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 2 & 4 & 6 \\ 4 & 6 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

$\therefore A_1(1, 2), B_1(2, 3), C_1(3, 1)$

$$\begin{pmatrix} -4 & 0 \\ 0 & -4 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} -4 & -8 & -12 \\ -8 & -12 & -4 \end{pmatrix}$$

$\therefore A_2(-4, -8), B_2(-8, -12), C_2(-12, -4)$

The single matrix that will map $\triangle ABC$ onto $\triangle A_2B_2C_2$ is $\begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$. The matrix represents an enlargement centre at origin and scale factor -2.



Revision 9H

1. Evaluate each of the following if possible. If it is not possible, explain why.

   (a) $\begin{pmatrix} -1 & 1 \\ -2 & 5 \end{pmatrix} + \begin{pmatrix} 6 & 0 \\ 1 & 4 \end{pmatrix}$

   (b) $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ -2 \end{pmatrix}$

   (c) $\begin{pmatrix} 1 & 2 \end{pmatrix} + \begin{pmatrix} -1 & -3 \end{pmatrix} - \begin{pmatrix} -2 & 5 \end{pmatrix}$

   (d) $\begin{pmatrix} 3 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 2 \end{pmatrix} - \begin{pmatrix} 0 \end{pmatrix}$

2. Evaluate each of the following matrix products, if possible. If not possible, explain why.

   (a) $\begin{pmatrix} 5 \\ -1 \end{pmatrix}\begin{pmatrix} 3 & 4 \end{pmatrix}$

   (b) $\begin{pmatrix} 3 & 2 \end{pmatrix}\begin{pmatrix} 4 \\ -1 \end{pmatrix}$

   (c) $\begin{pmatrix} 3 & 4 \\ 1 & 5 \\ 2 & 6 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \\ 5 \end{pmatrix}$

   (d) $\begin{pmatrix} 2 & -1 & 4 \end{pmatrix}\begin{pmatrix} 0 \\ -3 \\ \frac{1}{2} \end{pmatrix}$

   (e) $\begin{pmatrix} 4 & 1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

3. Find the value of the unknowns in each of the following.

   (a) $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 5 \\ 2 \end{pmatrix}$

   (b) $2\begin{pmatrix} x \\ y \end{pmatrix} - 4\begin{pmatrix} 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 4x \\ 3y \end{pmatrix}$

   (c) $x\begin{pmatrix} 2 \\ 4 \end{pmatrix} + y\begin{pmatrix} 5 \\ 7 \end{pmatrix} = \begin{pmatrix} -5 \\ -1 \end{pmatrix}$

4. Find the value of the unknowns in each of the following.

   (a) $\begin{pmatrix} 0 & 2 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

   (b) $\begin{pmatrix} 2 & 1 \\ 4 & x \end{pmatrix}\begin{pmatrix} y \\ 2 \end{pmatrix} = \begin{pmatrix} 6 \\ 10 \end{pmatrix}$

   (c) $\begin{pmatrix} 3 & 1 \\ x & 4 \end{pmatrix}\begin{pmatrix} y \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

5. Find the values of p and q in the equation

   $$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \end{pmatrix}\begin{pmatrix} p \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 2q \end{pmatrix}.$$

6. **(a)** Given that

   $$\begin{pmatrix} 4 & 2 \\ -1 & p \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} q & 14 \\ -2 & 7 \end{pmatrix},$$

   find the value of p and of q.

   (b) Given that $\mathbf{A} = \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 2 & 6 \\ 0 & k \end{pmatrix}$

   (i) find $\mathbf{AB}$ and $\mathbf{BA}$,

   (ii) if $\mathbf{AB} = \mathbf{BA}$, find the value of k.

7. Given that $\mathbf{A} = \begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 5 & w \\ y & 4 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} x & 6 \\ 4 & z \end{pmatrix}$, find the values of w, x, y and z when

(i) $2\mathbf{A} + \mathbf{B} = \mathbf{C}$, **(ii)** $3\mathbf{A} - 2\mathbf{B} = 4\mathbf{C}$.

8. Given that $\mathbf{A} = \begin{pmatrix} -2 & 1 \\ -1 & 4 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 3 & 1 \\ -2 & -5 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 3 & -1 \\ 4 & -2 \end{pmatrix}$, write each of the following as a single matrix.

(i) $2\mathbf{A} + \mathbf{B} - \mathbf{C}$ **(ii)** $\mathbf{B} + \mathbf{AC}$

(iii) $\mathbf{A} + \mathbf{BC}$

9. Given that $\mathbf{A} = \begin{pmatrix} -4 & p \\ -1 & 2 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} q & 0 \\ 2 & 3 \end{pmatrix}$ and $\mathbf{AB} = \mathbf{BA}$, find the value of p and of q.

10. The monthly fee charged by three cable television providers is given in the matrix $\mathbf{A}$ below.

$$\mathbf{A} = \begin{pmatrix} 120 \\ 95 \\ 102 \end{pmatrix} \begin{array}{l} \text{Star TV} \\ \text{Tech Cable} \\ \text{ABC TV} \end{array}$$

(i) Evaluate $12\mathbf{A}$.

During the World Cup, all three cable television providers offer a World Cup special package for an additional cost shown in the matrix $\mathbf{W}$.

$$\mathbf{W} = \begin{pmatrix} 60 \\ 75 \\ 62 \end{pmatrix} \begin{array}{l} \text{Star TV} \\ \text{Tech Cable} \\ \text{ABC TV} \end{array}$$

(ii) Evaluate $\mathbf{B} = 12\mathbf{A} + \mathbf{W}$ and explain what this matrix represents.

11. If $\mathbf{A} = \begin{pmatrix} 0.6 & 0.8 \\ 0.8 & -0.6 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ and $\mathbf{ABC} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, find the matrix $\mathbf{C}$.

12. Given that $\mathbf{A} = \begin{pmatrix} 1 & 3 \\ -2 & 2 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} \frac{1}{4} & -1 \\ 1\frac{1}{4} & 1 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 1\frac{3}{4} & -2 \\ \frac{3}{4} & 0 \end{pmatrix}$, evaluate $\mathbf{AB}$ and $\mathbf{AC}$. Use the sum of these products to determine $\mathbf{A}^{-1}$. Hence, evaluate x and y if $\mathbf{A}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$.

13. If $\mathbf{A}$ and $\mathbf{B}$ are the matrices $\begin{pmatrix} 1 & 2 & 1 \\ 3 & 4 & 2 \end{pmatrix}$ and $\begin{pmatrix} 1 & 3 \\ 0 & 2 \\ 2 & 1 \end{pmatrix}$ respectively, find the matrices $\mathbf{AB}$ and $\mathbf{BA}$.

14. Given that $\mathbf{A} = \begin{pmatrix} 7 & 5 \\ 8 & 9 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}$, write down the inverse matrix of $\mathbf{B}$ and use it to find the matrices $\mathbf{P}$ and $\mathbf{Q}$ such that

(i) $\mathbf{PB} = \mathbf{A}$ **(ii)** $\mathbf{BQ} = 2\mathbf{A}$

15. Solve the following sets of equations using the matrix method.

(a) $2x - 3y = 8$
$4x - y = 4$

(b) $5x - 6y = 12$
$10x - 12y = 24$

(c) $6x - 2y = 9$
$3x - y = 1$

Explain what your answers represent.

16. Lixin and Rui Feng took a driving theory test. The matrices below show the breakdown of their results and the marks awarded for the test.

$$\mathbf{A} = \begin{array}{c} \\ \\ \end{array}\begin{array}{c} \begin{array}{ccc} \text{Correct} & \text{No Attempt} & \text{Incorrect} \end{array} \\ \begin{pmatrix} 39 & 5 & 6 \\ 29 & 8 & 13 \end{pmatrix} \end{array}\begin{array}{l} \\ \text{Lixin} \\ \text{Rui Feng} \end{array}$$

$$\mathbf{M} = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}\begin{array}{l} \text{Correct} \\ \text{No Attempt} \\ \text{Incorrect} \end{array}$$

(i) Evaluate the matrix $\mathbf{R} = \mathbf{AM}$.
(ii) Explain what the answer in **(i)** represents.

17. **(a)** Given that $\begin{pmatrix} 0 & 5 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -15 \\ 28 \end{pmatrix}$, find the value of x and of y.

(b) Given that $\mathbf{A} = \begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 2 & 1 \end{pmatrix}$. Evaluate each of the following matrix products if possible.

(i) **AB** **(ii)** **AC**
(iii) **BC** **(iv)** **CB**

18. The matrices $\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ are such that $\mathbf{AB} = \mathbf{A} + \mathbf{B}$. Find the values of a, b and c.

19. The matrix below shows the results for three soccer teams.

$$\mathbf{R} = \begin{array}{c} \begin{array}{ccc} \text{Win} & \text{Draw} & \text{Lose} \end{array} \\ \begin{pmatrix} 12 & 5 & 3 \\ 3 & 8 & 7 \\ 9 & 4 & 4 \end{pmatrix} \end{array}\begin{array}{l} \\ \text{Wanderers} \\ \text{United} \\ \text{Saints} \end{array}$$

(i) A win gains 3 points, a draw 1 point and a loss 0 point. Represent this information with a 3×1 column matrix **P**.
(ii) Evaluate the matrix **RP**.
(iii) Explain what the answer in **(ii)** represents.

20. A school bus transports students to their school every weekday. Students in the morning session take the morning bus and students in the afternoon session take the afternoon bus. The matrix **A** shows the number of boys and girls who take the school bus.

$$\mathbf{A} = \begin{array}{c} \begin{array}{cc} \text{Morning} & \text{Afternoon} \end{array} \\ \begin{pmatrix} 13 & 14 \\ 10 & 12 \end{pmatrix} \end{array}\begin{array}{l} \\ \text{Boys} \\ \text{Girls} \end{array}$$

(i) Evaluate the matrix $\mathbf{K} = 5\mathbf{A}$.
(ii) The fee for a morning bus ride is \$2.50 and the fee for an afternoon bus ride is \$1.80.

Evaluate the matrix $\mathbf{L} = \mathbf{K}\begin{pmatrix} 2.50 \\ 1.80 \end{pmatrix}$.

(iii) State what the elements of **L** represent.

21. Two cafés sold the following cups of beverages on a particular day.

	Latte	Mocha	Iced Lemon Tea
Café *A*	75	84	135
Café *B*	88	95	140

The cost price for a cup of latte is \$0.70, a cup of mocha \$0.85 and a cup of iced lemon tea \$1.00. The selling prices of a cup of latte, a cup of mocha and a cup of iced lemon tea are \$3.20, \$4.00 and \$4.50 respectively.

Given that $\mathbf{P} = \begin{pmatrix} 75 & 84 & 135 \\ 88 & 95 & 140 \end{pmatrix}$, $\mathbf{Q} = \begin{pmatrix} 0.7 \\ 0.85 \\ 1 \end{pmatrix}$ and $\mathbf{R} = \begin{pmatrix} 3.2 \\ 4 \\ 4.5 \end{pmatrix}$, calculate

(i) **PQ**, and explain what it represents,
(ii) $\mathbf{R} - \mathbf{Q}$, and explain what it represents,
(iii) $\mathbf{P}(\mathbf{R} - \mathbf{Q})$, and explain what it represents.

22. For a charity event, Raj went around with a donation box requesting for donations. The charity event was held from Friday to Sunday. The two matrices below show the amount of money, in different denominations, collected for the three days.

$$\mathbf{A} = \begin{array}{c} \begin{array}{ccc} 10¢ & 20¢ & 50¢ \end{array} \\ \begin{pmatrix} 65 & 45 & 46 \\ 60 & 56 & 58 \\ 78 & 56 & 54 \end{pmatrix} \end{array} \begin{array}{l} \\ \text{Friday} \\ \text{Saturday} \\ \text{Sunday} \end{array}$$

$$\mathbf{B} = \begin{array}{c} \begin{array}{cccc} \$1 & \$2 & \$5 & \$10 \end{array} \\ \begin{pmatrix} 32 & 26 & 18 & 16 \\ 45 & 34 & 20 & 10 \\ 38 & 22 & 25 & 24 \end{pmatrix} \end{array} \begin{array}{l} \\ \text{Friday} \\ \text{Saturday} \\ \text{Sunday} \end{array}$$

(i) Evaluate the product $\mathbf{X} = \mathbf{A}\begin{pmatrix} 10 \\ 20 \\ 50 \end{pmatrix}$ and state what the elements of **X** represent.

(ii) Evaluate the product $\mathbf{Y} = \mathbf{B}\begin{pmatrix} 1 \\ 2 \\ 5 \\ 10 \end{pmatrix}$ and state what the elements of **Y** represent.

(iii) Evaluate the matrix $\mathbf{S} = \frac{1}{100}\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}\mathbf{X}$ and state what **S** represents.

(iv) Evaluate the matrix $\mathbf{T} = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}\mathbf{Y} + \mathbf{S}$ and state what **T** represents.

23. An ice-cream factory produces tubs of ice-cream in four different flavours – Chocolate (C), Mint (M), Strawberry (S) and Vanilla (V). Deliveries are made to three different ice-cream shops, Shop 1, Shop 2 and Shop 3.

> Shop 1 receives 50 of C, 60 of M, 70 of S and 40 of V.
>
> Shop 2 receives 30 of C, 40 of M, 50 of S and 30 of V.
>
> Shop 3 receives 40 of C, 30 of M, 60 of S and 50 of V.

(i) The cost of one tub of ice-cream C, M, S and V are \$3.20, \$3.10, \$3.00 and \$3.30 respectively. Write down two matrices only, such that the elements of their product under matrix multiplication, give the cost of the ice-cream delivered to each shop. Evaluate this product.

(ii) During the month of June, Shop 1 has 25 of such deliveries, Shop 2 has 12 deliveries and Shop 3 has 15 deliveries. Write down two matrices only, such that the elements of their product give the total number of tubs of ice-cream, of each flavour leaving the factory in June. Evaluate this product.

(iii) Calculate the total number of tubs of ice-cream supplied by the factory to the three ice-cream shops.

24. There are three types of tour coaches. A Luxury coach has 48 seats, a Comfort coach has 32 seats and a Mini coach has 26 seats.

> Robert Tours has 12 Luxury, 8 Comfort and 11 Mini coaches.
>
> Star Tours has 18 Luxury, 11 Comfort and 7 Mini coaches.
>
> Mido Tours has 8 Luxury, 9 Comfort and 15 Mini coaches.

(i) Write down two matrices whose products will show the greatest number of passengers that each tour company can accommodate on any particular day when each coach is used once. Evaluate this matrix product.

(ii) Robert Tours charges \$9.80 per seat, Star Tours charges \$10.40 per seat and Mido Tours charges \$9.90 per seat. Express the maximum total earnings for the three companies as a product of two matrices and hence, find the total earnings.

25. On a particular day, Best Money Exchange offered the following rates for three currencies, Singapore dollar (S\$), Hong Kong dollar (HK\$) and Malaysian ringgit (RM).

> HK\$100 for S\$16
> RM100 for S\$40

The matrix **S** below shows the amount of Hong Kong dollars and Malaysian ringgit owned by three friends after exchanging their currencies. On the same day, all three of them went to Best Money Exchange to exchange **all** their currencies for Singapore dollars.

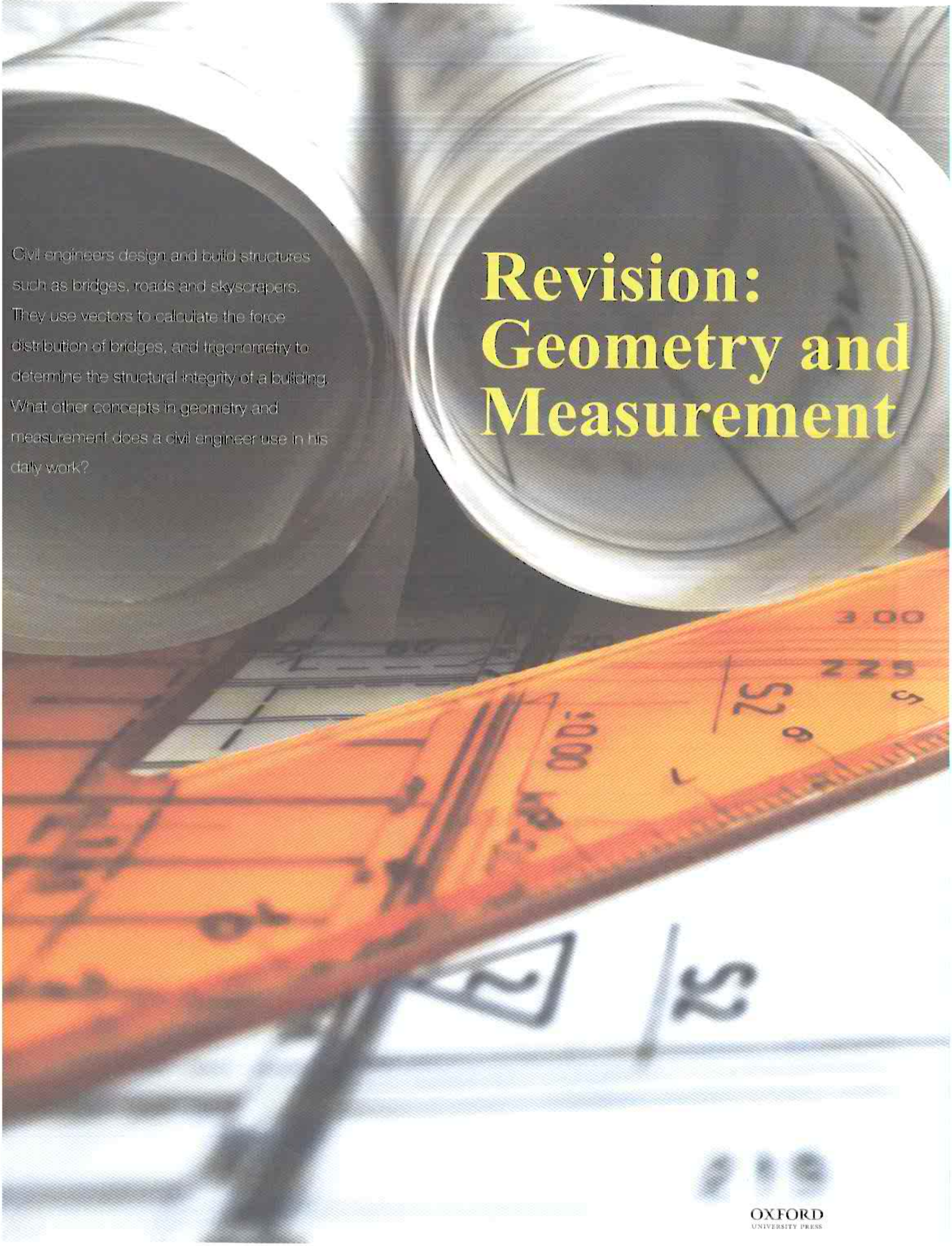
$$\mathbf{S} = \begin{array}{c} \begin{array}{cc} \text{HK\$} & \text{RM} \end{array} \\ \begin{pmatrix} 125 & 320 \\ 200 & 160 \\ 90 & 450 \end{pmatrix} \end{array} \begin{array}{l} \\ \text{Nora} \\ \text{Shirley} \\ \text{Priya} \end{array}$$

(i) Write down two matrices only, such that the elements of their matrix product give the amount of Singapore dollars owned by each friend. Evaluate this product.

(ii) Write down two matrices only, such that the elements of their matrix product, give the total amount of Singapore dollars owned by the three friends. Calculate this total amount.

Revision: Geometry and Measurement

Civil engineers design and build structures such as bridges, roads and skyscrapers. They use vectors to calculate the force distribution of bridges, and trigonometry to determine the structural integrity of a building. What other concepts in geometry and measurement does a civil engineer use in his daily work?



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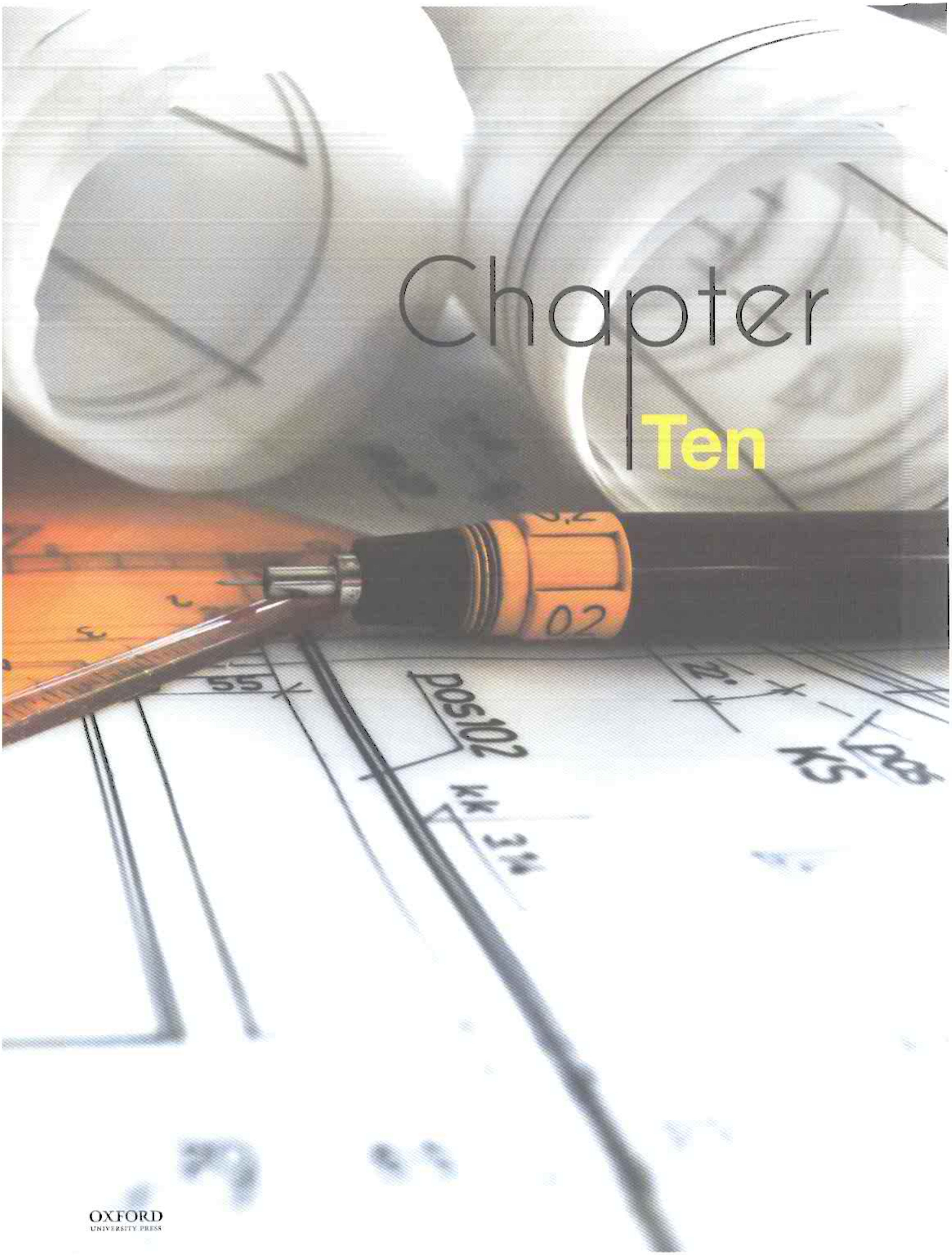
Chapter Ten



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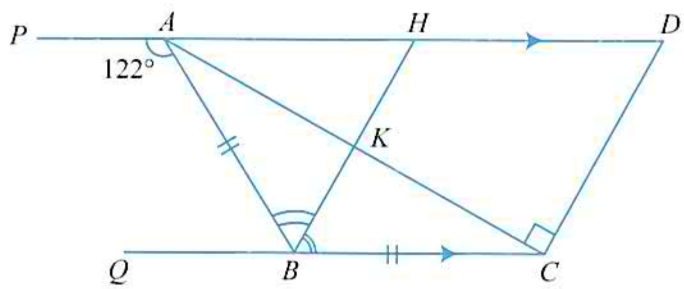
10.1 Angles, Triangles and Polygons

Worked Example 1

(Alternate and Interior Angles)

In the diagram, $PAHD$ is parallel to $Q\hat{B}C$, $AB = BC$, $P\hat{A}B = 122°$, $A\hat{C}D = 90°$ and BH bisects $A\hat{B}C$.



Find

(i) $A\hat{B}H$,

(ii) $B\hat{K}C$,

(iii) $A\hat{D}C$.

Solution:

(i) $A\hat{B}C = P\hat{A}B$ (alt. $\angle$s, $PA \parallel BC$)

$= 122°$

$A\hat{B}H = \dfrac{122°}{2}$

$= 61°$

(ii) $B\hat{C}A = \dfrac{180° - 122°}{2}$ (base $\angle$s of isos. $\triangle ABC$)

$= 29°$

$B\hat{K}C = 180° - 29° - 61°$ ($\angle$ sum of a $\triangle$)

$= 90°$

(iii) $A\hat{D}C = 180° - B\hat{C}D$ (int. $\angle$s, $HD \parallel BC$)

$= 180° - (90° + 29°)$

$= 61°$

Problem Solving Tip

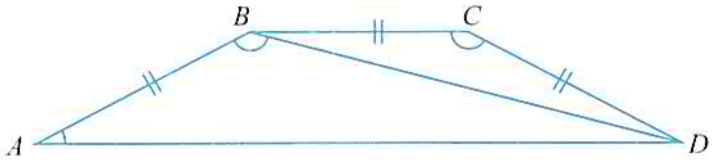
Alternatively, as BH bisects $A\hat{B}C$, BK is the height of isosceles triangle ABC.

$\therefore B\hat{K}C = 90°$

Worked Example 2

(Problem involving a Regular Polygon)

AB, BC and CD are adjacent sides of a 12-sided regular polygon.



Find

(i) $B\hat{C}D$, **(ii)** $A\hat{B}D$, **(iii)** $B\hat{A}D$.

Solution:

(i) $B\hat{C}D = \dfrac{(12-2)\times 180°}{12}$

$= 150°$

RECALL

The sum of interior angles of a n-sided polygon is $(n-2) \times 180°$.

(ii) $C\hat{B}D = \dfrac{180° - 150°}{2}$ (base $\angle$s of isos. $\triangle CBD$)

$= 15°$

$A\hat{B}D = 150° - 15°$

$= 135°$

(iii) $B\hat{A}D + A\hat{D}C + B\hat{C}D + A\hat{B}C = 360°$ ($\angle$ sum of quadrilateral)

By symmetry,

$B\hat{A}D = A\hat{D}C$.

$B\hat{C}D = A\hat{B}C = 150°$ (int. $\angle$ of regular polygon)

$\therefore\ B\hat{A}D = \dfrac{360° - 2 \times 150°}{2}$

$= 30°$

Worked Example 3

(Problem involving an n-sided Polygon)

The sum of the interior angles of an n-sided polygon is 1080°. Find

(i) the value of n,

(ii) the size of an exterior angle of the polygon.

Solution:

(i) $(n-2) \times 180° = 1080°$

$n - 2 = 6$

$n = 8$

RECALL

The sum of exterior angles of **all** polygons is 360°.

(ii) Size of an exterior angle $= \dfrac{360°}{8}$

$= 45°$

Worked Example 4

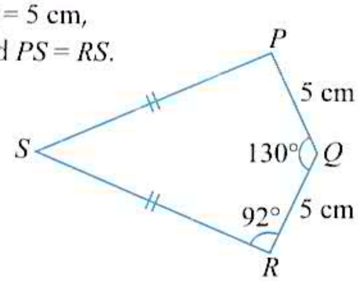
(Symmetric Properties of a Polygon)

In the kite $PQRS$, $PQ = QR = 5$ cm, $P\hat{Q}R = 130°$, $Q\hat{R}S = 92°$ and $PS = RS$.

Find

(i) $P\hat{S}R$,

(ii) the length of PR.



Solution:

(i) $P\hat{S}R = 360° - S\hat{P}Q - P\hat{Q}R - Q\hat{R}S$

$= 360° - 92° - 130° - 92°$ ($S\hat{P}Q = Q\hat{R}S$, symmetric property of a kite)

$= 46°$

RECALL

Angle sum of a quadrilateral $= 360°$

(ii) Consider $\triangle PQR$.

Using cosine rule,

$$PR^2 = PQ^2 + QR^2 - 2(PQ)(QR)\cos P\hat{Q}R$$
$$= 5^2 + 5^2 - 2(5)(5)\cos 130°$$
$$= 82.139 \text{ (to 5 s.f.)}$$
$$PR = \sqrt{82.139}$$
$$= 9.06 \text{ cm (to 3 s.f.)}$$

ATTENTION

In order for the final answer to be accurate to three significant figures, any intermediate working must be correct to at least four significant figures.

Revision 10A

1. **(a)** Each interior angle of a regular polygon is 168°. Find the number of sides of the polygon.

   (b) In a regular polygon, each interior angle is 160° greater than the exterior angle. Find the number of sides of the polygon.

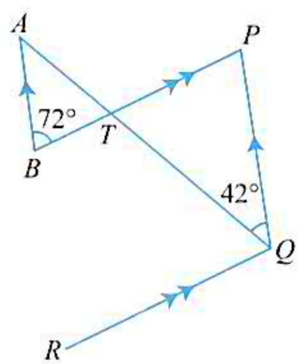
2. **(a)** The exterior angles of a hexagon are in the ratio 4 : 5 : 6 : 7 : 7 : 7. Calculate the largest interior angle of the hexagon.

   (b) In a heptagon, one interior angle is 126° and the other angles are x° each. Find the value of x.

3. **(a)** Find the number of sides of a polygon if the sum of its interior angles is 2700°.

   (b) A polygon has n sides. Three of its exterior angles are 36°, 55° and 65°. The remaining $(n - 3)$ exterior angles are each equal to 8.5°. Find the value of n.

4. In the diagram, AB is parallel to PQ, BP is parallel to RQ, $A\hat{B}T = 72°$ and $P\hat{Q}T = 42°$.



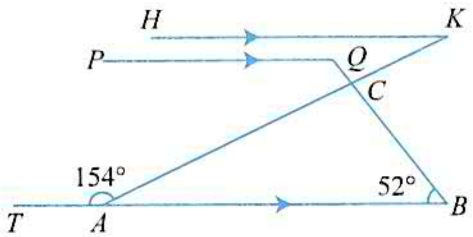
Find

(i) $B\hat{P}Q$,

(ii) $P\hat{T}Q$,

(iii) $R\hat{Q}T$.

5. In the diagram, HK, PQ and AB are parallel, $C\hat{A}T = 154°$ and $A\hat{B}C = 52°$.

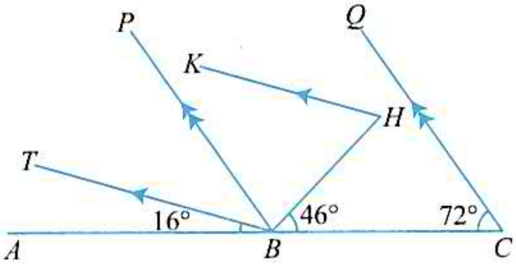


Find

(i) $H\hat{K}C$, (ii) $Q\hat{C}K$,

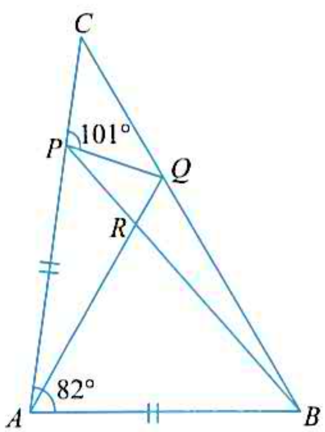
(iii) $P\hat{Q}B$.

6. In the diagram, BT is parallel to HK, BP is parallel to CQ, $A\hat{B}T = 16°$, $C\hat{B}H = 46°$ and $B\hat{C}Q = 72°$.



Find $P\hat{B}T$ and $B\hat{H}K$.

7. In the diagram, $AP = AB$, $\triangle ABQ$ is equilateral, $C\hat{P}Q = 101°$ and $P\hat{A}B = 82°$.

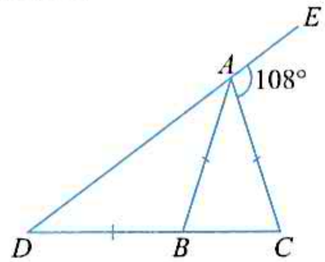


Find

(i) $A\hat{C}B$, (ii) $P\hat{Q}R$,

(iii) $A\hat{R}B$.

8. In the figure, $AB = BD = AC$ and $E\hat{A}C = 108°$. DAE is a straight line.



Find

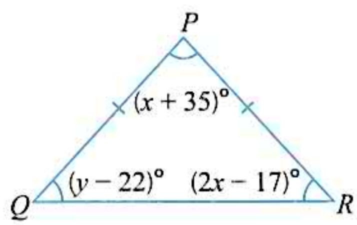
(i) $A\hat{C}B$, (ii) $A\hat{D}B$.

9. The interior angles of a quadrilateral $ABCD$ taken in order are in the ratio 1 : 2 : 3 : 4. Show that $ABCD$ is a trapezium.

10. $ABCDE$ is a pentagon in which AB is parallel to ED. Given that $A\hat{B}C = 155°$, $B\hat{C}D = 3x°$, $C\hat{D}E = 2x°$ and $D\hat{E}A = 75°$, find

(i) $E\hat{A}B$, (ii) the value of x.

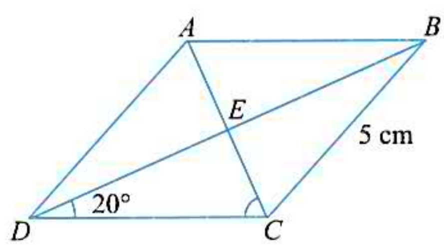
11. Triangle PQR is isosceles with $PQ = PR$. The angles are as shown in the diagram.



(i) Write down a pair of simultaneous equations, in terms of x and y, to represent this information.

(ii) Solve the simultaneous equations to find the value of x and of y.

12. The diagram shows a sketch of a rhombus.



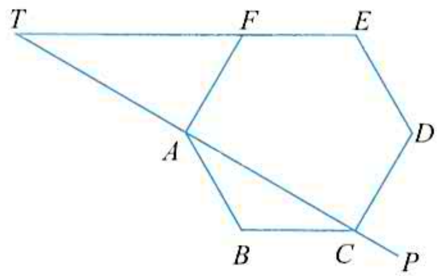
Find

(i) $E\hat{C}D$, (ii) the length of DE.

13. AB, BC and CD are adjacent sides of an 18-sided regular polygon. Find

(i) $A\hat{B}C$, (ii) $A\hat{C}D$.

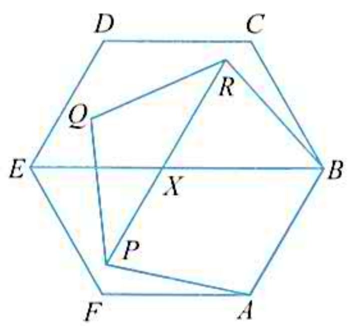
14. In the diagram, $ABCDEF$ is a regular hexagon. PCA produced meets EF produced at T.



Find

(i) $A\hat{T}E$, (ii) $B\hat{C}P$.

15. In the diagram, $ABCDEF$ is a regular hexagon and $APQRB$ is a regular pentagon. EXB and PXR are straight lines.

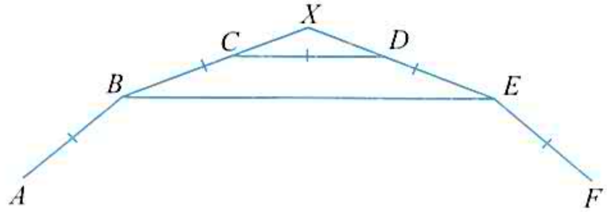


Find

(i) $B\hat{A}P$, (ii) $A\hat{B}X$,

(iii) $E\hat{X}R$.

16. The diagram shows part of a regular 15-sided polygon $ABCDEF$.... BC produced meets ED produced at X.



(i) Find $B\hat{C}D$ and $C\hat{X}D$.

(ii) Determine whether $\triangle XBE$ is isosceles.

(iii) Find $A\hat{B}E$.

17. Construct triangle ABC where $AB = 6$ cm, $BC = 7.5$ cm and $A\hat{B}C = 75°$.

(a) Construct

(i) the perpendicular bisector of BC,

(ii) the bisector of angle ABC.

(b) These two bisectors meet at X.
Complete the statement below.
The point X is equidistant from the lines ___ and ___ and equidistant from the points ___ and ___.

18. Construct triangle PQR where $PQ = 10$ cm, $QR = 8$ cm and $PR = 7$ cm.

(a) Construct

(i) the perpendicular bisector of PQ,

(ii) the bisector of angle PQR.

(b) Mark clearly a possible point which lies inside the triangle, equidistant from P and Q, and is nearer to QR than PQ. Label this point X.

19. Construct a parallelogram $ABCD$ such that $AB = 6.4$ cm, $BC = 4.4$ cm and $A\hat{B}C = 120°$.

(i) Measure and write down the length of the diagonal BD.

(ii) Construct the perpendicular bisector of BD such that it cuts AB and CD. Measure and write down the length of ST, such that S and T are the points where the perpendicular bisector of BD cuts AB and CD respectively.

20. Construct a circle with diameter $AC = 8$ cm.

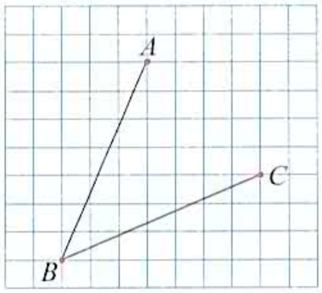
(i) Find and label a point B on the circumference of the circle such that $AB = BC$.

(ii) Find and label the point D on the circumference of the circle such that it is a reflection of the point B in the line AC.

(iii) Join the points to form a quadrilateral $ABCD$. State the name of this quadrilateral.

21. Draw the locus of a point exactly 3.5 cm away from straight line AB.

22. In the diagram in the answer space, draw the locus of points (in the plane of the paper) which are equidistant from BA and BC.



23. Answer the whole of this question on a sheet of plain paper.

(i) Draw accurately the triangle PQR with $PQ = 6$ cm, $PR = 8$ cm and $RQ = 10$ cm. Measure and write down the size of $P\hat{Q}R$.

(ii) On the same diagram,

(a) construct the locus of points which are equidistant from P and Q,

(b) construct the locus of point X inside triangle PQR such that the area of triangle PQX is 9 cm^2.

(iii) Y is the point inside triangle PQR such that $PY = QY$ and area of triangle PQY is 9 cm^2. Mark the position of Y and measure and write down the length of PY.

24. Answer the whole of this question on a sheet of plain paper.

Construct, in a single diagram,

(i) the quadrilateral $PQRS$ where $PQ = 11.5$ cm, $QR = 8$ cm, $Q\hat{P}S = 80°$, $QS = 12.5$ cm and $Q\hat{R}P = 65°$,

(ii) the locus of the points A which is on the same side of PQ as S such that the areas of triangles PQS and PQA are equal,

(iii) a circle passing through the points P, Q and R,

(iv) the locus of the points B such that $Q\hat{R}B$ is always a right angle.

25. Answer the whole of this question on a sheet of plain paper.

(i) Construct, in a single diagram,

(a) the triangle ABC in which $AB = 7$ cm, $AC = 6$ cm and $B\hat{A}C = 60°$,

(b) the locus of points equidistant from A and B,

(c) the locus of points equidistant from AC and AB.

(ii) Mark clearly on your diagram, the point X which is the point of intersection of the two loci constructed in **(i)**.

(iii) The point P, inside the triangle, is such that $C\hat{A}P \leqslant B\hat{A}P$ and $AP \leqslant BP$. Shade the locus of P.

26. Answer the whole of this question on a sheet of plain paper.

(i) Draw accurately the triangle ABC with base $AB = 7$ cm, $C\hat{A}B = 82°$ and $AC = 5$ cm. Measure, and write down, the length of BC.

(ii) On the same diagram,

(a) construct the locus of points which are 2.4 cm from A,

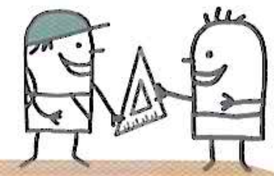
(b) draw the locus of points (on the same side of BC as A), within the triangle, which are 2.4 cm from BC,

(c) construct the circle of radius 2.4 cm which passes through A and touches BC, and whose centre is inside triangle ABC,

(d) construct the locus of points equidistant from AC and BC.

(iii) A point P lies inside the triangle ABC. The point P is such that it is less than 2.4 cm from BC but more than 2.4 cm from A. Its distance from BC is more than its distance from AC. The point P also lies inside the circle passing through A. Indicate clearly, by shading, the region in which P must lie.

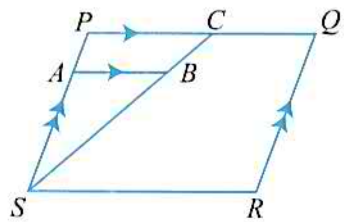
10.2 Congruence and Similarity



Worked Example 5

(Applications of Similar Triangles)

The diagram shows a parallelogram $PQRS$. C is a point on PQ such that $2PC = 3CQ$. A is a point on PS such that $SA = 3AP$ and AB is parallel to PQ.



(a) Show that $\triangle SAB$ is similar to $\triangle SPC$.

(b) Given that the area of $\triangle SPC$ is 24 cm², find

(i) the area of $\triangle SAB$,

(ii) the area of parallelogram $PQRS$.

Solution:

(a) $S\hat{A}B = S\hat{P}C$ (corr. ∠s, $AB \parallel PC$)

$A\hat{S}B = P\hat{S}C$ (common ∠)

∴ $\triangle SAB$ is similar to $\triangle SPC$ (2 pairs of corr. ∠s equal)

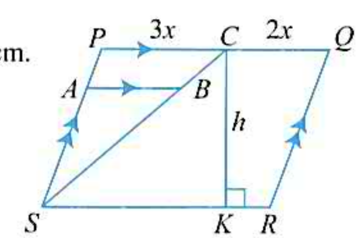
(b) (i) $\dfrac{\text{Area of } \triangle SAB}{\text{Area of } \triangle SPC} = \left(\dfrac{3}{4}\right)^2$ (Since $\dfrac{SA}{AP} = \dfrac{3}{1}$, then $\dfrac{SA}{SP} = \dfrac{3}{3+1} = \dfrac{3}{4}$.)

$\dfrac{\text{Area of } \triangle SAB}{24} = \dfrac{9}{16}$

Area of $\triangle SAB = \dfrac{9}{16} \times 24$

$= 13.5 \text{ cm}^2$

(ii) Let $PC = 3x$ cm, $CQ = 2x$ cm,
and the height of the parallelogram CK be h cm.



Area of $\triangle SPC = \dfrac{1}{2}(3x)(h)$

$24 = \dfrac{3}{2}hx$

$hx = 16$

Area of parallelogram $PQRS = (5x)(h)$

$= 5hx$

$= 5(16)$

$= 80 \text{ cm}^2$

ATTENTION

The vertices of the 2 triangles must match:

$S \leftrightarrow S$

$A \leftrightarrow P$

$B \leftrightarrow C$

INFORMATION

Alternatively, since the heights of $\triangle SPC$ and $\triangle SPQ$ are equal,

$\dfrac{\text{Area of } \triangle SPC}{\text{Area of } \triangle SPQ} = \dfrac{PC}{PQ} = \dfrac{3}{5}$.

Area of $\triangle SPQ = \dfrac{5}{3} \times 24$

$= 40 \text{ cm}^2$

Area of $PQRS = 2 \times \text{area of } \triangle SPQ$

$= 2 \times 40$

$= 80 \text{ cm}^2$

Worked Example 6

(Applications of Similar Solids)

The ratio of the surface areas of two similar solid chocolate cones is 4 : 25. If the smaller cone has a height of 6.8 cm and is made up of 500 cm³ of chocolate, find

(i) the height of the larger cone,

(ii) the volume of chocolate used to make the larger cone.

Solution:

(i) Let the height of the larger cone be H cm.

$$\left(\frac{H}{6.8}\right)^2 = \frac{25}{4}$$

$$\frac{H}{6.8} = \frac{5}{2}$$

$$H = \frac{5}{2} \times 6.8$$

$$= 17$$

∴ Height of the larger cone is 17 cm.

RECALL

The ratio of the surface areas of two similar figures is the square of the ratio of the corresponding lengths, i.e. $\frac{A_1}{A_2} = \left(\frac{l_1}{l_2}\right)^2$, where A_1 and l_1 are the surface area and the length of the first figure respectively, and A_2 and l_2 are the surface area and the length of the second figure, respectively.

(ii) Let the volume of chocolate used to make the larger cone be V cm³.

$$\frac{V}{500} = \left(\frac{5}{2}\right)^3$$

$$V = \left(\frac{5}{2}\right)^3 \times 500$$

$$= 7812.5$$

∴ Volume of chocolate used to make the larger cone is 7812.5 cm³.

RECALL

Similar to **(i)**, $\frac{V_1}{V_2} = \left(\frac{l_1}{l_2}\right)^3$, where V_1 and V_2 are the volumes of the first and second solids, respectively.

Revision 10B

1. In a garden, there are two ponds which are geometrically similar. The depth of the larger pond is twice that of the smaller pond. Given that the capacity of the larger pond is 3360 litres, find the capacity of the smaller pond.

2. A pet shop sells two sizes of bathtubs which are geometrically similar. The ratio of the lengths of the bathtubs is 3 : 4.

   (i) The height of the small bathtub is 0.75 m. Find the height of the large bathtub.

   (ii) Given that the large bathtub has a capacity of V litres, find the capacity of the small bathtub.

3. The total surface area of two glass spheres are 640 cm^2 and 1210 cm^2.

(a) Find, in its simplest form, the ratio of
  (i) the smaller radius to the larger radius,
  (ii) the smaller volume to the larger volume.

(b) It costs \$3.20 to paint the smaller sphere. Find the cost of painting the larger sphere.

4. The volumes of two similar bags are 108 cm^3 and 500 cm^3. Find the ratio of

(i) their heights,

(ii) their total surface areas.

5. A conical flask has an external surface area of 50 cm^2 and a capacity of 845 cm^3. Find the volume of a similar conical flask which has a surface area of 32 cm^2.

6. The volume of a solid stone statue 3 m high is V_1 and its surface area is A_1. A model of the statue, 20 cm high, has a volume of V_2 and a surface area of A_2. Find the ratio of

(i) $V_1 : V_2$, (ii) $A_1 : A_2$.

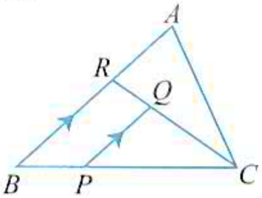
7. A model of a suspension bridge is made on a scale of 1 : 600.

(i) Given that the supporting towers in the model have a height of 34 cm, find their actual height in metres.

(ii) It is given that the costs of painting the model and the actual bridge are proportional to their respective surface areas. If it costs \$4 to paint the model, calculate the cost of painting the actual bridge.

(iii) A steel section has a mass of 432 tonnes. Find the mass of the section in the model, if it is constructed from the same material.

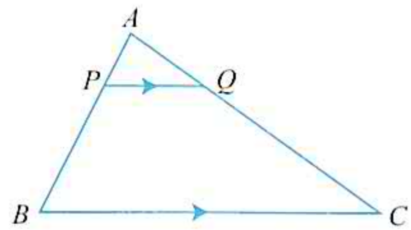
8. In the diagram, PQ is parallel to BA. P lies on BC such that $BP : PC = 2 : 5$ and R lies on BA such that $BR : RA = 4 : 3$.



Given that the area of ΔBCR is 98 cm^2, find the area of

(i) $\triangle ACR$, (ii) $\triangle CPQ$.

9. In the diagram, ABC is a triangle. P and Q lie on the sides AB and AC respectively. PQ is parallel to BC.

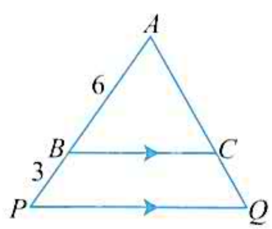


Given that $AB = 8$ cm, $AP = 2$ cm and $AC = 12$ cm, find

(i) the length of QC, (ii) $\dfrac{\text{area of}\triangle APQ}{\text{area of}\triangle PQB}$,

(iii) $\dfrac{\text{area of } PBCQ}{\text{area of}\triangle ABC}$.

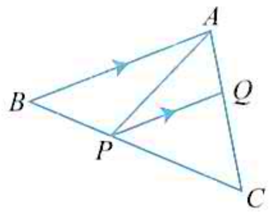
10. In the diagram, B and C lie on AP and AQ respectively such that BC is parallel to PQ.



Given that $AB = 6$ cm, $BP = 3$ cm and the area of $\triangle ABC$ is 20 cm^2, find

(i) the area of $\triangle APQ$, (ii) the area of $BPQC$.

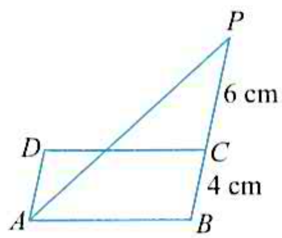
11. In the diagram, $BP : PC = 2 : 3$ and the area of $\triangle APC$ is 36 cm².



(i) Find the area of $\triangle ABP$.

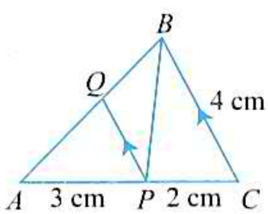
(ii) Given that PQ is parallel to BA, find the area of $\triangle CPQ$.

12. The diagram shows a parallelogram $ABCD$. $BC = 4$ cm, and P lies on BC produced such that $CP = 6$ cm. The area of $ABCD$ is 40 cm².



Find the area of $\triangle ABP$.

13. In the diagram, ABC is a triangle. P and Q lie on AC and AB respectively. $AP = 3$ cm, $PC = 2$ cm, $BC = 4$ cm and PQ is parallel to CB.



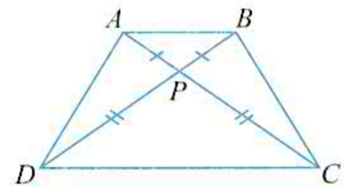
Find

(i) the length of PQ,

(ii) the ratio of the area of $\triangle ABP$ to the area of $\triangle BPC$,

(iii) the ratio of the area of $\triangle APQ$ to the area of $PQBC$.

14. In the quadrilateral $ABCD$, the diagonals AC and BD intersect at P. $\triangle ABP$ and $\triangle PCD$ are isosceles.

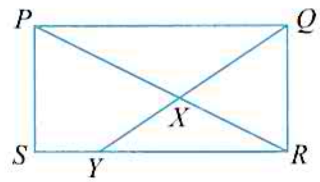


(i) Show that $\triangle APD$ and $\triangle BPC$ are congruent.

(ii) Name two other triangles that are congruent.

(iii) Name two triangles that are similar but not congruent.

15. In the diagram, $PQRS$ is a rectangle. Y lies on SR such that PR and QY intersect at X.



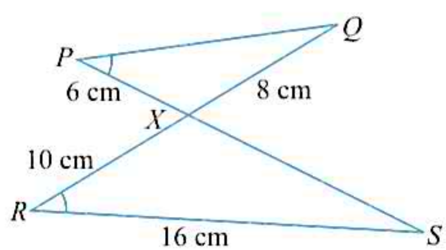
(a) Prove that $\triangle PQX$ and $\triangle RYX$ are similar.

(b) Given that $SY = \frac{1}{4}SR$, find

(i) area of $\triangle RYX$: area of $\triangle PQX$,

(ii) area of $\triangle QXR$: area of rectangle $PQRS$.

16. In the diagram, triangles PXQ and RXS are similar. PXS and QXR are straight lines. Angle QPX = Angle SRX.

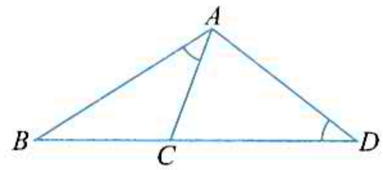


(a) Find

(i) XS, (ii) PQ.

(b) Given that the area of triangle PXQ is k cm², find the area of triangle RXS, in terms of k.

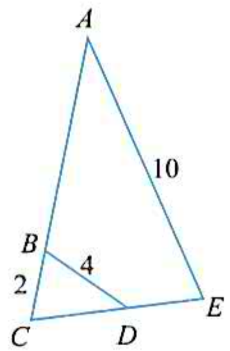
17. In the diagram, ABD is a triangle. C lies on BD such that $B\hat{A}C = B\hat{D}A$.



(i) Show that triangles ABC and DBA are similar.

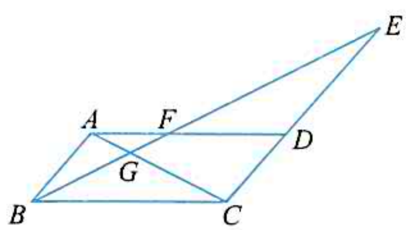
(ii) Given that $AB = 6$ cm and $BC = 4$ cm, find BD.

18. In the diagram, triangles ACE and DCB are similar.



Given that $AE = 10$ cm, $BC = 2$ cm and $BD = 4$ cm, find CE.

19. In the diagram, $ABCD$ is a parallelogram. BF produced meets CD produced at E. BF and AC intersect at G and $AG = \frac{1}{2}GC$.



(a) Show that triangles ABG and CEG are similar.

(b) Name a triangle similar to triangle EFD.

(c) Name two triangles that are congruent and show that they are congruent.

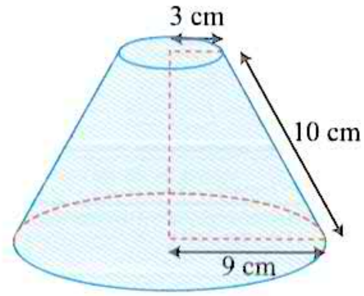
(d) Find

(i) $\dfrac{AB}{CE}$,

(ii) $\dfrac{\text{area of } \triangle ABG}{\text{area of } \triangle ABC}$,

(iii) $\dfrac{\text{area of } \triangle AFG}{\text{area of } \triangle CBG}$.

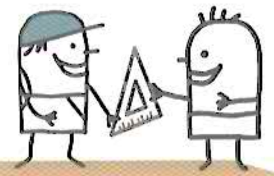
20. The solid figure shows the shape of a frustum. Its circular top and base have radii 3 cm and 9 cm respectively. The slant edge is 10 cm long.



(i) Find the height of the solid.

(ii) By using the difference in the volumes of two cones, calculate the volume of the frustum.

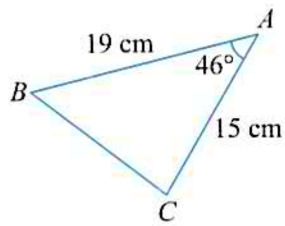
10.3 Pythagoras' Theorem and Trigonometry



Worked Example 7

(Finding the Area of a Triangle)

The figure shows a triangle ABC, where $AB = 19$ cm, $AC = 15$ cm and $\angle BAC = 46°$.



Find

(i) the length of BC, **(ii)** the area of $\triangle ABC$.

Solution:

(i) Using cosine rule,

$$BC^2 = AB^2 + AC^2 - 2 \times AB \times AC \times \cos B\hat{A}C$$
$$= 19^2 + 15^2 - 2 \times 19 \times 15 \times \cos 46°$$
$$= 190.0 \text{ (to 4 s.f.)}$$

$$BC = \sqrt{190.0}$$
$$= 13.8 \text{ cm (to 3 s.f.)}$$

(ii) Area of $\triangle ABC = \frac{1}{2} \times 19 \times 15 \times \sin 46°$

$$= 103 \text{ cm}^2 \text{ (to 3 s.f.)}$$

RECALL

Area of $\triangle ABC = \frac{1}{2}ab\sin C$

Worked Example 8

(Trigonometric Ratios)

In $\triangle PQR$, where $PQ = 8$ cm, $QR = 15$ cm and $\angle PQR = 90°$. Given that QR is produced to S, find

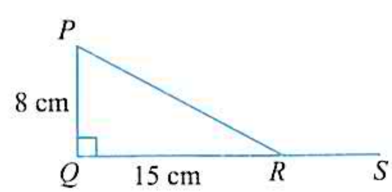
(i) the length of PR, **(ii)** $\sin \angle PRQ$,

(iii) $\cos \angle PRS$.

Solution:

(i) By Pythagoras' Theorem,

$$PR^2 = PQ^2 + QR^2$$
$$= 8^2 + 15^2$$
$$= 289$$
$$\therefore PR = \sqrt{289}$$
$$= 17 \text{ cm}$$



(ii) $\sin \angle PRQ = \dfrac{\text{opp}}{\text{hyp}}$

$= \dfrac{8}{17}$

(iii) $\cos \angle PRS = -\cos(180° - \angle PRS)$

$= -\cos \angle PRQ$

$= -\dfrac{\text{adj}}{\text{hyp}}$

$= -\dfrac{15}{17}$

RECALL

$\cos A = -\cos(180° - A)$

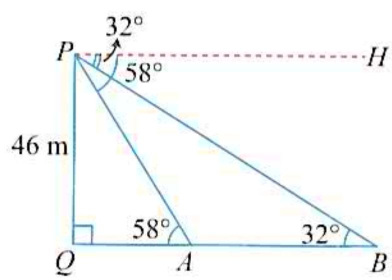
Worked Example 9

(Problem involving Angle of Depression)

From the top of a building 46 m high, the angles of depression of two points A and B on the ground due east of the building are 58° and 32°, respectively. Calculate the distance between the two points.

Solution:

Let PQ be the height of the building.



$\angle PAQ = 58°$ (alt. $\angle$s, $PH // QB$)

$\tan 58° = \dfrac{46}{AQ}$

$AQ = \dfrac{46}{\tan 58°}$

$= 28.74$ m (to 4 s.f.)

$\angle PBQ = 32°$ (alt. $\angle$s, $PH // QB$)

$\tan 32° = \dfrac{46}{BQ}$

$BQ = \dfrac{46}{\tan 32°}$

$= 73.62$ m (to 4 s.f.)

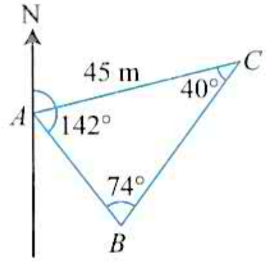
$\therefore$ Distance between the two points A and $B = 73.62 - 28.74$

$= 44.9$ m (to 3 s.f.)

Worked Example 10

(Problem involving Bearings)

In the figure, A, B and C are three points on a map on level ground.



Given that $\angle ABC = 74°$, $\angle ACB = 40°$ and the bearing of B from A is 142°, find

(i) the bearing of C from A,

(ii) the bearing of C from B.

Solution:

(i) $\angle BAC = 180° - 74° - 40°$ ($\angle$ sum of a $\triangle$)

$= 66°$

$142° - 66° = 76°$

$\therefore$ Bearing of C from A is 076°

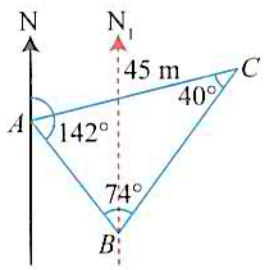
(ii) $\angle ABN_1 = 180° - 142°$ (int. $\angle$s, AN // BN_1)

$= 38°$

$\angle N_1BC = 74° - 38°$

$= 36°$

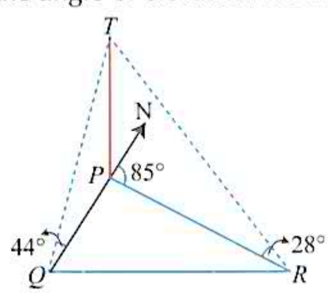
$\therefore$ Bearing of C from B is 036°



Worked Example 11

(Three-dimensional Problem)

Three points P, Q and R are on level ground. Q is due south of P and the bearing of R from P is 085°. A vertical mast PT stands at P. The angle of elevation of T from Q is 44° and the angle of elevation of T from R is 28°.



(a) Given that the height of the mast is 50 m, find

(i) the distance between P and Q,

(ii) the distance between Q and R,

(iii) the bearing of R from Q.

(b) A man at Q walks in a straight line towards R. Find the greatest angle of elevation of T from any point along QR.

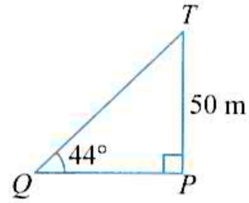
Solution:

(a) (i) In $\triangle PQT$,

$$\tan 44° = \frac{50}{PQ}$$

$$PQ = \frac{50}{\tan 44°}$$

$$= 51.8 \text{ m (to 3 s.f.)}$$

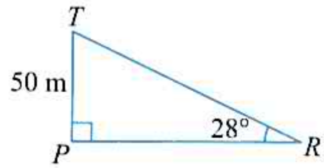


(ii) In $\triangle PRT$,

$$\tan 28° = \frac{50}{PR}$$

$$PR = \frac{50}{\tan 28°}$$

$$= 94.04 \text{ m (to 4 s.f.)}$$



In $\triangle QPR$,

$\angle QPR = 180° - 85°$

$= 95°$ ($\angle$s on a str. line)

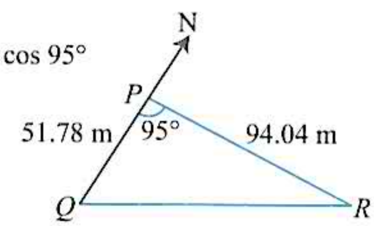
Using cosine rule,

$$QR^2 = 51.78^2 + 94.04^2 - 2 \times 51.78 \times 94.04 \times \cos 95°$$

$$= 12\ 370 \text{ (to 4 s.f.)}$$

$$\therefore QR = \sqrt{12\ 370}$$

$$= 111 \text{ m (to 3 s.f.)}$$



(iii) Using sine rule,

$$\frac{\sin \angle PQR}{PR} = \frac{\sin \angle QPR}{QR}$$

$$\frac{\sin \angle PQR}{94.04} = \frac{\sin 95°}{111.2}$$

$$\sin \angle PQR = \frac{94.04 \times \sin 95°}{111.2}$$

$$\angle PQR = \sin^{-1}\left(\frac{94.04 \times \sin 95°}{111.2}\right)$$

$$= 57.4° \text{ (to 1 d.p.)}$$

$\therefore$ Bearing of R from Q is 057.4°

ATTENTION

In order for the final answer to be accurate to three significant figures, any intermediate working must be correct to at least four significant figures.

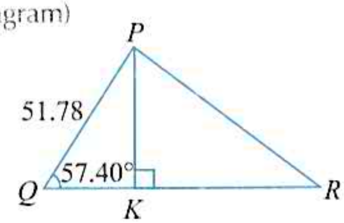
Alternatively, you can store the values in your calculator and recall them for use in subsequent workings.

(b) The greatest angle of elevation of T from any point along QR occurs at the point K on QR where PK is perpendicular to QR. (see diagram)

In $\triangle PQK$,

$$\sin 57.40° = \frac{PK}{51.78}$$

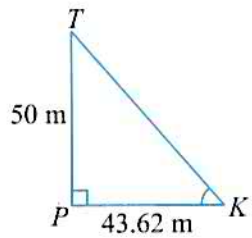
$$PK = 51.78 \times \sin 57.40°$$
$$= 43.62 \text{ m (to 4 s.f.)}$$



$$\tan \angle PKT = \frac{50}{43.62} \text{ (see diagram)}$$

$$\angle PKT = \tan^{-1}\frac{50}{43.62}$$
$$= 48.9° \text{ (to 1 d.p.)}$$

$\therefore$ The greatest angle of elevation of T from any point along QR is 48.9°.



ATTENTION

The greatest angle of elevation of T from a point along QR occurs when the man is nearest to T, i.e. PK is a minimum, hence K is the point on QR such that PK is perpendicular to QR.

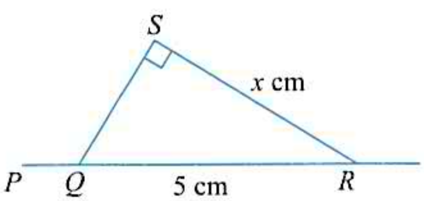
Revision 10C

1. Solve the following equations for $0° < x < 180°$.

(i) $\sin x = \sin 35°$ **(ii)** $\cos x = -\cos 25°$

(iii) $\sin x = 0.5$ **(iv)** $2 \cos x = -1$

2.



In the diagram, PQR is a straight line. $QR = 5$ cm, $SR = x$ cm and $\angle QSR = 90°$. Find, in terms of x, the value of

(i) QS, **(ii)** $\sin \angle QRS$,

(iii) $\cos \angle PQS$.

3. **(a)** A ladder 12 m long leans against a wall. Its foot on the ground is 8 m from the wall. Find

(i) the angle the ladder makes with the ground,

(ii) the height of the ladder above the ground.

(b) A tower stands at 60 m high and a man stands 75 m away from the tower on ground level. Find the angle of elevation of the tower from the man.

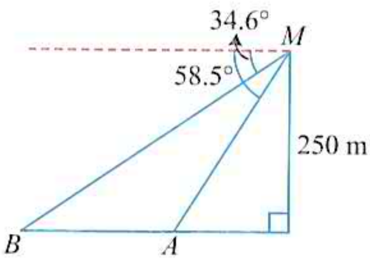
4. In $\triangle ABD$, $\angle ABD = 90°$. AC bisects $\angle BAD$. Given that $AC = 6$ cm and $\angle ACB = 60°$, find the lengths of AB and AD.

5. In $\triangle ABC$, $\angle ABC = 42°$, $\angle BAC = 41°$ and $BC = 8.6$ cm. Calculate

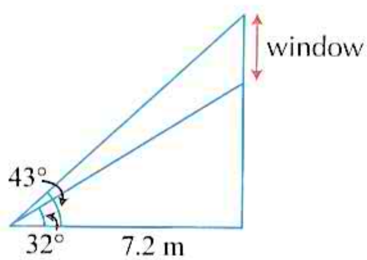
(i) the length of AB,

(ii) the area of $\triangle ABC$.

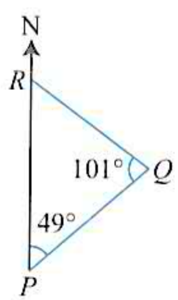
6. **(a)** The diagram shows a man standing at the point M on top of a cliff 250 m high. He observes two ships A and B, where the angle of depression of ship A from the man is 58.5° and the angle of depression of ship B from the man is 34.6°. Find the distance between the two ships.



(b) The angles of elevation of the top and bottom of a window from a point on the ground are 43° and 32°, respectively. Given that the point is 7.2 m from the foot of the window, find the height of the window.



7. The points P, Q and R are on level ground and $\angle PQR = 101°$. Given that R is due north of P and the bearing of Q from P is 049°.



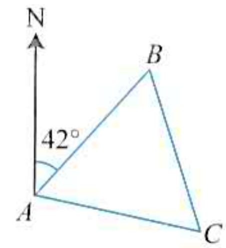
Find

(a) **(i)** the bearing of Q from R,

(ii) the bearing of R from Q.

(b) If the distance between P and Q is 1.45 km, find the distance between P and R.

8. In the diagram, the points A, B and C form an equilateral triangle. The bearing of B from A is 042°.

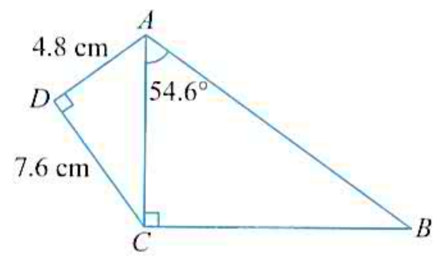


Find

(i) the bearing of C from A,

(ii) the bearing of C from B.

9. In the diagram, $\angle ACB = \angle ADC = 90°$, $\angle BAC = 54.6°$, $AD = 4.8$ cm and $DC = 7.6$ cm.



(a) Find

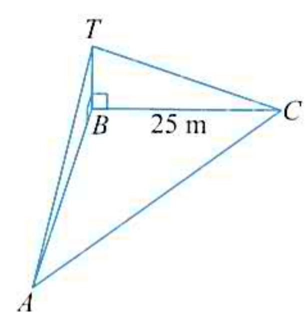
(i) $\angle ACD$,

(ii) the length of AC,

(iii) the length of AB.

(b) Given that E is the point on AB such that $AE = 7$ cm, find the area of $\triangle ACE$.

10. Three points A, B and C are on level ground. A is due south of B and C is due east of B. BT is a vertical flagpole and the distance between B and C is 25 m.



(i) Given that the angle of elevation of T from C is 18°, find the height of the flagpole.

(ii) The bearing of C from A is 036°. Find the angle of elevation of T from A.

11. Two ships V and W leave a port Y at 1200. Ship V sails at a speed of 15 km/h on a bearing of 040° while ship W sails at 24 km/h on a bearing of 100°. Find, at the time 1400,

(i) the distance between the two ships,

(ii) $\angle YVW$, giving your answer to the nearest degree,

(iii) the bearing of ship W from ship V.

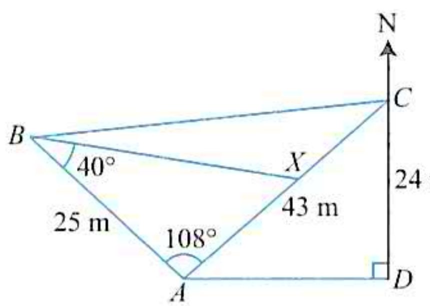
12. Three points A, B and C are on level ground such that the bearings of B and C from A are 195° and 305° respectively. Given that AB = 3.4 km and AC = 4.5 km, find

(i) the distance between B and C in km,

(ii) the bearing of C from B,

(iii) the area of $\triangle ABC$ in km².

13. In the diagram, the points A, B, C and D are on level ground such that CD = 24 m, AC = 43 m, AB = 25 m and $\angle BAC$ = 108°. D is due south of C and A is due west of D.



Given that X lies on AC such that $\angle ABX$ = 40°, find

(i) the bearing of C from A,

(ii) the length of CX,

(iii) the length of BC,

(iv) the area of the quadrilateral $ABCD$.

14. The diagram below is a scale drawing showing the positions of two harbours, X and Y.
Scale: 1 cm represents 5 km.

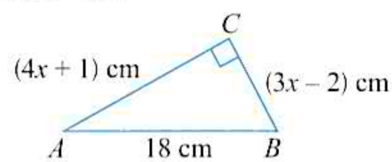
A cruise ship is 27.5 km from Harbour Y on a bearing of 290°.

(i) Mark and label on the diagram the position, C, of the cruise ship.

(ii) Find the actual distance of the cruise ship from Harbour X.

(iii) Find the bearing of the cruise ship from Harbour X.

15. In $\triangle ABC$, AB = 18 cm, AC = $(4x + 1)$ cm, BC = $(3x - 2)$ cm and $\angle ACB$ = 90°.



(a) Form an equation in x and show that it reduces to $25x^2 - 4x - 319 = 0$.

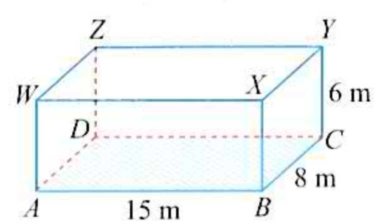
(b) Solve the equation $25x^2 - 4x - 319 = 0$, giving your answer correct to 2 decimal places.

(c) Use your answer in **(b)** to find

(i) the perimeter of $\triangle ABC$,

(ii) the area of $\triangle ABC$.

16. The diagram below shows a cuboid with dimensions 15 m by 8 m by 6 m.



Find

(i) $\angle ABW$, **(ii)** $\angle BDX$, **(iii)** $\angle AZC$.

17. A harbour H and an oil rig P are 62 km apart, with P due east of H. A supply ship leaves H for a second oil rig Q which is 44 km from P on a bearing of 048° from H. Angle HQP is acute.

(a) Find

(i) the bearing of Q from P,

(ii) the distance HQ.

A seaside resort R is situated west of the line HQ. R is 45 km and 61 km from H and Q respectively. The supply ship leaves H at 1115 and travels towards R, where it stops for 40 minutes before it returns to H.

(b) Given that the supply ship travels at a constant speed of 15 km/h, find the time it returns to H.

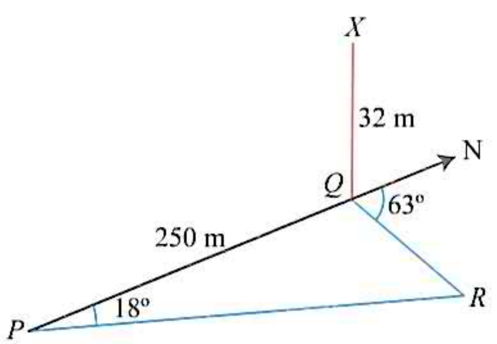
(c) Find

(i) $\angle HQR$,

(ii) the shortest distance from R to HQ,

(iii) the area of $HPQR$.

18. The diagram show three points P, Q and R on level ground such that Q is due north of P. The bearing of R from P is 018° and the bearing of R from Q is 063°.



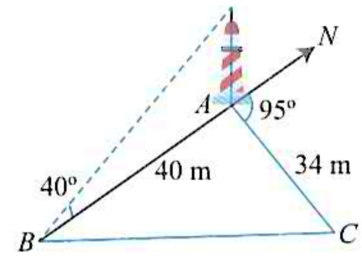
(a) Given that the distance $PQ = 250$ m, find

(i) the distance QR,

(ii) the bearing of P from R.

(b) A vertical post QX, 32 m high stands at point Q. Find the angle of elevation of X from P.

19. Three points A, B and C are on level ground. A vertical lighthouse stands at A as shown in the diagram. Priya stands at point B, due south of A and the bearing of another point C from A is 95°. It is given that $AC = 34$ m and $AB = 40$ m.



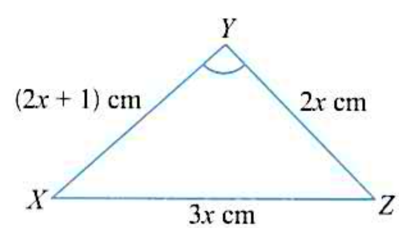
(a) Priya observes that the angle of elevation of the top of the lighthouse is 40°. Calculate

(i) the height of the lighthouse,

(ii) the angle of elevation of the top of the lighthouse from C.

(b) Priya jogs along BC and reaches a point D where the angle of depression of D from the top of the lighthouse is the greatest. Find this angle of depression.

20. In $\triangle XYZ$, it given that $XY = (2x + 1)$ cm, $XZ = 3x$ cm and $YZ = 2x$ cm.

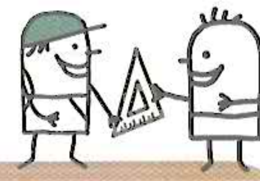


(i) Given that $\cos \angle XYZ = -\frac{1}{21}$, form an equation in x and show that it reduces to $13x^2 - 88x - 21 = 0$.

(ii) Solve the equation $13x^2 - 88x - 21 = 0$, and explain why one of the values of x has to be rejected.

(iii) Calculate the area of $\triangle XYZ$, giving your answer to 3 significant figures.

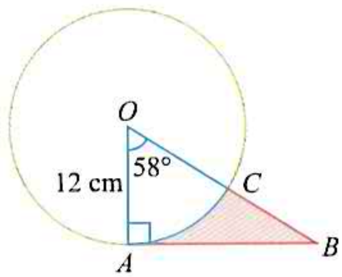
10.4 Mensuration



Worked Example 12

(Finding the Arc Length)

The figure shows a circle with centre O and radius 12 cm. The points A and C lie on the circumference of the circle and OCB is a straight line.



Given that $\angle AOB = 58°$ and OA is perpendicular to AB, find

(i) the arc length AC,

(ii) the perimeter of the shaded region ABC.

Solution:

(i) Length of arc $AC = \frac{58°}{360°} \times 2\pi \times 12$

$= 12.1$ cm (to 3 s.f.)

(ii) $\tan 58° = \frac{AB}{12}$

$AB = 12 \tan 58°$

$= 19.20$ cm (to 4 s.f.)

$\cos 58° = \frac{12}{OB}$

$OB = \frac{12}{\cos 58°}$

$= 22.64$ cm (to 4 s.f.)

$BC = 22.64 - 12$

$= 10.64$ cm

$\therefore$ Perimeter of shaded region $ABC = AB + BC +$ length of arc AC

$= 19.20 + 10.64 + 12.15$

$= 42.0$ cm (to 3 s.f.)

RECALL

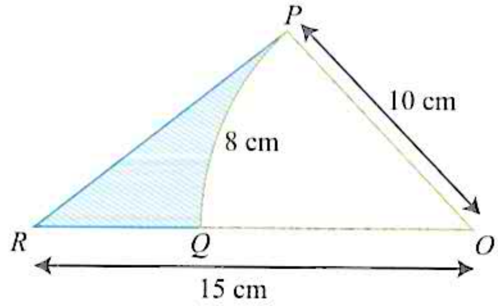
For a sector subtending an angle $x°$ at the centre of a circle of radius r,

length of arc $= \frac{x°}{360°} \times 2\pi r$.

Worked Example 13

(Problem involving Arc Length and Area of Sector)

In the figure, OPQ is a sector of a circle with centre O and radius 10 cm.



Given that arc PQ = 8 cm and OR = 15 cm, find

(i) the angle POQ in **radians**,

(ii) the length of PR, giving your answer correct to 2 decimal places,

(iii) the area of shaded region PQR.

Solution:

(i) Given that arc length = 8 cm,

$$10\theta = 8$$
$$\theta = 0.8$$

$\therefore$ Angle POQ = 0.8 radians

(ii) Using cosine rule,

$$PR^2 = 10^2 + 15^2 - 2 \times 10 \times 15 \times \cos 0.8$$
$$= 115.988 \text{ (to 3 d.p.)}$$
$$PR = \sqrt{115.988}$$
$$= 10.77 \text{ cm (to 2 d.p.)}$$

(iii) Area of shaded region PQR = area of $\triangle OPR$ – area of sector POQ

$$= \frac{1}{2} \times 10 \times 15 \times \sin 0.8 - \frac{1}{2} \times 10^2 \times 0.8$$
$$= 13.8 \text{ cm}^2 \text{ (to 3 s.f.)}$$

RECALL

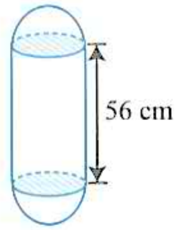
For a sector subtending an angle θ radians at the centre of a circle of radius r,
length of arc = $r\theta$ and
area of sector = $\frac{1}{2}r^2\theta$.

ATTENTION

Ensure that your calculator has been set to the Radian mode.

Worked Example 14

(Volume and Surface Area of a Composite Solid)
The diagram shows a solid which is cylindrical with hemispherical ends. The height of the cylinder is 56 cm and the area of the base of the cylinder is 1386 cm^2.



(a) Calculate
  (i) the volume of the cylinder,
  (ii) the radius of the base of the cylinder,
  (iii) the volume of the solid.

(b) Given that the solid is made from a material of density 0.05 g/cm^3, find the mass of the solid.

(c) Find the total surface area of the solid.

(d) Given that the solid is melted and made into a cone with a base radius of 28 cm, calculate the height of the cone, giving your answer in metres.

Solution:

(a) (i) Volume of cylinder = area of base × height
$= 1386 \times 56$
$= 77\ 616 \text{ cm}^3$

(ii) Let r cm be the radius of the base of the cylinder.
Area of base $= \pi r^2$

$$\pi r^2 = 1386$$
$$r^2 = \frac{1386}{\pi}$$
$$r = \sqrt{\frac{1386}{\pi}}$$
$$= 21.0 \text{ (to 3 s.f.)}$$

∴ The radius of the base of the cylinder is 21.0 cm.

(iii) Volume of two hemispheres = Volume of a sphere
$$= \frac{4}{3} \times \pi \times 21.00^3$$
$$= 38\ 790 \text{ cm}^3 \text{ (to 4 s.f.)}$$

∴ Volume of solid = 77 616 + 38 790
$= 116\ 000 \text{ cm}^3$ (to 3 s.f.)

RECALL

Volume of sphere $= \frac{4}{3}\pi r^3$, where r is the radius of the sphere

(b) Mass of solid = 116 400 × 0.05
$= 5820$ g (to 3 s.f.)

(c) Curved surface area of cylinder $= 2 \times \pi \times 21.00 \times 56$
$= 7389 \text{ cm}^2$ (to 4 s.f.)

Surface area of sphere $= 4 \times \pi \times 21.00^2$
$= 5542 \text{ cm}^2$ (to 4 s.f.)

Total surface area of solid $= 7389 + 5542$
$= 12\,900 \text{ cm}^2$ (to 3 s.f.)

(d) Let h cm be the height of the cone.
Volume of cone $= 116\,400 \text{ cm}^3$

$$\frac{1}{3} \times \pi \times 28^2 \times h = 116\,400$$
$$h = \frac{3 \times 116\,400}{784\pi}$$
$$= 142 \text{ cm (to 3 s.f.)}$$
$$= 1.42 \text{ m}$$

$\therefore$ The height of the cone is 1.42 m.

RECALL

Curved surface area of cylinder $= 2\pi rh$, where r and h are the radius and height of the cylinder, respectively.
Surface area of sphere $= 4\pi r^2$, where r is the radius of the sphere.

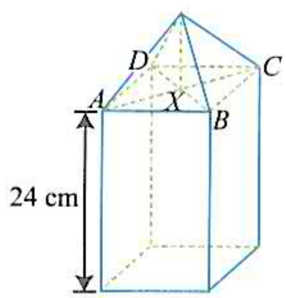
RECALL

Volume of cone $= \frac{1}{3}\pi r^2 h$, where r and h are the radius and height of the cone, respectively.

Worked Example 15

(Volume and Dimensions of a Composite Solid)

A model consists of a solid cuboid attached to a solid pyramid as shown in the diagram. The height of the cuboid is 24 cm and the area of its base is 96 cm².



(i) Calculate the volume of the cuboid.

(ii) Given that the volume of the pyramid is 144 cm³, find the height of the pyramid.

(iii) The model is made from a material with a density of 0.5 g/cm³. Find its mass.

(iv) Given that the width and length of the base of the cuboid are in the ratio 2 : 3, find the width and length of the cuboid.

(v) Hence, find the slant height of the pyramid.

Solution:

(i) Volume of cuboid = area of base × height

$= 96 \times 24$

$= 2304 \text{ cm}^3$

(ii) Let h cm be the height of the pyramid.

Volume of pyramid = 144 cm^3

$$\frac{1}{3} \times \text{area of base} \times \text{height} = 144$$

$$\frac{1}{3} \times 96 \times h = 144$$

$$h = \frac{144 \times 3}{96}$$

$$= 4.5$$

$\therefore$ The height of the pyramid is 4.5 cm.

(iii) Volume of model = volume of cuboid + volume of pyramid

$= 2304 + 144$

$= 2448 \text{ cm}^2$

Mass of model $= 2448 \times 0.5$

$= 1224$ g

(iv) Let the width of the base of the cuboid be $2x$ cm.
Then the length of the base is $3x$ cm.

Area of base = 96 cm^2

$$2x \times 3x = 96$$

$$6x^2 = 96$$

$$x^2 = 16$$

$$x = 4$$

$\therefore$ Width of base $= 2 \times 4 = 8$ cm
$\therefore$ Length of base $= 3 \times 4 = 12$ cm

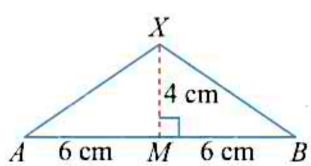
(v) Let M be the midpoint of AB.
In $\triangle XMB$,
using Pythagoras' Theorem,

$$XB^2 = XM^2 + MB^2$$

$$= 4^2 + 6^2$$

$$= 52$$

$$XB = \sqrt{52} \text{ cm}$$



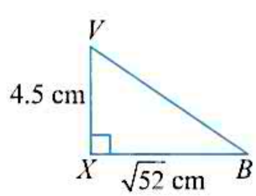
Let V be the vertex of the pyramid.
In $\triangle VXB$,
using Pythagoras' Theorem,

$$VB^2 = VX^2 + XB^2$$

$$= 4.5^2 + \left(\sqrt{52}\right)^2$$

$$= 72.25$$

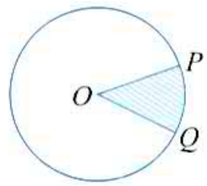
$$VB = 8.5 \text{ cm}$$



$\therefore$ The slant height of the pyramid is 8.5 cm.

Revision 10D

1. In the figure, the area of the shaded sector POQ is $\frac{3}{20}$ of the area of the whole circle.



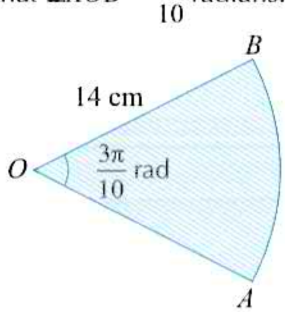
(a) Find $\angle POQ$, giving your answer in degrees.

(b) Given that the area of the circle is 616 cm², find

(i) the radius of the circle,

(ii) the area of the shaded sector POQ.

2. The figure shows a sector of a circle of radius 14 cm. It is given that $\angle AOB = \frac{3\pi}{10}$ radians.



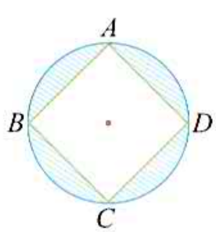
Find

(i) the perimeter of the sector,

(ii) the area of the sector,

expressing your answers in terms of π.

3. The square $ABCD$ is inscribed in the circle of radius 8 cm.



Giving your answers correct to the nearest cm², find

(i) the area of the circle,

(ii) the area of the shaded portion.

4. A hollow sphere is made of a metal that is 1.5 cm thick with an external diameter of 22 cm.

(i) Find the volume of metal used to make the sphere.

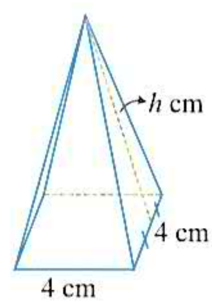
(ii) If the density of the metal is 10.7 g/cm³, find the mass of the sphere.

5. A cylinder of base radius 8 cm contains water to a height of 9 cm. Nora drops 18 spherical marbles each of radius 1 cm into the cylinder.

(i) Find the rise in the water level.

(ii) Hence, calculate the total surface area of the cylinder that is in contact with water, giving your answer in terms of π.

6. A pyramid has a square base of sides 4 cm each, and a total surface area of 176 cm².



Calculate its

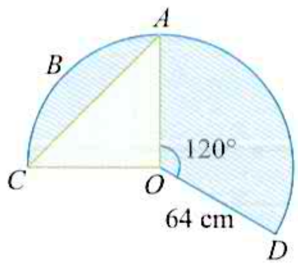
(i) slant height h,

(ii) volume.

7. (a) A cylindrical jar of diameter 14 cm and depth 20 cm is half-filled with water. When 300 spherical ball bearings of the same size are dropped into the jar, the water level rises by 2.8 cm. Find the radius of each ball bearing, expressing your answer in millimetres.

(b) A metallic sphere of radius $10\frac{1}{2}$ cm is melted down and recast into small cones of radius $3\frac{1}{2}$ cm and height 3 cm. How many of such cones can be made?

8. The diagram shows the top face of a company logo. It consists of a quadrant $OABC$, where ABC is a segment, and a sector OAD of radius 64 cm.



(a) Given that sector OAD has an angle of 120° at the centre of the circle, find

(i) the length of the arc of sector OAD,

(ii) the area of sector OAD,

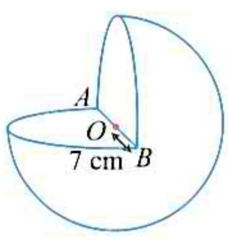
(iii) the area of segment ABC.

(b) The logo is to be made from concrete. If the logo has a uniform thickness of 5 cm, find the volume of concrete needed to build the logo, expressing your answer in m^3.

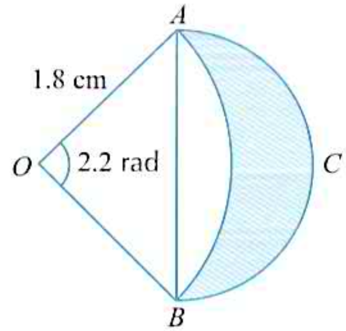
(c) If the cost of concrete is \$250.50 per m^3, find the total cost of building this logo.

9. Two cylindrical jars, A and B, have diameters $2x$ cm and $5x$ cm respectively. Initially, B is empty and A contains water to a depth of 20 cm. If all the water in A is poured into B, find the height of the water in jar B.

10. The figure shows a sphere with a quarter of it removed. Given that the radius of the sphere is 7 cm, find the total surface area of the figure.



11. The figure shows a sector OAB with centre O and a semicircle ACB with diameter AB.

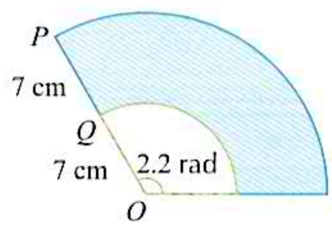


Given that $OA = 1.8$ cm and $\angle AOB = 2.2$ radians, find

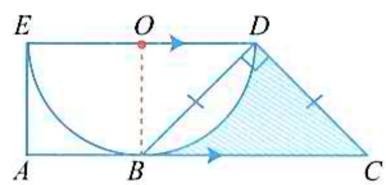
(i) the perimeter of the shaded region,

(ii) the area of the shaded region.

12. The windscreen wiper of a car sweeps through an angle of 2.2 radians. The shaded region in the figure below represents the area of the windscreen swept by the wiper. Given that $OQ = PQ = 7$ cm, find the area of the shaded region.

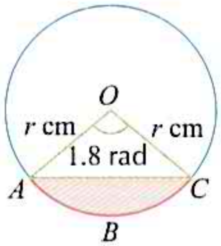


13. In the figure, $ACDE$ is a trapezium with parallel sides ED and AC. DBE is a semicircle with centre O and with DE as a diameter, and its area is 77 cm^2.



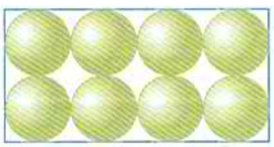
If $BD = CD$ and O is directly above B, find the area of the shaded region CBD.

14. The figure shows the cross section of a circular oil pipeline, with centre O and radius r cm. The minor segment ABC represents the part of the pipeline which is filled with oil.



Given that $\angle AOC = 1.8$ radians, find the percentage of the area of the cross section which is filled with oil, giving your answer correct to the nearest integer.

15. 80 spheres, each of radius 35 mm are packed in a rectangular wooden box. There are 8 spheres in a single layer as shown in the diagram.



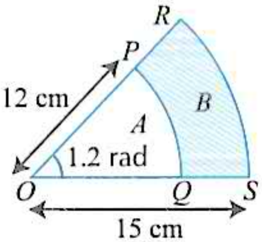
Calculate

(a) **(i)** the volume of the smallest box required, in cm^3,

(ii) the percentage of the total volume of the box filled by the spheres.

(b) When the box is unpacked, each sphere is coated with a layer of paint 0.002 mm thick. How many boxes of spheres can be painted with 1 litre of paint?

16. The figure shows an arc PQ of a circle with centre O and radius 12 cm, and an arc RS of a circle with centre O and radius 15 cm.

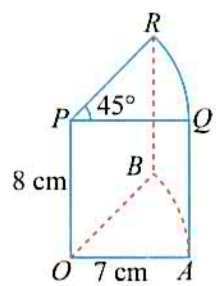


Given that the angle POQ is 1.2 radians, find

(i) the perimeter of the shaded region B,

(ii) the difference between the areas of the regions A and B.

17. The figure shows a prism in which each cross section of the prism is a sector of a circle of radius 7 cm. OAB and PQR are cross sections of the prism, where A, B, Q and R lie on the curved surface of the prism. The two cross sections are horizontal and are 8 cm apart. The vertical planes $OAQP$ and $OBRP$ are rectangular.



Given that $\angle RPQ = 45°$, find

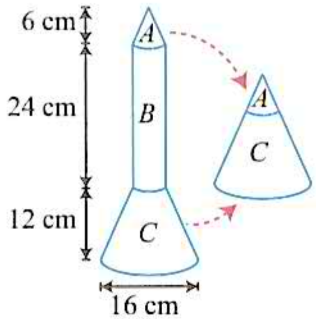
(i) the length of the arc AB,

(ii) the area of sector PQR,

(iii) the volume of the prism,

(iv) the total surface area of the prism.

18. In the figure, the rocket model consists of three parts, A, B and C. Parts A and C can be joined together to form a right circular cone while part B is a right cylinder.

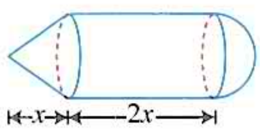


Find

(i) the volume of the rocket model,

(ii) the total curved surface area, excluding the base, of the rocket model.

19. The diagram shows a container which consists of a cylinder with a cone attached to one end and a hemisphere attached to the other end.



The height of the cone is x cm while the length of the cylinder is $2x$ cm.

(a) Find $\dfrac{\text{volume of cone}}{\text{volume of cylinder}}$, expressing your answer as a fraction in the simplest form.

(b) If the volume of the cylinder is 485 cm^3 and its height is 12 cm,

(i) find the radius of the cylinder.

(ii) Find the curved surface area of the cone.

(iii) The exterior of the container is to be painted with a coat of paint of thickness 0.3 mm. Find the volume of paint needed to paint the container.

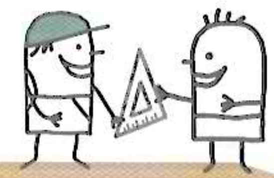
20. Rainwater, collected in a rectangular container with a base measuring 5 m by 8 m, reached a height of 4.5 cm. All the water was then allowed to run into a cylindrical tank of internal diameter 2.4 m.

(i) Find the depth of water in the cylindrical tank.

(ii) Part of the water in the cylindrical tank was used to completely fill 5 hemispherical containers of internal radius 26 cm. Find the decrease in the water level in the cylindrical tank.

(iii) The remaining water in the cylindrical tank was then drained through a valve at a rate of 2.5 litres per minute. Find the time taken to drain all the remaining water, giving your answer correct to the nearest minute.

10.5 Geometrical Transformation and Symmetry



Worked Example 16

(Geometrical Transformation)

T_1 is the translation $\begin{pmatrix} 3 \\ 5 \end{pmatrix}$ and T_2 is a translation that will move the point (5, 2) to (7, 9).

(i) Find the image of the point $A(-1, -5)$ under T_1.

(ii) Find the translation vector represented by T_2.

(iii) What is the image of the point $B(3, -7)$ under T_2?

The image of $B(3, -7)$ under T_2 is then reflected in the line $x = 7$ to give B''.

(iv) Find the coordinates of B''.

Solution:

(i) $A' = T_1(A)$

$$= \begin{pmatrix} 3 \\ 5 \end{pmatrix} + \begin{pmatrix} -1 \\ -5 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

$\therefore$ Image of A under T_1 is (2, 0).

(ii) Let the translation vector of T_2 be $\begin{pmatrix} a \\ b \end{pmatrix}$.

$$\begin{pmatrix} 7 \\ 9 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} 5 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 7 \\ 9 \end{pmatrix} - \begin{pmatrix} 5 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ 7 \end{pmatrix}$$

$\therefore$ The translation vector of T_2 is $\begin{pmatrix} 2 \\ 7 \end{pmatrix}$.

(iii) $B' = T_2(B)$

$$= \begin{pmatrix} 2 \\ 7 \end{pmatrix} + \begin{pmatrix} 3 \\ -7 \end{pmatrix}$$

$$= \begin{pmatrix} 5 \\ 0 \end{pmatrix}$$

$\therefore$ The image of B under T_2 is (5, 0).

(iv) x-coordinate of $B'' = 7 + 2$

$= 9$

$\therefore B''(9, 0)$

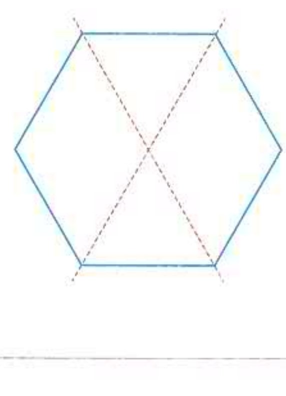
Worked Example 17

(Symmetry)

The diagram shows a regular hexagon with 2 lines of symmetry.

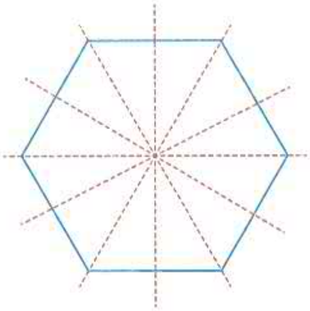
(i) Complete the diagram by drawing all the lines of symmetry.

(ii) Determine the order of rotational symmetry for the hexagon.



Solution:

(i)



(ii) Order of rotational symmetry = 6

Revision 10E

1. R represents a reflection in the x-axis and T represents a translation $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$. Find the image of the point (2, −5) under

(i) R, **(ii)** T^2,
(iii) TR, **(iv)** RT.

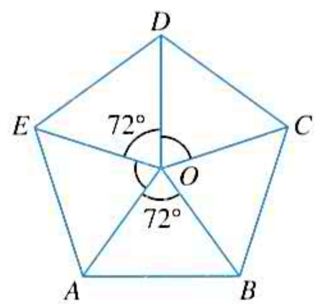
2. R is a 90° anticlockwise rotation about the point (1, 2) and M is a reflection in the line $y = -1$. Find the image of the point (3, 0) under

(i) R^2, **(ii)** M^2,
(iii) MR, **(iv)** RM.

3. T is a translation which maps the point (2, −3) onto the point (3, 1). M is a reflection in the line $x = 1$. Write down the coordinates of the point onto which the point (4, −1) is mapped under

(i) T, **(ii)** M,
(iii) TM, **(iv)** MT.

4.



In the diagram, O is the centre of the regular hexagon $ABCDE$. Describe completely the single transformation which maps

(i) $\triangle OAB$ onto $\triangle OCD$,
(ii) $\triangle AEO$ onto $\triangle ABO$,
(iii) $\triangle AEO$ onto $\triangle BCO$.

5. **Answer the whole of this question on a sheet of graph paper.**

(a) Using a scale of 1 cm to represent 1 unit on each axis, draw the x- and y-axes for $-8 \leqslant x \leqslant 10$. Draw and label $\triangle ABC$ with vertices $A(1, 4)$, $B(5, 5)$ and $C(4, 7)$.

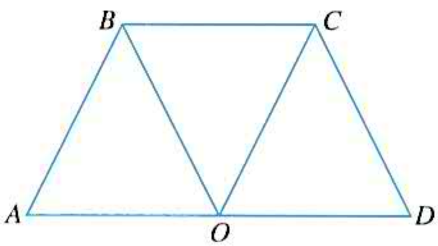
(b) $\triangle ABC$ is rotated through 90° anticlockwise about the point (0, 4) to give $\triangle A_1B_1C_1$. Draw and label $\triangle A_1B_1C_1$.

(c) $\triangle ABC$ is reflected in the x-axis to give $\triangle A_2B_2C_2$. Draw and label $\triangle A_2B_2C_2$.

(d) $\triangle A_2B_2C_2$ is reflected in the line $y = x$ to give $\triangle A_3B_3C_3$. Draw and label the line $y = x$ in the graph and hence draw and label $\triangle A_3B_3C_3$.

(e) $\triangle A_1B_1C_1$ can be mapped onto $\triangle A_3B_3C_3$ by a single transformation T. Describe the transformation T clearly.

6.

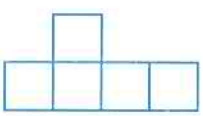


In the diagram, triangles OAB, OBC and OCD are equilateral and AOD is a straight line.

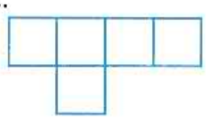
(i) Find single transformation which will map $\triangle OAB$ onto $\triangle OCD$.

(ii) Find two successive transformations that will map $\triangle OBC$ onto $\triangle ODC$.

7. **(a)** Add one square to the figure so that the resulting figure will have exactly one line of symmetry.



(b) Add one square to the figure so that the resulting figure will have rotational symmetry of order 2.

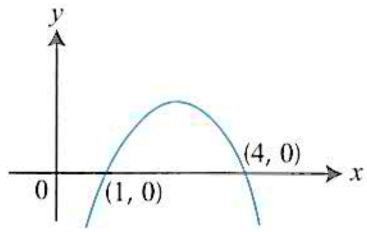


(c) Mark with an X on the resulting figure in **(b)** for the centre of rotational symmetry.

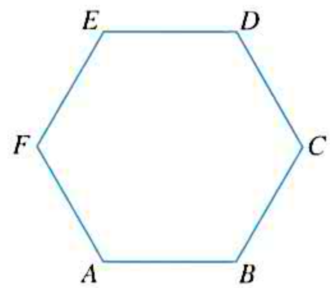
8. The diagram shows a quadratic curve. If the equation of the curve is $y = a + bx - x^2$, find

(a) the values of a and b,

(b) the equation of the line of symmetry.



9.



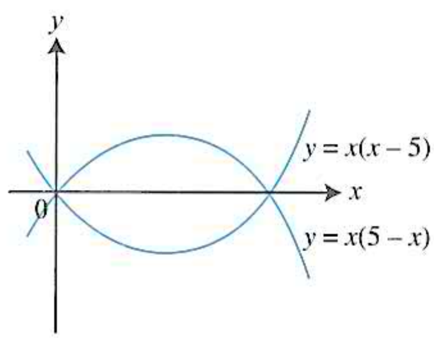
$ABCDEF$ is a regular hexagon.

(a) State the number of axes of symmetry of the hexagon.

(b) Calculate the size of the angles

(i) ABC, **(ii)** ACD.

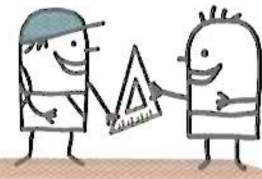
10.



The diagram shows the curves $y = x(x - 5)$ and $y = x(5 - x)$.

(a) Write down the equations of the lines of symmetry.

(b) State the order of rotational symmetry of the figure and write down the coordinates of the centre of rotation.

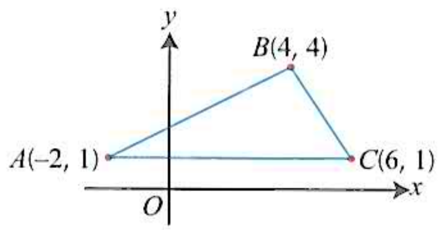


10.6 Coordinate Geometry

Worked Example 18

(Finding the Length and Equation of a Straight Line)

The coordinates of $\triangle ABC$ are $A(-2, 1)$, $B(4, 4)$ and $C(6, 1)$, as shown in the diagram.



(i) Find the equation of AB.

(ii) Find the area of $\triangle ABC$.

(iii) Given that AC is the axis of symmetry of the quadrilateral $ABCD$, find the coordinates of D.

Solution:

(i) Gradient of $AB = \dfrac{4-1}{4-(-2)}$

$= \dfrac{1}{2}$

Given that the line passes through $A(-2, 1)$,

$$y = mx + c$$
$$1 = \frac{1}{2}(-2) + c$$
$$1 = -1 + c$$
$$c = 2$$

$\therefore$ Equation of AB is $y = \dfrac{1}{2}x + 2$

(ii) Area of $\triangle ABC = \dfrac{1}{2} \times \text{base} \times \text{height}$

$= \dfrac{1}{2} \times AC \times \text{height}$

$= \dfrac{1}{2} \times [6-(-2)] \times (4-1)$

$= \dfrac{1}{2} \times 8 \times 3$

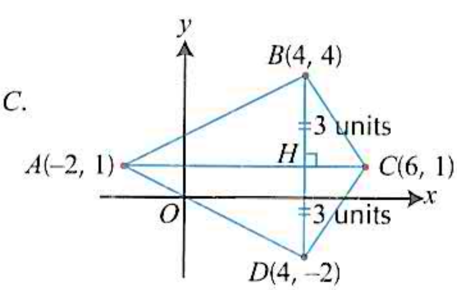
$= 12 \text{ units}^2$

(iii) Given that AC is the axis of symmetry, $BH = HD$ where BH is perpendicular to AC.

Hence, the y-coordinate of $D = 1 - 3$

$= -2$

$\therefore$ The coordinates of D are $(4, -2)$.



RECALL

For 2 points $A(x_1, y_1)$ and $B(x_2, y_2)$, gradient of $AB = \dfrac{y_2 - y_1}{x_2 - x_1}$.

$y = mx + c$ is the equation of a straight line with gradient m and y-intercept c.

ATTENTION

Since AC is parallel to the x-axis, the length of AC is given by the difference in the x-coordinates of A and C. The height of $\triangle ABC$, i.e. from B to base AC, is given by the difference in the y-coordinates of B and A or C.

ATTENTION

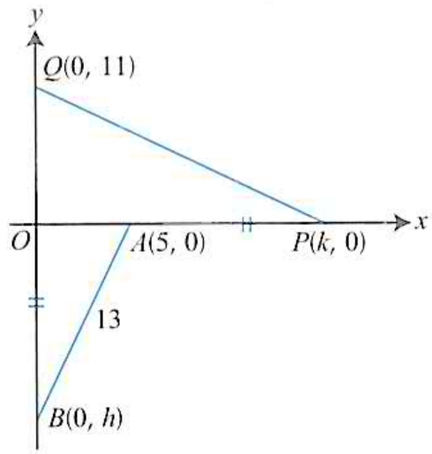
BH = difference in the y-coordinates of B and $C = 4 - 1$
$= 3$

Since BD is parallel to the y-axis, then the x-coordinates of B, H and D are the same.

Worked Example 19

(Using the Area of a Triangle to find Length)

The coordinates of A, B, P and Q are (5, 0), (0, h), (k, 0) and (0, 11), as shown in the diagram.



Given that $OB = AP$ and $AB = 13$ units, find

(i) the value of h and of k,

(ii) the length of PQ,

(iii) the length of OR such that OR is perpendicular to PQ and R lies on PQ.

RECALL

For 2 points $A(x_1, y_1)$ and $B(x_2, y_2)$, length of AB $= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

Solution:

(i)

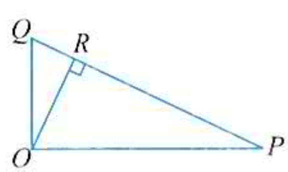
$$AB = 13 \text{ units}$$
$$\sqrt{(0-5)^2 + (h-0)^2} = 13$$
$$\sqrt{5^2 + h^2} = 13$$
$$25 + h^2 = 169$$
$$h^2 = 144$$
$$h = -12 \quad \text{(since } h \text{ is below the } x\text{-axis)}$$
$$\therefore k = 5 + 12$$
$$= 17$$

(ii)

$$PQ = \sqrt{(11-0)^2 + (0-17)^2}$$
$$= \sqrt{11^2 + 17^2}$$
$$= \sqrt{410}$$
$$= 20.2 \text{ units (to 3 s.f.)}$$

(iii) Area of $\triangle OPQ = \frac{1}{2} \times PQ \times OR$ and $\frac{1}{2} \times OP \times OQ$

$$\frac{1}{2} \times PQ \times QR = \frac{1}{2} \times OP \times OQ$$
$$\frac{1}{2} \times 20.25 \times QR = \frac{1}{2} \times 17 \times 11$$
$$QR = \frac{\frac{1}{2} \times 17 \times 11}{\frac{1}{2} \times 20.25}$$
$$= 9.23 \text{ units (to 3 s.f.)}$$



Worked Example 20

(Collinear Points, Parallel Lines and Perpendicular Lines)

The coordinates of 4 points are $O(0, 0)$, $A(k, 2)$, $B(16, 2k)$ and $C(22, 3k + 1)$. Find the value(s) of k if

(a) the points O, A and B are collinear,

(b) OA is parallel to BC,

(c) OA is perpendicular to BC.

Solution:

(a) If O, A and B are collinear, they lie on the same straight line,

i.e. gradient of OA = gradient of OB

$$\frac{2-0}{k-0} = \frac{2k-0}{16-0}$$

$$\frac{2}{k} = \frac{k}{8}$$

$$k^2 = 16$$

$$k = \pm 4$$

(b) OA parallel to BC,

i.e. gradient of OA = gradient of BC.

$$\frac{2-0}{k-0} = \frac{2k-(3k+1)}{16-22}$$

$$\frac{2}{k} = \frac{-k-1}{-6}$$

$$-12 = -k^2 - k$$

$$k^2 + k - 12 = 0$$

$$(k-3)(k+4) = 0$$

$$k = 3 \text{ or } k = -4$$

(c) OA is perpendicular to BC,

i.e. gradient of OA × gradient of $BC = -1$

$$\frac{2}{k} \times \frac{-k-1}{-6} = -1$$

$$\frac{-k-1}{-3k} = -1$$

$$-k - 1 = 3k$$

$$4k = -1$$

$$k = -\frac{1}{4}$$

Revision 10F

1. **(a)** Find the gradient of the straight line passing through the points (1, 2) and (9, 10).

   (b) Given that the gradient of the straight line passing through the point (2, 3) is 5, find the equation of the line.

2. The gradient of the line joining $(5, k)$ and $(4, -3)$ is 2. Find the value of k and the equation of the line.

3. The straight line l cuts the axes at the points E and F. The equation of the line l is $3x + 4y = 24$.

   (a) Find the length of EF.

   (b) Given that another line passes through the point $(-2, 1)$ and has the same gradient as l, find the equation of this line.

4. The line $2x + 3y = 18$ intersects the x-axis at P and the y-axis at Q. Find

   (i) the coordinates of P and Q,

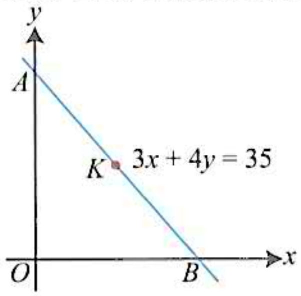
   (ii) the length of PQ.

5. The line $\frac{x}{4}+\frac{y}{6}=1$ cuts the x-axis at H and the y-axis at J. Find the length of HJ.

6. The points $A(-3, 8)$, $B(0, 23)$ and $C(2, k)$ are collinear, i.e. they lie on a straight line. Find the value of k.

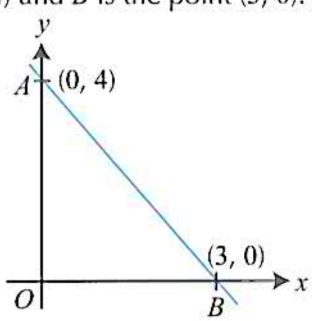
7. (a) It is given that the straight line $3y = k - 2x$ passes through the point $(-1, -5)$. Find the value of k.
   (b) The gradient of the straight line $(2k-1)y + (k+1)x = 3$ is equal to the gradient of the line $y = 3x - 7$. Find the value of k.
   (c) A straight line $\frac{x}{a}+\frac{y}{b}=1$ passes through $(-1, 3)$ and $\left(1, \frac{1}{6}\right)$. Find the value of a and of b, and the gradient of the line.

8. The figure shows the straight line $3x + 4y = 35$, which cuts the x- and y-axes at A and B, respectively.



   (i) Write down the gradient of AB.
   (ii) Find the coordinates of K if the point K lies on the line and is equidistant from both axes.

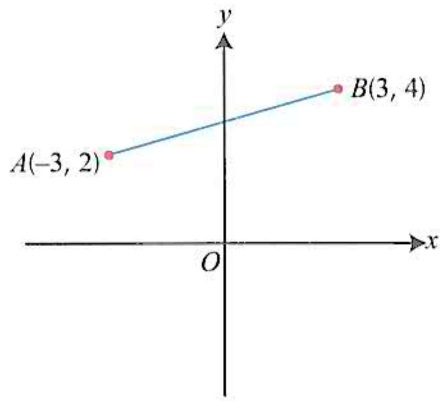
9. The figure shows a line segment AB, where A is the point $(0, 4)$ and B is the point $(3, 0)$.



   (a) Find
      (i) $\angle ABO$,
      (ii) the equation of the line AB.
   (b) If the line AB is reflected in the y-axis, find the equation of the reflected line.

10. Find the equation of the line which has a gradient of $-1\frac{1}{2}$ and passes through the point of intersection of the lines $5x + 3y = 2$ and $x - y = 6$.

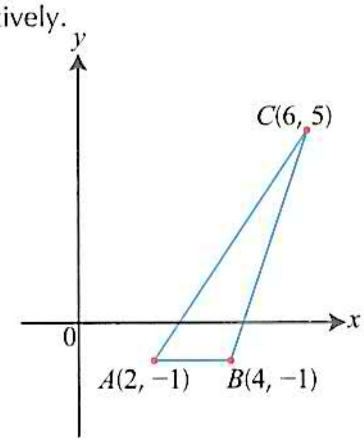
11. The diagram shows the line passing through the points $A(-3, 2)$ and $B(3, 4)$.



   (a) Find the equation of AB.
   (b) Given that C is the point $(3, -2)$, find
      (i) the area of ΔABC,
      (ii) the equation of AC.

12. The distance between the points $A(4 - k, 1)$ and $B(2, k - 2)$ is $\sqrt{13-4k}$. Find the possible values of k.

13. A, B and C are the points $(2, -1)$, $(4, -1)$ and $(6, 5)$ respectively.

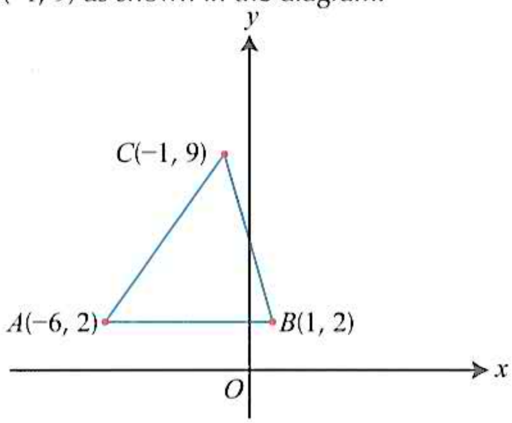


   (a) (i) Find the lengths of AB, BC and AC.
      (ii) Find the area of $\triangle ABC$.
   (b) The coordinates of a point D are $(3, t)$ such that $\triangle ABD$ is an isosceles triangle with an area of 3 units². Find the possible values of t.

14. Given that the coordinates of the points P and Q are $(-4, 2)$ and $(7, 5)$ respectively, find
   (a) the coordinates of the point R such that it lies on the y-axis and $PR = QR$,
   (b) the coordinates of the point S such that it lies on the x-axis and $PS = QS$.

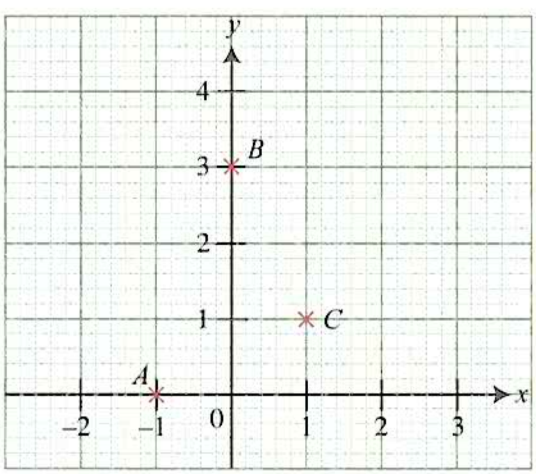
15. By showing that the points $A(-2, 6)$, $B(-2, 1)$ and $C(4, 1)$ are the vertices of a right-angled triangle, find the length of the perpendicular from B to AC.

16. The vertices of $\triangle ABC$ are $A(-6, 2)$, $B(1, 2)$ and $C(-1, 9)$ as shown in the diagram.



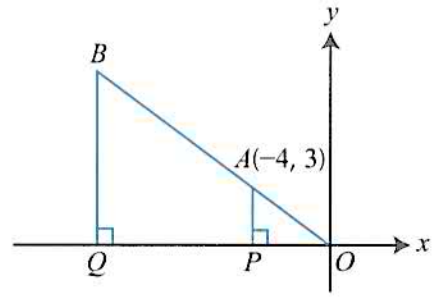
   (i) Find the area of $\triangle ABC$.
   (ii) Calculate the length of BC.
   (iii) Hence, find the shortest distance from A to the line BC.

17. The diagram shows the points $A(-1, 0)$, $B(0, 3)$ and $C(1, 1)$.



   (a) Determine if triangle ABC is a right-angled triangle.
   (b) Given that $ABDC$ is a parallelogram, find the coordinates of D.

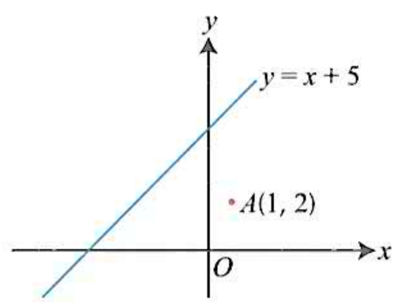
18. In the figure, OAB is a straight line such that $OB = 3OA$ and the coordinates of A are $(-4, 3)$. P and Q lie on the x-axis such that AP and BQ are perpendicular to the x-axis.



   Find
   (i) the length of OB,
   (ii) the coordinates of B,
   (iii) the area of $ABQP$,
   (iv) the length of AQ.

19. The coordinates of the points X and Y are $(1, 2)$ and $(3, 10)$ respectively.
   (i) Find the equation of XY.
   (ii) Given that $x = 3$ is the line of symmetry of $\triangle XYZ$, find the coordinates of Z.
   (iii) Hence, find the length of the perpendicular from Z to XY.

20. The diagram below shows the line l, with equation $y = x + 5$ and the point $A(1, 2)$.



   Show that the general expression for the distance between A and any point (x, y) on the line l is $\sqrt{2(x^2 + 2x + 5)}$.

21. Find the gradient of a line parallel to
   (a) $y = 3x - 2$, (b) $2y = -\frac{3}{4}x - 6$

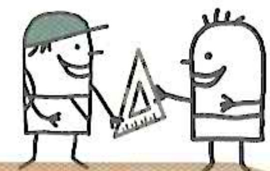
22. Find the gradient of a line perpendicular to
   (a) $y = 4x + 1$, (b) $4y = -\frac{1}{2}x + 6$.

23. Show that the line $4y = 2x - 16$ is perpendicular to $5y = -10x + 2$.

24. Find the equation of the line passing through the point $(-3, 4)$ and perpendicular to the line $2x + 6y = 1$.

25. The coordinates of A, B and D are $(2, 1)$, $(12, 6)$ and $(6, 9)$ respectively. Given that AB is parallel to DC and $\angle ABC = 90°$, find the
(a) equation of BC and of DC,
(b) coordinates of C.

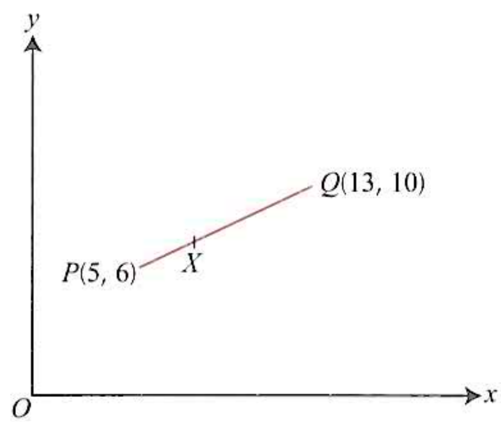
10.7 Vectors in Two Dimensions



Worked Example 21

(Position Vectors)

P is the point $(5, 6)$, Q is the point $(13, 10)$ and X is the point on $\overrightarrow{PQ}$ such that $\overrightarrow{PX} = \frac{1}{3}\overrightarrow{XQ}$.



(a) Express the following as column vectors.
(i) $\overrightarrow{PQ}$ **(ii)** $\overrightarrow{PX}$
(iii) The position vector of X relative to the origin O

(b) If O, P and Q are three of the vertices of a parallelogram, find the coordinates of the two possible positions of the fourth vertex.

Solution:

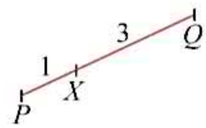
(a) $\overrightarrow{OP} = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$ and $\overrightarrow{OQ} = \begin{pmatrix} 13 \\ 10 \end{pmatrix}$.

(i) $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$

$$= \begin{pmatrix} 13 \\ 10 \end{pmatrix} - \begin{pmatrix} 5 \\ 6 \end{pmatrix}$$

$$= \begin{pmatrix} 8 \\ 4 \end{pmatrix}$$

(ii) Given that $\overrightarrow{PX} = \frac{1}{3}\overrightarrow{XQ}$, then $\frac{PX}{XQ} = \frac{1}{3}$, i.e. $\frac{PX}{PQ} = \frac{1}{4}$.



We have $\overrightarrow{PX} = \frac{1}{4}\overrightarrow{PQ} = \frac{1}{4}\begin{pmatrix} 8 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

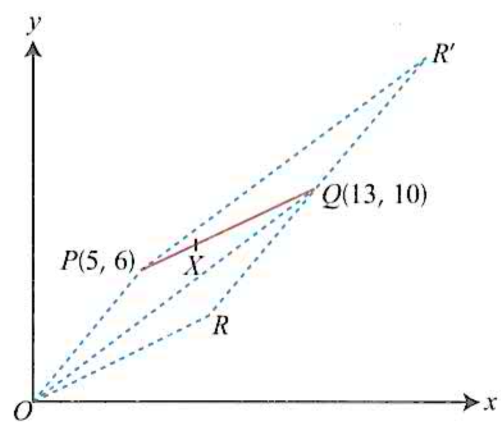
(iii) $\overrightarrow{OX} = \overrightarrow{OP} + \overrightarrow{PX}$

$$\overrightarrow{OX} = \begin{pmatrix} 5 \\ 6 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 7 \\ 7 \end{pmatrix}$$

$\therefore$ The position vector of X relative to the origin O is $\overrightarrow{OX} = \begin{pmatrix} 7 \\ 7 \end{pmatrix}$.

(b) In the figure shown, the two possible positions of the fourth vertex, R and R', are marked.



For parallelogram $OPQR$, $\overrightarrow{OR} = \overrightarrow{PQ} = \begin{pmatrix} 8 \\ 4 \end{pmatrix}$.

For the parallelogram $OPR'Q$, $\overrightarrow{OR'} = \overrightarrow{OP} + \overrightarrow{OQ}$ (Parallelogram Law of Vector Addition)

$$= \begin{pmatrix} 5 \\ 6 \end{pmatrix} + \begin{pmatrix} 13 \\ 10 \end{pmatrix}$$

$$= \begin{pmatrix} 18 \\ 16 \end{pmatrix}$$

$\therefore$ The coordinates of the two possible positions of the fourth vertex are (8, 4) and (18, 16).

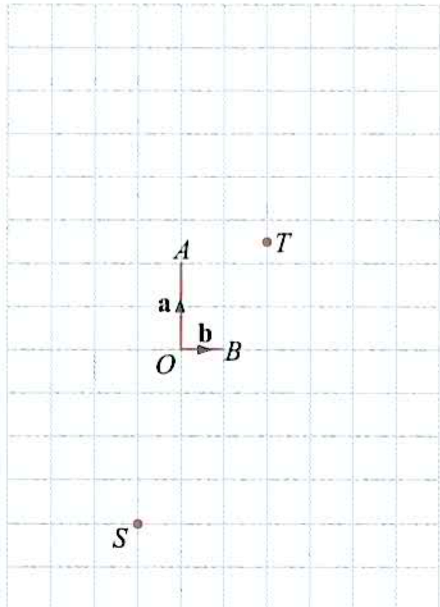
ATTENTION

For $OPQR$, $\overrightarrow{PQ}$ and $\overrightarrow{OR}$ are equal vectors.

Worked Example 22

(Drawing Vectors on a Cartesian Plane)

In the given diagram, $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$.



(a) Mark clearly on the diagram,

(i) the point P, such that $\overrightarrow{OP} = 3\mathbf{a} + 2\mathbf{b}$,

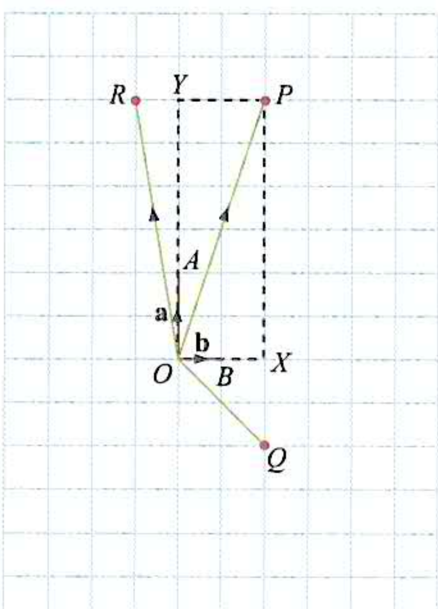
(ii) the point Q, such that $\overrightarrow{OQ} = 2\mathbf{b} - \mathbf{a}$,

(iii) the point R, such that $\overrightarrow{OR} = -(\mathbf{b} - 3\mathbf{a})$.

(b) Write down $\overrightarrow{OS}$ and $\overrightarrow{OT}$ in terms of $\mathbf{a}$ and/or $\mathbf{b}$.

Solution:

(a)



(b) $\overrightarrow{OS} = -2\mathbf{a} - \mathbf{b}$

$= -(2\mathbf{a} + \mathbf{b})$

$\overrightarrow{OT} = 1\frac{1}{4}\mathbf{a} + 2\mathbf{b}$

$= \frac{5}{4}\mathbf{a} + 2\mathbf{b}$

$= \frac{1}{4}(5\mathbf{a} + 8\mathbf{b})$

ATTENTION

In (a), $\overrightarrow{OP} = 3\mathbf{a} + 2\mathbf{b}$

$= \overrightarrow{OY} + \overrightarrow{OX}$

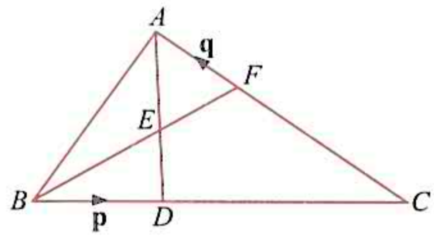
(Parallelogram Law of Vector Addition)

OP is a diagonal of $OXPY$, where $\overrightarrow{OX} = 2\mathbf{b}$ and $\overrightarrow{OY} = 3\mathbf{a}$.

Worked Example 23

(Geometric Problem involving Vectors)

In the diagram, $BC = 3BD$ and $CA = 4FA$. E is the midpoint of DA. $\overrightarrow{BD} = \mathbf{p}$ and $\overrightarrow{FA} = \mathbf{q}$.



(a) Express, as simply, as possible, in terms of $\mathbf{p}$ and/or $\mathbf{q}$,

(i) $\overrightarrow{DC}$, **(ii)** $\overrightarrow{DA}$, **(iii)** $\overrightarrow{DE}$.

(b) Show that $\overrightarrow{BE} = 2(\mathbf{p} + \mathbf{q})$.

(c) Express $\overrightarrow{BF}$ as simply as possible, in terms of $\mathbf{p}$ and/or $\mathbf{q}$.

(d) Calculate the value of

(i) $\dfrac{BE}{BF}$,

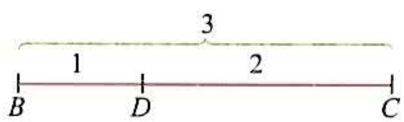
(ii) $\dfrac{\text{area of } \triangle ABE}{\text{area of } \triangle ABF}$,

(iii) $\dfrac{\text{area of } \triangle ABE}{\text{area of } \triangle ABC}$.

Solution:

(a) (i) $DC = 2BD$ ($BC = 3BD$)

$\therefore \overrightarrow{DC} = 2\mathbf{p}$



(ii) $CA = 4FA$

$\therefore \overrightarrow{CA} = 4\mathbf{q}$ ($\overrightarrow{CA} + 4\overrightarrow{FA}$)

$\overrightarrow{DA} = \overrightarrow{DC} + \overrightarrow{CA}$ (Triangle Law of Vector Addition)

$= 2\mathbf{p} + 4\mathbf{q}$

$= 2(\mathbf{p} + 2\mathbf{q})$

(iii) $DE = \frac{1}{2}DA$ (E is the midpoint of DA)

$\overrightarrow{DE} = \frac{1}{2} \times 2(\mathbf{p} + 2\mathbf{q})$

$= \mathbf{p} + 2\mathbf{q}$

(b) $\overrightarrow{BE} = \overrightarrow{BD} + \overrightarrow{DE}$ (Triangle Law of Vector Addition)

$= \mathbf{p} + \mathbf{p} + 2\mathbf{q}$

$= 2\mathbf{p} + 2\mathbf{q}$

$= 2(\mathbf{p} + \mathbf{q})$

(c) $\overrightarrow{BF} = \overrightarrow{BC} + \overrightarrow{CF}$

$= 3\mathbf{p} + 3\mathbf{q}$

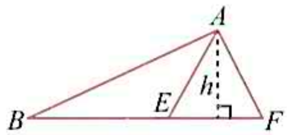
$= 3(\mathbf{p} + \mathbf{q})$

(d) **(i)** From **(b)** and **(c)**, we have $\overrightarrow{BE} = \frac{2}{3}\overrightarrow{BF}$.

$\therefore \frac{BE}{BF} = \frac{2}{3}$

(ii) $\frac{\text{Area of }\triangle ABE}{\text{Area of }\triangle ABF} = \frac{BE}{BF}$

$= \frac{2}{3}$



(iii) $\frac{\text{Area of }\triangle ABF}{\text{Area of }\triangle ABC} = \frac{1}{4}$

$= \frac{3}{12}$

$\therefore \frac{\text{Area of }\triangle ABE}{\text{Area of }\triangle ABC} = \frac{2}{12}$

$= \frac{1}{6}$

ATTENTION

For **(d)(ii)**, $\triangle ABE$ and $\triangle ABF$ share a common height, h.

Similarly, for **(iii)**, $\triangle ABF$ and $\triangle ABC$ share a common height.

Revision 10G

1. Given that $\mathbf{a} = \begin{pmatrix} 12 \\ 5 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} s \\ 0 \end{pmatrix}$, where $s > 0$, find the value of s such that $|\mathbf{a}| = |\mathbf{b}|$.

2. X is the point $(-1, -2)$ and Y is the point $(2, 4)$.

(i) Write down the column vector $\overrightarrow{XY}$.

(ii) Find $|\overrightarrow{XY}|$.

(iii) Z is the point such that $2\overrightarrow{XZ} = 5\overrightarrow{XY}$. Find the coordinates of Z.

3. Given that $\overrightarrow{AB} = \begin{pmatrix} -5 \\ 0 \end{pmatrix}$ and $\overrightarrow{PQ} = \begin{pmatrix} t \\ -3 \end{pmatrix}$, find

(i) $|\overrightarrow{AB}|$,

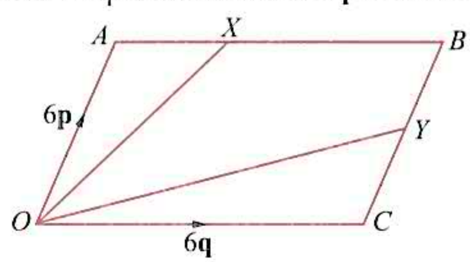
(ii) two possible values of t if $|\overrightarrow{AB}| = |\overrightarrow{PQ}|$.

4. It is given that P is the point $(-2, -1)$, Q is the point $(4, 2)$ and R is the point $(2, 6)$. Express as column vectors the position of

(i) the point M, which is the midpoint of QR,

(ii) the point N on PM such that $4\overrightarrow{PN} = \overrightarrow{NM}$.

5. In the diagram below, $OABC$ is a parallelogram. X is a point on AB such that $AX : XB = 1 : 2$ and Y is the midpoint of BC. $OA = 6\mathbf{p}$ and $OC = 6\mathbf{q}$.

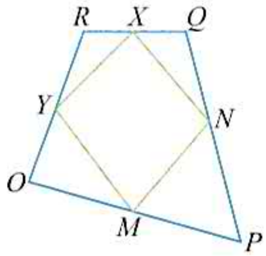


Express the following in terms of $\mathbf{p}$ and/or $\mathbf{q}$.

(i) $\overrightarrow{AX}$, **(ii)** $\overrightarrow{OX}$, **(iii)** $\overrightarrow{OY}$,

(iv) $\overrightarrow{XY}$, **(v)** $\overrightarrow{AY}$.

6. In the quadrilateral $OPQR$, the points M, N, X and Y are the midpoints of OP, PQ, QR and RO, respectively.



Given that $\overrightarrow{OP} = \begin{pmatrix} 6 \\ -1 \end{pmatrix}$, $\overrightarrow{OQ} = \begin{pmatrix} 5 \\ 7 \end{pmatrix}$ and $\overrightarrow{OR} = \begin{pmatrix} 2 \\ 6 \end{pmatrix}$, find the vectors $\overrightarrow{PQ}$, $\overrightarrow{RQ}$, $\overrightarrow{MN}$ and $\overrightarrow{XY}$.

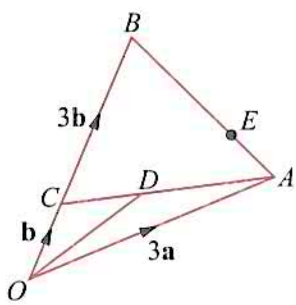
State the geometrical relationship between MN and XY. Justify your answer using vectors.

7. Given that $\overrightarrow{PQ} = \begin{pmatrix} 13 \\ 0 \end{pmatrix}$ and $\overrightarrow{RS} = \begin{pmatrix} -5 \\ 12 \end{pmatrix}$, show that $|\overrightarrow{PQ}| = |\overrightarrow{RS}|$.
Explain why $\overrightarrow{PQ} \neq \overrightarrow{RS}$ although $|\overrightarrow{PQ}| = |\overrightarrow{RS}|$.

8. It is given that P is the point $(0, 2)$, Q is the point $(8, 0)$ and $\overrightarrow{QR} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$. Find

(i) $|\overrightarrow{PQ}|$,
(ii) the coordinates of the point R,
(iii) $|\overrightarrow{PR}|$.

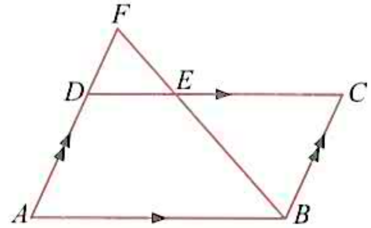
9. In the figure, $\overrightarrow{OA} = 3\mathbf{a}$, $\overrightarrow{OC} = \mathbf{b}$ and $\overrightarrow{CB} = 3\mathbf{b}$. D is a point on AC such that $\frac{CD}{CA} = \frac{1}{3}$ and E is a point on AB such that $\frac{AE}{AB} = \frac{1}{3}$.



Express the following in terms of $\mathbf{a}$ and $\mathbf{b}$.

(i) $\overrightarrow{AB}$ **(ii)** $\overrightarrow{AC}$
(iii) $\overrightarrow{OD}$ **(iv)** $\overrightarrow{OE}$

10. In the diagram below, $ABCD$ is a parallelogram. The point E, on DC, is such that $DE = \frac{1}{3}DC$.
The lines AD and BE, when produced, meet at F.



Given that $\overrightarrow{AB} = \begin{pmatrix} 6 \\ 0 \end{pmatrix}$ and $\overrightarrow{AD} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$,

(a) find the value of $|\overrightarrow{AB}|$,
(b) express each of the following as a column vector.

(i) $\overrightarrow{CB}$ **(ii)** $\overrightarrow{EC}$ **(iii)** $\overrightarrow{FE}$

11. Relative to an origin O, the position vectors of points P, Q and R are $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$, respectively. The midpoint of PQ is X.

(i) Find the position vector of X, in terms of $\mathbf{p}$ and $\mathbf{q}$.

The point S lies on RX, such that $\overrightarrow{RS} = 2\overrightarrow{SX}$.

(ii) Find the position vector of S, in terms of $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$.

12. It is given that P is the point $(0, 2)$, Q is the point $(3, 3)$ and O is the origin.

(a) Express $\overrightarrow{PQ}$ as a column vector.
(b) Calculate the coordinates of the point R, where $\overrightarrow{QR} = 2\overrightarrow{PQ}$.
(c) It is given that $\overrightarrow{PS} = \begin{pmatrix} 6 \\ -1 \end{pmatrix}$.

(i) Calculate the length of PS.
(ii) Write down the gradient of the line PS.
(iii) Write down the equation of the line PS.
(iv) The point T lies on PS and TQ is parallel to the y-axis. Calculate the coordinates of T.

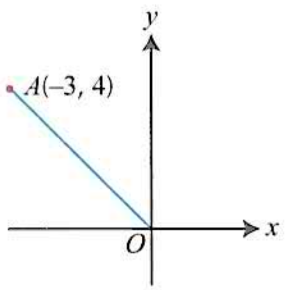
13. It is given that $\mathbf{a} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} -5 \\ 12 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} p \\ q \end{pmatrix}$.

(i) Find $|\mathbf{b}|$.

(ii) Express $3\mathbf{a} + 2\mathbf{b}$ as a column vector.

(iii) Given that $2\mathbf{a} - \mathbf{b} = 2\mathbf{c}$, find the value of p and the value of q.

14. A is the point $(-3, 4)$ and O is the origin.

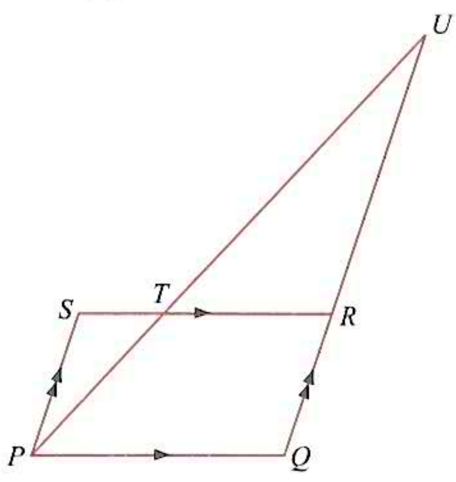


(i) The point B lies on OA produced. Given that $\overrightarrow{OB} = 3\overrightarrow{OA}$, express $\overrightarrow{OB}$ as a column vector.

(ii) C is the point $(2, 16)$. Express $\overrightarrow{AC}$ as a column vector and find $|\overrightarrow{AC}|$.

(iii) The point D is the result of the translation of point A by $\begin{pmatrix} 5 \\ -2 \end{pmatrix}$. Find the coordinates of point D.

15. $PQRS$ is a parallelogram. The point T, on SR, is such that $TR = \frac{2}{3}SR$. The lines PT and QR are produced to meet at U. It is given that $\overrightarrow{PQ} = \begin{pmatrix} 6 \\ 0 \end{pmatrix}$ and $\overrightarrow{QR} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$.



(a) Find $|\overrightarrow{QR}|$, giving your answer correct to the nearest whole number.

(b) Express each of the following as a column vector.

(i) $\overrightarrow{SP}$ **(ii)** $\overrightarrow{ST}$

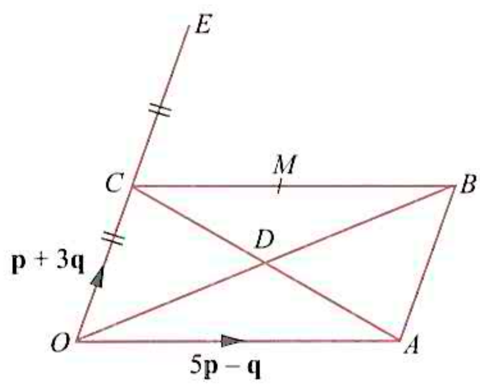
(iii) $\overrightarrow{RT}$ **(iv)** $\overrightarrow{UT}$

16. The quadrilateral $PQRS$ is such that $\overrightarrow{PQ} = 3\mathbf{a}$, $\overrightarrow{QR} = \mathbf{b}$ and $\overrightarrow{RS} = -2\mathbf{a}$.

(i) What is the special name given to the quadrilateral $PQRS$? Justify your answer using vectors.

(ii) Express $\overrightarrow{SP}$ in terms $\mathbf{a}$ and $\mathbf{b}$, giving your answer in the simplest form.

17. (a) In the figure, $OABC$ is a parallelogram whose diagonals meet at D. M is the midpoint of BC. It is given that $\overrightarrow{OA} = 5\mathbf{p} - \mathbf{q}$ and $\overrightarrow{OC} = \mathbf{p} + 3\mathbf{q}$.

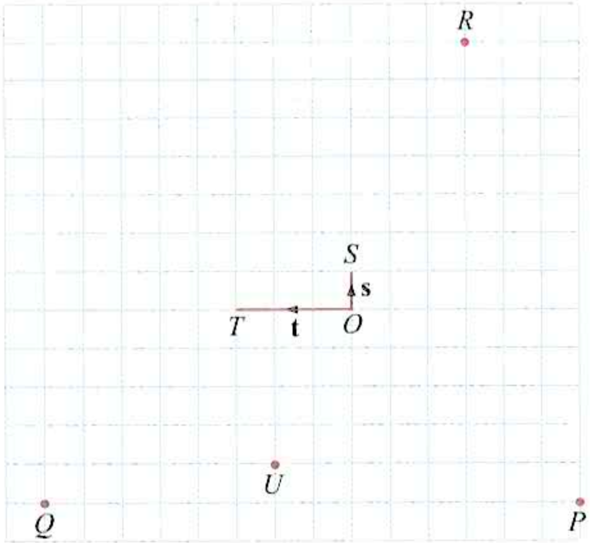


Express the following in terms of $\mathbf{p}$ and $\mathbf{q}$.

(i) $\overrightarrow{OD}$ **(ii)** $\overrightarrow{AC}$ **(iii)** $\overrightarrow{AM}$

(b) Express $\overrightarrow{AE}$ in terms of $\mathbf{p}$ and $\mathbf{q}$ if OC is produced to E such that $OC = CE$.

18. In the diagram, $\overrightarrow{OS} = \mathbf{s}$ and $\overrightarrow{OT} = \mathbf{t}$.



(a) Indicate clearly on the diagram

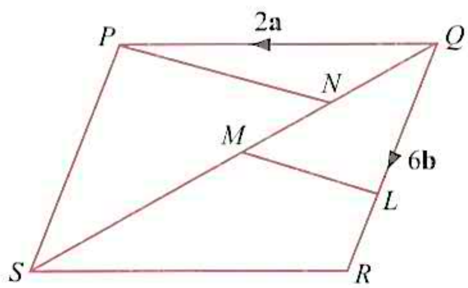
(i) the point A, such that $\overrightarrow{OA} = 2\mathbf{s} + 3\mathbf{t}$,

(ii) the point B, such that $\overrightarrow{OB} = -2(\mathbf{t} - \mathbf{s})$,

(iii) the point C, such that $\overrightarrow{OC} = 3\mathbf{t} - \mathbf{s}$.

(b) Write down $\overrightarrow{OP}$, $\overrightarrow{OQ}$, $\overrightarrow{OR}$ and $\overrightarrow{OU}$ in terms of $\mathbf{s}$ and $\mathbf{t}$.

19. In the diagram, $PQRS$ is a parallelogram, M is the midpoint of QS, N is the midpoint of QM and L is the point on QR such that $QL = \frac{2}{3}QR$.



(a) Given that $\overrightarrow{QP} = 2\mathbf{a}$ and $\overrightarrow{QR} = 6\mathbf{b}$, express, as simply as possible, in terms of $\mathbf{a}$ and/or $\mathbf{b}$,

(i) $\overrightarrow{QS}$, **(ii)** $\overrightarrow{QM}$,

(iii) $\overrightarrow{PN}$, **(iv)** $\overrightarrow{ML}$.

(b) What do your answers in **(a)(iii)** and **(a)(iv)** tell you about PN and ML?

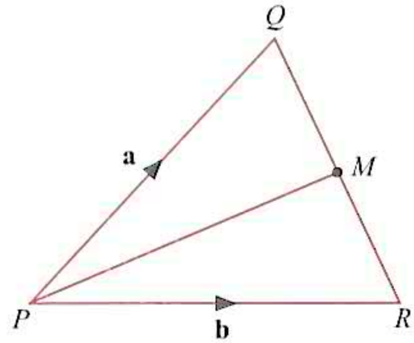
(c) What is the special name given to the quadrilateral $PMLN$?

(d) Write down the value of each of the following:

(i) $\dfrac{\text{Area of } \triangle PNQ}{\text{Area of } \triangle PSN}$

(ii) $\dfrac{\text{Area of } \triangle PSN}{\text{Area of } \triangle QML}$

20. In the diagram, $\overrightarrow{PQ} = \mathbf{a}$, $\overrightarrow{PR} = \mathbf{b}$ and M is the midpoint of QR.



(a) Express the following as simply as possible, in terms of $\mathbf{a}$ and/or $\mathbf{b}$.

(i) $\overrightarrow{QR}$ **(ii)** $\overrightarrow{QM}$

(iii) $\overrightarrow{PM}$

(b) Given further that $\overrightarrow{PM} = \lambda\mathbf{a} + \mu\mathbf{b}$, where λ and μ are real constants, and N is the point such that $\overrightarrow{PN} = \mu\mathbf{b}$, mark and label the point N on the diagram.

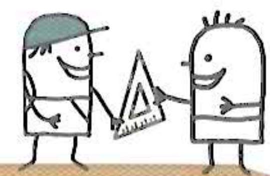
(c) The point S lies on PM produced. Given that $\overrightarrow{PS} = 2\overrightarrow{PM}$, express the following as simply as possible, in terms of $\mathbf{a}$ and/or $\mathbf{b}$,

(i) $\overrightarrow{PS}$, **(ii)** $\overrightarrow{QS}$,

(iii) $\overrightarrow{RS}$.

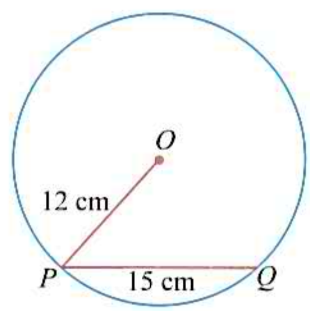
(d) State what type of quadrilateral $PRSQ$ is.

10.8 Properties of Circles



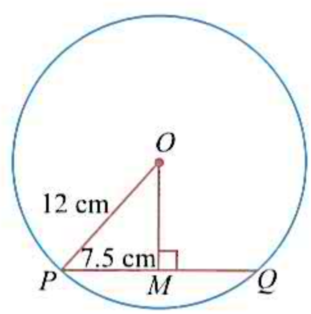
Worked Example 24

(Application of Symmetric Properties of Circles)
A chord PQ of length 15 cm is drawn in a circle of radius 12 cm as shown in the figure.



Find the perpendicular distance from the centre of the circle O to the chord.

Solution:



RECALL

The perpendicular bisector of a chord of a circle passes through the centre of the circle, i.e. $PM = MQ \Leftrightarrow OM \perp PQ$.

The perpendicular from O cuts PQ at its midpoint, M, i.e. $PM = 7.5$ cm.

In $\triangle OPM$,

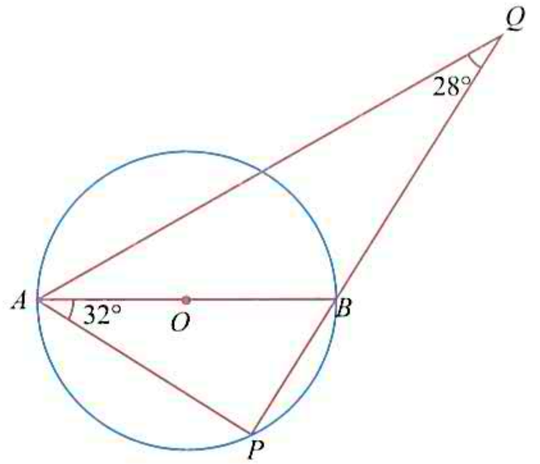
$$\begin{aligned} OM^2 &= OP^2 - PM^2 \text{ (Pythagoras' Theorem)} \\ &= 12^2 - 7.5^2 \\ &= 87.75 \\ OM &= 9.37 \text{ cm (to 3 s.f.)} \end{aligned}$$

$\therefore$ The perpendicular distance from O to PQ is 9.37 cm.

Worked Example 25

(Application of Angle Properties of Circles)

In the figure, AB is a diameter of the circle, centre O. P is a point on the circle such that $P\hat{A}B = 32°$ and $A\hat{Q}P = 28°$.



(a) Find

(i) $P\hat{O}B$, **(ii)** $B\hat{A}Q$.

(b) Given that $AP = 10$ cm, find the radius of the circle.

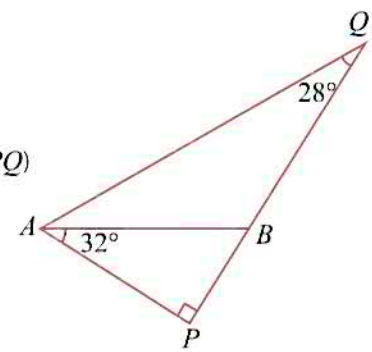
Solution:

RECALL

An angle at the centre of a circle is twice that of any angle at the circumference subtended by the same arc, i.e. $P\hat{O}B = 2 \times P\hat{A}B$.

(a) (i) $P\hat{O}B = 2 \times P\hat{A}B$ (∠ at centre = 2 ∠ at ⊙^{ce})
$= 2 \times 32°$
$= 64°$

(ii) $A\hat{P}B = 90°$ (rt. ∠ in semicircle)
$B\hat{A}Q = 180° - 28° - 90° - 32°$ (∠ sum of $\triangle APQ$)
$= 30°$



(b) In $\triangle APB$,

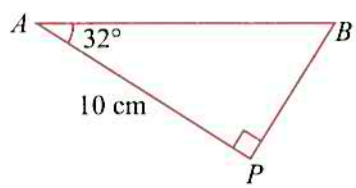
$$\cos 32° = \frac{10}{AB}$$

$$AB = \frac{10}{\cos 32°}$$

$= 11.79$ cm (to 4 s.f.)

$$\text{Radius} = \frac{11.79}{2}$$

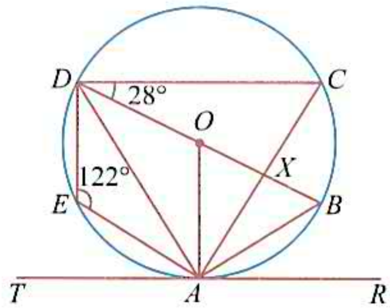
$= 5.90$ cm (to 3 s.f.)



Worked Example 26

(Applications of Symmetric and Angle Properties of Circles)

In the figure, TAR is a tangent to the circle $ABCDE$ with centre O. The chord AC intersects the diameter BD at X. $A\hat{E}D = 122°$ and $B\hat{D}C = 28°$.



Find

(i) $A\hat{C}D$, (ii) $O\hat{A}B$,

(iii) $B\hat{A}R$, (iv) $A\hat{D}C$.

Solution:

(i) $A\hat{C}D = 180° - A\hat{E}D$ (∠s in opp. segments)
$= 180° - 122°$
$= 58°$

(ii) $D\hat{B}A = D\hat{C}A$ (∠s in same segment)
$= 58°$

$O\hat{A}B = D\hat{B}A$ (base ∠s of isos. △ as $OA = OB$, radii of circle)
$= 58°$

(iii) $O\hat{A}R = 90°$ (tangent ⊥ radius)

$B\hat{A}R = O\hat{A}R - O\hat{A}B$
$= 90° - 58°$
$= 32°$

(iv) $B\hat{A}C = B\hat{D}C$
$= 28°$ (∠s in same segment)

$C\hat{A}R = B\hat{A}R + B\hat{A}C$
$= 32° + 28°$
$= 60°$

$A\hat{D}C = C\hat{A}R$
$= 60°$ (∠s in alt. segment)

RECALL

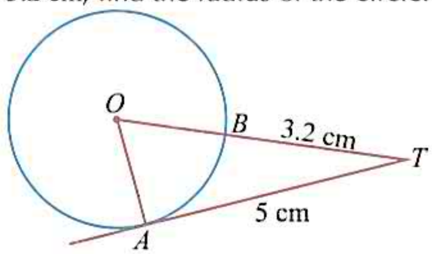
Angles in opposite segments are supplementary, i.e. $A\hat{C}D + A\hat{E}D = 180°$ and $C\hat{D}E + C\hat{A}E = 180°$.

RECALL

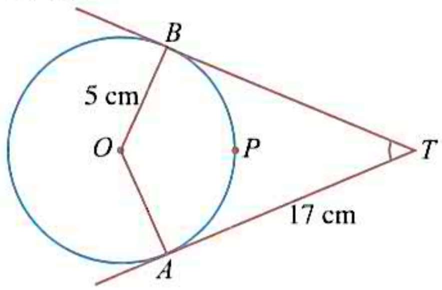
The tangent at the point of contact is perpendicular to the radius of a circle, i.e. $TAR \perp OA$.

Revision 10H

1. In the diagram, O is the centre of the circle and TA is the tangent to the circle at A. Given that $TA = 5$ cm and OT cuts the circle at B such that $TB = 3.2$ cm, find the radius of the circle.



2. In the diagram, TA and TB are the tangents to the circle with centre O and radius 5 cm. It is given that $TA = 17$ cm.

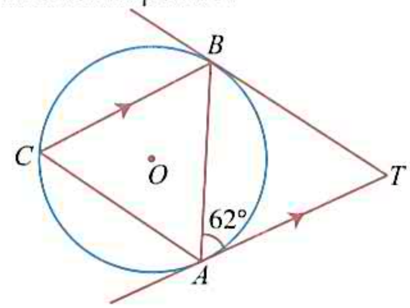


Find
(i) the area of the quadrilateral $ATBO$,
(ii) the length of the minor arc APB.

3. A chord of length 18 cm is drawn in a circle of radius 16 cm. Calculate the perpendicular distance from the centre of the circle to the chord.

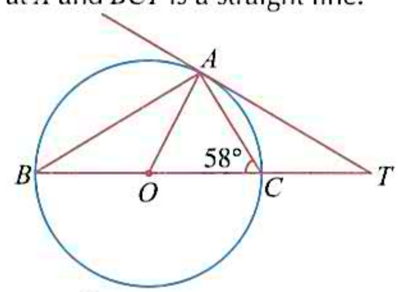
4. The perpendicular distance from the centre of a circle to a chord drawn in the circle is 7.5 cm. Given that the chord has a length of 12 cm, calculate the radius of the circle.

5. In the diagram, TA and TB are tangents to the circle from an external point T.



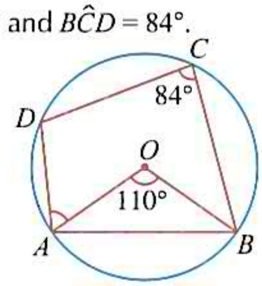
C is a point on the circle such that CB is parallel to AT, $B\hat{A}T = 62°$. Find
(i) $A\hat{T}B$, **(ii)** $B\hat{C}A$,
(iii) $B\hat{A}C$.

6. The diagram shows a circle with centre O. BC is a diameter of the circle. TA is the tangent to the circle at A and BCT is a straight line.



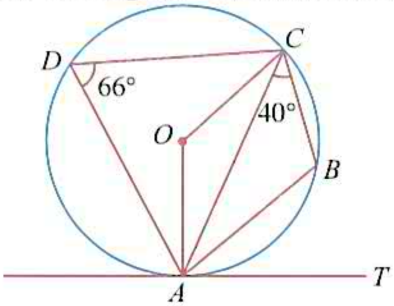
Given that $A\hat{C}O = 58°$, find
(i) $A\hat{O}C$, **(ii)** $A\hat{B}C$,
(iii) $C\hat{A}T$, **(iv)** $A\hat{T}C$.

7. The diagram shows a circle $ABCD$ with centre O. $A\hat{O}B = 110°$ and $B\hat{C}D = 84°$.



Find $O\hat{A}D$.

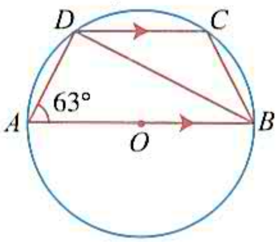
8. In the diagram, O is the centre of the circle $ABCD$ and TA is the tangent to the circle at A.



Given that $A\hat{D}C = 66°$ and $A\hat{C}B = 40°$, find

(i) $A\hat{O}C$, **(ii)** $B\hat{A}C$,
(iii) $A\hat{C}O$, **(iv)** $T\hat{A}B$.

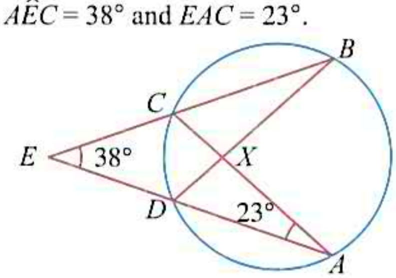
9. In the diagram, AB is a diameter of the circle $ABCD$ with centre O. DC is parallel to AB and $B\hat{A}D = 63°$.



Find

(i) $A\hat{B}D$, **(ii)** $C\hat{B}D$,
(iii) $B\hat{O}C$.

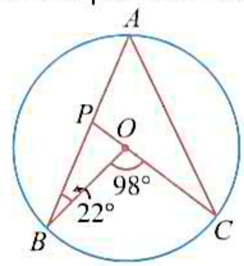
10. In the diagram, a circle passes through the points A, B, C and D. ADE, AXC, BXD and BCE are straight lines. $A\hat{E}C = 38°$ and $EAC = 23°$.



Find

(i) $B\hat{D}A$, **(ii)** $B\hat{X}C$.

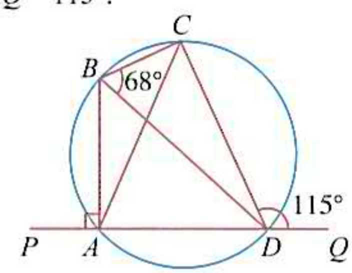
11. The diagram shows a circle with centre O. A, B and C lie on the circumference such that $B\hat{O}C = 98°$ and $A\hat{B}O = 22°$. CO produced meets AB at P.



Find

(i) $B\hat{A}C$, **(ii)** $A\hat{C}P$.

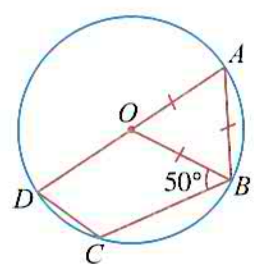
12. In the diagram, A, B, C and D are points on the circle and $PADQ$ is a straight line. $P\hat{A}B = 90°$, $C\hat{B}D = 68°$ and $C\hat{D}Q = 115°$.



Find

(i) $A\hat{C}B$, **(ii)** $A\hat{C}D$.

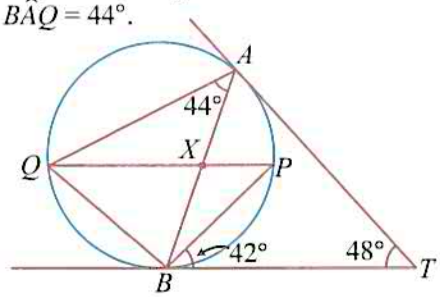
13. In the diagram, AD is a diameter of the circle $ABCD$ and centre O. Triangle AOB is equilateral and $O\hat{B}C = 50°$.



Find

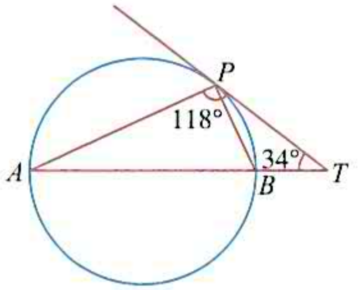
(i) $B\hat{C}D$, **(ii)** $O\hat{D}C$,
(iii) $C\hat{B}D$, **(iv)** $C\hat{O}D$.

14. In the diagram, TA and TB are tangents to the circle at A and B respectively. The lines PQ and AB intersect at X. It is given that $A\hat{T}B = 48°$, $P\hat{B}T = 42°$ and $B\hat{A}Q = 44°$.



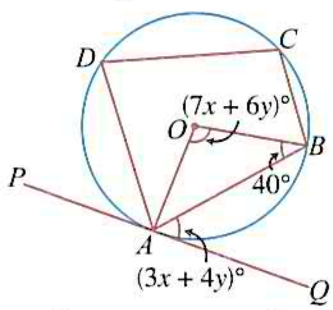
Find
(i) $P\hat{B}A$, **(ii)** $A\hat{X}P$,
(iii) $A\hat{Q}B$.

15. In the diagram, AB is a diameter of the circle APB and PT is the tangent to the circle at P.



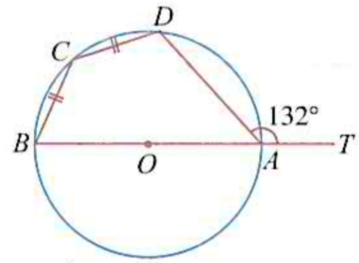
Given that $A\hat{P}T = 118°$ and $B\hat{T}P = 34°$, find $A\hat{B}P$.

16. In the diagram, O is the centre of the circle $ABCD$ and PAQ is the tangent to the circle at A.



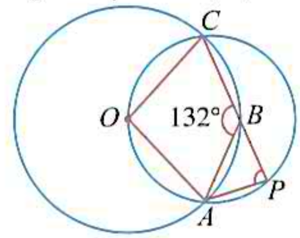
Given that $B\hat{A}Q = (3x + 4y)°$, $A\hat{O}B = (7x + 6y)°$ and $A\hat{B}O = 40°$, find the value of x and of y.

17. The diagram shows a circle $ABCD$ with centre O. $BOAT$ is a straight line.



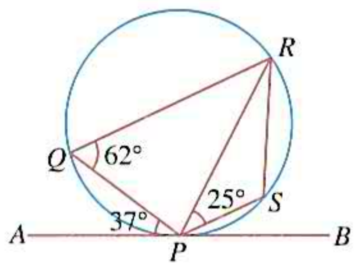
Given that $T\hat{A}D = 132°$ and $BC = CD$, find $A\hat{D}C$.

18. In the diagram, O is the centre of the larger circle and CBP is a straight line. The smaller circle passes through the points A, P, C and O.



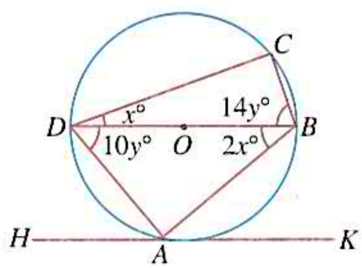
Given that $A\hat{B}C = 132°$, find
(i) $A\hat{O}C$, **(ii)** $A\hat{P}C$.

19.



In the figure, APB is a tangent at P to the circle $PQRS$. Given that $A\hat{P}Q = 37°$, $P\hat{Q}R = 62°$ and $R\hat{P}S = 25°$, find
(i) $Q\hat{R}P$, **(ii)** $Q\hat{P}R$,
(iii) $P\hat{R}S$.

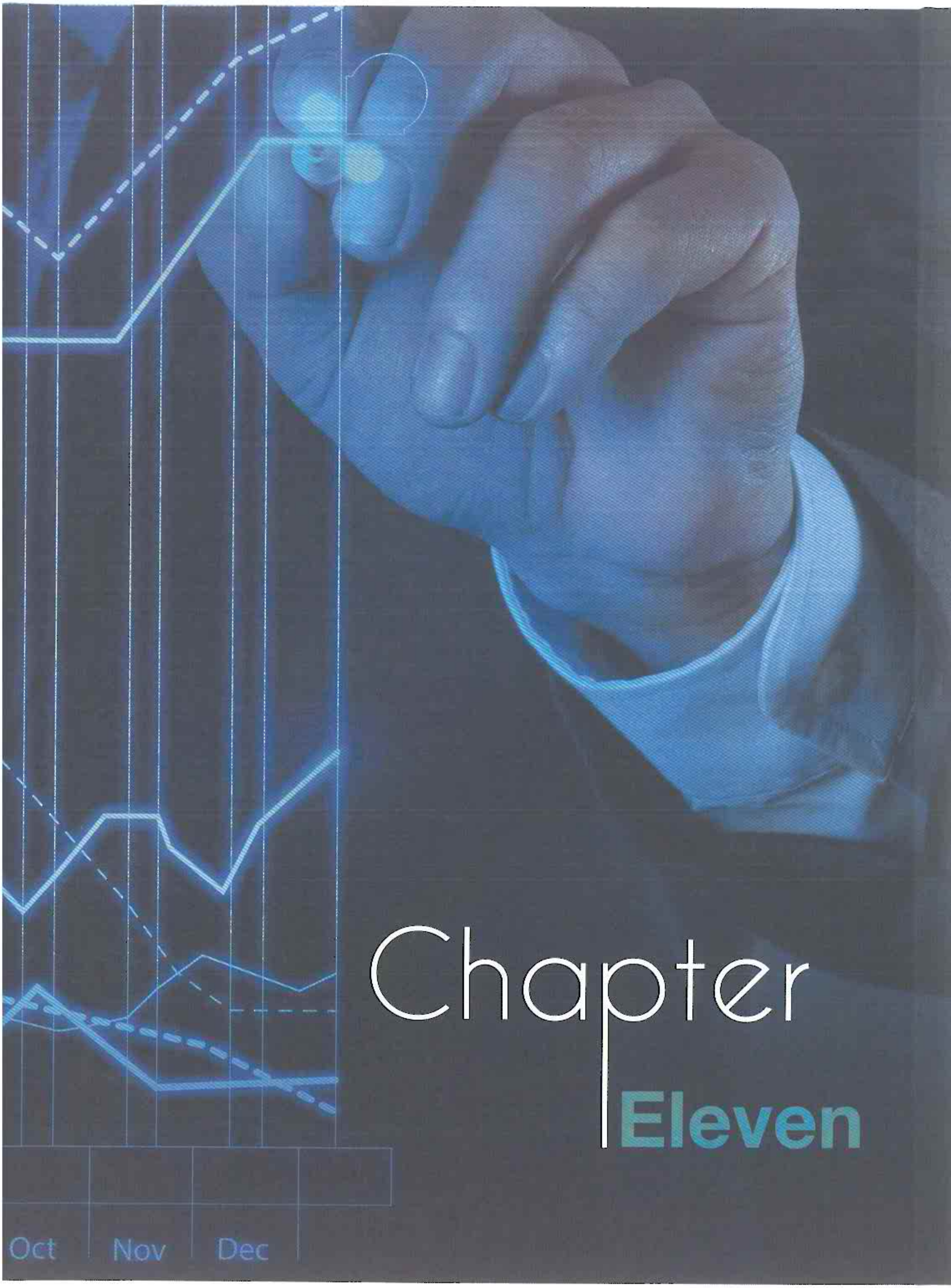
20.



In the figure, O is the centre of the circle and HK is a tangent to the circle at A. Given that $A\hat{D}B = 10y°$, $A\hat{B}D = 2x°$, $C\hat{D}B = x°$ and $D\hat{B}C = 14y°$, calculate, in degrees,
(i) $C\hat{O}B$, **(ii)** $A\hat{C}D$,
(iii) $K\hat{A}B$.

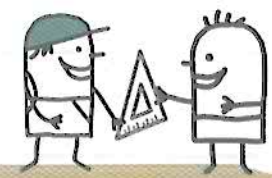
Revision: Probability and Statistics

'It is easy to lie with statistics. It is hard to tell the truth without it.' In Books 1 to 4, we have learnt a variety of methods to analyse data, obtain statistics from the data and evaluate the statistics. In addition, we have also learnt the basic laws of probability, which is a closely related field to the study of Statistics. This chapter provides you with a summary of the concepts you have learnt and opportunities to practise questions from these topics.



Chapter Eleven

11.1 Probability



Worked Example 1

(Probability of Single Events)

A bag contains 52 cards numbered from and 1 to 52 inclusive. A card is chosen at random from the bag. Find the probability that the number on the chosen card

(i) is a single digit,

(ii) is a perfect square,

(iii) contains at least one '4',

(iv) does not contain a '4'.

Solution:

(i) Total number of outcomes = 52

There are 9 numbers which are single digits, i.e. 1, 2, 3, ..., 9.

$\therefore$ P(number is a single digit) $= \dfrac{9}{52}$

(ii) There are 7 numbers which are perfect squares, i.e. 1, 4, 9, 16, 25, 36, 49.

$\therefore$ P(number is a perfect square) $= \dfrac{7}{52}$

(iii) There are 14 numbers which contains at least one '4' i.e. 4, 14, 24, 34, 40, 41, 42, ..., 49.

$\therefore$ P(number contains at least one '4') $= \dfrac{14}{52}$

$= \dfrac{7}{26}$

(iv) P(number does not contain a '4') = 1 – P(number contains at least one '4')

$= 1 - \dfrac{7}{26}$

$= \dfrac{19}{26}$

RECALL

For **(iv)**, we use the formula P(not E) = 1 – P(E).

Worked Example 2

(Probability of Simple Combined Events)

Bag A contains four cards bearing the numbers 2, 3, 4 and 5. Bag B contains six cards bearing the numbers 4, 5, 6, 7, 8 and 9. A card is drawn at random from each bag. Find the probability that

(i) the two cards bear the same number,

(ii) the larger of the two numbers is 5,

(iii) the sum of the two numbers on the cards is equal to 10,

(iv) the product of the two numbers on the cards is less than 15.

Solution:

The table below represents the outcomes of this probability experiment.

Bag A \ Bag B	4	5	6	7	8	9
2	2, 4	2, 5	2, 6	2, 7	2, 8	2, 9
3	3, 4	3, 5	3, 6	3, 7	3, 8	3, 9
4	4, 4	4, 5	4, 6	4, 7	4, 8	4, 9
5	5, 4	5, 5	5, 6	5, 7	5, 8	5, 9

(i) Total number of outcomes $= 4 \times 6$
$= 24$

From the table, there are 2 outcomes with the two cards bearing the same number, i.e. (4, 4) and (5, 5).

$$\therefore \text{P(two cards bear the same number)} = \frac{2}{24} = \frac{1}{12}$$

(ii) From the table, there are 4 outcomes where the larger of the two numbers is 5, i.e. (5, 4), (2, 5), (3, 5) and (4, 5).

$$\therefore \text{P(larger of the two numbers is 5)} = \frac{4}{24} = \frac{1}{6}$$

(iii) From the table, there are 4 outcomes where the sum of the two numbers is 10, i.e. (5, 5), (4, 6), (3, 7) and (2, 8).

$\therefore$ P(sum of the two numbers is 10) $= \frac{4}{24}$

$= \frac{1}{6}$

(iv) From the table, there are 5 outcomes where the product of the two numbers is less than 15, i.e. (2, 4), (3, 4), (2, 5), (2, 6) and (2, 7).

$\therefore$ P(product of the two numbers is less than 15) $= \frac{5}{24}$

Worked Example 3

(Probability involving Independent Events)

On any particular school day, the probability that Lixin will be late for school is 0.2 and the probability that Shirley is late for school is 0.15. Find the probability that for a particular school day,

(i) both Lixin and Shirley are late for school,

(ii) exactly one of them is late for school.

Solution:

(i) P(both Lixin and Shirley are late) $= 0.2 \times 0.15$

$= 0.03$

(ii) P(exactly one of them is late) = P(Lixin is late and Shirley is not late) + P(Lixin is not late and Shirley is late)

$= 0.2 \times (1 - 0.15) + (1 - 0.2) \times 0.15$

$= 0.2 \times 0.85 + 0.8 \times 0.15$

$= 0.29$

ATTENTION

For **(i)**, the event that Lixin is late and the event that Shirley is late are **independent** of each other. Hence, we use the **Multiplication Law of Probability**, i.e. P(Lixin and Shirley are late) = P(Lixin is late) $\times$ P(Shirley is late).

Worked Example 4

(Use of Tree Diagram)

A box contains 15 red balls and 6 blue balls. The balls are identical except for their colour. Two balls are drawn at random, in succession from the box.

(a) Draw a tree diagram to represent the information given.

(b) Using the tree diagram, find the probability that

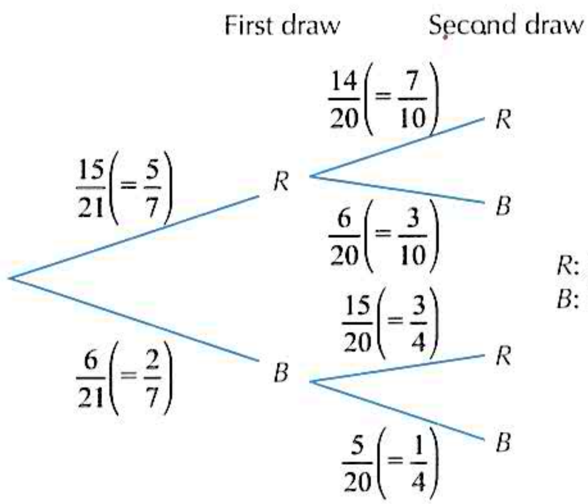
(i) both balls drawn are red,

(ii) the second ball drawn is blue,

(iii) both balls drawn are different in colour.

Solution:

(a)



R: Red ball
B: Blue ball

ATTENTION

The probability experiment in Worked Example 4 is an example of **dependent events**. In the second draw, the probability of drawing a ball of either colour **depends** on which coloured ball was drawn in the first draw.

(b) (i) From the diagram, P(both balls drawn are red) = P(RR)

$$= \frac{5}{7} \times \frac{7}{10}$$
$$= \frac{1}{2}$$

(ii) From the diagram, P(second ball drawn is blue) = P(RB) + P(BB)

$$= \left(\frac{5}{7} \times \frac{3}{10}\right) + \left(\frac{2}{7} \times \frac{1}{4}\right)$$
$$= \frac{3}{14} + \frac{1}{14}$$
$$= \frac{2}{7}$$

Problem Solving Tip

For **(ii)**, the two probabilities P(RB) and P(BB) can be added together as they are mutually exclusive. It is not possible for the events to occur simultaneously.

(iii) From the diagram,

P(both balls drawn are different in colour) = P(RB) + P(BR)

$$= \left(\frac{5}{7} \times \frac{3}{10}\right) + \left(\frac{2}{7} \times \frac{3}{4}\right)$$
$$= \frac{3}{14} + \frac{3}{14}$$
$$= \frac{3}{7}$$

Revision 11A

1. In an election with only two candidates, x voters voted for candidate A and 36 voters voted for candidate B. A voter is chosen at random.
   (i) Write down the expression for the probability that a voter who voted for Candidate A is chosen.
   (ii) Given that the probability in **(i)** is $\frac{2}{5}$, find the value of x.

2. A bag contains 5 red marbles, 4 blue marbles and 7 green marbles. The marbles are identical except for their colour.
   (i) A marble is chosen at random. What is the probability that it is a blue marble?
   (ii) How many more blue marbles must be added to the bag such that the probability of choosing a blue marble is $\frac{1}{3}$?
   (iii) How many red or green marbles must be added to the bag such that the probability of choosing a blue marble is $\frac{1}{6}$?

3. The faces of a 6-sided die are numbered from 3 to 8. Two of such dice are rolled and the resulting product is shown in the possibility diagram below.

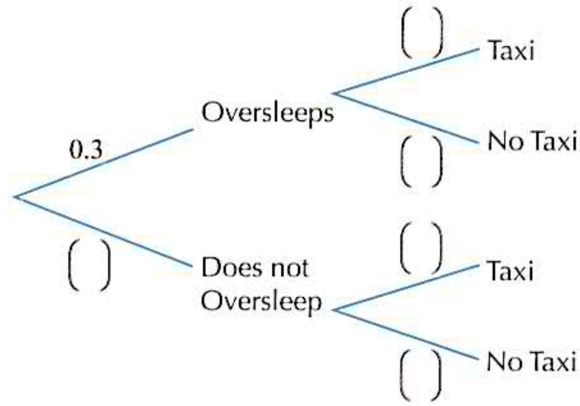
First die

Second die

×	3	4	5	6	7	8
3			15			
4						
5		20				
6						
7						56
8						

   (a) Copy and complete the possibility diagram.
   (b) Using the diagram in **(a)**, find the probability that the product is
   (i) odd, **(ii)** less than or equal to 23,
   (iii) prime, **(iv)** divisible by 14.

4. The probability that Michael oversleeps is 0.3. If he oversleeps, the probability that he takes a taxi to work is 0.8. If he does not oversleep, the probability that he takes a taxi to work is 0.2.
   (i) Copy and complete the tree diagram below to represent the information given.



   (ii) Using the tree diagram, calculate the probability that Michael takes a taxi to work.

5. Two bags, X and Y, contain red and black marbles only. Bag X contains 4 red marbles and 2 black marbles. Bag Y contains 3 red marbles and 1 black marble.
   A bag is chosen by flipping a fair coin. If the coin shows a head, bag X is chosen; otherwise, bag Y is chosen. A marble is then selected at random from the chosen bag.
   (i) Draw a tree diagram to illustrate this experiment.
   (ii) Hence, or otherwise, calculate the probability of selecting a red marble.

6. A pencil case contains two erasers and one pen. A second pencil case contains one eraser, two pens and a ruler. Kate picks one item from each pencil case.

(a) Complete the table to show all the possible outcomes.

Second pencil case

First pencil case		E	P	P	R
	E	EE	EP		
	E				
	P				

E: eraser, P: pen, R: ruler

(b) Find the probability that Kate picks

(i) two pens,

(ii) at least one eraser.

7. The letters of the word 'EXCELLENCE' are each printed on separate cards and the cards are put in a box. Two cards are drawn at random from the box, without replacement. Find the probability that

(i) both cards contain vowels,

(ii) the second card drawn contains a vowel,

(iii) the letters on the two cards drawn are the same.

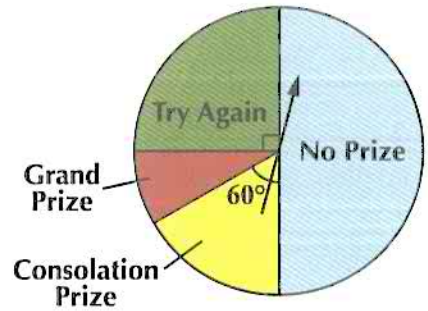
8. A box contains 33 table-tennis balls. 12 of the balls are white and the remaining balls are orange. The balls are identical except for their colour. A ball is taken out at random from the box and not replaced. A second ball is then taken out at random from the box. With the help of a tree diagram, find the probability that

(i) both balls are orange,

(ii) both balls are of the same colour,

(iii) the balls are of different colours.

9. In a game of 'Wheel of Fortune', a fair pointer is spun. The wheel is shown in the diagram below. The player can only spin the wheel once, unless the pointer lands on 'Try Again', for which he gets another chance to spin the wheel.



Find the probability that a player

(i) wins the grand prize on the first spin,

(ii) wins the consolation prize on his second spin,

(iii) spins the pointer three times and wins nothing.

10. A bag contains one red ball, two black balls and three white balls.

(i) A ball is drawn at random from the bag. What is the probability that the ball drawn is white?

(ii) The first ball drawn is replaced and a second ball is drawn at random. What is the probability of drawing two balls of different colours?

(iii) If instead, the first ball drawn is not replaced and a second ball is drawn at random, what is the probability of drawing two balls of the same colour?

11. A bag contains 4 cards, one marked with the letter 'A', one with the letter 'B' and two with the letter 'L'. The cards are drawn at random from the bag, one at a time, without replacement. Calculate the probability that

(i) the first two cards will each have the letter 'L' marked on them,

(ii) the second card to be drawn will have the letter 'B' marked on it,

(iii) the order in which the cards are drawn will spell out the word 'BALL'.

12. Two national servicemen, Rui Feng and Farhan, take part in an annual fitness proficiency test. The probability that Rui Feng will pass the test is $\frac{2}{3}$ and the probability that Farhan will pass the test is $\frac{5}{6}$. Find the probability that

(i) Rui Feng passes the test and Farhan fails the test,

(ii) both of them fail the test,

(iii) at least one of them passes the test.

13. Three fair dice are tossed simultaneously. Find the probability that

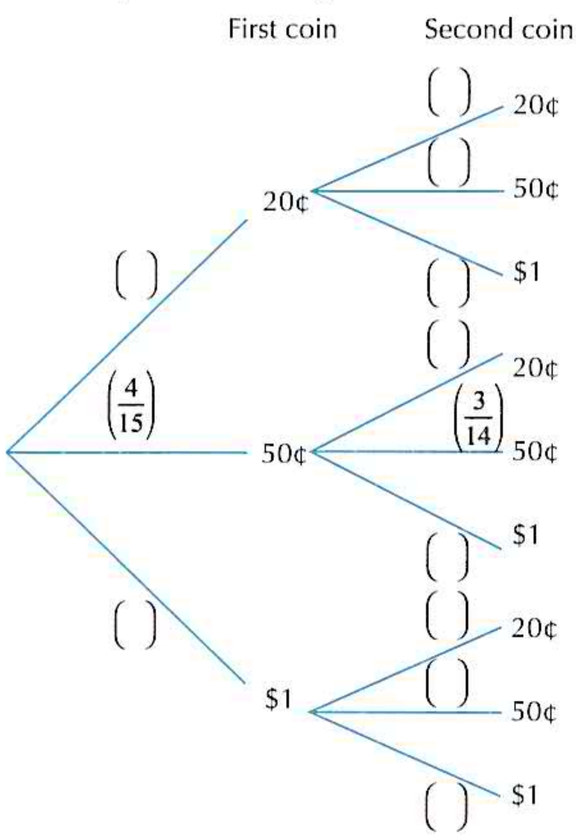
(i) the sum of the three numbers is 3,

(ii) the sum of the three numbers is 4,

(iii) the three dice show different numbers.

14. Nora has 15 coins in her wallet. She has eight 20¢ coins, four 50¢ coins and three \$1 coins. Two coins are taken out of her wallet at random, without replacement.

(a) Complete the tree diagram below.



(b) Find the probability that

(i) the first coin is a 20¢ coin and the second coin is a \$1 coin,

(ii) the two coins taken out are in different denominations,

(iii) the second coin is a 50¢ coin,

(iv) the total value of the two coins is more than one dollar.

15. The following table shows the number of boys and girls who are either Science or Arts students.

	Science	Arts
Boys	7	18
Girls	9	14

A boy and a girl are selected at random. Find the probability that

(i) the boy selected is a Science student and the girl selected is an Arts student,

(ii) both the boy and the girl are Arts students,

(iii) exactly one of them is a Science student.

16. Nora, Amirah and Shirley are three contestants taking part in a chess competition. The probability that one of them wins the competition is $\frac{2}{3}$. It is also given that the probability of Nora winning the competition is $\frac{1}{3}$ and the probability of Amirah winning is $\frac{1}{8}$. Find that probability that

(i) none of them wins the competition,

(ii) Shirley wins the competition,

(iii) Nora or Shirley wins the competition.

17. A biased coin is such that on any single flip, it is more likely to obtain a head than a tail. The probability of obtaining a head is x. The biased coin is flipped twice and the probability of obtaining a head and a tail is $\frac{3}{8}$.

(i) With the information given, form a quadratic equation in x and show that it reduces to $16x^2 - 16x + 3 = 0$.

(ii) Solve the equation $16x^2 - 16x + 3 = 0$, and explain why one of the answers should be rejected.

(iii) With the answer in (ii), find the probability of obtaining two heads, when the same biased coin is flipped twice.

18. The letters of the word 'ELEMENTARY' are written on individual cards and the cards are put in a box. A card is drawn at random from the box. If the letter on the card is an 'E', it will be put back into the box and a new card is then drawn at random. However, if the card drawn is not an 'E', the card will not be put back into the box and a second card will be drawn at random from the box. Calculate the probability that

(i) both of the cards drawn bear the letter 'E',

(ii) both of the cards drawn do not bear the letter 'E',

(iii) one card bears the letter 'E' while the other card does not bear the letter 'E'.

19. There are 24 white marbles, x red marbles and y blue marbles in a box. One marble is drawn at random.

(a) Given that the probability a red marble is drawn is $\frac{1}{5}$ and a blue marble is drawn is $\frac{2}{5}$, calculate the value of x and of y.

(b) With the values of x and y in (a), calculate the probability that two marbles drawn in succession without replacement are

(i) of the same colour,

(ii) a white marble followed by a red marble.

20. Bag P contains 6 yellow balls and 8 blue balls while Bag Q contains 5 yellow balls and 9 blue balls. A ball is selected at random from Bag P and placed into Bag Q. A ball is then selected from Bag Q and returned to Bag P. Find the probability that

(i) Bag P contains 7 yellow balls and 7 blue balls,

(ii) Bag Q contains 6 yellow balls and 8 blue balls,

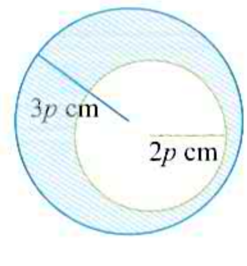
(iii) Bag P contains 6 yellow balls and 8 blue balls,

(iv) Bag Q contains 5 yellow balls and 10 blue balls,

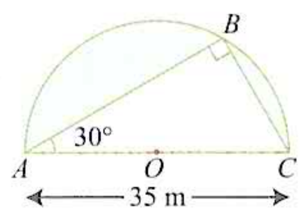
(v) Bag P contains 4 yellow balls and 10 blue balls,

(vi) Bag Q has more blue balls than yellow balls.

21. The figure shows two circles of radii $2p$ cm and $3p$ cm. A point is selected at random in the larger circle. Find the probability that the point lies in the blue region.



22. The figure shows a right-angled triangle ABC lying inside a semicircle with centre O. A point is selected at random in the semicircle. Given that $\angle BAC = 30°$ and $AC = 35$ m, find the probability that the point lies in the green region.

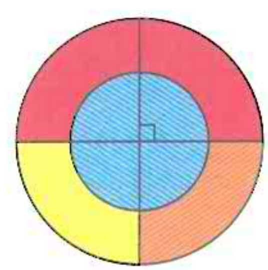


23. The figure shows a dart board which consists of two concentric circles of radius 3 cm and 6 cm, respectively. The board is coloured red, blue, yellow and orange as shown. Suppose each time Khairul throws a dart, the dart will land on the board and is equally likely to land on any part of the board.
Find the probability that a dart Khairul throws will land on

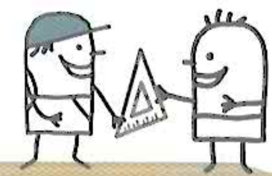
(i) the red region,

(ii) either the blue or the yellow region,

(iii) a region that is not yellow.



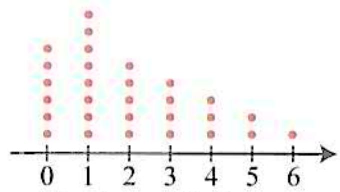
11.2 Statistics



Worked Example 5

(Dot Diagram)

The dot diagram shows the number of children in each family living in an apartment block.



(i) Find the modal and median number of children.

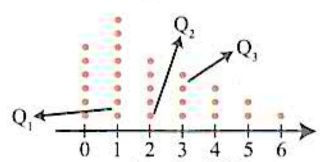
(ii) Find the lower quartile and upper quartile of the number of children.

(iii) Find the mean number of children in a family staying at the apartment block.

(iv) Two families are selected at random from the apartment block. Find the probability that one family has two children while the other family has five children.

Solution:

Problem Solving Tip



(i) Mode = 1 child

Total number of families = 6 + 8 + 5 + 4 + 3 + 2 + 1
= 29

For an odd number of data, the median is the value of the data in the middle position, i.e. 15th family in this case.

∴ Median = 2 children

(ii) Lower quartile = Mean of the number of children of the 7th and 8th family
= 1 child

Upper quartile = Mean of the number of children of the 22nd and 23rd family
= 3 children

(iii) Mean number of children in a family $= \dfrac{6 \times 0 + 8 \times 1 + 5 \times 2 + 4 \times 3 + 3 \times 4 + 2 \times 5 + 1 \times 6}{29}$
$= 2$

(iv) P(one family has two children and one family has five children)
= P(first family has two children and second family has five children)
+ P(first family has five children and second family has two children)

$$= \left(\frac{5}{29} \times \frac{2}{28}\right) + \left(\frac{2}{29} \times \frac{5}{28}\right)$$

$$= \frac{5}{203}$$

Worked Example 6

(Scatter Diagram)

Six identical baskets contain different number of apples of varying sizes. The combined weight of the basket and the apples is given in the table below.

Number of apples	8	15	20	27	32	40
Combined mass of apples and basket (kg)	2.2	2.8	3.2	4.1	4.7	4.9

(i) Using a scale of 2 cm to represent 5 apples on the horizontal axis and 4 cm to represent 1 kg on the vertical axis, plot the above data on a scatter diagram.

(ii) State whether the data display strong positive, weak positive, strong negative or weak negative correlation.

(iii) Draw a line of best fit.

(iv) Use the line of best fit to find the approximate weight of the basket.

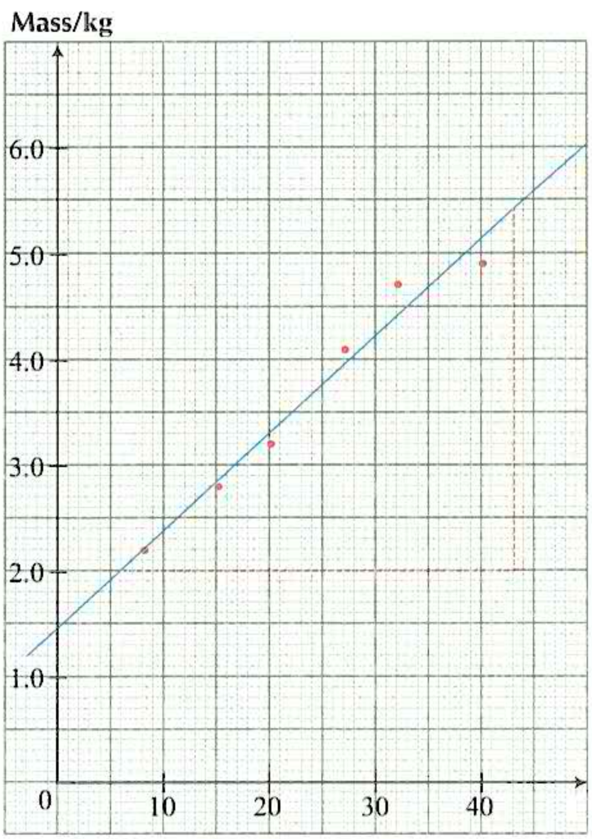
(v) Find the gradient of the line of best fit.

(vi) Write down the equation of the line of best fit.

(vii) Using the equation of the line of best fit, write down the estimated average weight of an apple.

Solution:

(i)



(ii) The data shows strong, positive correlation.

(iii) The line of best fit is drawn with approximately half of the plots lying above the line and the other half lying below the line of best fit.

(iv) 1.45 kg

(v) Gradient $= \dfrac{5.4-2}{43-6}$

$= 0.09$ (to 3 s.f.)

(vi) Gradient $= 0.09$

y-intercept $= 1.45$

$\therefore$ The equation of the line of best fit is

$y = 0.09x + 1.45$

(vii) From the equation, mass of the basket = 1.45 kg

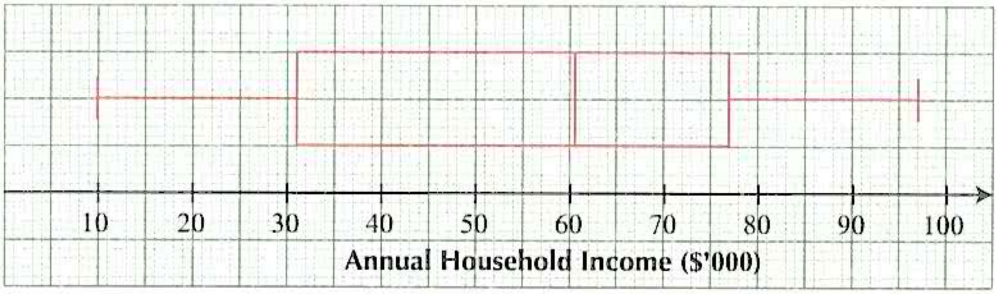
Average mass of an apple = 0.09 kg

= 90 g

Worked Example 7

(Interpretation of a Box-and-Whisker Plot)

The box-and-whisker plot shows the annual household incomes of 100 households in a certain housing district.



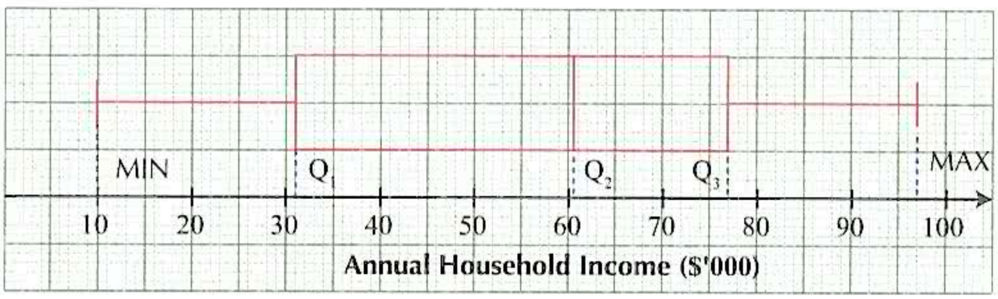
Using the plot, find

(i) the range,

(ii) interquartile range,

(iii) the median.

Solution:



From the plot,

(i) Range = MAX – MIN
= 97 000 – 10 000
= $87 000

(ii) Interquartile range = $Q_3 - Q_1$
= 77 000 – 31 000
= $46 000

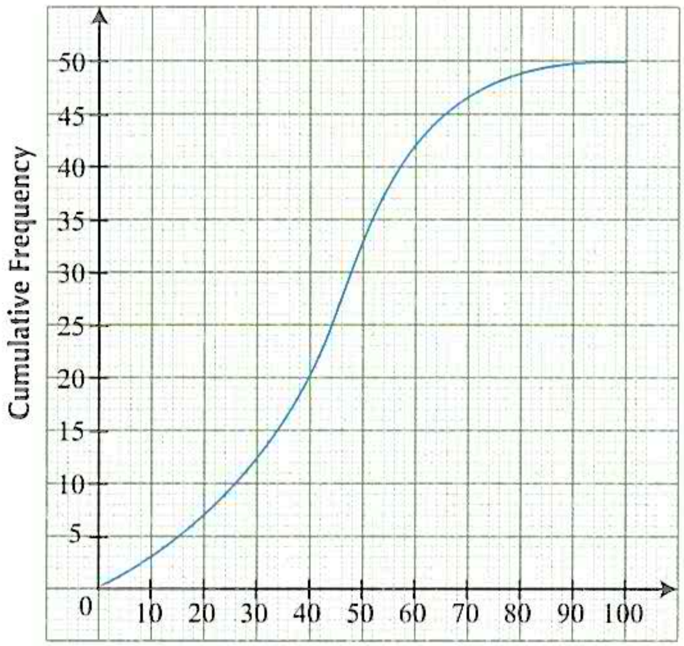
(iii) Median = $60 500

Worked Example 8

(Interpretation of a Cumulative Frequency Curve)

A class of 50 students sat for a History examination. The marks obtained by the students were tabulated and a cumulative frequency curve was drawn to illustrate the marks.

Cumulative Frequency Curve for the Marks of the History Examination



(a) Using the graph, find

  (i) the lower quartile,

  (ii) the median,

  (iii) the upper quartile,

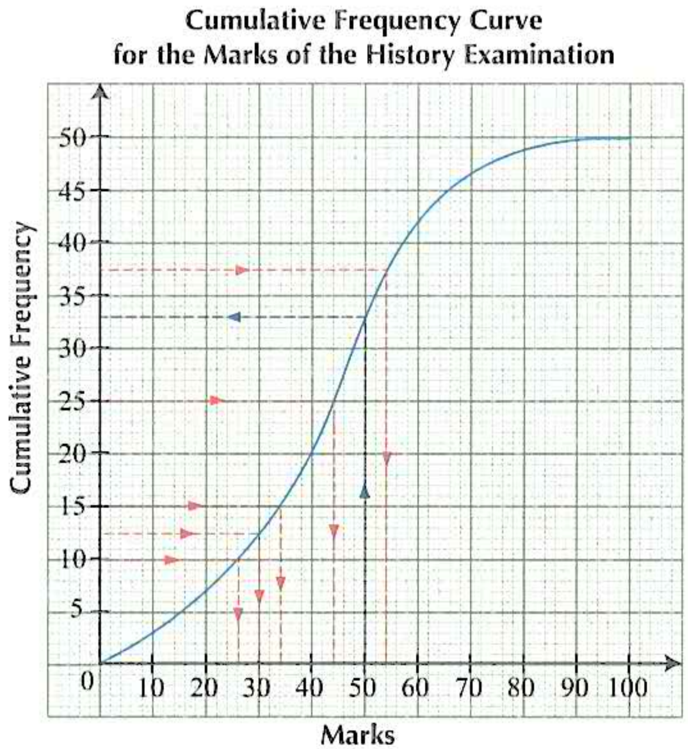
  (iv) the interquartile range.

(b) If 70% of the students passed the examination, what is the passing mark?

(c) If not more than 20% failed the examination, what is the passing mark?

(d) How many students passed the examination if the passing mark is 50?

Solution:



(a) From the graph,

(i) Lower quartile = 30

(ii) Median = 44

(iii) Upper quartile = 54

(iv) Interquartile Range = Upper quartile – Lower quartile
= 54 – 30
= 24

(b) If 70% of the students passed the examination, 30% of the students failed the examination.

30% means $\frac{30}{100} \times 50 = 15$ students failed the examination.

From the curve, passing mark = 34

(c) 20% of the students means $\frac{20}{100} \times 50 = 10$, i.e. not more than 10 students failed the examination.
From the curve, passing mark = 26

(d) From the curve, the number of students who scored less than 50 marks is 33.
∴ Number of students who passed the examination = 50 – 33
= 17

RECALL

The accuracy of the answer can only be accurate up to half of a small square grid.

Worked Example 9

(Comparing the Means and Standard Deviations of Two Sets of Data)

The waiting times, in minutes, for 100 patients admitted to the emergency departments of two hospitals are given below.

Vermont Hospital

Time (minutes)	Number of Patients
$10 < x \leqslant 15$	6
$15 < x \leqslant 20$	14
$20 < x \leqslant 25$	20
$25 < x \leqslant 30$	32
$30 < x \leqslant 35$	16
$35 < x \leqslant 40$	12

Innova Hospital

Mean	23.8 minutes
Standard Deviation	7.56 minutes

(i) Calculate the mean and standard deviation of the waiting times at the Emergency Department of Vermont Hospital. Show your working clearly.

(ii) Which hospital's emergency department is more efficient in handling the waiting times of its patients? Give a reason for your answer.

(iii) Which hospital's emergency department has more consistent waiting times? Give a reason for your answer.

Solution:

Time (minutes)	Frequency	Mid-value (x)	fx	fx^2
$10 < x \leqslant 15$	6	12.5	75	937.5
$15 < x \leqslant 20$	14	17.5	245	4287.5
$20 < x \leqslant 25$	20	22.5	450	10 125
$25 < x \leqslant 30$	32	27.5	880	24 200
$30 < x \leqslant 35$	16	32.5	520	16 900
$35 < x \leqslant 40$	12	37.5	450	16 875
Sum	$\sum f = 100$		$\sum fx = 2620$	$\sum fx^2 = 73\ 325$

RECALL

It is easier to use the formula $\sqrt{\frac{\sum fx^2}{\sum f} - \bar{x}^2}$ when finding the standard deviation.

You may also obtain the mean and the standard deviation by using the statistical functions in your calculator.

(i) Mean, $\bar{x} = \frac{\sum fx}{\sum f}$

$$= \frac{2620}{100}$$

$= 26.2$ minutes

Standard Deviation $= \sqrt{\frac{\sum fx^2}{\sum f} - \bar{x}^2}$

$$= \sqrt{\frac{73\ 325}{100} - 26.2^2}$$

$= 6.84$ minutes (to 3 s.f.)

(ii) Innova Hospital's Emergency Department is more efficient as the mean waiting time is shorter.

(iii) The waiting times at the emergency department of Vermont Hospital are more consistent as the standard deviation is smaller.

Revision 11B

1. **(a)** The distribution of marks scored by the students of a class for a quiz is shown in the table below.

Marks	4	5	6	7	8	9	10
Number of Students	7	5	6	3	a	8	7

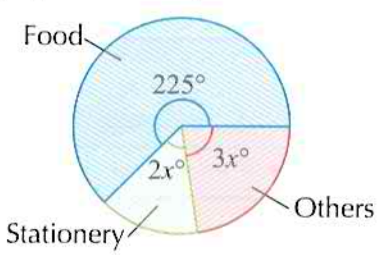
The median mark is 7. Write down the possible values of a.

(b) The ages of 6 people are 15, 42, 15, 21, x and $2x$. If the mean age is 18 years, find

(i) the value of x,

(ii) standard deviation.

2. The pie chart illustrates the value of various goods sold by a provision shop.



(a) Calculate the value of x.

(b) Given that the total value of the sales was \$21 600, find the sales value of

(i) food,

(ii) stationery.

3. The following data shows the number of push-ups done by 7 soldiers in 1 minute.

43, 30, 55, 21, 28, 32, 33

Find

(i) the median number of push-ups,

(ii) the range,

(iii) the interquartile range.

4. The masses (kg) of the school bags of a group of primary six students were weighed, and the data collected is shown in the stem-and-leaf diagram below.

3	0 1 4 6 7 9
4	3 4 7
5	0 1 2 4 4 5 6 7 9
6	1 2 3 3 3 3 5 6 7 8 9
7	0 0 1 2 4 4 4 6 7 8 8 9
8	0 2 3 4 6 7 8 9
9	4 6 7 8
10	0 2 3 8

Key: 3 | 9 means 3.9 kg

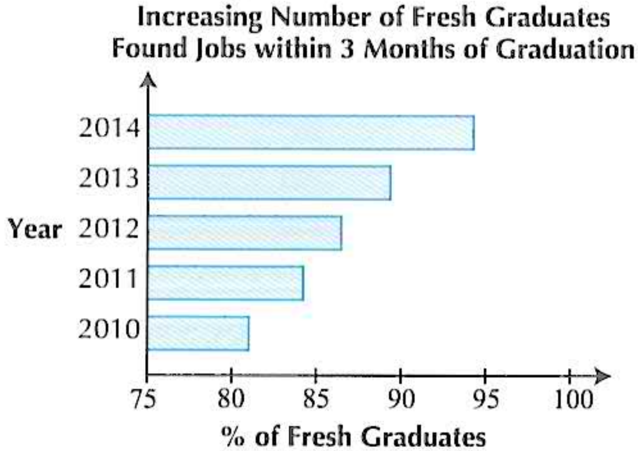
(i) How many primary six students are there in the group?

(ii) Find the median mass of the school bags.

(iii) Find the modal mass of the school bags.

(iv) The school encourages students to carry bags which weigh less than 7.5 kg. School bags with masses of at least 7.5 kg are considered 'overweight'. Find the percentage of school bags which are considered 'overweight'.

5. The graph given shows the percentage of fresh graduates from an university who found jobs within three months of graduation, from 2010 to 2014.



Explain one way in which the graph is misleading.

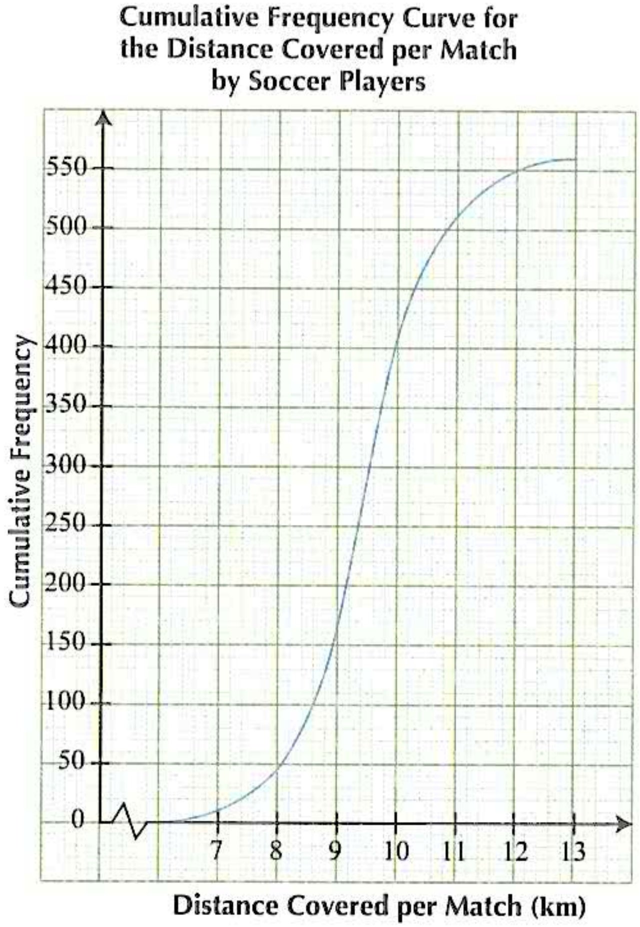
6. The mass of each of the 40 students in a class, corrected to nearest kg, are shown below.

52	67	65	57	52	60	58	59
53	42	51	72	69	57	54	54
58	52	44	47	73	58	62	56
63	57	68	59	63	47	68	58
48	50	64	54	57	59	44	55

(i) Construct a frequency table, using a class interval of 5 kg, starting with 41 kg.

(ii) Using the table, estimate the mean mass.

7. The cumulative frequency curve shows the average distance, d km covered per match by 560 soccer players for a soccer league season and the number of players who cover less than or equal to d km.



(a) Use the graph to estimate the number of players who cover more than 10.5 km per match.

(b) Use the graph to estimate, showing your method clearly,

(i) the median distance covered,

(ii) the interquartile range.

(c) Explain in words what the answer in **(b)(i)** means.

8. The fares paid by bus passengers in one day are shown in the following frequency table.

Fare in Cents	45	64	85	105	115	125
Number of Passengers	47	165	72	34	46	26

(a) (i) Calculate the mean fare per passenger, giving your answer to the nearest cent.

(ii) Write down the modal fare.

(b) 44 families were surveyed on how many children (aged below 13) there are in their families. The data below shows the raw survey results.

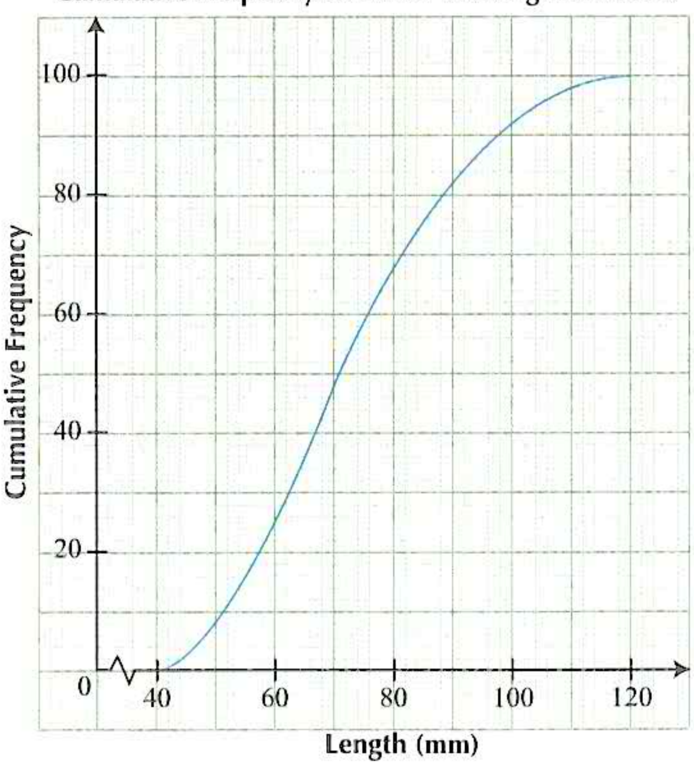
1	4	7	4	2	3	4	2	3	4	4
2	3	2	3	1	2	3	2	4	5	2
3	4	1	2	1	6	2	5	3	2	1
5	5	1	3	2	1	1	1	6	3	3

(i) Rewrite the data using a frequency distribution table.

(ii) Calculate the percentage of families with fewer than three children.

9. The diagram shows the cumulative frequency curve for the lengths (mm) of 100 leaves from a certain species of plant.

Cumulative Frequency Curve for the Length of Leaves



(a) Copy and complete the grouped frequency table of the lengths of the 100 leaves.

Lengths (mm)	Frequency
$30 < x \leqslant 50$	
$50 < x \leqslant 70$	
$70 < x \leqslant 90$	
$90 < x \leqslant 110$	
$110 < x \leqslant 130$	

(b) Using the grouped frequency table, calculate an estimate of

(i) the mean length of a leaf from the plant,

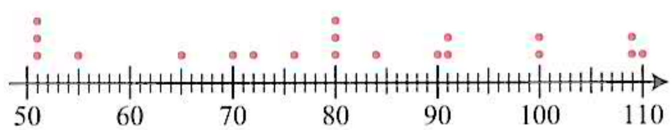
(ii) the standard deviation.

(c) Another 100 leaves from the same species of plant were collected and measured. These leaves have the same range but a smaller standard deviation.

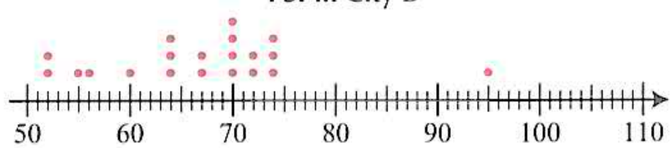
Describe how the cumulative frequency curve will differ from the given curve.

10. The following dot diagrams represent the Pollutant Standards Index (PSI) in two cities over a period of 20 days. The PSI is an indicator of the air quality. A lower PSI reading indicates better air quality.

PSI in City *A*



PSI in City *B*



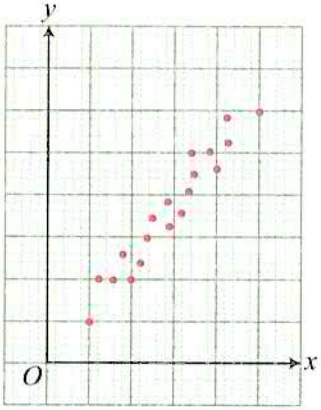
(i) Construct a frequency table for the PSI for each city, using the class intervals $50 < x \leqslant 70$ and so on.

(ii) Calculate estimates of the mean and standard deviation for the PSI for each city.

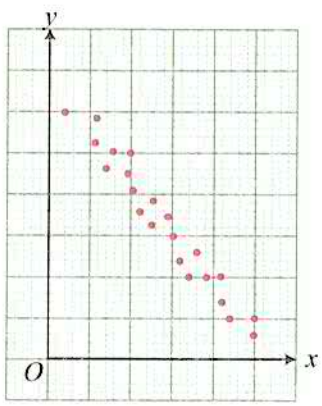
(iii) Compare and comment on the air qualities between the two cities.

11. For each of the following scatter diagrams and state whether there is positive, negative or zero correlation and whether the correlation is strong or weak.

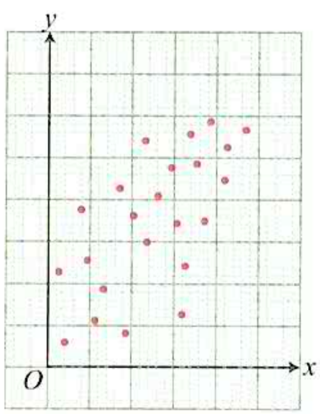
(a)



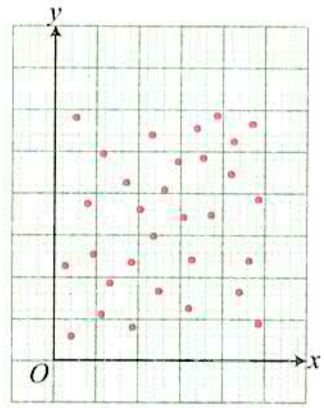
(b)



(c)



(d)



12. Values of two variables x and y obtained from surveys are given in the following tables. Draw a scatter diagram for each set of values and comment whether the data display positive, negative or no correlation. Also state whether the correlation is strong or weak.

(i)

x	3	6	10	12	15	15	19	21	24	25	28	30
y	6	13	7	12	10	19	25	15	10	20	27	23

(ii)

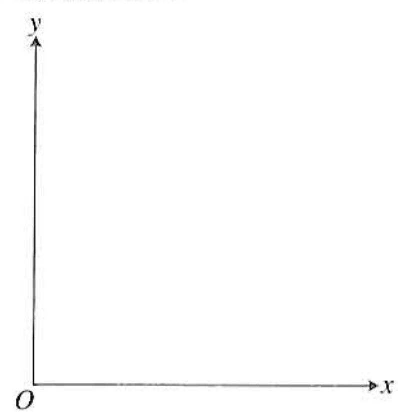
x	2	5	6	6	6	7	7.5	8	8	9	10	11
y	83	81	79	77	69	77	73	81	74	72	68	62

(iii)

x	5	10	20	26	29	35	38	40	42	42	45	45	50
y	6	11	16	9	3	8	2	6	7	14	8	6	11

13. The age x years, and the height y cm of 12 University students were measured and recorded.

(i) Sketch, using the axes below, the scatter diagram you would expect to obtain. You do not need to scale the axes.

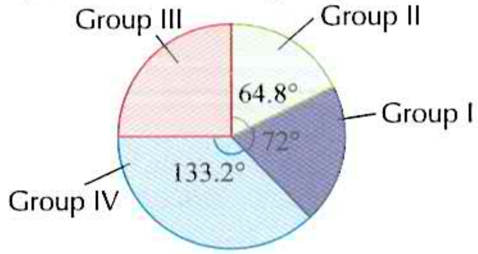


(ii) Describe fully the correlation (if any) shown by your sketch.

14. The diameters (d cm) of 100 tree trunks were measured and classified into four groups.

Group I:	$0 < d < 50$
Group II:	$50 \leqslant d < 60$
Group III:	$60 \leqslant d < 70$
Group IV:	$70 \leqslant d < 100$

Each of the 100 tree trunks is in one of the four groups, as shown in the given pie chart.



(i) Copy and complete and table given.

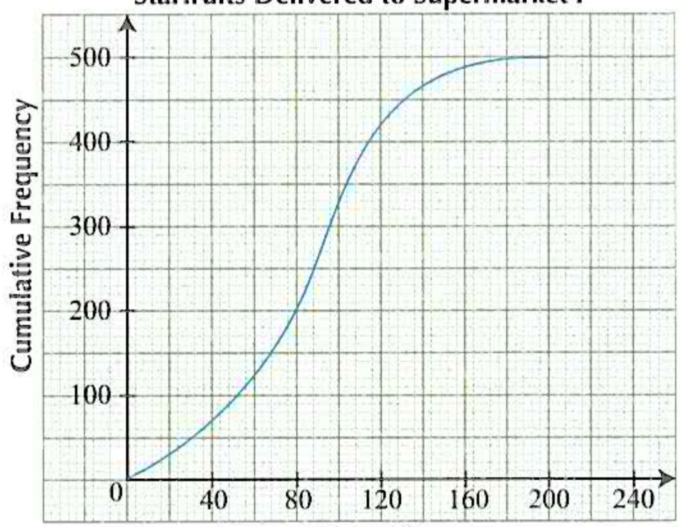
	Diameter (d cm)	Number of Tree Trunks
Group I	$0 < d < 50$	
Group II	$50 \leqslant d < 60$	18
Group III	$60 \leqslant d < 70$	
Group IV	$70 \leqslant d < 100$	

(ii) Calculate an estimate of the mean and standard deviation of the diameter of a tree trunk.

(iii) Two tree trunks are chosen at random. Find the probability that the diameters of both tree trunks lie in the range $50 \leqslant d < 60$.

15. The diagram shows the cumulative frequency curve for the mass (g) of 500 starfruits delivered to Supermarket P.

Cumulative Frequency Curve for the Mass of Starfruits Delivered to Supermarket P

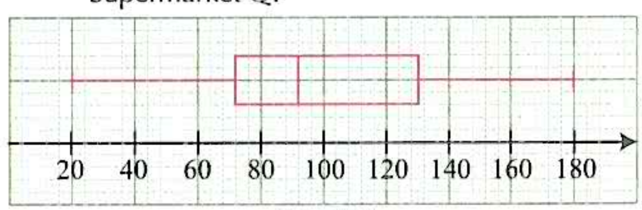


(a) Use the graph to
(i) estimate the median,
(ii) find the interquartile range,
(iii) estimate the 80th percentile.

(b) With reference to the context of this question, explain in words the meaning of the '80th percentile'.

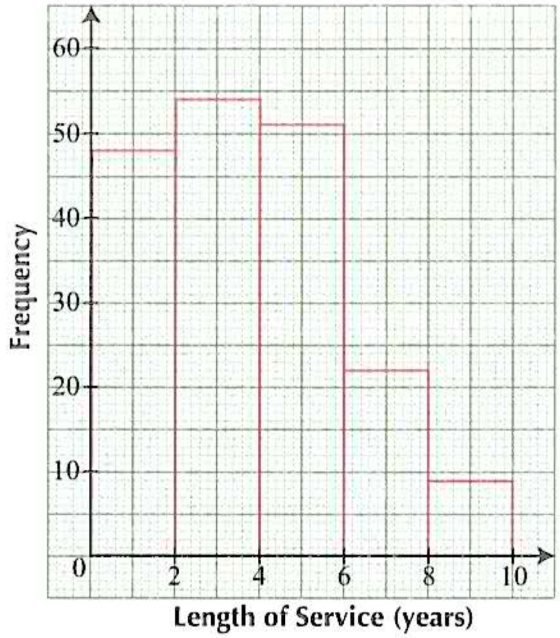
(c) Starfruits with a mass exceeding a certain value are graded as 'A'. Given that there are 150 grade 'A' starfruits, find the value of this mass.

(d) The box-and-whisker plot below shows the mass of another 500 starfruits delivered to Supermarket Q.



By comparing the median and interquartile range, state
(i) which delivery has bigger mass,
(ii) which delivery has a bigger spread of mass.

16. The histogram shows the length of service (in years) of a group of workers of an automobile company.



(i) Copy and complete the following table.

Length of Service (x years)	Frequency
$0 < x \leq 2$	48
$2 < x \leq 4$	
$4 < x \leq 6$	
$6 < x \leq 8$	22
$8 < x \leq 10$	

(ii) Find the total number of workers in the group.

(iii) Find an estimate of the mean number of years of service per worker.

(iv) Explain why the answer to **(iii)** is only an estimate of the mean.

(v) Calculate the **greatest** possible mean number of years of service.

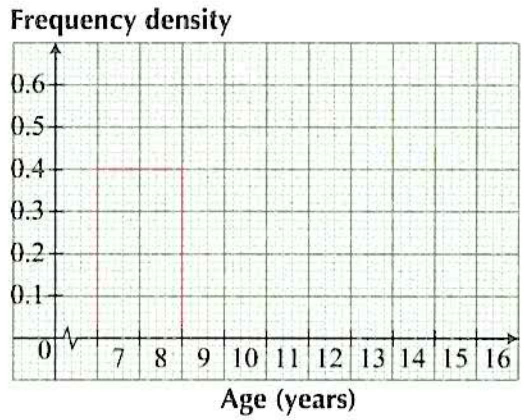
17. The number of participants in a chess competition was recorded.

Age in years	7–8	9–11	12–15
Number of participants	80	120	200

This information is to be represented on a histogram.

(a) Copy and complete the histogram.

(b) In the same diagram, draw the frequency polygon of the distribution.



18. The table below shows the marks of 160 students in the end-of-year Mathematics examination.

Marks	Frequency
$0 < x \leq 20$	12
$20 < x \leq 40$	25
$40 < x \leq 60$	30
$60 < x \leq 80$	55
$80 < x \leq 100$	38

(i) Find estimates of the mean and standard deviation of the marks.

The results of the mid-year examination, held earlier in the same year, of the same 160 students are summarised in the table below.

Mean	56.3 marks
Standard Deviation	19.8 marks

(ii) Which examination did the students perform better at? Give a reason for your answer.

(iii) Which examination were the marks more consistent? Give a reason for your answer.

19. There are 100 houses in a certain housing estate. The following table shows the number of children (under the age of 13) living in each house.

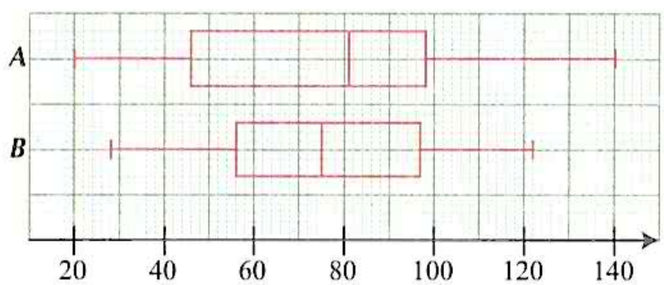
Number of Children	1	2	3	4	More than 4
Number of Houses	13	37	17	5	7

(i) How many houses in the estate do **not** have any children living in them?

(ii) Explain clearly why it is **not** possible to calculate the mean number of children living in each house in the housing estate.

(iii) There are 200 children in the estate. A house is considered 'overcrowded' if it has more than 4 children living in it. Find the mean number of children in 'crowded houses'.

20. The box-and-whisker plots below show the daily sales of two types of computer tablets, produced by two companies, A and B, during a particular month.



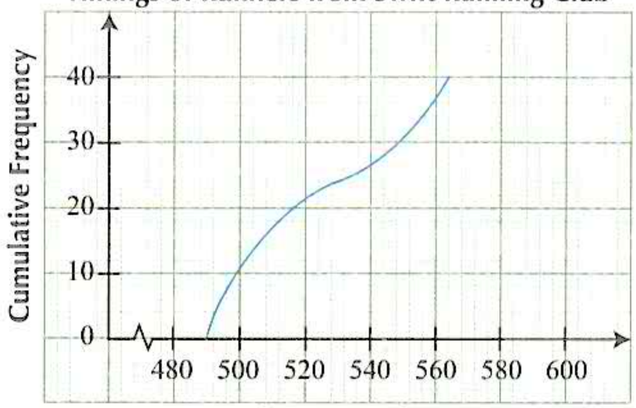
(a) Use the diagram to find, for Company A,
- **(i)** the median,
- **(ii)** the range,
- **(iii)** the interquartile range.

(b) Use the diagram to find, for Company B,
- **(i)** the median,
- **(ii)** the range,
- **(iii)** the interquartile range.

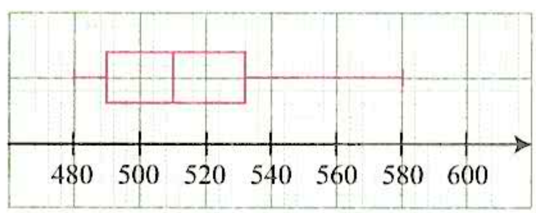
(c) Which company's overall sales performance was better? Give a reason for your answer.

(d) Which company's sales performance showed more consistency? Give a reason for your answer.

21. The cumulative frequency curve shows the distribution of the 2.4-km timings of 40 cross-country runners from Swift Running Club.



The box-and-whisker plot below shows the distribution of the 2.4-km timings of 40 cross-country runners from Ninja Running Club.



(a) The table below summarises the statistics for the 2.4-km timings of both running clubs. Copy and complete the table.

Running Club	Lower Quartile	Median	Upper Quartile	Interquartile Range
Swift		516		
Ninja	490		532	

(b) **(i)** 'The runners from Swift Running Club were generally faster in their 2.4-km runs'. Do you agree? Explain your answer.

(ii) 'The runners from Ninja Running Club were generally more consistent in their 2.4-km timings'. Do you agree? Explain your answer.

22. The frequency table below shows the lifespans of a sample of 160 batteries manufactured by factory A.

Lifespan (x hours)	Frequency
$4 < x \leqslant 6$	3
$6 < x \leqslant 8$	13
$8 < x \leqslant 10$	42
$10 < x \leqslant 12$	80
$12 < x \leqslant 14$	16
$14 < x \leqslant 16$	6

(i) Calculate an estimate of the mean and standard deviation of the lifespans of the batteries.

The mean and standard deviation of the lifespans of another sample of 160 batteries, manufactured by factory B is given in the table below.

Mean	12.5 hours
Standard Deviation	1.12 hours

(ii) Compare and comment on the lifespans of the batteries manufactured by factory A and factory B.

(iii) If the price of the batteries manufactured by each factory is the same, which battery would you buy? Justify your choice.

23. The number of times the employees in a small company took medical leave in 2014 are shown below.

7	4	6	4	2	5	6
6	9	8	5	7	9	1
4	3	5	1	5	1	4
1	3	4	8	7	5	5
4	4	4	2	1	2	6

(a) Construct a frequency table for the data given.

(b) Find the median number of times the employees took medical leave in 2014.

(c) Find the mean and standard deviation.

(d) Due to a systematic counting error, the number of times each employee took medical leave in 2014 has been overcounted by 1. Explain clearly how the

(i) standard deviation, and

(ii) mean

are affected by the overcounting.

24. The speeds, in km/h, of 20 cars measured by a speed camera at a certain point of an expressway on Saturday are shown below.

80	83	70	64	71	75	61	80	79	68
85	73	67	88	72	62	69	74	75	78

(i) Calculate the mean speed and the standard deviation.

The speeds, in km/h, of another group of 20 cars were measured by the same speed camera on Sunday. The following data were collected.

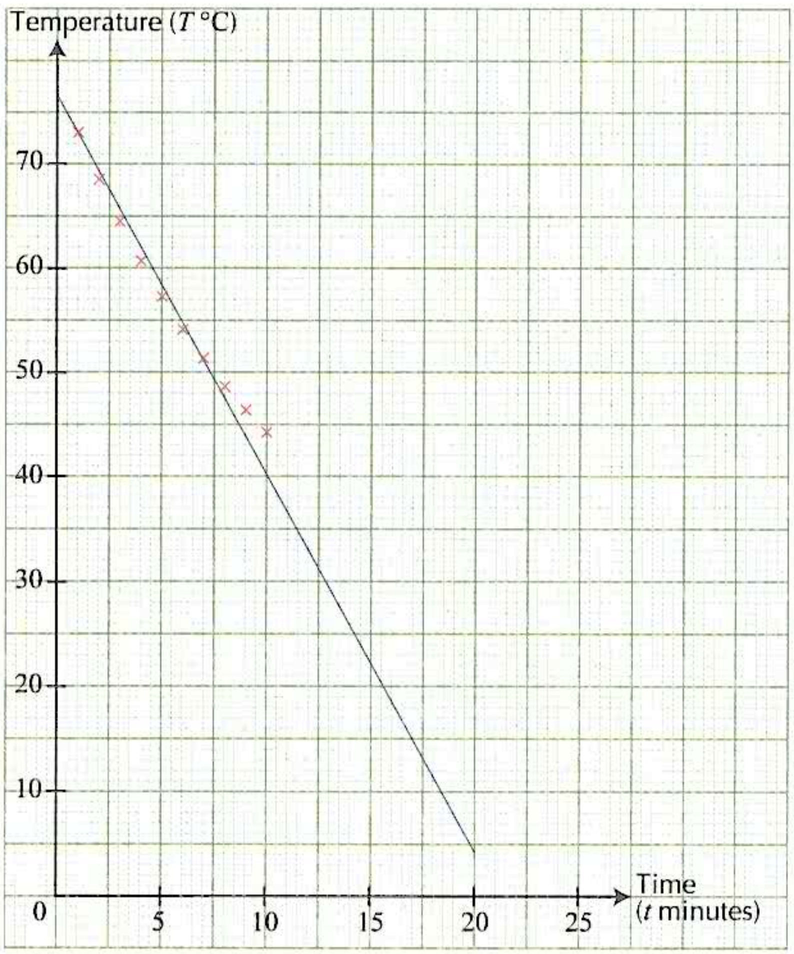
Mean	80.3 km/h
Standard Deviation	5.2 km/h

(ii) Make two comparisons between the two groups of cars on Saturday and Sunday.

(iii) Shirley suggested that the data should be grouped before calculating the mean and standard deviation. Should you follow her suggestion? Explain your answer.

PROBLEM 1: Cooling of a Cup of Tea

Very hot coffee, tea and water may cause serious burn injury. It had been reported that hot water at 56 °C could cause serious burns in 15 seconds. An experiment was done by a group of students to study the cooling of a cup of hot tea. The plotted points in the graph below represent the temperature of the tea over time. The students initially used a straight line graph to model the cooling of the tea.

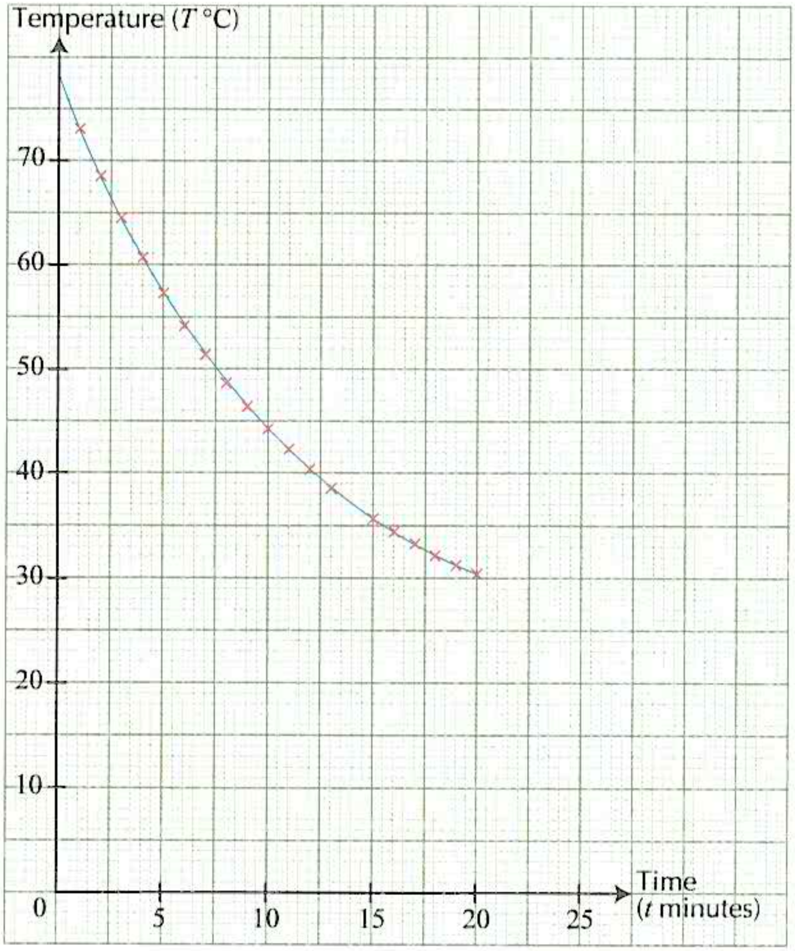


(a) Use the graph to find the equation of the straight line.

(b) Use the equation obtained in **(a)** to estimate

- **(i)** the time taken for the temperature of the tea to decrease to 56 °C,
- **(ii)** the temperature of the tea after 25 minutes. Explain why this is not a good estimate of the temperature of the tea.

(c) After further research and collection of more data, the students found that the best-fit curve to model the cooling of the tea is given by the equation $T = a(2.72)^{-0.09t} + 21$, where a is a constant.



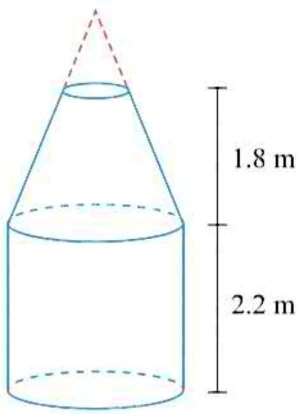
(i) Find the value of a.

(ii) Hence, use the equation to estimate the temperature of the tea after 50 minutes.
What do you think this temperature represents?

PROBLEM 2: The Bottle Tree

The photograph shows a 'bottle tree' (species: brachychiton) inside the Flower Dome at Gardens by the Bay, Singapore. The name comes from its thick trunk and it can grow to a height of 18 to 20 m. Its large span makes transportation difficult.

In this question, the trunk of the 'bottle tree' can be modelled as a frustum attached to the top of a cylinder, as shown in the figure below.



The circumference of the base of the tree trunk is about 2.5 m and the height of the cylindrical section is about 2.2 m. The section modelled by the frustum has an approximate height of 1.8 m and the circumference of the top surface is about 0.95 m. The frustum is obtained by removing a smaller cone of base circumference 0.95 m from a larger cone of base circumference 2.5 m.

(a) Calculate the radius, in metres, of the base of the tree trunk.

(b) Find the volume, in cubic metres, of the tree trunk.

(c) It is believed that the density of the tree trunk is not more than 600 kg/m^3. The tree is to be transported to another location by a truck. Given that the available trucks have maximum load capacities of $\frac{1}{2}$ tonne, 1 tonne, 3 tonnes and 5 tonnes, which is the best possible choice of truck to carry the tree? Show your working and give reasons to justify your answer. (1 tonne = 1000 kg)

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PROBLEM 3: Population Density

The table below shows the net population density of Singapore and of Hong Kong at the start of 2013.

Population Density

Land use	Land area (sq km)	
	Singapore	Hong Kong
Total land area	710	1108
Total undevelopable land area	210	791
Water bodies	37	30
Ports and airport	22	16
Defence requirements	133	-
Woodlands/shrubland/grassland/wetlands	18	738
Barren land	-	7
Total developable land area	500	317
Population	5 300 000	7 000 000
Density (people per sq km)	**10 600**	**22 110**

Problem Solving Tip

In **(i)**, think about significant figures or rounding off errors.

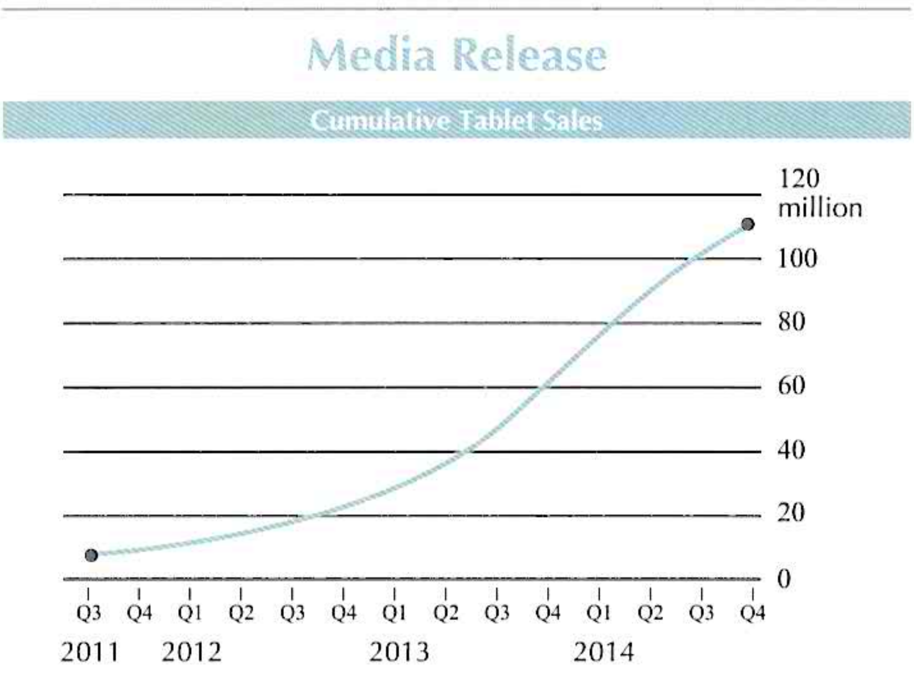
(i) The net population density is obtained by dividing the total population by the total developable land area. Verify the net population density of Singapore and for Hong Kong in the above table. Why is your answer for Hong Kong different from what is stated in the table?

(ii) Another method used to calculate population density is called the gross population density, which is obtained by dividing the total population by the total land area. Find the gross population density of Singapore and of Hong Kong, correct to 2 significant figures. Which city has a higher gross population density?

(iii) Explain whether the net population density or the gross population density is a better measure of the population density of a city.

PROBLEM 4: The Presentation of Data

Statville Technologies is a leading producer of tablet computers. Its flagship product, the 'Statville Tablet' was released in 2011. In its latest media event, the Chief Executive Officer of Statville Technologies proudly announced the sale of its 120 millionth tablet, and that 'quarterly sales performances have been very positive'. The following diagram was shown during the media event.



ATTENTION

A **quarter** is a 3-month period on a financial calendar and it is typically expressed as 'Q'. Company earnings are usually reported quarterly.

Study the diagram and answer the questions below.

(i) Suppose that for a particular quarter, zero sales were made. How will this be reflected on the curve shown in the diagram?

(ii) Does the curve in the diagram support the claim that 'quarterly sales performances have been very positive'? Explain your answer.

(iii) Name one feature of the curve which you think may **not** reflect the actual quarterly sales figures accurately.

(iv) Why do you think that Statville Technologies used a 'cumulative sales curve' to present its sales performance data?

PROBLEM 5: Box Office Earnings

The table below shows the top five worldwide highest grossing movies of all time. The amounts shown are in US dollars and the figures are not adjusted for inflation. The worldwide box office earnings of '\$2 781 606 847' for Blue Planet is calculated using the price of movie tickets in 2009.

ALL-TIME WORLDWIDE BOX OFFICE

Rank	Title	Year of Release	Worldwide Box Office Earnings
1	Blue Planet	2009	\$2 781 606 847
2	The Shipwreck	1997	\$2 185 372 302
3	Superheroes	2012	\$1 515 679 547
4	Platform 9.75	2011	\$1 327 655 619
5	Freeze	2013	\$1 259 136 600

INFORMATION

The box office is a place where tickets of admission are sold to the public.

(a) **(i)** How much more money did Blue Planet earn than The Shipwreck at the box office?

(ii) Is the comparison between Blue Planet and The Shipwreck fair? Why or why not? Explain your answer.

(b) We can also calculate and compare the earnings of the movies after *adjusting for inflation*.

The **inflation rate** is the rate of increase of the ticket prices. Assuming a yearly inflation rate of 3.5% in movie ticket prices, the box office earnings of Superheroes would increase by 3.5% to \$1 515 679 547 × 1.035 = \$1 568 728 331 in 2013.

Calculate the box office earnings of

(i) Blue Planet, **(ii)** The Shipwreck,

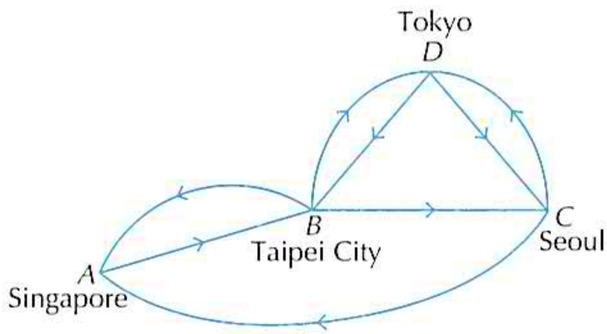
in 2013.

(c) Hence, compare the box office earnings of the two movies, Blue Planet and The Shipwreck in 2013, taking into account a yearly inflation rate of 3.5%. Which movie made more money and by how much? Why is this comparison a fairer one?

(d) The movie, The President, was released in 1991 and made \$205 400 000 at the box office. Give an equation that allows you to calculate the box office earnings of The President in 2013, taking into account a yearly inflation rate of 3.5%.

PROBLEM 6: Incidence Matrices

Sears Airways is a Singapore low-cost carrier (budget airline). On a particular day, Sears Airways has flights connecting four cities, i.e. Singapore, Taipei City, Seoul and Tokyo. The flights can be represented by the following network, with the arrows indicating the direction of the flights.



This **incidence matrix**, **M** (shown below) shows whether it is possible to travel directly from one city to another.

$$\mathbf{M} = \begin{array}{c} \text{End of flight} \\ \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \end{array} \begin{array}{c} \text{Start of flight} \\ A \\ B \\ C \\ D \end{array}$$

(a) By studying both the network and the matrix **M** closely, explain

(i) what the entry '1' in matrix **M** represents,

(ii) why the diagonal of **M** contain only the entries '0'.

(b) Using matrix multiplication, calculate $\mathbf{M}^2$. (Note: $\mathbf{M}^2 = \mathbf{MM}$)
Explain what the entries in the matrix $\mathbf{M}^2$ represent.

(c) Calculate $\mathbf{M}^2 + \mathbf{M}$. This matrix sum does not contain any '0' entry. What is the meaning of the absence of the '0' entry?

(d) What would the matrix $\mathbf{M}^0$ be? Explain your answer.

PROBLEM 7: 4-D Draw

Study the news report and answer the questions.

NEWS REPORT

6904 Strikes Twice!

'The four-digit number, 6904, has won both the first and second prizes in the 4-D draw on June 27, 2007. The probability of a particular number winning any prize in a 4-D draw is one out of 10 000 while the probability of any number appearing twice in the same draw is one out of 10 000 times 10 000, or one in 100 million.'

INFORMATION

Personal gambling habits, if not controlled, leads to problem gambling, which brings hardship to self and family. Stay away from gambling.

(a) A person can bet on any number from 0000 to 9999 in a 4-D draw. Find the probability of a particular number, e.g. 6904, winning the first prize in the draw.

(b) There are 23 prizes to be won in a 4-D draw: First prize, second prize, third prize, 10 starter prizes and 10 consolation prizes. Find the probability of a particular number, e.g. 6904, winning **any** prize in a 4-D draw, leaving your answer correct to 4 significant figures.

(c) The news report wrote, "The probability of a particular number winning any prize in a 4-D draw is one out of 10 000 ..." By comparing your solutions for parts **(a)** and **(b)**, do you agree with this statement in the news report? Explain.

(d) Find the probability of a particular number, e.g. 6904, winning both the first and second prizes in the same 4-D draw.

(e) Find the probability of **any** number winning both the first and second prizes in the same 4-D draw.

(f) The news report wrote, "... the probability of any number appearing twice in the same draw is one out of 10 000 times 10 000, or one in 100 million." By comparing your solutions for parts **(d)** and **(e)**, do you agree with this statement in the news report? Explain.

Problem Solving Tip

For part **(b)**, find the probability that a particular number will *not* appear in the draw first.

For part **(e)**, this is similar to tossing two dice and finding the probability that both dice will show the same number.

PROBLEM 8: Dengue Fever

The table below shows the number of cases of dengue fever in the first 10 weeks of 2012 to 2014.

Week \ Year	2012	2013	2014
1	74	132	436
2	64	204	479
3	60	219	402
4	50	264	336
5	84	292	234
6	87	322	273
7	65	246	369
8	50	294	193
9	55	247	186
10	45	272	209

Source: https://www.moh.gov.sg/content/moh_web/home/statistics/infectiousDiseasesStatistics/weekly_infectiousdiseasesbulletin.html

(a) By looking at the above table (without performing any computation), answer the following questions.

(i) Which year has the least number of cases of dengue fever in the first 10 weeks?

(ii) Which year has the greatest number of cases of dengue fever in the first 10 weeks? Are you able to observe this easily from the above table? What can you do to find the answer?

(b) Go to http://www.shinglee.com.sg/StudentResources/ and open the spreadsheet with the full set of data for all the 52 weeks for 2012 and 2013, and the first 42 weeks for 2014. The chart shows the line graph for the number of cases of dengue fever. It is obvious from the chart that the number of cases for 2012 is much lower than those for 2013 and for 2014. But are you able to tell from the chart whether 2013 or 2014 has more cases of dengue fever for the first 42 weeks? What is an alternative way of presenting the data to allow easier comparison between two or more sets of data?

In the same spreadsheet, insert a 2-D Line and an empty chart will appear. Right click on the chart and choose 'Select data'. Click on Cell E3, hold and drag to Cell G54. This will appear in the Chart Data Range. Select Series 1 under the Legend Entries, choose 'Edit', and change the Series Name to 2012. Similarly, change the name for Series 2 and 3 to 2013 and 2014 respectively. Click 'OK'. This will plot the cumulative frequency curves for the number of cases of dengue fever for 2012 to 2014. Resize the chart if necessary.

(c) Using the chart that you have created above, answer the following questions.

(i) Which year has the greatest number of cases of dengue fever for the first 42 weeks?

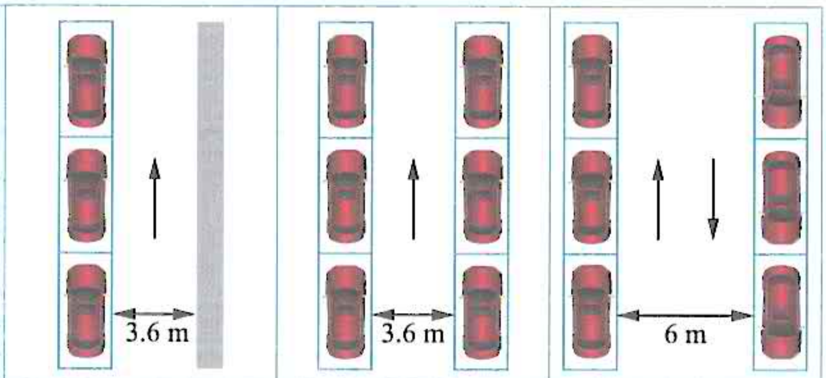
(ii) How do you describe the trend for the number of cases of dengue fever for the first 42 weeks of 2013 and of 2014?

(iii) Can you predict whether there will be more cases of dengue fever for the remaining 10 weeks of 2014 as compared to 2013? Explain.

PROBLEM 9: Car Park Design

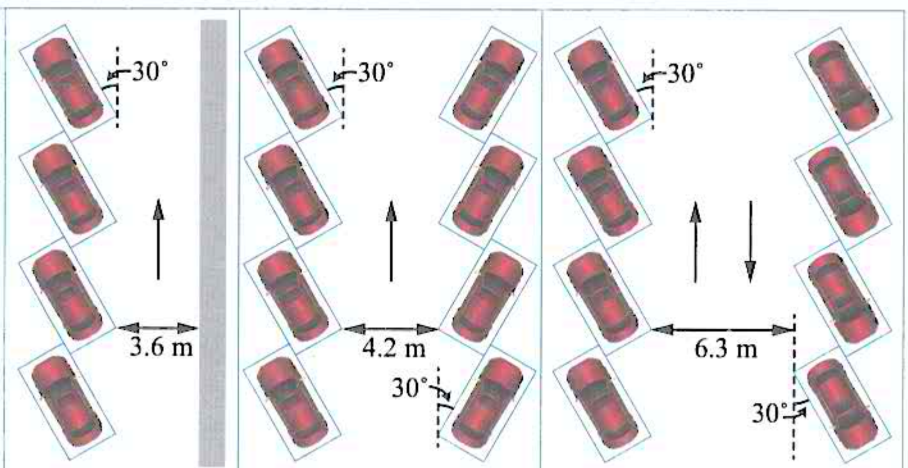
A school is holding an evening concert. The parade square, which is a rectangular plot of land with dimensions 40 m by 30 m, will be converted into a car park to provide additional parking space. You are required to calculate the maximum number of additional car park labels that can be issued.

Parking lots may be arranged to allow for parallel or angled parking. The figure below shows the arrangement of parking lots in parallel parking and the minimum width of the parking aisle. The proposed dimensions of a parking lot for parallel parking are 5.4 m by 2.4 m.



Arrangement of parking lots			
Parking lots	one side	both sides	one or both sides
Traffic flow	one-way	one-way	two-way

For angled parking, the figure below shows the arrangement of parking lots arranged at a parking angle of 30° to the line of traffic flow and the minimum width of the parking aisle. The proposed dimensions of a parking lot for angled parking are 4.8 m by 2.4 m.



Arrangement of parking lots			
Parking lots	one side	both sides	one or both sides
Traffic flow	one-way	one-way	two-way

Your teacher suggests the following guidelines for the minimum width of the parking aisle for the different parking angles, as shown in the table below.

Parking angle	Parking lots on one side, one-way traffic flow	Parking lots on both sides, one-way traffic flow	Parking lots on one or both sides, two-way traffic flow
0° (parallel)	3.6 m	3.6 m	6.0 m
30°	3.6 m	4.2 m	6.3 m
45°	4.2 m	4.8 m	6.3 m
60°	4.8 m	4.8 m	6.6 m
90°	6.0 m	6.0 m	6.6 m

Showing all relevant calculations, suggest the arrangement of car park lots in the parade square so as to maximise the number of lots.
What is the maximum number of additional car park labels that can be issued?

Guiding Questions:

(a) Should the traffic flow be restricted to one way or two ways?

(b) Do the entrance and exit of the car park affect your model?

(c) Does your model include handicap parking lots? If so, how will this affect the design of the car park?

SPECIMEN PAPER

Paper 1 (80 Marks) **Time: 2 hours**

Answer **all** questions.

All workings must be shown clearly. Omission of essential working will result in loss of marks.

1. Express each of the following as a single fraction.

(a) $\frac{7}{9} - \frac{3}{7}$ [1]

(b) $2\frac{4}{9} \div \frac{3}{11}$ [1]

2. An area of 36 cm^2 on a map represents an actual area of 144 km^2.

(a) Express the scale of the map in the form $1 : n$, where n is an integer. [2]

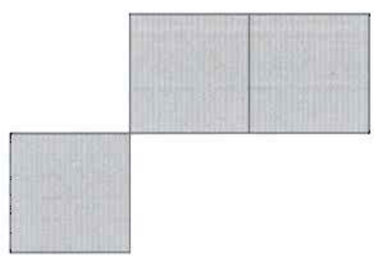
(b) Find the distance, in km, represented by a railway line of length 24 cm on the map. [2]

3. Factorise completely

(a) $3bx - 6ay - 3ab + 6xy$, [2]

(b) $x^4 - 10x^2 + 9$. [2]

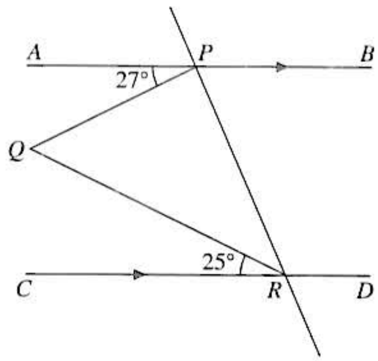
4. **(a)** Complete the diagram below by adding one more square to make a plane figure that has only one line of symmetry. Draw the line of symmetry on the final diagram. [2]



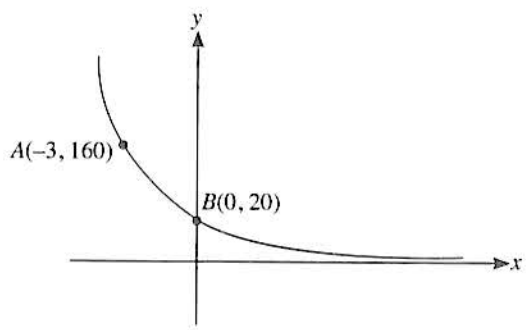
(b) Tony has 200 one-centimetre cubes. He forms a cuboid with all the cubes. The perimeter of the top of the cuboid is 18 cm. Each side of the cuboid has a length greater than 1 cm. Find the height of the cuboid.

[2]

5. In the figure, $AB \parallel CD$, $\angle APQ = 27°$ and $\angle CRQ = 25°$. Find the reflex angle PQR. [2]



6. The diagram shows the graph of $y = \frac{h}{a^x}$. The points $A(-3, 160)$ and $B(0, 20)$ lie on the graph. Find the value of h and of a. [2]



7. **(a)** Simplify $9a^{\frac{3}{4}}b^{\frac{4}{5}} \times \frac{b^{\frac{3}{4}}}{(-3a)^3}$ and express your answer in positive indices. [2]

(b) Is it possible to construct a regular polygon with an interior angle of 128°? Explain your answer clearly. [2]

8. When a shaft of a machine is turning at its maximum constant speed, the horsepower that it transmits is directly proportional to its diameter. If a 8 cm shaft turning at its maximum constant speed transmits 128 horsepower, how many horsepower can a 12 cm shaft turning at its maximum constant speed transmit? [2]

9. **(a)** Solve the inequalities $x + 5 < 37 - 1\frac{1}{2}x \leqslant 20$. [2]

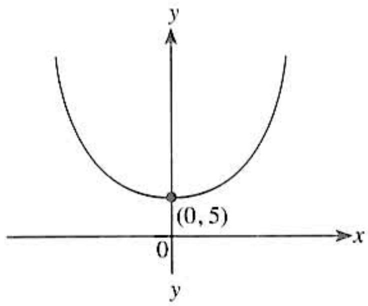
(b) Hence write down

(i) the least value of x, [1]

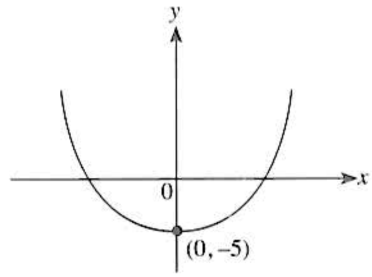
(ii) the greatest integer value of x. [1]

10. **(a)** Which of the following is the graph of $y = 5 - x^2$? [1]

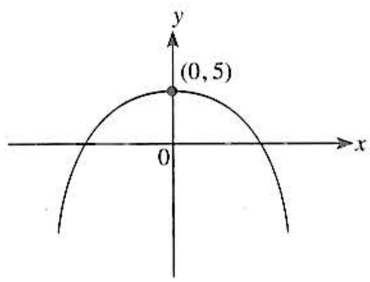
I



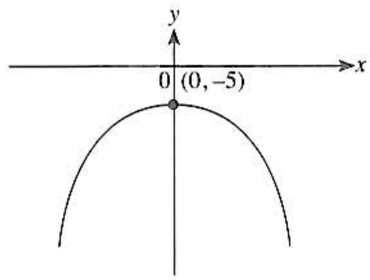
II



III



IV



(b) **(i)** Express $x^2 + 6x + 7$ in the form of $(x + p)^2 + q$. [1]

(ii) Sketch the graph of $y = x^2 + 6x + 7$, indicating the coordinates of the minimum point clearly. [2]

11. In the scale drawing, the quadrilateral $ABCD$ (vertices taken in an anticlockwise direction) is a field of a farmer. $AB = 12.5$ cm, $\angle ABC = 95°$, $BC = 10$ cm, $AD = 11$ cm and $CD = 9.5$ cm.

The scale is 1 cm to 2 km.

(a) Complete the quadrilateral $ABCD$. [2]

(b) Construct

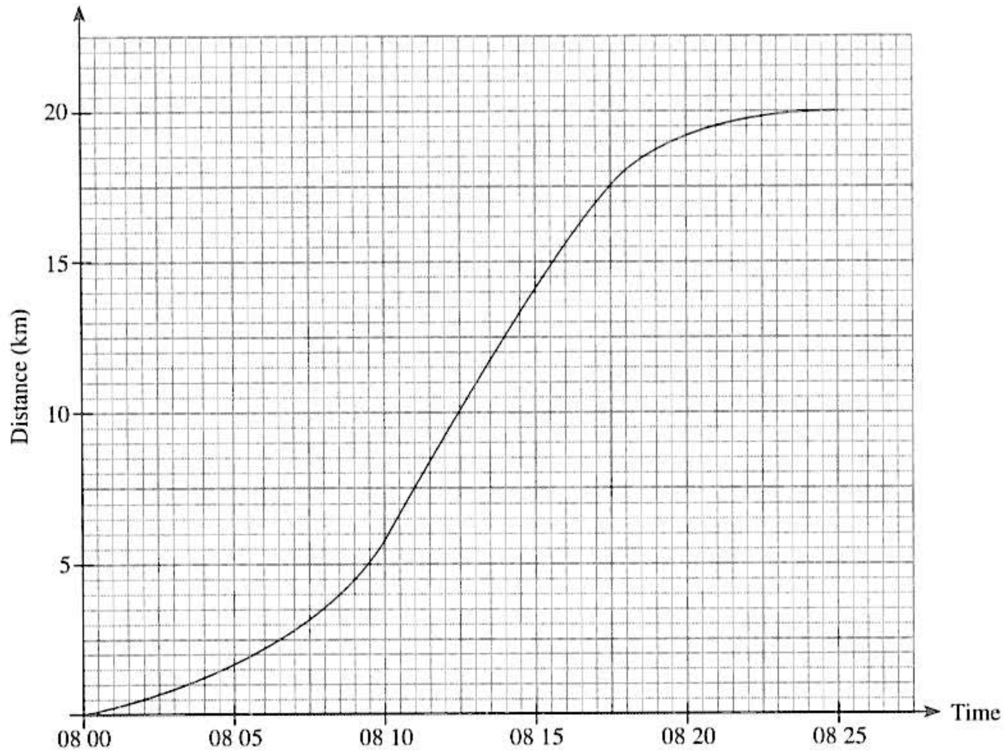
(i) the perpendicular bisector of AB, [1]

(ii) the bisector of $\angle BAD$. [1]

(c) The two lines drawn in **(b)** meet at the point T. The farmer wishes to place a scarecrow at point T. Measure and write down the actual distance the farmer has to walk from A before reaching point T.

[2]

12. The distance-time graph for a train journey from Singapore to Johor is as shown. The train started its journey at 08 00.



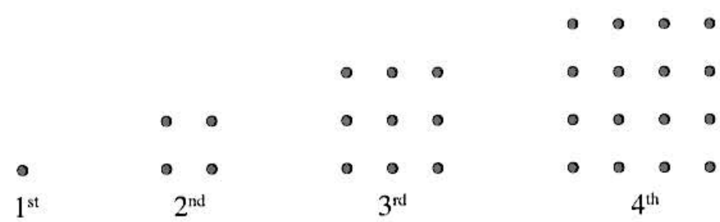
(a) Find the distance the train had travelled by 08 12. [1]

(b) Between what times was the train moving at its greatest speed? [1]

(c) What is the speed of the train at 08 08? [1]

(d) Give a brief description of the motion of the train between 08 17 and 08 25. [2]

13. The diagram shows a sequence of dots drawn on a piece of paper.



Find the difference in the number of dots between

(a) the 4th and 5th patterns, [1]

(b) the 50th and the 51st pattern, [1]

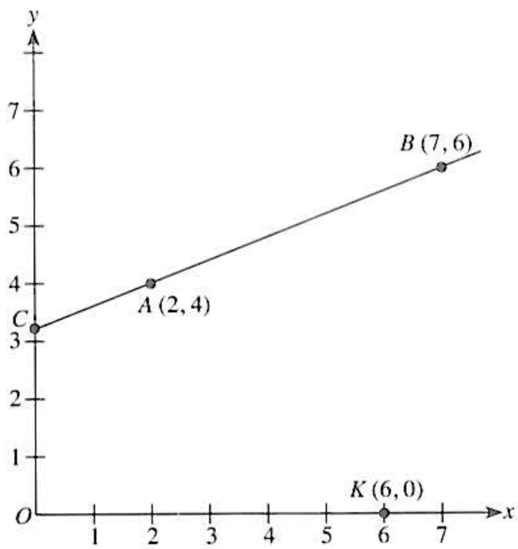
(c) the n^{th} and the $(n + 1)^{\text{th}}$ pattern. [2]

14. $\overrightarrow{AB} = \begin{pmatrix} -6 \\ 8 \end{pmatrix}$

(a) Find $|\overrightarrow{AB}|$. [1]

(b) The line AB cuts the y-axis at $(0, 5)$. Find the equation of the line AB. [2]

15. The coordinates of A and B are $(2, 4)$ and $(7, 6)$ respectively. BA is produced to cut the y-axis at C.



Calculate

(a) the coordinates of C, [1]

(b) the equation of AB, [1]

(c) the area of the quadrilateral $OCAK$, where K is the point $(6, 0)$. [2]

16. ξ, A, B and C are defined as:

ξ = {natural numbers less than 12}

A = {prime numbers}

B = {even numbers}

C = {natural numbers divisible by 3}

(a) Draw a Venn diagram to illustrate the relationship among the given sets. [2]

(b) List the elements contained in the set $(B \cap C)$. [1]

(c) Write down the elements in the set $(A \cap B')$. [1]

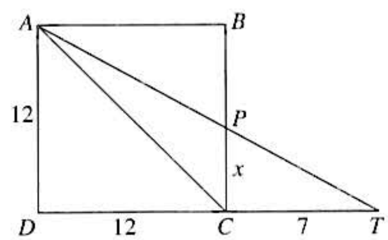
(d) Describe, as simply as possible, in words, the elements contained in the set $B \cap C$. [1]

17. Given that $y = \sqrt{\dfrac{4 + 5x}{3x - 11}}$,

(a) find the value of y when $x = 9$. [1]

(b) express x in terms of y. [2]

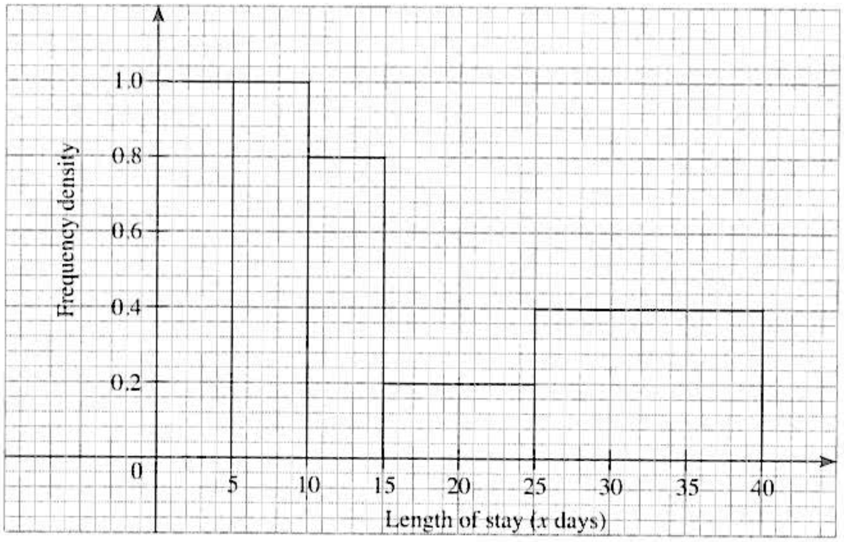
18. In the diagram, $ABCD$ is a square of side 12 cm. DC is extended to T such that $CT = 7$ cm. AT cuts BC at P and $PC = x$ cm.



(a) Name a pair of similar triangles and hence, or otherwise, calculate the value of x. [2]

(b) Calculate the value of tan $\angle APC$. [2]

19. The histogram illustrates the length of stay (in days) in Australia of a group of Singaporean tourists this year.



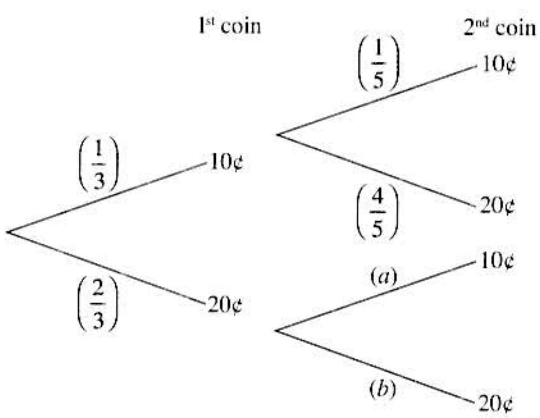
(a) Copy and complete the following table. [2]

Length of stay (x days)	$0 < x \leqslant 5$	$5 < x \leqslant 10$	$10 < x \leqslant 25$	$25 < x \leqslant 40$
Number of tourists	20	20		

(b) Find the fraction of tourists who stayed in Australia longer than 15 days. [2]

(c) Calculate an estimate of the average length of stay of this group of tourists, giving your answer correct to the nearest whole number. [3]

20. Kumar has two 10¢ coins and four 20¢ coins in his pocket. He takes two coins out of his pocket at random, one after the other. The tree diagram shows the possible outcomes and their probabilities.



(a) Find the value of a and of b. [2]

(b) Find the probability that the value of the two coins taken out is exactly 30 cents. [2]

(c) Kumar then takes out a third coin. Find, showing your working clearly, the probability that the total value of the three coins taken out is

(i) exactly 60 cents, [1]

(ii) not more than 45 cents. [2]

Paper 2 (100 marks) **Time: 2 hours 30 minutes**

Answer **all** questions.

Electronic calculators can be used in this paper.

All workings must be shown clearly. Omission of essential working will result in loss of marks.

1. A soft drink wholesaler supplies different types of drinks to three different coffee shops. The number of bottles of drinks supplied and the price per bottle are given in the table below.

	Drink C	Drink P	Drink S	Drink H
Coffee shop A	480	540	380	540
Coffee shop B	360	380	420	420
Coffee shop C	420	450	390	610
Price per bottle	32¢	31¢	33¢	35¢

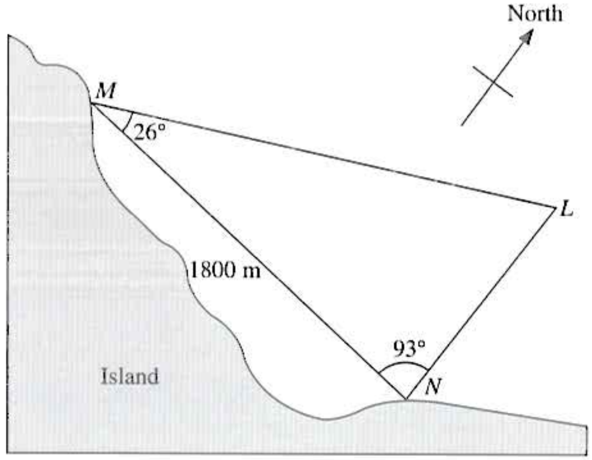
The delivery order for the above is represented by $\mathbf{D} = \begin{pmatrix} 480 & 540 & 380 & 540 \\ 360 & 380 & 420 & 420 \\ 420 & 450 & 390 & 610 \end{pmatrix}$.

(a) Given that $\mathbf{E} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, calculate $\mathbf{P} = \mathbf{DE}$ and explain what the elements in $\mathbf{P}$ represent. [2]

(b) Given that $\mathbf{F} = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$, calculate $\mathbf{Q} = \mathbf{FD}$ and explain what the elements in $\mathbf{Q}$ represent. [2]

(c) Given that $\mathbf{G} = \begin{pmatrix} 32 \\ 31 \\ 33 \\ 35 \end{pmatrix}$, calculate $\mathbf{S} = \mathbf{DG}$ and explain what the elements in $\mathbf{S}$ represent. [2]

2. The diagram shows part of the coast of a small island. The point M represents the position of Kampong Melayu, point N represents the position of Nurul Village and point L represents the position of Longlife lighthouse. It is given that $MN = 1800$ m, $\angle LMN = 26°$ and $\angle MNL = 93°$.



(a) Calculate, correct to the nearest 5 m, the distance LM. [2]

(b) A boat sails directly from M to L. Calculate, correct to the nearest 5 m, its closest distance to Nurul Village. [2]

(c) The boat sailing from M to L is moving at an average speed of 8.6 km/h. Calculate the time taken, correct to the nearest minute, for it to reach L. [1]

(d) Given that L is due north of N, calculate the bearing of

(i) M from N, [1]

(ii) L from M. [2]

3. Mr Tan makes a road trip from Singapore to Malacca via the Second Link at Tuas.

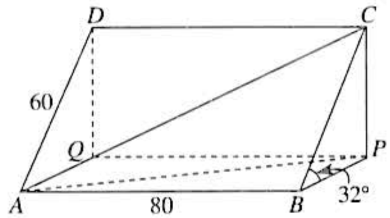
(a) Mr Tan finds that the trip of 210 km will take him x hours. Write down an expression for the average speed of the journey. [1]

(b) He finds that if he has to reduce his average speed by 4 km/h for the whole journey, he would need to take an extra 10 minutes. Form an equation in x and show that it reduces to $12x^2 + 2x - 105 = 0$. [3]

(c) Solve the equation $12x^2 + 2x - 105 = 0$ to find the time taken to travel from Singapore to Malacca and find the original speed of the trip. [3]

(d) Mr Tan fills up his petrol tank and pays a total of RM132 for 48 litres of petrol. Find the cost of 1 litre of petrol in Singapore dollars when the exchange rate is S\$1 = RM3.065. Give your answer to the nearest cent. [2]

4. In the diagram, $ABCD$ represents the rectangular sloping surface of a desk. $ABPQ$ is a rectangle and CP and DQ are vertical lines.



Given that $AB = DC = QP = 80$ cm, $BC = AD = 60$ cm, $\angle CBP = \angle DAQ = 32°$, calculate

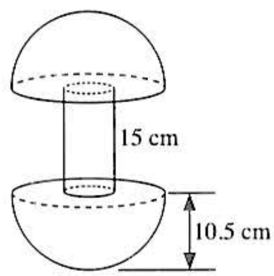
(a) CP, [2]

(b) AC, [2]

(c) $\angle CAP$, [2]

(d) $\angle PAQ$. [2]

5. **(a)** The diagram shows a solid made up of two identical hemispheres and a right cylinder.



The radii of the hemispheres are 10.5 cm. The cylinder has a radius of 3.5 cm and a length of 15 cm. Find

(i) the volume of the solid, [2]

(ii) the total surface area of the solid. [4]

[Give your answer in terms of π.]

(b) Fig. 1 shows an inverted cone of height h and radius r. It contains water to a depth of $\frac{1}{2}h$.

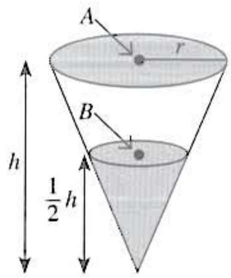


Fig. 1

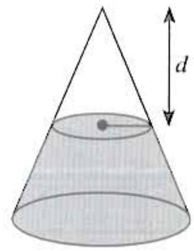


Fig. 2

(i) Find the ratio of area of surface B to the area of surface A. [1]

(ii) Find the volume of the water if the cone can hold 480 cm^3 of water when full. [2]

(iii) The cone is now inverted such that the liquid rests on the flat circular base of the cone as shown in Fig. 2. Find, in terms of h, an expression for d, the distance of the liquid surface from the vertex of the cone. [2]

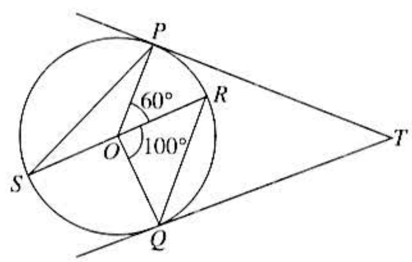
6. In the diagram, TP and TQ are tangents to the circle, centre O. Given that SOR is a straight line, $\angle POR = 60°$ and $\angle QOR = 100°$, calculate, stating reasons clearly,

(a) $\angle PTQ$, [1]

(b) $\angle PSR$, [1]

(c) $\angle RPT$, [2]

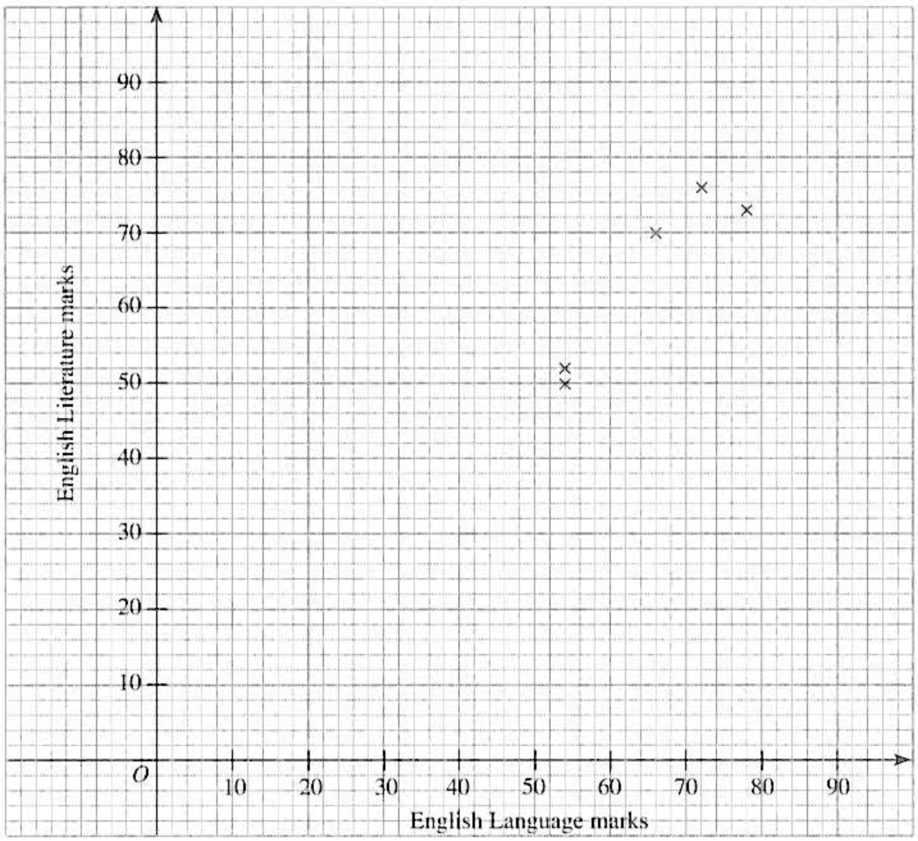
(d) $\angle RQT$. [2]



7. The table below shows the marks scored by 12 pupils in English Language and English Literature examinations.

Pupil	*A*	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*	*J*	*K*	*L*
English Language	72	66	54	78	54	79	81	73	44	61	82	48
English Literature	76	70	52	73	50	72	84	69	47	68	79	52

(i) On the grid, complete the scatter diagram to show the English Language and English Literature marks of the 12 pupils. The first five points have been plotted for you. [2]



(ii) What type of correlation is shown in the scatter diagram? [1]

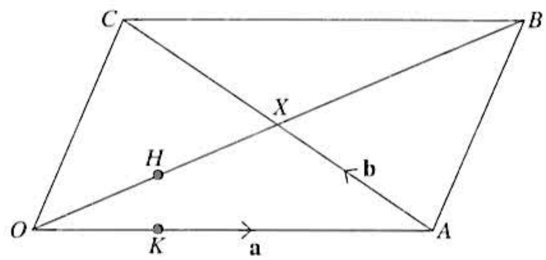
(iii) Draw a line of best fit on the scatter diagram. [1]

(iv) Use your line of best fit to estimate the number of marks that a pupil who scored 75 in English Language is likely to score in English Literature. [1]

(v) Use your line of best fit to estimate the number of marks that a pupil who scored 58 in English Literature is likely to score in English Language. [1]

(vi) Find the gradient of the line of best fit and hence find the equation of the line of best fit in the form of $y = mx + c$, where m represents gradient and c represents y-intercept. [2]

8.



In the diagram, $OABC$ is a parallelogram. OB and AC intersect at X, H is the midpoint of OX and K is the point on OA such that $\overrightarrow{KA} = 2\overrightarrow{OK}$. Given that $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{AX} = \mathbf{b}$, express each of the following in terms of $\mathbf{a}$ and/or $\mathbf{b}$.

(a) $\overrightarrow{OX}$ [1]

(b) $\overrightarrow{AB}$ [1]

(c) $\overrightarrow{CH}$ [1]

(d) $\overrightarrow{HK}$ [1]

Using your answers to **(c)** and **(d)**, state two facts about the points C, H and K. [2]

9. The total amount a household has to pay for its utilities bill is calculated by adding a fixed charge and the cost of the quantity of water and electricity used. A goods and services tax (GST) of 7% is also applicable for the services rendered.

(a) The fixed charge that Mr Singh has to pay for his house consists of domestic refuse removal charge of \$14.40 per month and sanitary appliance fee at \$3 per appliance per month. If Mr Singh has two sanitary appliances in his house, calculate the total fixed charge he has to pay for the whole year. [2]

(b) In January 2006, Mr Singh used 525 kilowatt-hours of electricity and 12.5 m^3 of water. The charges for electricity were 18.64 cents per kilowatt-hour and water was set at the rate of \$1.05 per cubic metre. Calculate the total cost, correct to the nearest cent, for the electricity and water bills for January 2006. [2]

(c) Mr Sidek lives in a private house which has four sanitary appliances and the domestic refuse charge for his house is \$22.70 per month. In January 2006, he used 954 kilowatt-hours of electricity and 18.6 m^3 of water. Calculate the total amount of utilities charges he had to pay in January 2006. [3]

(d) In January 2007, the service provider revised its rates. The domestic charge is to be increased by 10%. Electric charges were 19.87 cents per kilowatt-hour while water was charged at \$1.12 per m^3 for quantities less than 15 m^3 and at \$1.26 for quantities exceeding 15 m^3. Mr Sidek wishes to cut down on his consumption so that he will not pay more than the amount he paid in January 2006. Assuming that there is no change in the sanitary appliance fee, suggest to Mr Sidek a possible amount of electricity and water that he can use up to so as not to exceed the amount. [3]

10. The percentage of the maximum possible amount of chlorophyll produced by a leaf in the first 5 months of the year is given by $y_1 = 100 - 4(t - 5)^2$, where t is the time in months. The table below shows the percentage of the maximum possible amount of chlorophyll produced by the leaf in the first 5 months of the year.

Time (t months)	0	1	2	3	4	5
y_1 (%)	0	36	64	p	96	100

(a) Find the value of p. [1]

(b) Using a horizontal scale of 2 cm to represent 1 month and a vertical scale of 2 cm to represent 20%, draw the graph of $y_1 = 100 - 4(t - 5)^2$. [3]

(c) Use your graph to find

(i) the value of y_1 when $t = 2.4$, [1]

(ii) the value of t when $y_1 = 75$. [1]

(d) **(i)** By drawing a tangent, find the gradient of the curve when $t = 3.5$. [2]

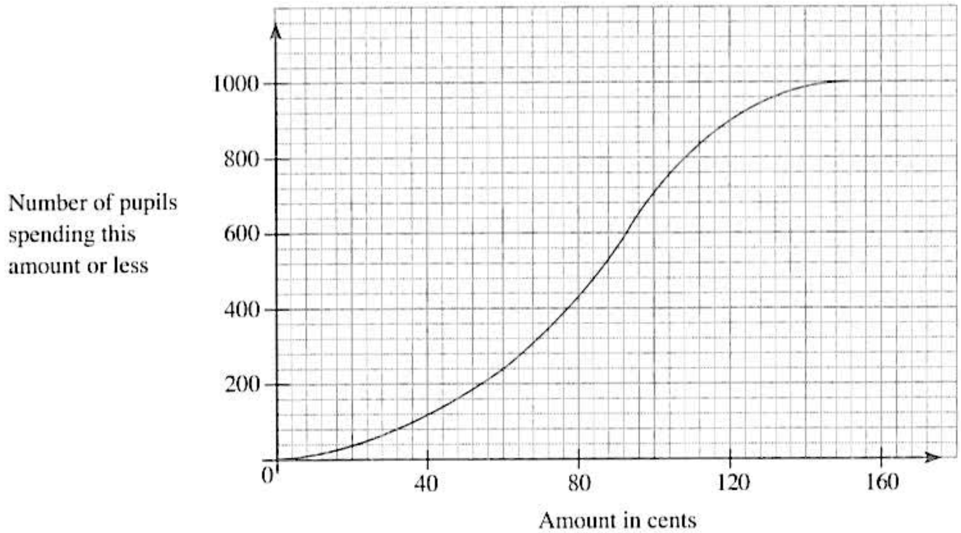
(ii) State what the value of the gradient represents. [1]

(e) The percentage of the maximum possible amount of chlorophyll produced by another leaf in the first 5 months of the year is given by $y_2 = 12t + 40$.

(i) On the same axes, draw the graph $y_2 = 12t + 40$. [2]

(ii) Find the values of t for which the percentage of the maximum possible amount of chlorophyll produced by the two leaves will be the same. [1]

11. The cumulative frequency curve shows the daily expenses of 1000 pupils in secondary school *ABC* during recess.



(a) Use the graph to estimate

(i) the median, [2]

(ii) the interquartile range. [3]

(b) If a pupil is selected at random, find the probability that he spends

(i) 80 cents or less, [1]

(ii) more than 50 cents but not more than 90 cents. [2]

12. Answer the whole of this question on a sheet of graph paper.

(a) Using a scale of 1 cm to represent 1 unit on each axis, draw x- and y-axes for $0 \leqslant x \leqslant 14$ and $-6 \leqslant y \leqslant 14$. Draw and label the rectangle whose vertices are $A(1, 2)$, $B(3, 2)$, $C(3, 3)$ and $D(1, 3)$. [1]

(b) The enlargement E has the origin as its centre and maps rectangle $ABCD$ onto $A_1B_1C_1D_1$. Given that A_1 is the point $(4, 8)$,

(i) draw and label the rectangle $A_1B_1C_1D_1$, [1]

(ii) write down the scale factor of E. [1]

(c) The point $C_2(7, -1)$ is the image of C under a reflection in the line m. Draw and label the line m and find its equation. [2]

(d) The transformation R, a clockwise rotation of 90° about the origin, maps rectangle $ABCD$ onto rectangle $A_4B_4C_4D_4$. Draw and label rectangle $A_4B_4C_4D_4$, and find the matrix represent R. [3]

Practise Now Answers

CHAPTER 1 – LINEAR INEQUALITIES IN TWO VARIABLES

Practise Now 2

$y < x, x + 2y \leqslant 8$

Practise Now 3

max = 50, min = 30

Practise Now 4

(a) $x + y \leqslant 1000, y \geqslant 2x, x \geqslant 100, y \leqslant 800$

(c) 330 cans of *Coola* and 670 cans of *Shiok*

CHAPTER 2 – FURTHER SETS

Practise Now 1

7

Practise Now 2

(i) 2 **(ii)** 4
(iii) 4 **(iv)** 42

Practise Now 3

(i) $\{-20, 20\}$ **(ii)** $\{8\}$
(iii) $\{x : -15 < x \leqslant 12\}$ **(iv)** $\varnothing$

Practise Now 4

$a = 5, b \neq -2$

Practise Now (Page 30)

3

Practise Now 5

(i) 32 **(ii)** 23

CHAPTER 3 – PROBABILITY OF COMBINED EVENTS

Practise Now 1

(a) $\{22, 23, 25, 32, 33, 35, 52, 53, 55\}$

(b) **(i)** $\frac{2}{9}$ **(ii)** $\frac{5}{9}$
(iii) $\frac{2}{9}$ **(iv)** $\frac{1}{9}$
(v) $\frac{8}{9}$

Practise Now 2

(i) $\frac{1}{3}$ **(ii)** $\frac{1}{3}$
(iii) 0 **(iv)** $\frac{2}{3}$

Practise Now 3

1. **(b)** **(i)** $\frac{1}{6}$ **(ii)** $\frac{5}{12}$
(iii) $\frac{1}{4}$

2. **(i)** $\frac{2}{5}$ **(ii)** $\frac{6}{25}$
(iii) $\frac{8}{25}$

Practise Now 4

1. **(a)** 2, 6, 7; 3, 4, 7, 8; 4, 5, 9; 5, 6, 9, 10; 6, 7, 10, 11; 8, 11, 12

1, 2, 5, 6; 2, 10, 12; 3, 6, 15; 4, 8, 20, 24; 5, 10, 25, 30; 6, 12, 36

(b) **(i)** $\frac{1}{2}$ **(ii)** $\frac{1}{3}$
(iii) $\frac{1}{6}$ **(iv)** 0

(c) **(i)** $\frac{1}{4}$ **(ii)** $\frac{1}{3}$
(iii) $\frac{5}{24}$ **(iv)** 1

2. **(a)** **(i)** $\frac{1}{9}$ **(ii)** $\frac{5}{9}$
(b) **(ii)** $\frac{1}{3}$ **(iii)** $\frac{8}{9}$

Practise Now 5

1. **(i)** $\frac{3}{8}$ **(ii)** $\frac{7}{8}$

2. **(b)** **(i)** $\frac{5}{8}$ **(ii)** $\frac{1}{4}$
(iii) $\frac{1}{2}$ **(iv)** $\frac{1}{8}$

Practise Now 6

(i) $\frac{4}{13}$ **(ii)** $\frac{4}{13}$
(iii) $\frac{2}{13}$ **(iv)** $\frac{11}{13}$

Practise Now 7

(i) $\frac{11}{30}$ **(ii)** $\frac{73}{168}$
(iii) $\frac{307}{840}$

Practise Now 8

(i) $\frac{7}{12}$ **(ii)** $\frac{5}{12}$
(iii) $\frac{25}{144}$ **(iv)** $\frac{5}{12}$
(v) $\frac{49}{144}$

Practise Now 9

1. **(i)** $\frac{2}{9}$ **(ii)** $\frac{1}{9}$

2. **(i)** 0.035 **(ii)** 0.065
(iii) 0.38

Practise Now 10

1. **(i)** $\frac{16}{63}$ **(ii)** $\frac{32}{63}$
(iii) $\frac{43}{63}$

2. **(i)** $\frac{7}{30}$ **(ii)** $\frac{7}{15}$
(iii) $\frac{49}{120}$

CHAPTER 4 – STATISTICAL DATA ANALYSIS

Practise Now (Page 76)

(b) **(i)** 32 **(ii)** 18
(iii) 28

Practise Now 1

(a) 35, 55, 65, 70

(c) **(i)** 50 **(ii)** $\frac{4}{7}$
(iii) 6.4

Practise Now 2

(i) 180 **(ii)** $\frac{4}{5}$
(iii) 29.8

Practise Now 3

1. (i) 14, 30.5, 44
   (ii) 49
   (iii) 30
2. (i) $Q_1 = 15.5, Q_2 = 26, Q_3 = 41$
   (ii) 45
   (iii) 25.5

Practise Now 4

(i) 50, 17, 81
(ii) 64
(iii) 10^{th} percentile = 8
80^{th} percentile = 84
(iv) 34

Practise Now 5

(a) (i) 62 (ii) 18
(b) (i) 70 (ii) 24
(c) School *B*
(d) School *A*

Practise Now 7

(a) (i) 28 (ii) 62
(iii) 14

Practise Now 8

(a) (i) 2500 (ii) 4700
(iii) 1500
(b) (i) 5000 (ii) 5500
(iii) 2200
(c) Brick *B*

Practise Now 9

7.18

Practise Now (Page 117)

3.74

Practise Now 10

(a) (i) 9.2 (ii) 4.18

Practise Now (Page 121)

12.1

CHAPTER 5 – MATRICES

Practise Now (Page 130)

1. (a) 2×4 (b) 2×2
   (c) 3×1 (d) 1×4
   (e) 1×1 (f) 1×1

2. (i) $\mathbf{M} = \begin{pmatrix} 14 & 3 & 5 & 2 \\ 1 & 8 & 3 & 4 \end{pmatrix}$
   (ii) 5
   (iii) 15
   (iv) 16

Practise Now 1

$a = 2, b = 3, c = 0, d = -3$

Practise Now 2

1. (a) $\begin{pmatrix} 5 & 5 \\ 8 & 13 \end{pmatrix}$
   (b) $\begin{pmatrix} -2 & 11 \\ 22 & -1 \end{pmatrix}$
   (c) Not possible
2. (i) $\mathbf{Q} = \begin{pmatrix} 46 & 40 & 31 \\ 42 & 38 & 35 \end{pmatrix}$
   (ii) $\begin{pmatrix} 95 & 68 & 70 \\ 89 & 83 & 56 \end{pmatrix}$

Practise Now 3

1. (i) $\begin{pmatrix} 14 & 11 \\ -9 & 0 \\ -5 & 46 \end{pmatrix}$
   (ii) $\begin{pmatrix} 10 & 31 \\ 9 & -54 \\ -55 & 20 \end{pmatrix}$
2. (a) $x = -7, y = 4$
   (b) $x = 5\frac{1}{2}, y = 22$

Practise Now 4

(i) $\begin{array}{cc} \begin{array}{cc} \text{Adults} & \text{Children} \end{array} & \\ \begin{pmatrix} 75 & 125 \\ 105 & 40 \end{pmatrix} & \begin{array}{l} \text{Morning} \\ \text{Afternoon} \end{array} \end{array}$

(ii) $\mathbf{E} = \begin{pmatrix} 14 & 10 \\ 18 & 7 \end{pmatrix}$

(iii) $\begin{pmatrix} 89 & 135 \\ 123 & 47 \end{pmatrix}$

Practise Now 5

(a) $\begin{pmatrix} 128 & -18 \\ 140 & 111 \end{pmatrix}$

(b) $\begin{pmatrix} 71 & -12 & -52 \\ -80 & 15 & 55 \\ 0 & -63 & 153 \end{pmatrix}$

(c) $\begin{pmatrix} -13 & 28 \end{pmatrix}$

(d) Not possible

(e) $\begin{pmatrix} -49 & 14 & 21 \\ -56 & 16 & 24 \\ 35 & -10 & -15 \end{pmatrix}$

(f) (-48)

Practise Now (Page 158)

6

Practise Now 7

$\mathbf{A}^{-1} = \begin{pmatrix} -\frac{2}{9} & \frac{1}{3} \\ \frac{7}{9} & -\frac{2}{3} \end{pmatrix}, \mathbf{B}^{-1} = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{6} \\ \frac{3}{2} & \frac{1}{6} \end{pmatrix}$

Practise Now 8

$\mathbf{A}^{-1} = \begin{pmatrix} -\frac{3}{8} & \frac{1}{4} \\ -\frac{7}{8} & \frac{1}{4} \end{pmatrix}$

(i) $\begin{pmatrix} -\frac{3}{2} & -\frac{13}{8} \\ -\frac{9}{2} & -\frac{33}{8} \end{pmatrix}$ (ii) $\begin{pmatrix} -\frac{53}{8} & \frac{11}{4} \\ -2 & 1 \end{pmatrix}$

Practise Now 9

1. (i) 20
   (ii) $\begin{pmatrix} 28 \\ 21 \end{pmatrix}$
2. (i) $\begin{pmatrix} 70 & 120 & 90 & 80 \\ 120 & 0 & 150 & 140 \\ 0 & 150 & 85 & 60 \\ 200 & 140 & 70 & 0 \\ 80 & 110 & 0 & 95 \end{pmatrix} \begin{pmatrix} 35 \\ 40 \\ 45 \\ 38 \end{pmatrix}$
   $= \begin{pmatrix} 14\,340 \\ 16\,270 \\ 12\,105 \\ 15\,750 \\ 10\,810 \end{pmatrix}$

(ii) $\begin{pmatrix} 45 & 42 & 38 & 55 & 52 \end{pmatrix}\begin{pmatrix} 70 & 120 & 90 & 80 \\ 120 & 0 & 150 & 140 \\ 0 & 150 & 85 & 60 \\ 200 & 140 & 70 & 0 \\ 80 & 110 & 0 & 95 \end{pmatrix}$

$= \begin{pmatrix} 23\,350 & 24\,520 & 17\,430 & 16\,700 \end{pmatrix}$

Total number of otahs = 82 000

(iii) $\begin{pmatrix} 45 & 42 & 38 & 55 & 52 \end{pmatrix}\begin{pmatrix} 14\,340 \\ 16\,270 \\ 12\,105 \\ 15\,750 \\ 10\,810 \end{pmatrix}$

$= \begin{pmatrix} 3\,217\,000 \end{pmatrix}$

Practise Now 10

(a) $x = 2$, $y = 3$, intersect at (2, 3)
(b) No solution, parallel lines
(c) Infinite solutions, same line

CHAPTER 6 – MATRICES OF TRANSFORMATION

Practise Now 3

(a) (–1, 0) **(b)** 2

Practise Now 4

(a) (1, 1) **(b)** 135 units^2

Practise Now 5

reflection in the line $y + x = 0$

Practise Now 6

90° clockwise rotation about the origin

Practise Now 7

$\begin{pmatrix} 3 & 4 \\ 1 & 3 \end{pmatrix}$

Practise Now 8

$A'(3, 3)$, $B'(15, 12)$, $C'(9, 18)$

Practise Now 10

(a) (–3, –2) **(b)** (1, 3)

Practise Now 11

A reflection in $y = 4$ followed by an enlargement scale factor $1\frac{1}{2}$ and centre (1, 4).

Practise Now 12

RM is the reflection in the line $y + x = 0$

CHAPTER 7 – VECTORS

Practise Now (Page 221)

$\overrightarrow{AB} = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$; 5 units

$\mathbf{c} = \begin{pmatrix} -2 \\ -2 \end{pmatrix}$; 2.83 units

$\overrightarrow{DE} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$; 5 units

$\mathbf{f} = \begin{pmatrix} 0 \\ -3 \end{pmatrix}$; 3 units

Practise Now 2

(a) (i) $x = 4, y = 4\frac{1}{2}$

(ii) $\begin{pmatrix} -6 \\ \frac{1}{2} \end{pmatrix}$

(b) (i) $y = \frac{105 - 24x}{2}$

Practise Now 3

(i) $\overrightarrow{PB} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$

(ii) $\overrightarrow{BQ} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

(iii) $\overrightarrow{PR} = \begin{pmatrix} -4 \\ -1 \end{pmatrix}$

(iv) Yes.
$\overrightarrow{PQ} \neq \overrightarrow{PR}$; Different direction.

Practise Now 4

1. (ii) $\mathbf{a} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$; $\mathbf{b} = \begin{pmatrix} 2 \\ -4 \end{pmatrix}$;

$\mathbf{a} + \mathbf{b} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$

(iv) $|\mathbf{a}| = \sqrt{34} = 5.83$ units
$|\mathbf{b}| = \sqrt{20} = 4.47$ units
$|\mathbf{a} + \mathbf{b}| = \sqrt{26} = 5.10$ units

2. (i) $\begin{pmatrix} 8 \\ 12 \end{pmatrix}$ **(ii)** $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$

Practise Now 5

(i) $\overrightarrow{PR}$ **(ii)** $\overrightarrow{PR}$
(iii) $\overrightarrow{PQ}$

Practise Now 6

(a) $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ **(b)** $\begin{pmatrix} -6 \\ -7 \end{pmatrix} + \begin{pmatrix} 6 \\ 7 \end{pmatrix}$

Practise Now 7

(ii) $\mathbf{r} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$, $\mathbf{s} = \begin{pmatrix} 6 \\ 2 \end{pmatrix}$, $\mathbf{r} - \mathbf{s} = \begin{pmatrix} -3 \\ 3 \end{pmatrix}$

(iii) $|\mathbf{r}| = 5.83$, $|\mathbf{s}| = 6.32$, $|\mathbf{r} - \mathbf{s}| = 4.24$

(iv) No

Practise Now 8

(a) $\mathbf{b} - \mathbf{a}$ **(b)** $\mathbf{a} - \mathbf{b}$
(c) $\mathbf{m} - \mathbf{n}$ **(d)** $\mathbf{v} + \mathbf{w}$
(e) $-\mathbf{v} - \mathbf{w}$

Practise Now 9

(a) $\mathbf{q}$ **(b)** $-\mathbf{p}$
(c) $\mathbf{q} + \mathbf{p}$ **(d)** $\mathbf{q} - \mathbf{p}$
(e) $\mathbf{p} - \mathbf{q}$

Practise Now 10

(a) $\overrightarrow{AC}$ **(b)** $\overrightarrow{CB}$
(c) Cannot simplify. **(d)** $\overrightarrow{RQ}$
(e) $\overrightarrow{PR}$ **(f)** $\overrightarrow{SQ}$

Practise Now 11

(a) (i) $\begin{pmatrix} -7 \\ 9 \end{pmatrix}$ **(ii)** $\begin{pmatrix} -3 \\ 0 \end{pmatrix}$

(b) (i) $x = 13, y = -10$
(ii) $x = -1, y = -10$

Practise Now 12

1. (a) (i) Parallel
(ii) Not parallel
(iii) Parallel

(b) Same direction: $\begin{pmatrix} 8 \\ -6 \end{pmatrix}$

Opposite direction: $\begin{pmatrix} -8 \\ 6 \end{pmatrix}$

2. $p = -9$

Practise Now 13

1. (i) $\begin{pmatrix} -5 \\ 5 \end{pmatrix}$ **(ii)** $\begin{pmatrix} 4 \\ 8 \end{pmatrix}$

2. $x = 1\frac{1}{2}, y = 3$

Practise Now 14

(a) $\overrightarrow{OP} = \begin{pmatrix} 8 \\ -2 \end{pmatrix}$, $\overrightarrow{OQ} = \begin{pmatrix} -1 \\ 7 \end{pmatrix}$

$\overrightarrow{PQ} = \begin{pmatrix} -9 \\ 9 \end{pmatrix}$

(b) Coordinates of B are $(2, -2)$.

Practise Now 15

Coordinates of C are $(1, -9)$.

Practise Now 16

(a) **(i)** $8\mathbf{a} + 4\mathbf{b}$ **(ii)** $2\mathbf{a}$

(iii) $6\mathbf{a}$ **(iv)** $\frac{2}{3}(3\mathbf{a} - 2\mathbf{b})$

(b) **(i)** $\frac{1}{9}$ **(ii)** $\frac{1}{3}$

Practise Now 17

(i) $\overrightarrow{PQ} = 3(\mathbf{q} - 3\mathbf{p})$; $\overrightarrow{RS} = \frac{3}{2}(\mathbf{q} - 2\mathbf{p})$

(ii) $\frac{3}{4}(\mathbf{q} - 3\mathbf{p})$

CHAPTER 8 – LOCI

Practise Now 1

(iii) 6.8 cm

Practise Now 2

(a) **(i)** 97.2° **(b)** **(iii)** 2.2 cm

Practise Now 3

(iv) 2.8 cm

Practise Now 6

(ii) 5.2 cm

Answers

CHAPTER 1 – LINEAR INEQUALITIES IN TWO VARIABLES

Exercise 1A

2. (a) $x \geqslant 0, y \geqslant 0, y \leqslant 3, x + y \leqslant 4$
   (b) $x \leqslant 3, 2y > x, 3x + 2y \geqslant 6,$
   $3y \leqslant x + 9$
   (c) $x < 4, 4y \geqslant x, 4x + y \geqslant 4,$
   $2y \leqslant x + 4$
   (d) $2y > x, 2x + y \leqslant 4, y \leqslant 2x + 2$
3. (a) $x \geqslant 0, y \leqslant 4, x + y \leqslant 5,$
   $2y \geqslant x$
   (b) $y \leqslant 2, x + y \leqslant 4, y \leqslant 4x + 4,$
   $4y \geqslant x$
   (c) $4x + 3y \leqslant 12, 2y \geqslant x - 2,$
   $y \leqslant 3x + 3$
   (d) $2x + 3y \leqslant 6, y \leqslant 2x + 2,$
   $2y \geqslant x - 2$
4. 16
5. max = 22, min = 8
6. max = $14\frac{1}{2}$, min = 4

Exercise 1B

1. (a) $x + y \leqslant 30, x \geqslant 8, y \geqslant 2x$
   (c) 600 cm
2. (a) $x + y \leqslant 40, x \geqslant 12,$
   $x \geqslant 2y$
   (c) 266 g
3. (a) $x + y \leqslant 200, x \geqslant 2y, y \geqslant 50,$
   $x \leqslant 140$
   (c) 140 bottles of *Power Clean* and 60 bottles of *Disappear*
4. (a) $40x + 50y \geqslant 300,$
   $3000x + 2000y \geqslant 20\,000$
   (c) 8
5. $\frac{x}{5}, \frac{2y}{5}, \frac{3y}{5}$
   (a) $\frac{4x}{5} + \frac{2y}{5} \leqslant 3200,$
   $\frac{x}{5} + \frac{3y}{5} \geqslant 3000$
   (b) $y > x, x \leqslant 2300, y \leqslant 5000$
   (d) 1500 kg of *Fragrant* and 5000 kg of *Instant*

Review Exercise 1

2. (a) $y \geqslant 0, x \leqslant 2, y \leqslant 2, 2x + y \geqslant 2$
   (b) $x \leqslant 3, 2y \leqslant x + 4, 3x + 2y \geqslant 6,$
   $2y \geqslant x$
   (c) $x + 2y \leqslant 10, 2y - 4x + 3 \leqslant 0,$
   $2y - x + 3 \geqslant 0$
   (d) $3x + y \geqslant 3, 3y \leqslant 4x + 9,$
   $5x + 3y \leqslant 36, 5y + 2 \geqslant 2x$
   (e) $x + 2y \leqslant 11, 11y + 7 \geqslant 8x,$
   $8x + y + 5 \geqslant 0, y \leqslant x + 4$
   (f) $y \leqslant 7x, x + y \leqslant 8, 2x + y \leqslant 12,$
   $3y + 19 \geqslant 5x, 3x + 2y \geqslant 0$
   (g) $x \leqslant 3, y \leqslant 2x + 1, x + y > 4$
3. Greatest = 3
4. max = 2
5. (a) $y > 10, 20 \leqslant x \leqslant 50,$
   $x + y < 70, x \geqslant y$
   (c) 15
6. (a) $x + y \geqslant 10, 240x + 160y \leqslant 2400,$
   $200x + 80y \geqslant 1600$
   (c) 12.5 kg

Challenge Yourself

\$675 000 in government bond fund, \$225 000 in bank's fund, \$100 000 in high-risk account

CHAPTER 2 – FURTHER SETS

Exercise 2A

1. 12
2. 7
3. 29
4. 13
5. (i) 17, 1 (ii) 40, 24
6. (b) (i) 10 (ii) 8
7. 32%
8. 72
9. (i) 3 (ii) 16
10. (i) 165 (ii) 25
    (iii) 20 (iv) 10
11. (b) (i) 9 (ii) 15
12. (i) {–20, 25} (ii) {7}
    (iii) $\{x : -5 < x < 15\}$
    (iv) $\varnothing$
13. $a = \frac{2}{5}, b \neq 7$
14. (i) 0 (ii) 100
15. (i) 42 (ii) 75
16. $p + q - r$
17. (i) 33 (ii) 28
    (iii) 56
18. 30
19. 36
20. No
21. $90 = 78 + x, x = 12$
22. (i) 15 (ii) 0
    (iii) 34

Review Exercise 2

1. 18
2. 15
3. (i) 362 (ii) 327
4 (i) 20% (ii) 52%
   (iii) 17%
5. (i) 15 (ii) 15
   (iii) 60
6. 29
7. (i) 12, 2 (ii) 25, 15
8. 7
9. 55, 35

Challenge Yourself

44%

CHAPTER 3 – PROBABILITY OF COMBINED EVENTS

Exercise 3A

1. $\{(H, 1), (H, 2), (H, 3), (H, 4), (H, 5), (H, 6), (T, 1), (T, 2), (T, 3), (T, 4), (T, 5), (T, 6)\}$
2. F: Faulty pen; N: Non-faulty pen
   $S = \{F_1, F_2, F_3, N_1, N_2, N_3, N_4\}$; $\frac{4}{7}$
   $\frac{1}{2}$
3. (i) $\frac{2}{11}$ (ii) $\frac{4}{11}$
   (iii) $\frac{4}{11}$ (iv) $\frac{7}{11}$
4. (b) (i) $\frac{1}{6}$ (ii) $\frac{5}{6}$
   (iii) $\frac{1}{4}$

5. **(a)** 0, 1, 2, 3, 4, 5, 6;
2, 3, 5, 6;
2, 3, 4, 5, 6, 7;
3, 4, 5, 6, 7, 8;
4, 6, 7, 8, 9;
5, 6, 7, 8, 9, 10;
(b) 36
(c) **(i)** $\frac{1}{9}$ **(ii)** $\frac{17}{36}$
(iii) $\frac{19}{36}$ **(iv)** $\frac{1}{2}$
(v) $\frac{1}{2}$
(d) Sum of 7

6. **(a)** 12, 13; 12, 13, 14; 13, 15
28, 35; 32, 48; 36, 45, 54
(b) **(i)** $\frac{4}{9}$ **(ii)** $\frac{2}{3}$
(iii) $\frac{8}{9}$
(c) **(i)** $\frac{2}{9}$ **(ii)** $\frac{7}{9}$
(iii) $\frac{5}{9}$

7. **(i)** $\frac{1}{8}$ **(ii)** $\frac{3}{8}$
(iii) $\frac{1}{2}$

8. {*RB*, *BB*, *WB*, *RR*, *BR*, *WR*}
(i) $\frac{1}{3}$ **(ii)** $\frac{1}{3}$
(iii) $\frac{2}{3}$

9. **(a)** {11, 12, 13, 21, 22, 23, 31, 32, 33}
(b) **(i)** $\frac{1}{3}$ **(ii)** 0
(iii) $\frac{4}{9}$ **(iv)** $\frac{5}{9}$

10. *B*: Boy; *G*: Girl
S = {*BBB*, *BBG*, *BGB*, *GBB*, *BGG*, *GBG*, *GGB*, *GGG*}
(i) $\frac{1}{8}$ **(ii)** $\frac{3}{8}$
(iii) $\frac{3}{8}$

11. **(a)** **(i)** $\frac{1}{5}$ **(ii)** $\frac{1}{5}$
(iii) $\frac{4}{5}$ **(iv)** $\frac{6}{25}$
(b) $\frac{2}{5}$

12. **(a)** 2, 3, 4, 5, 6;
2, 4, 8, 10, 12
(b) **(i)** $\frac{1}{4}$ **(ii)** $\frac{3}{4}$
(iii) $\frac{1}{4}$ **(iv)** $\frac{5}{6}$
(v) $\frac{1}{3}$

13. **(a)** 1, 2, 3, 4, 5;
1, 0, 2, 3;
2, 1, 0, 1, 2, 3;
3, 2, 1, 0, 1, 2;
4, 3, 2, 1, 0, 1;
5, 3, 2, 1
(b) **(i)** $\frac{5}{18}$ **(ii)** $\frac{5}{6}$
(iii) $\frac{1}{2}$ **(iv)** $\frac{4}{9}$
(v) $\frac{1}{3}$

14. **(i)** $\frac{3}{10}$ **(ii)** $\frac{11}{20}$
(iii) $\frac{1}{5}$ **(iv)** 1
(v) 0

15. **(i)** $\frac{1}{8}$ **(ii)** $\frac{1}{4}$
(iii) $\frac{1}{2}$ **(iv)** $\frac{1}{4}$
(v) $\frac{11}{16}$ **(vi)** $\frac{3}{4}$
(vii) $\frac{1}{4}$

16. **(i)** $\frac{1}{6}$ **(ii)** $\frac{1}{3}$

17. **(i)** $\frac{2}{3}$ **(ii)** $\frac{4}{9}$
(iii) $\frac{2}{3}$ **(iv)** $\frac{2}{3}$
(v) $\frac{4}{9}$ **(vi)** $\frac{2}{9}$
(vii) $\frac{1}{3}$

18. **(a)** **(i)** $\frac{1}{4}$ **(ii)** $\frac{1}{2}$

19. $\frac{1}{7}$

20. **(a)** **(i)** $\frac{4}{15}$ **(ii)** $\frac{3}{5}$
(iii) $\frac{11}{15}$
(b) $\frac{4}{15}$

21. $\frac{5}{24}$

Exercise 3B

1. **(i)** $\frac{5}{11}$ **(ii)** $\frac{4}{11}$
(iii) $\frac{9}{11}$ **(iv)** $\frac{4}{11}$
(v) $\frac{2}{11}$

2. **(i)** $\frac{7}{15}$ **(ii)** $\frac{1}{3}$
(iii) $\frac{4}{5}$ **(iv)** $\frac{1}{5}$

3. **(i)** $\frac{3}{17}$ **(ii)** $\frac{2}{17}$
(iii) $\frac{5}{17}$ **(iv)** $\frac{10}{17}$
(v) $\frac{13}{17}$ **(vi)** $\frac{8}{17}$

4. **(i)** $\frac{5}{6}$ **(ii)** $\frac{1}{6}$

5. **(i)** $\frac{2}{13}$ **(ii)** $\frac{5}{13}$
(iii) $\frac{5}{13}$ **(iv)** $\frac{11}{13}$

6. **(i)** $\frac{5}{14}$ **(ii)** $\frac{11}{14}$
(iii) $\frac{3}{14}$

7. **(i)** $\frac{7}{15}$ **(ii)** $\frac{17}{30}$
(iii) $\frac{8}{15}$ **(iv)** $\frac{13}{30}$

8. **(i)** $\frac{61}{120}$ **(ii)** $\frac{59}{120}$
(iii) $\frac{13}{24}$

9. **(b)** **(i)** Mutually exclusive
(ii) Not mutually exclusive
(iii) Not mutually exclusive
(iv) Mutually exclusive
(v) Not mutually exclusive
(vi) Not mutually exclusive

10. **(a)** 4
(b) Not mutually exclusive

Exercise 3C

1. **(a)** $\frac{5}{9}, \frac{4}{9}; \frac{5}{9}, \frac{4}{9}, \frac{5}{9}, \frac{4}{9}$
(b) **(i)** $\frac{5}{9}$ **(ii)** $\frac{4}{9}$
(iii) $\frac{20}{81}$ **(iv)** $\frac{4}{9}$

2. (a) $\frac{3}{5}, \frac{2}{5}; \frac{3}{5}, \frac{2}{5}, \frac{3}{5}, \frac{2}{5}$

(b) (i) $\frac{9}{25}$ (ii) $\frac{12}{25}$

(iii) $\frac{2}{5}$

3. (a) $\frac{1}{2}, \frac{1}{4}; \frac{1}{4}, \frac{3}{4}, \frac{1}{4}, \frac{3}{4}, \frac{1}{4}, \frac{3}{4}$

Sum: 40, 20, 30, 60

(b) (i) $\frac{3}{4}$ (ii) $\frac{1}{4}$

(c) (i) $\frac{3}{8}$ (ii) $\frac{9}{16}$

(iii) $\frac{15}{16}$ (iv) $\frac{1}{16}$

4. (i) $\frac{7}{12}$ (ii) $\frac{7}{24}$

(iii) $\frac{3}{8}$

5. $\frac{3}{8}$

(a) $\frac{2}{7}$

(b) $\frac{5}{8}; \frac{2}{7}, \frac{5}{7}, \frac{3}{7}, \frac{4}{7}$

(c) (i) $\frac{15}{56}$ (ii) $\frac{3}{28}$

(iii) $\frac{3}{8}$

6. (i) $\frac{2}{3}$ (ii) $\frac{15}{22}$

(iii) $\frac{5}{22}$ (iv) $\frac{5}{11}$

7. (a) $\frac{3}{5}$

(b) (i) $\frac{1}{3}$ (ii) $\frac{8}{15}$

(iii) $\frac{2}{3}$

8. (a) $\frac{1}{5}, \frac{4}{5}; \frac{1}{4}, \frac{3}{4}, \frac{1}{4}, \frac{3}{4}$

(b) Yes

9. (a) $\frac{1}{6}, \frac{1}{3}, \frac{1}{2};$

$\frac{5}{6}, \frac{1}{6}, \frac{5}{6}, \frac{1}{6}, \frac{5}{6}, \frac{1}{6};$

$\frac{5}{6}, \frac{1}{6}, \frac{5}{6}, \frac{1}{6}, \frac{5}{6}, \frac{1}{6}, \frac{5}{6}, \frac{1}{6},$

$\frac{5}{6}, \frac{1}{6}, \frac{5}{6}, \frac{1}{6}$

(b) (i) $\frac{5}{108}$ (ii) $\frac{1}{72}$

(iii) $\frac{1}{36}$ (iv) $\frac{5}{18}$

(v) $\frac{7}{24}$

10. (i) $\frac{5}{9}$ (ii) $\frac{25}{81}$

(iii) $\frac{40}{81}$

11. (i) $\frac{91}{855}$ (ii) $\frac{182}{1539}$

(iii) $\frac{5591}{15\ 390}$

12. (a) (i) $\frac{7}{30}$ (ii) $\frac{2}{5}$

(iii) $\frac{11}{30}$

(b) $\frac{1}{10}$

(c) No

13. (i) $\frac{3}{10}$ (ii) $\frac{1}{15}$

(iii) $\frac{2}{15}$ (iv) $\frac{1}{9}$

14. (a) $\frac{1}{5}$

(b) (i) $\frac{1}{10}$ (ii) $\frac{2}{5}$

(iii) $\frac{3}{10}$ (iv) $\frac{3}{5}$

15. (b) (i) 0 (ii) $\frac{17}{33}$

(iii) $\frac{5}{44}$

16. (i) $\frac{1}{37}$ (ii) $\frac{508}{1295}$

17. (i) 0.001 08 (ii) 0.598

(iii) 0.402 (iv) 0.0454

18. (i) $\frac{3}{65}$ (ii) $\frac{21}{520}$

(iii) $\frac{63}{260}$

19. (a) (i) $\frac{25}{216}$ (ii) $\frac{125}{1296}$

(iii) $\frac{671}{1296}$

(b) (i) $\frac{5}{108}$ (ii) $\frac{1}{36}$

Review Exercise 3

1. (i) $\frac{1}{12}$ (ii) $\frac{1}{4}$

2. (i) $\frac{1}{6}$ (ii) $\frac{1}{4}$

(iii) $\frac{1}{4}$ (iv) $\frac{1}{2}$

3. (i) $\frac{11}{25}$ (ii) $\frac{7}{25}$

(iii) $\frac{3}{10}$ (iv) $\frac{6}{25}$

4. (i) $\frac{1}{12}$ (ii) $\frac{143}{144}$

(iii) $\frac{1}{1728}$

5. (i) $\frac{6}{7}$ (ii) $\frac{1}{49}$

(iii) $\frac{12}{49}$ (iv) $\frac{216}{343}$

6. (i) $\frac{19}{24}$ (ii) $\frac{5}{24}$

(iii) $\frac{1}{2}$

7. (i) $\frac{1}{2}$ (ii) $\frac{1}{6}$

(iii) $\frac{7}{30}$ (iv) $\frac{2}{15}$

8. (i) $\frac{1}{15}$ (ii) $\frac{1}{6}$

(iii) $\frac{1}{180}$

9. (i) $\frac{y}{x+y}$

(ii) $\frac{xy}{(x+y)(x+y-1)}$

(iii) $\frac{2xy}{(x+y)(x+y-1)}$

10. $\frac{1}{4}$

11. (a) (i) $\frac{40}{87}$ (ii) $\frac{38}{87}$

(b) (i) $\frac{1}{3}$ (ii) $\frac{8}{15}$

(iii) $\frac{13}{15}$

12. (i) 0.36 (ii) 0.24

(iii) 0.32 (iv) 0.24

13. (i) $\frac{31}{45}$ (ii) $\frac{11}{120}$

(iii) $\frac{5}{36}$

14. (i) $\frac{1}{72}$ (ii) $\frac{31}{72}$

(iii) $\frac{5}{36}$

15. (i) $\frac{5}{39}$ (ii) $\frac{35}{156}$

(iii) 0 (iv) $\frac{47}{78}$

16. (a) (i) $\frac{8}{35}$ (ii) $\frac{2}{35}$

(iii) $\frac{46}{105}$ (iv) $\frac{33}{35}$

(b) (i) $\frac{1}{5}$ (ii) $\frac{8}{105}$

(iii) $\frac{33}{35}$

Challenge Yourself

1. $\frac{1}{11}$
2. Yes
3. (a) 1
   (b) 0.891; Yes
   (c) 22

CHAPTER 4 – STATISTICAL DATA ANALYSIS

Exercise 4A

1. (b) (i) 24 (ii) 15
   (iii) 80
2. (a) 132, 160, 184, 203, 217, 230
   (b) (i) 46 (ii) 52
   (iii) 63
3. (i) 26 (ii) 24
   (iii) 80%
4. (i) 33 (ii) 6
   (iii) 450.6
5. (a) (i) Soil *A*: 130; Soil *B*: 90
   (ii) Soil *A*: 9% ; Soil *B*: 3%
   (iii) Soil *A*: 41; Soil *B*: 46
   (b) (i) Soil *A*
   (c) (i) 36% (ii) 24%
6. (i) 57 (ii) $\frac{13}{60}$
   (iii) 51
7. (i) 13 (ii) $\frac{1}{5}$
   (iii) 23
8. (ii) $80 \leqslant x < 100$
9. (a) (i) 55% (ii) 52
   (iii) 24
   (b) (i) 2, 6, 18, 44, 10
   (ii) 55.9 g
10. (i) Examination *B*
    (ii) Examination *C*

Exercise 4B

1. (a) 8, 4, 6, 8, 4
   (b) 29, 58.5, 65, 71, 12.5
   (c) 31, 12, 18, 29.5, 17.5
   (d) 164, 102, 166, 207, 105
   (e) 19.7, 5.8, 10.4, 14.1, 8.3
2. (i) 7.5, 1, 24 (ii) 29, 23
3. (i) 45 (ii) 86
   (iii) 47
4. (a) (i) 97, 88, 105 (ii) 17
   (b) (i) 85 (ii) 110
5. (i) 50 (ii) 57
   (iii) 39 (iv) 14
6. (a) (i) 35.5 (ii) 6
   (b) 34
7. (a) (i) 23.5 (ii) 26.5
   (iii) 6.5 (iv) 16
   (b) 25
8. (a) (i) 42 (ii) 58
   (b) (i) 26 (ii) 24
   (c) 75
   (d) School *B*
9. (i) 21, 28, 34 (ii) 38
   (iii) 8.5 (iv) 2.63%
10. (a) (i) 50 (ii) 60
    (iii) 29 (iv) 195
    (v) 44
    (b) School *A*: 14%; School *B*: 29%
    (c) Agree
11. (a) (i) 84 (ii) 51.5
    (iii) 57
    (b) (i) 229 (ii) 146
    (iii) 97
    (c) City *Y*
12. (a) (i) 10, 13, 15.25
    (ii) 5.25
    (b) 73.3%
    (c) Median position and median waiting time

Exercise 4C

2. (i) 19, 21, 25 (ii) 11
3. (i) 169 (ii) 23
4. (i) 0.05, 0.07, 0.086
5. (i) $a = 168$, $b = 188.5$, $c = 195$, $d = 199$, $e = 213$
   (ii) 10.5; Interquartile range
   (iii) 45; Range
6. (a) (i) Type *A* (ii) Type *C*
   (b) Type *B*
   (c) Type *B*
7. (a) (i) 34 (ii) 63
   (iii) 8
   (b) (i) 41 (ii) 68
   (iii) 14
   (c) Agree
8. (a) (i) 64 (ii) 56
   (iii) 26
   (b) (i) 84 (ii) 42
   (iii) 36
   (c) Agree
   (d) History examination
9. (a) (i) 15 (ii) 7
   (iii) 6
   (b) (i) 34 (ii) 24
   (c) Agree
10. (a) (i) 40.5 (ii) 5
    (b) (i) 50 (ii) 9.5
    (d) Luxury
11. (a) (i) Set *X*: 30; Set *Y*: 21
    (ii) Set *X*: 40; Set *Y*: 25
    (iii) Set *X*: 20; Set *Y*: 7
    (b) Set *X*
    (c) Set *X*
    (d) Set *Y*
    (e) Curve *B*
    (f) Curve $A \rightarrow P$;
    Curve $B \rightarrow R$;
    Curve $C \rightarrow Q$
12. *X* to *B*; *Y* to *A*; *Z* to *C*

Exercise 4D

1. (a) 3.27
   (b) 4.02
   (c) 3.06
2. (a) 11.1
   (b) 9.35
   (c) 11.9
3. 1.96
4. 1.68
5. 6.20

6. 25.0

7. **(a)** 11.7

(b) 7.23

8. **(i)** Class A: Mean = 8
Standard deviation = 2.60

Class B: Mean = 7.875
Standard deviation = 8.87

9. **(i)** 19 **(ii)** 5.69

10. **(i)** 15.4, 7.23 **(ii)** 12.4, 7.23

11. **(i)** Train A: Mean = 5.28
Standard deviation = 1.55

Train B: Mean = 4.98
Standard deviation = 1.67

(ii) A **(iii)** B

12. **(a)** **(i)** 25 **(ii)** 2

13. **(a)** **(i)** City A: 51.5
City B: 48.6

(ii) City A: 5.20
City B: 6.19

(b) City A

(c) City A

14. $x = 1, y = 4$ or $x = 4, y = 1$

15. **(i)** A, C **(ii)** C

16. **(i)** Yes **(ii)** No

(iii) 55.3, 8.58

Review Exercise 4

1. **(a)** **(i)** 200 **(ii)** $\frac{1}{5}$

(iii) 23

(b) 31.5

2. **(i)** 17 **(ii)** 13

(iii) 27.8, 31.7

3. **(a)** 1.45

(b) **(i)** 0.5, 1, 3 **(ii)** 2.5

4. **(a)** **(i)** 44 **(ii)** 37, 50

(b) 13

(c) **(i)** 15 **(ii)** 6

5. **(a)** **(i)** Vishal: 25; Jun Wei: 17.5

(ii) Vishal: 14.9; Jun Wei: 12.5

6. **(a)** **(i)** 48 **(ii)** 13

(b) **(i)** 57 **(ii)** 32

(c) 12.5%

7. **(i)** $p = 990$, $q = 141$, $r = 24$, $t = 179$

8. **(i)** University A: 760
University B: 520

(ii) Agree

(iii) A

9. **(a)** **(i)** 49 **(ii)** 44

(b) $Q_1 = 36, Q_3 = 64$,
interquartile range = 28

(c) 68, 24

10. **(a)** 18, 42, 15, 4, 1

(b) **(i)** 212 **(ii)** 16.9

11. **(a)** **(i)** 54.5 **(ii)** 7.5

(iii) Grade 1: 17.5%;
Grade 2: 60%;
Grade 3: 22.5%

(b) **(i)** 50, 7

Challenge Yourself

2. $\{-2, -1, 0, 1, 2\}$ and $\{-\sqrt{5}, 0, 0, 0, \sqrt{5}\}$

CHAPTER 5 – MATRICES

Exercise 5A

1. **(a)** 3×2 **(b)** 1×3

(c) 3×3 **(d)** 2×1

(e) 1×1 **(f)** 2×2

2. **(a)** Equal **(b)** Not equal

(c) Not equal **(d)** Not equal

3. **(i)** $\mathbf{F} = \begin{pmatrix} 4 & 0 & 5 & 6 \\ 8 & 7 & 5 & 3 \end{pmatrix}$

(ii) Banana

(iii) 15; Total number of boys

(iv) 9

4. **(a)** Not equal **(b)** Equal

5. **B** and **Q**; **C** and **O**; **D** and **I**; **E** and **H**; **F** and **P**; **G** and **L**; **J** and **N**;

6. **(a)** $a = 1, b = 3, c = 5, k = 7$

(b) $a = 3, b = 13, c = 6, d = 7$

(c) $x = 7, y = 5, k = 9, h = 6$

(d) $x = 5, y = 2$

(e) $x = 6, h = 10, k = 14, y = 9$

(f) $x = \frac{5}{2}, y = 4, z = -3, k = 0$

7. **(i)** $\mathbf{S} = \begin{pmatrix} 0 & 3 & 1 & 7 \\ 3 & 0 & 4 & 2 \\ 1 & 4 & 0 & 5 \\ 7 & 2 & 5 & 0 \end{pmatrix}$

(ii) 5

(iv) 9; Total number of goals scored by Team B

Exercise 5B

1. **(a)** $\begin{pmatrix} 7 & 10 \\ 11 & -5 \end{pmatrix}$

(b) $\begin{pmatrix} 12 \\ -17 \end{pmatrix}$

(c) $\begin{pmatrix} -2 & 15 & -3 \end{pmatrix}$

(d) Not possible

(e) $\begin{pmatrix} -3 & -9 & 1 \\ 13 & 5 & -8 \end{pmatrix}$

(f) $\begin{pmatrix} 4 \\ -10 \\ 4 \end{pmatrix}$

(g) Not possible

(h) $\begin{pmatrix} 12 & 9 \\ 1 & 6 \end{pmatrix}$

2. **(a)** $\begin{pmatrix} -4 \\ 2 \end{pmatrix}$

(b) $\begin{pmatrix} 13 & -3 \\ -4 & 5 \end{pmatrix}$

(c) $\begin{pmatrix} -4 & 5 \end{pmatrix}$

(d) $\begin{pmatrix} 8 & 7 & 13 \\ -14 & 14 & -17 \end{pmatrix}$

(e) Not possible

(f) $\begin{pmatrix} 11 & 0 \\ -5 & 0 \\ -12 & 20 \end{pmatrix}$

(g) Not possible

(h) $\begin{pmatrix} 14 \end{pmatrix}$

3. **(i)** $\mathbf{Q} = \begin{pmatrix} 42 & 35 & 38 \\ 33 & 40 & 37 \end{pmatrix}$

(ii) $\begin{pmatrix} 83 & 73 & 67 \\ 72 & 73 & 73 \end{pmatrix}$

4. **(a) (i)** $\begin{pmatrix} 6 & -2 \\ -6 & 13 \end{pmatrix}$

(ii) $\begin{pmatrix} 6 & -2 \\ -6 & 13 \end{pmatrix}$

(iii) $\begin{pmatrix} 1 & 5 \\ -3 & 8 \end{pmatrix}$

(iv) $\begin{pmatrix} 1 & 5 \\ -3 & 8 \end{pmatrix}$

(v) $\begin{pmatrix} 6 & 0 \\ -7 & 17 \end{pmatrix}$

(vi) $\begin{pmatrix} 6 & 0 \\ -7 & 17 \end{pmatrix}$

5. **(i)** $\begin{pmatrix} -1 & 2 \\ 1 & 2 \end{pmatrix}$ **(ii)** $\begin{pmatrix} 1 & -2 \\ -1 & -2 \end{pmatrix}$

(iii) $\begin{pmatrix} 4 & -2 \\ 4 & -4 \end{pmatrix}$ **(iv)** $\begin{pmatrix} -1 & 3 \\ 0 & 2 \end{pmatrix}$

(v) $\begin{pmatrix} -1 & 1 \\ 2 & 2 \end{pmatrix}$

6. **(i)** $\begin{pmatrix} 216 & 197 & 190 & 293 & 168 \\ 83 & 102 & 107 & 85 & 87 \\ 90 & 80 & 120 & 111 & 102 \end{pmatrix}$

Exercise 5C

1. **(a)** $\begin{pmatrix} 2 & -4 & 6 \end{pmatrix}$

(b) $\begin{pmatrix} -8 \\ 4 \end{pmatrix}$

(c) $\begin{pmatrix} 3 \\ 2 \\ -4 \end{pmatrix}$

(d) $\begin{pmatrix} 2 & 5 \\ 7 & -8 \end{pmatrix}$

(e) $\begin{pmatrix} 2 & -1 & -6 \\ 1.6 & -4 & -2.4 \end{pmatrix}$

(f) $\begin{pmatrix} 5 & 25 \\ -20 & 15 \\ -5 & 10 \end{pmatrix}$

(g) $\begin{pmatrix} 18 & 1\frac{1}{2} & 3 \\ 0 & 6 & 1 \\ 15 & -12 & -6 \end{pmatrix}$

2. **(a)** $\begin{pmatrix} 10 \\ 1 \end{pmatrix}$

(b) $\begin{pmatrix} 10 & -10 & 2 \end{pmatrix}$

(c) $\begin{pmatrix} 11 & 17 \\ -28 & 26 \end{pmatrix}$

(d) $\begin{pmatrix} 4 & 0 & 3 \\ 23 & -4 & 1 \end{pmatrix}$

3. **(i)** $\begin{pmatrix} 5 & 6 \\ 1 & 10 \end{pmatrix}$ **(ii)** $\begin{pmatrix} 6 & 8 \\ 0 & 13 \end{pmatrix}$

(iii) $\begin{pmatrix} 2 & -2 \\ 0 & 9 \end{pmatrix}$ **(iv)** $\begin{pmatrix} 9 & 6 \\ -5 & 33 \end{pmatrix}$

4. **(a)** $a = 2, b = -2$

(b) $x = 2, y = 3$

(c) $a = -1, b = -1, c = -1, d = 2$

(d) $a = -1, b = 6, c = 2, d = 0, e = 8, f = 11$

5. **(i)** $\begin{array}{l} \text{Centre } X \\ \text{Centre } Y \\ \text{Centre } Z \end{array} \begin{pmatrix} \$8160 \\ \$8640 \\ \$7620 \end{pmatrix}$

(ii) $\begin{pmatrix} 8490 \\ 8910 \\ 8020 \end{pmatrix}$ **(iii)** Centre Z

6. **(a)** $\begin{pmatrix} 10 \\ 32 \end{pmatrix}$

(b) $\begin{pmatrix} -10 & -8 \\ 64 & 56 \end{pmatrix}$

(c) $\begin{pmatrix} -6 & 18 \\ -7 & 21 \end{pmatrix}$

(d) (15)

(e) Not possible

(f) $\begin{pmatrix} -4 & 5 \\ 11 & 44 \\ 13 & -19 \end{pmatrix}$

(g) $\begin{pmatrix} 7 \\ 28 \\ -14 \end{pmatrix}$

(h) $\begin{pmatrix} \frac{1}{2} & 1 & 1\frac{1}{2} & 2 \end{pmatrix}$

7. $p = -1\frac{3}{7}, q = 15$

8. **(i)** $\begin{pmatrix} 2 & 0 \\ 11 & 5k \end{pmatrix}$

(ii) $\begin{pmatrix} 2 & 0 \\ 4+k & 5k \end{pmatrix}$

(iii) $k = 7$

9. **(i)** $\begin{pmatrix} 8 & -3 \\ 7 & 5 \end{pmatrix}$ **(ii)** $\begin{pmatrix} 8 & -3 \\ 7 & 5 \end{pmatrix}$

Yes

10. $\begin{pmatrix} 7 & 6 \\ 4 & 3 \end{pmatrix} = 7\mathbf{A} + 6\mathbf{B} + 4\mathbf{C} + 3\mathbf{D}$

11. **(i)** $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}; \mathbf{B} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

(ii) $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}; \mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

(iii) No

Exercise 5D

1. **(i)** $-2, \begin{pmatrix} -3 & 2 \\ \frac{5}{2} & -\frac{3}{2} \end{pmatrix}$

(ix) 0, no inverse

(v) $1, \begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix}$

(vi) $1, \begin{pmatrix} 7 & -10 \\ -2 & 3 \end{pmatrix}$

(xi) 0, no inverse

(xii) $12, \begin{pmatrix} \frac{1}{2} & -\frac{1}{4} \\ \frac{1}{6} & \frac{1}{12} \end{pmatrix}$

(xvi) $1, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

(xiii) 0, no inverse

2. $\mathbf{A} = \mathbf{B}^{-1}, \mathbf{B} = \mathbf{A}^{-1}$

3. $\mathbf{P} = \mathbf{Q}^{-1}, \mathbf{Q} = \mathbf{P}^{-1}$

4. **(a)** $\frac{1}{ps - qr}\begin{pmatrix} s & -q \\ -r & p \end{pmatrix}$

(b) $\begin{pmatrix} \frac{1}{a} & 0 \\ 0 & a \end{pmatrix}$

(c) $\begin{pmatrix} -\frac{1}{2a} & \frac{1}{2a} \\ \frac{1}{2b} & \frac{1}{2b} \end{pmatrix}$

(d) $\begin{pmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{pmatrix}$

(e) $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

(f) $\begin{pmatrix} \frac{1}{4} & 0 \\ 0 & 1 \end{pmatrix}$

(g) No inverse

5. $-2, 0$; $\mathbf{A}^{-1} = \begin{pmatrix} -\frac{5}{2} & 2 \\ \frac{3}{2} & -1 \end{pmatrix}$

6. $a = 2$, $\frac{1}{16}\begin{pmatrix} 5 & 4 \\ 1 & 4 \end{pmatrix}$

7. $\frac{1}{10}\begin{pmatrix} 4 & -2 \\ -1 & 3 \end{pmatrix}$

(i) $\frac{1}{10}\begin{pmatrix} 2 & 4 \\ 3 & -9 \end{pmatrix}$

(ii) $\begin{pmatrix} 2 & 2 \\ -\frac{3}{2} & -2 \end{pmatrix}$

8. $\mathbf{A}^{-1} = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$

$\mathbf{B}^{-1} = \frac{1}{10}\begin{pmatrix} -2 & 4 \\ 3 & -1 \end{pmatrix}$

(i) $\begin{pmatrix} -7 & 2 \\ 5 & 0 \end{pmatrix}$

(ii) $\begin{pmatrix} -2 & 5 \\ 4 & -5 \end{pmatrix}$

9. $\mathbf{A}^{-1} = \begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix}$

(i) $\begin{pmatrix} 3 & 8 \\ -7 & -19 \end{pmatrix}$

(ii) $\begin{pmatrix} -8 & 13 \\ 5 & -8 \end{pmatrix}$

10. $\mathbf{A}^{-1} = \begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix}$

(i) $\begin{pmatrix} 10 & 11 \\ -19 & -19 \end{pmatrix}$

(ii) $\begin{pmatrix} -31 & 15 \\ -33 & 14 \end{pmatrix}$

11. (i) $\begin{pmatrix} 4 & 18 \\ 0 & 16 \end{pmatrix}$

(ii) $k = \frac{3}{8}$

(iii) $\frac{1}{3}$

(iv) $-\frac{1}{2}$

Exercise 5E

1. (i) 12 matches

(ii) $\begin{pmatrix} 16 \\ 28 \\ 9 \\ 13 \end{pmatrix}$

2. (a) $x = 9, y = 2$

(b) $x = 1\frac{1}{5}, y = 1\frac{3}{5}$

(c) $x = 1, y = 2$ **(d)** $x = \frac{1}{3}, y = 4$

(e) $x = 3, y = -2$ **(f)** $x = 3, y = 1$

(g) parallel **(h)** identical

3. $\begin{pmatrix} 220 & 430 & 555 & 355 \\ 245 & 485 & 520 & 310 \\ 280 & 430 & 515 & 375 \end{pmatrix}\begin{pmatrix} 130 \\ 115 \\ 90 \\ 75 \end{pmatrix}$

$= \begin{pmatrix} 154\,625 \\ 157\,675 \\ 160\,325 \end{pmatrix}$

Total amount = \$472 625

4. (i) $\begin{pmatrix} 85 & 74 & 80 & 60 & 82 \\ 65 & 84 & 70 & 52 & 94 \\ 38 & 42 & 56 & 40 & 56 \end{pmatrix}\begin{pmatrix} 2.80 \\ 2.40 \\ 2.60 \\ 3.00 \\ 2.50 \end{pmatrix}$

$= \begin{pmatrix} 1008.60 \\ 956.60 \\ 612.80 \end{pmatrix}$

(ii) \$2578

5. (i) $\begin{pmatrix} 22 & 32 & 42 & 28 \\ 18 & 26 & 36 & 32 \\ 27 & 24 & 52 & 25 \end{pmatrix}\begin{pmatrix} 0.90 \\ 1.00 \\ 1.10 \\ 1.20 \end{pmatrix}$

$= \begin{pmatrix} 131.60 \\ 120.20 \\ 135.50 \end{pmatrix}$

(ii) $\begin{pmatrix} 26 & 29 & 30 \end{pmatrix}\begin{pmatrix} 131.60 \\ 120.20 \\ 135.50 \end{pmatrix}$

$= \begin{pmatrix} 10\,972.4 \end{pmatrix}$

Total amount = \$10 972.40

6. (i) $\begin{pmatrix} 220 & 240 & 180 & 85 \\ 50 & 60 & 210 & 135 \\ 10 & 40 & 200 & 250 \end{pmatrix}\begin{pmatrix} 15 \\ 13.5 \\ 12 \\ 10 \end{pmatrix}$

$= \begin{pmatrix} 9550 \\ 5430 \\ 5590 \end{pmatrix}$

(ii) $\begin{pmatrix} 725 \\ 455 \\ 500 \end{pmatrix}$

(iii) $\begin{pmatrix} 280 & 340 & 590 & 470 \end{pmatrix}$

(iv) $\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 9550 \\ 5430 \\ 5590 \end{pmatrix}$

$= \begin{pmatrix} 20\,570 \end{pmatrix}$

or

$\begin{pmatrix} 280 & 340 & 590 & 470 \end{pmatrix}\begin{pmatrix} 15 \\ 13.5 \\ 12 \\ 10 \end{pmatrix}$

$= \begin{pmatrix} 20\,570 \end{pmatrix}$

Total cost = \$20 570

7. $\frac{1}{10}\begin{pmatrix} 6 & -4 \\ -2 & 3 \end{pmatrix}$, $x = 2, y = 3$

8. $\begin{pmatrix} 3 & -7 \\ -2 & 5 \end{pmatrix}$, $(1, 2)$; parallel

9. $\begin{pmatrix} -3 & 11 \\ -2 & 7 \end{pmatrix}$, $x = 3, y = 1$; identical

10. $-4\frac{1}{2}$

11. (i) $\begin{pmatrix} 2 & 6 & 5 & 4 & 5 \\ 3 & 8 & 2 & 3 & 2 \\ 4 & 9 & 3 & 6 & 3 \\ 3 & 5 & 6 & 3 & 4 \end{pmatrix}\begin{pmatrix} 30 \\ 1.8 \\ 4.8 \\ 3.5 \\ 2.4 \end{pmatrix}$

$= \begin{pmatrix} 120.8 \\ 129.3 \\ 178.8 \\ 147.9 \end{pmatrix}$

(ii) $\begin{pmatrix} 85 & 90 & 80 & 120 \end{pmatrix}\begin{pmatrix} 120.8 \\ 129.3 \\ 178.8 \\ 147.9 \end{pmatrix}$

$= \begin{pmatrix} 53\,957 \end{pmatrix}$

(iii) $\begin{pmatrix} 1.3 & 0 & 0 & 0 \\ 0 & 1.25 & 0 & 0 \\ 0 & 0 & 1.2 & 0 \\ 0 & 0 & 0 & 1.15 \end{pmatrix}\begin{pmatrix} 120.8 \\ 129.3 \\ 178.8 \\ 147.9 \end{pmatrix}$

$= \begin{pmatrix} 157.04 \\ 161.63 \\ 214.56 \\ 170.09 \end{pmatrix}$

12. (i) $\begin{pmatrix} 4 & 5 & 6 \\ 3 & 6 & 7 \\ 5 & 8 & 6 \\ 6 & 4 & 5 \end{pmatrix}\begin{pmatrix} 12 \\ 15 \\ 24 \end{pmatrix}$

$= \begin{pmatrix} 267 \\ 294 \\ 324 \\ 252 \end{pmatrix}$

(ii) $\begin{pmatrix} 60 & 80 & 90 & 80 \end{pmatrix}\begin{pmatrix} 267 \\ 294 \\ 324 \\ 252 \end{pmatrix}$

$= \begin{pmatrix} 88\,860 \end{pmatrix}$

Total cost = 88 860 cents

13. (i)

$\begin{pmatrix} 280 & 320 & 360 \end{pmatrix}\begin{pmatrix} 1.2 & 0 & 1.4 & 2.6 & 5.2 \\ 0 & 1.6 & 1.6 & 2.8 & 4.7 \\ 1.4 & 1.8 & 0 & 3 & 4.4 \end{pmatrix}$

$= \begin{pmatrix} 840 & 1160 & 904 & 2704 & 4544 \end{pmatrix}$

(ii)

$\begin{pmatrix} 840 & 1160 & 904 & 2704 & 4544 \end{pmatrix}\begin{pmatrix} 12.50 \\ 5.20 \\ 7.80 \\ 1.40 \\ 1.10 \end{pmatrix}$

$= \begin{pmatrix} 32\,367.20 \end{pmatrix}$

Review Exercise 5

1. (a) $\begin{pmatrix} 2 & 3 \\ 6 & 3 \end{pmatrix}$

(b) $\begin{pmatrix} 6 & 4 & 1 \\ 3 & 2 & 11 \end{pmatrix}$

(c) $\begin{pmatrix} -5 & 2 \\ 5 & -7 \\ -7 & 0 \end{pmatrix}$

(d) $\begin{pmatrix} 4 \\ -5 \end{pmatrix}$

(e) $\begin{pmatrix} 4 & 6 \end{pmatrix}$

(f) $\begin{pmatrix} -3 & -5 & 16 \end{pmatrix}$

2. (i) $a = 8, b = 9, c = 2, d = 10$

(ii) $a = -13\frac{1}{2}, b = -1, c = -6\frac{1}{2},$
$d = \frac{1}{4}$

3. (a) $\begin{pmatrix} 3 & 1 \\ 9 & 3 \end{pmatrix}$

(b) $\begin{pmatrix} 9 \end{pmatrix}$

(c) $\begin{pmatrix} 10 \end{pmatrix}$

(d) $\begin{pmatrix} 3 & 0 & 1 \\ 6 & 0 & 2 \\ 9 & 0 & 3 \end{pmatrix}$

(e) Not possible

(f) $\begin{pmatrix} 3\frac{1}{2} \\ -3\frac{1}{2} \end{pmatrix}$

(g) $\begin{pmatrix} 2 \\ 7 \\ -1 \end{pmatrix}$

(h) $\begin{pmatrix} 0 & 8 \\ -2 & -5 \\ 6 & 7 \end{pmatrix}$

(i) $\begin{pmatrix} 1 \\ -7 \end{pmatrix}$

(j) $\begin{pmatrix} 4 & 14 \end{pmatrix}$

4. (a) $a = 0, b = -4$

(b) $x = 12, y = 8$

(c) $a = -3, b = 2, c = 3\frac{1}{2}$

5. -9; $\frac{1}{22}\begin{pmatrix} 9 & -2 \\ -7 & 4 \end{pmatrix}$

6. (a) $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

(b) $\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$

(c) No inverse

(d) $\begin{pmatrix} 2 & -\frac{5}{4} \\ -1 & \frac{3}{4} \end{pmatrix}$

(e) $\begin{pmatrix} \frac{3}{5} & \frac{4}{5} \\ -\frac{4}{5} & \frac{3}{5} \end{pmatrix}$

7. (a) $x = 36, y = 18$ **(b)** $x = 0, y = 2$

(c) identical **(d)** parallel

8. $\begin{pmatrix} 450 & 240 & 120 & 80 & 60 \\ 250 & 140 & 80 & 60 & 20 \\ 280 & 120 & 50 & 30 & 24 \end{pmatrix}\begin{pmatrix} 1 \\ 1.5 \\ 6.5 \\ 5.5 \\ 4.8 \end{pmatrix}$

$= \begin{pmatrix} 2318 \\ 1406 \\ 1065.2 \end{pmatrix}$

9. (i) $\begin{pmatrix} 12 & 8 & 12 & 15 \\ 15 & 0 & 16 & 14 \\ 0 & 20 & 25 & 16 \end{pmatrix}\begin{pmatrix} 8.4 \\ 7.8 \\ 8.8 \\ 8.2 \end{pmatrix}$

$= \begin{pmatrix} 391.8 \\ 381.6 \\ 507.2 \end{pmatrix}$

(ii) $\begin{pmatrix} 22 & 18 & 25 \end{pmatrix}\begin{pmatrix} 391.8 \\ 381.6 \\ 507.2 \end{pmatrix}$

$= \begin{pmatrix} 28\,168.4 \end{pmatrix}$

Total amount = \$28 168.40

10. Total amount = \$3452.90

11. (i) $\begin{pmatrix} 11 & 2 & 5 \\ 7 & 2 & 11 \\ 4 & 5 & 10 \\ 7 & 4 & 7 \\ 12 & 1 & 9 \\ 9 & 2 & 8 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$

$= \begin{pmatrix} 35 \\ 23 \\ 17 \\ 25 \\ 37 \\ 29 \end{pmatrix}$

(ii) $\begin{pmatrix} 18 & 11 & 2 & 5 \\ 20 & 7 & 2 & 11 \\ 19 & 4 & 5 & 10 \\ 18 & 7 & 4 & 7 \\ 22 & 12 & 1 & 9 \\ 19 & 9 & 2 & 8 \end{pmatrix}\begin{pmatrix} 300 \\ 500 \\ 200 \\ -300 \end{pmatrix}$

$= \begin{pmatrix} 9800 \\ 6600 \\ 5700 \\ 7600 \\ 10\ 100 \\ 8200 \end{pmatrix}$

Challenge Yourself

1. **(a)** $\begin{pmatrix} -7 \\ -1 \end{pmatrix}$

(b) $\begin{pmatrix} 6 & 15 \\ 0 & 4 \end{pmatrix}$

2. **(i)** $\mathbf{X} = \begin{pmatrix} 2 & -9 \\ -1 & 5 \end{pmatrix}$

(ii) $\mathbf{Y} = \begin{pmatrix} 2 & -9 \\ -1 & 5 \end{pmatrix}$

(iii) Yes

3. **(a)** $\mathbf{X} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$; $\mathbf{Y} = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}$

4. $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$; $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

CHAPTER 6 – FURTHER GEOMETRICAL TRANSFORMATIONS

Exercise 6A

5. $P\left(2\frac{1}{2}, 2\frac{1}{2}\right), Q\left(3\frac{1}{2}, 1\frac{1}{2}\right), R(2, 2)$

6 **(a)** (2, 1)

(b) $P(12, 1)$, $Q(6, 7)$

7. $L\left(1\frac{1}{2}, 3\right), M(2, 5), N\left(5, 2\frac{1}{2}\right)$

8. **(a)** $A'(4, 4)$ **(b)** $B'(10, 6)$

(c) $C'(4, 2)$ **(d)** $D'(4.5, 5)$

9. **(a)** (5, 5), $k = 2$

(b) (0, 1), $k = 2$

(c) (4, 3), $k = -\frac{1}{3}$

(d) (5.5, 3), $k = \frac{1}{3}$

12. **(a)** (7, 6), 2 **(b)** (2, 1), 3

(c) (4, 6), $1\frac{1}{2}$ **(d)** (4, 6), $-\frac{1}{3}$

(e) (3, 2), 2 **(f)** (4, 5), –2

13. **(a)** $\triangle ABC$ **(b)** $PBQR$

14. **(a)** $P(-6, 5)$, $Q(0, -3)$

(b) 10 units

15. $A(1, 1)$, $B(5, 2)$, $C(2, 3)$

16. $A(1, 1)$, $C(0, 3)$

17. 20 cm

19. 6 cm

20. **(a)** –3, centre O

Exercise 6B

1. reflection in the x-axis

2. (0, 0) and (1, 0); invariant points

3. 180° rotation about the origin

4. $A'(-2, 3)$, $B'(-4, 3)$, $C'(-4, 6)$; reflection in y-axis; $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

5. $A'(7, 6)$, $B'(12, 11)$, $C'(11, 18)$, $D'(-18, -19)$, $E'(-23, -19)$

6. $A'(0, -3)$, $B'(-8, 4)$, $C'(6, -7)$, $D'(-14, 11)$, $E'(4, 3)$

7. T is a 90° anticlockwise rotationabout the origin. T_2 is a half-turnabout the origin, $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$; T^{-1} is a 90° clockwise rotation about the origin; $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

8. $\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$

9. $\begin{pmatrix} 3 & 2 \\ 0 & -1 \end{pmatrix}$

10. (4, –6), (8, –6), (6, –10);

$\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$

11. **(i)** (3, 1) **(ii)** (5, 2)

(iii) (7, 3)

(iv) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

12. **(i)** **(a)** (–5, 3) **(b)** (4, 7)

(ii) **(a)** (4, –3) **(b)** (–3, 2)

(iii) **(a)** (–5, 2) **(b)** (2, 5)

(iv) **(a)** $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

(c) $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

Exercise 6C

1. **(i)** $A'\left(2\frac{1}{2}, 5\right), B'\left(2\frac{1}{2}, 10\right)$, $C'\left(12\frac{1}{2}, 5\right)$

(ii) 4

(iii) 25

2. **(i)** $A'(-2, -2)$, $B'(-6, -2)$, $C'(-8, -6)$, $D'(-4, -8)$; 4

(ii) $A'\left(\frac{1}{2}, \frac{1}{2}\right), B'\left(1\frac{1}{2}, \frac{1}{2}\right), C'\left(2, 1\frac{1}{2}\right)$, $D'(1, 2)$; $\frac{1}{4}$

4; $\frac{1}{4}$

3. $\begin{pmatrix} 1\frac{1}{2} & 0 \\ 0 & 1\frac{1}{2} \end{pmatrix}$

4. 2; 3; $\begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}$

5. $A_1(1, 2)$, $B_1(4, 2)$, $C_1(2, 3)$; $A_2(-6, -12)$, $B_2(-24, -12)$, $C_2(-12, -18)$; $\begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$

Exercise 6D

1. **(a)** (–3, 7) **(b)** (1, 7)

2. (–2, –5)

3. **(a)** (3, –2) **(b)** (1, 4)

4. (–3, –1)

5. **(a)** (8, 4) **(b)** (6, 3)

6. **(a)** (–5, 3) **(b)** (–1, 3)

7. $A_1(5, 5)$, $A_2(5, 11)$; $k = 6$

8. A half-turn about the origin.

9. **(a)** (6, 11) **(b)** (2, 5)

(c) (2, 5) **(d)** (–8, – 4)

(e) (–3, 1)

10. A reflection in the line $y + x = 0$.

12. (3, –2), (6, –1)

13. $B(1, 8)$, $C(2, 3)$, $D(-5, -1)$

14. $(3, 3)$

15. (b) reflection in the line POR

16. $A'(1, 1)$, $B'(3, 2)$, $C'(4, -1)$;
$P(5, 0)$, $Q(5, 3)$, $R(8, 3)$, $S(8, 0)$

17. $C_1(12, 8)$, $C_2(-28, -8)$

Review Exercise 6

1. (a) $(-2, -1)$ **(b)** $(1, 2)$
(c) $(3, 4)$

2. (a) $(-3, 1)$ **(b)** $(-3, 1)$
(c) $(-1, 3)$ **(d)** $(1, 5)$

3. $x = 9$, $y = 0$

5. (a) Enlargement centre at A and scale factor 2
(b) Translation parallel to AE with length AE
(c) A 180° rotation about E
(d) Reflection in BH followed by 180° rotation about E
(e) Reflection in FD followed by 180° rotation about E

6. $p = 7$, $m = 6$, $n = -1$

7. H is a reflection in the line $x = 4$. K is an enlargement centre at (4, 2) and scale factor 2.

8. (a) 4 **(b)** 1
(c) 16

9. (a) 1 : 2 **(b)** 4 : 1

10. (a) $(1, 1)$ **(b)** 3
(c) $(4, 4)$ **(d)** $(2, 3)$

11. (a) $(-4, 0)$, $(-4, -4)$, $(-12, -4)$
(b) 2
(c) $\frac{1}{4}$

12. (a) $(-3, 4)$, $x = 0$ **(b)** $(-4, -3)$
(c) reflection in $y + x = 0$

13. (a) Enlargement centre at A and scale factor 4
(b) Translation parallel to AP with length AP
(c) A 180° rotation about P
(d) Enlargement centre at A and scale factor 2
(e) Enlargement centre at X where $2HX = XK$ and scale factor 3
(f) Enlargement centre at B and scale factor 2

Challenge Yourself

1. A 90° clockwise rotation about O

2. (a) A 180°rotation about O
(b) A 180°rotation about O

3. (a) A reflection in the line $y + x = 0$
(b) A reflection in the line $y = x$

CHAPTER 7 – VECTORS

Exercise 7A

1. (a) 5 units **(b)** 13 units
(c) 7.28 units **(d)** 6.5 units
(e) 8 units

2. (a) $\begin{pmatrix} -12 \\ 7 \end{pmatrix}$ **(b)** $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$
(c) $\begin{pmatrix} -4 \\ -8 \end{pmatrix}$ **(d)** $\begin{pmatrix} 3 \\ 1.2 \end{pmatrix}$
(e) $\begin{pmatrix} 0 \\ -3\frac{1}{4} \end{pmatrix}$

3. $a = -2$, $b = 2.5$

4. (a) 7 units
(b) (i) $\begin{pmatrix} 7 \\ 0 \end{pmatrix}$ **(ii)** $\begin{pmatrix} 3 \\ -4 \end{pmatrix}$

5. $\overrightarrow{AB} = \begin{pmatrix} -4 \\ 3 \end{pmatrix}$, 5 units;
$\overrightarrow{CD} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$, 2.24 units;
$\mathbf{p} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$, 4.24 units;
$\mathbf{q} = \begin{pmatrix} -2 \\ -1 \end{pmatrix}$, 2.24 units;
$\overrightarrow{RS} = \begin{pmatrix} -2 \\ 0 \end{pmatrix}$, 2 units;
$\overrightarrow{TU} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}$, 4 units

7. (a) (i) $x = 4$, $y = 5.5$
(ii) $\begin{pmatrix} -1 \\ 3\frac{1}{2} \end{pmatrix}$
(b) (i) $y = \dfrac{93 - 4x}{14}$

9. (i) $\overrightarrow{AY} = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$ **(ii)** $\overrightarrow{YB} = \begin{pmatrix} 2 \\ 6 \end{pmatrix}$
(iii) $\overrightarrow{AC} = \begin{pmatrix} -3 \\ -2 \end{pmatrix}$
(iv) Yes; $\overrightarrow{AB} \neq \overrightarrow{AC}$

10. $n = \pm\sqrt{40}$

12. $s = 2$, $t = 1$

13. (a) (ii) $\overrightarrow{DC}$, $\overrightarrow{HG}$
(b) (i) $\overrightarrow{JA}$, $\overrightarrow{GD}$, $\overrightarrow{FE}$
(ii) $\overrightarrow{GF}$, $\overrightarrow{KJ}$, $\overrightarrow{LA}$
(iii) $\overrightarrow{AD}$, $\overrightarrow{JG}$, $\overrightarrow{IH}$
(iv) $\overrightarrow{EG}$
(c) (i) $\overrightarrow{CB}$, $\overrightarrow{DA}$, $\overrightarrow{GJ}$, $\overrightarrow{HI}$ (any one)
(ii) $\overrightarrow{FE}$, $\overrightarrow{JA}$, $\overrightarrow{GD}$, $\overrightarrow{KL}$ (any one)
(iii) $\overrightarrow{AL}$, $\overrightarrow{FG}$, $\overrightarrow{JK}$, $\overrightarrow{ED}$ (any one)

Exercise 7B

1. (a) $\begin{pmatrix} 9 \\ 9 \end{pmatrix}$ **(b)** $\begin{pmatrix} -1 \\ -4 \end{pmatrix}$
(c) $\begin{pmatrix} -12 \\ -3 \end{pmatrix}$

2. (a) Yes
(b) Yes

3. (a) $\overrightarrow{LN}$ **(b)** $\overrightarrow{LN}$
(c) $\overrightarrow{LP}$

4. (a) $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ **(b)** $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$
(c) $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

5. (a) $\begin{pmatrix} 9 \\ 1 \end{pmatrix} + \begin{pmatrix} -9 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$
(b) $\begin{pmatrix} -3 \\ -7 \end{pmatrix} + \begin{pmatrix} 3 \\ 7 \end{pmatrix} = \mathbf{0}$
(c) $\begin{pmatrix} q \\ p \end{pmatrix} + \begin{pmatrix} -q \\ -p \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

6. (a) $\mathbf{0}$ **(b)** $\mathbf{0}$
(c) $\mathbf{0}$

7. (a) $\mathbf{p} - \mathbf{q}$ **(b)** $\mathbf{q} - \mathbf{p}$
(c) $\mathbf{b} - \mathbf{a}$ **(d)** $\mathbf{a} + \mathbf{b}$
(e) $\mathbf{s} - \mathbf{r}$ **(f)** $\mathbf{r} + \mathbf{s}$
(g) $-\mathbf{m} - \mathbf{n}$ **(h)** $\mathbf{n} - \mathbf{m}$

8. (a) $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$ (b) $\begin{pmatrix} 2 \\ 7 \end{pmatrix}$

(c) $\begin{pmatrix} 0 \\ 4 \end{pmatrix}$ (d) $\begin{pmatrix} 3 \\ 8 \end{pmatrix}$

10. (a) $\overrightarrow{PR}$ (b) $\overrightarrow{SR}$

(c) $\overrightarrow{SR}$ (d) $\overrightarrow{ST}$

(e) $\overrightarrow{PR}$ (f) $\overrightarrow{RS}$

12. (a) **s** (b) **−r**

(c) **r + s** (d) **s − r**

(e) **r − s**

13. (a) $\overrightarrow{RT}$ (b) $\overrightarrow{TS}$

(c) $\overrightarrow{ST}$

(d) Cannot simplify further

(e) $\overrightarrow{RT}$ (f) $\overrightarrow{US}$

14. (a) $x = 10, y = -7$ (b) $x = 9, y = 1$

(c) $x = -3, y = 10$ (d) $x = 3, y = 5$

16. (b) Yes. Vector addition is commutative.

(c) Yes. Vector addition is associative.

17. (a) (i) $\overrightarrow{PR}$ (ii) $\overrightarrow{RQ}$

(iii) $\overrightarrow{PQ}$

(b) (i) **a** (ii) **a + b**

(iii) **a − b**

18. (a) $\overrightarrow{KS}$ (b) $\overrightarrow{QS}$

(c) $\overrightarrow{PR}$ (d) $\overrightarrow{PS}$

(e) $\overrightarrow{PR}$ (f) **0**

Exercise 7C

1. (a) Parallel

(b) Not parallel

(c) Parallel

2. (a) $\begin{pmatrix} 16 \\ -14 \end{pmatrix}$; $\begin{pmatrix} -16 \\ 14 \end{pmatrix}$

(b) $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$; $\begin{pmatrix} -1 \\ -3 \end{pmatrix}$

(c) $\begin{pmatrix} -3 \\ -1 \end{pmatrix}$; $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$

3. (a) $\begin{pmatrix} 17 \\ -4 \end{pmatrix}$ (b) $\begin{pmatrix} 12 \\ 7\frac{1}{2} \end{pmatrix}$

(c) $\begin{pmatrix} -1 \\ 13 \end{pmatrix}$

4. (a) $\begin{pmatrix} 4 \\ 7 \end{pmatrix}$ (b) $\begin{pmatrix} -2 \\ 5 \end{pmatrix}$

(c) $\begin{pmatrix} 6 \\ -1 \end{pmatrix}$ (d) $\begin{pmatrix} -4 \\ -9 \end{pmatrix}$

5. (i) $\begin{pmatrix} -1 \\ -2 \end{pmatrix}$ (ii) $\begin{pmatrix} 0 \\ 7 \end{pmatrix}$

(iii) $\begin{pmatrix} 1 \\ -5 \end{pmatrix}$ (iv) $\begin{pmatrix} -1 \\ 5 \end{pmatrix}$

6. (a) Parallel (b) Parallel

(c) Not parallel

7. (a) 8 (b) −4

8. (a) $x = 2, y = 1$ (b) $x = -7, y = 4$

(c) $x = 3, y = 1$

10. $\overrightarrow{LM} = 2\mathbf{a} + \mathbf{b}$

$\overrightarrow{PR} = -\mathbf{a} + 1\frac{1}{2}\mathbf{b}$

$\overrightarrow{ST} = 2\mathbf{a} + 2\mathbf{b}$

$\overrightarrow{XY} = 2\frac{1}{2}\mathbf{a} - 2\mathbf{b}$

11. Coordinates of B are (−5, 4).

12. (i) $\begin{pmatrix} 6 \\ -10 \end{pmatrix}$ (ii) (7, −8)

(iii) (2, 5)

14. $k = \frac{1}{3}$; $\mathbf{v} = \begin{pmatrix} -45 \\ 24 \end{pmatrix}$

15. (i) $\begin{pmatrix} -1 \\ 30 \end{pmatrix}$ (ii) −4.5

(iii) $\overrightarrow{CD} = 4\overrightarrow{PQ}$

16. (i) $\begin{pmatrix} t+3 \\ 4 \end{pmatrix}$ (ii) 29

(iii) 4 or −10

17. (i) (10, −5) (ii) $-\frac{1}{4}$

(iii) $\frac{y}{x}$

(iv) $k\begin{pmatrix} x \\ y \end{pmatrix}$, for some real number k.

Exercise 7D

1. (9, 11)

2. (i) $-\frac{1}{2}\mathbf{q}$ (ii) $\mathbf{p} - \mathbf{q}$

(iii) $\mathbf{p} + \frac{1}{2}\mathbf{q}$ (iv) $\frac{1}{2}\mathbf{q} - \mathbf{p}$

3. (i) $\mathbf{v} - \mathbf{u}$ (ii) $\frac{1}{2}\mathbf{u}$

(iii) $\frac{1}{2}\mathbf{v}$ (iv) $\frac{1}{2}(\mathbf{v} - \mathbf{u})$

$\overrightarrow{BC}$ and $\overrightarrow{MN}$ are parallel; $BC = 2MN$

4. $\frac{1}{2}\mathbf{a} - \mathbf{b}$

5. (i) $\mathbf{b} - \mathbf{a}$ (ii) $\frac{2}{5}(\mathbf{b} - \mathbf{a})$

(iii) $\frac{1}{5}(3\mathbf{a} + 2\mathbf{b})$

6. (i) (2, 2) (ii) (0, −2)

7. (i) $\frac{4}{3}\mathbf{q}$ (ii) $\mathbf{q} - \mathbf{p}$

(iii) $\mathbf{p} - \frac{4}{3}\mathbf{q}$

8. (i) $\mathbf{a} + 2\mathbf{b}$ (ii) $-\frac{8}{3}\mathbf{b}$

(iii) $\frac{2}{3}\mathbf{b} - \mathbf{a}$

9. (i) $\mathbf{v} - \mathbf{u}$ (ii) $\frac{2}{5}(\mathbf{v} - \mathbf{u})$

(iii) $\frac{3}{2}\mathbf{u} + \mathbf{v}$ (iv) $\frac{1}{5}(3\mathbf{u} + 2\mathbf{v})$

(v) $\frac{1}{2}\mathbf{u} + \mathbf{v}$

10. (i) $15\mathbf{b} - 15\mathbf{a}$ (ii) $\frac{15}{4}(\mathbf{b} - \mathbf{a})$

(iii) $\frac{15}{4}(3\mathbf{a} + \mathbf{b})$ (iv) $15\mathbf{a} + 5\mathbf{b}$

11. (i) $20\mathbf{q} - 8\mathbf{p}$ (ii) $5\mathbf{q} - 2\mathbf{p}$

(iii) $6\mathbf{p} + 5\mathbf{q}$ (iv) $6\mathbf{p} - 3\mathbf{q}$

12. (a) (i) $\begin{pmatrix} 1 \\ -3 \end{pmatrix}$ (ii) $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

(iii) $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ (iv) $\begin{pmatrix} 6 \\ 2 \end{pmatrix}$

(b) $\frac{1}{2}$

13. (i) $\begin{pmatrix} 2 \\ 8 \end{pmatrix}$ (ii) $\begin{pmatrix} 5 \\ -1 \end{pmatrix}$

(iii) (3, 10)

14. (a) (i) $\frac{2}{3}\mathbf{b}$ (ii) $\frac{2}{3}\mathbf{a}$

(iii) $\frac{2}{3}\mathbf{a} + \mathbf{b}$ (iv) $\mathbf{a} - \mathbf{b}$

(v) $\frac{1}{3}(\mathbf{a} - \mathbf{b})$

(b) (i) $\frac{1}{3}$ (ii) $\frac{1}{9}$

(iii) $\frac{1}{18}$

15. (a) (i) $-3\mathbf{a}-7\mathbf{b}$ (ii) $-2\mathbf{a}-8\mathbf{b}$
(iii) $\mathbf{a}-11\mathbf{b}$
(c) $4\mathbf{b}-4\mathbf{a}$
(d) (i) $\frac{3}{4}$ (ii) $\frac{3}{4}$
(iii) $\frac{1}{2}$

16. (a) (i) $4\mathbf{q}-3\mathbf{p}$ (ii) $4\mathbf{q}-4\mathbf{p}$
(iii) $\mathbf{p}+\mathbf{q}$ (iv) $3\mathbf{q}-4\mathbf{p}$
(v) $\frac{3}{4}(4\mathbf{p}-3\mathbf{q})$
(c) (i) $\frac{3}{4}$ (ii) $\frac{9}{16}$

17. (a) (i) $\mathbf{p}-\mathbf{q}$ (ii) $\frac{2}{5}(\mathbf{p}-\mathbf{q})$
(iii) $\frac{1}{5}(2\mathbf{p}+3\mathbf{q})$ (iv) $\frac{1}{10}(9\mathbf{q}-4\mathbf{p})$
(b) (ii) R, S and T are collinear.
$RS : ST = 2 : 3$

Review Exercise 7

1. (a) 13 units (b) 10 units
(c) 5.39 units (d) 7.07 units
(e) 3 units
2. $p = \pm 4.58$
3. (i) 4.47 units (ii) 6
4. $p = 2, q = 1$
5. (a) (i) $\overrightarrow{HK}$, $\overrightarrow{GL}$, $\overrightarrow{FE}$, $\overrightarrow{KB}$ (any two)
(ii) $\overrightarrow{CL}$, $\overrightarrow{DE}$
(iii) $\overrightarrow{GH}$, $\overrightarrow{FG}$, $\overrightarrow{KJ}$, $\overrightarrow{LK}$, $\overrightarrow{EL}$ (any two)
(iv) $\overrightarrow{AB}$, $\overrightarrow{CD}$
(v) $\overrightarrow{JH}$, $\overrightarrow{KG}$, $\overrightarrow{CE}$ (any two)
(b) (i) $\overrightarrow{LB}$, $\overrightarrow{FL}$ (any one)
(ii) $\overrightarrow{KJ}$, $\overrightarrow{LK}$, $\overrightarrow{EL}$, $\overrightarrow{HI}$, $\overrightarrow{GH}$, $\overrightarrow{FG}$ (any one)
(iii) $\overrightarrow{KB}$, $\overrightarrow{IJ}$, $\overrightarrow{HK}$, $\overrightarrow{GL}$, $\overrightarrow{FE}$ (any one)
6. (a) $\overrightarrow{NM} = \overrightarrow{KL}$
(b) $\overrightarrow{RQ} = \overrightarrow{TU}$; $\overrightarrow{QP} = \overrightarrow{ST}$
(c) $\overrightarrow{AB} = \overrightarrow{DC}$; $\overrightarrow{BC} = \overrightarrow{AD}$
(d) $\overrightarrow{LM} = \overrightarrow{QP}$; $\overrightarrow{MN} = \overrightarrow{RQ}$;
$\overrightarrow{NO} = \overrightarrow{SR}$; $\overrightarrow{OP} = \overrightarrow{LS}$
8. (i) $\overrightarrow{AC}$ (ii) $\overrightarrow{AD}$
(iii) $\overrightarrow{AD}$ (iv) $\mathbf{0}$

9. (i) $\overrightarrow{OP}$ (ii) $\overrightarrow{OR}$
(iii) $\overrightarrow{RS}$ (iv) $\overrightarrow{RP}$
(v) $\overrightarrow{QR}$ (vi) $\overrightarrow{PQ}$
10. (i) $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$ (ii) $\begin{pmatrix} 5 \\ 0 \end{pmatrix}$
(iii) $\begin{pmatrix} 5 \\ 1 \end{pmatrix}$ (iv) $\begin{pmatrix} -1 \\ 3 \end{pmatrix}$
11. $k = \frac{1}{4}$; $\mathbf{v} = \begin{pmatrix} 16 \\ -12 \end{pmatrix}$
12. (a) $\begin{pmatrix} 16 \\ -15 \end{pmatrix}$
(b) (i) 13 units (ii) 18 units
(c) -48
13. (i) $B(1, 4)$; $D(3, -6)$
(ii) $\overrightarrow{BC} = \begin{pmatrix} 6 \\ 0 \end{pmatrix}$; $\overrightarrow{CD} = \begin{pmatrix} -4 \\ -10 \end{pmatrix}$
14. (i) $\begin{pmatrix} 4 \\ 6 \end{pmatrix}$ (ii) $(3, -2)$
15. (i) $\mathbf{u}+\frac{3}{2}\mathbf{v}$ (ii) $\mathbf{u}+\frac{1}{2}\mathbf{v}$
(iii) $\frac{1}{8}(2\mathbf{u}+9\mathbf{v})$ (iv) $\frac{1}{8}(9\mathbf{v}-2\mathbf{u})$
16. (a) (i) $2\mathbf{b}-2\mathbf{a}$ (ii) $\mathbf{b}-\mathbf{a}$
(iii) $\frac{12}{7}(\mathbf{a}+\mathbf{b})$ (iv) $3\mathbf{a}$
(v) $4\mathbf{b}$
(b) $4\mathbf{b}-3\mathbf{a}$
(d) $\frac{3}{7}$
(e) (i) $\frac{3}{7}$ (ii) $\frac{2}{7}$
17. $\frac{1}{8}(3\mathbf{a}+2\mathbf{b})$
18. (i) $4\mathbf{a}+4\mathbf{b}$ (ii) $2\mathbf{a}+4\mathbf{b}$
(iii) $8\mathbf{b}-4\mathbf{a}$
19. (a) (i) $\frac{2}{3}\mathbf{b}$ (ii) $\frac{2}{3}\mathbf{b}-2\mathbf{a}$
(iii) $\frac{1}{2}(\mathbf{a}+\mathbf{b})$ (iv) $\frac{1}{2}\mathbf{a}-\frac{1}{6}\mathbf{b}$
(b) $\frac{3}{1}$ or 3
20. (a) (i) $2\mathbf{p}-\mathbf{q}$ (ii) $3\mathbf{p}$
(iii) $3\mathbf{p}-2\mathbf{q}$ (iv) $6\mathbf{p}-4\mathbf{q}$
(b) S, R and T are collinear.
$SR = RT$ (or $ST = 2SR$)
(c) $\frac{1}{2}$

Challenge Yourself

2. (a) (i) $-\mathbf{p}$ (ii) $\mathbf{q}-\mathbf{p}$
(b) $\sqrt{2}$ units (c) $\frac{1}{\sqrt{2}}\mathbf{q}$

CHAPTER 8 – LOCI

Exercise 8A

1. A circle centre O, radius 4 cm
4. Two straight lines parallel to l and a distance 5 cm from l.
11. A circle of radius 4 cm

Exercise 8B

1. (i) 9 cm (iii)(b) 4.8 cm
2. (iii) (b) 6.7 cm
3. (iv) 5 cm, 50 m
4. (i) 81° (iv) 27 m
5. (iv) 45 m
7. (ii) 84 m

Exercise 8C

5. (i) 13.2 cm
7. 3.7 cm
9. (i) 103°
11. 9.9 cm
14. 7.7 cm

Review Excercise 8

2. (i) 100° (v) 1.2 m
3. (i) 7.7 cm, 29 m (iv) 20 m
4. (i) 040° (iv) 55 m
5. (i) 10.4 cm
6. (i) (a) 14.9 cm (b) 32°
(ii) (c) 7.6 cm
7. (i) 109°
8. (i) 10.9 cm (iii) 4 cm
9. (iii) 5.7 cm (v) 4.4 cm
10. (iv) 25 m
11. (iv) 24 m (v) 22.4 m

CHAPTER 9 – REVISION: NUMBERS AND ALGEBRA

Revision 9A – Numbers and Percentages

1. (i) 4.274 443 098 (ii) 4.3

2. (a) 120.5 g (b) 9 g
3. 103 m
4. (i) 108.6°C (ii) –34.9°C
5. $108
6. (a) $2^3 \times 3^2 \times 11$
(b) 60
7. (a) $3 \times 5^2 \times 7$
(b) 25, 75
(c) 21
8. (a) $2^2 \times 3^3$
(b) $2^4 \times 3^6 \times 5 \times 7$
(c) 2×3
9. 2100
10. (a) $28 000
(b) 1.7 gigabytes
11. (i) $180 (ii) $11.78
12. (i) 9.85×10^{11} (ii) $980 billion
13. 5×10^{-11} m
14. (i) 1.56×10^6 (ii) 2.39×10^5
(iii) 2×10^2 (iv) 5.48×10^{-4}
15. (i) 11^8 (ii) 11^{-6}
(iii) $11^{\frac{1}{5}}$
16. (a) $\frac{1}{2}$ (b) $\frac{x^5}{2}$
(c) –3 (d) 0
17. (i) 4.2×10^{12} cm³ (ii) 305%
18. (a) 238
(b) (i) 2.62 (ii) 0.996
19. (a) 20% (b) 4% decrease
20. Chan and Partners

Revision 9B – Proportion, Ratio, Rate and Speed

1. 15
2. 7.5 days
3. Nora: 120; Michael: 48; Ethan: 12
4. $180
5. $1527.60
6. (i) NZ$1728 (ii) S$357.14
7. $15 055
8. (i) 3 (ii) –1
9. 25, 2; $y = \frac{20}{x}$

10. (a) (i) $V = 9.6r^2$ (ii) 614 cm³
(b) 25 : 16
11. $y = \frac{32}{21}x^2$, $13\frac{5}{7}$
12. (a) $F = \frac{640}{d^2}$
(b) 5.06 cm
(c) 3 Newtons
13. (a) 0.2
(b) 3.6 km
(c) 80 cm²
14. (a) 24 km
(b) 4 000 000
15. (a) 4.8 m
(b) 250 cm²
(c) 240 litres
16. (i) $194\frac{4}{9}$ m/s
(ii) 4 h 38 min
17. 227
18. $4\frac{8}{11}$ km/h
19. (i) $(3.5x - 4.5)$ km
(ii) $3(x + 1)$ km
(iii) 48 km
(iv) $13\frac{5}{7}$ km/h
20. (i) 8.28×10^4 m/s
(ii) 9.00×10^{-3} s

Revision 9C – Algebraic Manipulation and Formulae

1. (a) $2x + 25$ (b) $18x - 1$
(c) $18y + 9$ (d) $-7p - 26$
(e) $9q - 21r + 5$ (f) $8ab$
2. (a) $5x^2(1 - 4y)$
(b) $(x - 2y)^2$
(c) $(3x + 4y + 3z)(3x + 4y - 3z)$
(d) $(3x - 5)(2x - 7)$
(e) $(5p + 1)(p + 2)$
3. (a) $2a + 12$ (b) $4x - 4y - 3z$
(c) $ab + 5bc$ (d) $22b - 9a - 3c$

4. (a) $\frac{c^4}{60ab}$ (b) $2a^3c$
(c) $\frac{5ab^4}{2}$ (d) $-a - b$
(e) $\frac{2x+5}{x-3}$
5. (a) $(x + 3)(x + y)$
(b) $(a - c)(b - c)$
(c) $(a - k)(x - h)$
(d) $(5c - d)(4a - 3k)$
(e) $(2a + b)(3a - 4k)$
6. (a) $b = -\frac{ax^2 + c}{x}$
(b) $c = \frac{6a}{2ak - 2 - ab}$
(c) $x = -\frac{5k+9}{12}$
(d) $u = \pm\sqrt{v^2 - 2as}$
(e) $a = \frac{bx^3}{1+x^3}$
7. (a) $\frac{5x-6}{4x}$
(b) $-\frac{13y+2}{(3y+1)(3y-1)}$
(c) $\frac{3a^2+14a-6}{(a-3)(a+4)}$
(d) $-\frac{p+3}{(p+5)(p-1)}$
(e) $\frac{15x^2-3xy-y^2}{(2x-y)(3x-y)}$
8. (a) $\frac{3}{x+1}$
(b) $\frac{3x+1}{(x+2)(x-2)}$
(c) $\frac{x^2-5x+8}{(2x-3)(x+2)}$
(d) $\frac{5}{(x-2)(x+1)}$
9. (i) $b = \frac{2ac}{3c-a}$ (ii) $1\frac{5}{28}$
10. (a) $h = \frac{2V}{3b^2} - \frac{a}{3}$
(b) (i) 3.384 m³
(ii) 49.2 m
(iii) $b = -2.01$ or 2.01
11. (i) $y = \pm\frac{b}{a}\sqrt{x^2 - a^2}$
(ii) $y = -2\frac{2}{3}$ or $2\frac{2}{3}$

12. (i) $\frac{1}{x}$ (ii) $\frac{2}{x+1.5}$

(iii) $\frac{1}{x}+\frac{1}{x+1.5}$

13. (a) $1\frac{1}{6}$, $1\frac{5}{12}$

(b) $(1-n)^3$

14. (i) 20, 30

(ii) $u_n = n(n-1)$

(iii) $n = 11$

15. (a) $\frac{q(x+y+3)-p(x+1)}{y+2}$

(b) $\frac{2x-18}{3}$

16. (i) $S_3 = 36 = 6^2$,

$S_4 = 100 = 10^2$,

$S_5 = 225 = 15^2$

(ii) $S_n = \left[\frac{n(n+1)}{2}\right]^2$

(iii) 20

17. (a) $a = 8, b = 24, c = 24, d = 8$

(b) 8

(c) 27

(d) (i) $12(n-2)$ (ii) $6(n-2)^2$

(iii) $(n-2)^3$

18. (i) $l = 60, m = 25, n = 36$

(ii) $T = 2\sqrt{SP}$ or $T^2 = 4SP$ or $T = S + P - 1$

(iii) 196

(iv) 112 is not a perfect square.

(v) 4442 is not a multiple of 4.

19. (a) $p = 4, q = 5$

(b) $r = 14, s = 20$

(c) $v = n - 3$

(d) (i) $d = \frac{nv}{2}$

(ii) $d = \frac{n(n-3)}{2}$

(iii) 405

20. (i) 128

(ii) 2^{11-n}

(iii) 13

Revision 9D – Equations and Inequalities

1. (i) 7 (ii) –2

(iii) 7

2. (a) $x \leqslant 2\frac{1}{9}$ (b) $x < 2\frac{1}{2}$

(c) $x \geqslant 5\frac{4}{5}$ (d) $x \geqslant \frac{3}{4}$

(e) $4\frac{1}{3} < x < 9\frac{1}{3}$

3. (a) $x = 2, y = 3$ (b) $x = \frac{1}{2}, y = \frac{1}{3}$

(c) $a = -1, b = -2$ (d) $p = 4, q = -3$

(e) $x = 8, y = -10$

4. (a) $x = 5.80$ or -3.80

(b) $x = \frac{2}{3}$ or $-3\frac{1}{2}$

(c) $x = 1\frac{1}{2}$ or -4

(d) $x = 1\frac{3}{4}$ or -1

(e) $x = 0.184$ or -2.33

5. (a) $x = -11$ or $-2\frac{1}{3}$

(b) $x = 1\frac{3}{4}$ or $-7\frac{1}{2}$

(c) $x = 5$ or -3

(d) $x = 2$ or $-\frac{2}{3}$

(e) $x = 3$ or $-\frac{1}{5}$

(f) $x = -1\frac{1}{3}$ or 6

(g) $x = 3$

6. (i) 3 (ii) 0.5

7. (a) 28 cm (b) 61.75 cm^2

8. (a) 5 (b) 5, 7, 9, 11

(c) $\frac{9}{14} < x < \frac{21}{22}$

9. (a) 6

(b) –25

10. (i) 1 second and 2.5 seconds

(ii) 6.28 seconds

11. (i) 11 (ii) –10

(iii) 7

12. 18

13. 17.6 cm

14. (i) $x^2 + 3.4x - 125 = 0$

(ii) 13.01 cm

15. Section A: 1.4 km and Section B: 1 km

16. 74

17. (i) $\frac{200}{x}$ (ii) $\frac{200}{x+5}$

(iv) 30.9 km/h

18. $\frac{3}{5}$

19. (a) (i) $\$\left(\frac{32}{x}\right)$

(ii) $\$\left(\frac{640+x}{20x}\right)$

(c) 160

20. (b) (i) $x = -24$ or 80

(c) 2 hours

21. 9.5 kg, 8.5 kg

22. 2.07 g/cm^3, 1.94 g/cm^3

23. 40.4, 18.5

25. (a) $x \leqslant 5, y \leqslant 5, y \leqslant \frac{3}{2}x, y \geqslant 5 - x$

(b) $y \leqslant 4 - \frac{1}{4}x$, $2y \geqslant 3x - 6$, $y \leqslant 2x + 4$, $y \geqslant -x - 2$

26. (ii) 8

27. (a) $x > 20, y \geqslant 40, 70 \leqslant x + y \leqslant 80$

(c) 40 hardback copies and 40 paperback copies

28. 28

29. 26

30. max = 16, min = –5

31. $y \geqslant 0, 4y + 3x \leqslant 12, y \leqslant 3x + 3$

32. (i) $3y + 4x = 24$ (ii) $3y + 4x \leqslant 24$

Revision 9E – Functions and Graphs

1. (a) No (b) Yes

2. (i) $7a - \frac{1}{2}$ (ii) $8a + 9$

(iii) $9a - 7\frac{1}{2}$

3. $74\frac{3}{4}, -\frac{13}{64}, 146\frac{3}{4}$

4. $f^{-1}(x) = \frac{x-11}{2}$

5. $-\frac{3}{2}, \frac{5}{2}, 4, \frac{8}{3}$

6. $a = 2, b = -2, f^{-1}(x) = \frac{x+2}{2}$

7. $f^{-1}(x) = \frac{4x-7}{x-5}, x = 5$

8. (i) $A(-1, 0), B(0, 2), C(2, 0)$

(ii) $x = 0.5$

9. (i) $(0, -5)$ (ii) 2

10. $p = -5, q = 4$

11. (a) $y = x^3 + 4$ (b) $y = 4^x$

(c) $y = \frac{4}{x}$

12. -2

13. $k = 3, a = 2$

15. $P(0, 9), Q(-1.5, 0), R(1.5, 0)$

16. (b) $x = 2$ (c) $(2, -9)$

17. (i) x-intercepts: $(-4, 0)$ and $(-2, 0)$
y-intercept: $(0, -8)$

(ii) $(-3, 1)$ (iv) $x = -3$

18. (i) $(x-3)^2 - 5$ (ii) $(3, -5)$

(iv) $x = 3$

19. (i) $\left(x+\frac{1}{2}\right)^2 + 2\frac{3}{4}$

(ii) $\left(-\frac{1}{2}, 2\frac{3}{4}\right)$

(iv) $x = -\frac{1}{2}$

20. (a) $-4, -2, 2$

(c) (ii) 3 units^2

21. (b) 1.8

22. (b) 4

(c) (ii) $x = -0.3$ and 3.3

(iii) $-0.3 \leqslant x \leqslant 3.3$

23. (b) 1.4

(c) (i) $x = 0.8$ or 3.2

(ii) $x = 1$ or 5

24. (a) $-3, -1, 9, 32$ (c) 0.75

(d) $-0.6, 1, 1.6$

(f) (i) -1, 1 and 2

(ii) $A = -2, B = -1, C = 2$

25. (b) (i) $x = 1.7$ or -4.7

(ii) $x = 0.4$ or -4.4

(iii) $-3 \leqslant x \leqslant 1$

(c) $(-1, -5)$

26. (a) $\frac{35}{x^2}$

(c) 62.3, 55.7, 53, 59.3

(e) (i) 3.05 m, 5.40 m

(ii) 51 (iii) 2.1 m

27. (a) 6.4 s (b) 0.68 m/s^2

Revision 9F – Graphs in Practical Situations

1. (a) (i) S\$8.80 (ii) HK\$125

(b) $4y = 25x$

2. (a) (i) 36.7 (ii) 122

(c) 27.8

3. (b) (i) 106 N (ii) 240 N

(c) 50 N

4. (a) (i) \$1.60 (ii) \$4.90

(iii) Company P

(b) Company Q

5. (i) 48 km (ii) 2 hours

(iii) 28 km (iv) 0900 to 1000

(v) 8 km/h

6. (i) 20 minutes

(ii) 9.6 mg/dl per minute

(iii) 6.4 mg/dl per minute

7. (i) 0.75 m/s^2 (ii) 1125 m

(iii) 1000 m (iv) $t = 30$ s

8. (i) 4 minutes

(ii) 0.8°C per minute

(iii) 74°C

(iv) 15°C per minute

9. (i) 40 (ii) -0.8 m/s^2

(iii) 360 m

10. (i) 10.8 m/s (ii) $12\frac{1}{3}$ m/s

(iii) 16

11. (i) Gradient of (a): $-\frac{1}{12}$

Gradient of (b): $-\frac{19}{300}$

(iii) S\$48.41

12. (i) 6.75 s

13. (i) Car A: 60 km/h; Car B: $65\frac{5}{11}$ km/h

(ii) 40 km, 40 minutes

(iii) 64 km, 1118

(iv) 40 km

(v) 80 km/h

14. (b) (i) 2 m/s^2 (ii) 41.1 s

15. (b) (i) 15 cm (ii) 12.5 cm

16. (a) \$1.84

(c) (i) Optimus Bus Services

(ii) 8 cents

17. (a) (i) $8\frac{1}{3}$ m/s^2

(ii) $A(4.8, 40), B(10.8, 40), C(14, 0)$

(iii) 12.5 m/s^2

(b) 400 m

18. (a) $a = 7, b = 115$

(c) (i) 7.1 (ii) 10 m/s^2

19. (ii) 3.13

Revision 9G – Sets

1. (i) {2, 3, 5, 7, 9} (ii) {2}

2. (i) {a} (ii) {a, c, e, f}

(iii) {e}

3. (i) {2}

(ii) {1, 2, 3, 5, 7, 9}

(iii) {2, 5, 7}

4. (i) {15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32}

(ii) {11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31, 33}

(iii) {11, 13, 17, 19, 23, 29, 31}

(iv) {15, 17, 19, 21, 23, 25, 27, 29, 31}

(v) {11, 13, 17, 19, 23, 29, 31}

(vi) {15, 21, 25, 27, 33}

5. (i) A (ii) B

6. (i) {64, 81, 100} (ii) {8, 9, 10}

7. (i) {1, 2, 3, 4, 6, 12}

(ii) {5, 7, 9, 11}

(iii) {1, 2, 3, 4, 5, 6, 7, 9, 11, 12}

8. (b) (i) {2, 4, 14}

(ii) {1, 2, 3, 4, 5, 7, 9, 11, 13, 14, 15, 17, 19}

(iii) {3, 5, 9, 11, 13, 15, 17, 19}

9. (i) {1, 2, 3, 4, 6, 8, 12, 24}

(ii) {2, 3}

(iii) {1, 2, 4, 8}

(iv) {2, 5, 7, 11, 13, 17, 19, 23}

12. (a) $A \cap B'$

(b) $A' \cap B'$ or $(A \cup B)'$

(c) $A \cap B'$

(d) $A' \cup B'$ or $(A \cap B)'$

14. **(ii)** $A \subset B$

15. **(i)** $\{\ \}$

(ii) $\{-4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

(iii) $\{-10, -9, -8, -7, -6, -5\}$

16. **(a)** **(i)** No **(ii)** No

(c) **(i)** π or $\sqrt{2}$ **(ii)** $\frac{1}{3}$

17. 12, 0, 26

18. 14

19. 9

20. **(i)** 9, 1 **(ii)** 20, 12

21. **(i)** $\varnothing$ **(ii)** 17

22. 3

Revision 9H – Matrices

1. **(a)** $\begin{pmatrix} 5 & 1 \\ -1 & 9 \end{pmatrix}$

(b) Not possible

(c) $\begin{pmatrix} 2 & -6 \end{pmatrix}$

(d) $\begin{pmatrix} 4 \end{pmatrix}$

2. **(a)** $\begin{pmatrix} 15 & 20 \\ -3 & -4 \end{pmatrix}$

(b) $\begin{pmatrix} 10 \end{pmatrix}$

(c) Not possible

(d) $\begin{pmatrix} 5 \end{pmatrix}$

(e) $\begin{pmatrix} 7 & 12 \\ 6 & 8 \end{pmatrix}$

3. **(a)** $x = 8, y = 8$

(b) $x = -4, y = -20$

(c) $x = 5, y = -3$

4. **(a)** $x = 1, y = 1$

(b) $x = 1, y = 2$

(c) $x = 3, y = -1$

5. $p = -4, q = 4$

6. **(a)** $p = 10, q = 8$

(b) **(i)** $\mathbf{AB} = \begin{pmatrix} 6 & 18+k \\ 0 & 2k \end{pmatrix}$;

$\mathbf{BA} = \begin{pmatrix} 6 & 14 \\ 0 & 2k \end{pmatrix}$

(ii) $k = -4$

7. **(i)** $w = 8, x = 9, y = 2, z = 10$

(ii) $w = -13\frac{1}{2}, x = -1, y = -6\frac{1}{2}$,

$z = \frac{1}{4}$

8. **(i)** $\begin{pmatrix} -4 & 4 \\ -8 & 5 \end{pmatrix}$ **(ii)** $\begin{pmatrix} 1 & 1 \\ 11 & -12 \end{pmatrix}$

(iii) $\begin{pmatrix} 11 & -4 \\ -27 & 16 \end{pmatrix}$

9. $p = 0, q = 15$

10. **(i)** $\begin{pmatrix} 1440 \\ 1140 \\ 1224 \end{pmatrix}$ **(ii)** $\begin{pmatrix} 1500 \\ 1215 \\ 1286 \end{pmatrix}$

11. $\begin{pmatrix} 0.3 & 0.4 \\ 0.4 & -0.3 \end{pmatrix}$

12. $\mathbf{AB} = \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}$, $\mathbf{AC} = \begin{pmatrix} 4 & -2 \\ -2 & 4 \end{pmatrix}$;

$\mathbf{A}^{-1} = \frac{1}{8}\begin{pmatrix} 2 & -3 \\ 2 & 1 \end{pmatrix}$; $x = -1, y = 0$

13. $\mathbf{AB} = \begin{pmatrix} 3 & 8 \\ 7 & 19 \end{pmatrix}$,

$\mathbf{BA} = \begin{pmatrix} 10 & 14 & 7 \\ 6 & 8 & 4 \\ 5 & 8 & 4 \end{pmatrix}$

14. $\mathbf{B}^{-1} = \begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix}$

(i) $\begin{pmatrix} 9 & -20 \\ 7 & -13 \end{pmatrix}$

(ii) $\begin{pmatrix} -52 & 70 \\ 34 & 44 \end{pmatrix}$

15. **(a)** $x = 0.4, y = -2.4$, intersect at $(0.4, -2.4)$

(b) Infinite solutions, same line

(c) No solution, parallel lines

16. **(i)** $\begin{pmatrix} 72 \\ 45 \end{pmatrix}$

17. **(a)** $x = -20, y = -3$

(b) **(i)** $\begin{pmatrix} 9 \\ 13 \end{pmatrix}$ **(ii)** Not possible

(iii) $\begin{pmatrix} 2 & 1 \\ 6 & 3 \end{pmatrix}$

(iv) $\begin{pmatrix} 5 \end{pmatrix}$

18. $a = 2, b = 0, c = 1.5$

19. **(i)** $\mathbf{P} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$ **(ii)** $\begin{pmatrix} 41 \\ 17 \\ 31 \end{pmatrix}$

20. **(i)** $\begin{pmatrix} 65 & 70 \\ 50 & 60 \end{pmatrix}$ **(ii)** $\begin{pmatrix} 288.50 \\ 233 \end{pmatrix}$

21. **(i)** $\begin{pmatrix} 258.9 \\ 282.35 \end{pmatrix}$ **(ii)** $\begin{pmatrix} 2.5 \\ 3.15 \\ 3.5 \end{pmatrix}$

(iii) $\begin{pmatrix} 924.6 \\ 1009.25 \end{pmatrix}$

22. **(i)** $\begin{pmatrix} 3850 \\ 4620 \\ 4600 \end{pmatrix}$ **(ii)** $\begin{pmatrix} 334 \\ 313 \\ 447 \end{pmatrix}$

(iii) $\begin{pmatrix} 130.7 \end{pmatrix}$ **(iv)** $\begin{pmatrix} 1224.7 \end{pmatrix}$

23. **(i)** $\begin{pmatrix} 50 & 60 & 70 & 40 \\ 30 & 40 & 50 & 30 \\ 40 & 30 & 60 & 50 \end{pmatrix}\begin{pmatrix} 3.20 \\ 3.10 \\ 3.00 \\ 3.30 \end{pmatrix}$

$= \begin{pmatrix} 688 \\ 469 \\ 566 \end{pmatrix}$

(ii) $\begin{pmatrix} 25 & 12 & 15 \end{pmatrix}\begin{pmatrix} 50 & 60 & 70 & 40 \\ 30 & 40 & 50 & 30 \\ 40 & 30 & 60 & 50 \end{pmatrix}$

$= \begin{pmatrix} 2210 & 2430 & 3250 & 2110 \end{pmatrix}$

(iii) 10 000

24. **(i)** $\begin{pmatrix} 12 & 8 & 11 \\ 18 & 11 & 7 \\ 8 & 9 & 15 \end{pmatrix}\begin{pmatrix} 48 \\ 32 \\ 26 \end{pmatrix}$

$= \begin{pmatrix} 1118 \\ 1398 \\ 1062 \end{pmatrix}$

(ii) $\begin{pmatrix} 9.80 & 10.40 & 9.90 \end{pmatrix}\begin{pmatrix} 1118 \\ 1398 \\ 1062 \end{pmatrix}$

$= \begin{pmatrix} 36\ 009.40 \end{pmatrix}$

25. **(i)** $\begin{pmatrix} 125 & 320 \\ 200 & 160 \\ 90 & 450 \end{pmatrix}\begin{pmatrix} 0.16 \\ 0.4 \end{pmatrix}$

$= \begin{pmatrix} 148 \\ 96 \\ 194.4 \end{pmatrix}$

(ii) $\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 148 \\ 96 \\ 194.4 \end{pmatrix}$

$= \begin{pmatrix} 438.40 \end{pmatrix}$

CHAPTER 10 – REVISION: GEOMETRY AND MEASUREMENT

Revision 10A – Angles, Triangles and Polygons

1. **(a)** 30 **(b)** 36
2. **(a)** 140° **(b)** 129
3. **(a)** 17 **(b)** 27
4. **(i)** 72° **(ii)** 66° **(iii)** 66°
5. **(i)** 26° **(ii)** 102° **(iii)** 128°
6. 56°, 62°
7. **(i)** 38° **(ii)** 79° **(iii)** 71°
8. **(i)** 72° **(ii)** 36°
10. **(i)** 105° **(ii)** 41
11. **(i)** $y - 2x = 5$, $3x + y = 184$ **(ii)** $x = 35.8$, $y = 76.6$
12. **(i)** 70° **(ii)** 4.70 cm
13. **(i)** 160° **(ii)** 150°
14. **(i)** 30° **(ii)** 150°
15. **(i)** 108° **(ii)** 60° **(iii)** 120°
16. **(i)** 156°, 132° **(ii)** Yes **(iii)** 132°
17. **(b)** AB; AC; B; C
19. **(i)** 5.7 cm **(ii)** 5.1 cm
20. **(iii)** Square
23. **(i)** 53° **(iii)** 4.2 cm
26. **(i)** 8 cm

Revision 10B – Congruence and Similarity

1. 420 litres
2. **(i)** 1 m **(ii)** $\frac{27}{64}V$ litres
3. **(a)** **(i)** 8 : 11 **(ii)** 512 : 1331 **(b)** \$6.05
4. **(i)** 3 : 5 **(ii)** 9 : 25
5. 432.64 cm^3
6. **(i)** 3375 : 1 **(ii)** 225 : 1
7. **(i)** 204 m **(ii)** \$1 440 000 **(iii)** 2 g
8. **(i)** 73.5 cm^2 **(ii)** 50 cm^2
9. **(i)** 9 cm **(ii)** $\frac{1}{3}$ **(iii)** $\frac{15}{16}$
10. **(i)** 45 cm^2 **(ii)** 25 cm^2
11. **(i)** 24 cm^2 **(ii)** 21.6 cm^2
12. 50 cm^2
13. **(i)** 2.4 cm **(ii)** 3 : 2 **(iii)** 9 : 16
14. **(ii)** $\triangle ADC$ and $\triangle BCD$ **(iii)** $\triangle PAB$ and $\triangle PDC$
15. **(b)** **(i)** 9 : 16 **(ii)** 3 : 14
16. **(a)** **(i)** $13\frac{1}{3}$ cm **(ii)** $9\frac{3}{5}$ cm **(b)** $\frac{25}{9}k$ cm^2
17. **(ii)** 9 cm
18. 5 cm
19. **(b)** $\triangle EBC$ or $\triangle BFA$ **(c)** $\triangle ABC$ and $\triangle CDA$ **(d)** **(i)** $\frac{1}{2}$ **(ii)** $\frac{1}{3}$ **(iii)** $\frac{1}{4}$
20. **(i)** 8 cm **(ii)** 980 cm^3

Revision 10C – Pythagoras' Theorem and Trigonometry

1. **(i)** 35° or 145° **(ii)** 155° **(iii)** 30° or 150° **(iv)** 120°
2. **(i)** $\sqrt{25-x^2}$ **(ii)** $\frac{\sqrt{25-x^2}}{5}$ **(iii)** $-\frac{\sqrt{25-x^2}}{5}$
3. **(a)** **(i)** 48.2° **(ii)** 8.94 m **(b)** 38.7°
4. $AB = 5.20$ cm; $AD = 10.4$ cm
5. **(i)** 13.0 cm **(ii)** 37.4 cm^2
6. **(a)** 209 m **(b)** 2.22 m
7. **(a)** **(i)** 150° **(ii)** 330° **(b)** 2.85 km
8. **(i)** 102° **(ii)** 162°
9. **(a)** **(i)** 32.3° **(ii)** 8.99 cm **(iii)** 15.5 cm **(b)** 25.6 cm^2
10. **(i)** 8.12 m **(ii)** 13.3°
11. **(i)** 42 km **(ii)** 82° **(iii)** 138.2°
12. **(i)** 6.50 km **(ii)** 334.4° **(iii)** 7.19 km^2
13. **(i)** 056.1° **(ii)** 12.7 m **(iii)** 56.0 m **(iv)** 939 m^2
14. **(ii)** 15 km **(iii)** 030°
15. **(b)** 3.65 or –3.49 **(c)** **(i)** 42.6 cm **(ii)** 69.8 cm^2
16. **(i)** 21.8° **(ii)** 19.4° **(iii)** 77.1°
17. **(a)** **(i)** 337.5° **(ii)** 60.7 km **(b)** 1755 or 5.55 pm **(c)** **(i)** 43.4° **(ii)** 41.9 km **(iii)** 2530 km^2
18. **(a)** **(i)** 109 m **(ii)** 198° **(b)** 7.3°
19. **(a)** **(i)** 33.6 m **(ii)** 44.6° **(b)** 51.2°
20. **(ii)** 7 or $-\frac{3}{13}$ **(iii)** 105 cm^2

Revision 10D – Mensuration

1. **(a)** 54° **(b)** **(i)** 14 cm **(ii)** 92.4 cm^2
2. **(i)** $(28 + 4.2\pi)$ cm **(ii)** $29\frac{2}{5}\pi$ cm^2
3. **(i)** 201 cm^2 **(ii)** 73 cm^2
4. **(i)** 1980 cm^3 **(ii)** 21 200 g
5. **(i)** $\frac{3}{8}$ cm **(ii)** 214π cm^2
6. **(i)** 20 cm **(ii)** 106 cm^3
7. **(a)** 7 mm **(b)** 126
8. **(a)** **(i)** 134 cm **(ii)** 4290 cm^2 **(iii)** 1170 cm^2 **(b)** 0.0375 m^3 **(c)** \$9.40
9. $3\frac{1}{5}$ cm
10. 616 cm^2
11. **(i)** 9.00 cm **(ii)** 1.79 cm^2
12. 162 cm^2
13. 35.0 cm^2

14. 13.1%

15. (a) (i) 27 440 cm^3

(ii) 52.4%

(b) 406

16. (i) 38.4 cm

(ii) 37.8 cm^2

17. (i) 5.50 cm

(ii) 19.2 cm^2

(iii) 154 cm^3

(iv) 194 cm^2

18. (i) 1740 cm^3 (ii) 897 cm^2

19. (a) $\frac{1}{6}$

(b) (i) 3.59 cm (ii) 78.7 cm^2

(iii) 12.9 cm^3

20. (i) 39.8 cm (ii) 4.07 cm

(iii) 10 h 46 min

Revision 10E – Geometrical Transformation and Symmetry

1. (i) (2, 5) (ii) (6, −7)

(iii) (4, 4) (iv) (4, 6)

2. (i) (−1, 4) (ii) (3, 0)

(iii) (3, −6) (iv) (5, 4)

3. (i) (5, 3) (ii) (−2, −1)

(iii) (−1, 3) (iv) (−3, 3)

4. (i) 144° anticlockwise rotation about O

(ii) reflection in AO

(iii) reflection in OD

5. (e) A translation $\begin{pmatrix} -4 \\ -4 \end{pmatrix}$

6. (i) 120° clockwise rotation about O

(ii) Reflection in AD followed by 120° anticlockwise rotation about O.

8. (a) $a = -4$, $b = 5$ (b) $x = 2.5$

9. (a) 6

(b) (i) 120° (ii) 90°

10. (a) $y = 0$, $x = 2.5$ (b) 2, (2.5, 0)

Revision 10F – Coordinate Geometry

1. (a) 1 (b) $y = 5x - 7$

2. $k = -1$; $y = 2x - 11$

3. (a) 10 units (b) $y = -\frac{3}{4}x - \frac{1}{2}$

4. (i) $P(9, 0)$, $Q(0, 6)$

(ii) 10.8 units

5. 7.21 units

6. 33

7. (a) −17 (b) $\frac{2}{7}$

(c) $a = 1\frac{2}{17}$, $b = 1\frac{7}{12}$, gradient $= -1\frac{5}{12}$

8. (i) $-\frac{3}{4}$ (ii) (5, 5)

9. (a) (i) $\tan^{-1}\left(\frac{4}{3}\right) = 53.1°$

(ii) $y = -\frac{4}{3}x + 4$

(b) $y = \frac{4}{3}x + 4$

10. $y = -\frac{3}{2}x + \frac{1}{4}$

11. (a) $y = \frac{1}{3}x + 3$

(b) (i) 18 units² (ii) $y = -\frac{2}{3}x$

12. $k = 0$ or 3

13. (a) (i) 2 units; 6.32 units; 7.21 units

(ii) 6 units²

(b) $t = 2$ or −4

14. (a) (0, 9) (b) $\left(2\frac{5}{11}, 0\right)$

15. 3.84 units

16. (i) 24.5 units² (ii) 7.28 units

(iii) 6.73 units

17. (b) (2, 4)

18. (i) 15 units (i) (−12, 9)

(iii) 48 units² (iv) 8.54 cm

19. (i) $y = 4x - 2$ (i) (5, 2)

(iii) 3.88 units

21. (a) 3 (b) $-\frac{3}{8}$

22. (a) $-\frac{1}{4}$ (b) 8

24. $y = 3x + 13$

25. (a) $y + 2x = 30$, $2y - x = 12$

(b) $C\left(9\frac{3}{5}, 10\frac{4}{5}\right)$

Revision 10G – Vectors

1. 13

2. (i) $\begin{pmatrix} 3 \\ 6 \end{pmatrix}$ (ii) 6.71 units

(iii) $\left(\frac{13}{2}, 13\right)$

3. (i) 5 units (ii) ±4

4. (i) $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ (ii) $\begin{pmatrix} -1 \\ 0 \end{pmatrix}$

5. (i) $2\mathbf{q}$ (ii) $6\mathbf{p} + 2\mathbf{q}$

(iii) $6\mathbf{q} + 3\mathbf{p}$ (iv) $4\mathbf{q} - 3\mathbf{p}$

(v) $6\mathbf{q} - 3\mathbf{p}$

6. $\overrightarrow{PQ} = \begin{pmatrix} -1 \\ 8 \end{pmatrix}$, $\overrightarrow{RQ} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$,

$\overrightarrow{MN} = \begin{pmatrix} \frac{5}{2} \\ \frac{7}{2} \end{pmatrix}$, $\overrightarrow{XY} = \begin{pmatrix} -\frac{5}{2} \\ -\frac{7}{2} \end{pmatrix}$

8. (i) 8.25 units (ii) (6, 4)

(iii) 6.32 units

9. (i) $4\mathbf{b} - 3\mathbf{a}$ (ii) $\mathbf{b} - 3\mathbf{a}$

(iii) $\frac{1}{3}(3\mathbf{a} + 2\mathbf{b})$ (iv) $\frac{2}{3}(3\mathbf{a} + 2\mathbf{b})$

10. (a) 6 units

(b) (i) $\begin{pmatrix} -2 \\ -4 \end{pmatrix}$ (ii) $\begin{pmatrix} 4 \\ 0 \end{pmatrix}$

(iii) $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$

11. (i) $\frac{1}{2}(\mathbf{p} + \mathbf{q})$ (ii) $\frac{1}{3}(\mathbf{p} + \mathbf{q} + \mathbf{r})$

12. (a) $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ (b) (9, 5)

(c) (i) 6.08 units

(ii) $-\frac{1}{6}$

(iii) $y = -\frac{1}{6}x + 2$

(iv) $\left(3, \frac{3}{2}\right)$

13. (i) 13 units

(ii) $\begin{pmatrix} 2 \\ 27 \end{pmatrix}$

(iii) $p = \frac{13}{2}$, $q = -5$

14. (i) $\begin{pmatrix} -9 \\ 12 \end{pmatrix}$

(ii) $\begin{pmatrix} 5 \\ 12 \end{pmatrix}$, 13 units

(iii) $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$

15. (a) 5 units

(b) (i) $\begin{pmatrix} -1 \\ -5 \end{pmatrix}$ (ii) $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$

(iii) $\begin{pmatrix} -4 \\ 0 \end{pmatrix}$ (iv) $\begin{pmatrix} -6 \\ -10 \end{pmatrix}$

16. **(i)** trapezium **(ii)** $-(\mathbf{a}+\mathbf{b})$

17. **(a)** **(i)** $3\mathbf{p}+\mathbf{q}$ **(ii)** $4\mathbf{q}-4\mathbf{p}$

(iii) $\frac{1}{2}(7\mathbf{q}-3\mathbf{p})$

(b) $7\mathbf{q}-3\mathbf{p}$

18. **(b)** $\overrightarrow{OP}=-2\mathbf{t}-5\mathbf{s}$, $\overrightarrow{OQ}=\frac{8}{3}\mathbf{t}-5\mathbf{s}$

$\overrightarrow{OR}=-\mathbf{t}+7\mathbf{s}$; $\overrightarrow{OU}=\frac{2}{3}\mathbf{t}-4\mathbf{s}$

19. **(a)** **(i)** $2(\mathbf{a}+3\mathbf{b})$ **(ii)** $\mathbf{a}+3\mathbf{b}$

(iii) $\frac{3}{2}(\mathbf{b}-\mathbf{a})$ **(iv)** $\mathbf{b}-\mathbf{a}$

(c) trapezium

(d) **(i)** $\frac{1}{3}$ **(ii)** $\frac{9}{4}$

20. **(a)** **(i)** $\mathbf{b}-\mathbf{a}$

(ii) $\frac{1}{2}(\mathbf{b}-\mathbf{a})$

(iii) $\frac{1}{2}(\mathbf{a}+\mathbf{b})$

(c) **(i)** $\mathbf{a}+\mathbf{b}$ **(ii)** $\mathbf{b}$

(iii) $\mathbf{a}$

(d) parallelogram

Revision 10H – Properties of Circles

1. 2.31 cm
2. **(i)** 85 cm^2 **(ii)** 12.8 cm
3. 13.2 cm
4. 9.60 cm
5. **(i)** 56° **(ii)** 62°
   (iii) 56°
6. **(i)** 64° **(ii)** 32°
   (iii) 32° **(iv)** 26°
7. 61°
8. **(i)** 132° **(ii)** 26°
   (iii) 24° **(iv)** 40°
9. **(i)** 27° **(ii)** 36°
   (iii) 54°
10. **(i)** 61° **(ii)** 96°
11. **(i)** 49° **(ii)** 27°
12. **(i)** 43° **(ii)** 47°
13. **(i)** 120° **(ii)** 70°
    (iii) 20° **(iv)** 40°
14. **(i)** 24° **(ii)** 68°
    (iii) 66°
15. 62°
16. $x=10, y=5$
17. 114°
18. **(i)** 96° **(ii)** 84°
19. **(i)** 37° **(ii)** 81°
    (iii) 37°
20. **(i)** 40° **(ii)** 40°
    (iii) 50°

CHAPTER 11 – REVISION: PROBABILITY AND STATISTICS

Revision 11A – Probability

1. **(i)** $\frac{x}{x+36}$ **(ii)** 24
2. **(i)** $\frac{1}{4}$ **(ii)** 2
   (iii) 8
3. **(a)** 9, 12, 18, 21, 24;
   12, 16, 20, 24, 28, 32;
   15, 25, 30, 35, 40;
   18, 24, 30, 36, 42, 48;
   21, 28, 35, 42, 49;
   24, 32, 40, 48, 56, 64

   (b) **(i)** $\frac{1}{4}$ **(ii)** $\frac{1}{3}$
   (iii) 0 **(iv)** $\frac{1}{6}$
4. **(i)** 0.7; 0.8; 0.2; 0.2; 0.8
   (ii) 0.38
5. **(ii)** $\frac{17}{24}$
6. **(a)** EP, ER; EE, EP, EP, ER; PE, PP, PP, PR

   (b) **(i)** $\frac{1}{6}$ **(ii)** $\frac{3}{4}$
7. **(i)** $\frac{2}{15}$ **(ii)** $\frac{2}{5}$
   (iii) $\frac{8}{45}$
8. **(i)** $\frac{35}{88}$ **(ii)** $\frac{23}{44}$
   (iii) $\frac{21}{44}$
9. **(i)** $\frac{1}{12}$ **(ii)** $\frac{1}{24}$
   (iii) $\frac{1}{32}$
10. **(i)** $\frac{1}{2}$ **(ii)** $\frac{11}{18}$
    (iii) $\frac{4}{15}$
11. **(i)** $\frac{1}{6}$ **(ii)** $\frac{1}{4}$
    (iii) $\frac{1}{12}$
12. **(i)** $\frac{1}{9}$ **(ii)** $\frac{1}{18}$
    (iii) $\frac{17}{18}$
13. **(i)** $\frac{1}{216}$ **(ii)** $\frac{1}{72}$
    (iii) $\frac{5}{9}$
14. **(a)** $\frac{8}{15}, \frac{1}{5}, \frac{1}{2}, \frac{2}{7}; \frac{3}{14}, \frac{4}{7}, \frac{3}{14}, \frac{4}{7}, \frac{2}{7}, \frac{1}{7}$

    (b) **(i)** $\frac{4}{35}$ **(ii)** $\frac{68}{105}$
    (iii) $\frac{4}{15}$ **(iv)** $\frac{13}{35}$
15. **(i)** $\frac{98}{575}$ **(ii)** $\frac{252}{575}$
    (iii) $\frac{52}{115}$
16. **(i)** $\frac{1}{3}$ **(ii)** $\frac{5}{24}$
    (iii) $\frac{13}{24}$
17. **(ii)** $\frac{1}{4}$ or $\frac{3}{4}$ **(iii)** $\frac{9}{16}$
18. **(i)** $\frac{9}{100}$ **(ii)** $\frac{7}{15}$
    (iii) $\frac{133}{300}$
19. **(a)** $x=12, y=24$

    (b) **(i)** $\frac{103}{295}$ **(ii)** $\frac{24}{295}$
20. **(i)** $\frac{4}{21}$ **(ii)** $\frac{9}{35}$
    (iii) $\frac{58}{105}$ **(iv)** 0
    (v) 0 **(vi)** 1
21. $\frac{5}{9}$
22. 0.449
23. **(i)** $\frac{3}{8}$ **(ii)** $\frac{7}{16}$
    (iii) $\frac{13}{16}$

Revision 11B – Statistics

1. (a) 1, 2, 3, 4 or 5
 (b) (i) 5 (ii) 11.8
2. (a) 27
 (b) (i) \$13 500 (ii) \$3240
3. (i) 32 (ii) 34
 (iii) 15
4. (i) 57 (ii) 6.9 kg
 (iii) 6.3 kg (iv) 36.8%
6. (ii) 56.9 kg
7. (a) 85
 (b) (i) 9.5 km (ii) 1.2 km
8. (a) (i) 79 cents (ii) 64 cents
 (b) (ii) 45.5%
9. (a) 8; 40; 34; 16; 2
 (b) (i) 72.8 mm (ii) 18.2 mm
10. (ii) City A: Mean = 81
 Standard deviation = 16.1
 City B: Mean = 67
 Standard deviation = 11.4
11. (a) positive, strong
 (b) negative, strong
 (c) positive, weak
 (d) zero correlation
12. (i) positive, weak correlation
 (ii) negative, strong correlation
 (iii) zero correlation
14. (i) 20, 25, 37
 (ii) 62.6, 21.9
 (iii) $\frac{17}{550}$
15. (a) (i) 88 (ii) 48
 (iii) 114
 (c) 104
 (d) (i) Supermarket Q
 (ii) Supermarket Q
16. (i) 54, 51, 9 (ii) 184
 (iii) 3.80 (v) 4.80
18. (i) 60.3, 24.4
 (ii) End-of-year examination
 (iii) Mid-year examination
19. (i) 21 (iii) 6
20. (a) (i) 81 (ii) 120
 (iii) 52
 (b) (i) 75 (ii) 94
 (iii) 41
 (c) Company A
 (d) Company B
21. (a) 500, 548, 48; 510, 42
 (b) (i) No (ii) Yes
22. (i) 10.4, 1.91 (iii) Battery from factory B
23. (b) 4
 (c) 4.51, 2.27
24. (i) 73.7, 7.29 (iii) No

PROBLEMS IN REAL-WORLD CONTEXTS

1. (a) $T = -3.6t + 76.5$
 (b) (i) 5.7 minutes
 (ii) −13.5 °C
 (c) (i) 57
 (ii) 21.6 °C
2. (a) 0.398 m
 (b) 1.55 m^3
 (c) 929 kg
3. (ii) Singapore
5. (a) (i) \$596 234 545
 (b) (i) \$3 191 957 836
 (ii) \$3 789 405 063
 (c) The Shipwreck
 (d) $205\,400\,000(1.035)^{22}$
6. (b) $\begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 0 & 2 & 1 & 0 \\ 2 & 0 & 1 & 2 \end{pmatrix}$
 (c) $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 1 & 2 & 1 & 1 \\ 2 & 1 & 2 & 2 \end{pmatrix}$
 (d) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$
7. (a) $\frac{1}{10\,000}$
 (b) 0.002 297
 (c) No
 (d) $\frac{1}{100\,000\,000}$
 (e) $\frac{1}{10\,000}$
 (f) No
8. (a) (i) 2012
 (b) (i) No
 (c) (i) 2013
 (ii) No

SPECIMEN PAPER

Paper 1

1. (a) $\frac{22}{63}$ (b) $\frac{26}{27}$
2. (a) 1 : 200 000 (b) 48 km
3. (a) $3(x - a)(b + 2y)$
 (b) $(x + 1)(x - 1)(x + 3)(x - 3)$
4. (b) 10
5. 308°
6. $a = 2, h = 20$
7. (a) $-\frac{b^{1\frac{11}{20}}}{3a^{2\frac{1}{4}}}$ (b) No
8. 192 horsepower
9. (a) $11\frac{1}{3} < x \text{ , } 12$
 (b) (i) $11\frac{1}{3}$ (ii) 12
10. (a) III
 (b) (i) $(x + 3)^2 - 2$
 (iii) $x = -3$
11. (c) 15.2 km
12. (a) 9.25 km (b) 08 10 – 08 15
 (c) 1 km/min
13. (a) 9 (b) 101
 (c) $2n + 1$
14. (a) 10 units (b) $3y + 4x = 15$
15. (a) $(0, 3\frac{1}{5})$ (b) $5y = 2x + 16$
 (c) $15\frac{1}{5}$ units^2
16. (b) {6}
 (c) {3, 5, 7, 11}
17. (a) $1\frac{3}{4}$ (b) $x = \frac{11y^2 + 4}{3y^2 - 5}$
18. (a) $\triangle CPT$ and $\triangle DAT$, $x = 4\frac{8}{19}$
 (b) $-1\frac{7}{12}$
19. (a) 24, 24 (b) $\frac{4}{11}$
 (c) 16

20. **(a)** $a = \frac{2}{5}$, $b = \frac{3}{5}$ **(b)** $\frac{8}{15}$

(c) **(i)** $\frac{1}{5}$ **(ii)** $\frac{1}{5}$

SPECIMEN PAPER

Paper 2

1. **(a)** $\begin{pmatrix} 1940 \\ 1580 \\ 1870 \end{pmatrix}$

(b) $\begin{pmatrix} 1260 & 1370 & 1190 & 1570 \end{pmatrix}$

(c) $\begin{pmatrix} 64\ 080 \\ 51\ 860 \\ 61\ 610 \end{pmatrix}$

2. **(a)** 2055 m **(b)** 790 m
(c) 14 min
(d) **(i)** 267° **(ii)** 061°

3. **(a)** $\frac{210}{x}$
(c) $x = 2.876$, 2 h 53 min, 73 km/h
(d) S\$0.90/litre

4. **(a)** 31.8 cm **(b)** 100 cm
(c) 18.5° **(d)** 57.5°

5. **(a)** **(i)** $1727\frac{1}{4}\pi$ cm^3
(ii) 742π cm^2
(b) **(i)** 1 : 4 **(ii)** 60 cm^3
(iii) $\frac{1}{2}\sqrt[3]{7}h$

6. **(a)** 20° **(b)** 30°
(c) 30° **(d)** 50°

7. **(ii)** Strong, positive correlation
(iv) 74
(v) 59
(vi) 0.9, $y = 0.9x + 8.5$

8. **(a)** $\mathbf{a} + \mathbf{b}$ **(b)** $\mathbf{a} + 2\mathbf{b}$
(c) $-\frac{1}{2}\mathbf{a} - 1\frac{1}{2}\mathbf{b}$ **(d)** $-\frac{1}{6}\mathbf{a} - \frac{1}{2}\mathbf{b}$

9. **(a)** \$261.94 **(b)** \$118.75
(c) \$248.30 **(d)** water: 15 m^3 electricity: 897 kWh

10. **(a)** $p = 84$
(c) **(i)** 73 **(ii)** 2.5
(d) **(i)** 12
(e) **(ii)** $t = 2$ or 5

11. **(a)** **(i)** 86 cents **(ii)** 42 cents
(b) **(i)** $\frac{43}{100}$ **(ii)** $\frac{19}{50}$

12. **(b)** **(ii)** 4
(c) $y = x - 4$
(d) $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

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