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7th EDITION

NEW SYLLABUS
# MATHEMATICS

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Consultant • Dr Yeap Ban Har   Authors • Dr Joseph Yeo • Teh Keng Seng • Loh Cheng Yee
• Ivy Chow • Ong Chan Hong • Jacinth Liew

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PREFACE

New Syllabus Mathematics (NSM)

is a series of textbooks specially designed to provide valuable learning experiences to engage the hearts and minds of students sitting for the GCE O level examination in Mathematics. Included in the textbooks are **Investigation**, **Class Discussion**, **Thinking Time**, **Journal Writing**, **Performance Task** and **Problems in Real-World Contexts** to support the teaching and learning of Mathematics.

Every chapter begins with a chapter opener which motivates students in learning the topic. Interesting stories about Mathematicians, real-life examples and applications are used to arouse students' interest and curiosity so that they can appreciate the beauty of Mathematics in their surroundings.

The use of ICT helps students to visualise and manipulate mathematical objects more easily, thus making the learning of Mathematics more interactive. Ready-to-use interactive ICT templates are available at http://www.shinglee.com.sg/StudentResources/

KEY FEATURES

CHAPTER OPENER
Each chapter begins with a chapter opener to arouse students' interest and curiosity in learning the topic.

LEARNING OBJECTIVES
Learning objectives help students to be more aware of what they are about to study so that they can monitor their own progress.

RECAP
Relevant prerequisites will be revisited at the beginning of the chapter or at appropriate junctures so that students can build upon their prior knowledge, thus creating meaningful links to their existing schema.

WORKED EXAMPLE
This shows students how to apply what they have learnt to solve related problems and how to present their working clearly. A suitable heading is included in brackets to distinguish between the different Worked Examples.

PRACTISE NOW
At the end of each Worked Example, a similar question will be provided for immediate practice. Where appropriate, this includes further questions of progressive difficulty.

SIMILAR QUESTIONS
A list of similar questions in the Exercise is given here to help teachers choose questions that their students can do on their own.

EXERCISE
The questions are classified into three levels of difficulty – Basic, Intermediate and Advanced.

SUMMARY
At the end of each chapter, a succinct summary of the key concepts is provided to help students consolidate what they have learnt.

REVIEW EXERCISE
This is included at the end of each chapter for the consolidation of learning of concepts.

CHALLENGE YOURSELF
Optional problems are included at the end of each chapter to challenge and stretch high-ability students to their fullest potential.

REVISION EXERCISE
This is included after every few chapters to help students assess their learning.

Learning experiences have been infused into Investigation, Class Discussion, Thinking Time, Journal Writing and Performance Task.
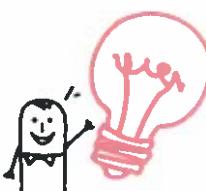
Investigation

Activities are included to guide students to investigate and discover important mathematical concepts so that they can construct their own knowledge meaningfully.

Class Discussion

Questions are provided for students to discuss in class, with the teacher acting as the facilitator. The questions will assist students to learn new knowledge, think mathematically, and enhance their reasoning and oral communication skills.

Thinking Time

Key questions are also included at appropriate junctures to check if students have grasped various concepts and to create opportunities for them to further develop their thinking.
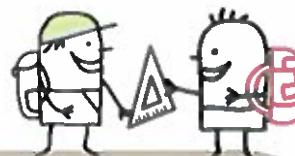
Journal Writing

Opportunities are provided for students to reflect on their learning and to communicate mathematically. It can also be used as a formative assessment to provide feedback to students to improve on their learning.

Performance Task

Mini projects are designed to develop research and presentation skills in the students.

MARGINAL NOTES

ATTENTION

This contains important information that students should know.

Problem Solving Tip

This guides students on how to approach a problem.

INFORMATION

This includes information that may be of interest to students.

RECALL
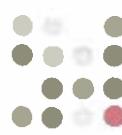
This contains certain mathematical concepts or rules that students have learnt previously.

Just For Fun

This contains puzzles, fascinating facts and interesting stories about Mathematics as enrichment for students.

Internet Resources

This guides students to search on the Internet for valuable information or interesting online games for their independent and self-directed learning.

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Quadratic Equations and Functions

A ball is thrown over the net. What do you notice about the path of the ball? The path of the ball can be described by a quadratic function. We can use the formula $h = ut + \frac{1}{2}at^2$ to find the height of the ball t seconds after leaving the hand. This idea is used in the study of mechanics.

Chapter
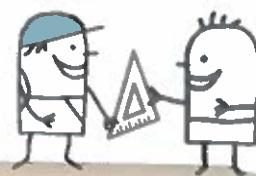
One

LEARNING OBJECTIVES

At the end of this chapter, you should be able to:

- solve quadratic equations in one variable by
  - completing the square for equations of the form $x^2 + px + q = 0$,
  - use of formula,
  - graphical method.
- solve fractional equations that can be reduced to quadratic equations,
- formulate a quadratic equation in one variable to solve problems,
- sketch the graphs of quadratic equations of the form
  $y = (x - h)(x - k)$, $y = -(x - h)(x - k)$, $y = (x - p)^2 + q$ and $y = -(x - p)^2 + q$.

1.1 Solving Quadratic Equations by Completing the Square

Recap

In Book 2, we have learnt that a quadratic equation is of the form

$ax^2 + bx + c = 0$, where a, b and c are real numbers and $a \neq 0$.

Worked Example 1

(Solving a Quadratic Equation by Factorisation)

Solve the equation $x^2 - 5x - 6 = 0$.

Solution:

$x^2 - 5x - 6 = 0$

$(x - 6)(x + 1) = 0$ (factorise by using the multiplication frame)

$x - 6 = 0$ or $x + 1 = 0$

$x = 6$ $x = -1$

$\therefore x = 6$ or $x = -1$

> **RECALL**
>
> If two factors P and Q are such that $P \times Q = 0$, then either $P = 0$ or $Q = 0$ or both P and Q are equal to 0.

PRACTISE NOW 1

Solve each of the following equations.

(a) $x^2 + 7x - 8 = 0$

(b) $6y^2 + 7y - 20 = 0$

> **SIMILAR QUESTIONS**
>
> Exercise 1A Questions 1(a)–(f)

Solving Quadratic Equations of the Form $(x + a)^2 = b$

In Worked Example 1, we solved the equation by factorisation. However, the solutions of some quadratic equations cannot be obtained by factorisation. An example of this type of quadratic equation is $x^2 + 6x - 5 = 0$. If this equation can be written in the form $(x + a)^2 = b$, where a and b are real numbers, then it can be solved easily by taking the square roots on both sides of the equation to obtain the solutions.

Worked Example 2

(Solving a Quadratic Equation of the Form $(x + a)^2 = b$)

Solve the equation $(x + 3)^2 = 14$.

Solution:

$(x + 3)^2 = 14$

$x + 3 = \pm\sqrt{14}$ (take the square roots on both sides)

$x + 3 = \sqrt{14}$ or $x + 3 = -\sqrt{14}$

$x = \sqrt{14} - 3$ $x = -\sqrt{14} - 3$

$= 0.742$ (to 3 s.f.) $= -6.74$ (to 3 s.f.)

$\therefore x = 0.742$ or $x = -6.74$

PRACTISE NOW 2

Solve each of the following equations.

(a) $(x + 7)^2 = 100$ (b) $(y - 5)^2 = 11$

Completing the Square for a Quadratic Expression

To express a quadratic equation of the form $x^2 + px + q = 0$ in the form $(x + a)^2 = b$, we first need to learn how to complete the square for a quadratic expression $x^2 + px$.

Let us consider the expansion of $(x + 3)^2$.

As shown in Book 2, we can use algebra discs to represent the expansion $(x + 3)^2 = x^2 + 6x + 9$ in the form of a square array (equal rows and columns of discs) as shown in Fig. 1.1(a) or a multiplication frame as shown in Fig. 1.1(b).

We observe that in the square array,

- the x^2 disc is in the top left-hand corner,
- the nine 1 discs are arranged as a 3 by 3 square at the bottom right-hand corner,
- the six x discs are divided equally into 2 parts, i.e. $6x$ is divided into 2 parts of $3x$.
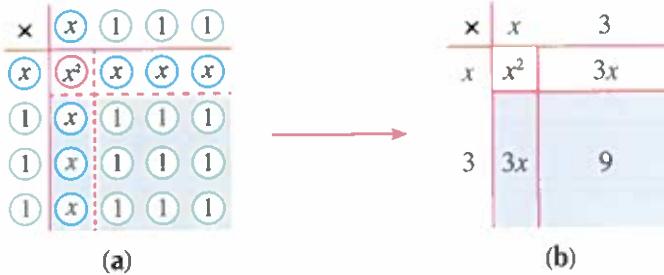
Fig 1.1

Quadratic expressions of the form $(x + a)^2$ can be arranged into a multiplication frame similar to the example in Fig. 1.1.

However, not all quadratic expressions can be expressed in the form $(x + a)^2$. For example, the expression $x^2 + 6x$ can only be arranged as shown in Fig. 1.2.

×	x	3
x	x^2	$3x$
3	$3x$	

Fig. 1.2

Comparing Fig. 1.1 and Fig. 1.2, what number must be added to complete the square? We observe that 9 must be added to $x^2 + 6x$ to make it into $(x + 3)^2$.

However, $x^2 + 6x \neq (x + 3)^2$.

×	x	3
x	x^2	$3x$
3	$3x$	

$\neq$

×	x	3
x	x^2	$3x$
3	$3x$	9
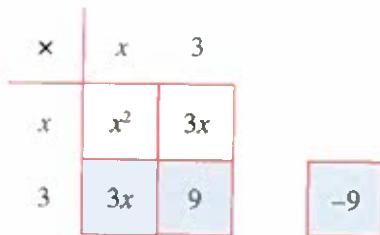
Since we add 9 to $x^2 + 6x$, we must subtract 9 as follows:

$$x^2 + 6x = x^2 + 6x + 9 - 9$$
$$= (x + 3)^2 - 9$$

Pictorially, it looks like

×	x	3
x	x^2	$3x$
3	$3x$	9

-9

ATTENTION

Essentially, we add $9 - 9 = 0$ to $x^2 + 6x$ so that the equality will still hold.

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Investigation

Completing the Square for Quadratic Expressions of the Form $x^2 + px$

To make a quadratic expression of the form $x^2 + px$ into a perfect square $(x + a)^2$, we have to add a number, b, to $x^2 + px$. In this investigation, we will find a relationship between b and p.

Copy and complete Table 1.1. The second one has been done for you.
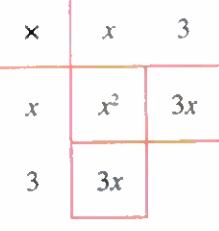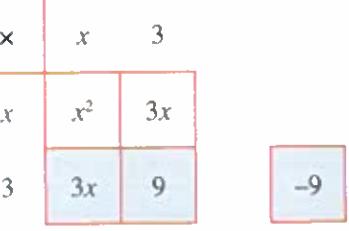
	Quadratic Expression $x^2 + px$	Number that must be added to complete the square, b	$\frac{1}{2} \times$ coefficient of x, $\frac{p}{2}$	Quadratic expression of the form $(x + a)^2 - b$
(a)	$x^2 + 4x$			
(b)	$x^2 + 6x$	$3^2 = 9$	$\frac{6}{2} = 3$	$x^2 + 6x$ $= x^2 + 6x + 3^2 - 3^2$ $= (x + 3)^2 - 9$
(c)	$x^2 + 8x$			
(d)	$x^2 + 10x$			

Table 1.1

1. What is the relationship between b and p?
2. To express $x^2 + px$ in the form $(x + a)^2 - b$, write down an expression of a and of b in terms of p.

From the investigation, on **completing the square**,

if $x^2 + px = (x + a)^2 - b$, then $a = \dfrac{p}{2}$ and $b = \left(\dfrac{p}{2}\right)^2$,

i.e. $x^2 + px = \left(x + \dfrac{p}{2}\right)^2 - \left(\dfrac{p}{2}\right)^2$.

For a quadratic expression of the form $x^2 + px + q$, we can express it as follows:

$$x^2 + px + q = (x^2 + px) + q$$
$$= \left(x + \dfrac{p}{2}\right)^2 - \left(\dfrac{p}{2}\right)^2 + q$$

Worked Example 3

(Completing the Square for Quadratic Expressions)
Express each of the following expressions in the form $(x + a)^2 + b$.

(a) $x^2 + 10x$ **(b)** $x^2 - 5x$

(c) $x^2 + 2x + 3$

Solution:

(a) The coefficient of x is 10. Half of this is 5.

$$\therefore x^2 + 10x = [x^2 + 10x + 5^2] - 5^2$$
$$= (x + 5)^2 - 25$$

(b) The coefficient of x is -5. Half of this is $-\dfrac{5}{2}$.

$$\therefore x^2 - 5x = \left[x^2 - 5x + \left(-\dfrac{5}{2}\right)^2\right] - \left(-\dfrac{5}{2}\right)^2$$
$$= \left(x - \dfrac{5}{2}\right)^2 - \dfrac{25}{4}$$

(c) $x^2 + 2x + 3 = (x^2 + 2x) + 3$

The coefficient of x is 2. Half of this is 1.

$$x^2 + 2x + 3 = [x^2 + 2x + 1^2] - 1^2 + 3$$
$$= (x + 1)^2 + 2$$

PRACTISE NOW 3

Express each of the following expressions in the form $(x + a)^2 + b$.

(a) $x^2 + 20x$ **(b)** $x^2 - 7x$

(c) $x^2 + \dfrac{1}{5}x$ **(d)** $x^2 + 6x - 9$

In Worked Example 4, we will show how to solve a quadratic equation by completing the square.

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Worked Example 4

(Solving a Quadratic Equation by Completing the Square)

Solve the equation $x^2 + 4x - 3 = 0$, giving your answers correct to 2 decimal places.

Solution:

As $x^2 + 4x - 3$ cannot be easily factorised, we need to transform the equation $x^2 + 4x - 3 = 0$ into the form $(x + a)^2 = b$ as follows:

$x^2 + 4x - 3 = 0$

$x^2 + 4x = 3$ (rewrite the equation such that the constant term is on the RHS of the equation)

$x^2 + 4x + \left(\dfrac{4}{2}\right)^2 = 3 + \left(\dfrac{4}{2}\right)^2$ (add $\left(\dfrac{4}{2}\right)^2$ to both sides of the equation to complete the square for the LHS)

$x^2 + 4x + 2^2 = 3 + 2^2$

$(x + 2)^2 = 7$ (factorise the expression on the LHS and simplify the RHS)

$x + 2 = \pm\sqrt{7}$ (take the square roots on both sides)

$x + 2 = \sqrt{7}$ or $x + 2 = -\sqrt{7}$

$x = \sqrt{7} - 2$ $x = -\sqrt{7} - 2$

$= 0.65$ (to 2 d.p.) $= -4.65$ (to 2 d.p.)
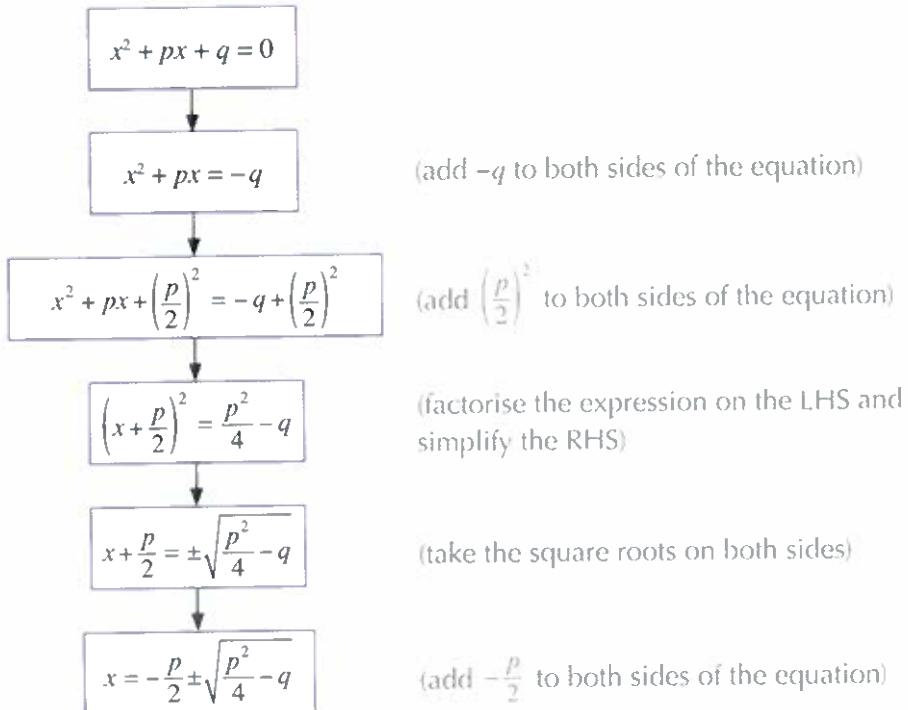
$\therefore x = 0.65$ or $x = -4.65$

> **ATTENTION**
>
> Alternatively, we can complete the square by:
>
> $x^2 + 4x - 3 = 0$
>
> $\left[x^2 + 4x + \left(\dfrac{4}{2}\right)^2\right] - \left(\dfrac{4}{2}\right)^2 - 3 = 0$
>
> $\left(x + \dfrac{4}{2}\right)^2 - 4 - 3 = 0$
>
> $(x + 2)^2 = 7$

In general, the steps taken to solve a quadratic equation $x^2 + px + q = 0$, where p and q are real numbers, by completing the square are as follows:

$$\boxed{x^2 + px + q = 0}$$

$$\downarrow$$

$$\boxed{x^2 + px = -q}$$ (add $-q$ to both sides of the equation)

$$\downarrow$$

$$\boxed{x^2 + px + \left(\dfrac{p}{2}\right)^2 = -q + \left(\dfrac{p}{2}\right)^2}$$ (add $\left(\dfrac{p}{2}\right)^2$ to both sides of the equation)

$$\downarrow$$

$$\boxed{\left(x + \dfrac{p}{2}\right)^2 = \dfrac{p^2}{4} - q}$$ (factorise the expression on the LHS and simplify the RHS)

$$\downarrow$$

$$\boxed{x + \dfrac{p}{2} = \pm\sqrt{\dfrac{p^2}{4} - q}}$$ (take the square roots on both sides)

$$\downarrow$$

$$\boxed{x = -\dfrac{p}{2} \pm \sqrt{\dfrac{p^2}{4} - q}}$$ (add $-\dfrac{p}{2}$ to both sides of the equation)

1. Solve each of the following equations, giving your answers correct to 2 decimal places.

Exercise 1A Questions 4(a)–(h), 5(a)–(d), 6

(a) $x^2 + 6x - 4 = 0$

(b) $x^2 + 7x + 5 = 0$

(c) $x^2 - x - 1 = 0$

2. Solve the equation $(x + 4)(x - 3) = 15$.

Exercise
1A

BASIC LEVEL

1. Solve each of the following equations.

(a) $2x^2 + 5x - 7 = 0$

(b) $4x^2 - 5x - 6 = 0$

(c) $7x + x^2 - 18 = 0$

(d) $4 - 3x - x^2 = 0$

(e) $x(3x - 1) = 2$

(f) $(7 - 3x)(x + 2) = 4$

2. Solve each of the following equations, giving your answers correct to 2 decimal places where necessary.

(a) $(x + 1)^2 = 9$

(b) $(2x + 1)^2 = 16$

(c) $(5x - 4)^2 = 81$

(d) $(7 - 3x)^2 = \dfrac{9}{16}$

(e) $(x + 3)^2 = 11$

(f) $(2x - 3)^2 = 23$

(g) $(5 - x)^2 = 7$

(h) $\left(\dfrac{1}{2} - x\right)^2 = 10$

INTERMEDIATE LEVEL

3. Express each of the following expressions in the form $(x + a)^2 + b$.

(a) $x^2 + 12x$

(b) $x^2 - 6x + 1$

(c) $x^2 + 3x - 2$

(d) $x^2 + 9x - 1$

(e) $x^2 + \dfrac{1}{2}x$

(f) $x^2 - \dfrac{2}{9}x$

(g) $x^2 + 0.2x$

(h) $x^2 - 1.4x$

4. Solve each of the following equations, giving your answers correct to 2 decimal places.

(a) $x^2 + 2x - 5 = 0$

(b) $x^2 + 17x - 30 = 0$

(c) $x^2 - 12x + 9 = 0$

(d) $x^2 - 5x - 5 = 0$

(e) $x^2 + \dfrac{1}{4}x - 3 = 0$

(f) $x^2 - \dfrac{6}{7}x + \dfrac{2}{49} = 0$

(g) $x^2 + 0.6x - 1 = 0$

(h) $x^2 - 4.8x + 2 = 0$

5. Solve each of the following equations.

(a) $x(x - 3) = 5x + 1$

(b) $(x + 1)^2 = 7x$
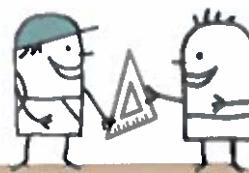
(c) $(x + 2)(x - 5) = 4x$

(d) $x(x - 4) = 2(x + 7)$

ADVANCED LEVEL

6. Given the equation $y^2 - ay - 6 = 0$, where a is a constant, find the expressions for y in terms of a.

1.2 Solving Quadratic Equations by using Formula

The general form of a quadratic equation is $ax^2 + bx + c = 0$, where a, b and c are real numbers and $a \neq 0$. Now, we shall use the method of completing the square to derive a formula for the solution to all quadratic equations.

$$ax^2 + bx + c = 0$$

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \qquad \text{(divide throughout by } a\text{)}$$

$$x^2 + \frac{b}{a}x = -\frac{c}{a} \qquad \text{(rewrite the equation such that the constant term is on the RHS of the equation)}$$

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2 \qquad \text{(add } \left(\frac{b}{2a}\right)^2 \text{ to both sides of the equation to make the LHS a perfect square)}$$

$$\left(x + \frac{b}{2a}\right)^2 = -\frac{c}{a} + \frac{b^2}{4a^2} \qquad \text{(factorise the expression on the LHS and simplify the RHS)}$$

$$= \frac{b^2 - 4ac}{4a^2}$$

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} \qquad \text{(take the square roots on both sides)}$$

$$= \pm\frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\therefore x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{or} \quad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

In general,

> if $ax^2 + bx + c = 0$, where a, b and c are real numbers and $a \neq 0$, then
> $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The above formula for solving quadratic equations is usually used when the quadratic expression cannot be factorised easily.

Worked Example 5

Solve the equation $3x^2 + 4x - 5 = 0$.

Solution:

Comparing $3x^2 + 4x - 5 = 0$ with $ax^2 + bx + c = 0$, we have $a = 3$, $b = 4$ and $c = -5$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-4 \pm \sqrt{4^2 - 4(3)(-5)}}{2(3)}$$

$$= \frac{-4 \pm \sqrt{16 - (-60)}}{6}$$

$$= \frac{-4 \pm \sqrt{16 + 60}}{6}$$

$$= \frac{-4 \pm \sqrt{76}}{6}$$

$= 0.786$ (to 3 s.f.) or -2.12 (to 3 s.f.)

$\therefore x = 0.786$ or $x = -2.12$

> **ATTENTION**
>
> Always ensure that the equation is in the form $ax^2 + bx + c = 0$ before substituting the values of a, b and c into the formula.

PRACTISE NOW 5

Solve each of the following equations.

(a) $2x^2 + 3x - 7 = 0$

(b) $5x^2 - 8x - 1 = 0$

(c) $(x - 1)^2 = 4x - 5$

(d) $(x + 3)(x - 1) = 8x - 7$

> **SIMILAR QUESTIONS**
>
> Exercise 1B Questions 1(a)–(f), 2(a)–(f), 3(a)–(f)

Class Discussion

Solutions to Quadratic Equations

Work in pairs.

Consider each of the following equations.

(a) $4x^2 - 12x + 9 = 0$

(b) $2x^2 + 5x + 8 = 0$

(c) $3x^2 + 5x - 4 = 0$

1. Find the value of $b^2 - 4ac$.

2. Use the quadratic formula to solve the equation. Are there any real solutions? Explain your answer.

3. What can you say about the sign of $b^2 - 4ac$ and the number of real solutions of a quadratic equation?

> **ATTENTION**
>
> The equation $x^2 - 6x + 9 = 0$ has only 1 real solution $x = 3$, i.e. 3 is the only real number that satisfies the equation.
>
> The equation $x^2 + 9 = 0$ has no real solutions as there is no real number that satisfies the equation.

From the class discussion, we observe that

For a quadratic equation $ax^2 + bx + c = 0$,
- if $b^2 - 4ac > 0$, the equation has *two real solutions*,
- if $b^2 - 4ac = 0$, the equation has *one real solution*,
- if $b^2 - 4ac < 0$, the equation has *no real solutions*.

Exercise 1B

BASIC LEVEL

1. Solve each of the following equations.

   (a) $x^2 + 4x + 1 = 0$ (b) $3x^2 + 6x - 1 = 0$

   (c) $2x^2 - 7x + 2 = 0$ (d) $3x^2 - 5x - 17 = 0$

   (e) $-3x^2 - 7x + 9 = 0$ (f) $-5x^2 + 10x - 2 = 0$

2. Solve each of the following equations.

   (a) $x^2 + 5x = 21$ (b) $10x^2 - 12x = 15$

   (c) $8x^2 = 3x + 6$ (d) $4x^2 - 7 = 2x$

   (e) $9 - 5x^2 = -3x$ (f) $16x - 61 = x^2$

INTERMEDIATE LEVEL
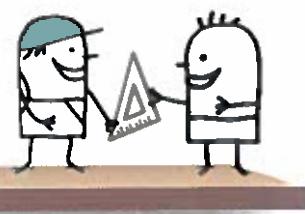
3. Solve each of the following equations.

   (a) $x(x + 1) = 1$ (b) $3(x + 1)(x - 1) = 7x$

   (c) $(x - 1)^2 - 2x = 0$ (d) $x(x - 5) = 7 - 2x$

   (e) $(2x + 3)(x - 1) - x(x + 2) = 0$

   (f) $(4x - 3)^2 + (4x + 3)^2 = 25$

4. Solve each of the following equations if possible.

   (a) $0.5(x^2 + 1) = x$ (b) $\frac{3}{4}x^2 + 2x - \frac{1}{2} = 0$

   (c) $5x - 7 = x^2$ (d) $3x - 4 = (4x - 3)^2$

1.3 Solving Quadratic Equations by Graphical Method

In Sections 1.1 and 1.2, we have learnt how to solve quadratic equations by completing the square and by using the quadratic formula. Another method that can be used to find the solutions of the quadratic equation $ax^2 + bx + c = 0$ is by drawing the corresponding quadratic graph of $y = ax^2 + bx + c$ and to find the x-coordinates of the points of intersection of this graph with the x-axis ($y = 0$).

RECALL

When solving a pair of simultaneous equations
$$y = ax^2 + bx + c \text{ --- (1)}$$
and $\quad y = 0, \text{ --- (2)}$
we obtain the quadratic equation $ax^2 + bx + c = 0$.

Worked Example 6

(Solving a Quadratic Equation by Graphical Method)

The variables x and y are connected by the equation $y = 2x^2 - 5x - 6$.

(i) Complete the table for $y = 2x^2 - 5x - 6$.

x	-2	-1	0	1	2	3	4
y							
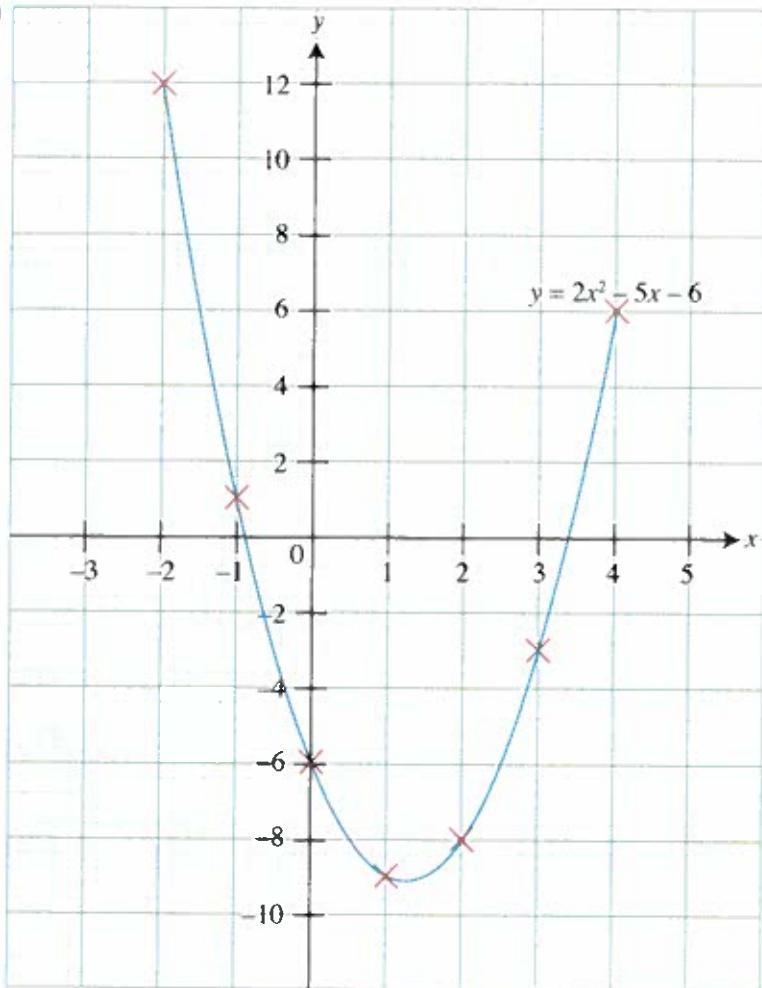
(ii) Draw the graph of $y = 2x^2 - 5x - 6$ for $-2 \leqslant x \leqslant 4$.

(iii) Hence, solve the equation $2x^2 - 5x - 6 = 0$.

Solution:

(i)

x	-2	-1	0	1	2	3	4
y	12	1	-6	-9	-8	-3	6

(ii)

$y = 2x^2 - 5x - 6$

> **RECALL**
>
> The solution of a pair of simultaneous linear equations is given by the coordinates of the *point of intersection* of the graphs of the two equations.

> **ATTENTION**
>
> For $2x^2 - 5x - 6 = 0$, the value of $b^2 - 4ac$ is $73 > 0$. Hence, there are two real solutions.

> **ATTENTION**
>
> The answers obtained by the graphical method can only be accurate up to half of a small square grid. In Worked Example 6, the solutions are accurate to the nearest 0.1.

(iii) From the graph, the x-coordinates of the points of intersection of $y = 2x^2 - 5x - 6$ and the x-axis (i.e. $y = 0$) are $x = -0.9$ and $x = 3.4$.

$\therefore$ The solutions of the equation $2x^2 - 5x - 6 = 0$ are $x = -0.9$ and $x = 3.4$.

1. The variables x and y are connected by the equation $y = 2x^2 - 4x - 1$.
   (i) Complete the table for $y = 2x^2 - 4x - 1$.

x	-2	-1	0	1	2	3	4
y							

   (ii) Draw the graph of $y = 2x^2 - 4x - 1$ for $-2 \leqslant x \leqslant 4$.
   (iii) Hence, solve the equation $2x^2 - 4x - 1 = 0$.
2. By drawing the graph of $y = 7 - 4x - 3x^2$ for $-3 \leqslant x \leqslant 2$, solve the equation $7 - 4x - 3x^2 = 0$ graphically.

Worked Example 7

(Solving a Quadratic Equation by Graphical Method)
The variables x and y are connected by the equation $y = x^2 - 4x + 4$.
(i) Complete the table for $y = x^2 - 4x + 4$.

x	-1	0	1	2	3	4	5
y							

(ii) Draw the graph of $y = x^2 - 4x + 4$ for $-1 \leqslant x \leqslant 5$.
(iii) Hence, solve the equation $x^2 - 4x + 4 = 0$.
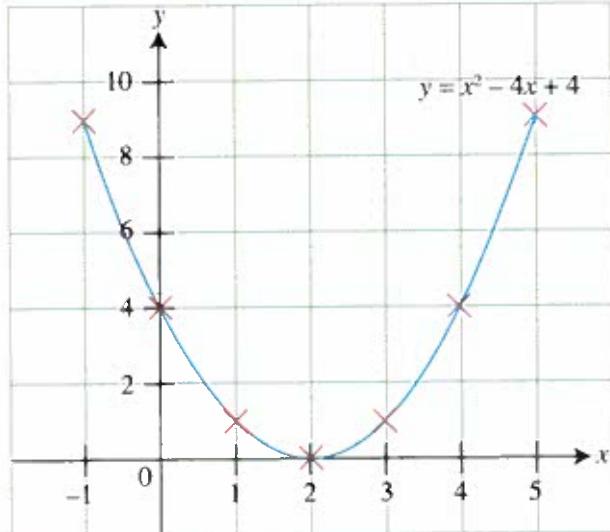
Solution:
(i)

x	-1	0	1	2	3	4	5
y	9	4	1	0	1	4	9

(ii)

$y = x^2 - 4x + 4$

ATTENTION

Equations such as $x^2 - 4x + 4 = 0$ are usually solved by factorisation as they can be easily factorised.

(iii) From the graph, the curve $y = x^2 - 4x + 4$ touches the x-axis (i.e. $y = 0$) at $x = 2$ only.

ATTENTION

For $x^2 - 4x + 4 = 0$, the value of $b^2 - 4ac$ is 0. Hence, there is only one real solution.

∴ The solution of the equation $x^2 - 4x + 4 = 0$ is $x = 2$.

1. The variables x and y are connected by the equation $y = x^2 - 6x + 9$.

   (i) Complete the table for $y = x^2 - 6x + 9$.

x	-1	0	1	2	3	4	5	6
y								

   (ii) Draw the graph of $y = x^2 - 6x + 9$ for $-1 \leqslant x \leqslant 6$.

   (iii) Hence, solve the equation $x^2 - 6x + 9 = 0$.

2. By drawing the graph of $y = 8x - x^2 - 16$ for $0 \leqslant x \leqslant 8$, solve the equation $8x - x^2 - 16 = 0$ graphically.

Thinking Time

Draw the graph of $y = 2x^2 + 4x + 3$ for $-2 \leqslant x \leqslant 4$.

(i) State the number of points of intersection between the graph and the x-axis.

(ii) How many real solutions are there to the equation $2x^2 + 4x + 3 = 0$? Explain your answer.

(iii) Find the value of $b^2 - 4ac$ for the equation $2x^2 + 4x + 3 = 0$.

(iv) How does the value of $b^2 - 4ac$ obtained in (iii) relate to the number of points of intersection of the graph with the x-axis?

Journal Writing

To solve a quadratic equation, you have learnt the following 4 methods:

(i) Factorisation

(ii) Completing the square

(iii) Use of the quadratic formula

(iv) Graphical method

Write down the advantages and disadvantages of using each method. When solving a quadratic equation, how would you choose which method to use?

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Exercise 1C

1. The variables x and y are connected by the equation $y = 2x^2 - 5x + 1$.

   (i) Complete the table for $y = 2x^2 - 5x + 1$.

x	-1	0	1	2	3	4
y						

   (ii) Draw the graph of $y = 2x^2 - 5x + 1$ for $-1 \leqslant x \leqslant 4$.

   (iii) Hence, solve the equation $2x^2 - 5x + 1 = 0$.

2. The variables x and y are connected by the equation $y = 7 - 5x - 3x^2$.

   (i) Complete the table for $y = 7 - 5x - 3x^2$.

x	-3	-2	-1	0	1	2
y						

   (ii) Draw the graph of $y = 7 - 5x - 3x^2$ for $-3 \leqslant x \leqslant 2$.

   (iii) Hence, solve the equation $7 - 5x - 3x^2 = 0$.

3. The variables x and y are connected by the equation $y = x^2 + 6x + 9$.

   (i) Complete the table for $y = x^2 + 6x + 9$.

x	-5	-4	-3	-2	-1	0
y						

   (ii) Draw the graph of $y = x^2 + 6x + 9$ for $-5 \leqslant x \leqslant 0$.

   (iii) Hence, solve the equation $x^2 + 6x + 9 = 0$.

4. (i) Draw the graph of $y = 3x^2 + 4x - 5$ for $-3 \leqslant x \leqslant 2$.

   (ii) Hence, solve the equation $3x^2 + 4x - 5 = 0$ graphically.

5. By drawing the graph of $y = 5 - 2x - x^2$ for $-4 \leqslant x \leqslant 2$, solve the equation $5 - 2x - x^2 = 0$ graphically.

6. (i) Draw the graph of $y = 4x^2 + 12x + 9$ for $-4 \leqslant x \leqslant 2$.

   (ii) Hence, solve the equation $4x^2 + 12x + 9 = 0$ graphically.

7. By drawing the graph of $y = 10x - 25 - x^2$ for $0 \leqslant x \leqslant 10$, solve the equation $10x - 25 - x^2 = 0$ graphically.

8. The profit, $\$P$ million, of a manufacturing company in its first 10 years of operation can be modelled by the equation $P = 2 - 0.1(x - 3)^2$, where x is the number of years of operation.

   (a) Using a scale of 1 cm to represent 1 year, draw a horizontal x-axis for $0 \leqslant x \leqslant 10$. Using a scale of 2 cm to represent $\$1$ million, draw a vertical P-axis for $-4 \leqslant P \leqslant 3$. On your axes, plot the points given in the table and join them with a smooth curve.

   (b) Use your graph to find the value of x when the profit of the company is zero.

9. During an annual carnival, participants are each expected to throw a balloon filled with water from the top of a platform onto a sandpit. Points are allocated based on the horizontal distance from the foot of the platform to where the balloon lands. Huixian throws a balloon. During the flight, its height above ground level, y cm, is represented by the equation $y = 200 + 7x - 6x^2$, where x is the horizontal distance, in metres, from the foot of the platform.

The table shows some values of x and the corresponding values of y.

x	0	1	2	3	4	5	6
y	200	201	190	167	132	85	26

(a) Using a scale of 1 cm to represent 1 m, draw a horizontal x-axis for $0 \leqslant x \leqslant 6$.

Using a scale of 2 cm to represent 50 cm, draw a vertical y-axis for $0 \leqslant y \leqslant 250$.

On your axes, plot the points given in the table and join them with a smooth curve.
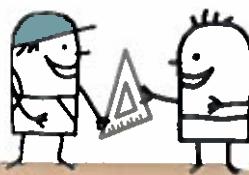
(b) Use your graph to find

    (i) the positive solution of $200 + 7x - 6x^2 = 0$,

    (ii) the horizontal distance from the foot of the platform when the balloon is 50 cm above the ground.

(c) Given that the flight of the balloon above ground level can only be modelled by the equation $y = 200 + 7x - 6x^2$ for $0 \leqslant x \leqslant t$, state the value of t. Explain your answer.

Solving Fractional Equations that can be reduced to Quadratic Equations

1.4

In Book 2, we have learnt that algebraic fractions are of the form $\frac{A}{B}$, where A and/or B are algebraic expressions, and $B \neq 0$. Equations that have one or more algebraic fractions are known as fractional equations. Examples of **fractional equations** are $\frac{2}{x + 2} = 5x - 1$ and $\frac{3}{x + 2} + \frac{x - 1}{x - 5} = 2$.

In this section, we will learn how to solve fractional equations that can be reduced to quadratic equations.

Worked Example 8 · (Solving a Fractional Equation by reducing it to a Quadratic Equation)

Solve the equation $\dfrac{2}{x+2} = 5x - 1$.

Solution:

$$\frac{2}{x+2} = 5x - 1$$

$$\frac{2}{x+2} \times (x+2) = (5x-1) \times (x+2) \quad \text{(multiply both sides by } (x+2))$$

$$2 = (5x-1)(x+2)$$

$$2 = 5x^2 + 10x - x - 2$$

$$2 = 5x^2 + 9x - 2$$

$$0 = 5x^2 + 9x - 4$$

$$5x^2 + 9x - 4 = 0 \quad \text{(rewrite the equation in the form } ax^2 + bx + c = 0)$$

> **ATTENTION**
>
> In an equation, if $x = y$, then $y = x$. Hence, $0 = 5x^2 + 9x - 4$ is equivalent to $5x^2 + 9x - 4 = 0$.

Comparing $5x^2 + 9x - 4 = 0$ with $ax^2 + bx + c = 0$, we have $a = 5$, $b = 9$ and $c = -4$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-9 \pm \sqrt{9^2 - 4(5)(-4)}}{2(5)}$$

$$= \frac{-9 \pm \sqrt{81 - (-80)}}{10}$$

$$= \frac{-9 \pm \sqrt{81 + 80}}{10}$$

$$= \frac{-9 \pm \sqrt{161}}{10}$$

$$= 0.369 \text{ (to 3 s.f.) or } -2.17 \text{ (to 3 s.f.)}$$

$\therefore x = 0.369$ or $x = -2.17$

PRACTISE NOW 8

1. Solve each of the following equations.

   (a) $\dfrac{6}{x+4} = x + 3$

   (b) $\dfrac{3}{x+2} = 3x - 1$

2. Solve the equation $\dfrac{4}{x} = 2x - 3$.

Worked Example 9

(Solving a Fractional Equation by reducing it to a Quadratic Equation)

Solve the equation $\dfrac{3}{x+2} + \dfrac{x-1}{x-5} = 2$.

Solution:

$$\frac{3}{x+2} + \frac{x-1}{x-5} = 2$$

$$\left[\frac{3}{x+2} + \frac{x-1}{x-5}\right] \times (x+2)(x-5) = 2 \times (x+2)(x-5) \text{ (multiply both sides}$$
$$\text{by } (x+2)(x-5))$$

$$\frac{3}{x+2} \times (x+2)(x-5) + \frac{x-1}{x-5} \times (x+2)(x-5) = 2(x+2)(x-5)$$

$$3(x-5) + (x-1)(x+2) = 2(x+2)(x-5)$$
$$3x - 15 + x^2 + 2x - x - 2 = 2(x^2 - 5x + 2x - 10)$$
$$x^2 + 4x - 17 = 2(x^2 - 3x - 10)$$
$$x^2 + 4x - 17 = 2x^2 - 6x - 20$$
$$0 = x^2 - 10x - 3$$
$$x^2 - 10x - 3 = 0 \text{ (rewrite the equation in the form}$$
$$ax^2 + bx + c = 0)$$

> **ATTENTION**
>
> The LCM of the denominators of $\dfrac{3}{x+2}$ and $\dfrac{x-1}{x-5}$ is $(x+2)(x-5)$.

Comparing $x^2 - 10x - 3 = 0$ with $ax^2 + bx + c = 0$, we have $a = 1$, $b = -10$ and $c = -3$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-10) \pm \sqrt{(-10)^2 - 4(1)(-3)}}{2(1)}$$

$$= \frac{10 \pm \sqrt{100 - (-12)}}{2}$$

$$= \frac{10 \pm \sqrt{100 + 12}}{2}$$

$$= \frac{10 \pm \sqrt{112}}{2}$$

$$= 10.3 \text{ (to 3 s.f.) or } -0.292 \text{ (to 3 s.f.)}$$

$$\therefore x = 10.3 \text{ or } x = -0.292$$

1. Solve each of the following equations.

(a) $\dfrac{1}{x-2} + \dfrac{2}{x-3} = 5$

(b) $\dfrac{5}{x-3} - \dfrac{x-1}{x-2} = 7$

2. Solve the equation $\dfrac{3}{x-2} - \dfrac{1}{(x-2)^2} = 2$.

Thinking Time

Lixin is given the following fractional equation to solve:

$$\frac{7}{x-3} - \frac{4}{x} = \frac{21}{x(x-3)}$$

Her working is as shown:

$$\frac{7}{x-3} \times x(x-3) - \frac{4}{x} \times x(x-3) = \frac{21}{x(x-3)} \times x(x-3)$$
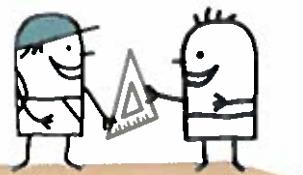
$$7x - 4(x-3) = 21$$
$$7x - 4x + 12 = 21$$
$$3x = 9$$
$$x = 3$$

Verify if the solution $x = 3$ is valid. Explain your answer.

1.5 Applications of Quadratic Equations in Real-World Contexts
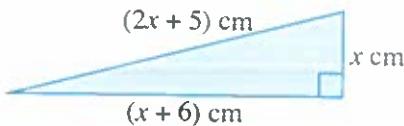
In order for quadratic equations to be applied to solve problems, we may have to formulate the quadratic equations first. Worked Examples 10 and 11 illustrate this.

Worked Example 10

(Application of Quadratic Equations in Real-World Contexts)

On a map, a piece of land is in the shape of a right-angled triangle with sides of length x cm, $(x + 6)$ cm and $(2x + 5)$ cm.

(i) From the information given, formulate an equation and show that it simplifies to $2x^2 + 8x - 11 = 0$.

(ii) Solve the equation $2x^2 + 8x - 11 = 0$, giving both answers correct to 3 decimal places.

(iii) Hence, find the perimeter of the triangle.

Solution:

(i) Using Pythagoras' Theorem,

$$x^2 + (x + 6)^2 = (2x + 5)^2$$
$$x^2 + x^2 + 12x + 36 = 4x^2 + 20x + 25$$
$$2x^2 + 8x - 11 = 0 \text{ (shown)}$$

(ii) Comparing $2x^2 + 8x - 11 = 0$ with $ax^2 + bx + c = 0$, we have $a = 2$, $b = 8$ and $c = -11$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-8 \pm \sqrt{8^2 - 4(2)(-11)}}{2(2)}$$
$$= \frac{-8 \pm \sqrt{64 - (-88)}}{4}$$
$$= \frac{-8 \pm \sqrt{64 + 88}}{4}$$
$$= \frac{-8 \pm \sqrt{152}}{4}$$
$$= 1.082 \text{ (to 3 d.p.) or } -5.082 \text{ (to 3 d.p.)}$$

(iii) Perimeter of the triangle $= x + (x + 6) + (2x + 5)$

$$= x + x + 6 + 2x + 5$$
$$= 4x + 11$$

Since the length of a triangle cannot be a negative value, $x = 1.082$.
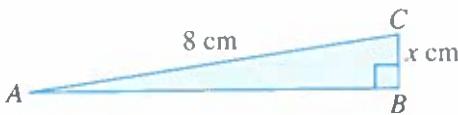
$\therefore$ Perimeter of the triangle $= 4(1.082) + 11$

$$= 15.3 \text{ cm (to 3 s.f.)}$$

Problem Solving Tip

**Polya's 4-step Problem Solving Model**

1. Understand the problem.
   - A right-angled triangle with 3 sides is given in terms of x.

2. Devise a plan.
   - Use Pythagoras' Theorem to formulate an equation in x.

3. Implement the plan.
   - Simplify and solve the equation.

4. Check your answers.
   - For answers in 3 significant figures, intermediate answers must be given to at least 4 significant figures.
   - Since length is positive, the negative value of x is rejected.
   - Use the value of x to compute the perimeter.

The figure shows a right-angled triangle *ABC* with dimensions as shown.

(i) If the perimeter of the triangle is 17 cm, write down an expression, in terms of x, for the length of *AB*.

(ii) Hence, formulate an equation in x and show that it simplifies to $2x^2 - 18x + 17 = 0$.

(iii) Solve the equation $2x^2 - 18x + 17 = 0$, giving both answers correct to 3 decimal places.

(iv) Hence, find the area of the triangle.

Worked Example 11

(Application of Quadratic Equations in Real-World Contexts)

A family decides to travel from Singapore to Kuala Lumpur, which are 315 km apart. The average speed of an aeroplane is 350 km/h more than the speed of a car. Let the average speed of the car be x km/h.

(i) Write down an expression, in terms of x, for the number of hours taken by the family if they choose to travel by aeroplane.

If they choose to travel by aeroplane instead of by car, they will be able to reach Kuala Lumpur 3 hours and 15 minutes earlier.

(ii) From the information given, formulate an equation in x and show that it reduces to
$$13x^2 + 4550x - 441\,000 = 0.$$

(iii) Solve the equation $13x^2 + 4550x - 441\,000 = 0$, giving both your answers correct to 2 decimal places.

(iv) Find the time taken by the family to travel by aeroplane, giving your answer correct to the nearest minute.

Problem Solving Tip

Polya's 4-step Problem Solving Model

1. Understand the problem.
   - The distance and average speed of the aeroplane and the difference in travel time are given.
   - What are the assumptions made? The distance travelled by the car and the aeroplane is exactly the same.

2. Devise a plan.
   - Use the algebraic method and the relationship between speed, distance and time to formulate an equation in x.

3. Implement the plan.
   - What is the time taken by the aeroplane? $\dfrac{315}{x + 350}$ hours
   - What is the time taken by the car? $\dfrac{315}{x}$ hours
   - Which has a larger value?
   - Hence, the difference is $\dfrac{315}{x} - \dfrac{315}{x + 350} = 3\dfrac{1}{4}$, because 3 hours 15 minutes $= 3\dfrac{1}{4}$ hours.

4. Check your answers.
   - For answers in 3 significant figures, intermediate answers must be given to at least 4 significant figures.
   - Since speed is positive, the negative value of x is rejected.
   - Use the value of x to compute the time taken to travel by aeroplane.

Solution:

(i) Number of hours taken by the family to travel by aeroplane $= \dfrac{315}{x + 350}$

(ii)

$$\dfrac{315}{x} - \dfrac{315}{x + 350} = 3\dfrac{15}{60}$$

$$\dfrac{315}{x} - \dfrac{315}{x + 350} = 3\dfrac{1}{4}$$

$$\dfrac{315}{x} - \dfrac{315}{x + 350} = \dfrac{13}{4}$$

$$\dfrac{315}{x} \times 4x(x + 350) - \dfrac{315}{x+350} \times 4x(x+350) = \dfrac{13}{4} \times 4x(x + 350)$$ (multiply both sides

by $4x(x + 350)$)

$$1260(x + 350) - 1260x = 13x(x + 350)$$

$$1260x + 441\,000 - 1260x = 13x^2 + 4550x$$

$$0 = 13x^2 + 4550x - 441\,000$$

$$13x^2 + 4550x - 441\,000 = 0 \text{ (shown)}$$

(iii) Comparing $13x^2 + 4550x - 441\,000 = 0$ with $ax^2 + bx + c = 0$, we have $a = 13$, $b = 4550$ and $c = -441\,000$.

$$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \dfrac{-4550 \pm \sqrt{4550^2 - 4(13)(-441\,000)}}{2(13)}$$

$$= \dfrac{-4550 \pm \sqrt{43\,634\,500}}{26}$$

$$= 79.06 \text{ (to 2 d.p.) or } -429.06 \text{ (to 2 d.p.)}$$

$\therefore x = 79.06$ or $x = -429.06$

(iv) Since the speed of the car cannot be a negative value, $x = 79.06$.

$\therefore$ Number of hours taken by the family to travel by aeroplane $= \dfrac{315}{79.06 + 350}$

$= 0.7342$ hours

$= 0.7342 \times 60$

$= 44$ minutes (to the nearest minute)

Mr Lee drove from City P to City Q, which are 600 km apart. During his return journey, his average speed was increased by 7 km/h and the time taken was 15 minutes less.

(i)   If he drove at an average speed of x km/h on his journey from City P to City Q, formulate an equation in x and show that it reduces to $x^2 + 7x - 16\,800 = 0$.

(ii) Solve the equation $x^2 + 7x - 16\,800 = 0$, giving both your answers correct to 2 decimal places.

(iii) Find the time taken for the return journey.

Exercise 1D

BASIC LEVEL

1. Solve each of the following equations.

    (a) $\dfrac{8}{x} = 2x + 1$

    (b) $3x - 1 = \dfrac{7}{x + 4}$

    (c) $\dfrac{x + 1}{5 - x} = x$

    (d) $x + \dfrac{7}{x} = 9$

    (e) $2x + 1 = \dfrac{x + 1}{x - 5}$

    (f) $\dfrac{5x}{x + 4} = 3x + 1$

2. The difference between two positive numbers, $\dfrac{12}{x + 1}$ and $\dfrac{12}{x}$, is 1.

    (i)   Form an equation in x and show that it reduces to $x^2 + x - 12 = 0$.

    (ii) Solve the equation $x^2 + x - 12 = 0$.

    (iii) Hence, find the two numbers.

INTERMEDIATE LEVEL

3. Solve each of the following equations.

    (a) $\dfrac{2}{x + 1} = \dfrac{5x}{3 - x}$

    (b) $\dfrac{(x - 2)(x - 3)}{(x - 1)(x + 2)} = \dfrac{2}{3}$

4. Find the value(s) of x that satisfy the equation $\dfrac{x(x - 3)}{(x + 1)^2} = \dfrac{3}{5}$.

5. Solve each of the following equations.

    (a) $\dfrac{x}{2} = \dfrac{4}{x} - 1$

    (b) $\dfrac{2}{x + 5} = 1 - \dfrac{x + 1}{5}$

    (c) $\dfrac{x - 2}{5} + \dfrac{1}{2x - 3} = 1$

    (d) $\dfrac{3}{x} + \dfrac{2}{x + 1} = 5$

    (e) $\dfrac{1}{x + 2} + \dfrac{1}{x - 2} = \dfrac{3}{8}$

    (f) $\dfrac{7}{x - 1} - \dfrac{x + 1}{x + 3} = \dfrac{1}{2}$

    (g) $\dfrac{5}{x - 2} = 2 - \dfrac{4}{(x - 2)^2}$

    (h) $\dfrac{5}{x - 1} + \dfrac{x}{(x - 1)^2} = 1$

17. Two weeks before Nora went to New York for a holiday, she exchanged S$2000 into US dollars (US$) at Samy's Money Exchange at a rate of US$1 = S$$x$.

    (i) Write down an expression, in terms of x, for the amount of US$ she received from Samy's Money Exchange.

    One week before her holiday, she exchanged another S$1000 into US$ at Chan's Money Exchange at a rate of US$1 = S$$(x + 0.05)$.

    (ii) Write down an expression, in terms of x, for the amount of US$ she received from Chan's Money Exchange.

    (iii) If Nora received a total of US$2370 from the two Money Exchanges, formulate an equation in x and show that it reduces to

    $$237x^2 - 288.15x - 10 = 0.$$

    (iv) Solve the equation $237x^2 - 288.15x - 10 = 0$, giving both your answers correct to 2 decimal places.

    (v) Find the exchange rate between S$ and US$ offered by Chan's Money Exchange.

18. Solve each of the following equations.

    (a) $\dfrac{4}{x-1} = \dfrac{x}{2x^2 + 3x - 5}$

    (b) $\dfrac{1}{x} + \dfrac{2}{x-1} + \dfrac{3}{x+1} = 0$
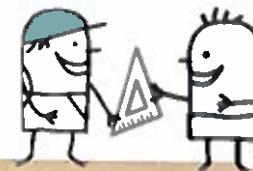
    (c) $\dfrac{1}{x^2 - 9} - \dfrac{2}{3-x} = 1$

    (d) $\dfrac{3}{x-3} + \dfrac{x+1}{x^2 - 5x + 6} = 1$

19. During a test flight, an aircraft flies from Sandy Land to White City and back to Sandy Land. The distance between Sandy Land and White City is 450 km and the total time taken for the whole journey is 5 hours and 30 minutes. Given that there is a constant wind blowing from Sandy Land to White City and that the speed of the aircraft in still air is 165 km/h, find the speed of the wind. State the assumptions you have made to solve this problem.

    Hint: Let the speed of the wind be x km/h.

1.6 Graphs of Quadratic Functions

∴ Recap

In Book 2, we have learnt how to draw the graphs of quadratic functions of the form $y = ax^2 + bx + c$, where a, b and c are constants and $a \neq 0$.


(a) $a > 0$ (b) $a < 0$
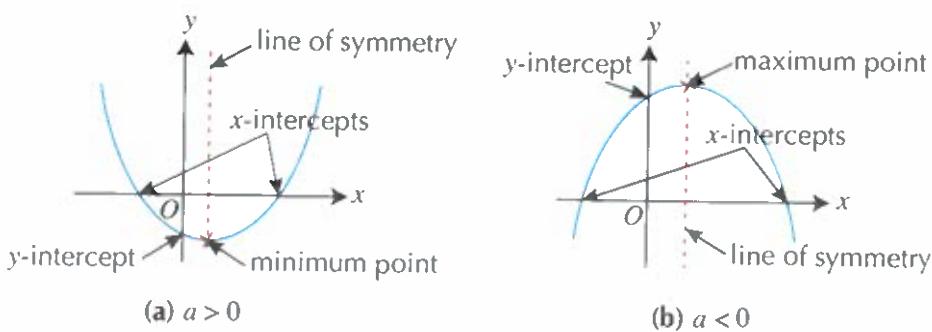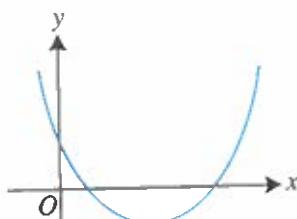
Fig. 1.3

In this section, we will learn how to sketch the graphs of quadratic functions of the form $y = (x - h)(x - k)$ or $y = -(x - h)(x - k)$ and $y = (x - p)^2 + q$ or $y = -(x - p)^2 + q$.
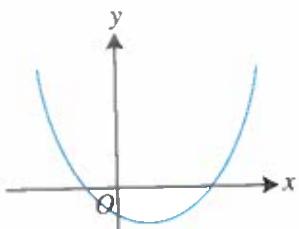
Determine which of the following sketches correspond to the quadratic functions
$y = x^2 + 4x - 5$ and $y = -x^2 - 4x + 5$.
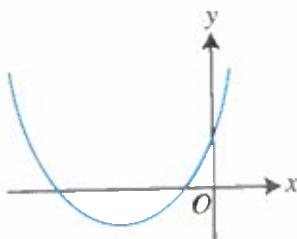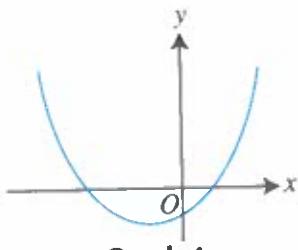
Graph 1
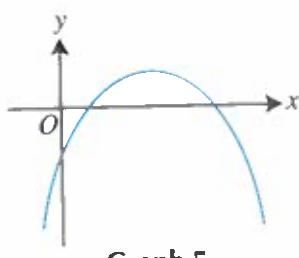
Graph 2
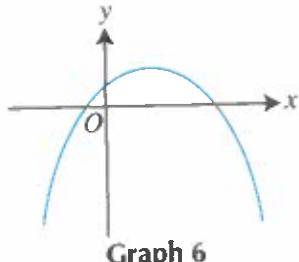
Graph 3
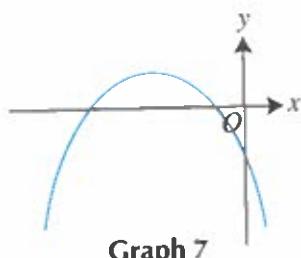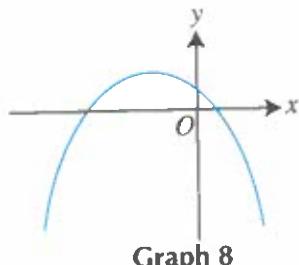
Graph 4

Graph 5

Graph 6

Graph 7

Graph 8

⠿ Graphs of the form $y = (x - h)(x - k)$ or $y = -(x - h)(x - k)$

Investigation

Graphs of $y = (x - b)(x - k)$ or $y = -(x - b)(x - k)$

1. Use a graphing software to plot the graph of $y = (x - 3)(x - k)$ for $k = -2, -1, 0,$ 1 and 2.

2. Study the graphs and answer each of the following questions.
   (a) Does the graph open upwards or downwards?
   (b) Write down the coordinates of the point(s) where the graph cuts the x-axis, i.e. the x-intercepts.
   (c) Write down the coordinates of the point where the graph cuts the y-axis, i.e. the y-intercept.
   (d) What is the relationship between the x-intercepts and the line of symmetry?
   (e) State the equation of the line of symmetry of the graph.
   (f) Write down the coordinates of the maximum or the minimum point of the graph.

3. Repeat Steps 1 and 2 for $y = -(x - 3)(x - k)$, $y = (x - 5)(x - k)$ and $y = -(x - 5)(x - k)$.

4. By looking at the equation of each graph, how do you determine if it opens upwards or downwards?

5. By looking at the equation of each graph, how do you determine the coordinates of the points where the graph cuts the x-axis?

6. What can you say about the line of symmetry of each graph?

From the investigation, we observe that:

- For the equation $y = (x - h)(x - k)$, the graph opens upwards. The graph cuts the x-axis at $(h, 0)$ and $(k, 0)$. The graph is symmetrical about the vertical line that passes through the minimum point.
- For the equation $y = -(x - h)(x - k)$, the graph opens downwards. The graph cuts the x-axis at $(h, 0)$ and $(k, 0)$. The graph is symmetrical about the vertical line that passes through the maximum point.

Worked Example 12

Sketch the graph of $y = (x - 1)(x - 5)$.

Solution:

Since the coefficient of x^2 is 1, the graph opens upwards.

When $y = 0$,

$(x - 1)(x - 5) = 0$

$x - 1 = 0$ or $x - 5 = 0$

     $x = 1$ $x = 5$
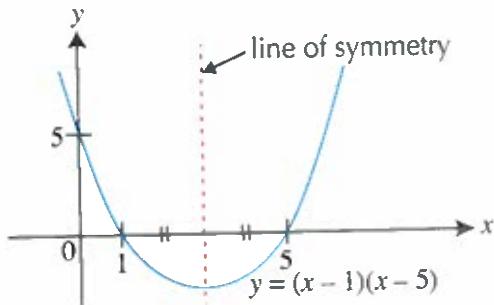
$\therefore$ The graph cuts the x-axis at $(1, 0)$ and $(5, 0)$.

When $x = 0$,

     $y = (-1)(-5)$

       $= 5$

$\therefore$ The graph cuts the y-axis at $(0, 5)$.

> **Problem Solving Tip**
>
> **Step 1:** State the coefficient of x^2 to determine if the graph opens upwards or downwards.
>
> **Step 2:** Obtain the x-intercepts by substituting $y = 0$ into the equation.
>
> **Step 3:** Obtain the y-intercept by substituting $x = 0$ into the equation.
>
> **Step 4:** Sketch the graph.

> **ATTENTION**
>
> The line of symmetry is halfway between the x-intercepts.
>
> How do we find the x-coordinate of the midpoint? In Worked Example 12, it is $x = 3$.
>
> With this information, how can we find the coordinates of the minimum point?

PRACTISE NOW 12

Sketch the graph of each of the following functions.

(a) $y = (x - 2)(x - 6)$

(b) $y = -(x - 3)(x + 1)$

(c) $y = (3 - x)(x + 5)$

> **SIMILAR QUESTIONS**
>
> Exercise 1E Questions 1(a)–(f), 3–6

Graphs of the form $y = (x - p)^2 + q$ or $y = -(x - p)^2 + q$

Investigation

Graphs of $y = (x - p)^2 + q$ or $y = -(x - p)^2 + q$

1. Use a graphing software to plot the graph of $y = (x - 2)^2 + q$ for $q = -4, -1, 0, 1$ and 4.

2. Study the graphs and answer each of the following questions.
   - **(a)** Does the graph open upwards or downwards?
   - **(b)** Write down the coordinates of the point(s) where the graph cuts the x-axis, i.e. the x-intercepts.
   - **(c)** Write down the coordinates of the point where the graph cuts the y-axis, i.e. the y-intercept.
   - **(d)** State the equation of the line of symmetry of the graph.
   - **(e)** Write down the coordinates of the maximum or the minimum point of the graph.

3. Repeat Steps 1 and 2 for $y = -(x - 2)^2 + q$, $y = (x + 3)^2 + q$ and $y = -(x + 3)^2 + q$.

4. By looking at the equation of each graph, how do you determine if it opens upwards or downwards?

5. By looking at the equation of each graph, how do you determine the coordinates of the maximum or the minimum point?

6. What can you say about the line of symmetry of each graph?

From the investigation, we observe that:

- For the equation $y = (x - p)^2 + q$, the graph opens upwards. The coordinates of the minimum point of the graph are (p, q) and the graph is symmetrical about the line $x = p$.
- For the equation $y = -(x - p)^2 + q$, the graph opens downwards. The coordinates of the maximum point of the graph are (p, q) and the graph is symmetrical about the line $x = p$.

INFORMATION

The graph of a quadratic function is a parabola. When it opens upwards, we say it is concave upwards.

Worked Example 13

(Sketching the Graph of $y = -(x - p)^2 + q$)

Given the quadratic function $y = -(x - 1)^2 + 4$,
- **(i)** find the coordinates of the x- and y-intercepts,
- **(ii)** write down the coordinates of the maximum point of the graph,
- **(iii)** sketch the graph,
- **(iv)** state the equation of the line of symmetry of the graph.

Solution:

(i) Since the coefficient of x^2 is -1, the graph opens downwards.

When $y = 0$,

$$-(x - 1)^2 + 4 = 0$$
$$-(x - 1)^2 = -4$$
$$(x - 1)^2 = 4$$
$$x - 1 = 2 \quad \text{or} \quad x - 1 = -2$$
$$x = 3 \qquad\qquad x = -1$$

∴ The graph cuts the x-axis at $(3, 0)$ and $(-1, 0)$.

When $x = 0$,
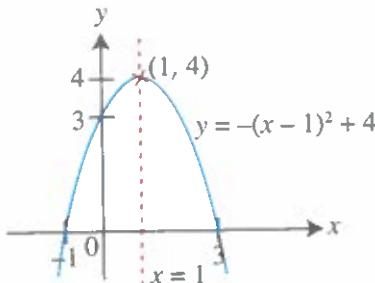
$$y = -(-1)^2 + 4$$
$$= 3$$

∴ The graph cuts the y-axis at $(0, 3)$.

(ii) The coordinates of the maximum point are $(1, 4)$.

(iii)

(iv) The equation of the line of symmetry is $x = 1$.

PRACTISE NOW 13

1. Given the quadratic function $y = -(x - 2)^2 + 9$,
   (i) find the coordinates of the x- and y-intercepts,
   (ii) write down the coordinates of the maximum point of the graph,
   (iii) sketch the graph,
   (iv) state the equation of the line of symmetry of the graph.

2. Given the quadratic function $y = (x + 1)^2 - 1$,
   (i) find the coordinates of the x- and y-intercepts,
   (ii) write down the coordinates of the minimum point of the graph,
   (iii) sketch the graph,
   (iv) state the equation of the line of symmetry of the graph.

Thinking Time

1. In Worked Example 13, we sketched the graph of $y = -(x-1)^2 + 4$.
   Express $y = -(x-1)^2 + 4$ in the factorised form $y = -(x-h)(x-k)$, and hence sketch the graph.

2. If we are given the following graph with x-intercepts at 1 and 3 and a minimum point at $(2, -1)$, can we express the equation of the curve in the form $y = (x-a)^2 + b$?

Worked Example 14

(Sketching the Graph of $y = ax^2 + bx + c$)

(i) Express $x^2 - 4x + 2$ in the form $(x-p)^2 + q$.

(ii) Write down the coordinates of the minimum point of the graph.

(iii) Hence, sketch the graph of $y = x^2 - 4x + 2$.

(iv) State the equation of the line of symmetry of the graph.

Solution:

(i) $x^2 - 4x + 2 = \left[x^2 - 4x + \left(-\frac{4}{2}\right)^2 \right] - \left(-\frac{4}{2}\right)^2 + 2$

$\qquad\qquad\qquad = (x-2)^2 - 2$

(ii) The coordinates of the minimum point are $(2, -2)$.
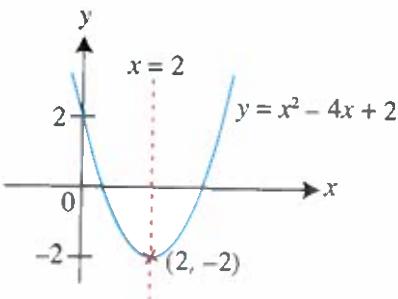
(iii) When $x = 0$,

$\qquad y = 2$

$\therefore$ The graph cuts the y-axis at $(0, 2)$.

> **Problem Solving Tip**
>
> Since the equation is of the form $y = (x-p)^2 + q$, the coordinates of the minimum point are (p, q).

(iv) The equation of the line of symmetry is $x = 2$.

1. (i)   Express $x^2 - 6x + 6$ in the form $(x - p)^2 + q$.
   (ii) Write down the coordinates of the minimum point of the graph.
   (iii) Hence, sketch the graph of $y = x^2 - 6x + 6$.
   (iv) State the equation of the line of symmetry of the graph.

2. (i)   Express $x^2 + x + 1$ in the form $(x + p)^2 + q$.
   (ii) Write down the coordinates of the minimum point of the graph.
   (iii) Hence, sketch the graph of $y = x^2 + x + 1$.
   (iv) State the equation of the line of symmetry of the graph.
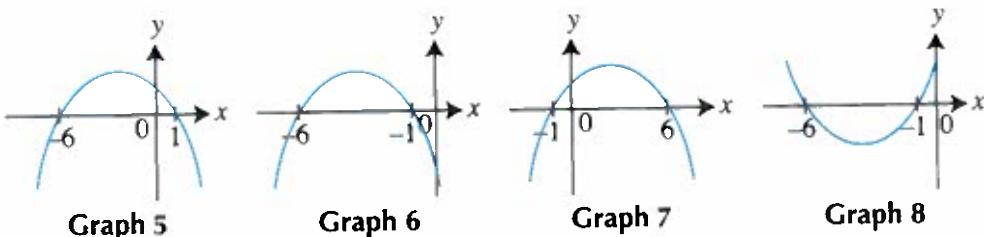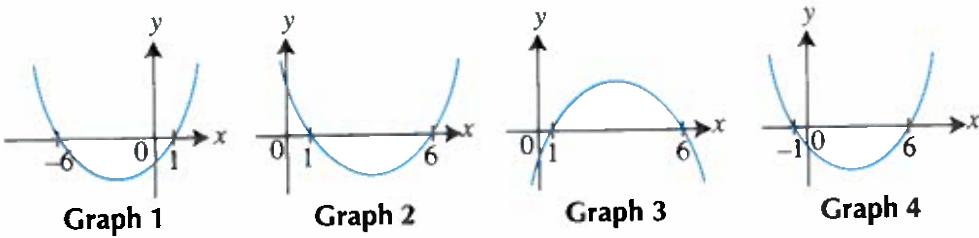
Class Discussion

Matching quadratic graphs with the corresponding functions

Work in pairs.

Match the graphs with their respective functions and justify your answers. If your classmate does not obtain the correct answer, explain to him what he has done wrong.

A: $y = -(x + 1)(x + 6)$	B: $y = (x + 1)(x + 6)$	C: $y = (x - 1)(x + 6)$	D: $y = -(x - 1)(x - 6)$
E: $y = x^2 - 7x + 6$	F: $y = -x^2 - 5x + 6$	G: $y = x^2 - 5x - 6$	H: $y = -x^2 + 5x + 6$

Graph 1   Graph 2   Graph 3   Graph 4

Graph 5   Graph 6   Graph 7   Graph 8

Exercise 1E

1. Sketch the graph of each of the following functions.
   - (a) $y = (x + 1)(x + 3)$
   - (b) $y = (x - 2)(x + 4)$
   - (c) $y = -(x + 1)(x - 5)$
   - (d) $y = -(x - 1)(x + 6)$
   - (e) $y = (3 - x)(x + 2)$
   - (f) $y = (2 - x)(4 - x)$

2. Sketch the graph of each of the following functions, stating the coordinates of the maximum or the minimum point and the equation of the line of symmetry.
   - (a) $y = x^2 + 2$
   - (b) $y = -x^2 - 6$
   - (c) $y = (x - 3)^2 + 1$
   - (d) $y = (x + 1)^2 - 3$
   - (e) $y = -(x + 2)^2 + 3$
   - (f) $y = -(x - 4)^2 - 1$

3. (i) Factorise $x^2 + \dfrac{3}{4}x$.

   (ii) Hence, sketch the graph of $y = x^2 + \dfrac{3}{4}x$.

4. Sketch the graph of $y = -(x^2 - x)$.

5. (i) Factorise $x^2 + x - 6$ completely.

   (ii) Hence, sketch the graph of $y = x^2 + x - 6$.

6. Sketch the graph of $y = x^2 - 4x + 3$.

7. (i) Express $x^2 - 8x + 5$ in the form $(x - p)^2 + q$.

   (ii) Hence, sketch the graph of $y = x^2 - 8x + 5$.

   (iii) Write down the coordinates of the minimum point of the graph.

   (iv) State the equation of the line of symmetry of the graph.

8. By first expressing $x^2 + 3x + 1$ in the form $(x + p)^2 + q$, sketch the graph of $y = x^2 + 3x + 1$. Write down the coordinates of the minimum point of the graph.

9. The graph of $y = (x - h)^2 + k$ has a minimum point at $\left(-\dfrac{1}{2}, \dfrac{3}{4}\right)$.
   - (i) State the value of h and of k.
   - (ii) Hence, sketch the graph of $y = (x - h)^2 + k$, indicating the coordinates of the point of intersection of the graph with the y-axis.

10. It is given that $-x^2 + 10x - 4$ can be expressed in the form $-(x - p)^2 + q$. By first finding the value of p and of q, sketch the graph of $y = -x^2 + 10x - 4$, indicating the coordinates of the maximum point of the graph.

1. A quadratic equation in one variable can be solved by
    *   completing the square,
    *   using formula,
    *   graphical method.
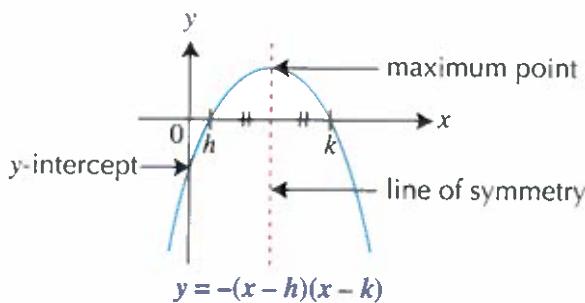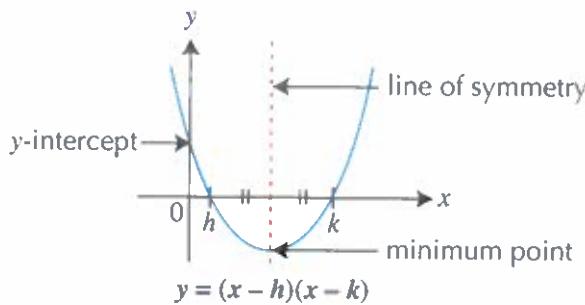
Completing the square	Using formula	Graphical method
To solve a quadratic equation in the form $x^2 + px + q = 0$ by **completing the square**: • Rewrite the equation such that the constant term is on the RHS of the equation, i.e. $x^2 + px = -q$. • Add $\left(\dfrac{p}{2}\right)^2$ to both sides of the equation to form $\left(x + \dfrac{p}{2}\right)^2 = \left(\dfrac{p}{2}\right)^2 - q$. • Take the square roots on both sides of the equation to solve for x.	The **formula** for solving a quadratic equation in the form $ax^2 + bx + c = 0$ is $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. Note that when $b^2 - 4ac < 0$, the equation has no real solutions.	To solve a quadratic equation in the form $ax^2 + bx + c = 0$ by **graphical method**: • Draw the graph of $y = ax^2 + bx + c$. • The solution(s) of the equation are given by the x-coordinate(s) of the point(s) of intersection of the graph with the x-axis.

2. Equations that have one or more algebraic fractions are known as **fractional equations**.
    To solve a fractional equation:
    *   Multiply both sides of the equation by the LCM of the denominators.
    *   Reduce it to a quadratic equation.
    *   Solve the equation either by factorisation or by using the quadratic formula.
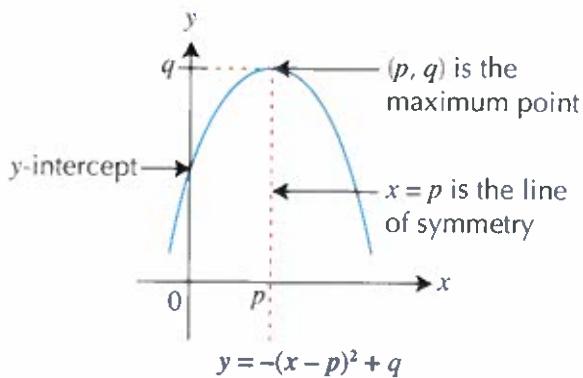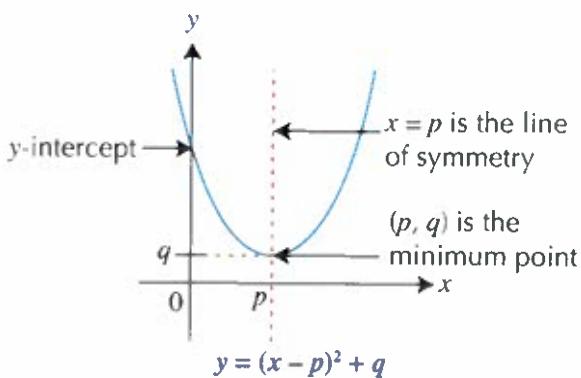
3. **Graphs of quadratic functions of the form $y = (x - h)(x - k)$ or $y = -(x - h)(x - k)$**

y-intercept →

line of symmetry

$y = (x - h)(x - k)$

minimum point

maximum point

y-intercept →

line of symmetry

$y = -(x - h)(x - k)$

The line of symmetry passes through the midpoint of the x-intercepts.

4. **Graphs of quadratic functions of the form $y = (x - p)^2 + q$ or $y = -(x - p)^2 + q$**

y-intercept →

$x = p$ is the line of symmetry

(p, q) is the minimum point

$y = (x - p)^2 + q$

(p, q) is the maximum point

y-intercept →

$x = p$ is the line of symmetry

$y = -(x - p)^2 + q$
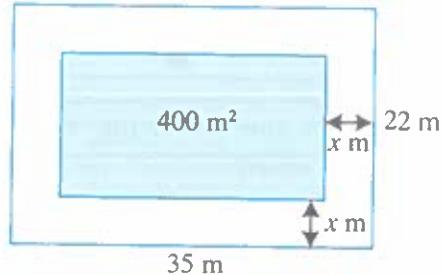
Review
Exercise
1

1. Solve each of the following equations by completing the square.
   (a) $x^2 + 8x + 5 = 0$
   (b) $x^2 + 7x - 3 = 0$
   (c) $x^2 - 11x - 7 = 0$
   (d) $x^2 + 1.2x = 1$

2. By using the quadratic formula, solve each of the following equations.
   (a) $2x^2 + 6x + 1 = 0$
   (b) $3x^2 - 7x - 2 = 0$
   (c) $-4x^2 + x + 5 = 0$
   (d) $3x^2 = 5x + 1$

3. Solve each of the following equations.
   (a) $(x - 3)^2 = \dfrac{4}{25}$
   (b) $(4 - x)^2 = 12$
   (c) $(x - 1)(x + 3) = 9$
   (d) $x(x + 4) = 17$

4. (i) Solve the equation $2x^2 - 7x + 4 = 0$, giving your answers correct to 2 decimal places.
   (ii) Hence, find the values of y that satisfy the equation $2(y - 1)^2 - 7(y - 1) + 4 = 0$.

5. Form a quadratic equation in the form $ax^2 + bx + c = 0$, where a, b and c are integers, given each of the following solutions.
   (a) $x = 2, x = \dfrac{6}{7}$
   (b) $x = -\dfrac{1}{2}, x = -\dfrac{2}{3}$

6. Solve each of the following equations.
   (a) $x - 1 = \dfrac{5}{x + 7}$
   (b) $\dfrac{x - 1}{x + 4} = \dfrac{2x}{x - 3}$
   (c) $\dfrac{1}{x} - 5x = 5$
   (d) $\dfrac{5}{x} = 3 - \dfrac{x}{x - 3}$
   (e) $\dfrac{2}{x + 1} + \dfrac{1}{x - 3} = 5$
   (f) $\dfrac{x}{x + 1} + \dfrac{1}{5} = \dfrac{3}{x - 2}$
   (g) $\dfrac{5}{x - 2} - \dfrac{3}{x^2 - 4} = \dfrac{2}{7}$
   (h) $\dfrac{1}{2x + 1} + \dfrac{x + 3}{2x^2 - 5x - 3} = 2$

7. (i) Express $y = x^2 - 7x + 12$ in the form $y = (x - h)(x - k)$.
   (ii) Hence, sketch the graph of $y = x^2 - 7x + 12$.

8. (i) Express $y = -x^2 + 5x - 4$ in the form $y = -(x - p)^2 + q$.
   (ii) Hence, sketch the graph of $y = -x^2 + 5x - 4$.

9. The difference between the reciprocals of two consecutive positive integers is $\dfrac{1}{12}$. Find the two numbers.

10. In November 2013, the exchange rate between Australian dollars (A\$) and Singapore dollars (S\$) offered by a money changer was A\$100 = S\$$x$. In December 2013, the exchange rate offered was A\$100 = S\$$(x - 5)$. Mr Neo found that, for every S\$650 he exchanged in December 2013, he would receive A\$20 more than if he exchanged in November 2013.
    (i) Formulate an equation in x.
    (ii) Hence, find the amount of Singapore dollars Mr Neo received if he exchanged A\$1250 in November 2013.

11. Farhan travelled by car from Town A to Town B, 40 km apart, at an average speed of x km/h.

    (i) Write down an expression, in terms of x, for the time taken by Farhan to travel from Town A to Town B.

    Khairul travelled by van from Town B to Town A at an average speed that was 30 km/h less than that of the car.

    (ii) Write down an expression, in terms of x, for the time taken by Khairul to travel from Town B to Town A.

    (iii) Given that Farhan took 10 minutes less than Khairul to complete the journey, form an equation in x and show that it reduces to $x^2 - 30x - 7200 = 0$.

    (iv) Solve the equation $x^2 - 30x - 7200 = 0$, giving both your answers correct to 2 decimal places.

    (v) Find the time taken by Khairul to travel from Town B to Town A.

12. In November 2013, the price of petrol was x cents per litre.

    (i) Write down an expression, in terms of x, for the number of litres of petrol that could be bought with $60 in November 2013.

    In December 2013, the price had increased by 10 cents per litre.

    (ii) Write down an expression, in terms of x, for the number of litres of petrol that could be bought with $60 in December 2013.

    (iii) Given that an additional $1\frac{3}{7}$ litres of petrol could be bought in November 2013 than in December 2013, form an equation in x and show that it reduces to $x^2 + 10x - 42\,000 = 0$.

    (iv) Solve the equation $x^2 + 10x - 42\,000 = 0$.

    (v) Find the number of litres of petrol that could be bought with $34 in December 2013.

13. A rectangular function room has dimensions 35 m by 22 m. Part of the floor is covered with ceramic tiles, as shown by the shaded rectangle in the figure.

    (i) Given that the part of the floor which is not covered by the tiles has a uniform width of x m, write down an expression, in terms of x, for the length and the breadth of the floor covered by the tiles.

    (ii) Given that the floor area covered by the tiles is 400 m², formulate an equation in x and show that it reduces to $2x^2 - 57x + 185 = 0$.

    (iii) Solve the equation $2x^2 - 57x + 185 = 0$, giving both your answers correct to 2 decimal places.

    (iv) State the width of the floor that is not covered by the tiles.

14. A stone was thrown from the top of a vertical tower into the sea. Its position during the flight is represented by the equation $y = 60 + 25x - x^2$, where y metres is the height of the stone above sea level and x metres is the horizontal distance from the foot of the tower.

(a) (i) Solve the equation $60 + 25x - x^2 = 0$, giving both your answers correct to 1 decimal place.

(ii) Explain briefly what the positive solution in **(a)(i)** represents.

(b) The table shows some values of x and the corresponding values of y.

x	0	2	4	6	8	10
y	60	106	144	174	196	210

x	12	14	16	18	20	22
y	216	214	204	186	160	126

Using a scale of 2 cm to represent 5 m, draw a horizontal x-axis for $0 \leqslant x \leqslant 22$.

Using a scale of 2 cm to represent 20 m, draw a vertical y-axis for $0 \leqslant y \leqslant 220$.

On your axes, plot the points given in the table and join them with a smooth curve.

(c) Use your graph to find

(i) the greatest height reached by the stone,

(ii) the horizontal distance from the foot of the tower when the stone is 180 m above sea level.

15. A water tank can be filled with water by two pipes in $11\frac{1}{9}$ minutes. If the smaller pipe takes 5 minutes longer than the larger pipe to fill the tank, find the time taken by each pipe to fill the tank.

16. A boat travels 12 km upstream and back in 1 hour and 30 minutes. Given that the speed of the current is 5 km/h, find the speed of the boat in still water.

Hint: Let the speed of the boat in still water be x km/h.

Challenge Yourself

1. A two-digit number is such that the sum of its digits is 6 while the product of its digits is $\frac{1}{3}$ of the original number. Find the original number.

Hint: Let x be one of the digits.

2. If $x = h$ and $x = k$ are the real solutions of the quadratic equation $ax^2 + bx + c = 0$, where a, b and c are constants and $a \neq 0$, we say that $x = h$ and $x = k$ are the roots of the equation. The sum of the roots, $h + k$, and the product of the roots, hk, can be expressed in terms of the coefficients a, b and c. Find an expression for $h + k$ and for hk in terms of a, b and/or c.

Further Functions

Given a function, every input gives only one output. Inverse functions reverse the process and give us the relevant inputs given the output. We can apply a function to convert one currency to another and similarly, an inverse function to convert from the new currency to the original one.

Chapter
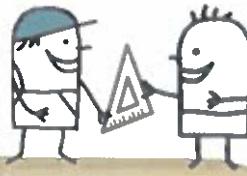
Two

2.1 Functions Involving Higher Order Expressions

:::: Recap

In Book 2, we have learnt how to solve simple questions on functions involving linear expressions.

For example,

Given the function $f : x \mapsto 6x + 3$, find the values of $f(1)$ and $f\left(\frac{1}{4}\right)$.

We have $f(1) = 6(1) + 3 = 9$ and $f\left(\frac{1}{4}\right) = 6\left(\frac{1}{4}\right) + 3 = 4\frac{1}{2}$.

In this section, we will learn how to solve questions on functions involving higher order expressions using quadratic expansions and indices.

Let us consider the function f mapping x to $x^2 + 2x$.

As shown in Book 2, we can write

$f : x \mapsto x^2 + 2x$ or $f(x) = x^2 + 2x$

When $x = 1$, $f(1) = 1^2 + 2(1) = 3$

When $x = 2$, $f(2) = 2^2 + 2(2) = 8$

When $x = -\frac{1}{2}$, $f\left(-\frac{1}{2}\right) = \left(-\frac{1}{2}\right)^2 + 2\left(-\frac{1}{2}\right) = -\frac{3}{4}$

PRACTISE NOW

If $f(x) = 6x^2 + 3x + 1$, find the value of $f(3)$.

Worked Example 1

(Problem involving Functions with Higher Order Expressions)
If $f(x) = x^2 + 3x + 2$, express each of the following in terms of x.
  (i) $f(2x)$ (ii) $f(2x + 1)$ (iii) $f(x^2 + 1)$

Solution:

(i) $f(2x) = (2x)^2 + 3(2x) + 2$
        $= 4x^2 + 6x + 2$

(ii) $f(2x + 1) = (2x + 1)^2 + 3(2x + 1) + 2$
          $= (4x^2 + 4x + 1) + (6x + 3) + 2$ (apply $(a + b)^2 = a^2 + 2ab + b^2$)
          $= 4x^2 + 10x + 6$

(iii) $f(x^2 + 1) = (x^2 + 1)^2 + 3(x^2 + 1) + 2$
          $= (x^4 + 2x^2 + 1) + (3x^2 + 3) + 2$ (apply $(a + b)^2 = a^2 + 2ab + b^2$ and Law 3 of Indices)
          $= x^4 + 5x^2 + 6$

PRACTISE NOW 1

If $f(x) = 4x^2 - 5x + 2$, express each of the following in terms of x.
(i) $f(3x)$ (ii) $f(2x + 3)$ (iii) $f(x^2 - 3)$

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Worked Example 2

(Problem involving Functions and Simultaneous Equations)

Given the function $g(x) = hx + d$ and that $g(2) = 6$ and $g(7) = 16$, find the value of h and of d.

Hence evaluate $g(5)$ and $g(-3)$.

Solution:

$g(2) = 2h + d = 6$ ——— (1)

$g(7) = 7h + d = 16$ ——— (2)

$(2) - (1)$: $5h = 10$

$\qquad h = 2$

Substitute $h = 2$ into (1): $2(2) + d = 6$

$\qquad\qquad\qquad d = 6 - 4$

$\qquad\qquad\qquad\quad = 2$

$\therefore h = 2$, $d = 2$

$g(5) = 2(5) + 2 = 12$

$g(-3) = 2(-3) + 2 = -4$

PRACTISE NOW 2

Given the function $f(x) = ax^2 + bx$ and that $f(3) = 15$ and $f(-2) = 8$, find the value of a and of b.

Hence evaluate $f(1)$ and $f(-5)$.

Exercise 2A

BASIC LEVEL

1. If $g(x) = x^2 + 5$, find the value of each of the following.
   (i) $g(a)$
   (ii) $g(a + 1)$
   (iii) $g(a + 1) - g(a - 1)$

2. If $F(x) = \frac{1}{2}x(x + 1)$, evaluate the following.
   (i) $F(1)$
   (ii) $F(2)$
   (iii) $F(3)$
   (iv) $F(x - 1)$
   (v) $F(5) - F(4)$
   (vi) $F(7) - F(6)$
   (vii) $F(x + 1)$
   (viii) $F(x) - F(x - 1)$
   (ix) $F(x^2)$
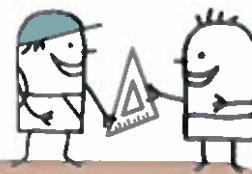
3. If $h(x) = x^2 - 5x + 4$,
   (i) express $h(2a) - h(a)$ in terms of a,
   (ii) find the values of a for which $h(a) = 0$,
   (iii) express $h(a^2) + h(a)$ in terms of a.

4. Given the function $g(x) = mx + c$ and that $g(1) = 5$ and $g(5) = -4$, find the value of m and of c. Hence, evaluate $g(3)$ and $g(-4)$.

INTERMEDIATE LEVEL

5. Given the function $h(x) = px^2 + qx + 2$ and that $h(2) = 34$ and $h(-3) = 29$, find the value of p and of q. Hence evaluate $h(4)$ and $h(-2)$.

2.2 Inverse Functions

Consider the function $f : x \mapsto x + 2$ for the domain $A = \{1, 2, 3, 4\}$.

$f(1) = 1 + 2 = 3$, $f(2) = 2 + 2 = 4$, $f(3) = 3 + 2 = 5$, $f(4) = 4 + 2 = 6$

The arrow diagram below shows the function f.
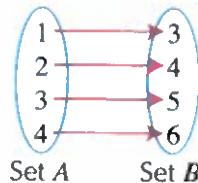
Set A   Set B

Fig. 2.1

If we reverse the direction of the arrows in Fig. 2.1, that is, map Set **B** into Set **A** instead, we get a new function called the **inverse function** of f denoted by f^{-1}.

The arrow diagram below shows the function f^{-1}.

Set B   Set A
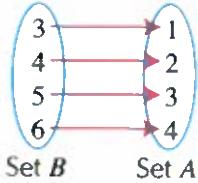
Fig. 2.2

In this case, the domain of f^{-1} is $\{3, 4, 5, 6\}$. Notice that

If $\begin{cases} f(1) = 3, \\ f(2) = 4, \\ f(3) = 5, \\ f(4) = 6, \end{cases}$ then $\begin{cases} f^{-1}(3) = 1 \\ f^{-1}(4) = 2 \\ f^{-1}(5) = 3 \\ f^{-1}(6) = 4 \end{cases}$

In general, if $y = f(x)$, then $f^{-1}(y) = x$

> **ATTENTION**
>
> Not all functions have inverses. The function $f(x) = x^2$ is an example of a function without an inverse. The inverse is not a function since every positive number, x^2, has two square roots, x and $-x$. Thus, f^{-1} cannot be defined.

Worked Example 3

(Finding the Inverse of a Function)

A function f is defined by $f : x \mapsto x + 2$, find the inverse function $f^{-1}(x)$.

Solution:

$f(x) = x + 2$

Let $y = x + 2$

$f(x) = y$ and $f^{-1}(y) = x$

Expressing x in terms of y, we have $x = y - 2$.

$\therefore f^{-1}(y) = y - 2$ or $f^{-1} : y \mapsto y - 2$.

Note: x and y are dummy variables and it is customary for us to write
$f^{-1}(x) = x - 2$ or $f^{-1} : x \mapsto x - 2$

> **ATTENTION**
>
> $f^{-1}(x) \neq \dfrac{1}{f(x)}$

PRACTISE NOW 3

A function f is defined by $f : x \mapsto 8x + 3$, find the inverse function $f^{-1}(x)$.

SIMILAR QUESTIONS

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Worked Example 4

(Finding the Inverse of a Function)

A function f is defined by $f : x \mapsto 5x - 4$, find the inverse function $f^{-1}(x)$.

Hence evaluate $f^{-1}(5)$, $f^{-1}(-10)$ and $f^{-1}\left(\dfrac{1}{2}\right)$.

Solution:

$f(x) = 5x - 4$

Let $y = 5x - 4$

$f(x) = y$ and $f^{-1}(y) = x$

Expressing x in terms of y, we have $x = \dfrac{y+4}{5}$.

$\therefore f^{-1}(y) = \dfrac{y+4}{5}$ or $f^{-1} : y \mapsto \dfrac{y+4}{5}$

Note: x and y are dummy variables and it is customary for us to write

$f^{-1}(x) = \dfrac{x+4}{5}$ or $f^{-1} : x \mapsto \dfrac{x+4}{5}$

$f^{-1}(5) = \dfrac{5+4}{5} = 1\dfrac{4}{5}$

$f^{-1}(-10) = \dfrac{-10+4}{5} = -1\dfrac{1}{5}$

$f^{-1}\left(\dfrac{1}{2}\right) = \dfrac{\frac{1}{2}+4}{5} = \dfrac{9}{10}$

Alternatively,

The image of x under f is $5x - 4$. Thus we need to find the value of x for which the image is 5, -10 and $\dfrac{1}{2}$.

$f(x) = 5x - 4$

If $x = f^{-1}(5)$, then $f(x) = 5$,

i.e. $5x - 4 = 5$

$\quad\quad 5x = 9$

$\quad\quad x = 1\dfrac{4}{5}$

$\therefore f^{-1}(5) = 1\dfrac{4}{5}$

If $x = f^{-1}(-10)$, then $f(x) = -10$,

i.e. $5x - 4 = -10$

$\quad\quad 5x = -6$

$\quad\quad x = -1\dfrac{1}{5}$

$\therefore f^{-1}(-10) = -1\dfrac{1}{5}$

If $x = f^{-1}\left(\dfrac{1}{2}\right)$, then $f(x) = \dfrac{1}{2}$,

i.e. $5x - 4 = \dfrac{1}{2}$

$\quad\quad 5x = 4\dfrac{1}{2}$

$\quad\quad x = \dfrac{9}{10}$

$\therefore f^{-1}\left(\dfrac{1}{2}\right) = \dfrac{9}{10}$

A function f is defined by $f : x \mapsto 7x - 4$, find the inverse function $f^{-1}(x)$.

Hence evaluate $f^{-1}(10)$, $f^{-1}(-4)$ and $f^{-1}\left(\dfrac{1}{7}\right)$.

Exercise 2B Questions 2–7

Worked Example 5

(Defining Inverse Functions)

A function f is defined by $f : x \mapsto \dfrac{x}{x-2}$ where $x \neq 2$. Find $f^{-1}(x)$ and state the value of x

for which f^{-1} is not defined. Evaluate $f^{-1}(5)$, $f^{-1}(-4)$ and $f^{-1}\left(-\dfrac{1}{2}\right)$.

Solution:

$f(x) = \dfrac{x}{x-2}$

Let $y = \dfrac{x}{x-2}$

$\therefore f(x) = y$ and $f^{-1}(y) = x$

$x = xy - 2y$

$2y = xy - x$

$2y = x(y-1)$

$x = \dfrac{2y}{y-1}$

$\therefore f^{-1}(y) = \dfrac{2y}{y-1}$

Hence $f^{-1}(x) = \dfrac{2x}{x-1}$ or $f^{-1} : x \mapsto \dfrac{2x}{x-1}$

f^{-1} is not defined when $x - 1 = 0$, i.e. $x = 1$.

$f^{-1}(5) = \dfrac{2(5)}{5-1} = \dfrac{10}{4} = 2\dfrac{1}{2}$

$f^{-1}(-4) = \dfrac{2(-4)}{(-4)-1} = \dfrac{-8}{-5} = 1\dfrac{3}{5}$

$f^{-1}\left(-\dfrac{1}{2}\right) = \dfrac{2\left(-\dfrac{1}{2}\right)}{\left(-\dfrac{1}{2}\right)-1} = \dfrac{-1}{-\dfrac{3}{2}} = \dfrac{2}{3}$

A function f is defined by $f : x \mapsto \dfrac{2}{x-5}$ where $x \neq 2$. Find $f^{-1}(x)$ and state the value of x

for which f^{-1} is not defined. Evaluate $f^{-1}(6)$, $f^{-1}(-3)$ and $f^{-1}\left(\dfrac{1}{4}\right)$.

Exercise 2B Questions 10–11

Worked Example 6

A function f is defined by $f : x \mapsto ax + b$.

Given that $f(2) = 10$ and $f^{-1}(3) = 1$, find the value of a and of b.

Solution:

$f(x) = ax + b$

$f^{-1}(3) = 1 \Rightarrow f(1) = 3$

Thus we have $f(1) = 3$ and $f(2) = 10$.

i.e. $\qquad 3 = a(1) + b$ —— (1)

and $\qquad 10 = a(2) + b$ —— (2)

$(2) - (1)$: $a = 7$

Substitute $a = 7$ into (1): $b = 3 - 7$

$\qquad\qquad\qquad\qquad = -4$

PRACTISE NOW 6

A function f is defined by $f : x \mapsto px + q$.

Given that $f(3) = 15$ and $f^{-1}(3) = 6$, find the value of p and of q.

SIMILAR QUESTIONS

Exercise 2B Questions 8, 9, 12–15

Exercise 2B

BASIC LEVEL

1. A function f is defined by $f : x \mapsto \dfrac{1}{4}x - 3$. Find the inverse function $f^{-1}(x)$.

2. A function f is defined by $f : x \mapsto x - 7$. Find the inverse function f^{-1} and hence evaluate $f^{-1}(3)$, $f^{-1}(7)$, $f^{-1}(-5)$ and $f^{-1}\left(\dfrac{1}{3}\right)$.

3. A function g is defined by $g : x \mapsto 3x + 4$. Express g^{-1} in similar form and hence evaluate $g^{-1}(3)$, $g^{-1}(-4)$, $g^{-1}\left(\dfrac{1}{2}\right)$ and $g^{-1}\left(-\dfrac{3}{4}\right)$.

4. A function h is defined by $h : x \mapsto 5x + 6$. Express h^{-1} in similar form and hence evaluate $h^{-1}(6)$, $h^{-1}(10)$, $h^{-1}\left(-\dfrac{2}{5}\right)$ and $h^{-1}\left(12\dfrac{1}{2}\right)$.

5. A function f is defined by $f : x \mapsto 8 - 3x$. Find the inverse function $f^{-1}(x)$. Hence evaluate $f^{-1}(9)$, $f^{-1}(-12)$, $f^{-1}\left(3\dfrac{1}{3}\right)$ and $f^{-1}\left(-\dfrac{3}{16}\right)$.

6. A function g is defined by $g : x \mapsto 6x - 8$ for all real values of x. What are the elements in the domain, whose images are 10, 40, −4 and −6?

7. Given the function $f(x) = 7 - \dfrac{3}{5}x$ and $g(x) = \dfrac{1}{4}x - 6$, evaluate each of the following.

   (i) $f^{-1}(3)$ 　　　　　　　　(ii) $f^{-1}(-17)$

   (iii) $g^{-1}(5)$ 　　　　　　　(iv) $g^{-1}(-6)$

   (v) $f^{-1}(2) + g^{-1}(1)$ 　　(vi) $f^{-1}(4) - g^{-1}(4)$

8. Given that $f(x) = ax^2 + bx$, $f(-2) = 20$ and $f(4) = 32$, find the value of a and of b.

12. Given that $f(x) = ax + b$, $f(1) = 1$ and $f^{-1}(5) = 2$, find the value of a and of b. Hence, find $f^{-1}(x)$ and evaluate $f^{-1}(7)$ and $f^{-1}\left(-5\frac{1}{2}\right)$.

9. A function h is defined by $h : x \mapsto px^2 + qx$. Given that $h(1) = 2$ and $h^{-1}(36) = 3$, find the value of p and of q. Hence evaluate $h(-1)$ and $h(2)$.

13. A function f is defined by $f : x \mapsto ax + b$. Given that $f(1) = 3$ and $f^{-1}(7) = 5$, find the value of a and of b. Hence find $f^{-1}(x)$.

10. Given the function $f(x) = \dfrac{5x}{2 - 4x}$ where $x \neq \dfrac{1}{2}$, find $f^{-1}(x)$ and state the value of x for which $f^{-1}(x)$ is not defined. Hence evaluate $f^{-1}(4)$ and $f^{-1}(-6)$.

14. A function f is defined by $f : x \mapsto px + q$. Given that $f(1) = -5$ and $f(-2) = -10$, find the value of p and of q. Hence find $f^{-1}(x)$.

11. Given the function $f : x \mapsto \dfrac{3x - 1}{x - 2}$ where $x \neq 2$, find $f^{-1}(x)$ and state the value of x for which $f^{-1}(x)$ is not defined. Hence evaluate $f^{-1}(5)$ and $f^{-1}(7)$.

15. A function g is defined by $g : x \mapsto mx + c$. Given that $g^{-1}(-3) = 0$ and $g^{-1}(1) = 2$, find the value of $g(5)$ and $g^{-1}(4)$.

Summary

1. If a function f maps x to y, then its inverse function f^{-1} maps y to x.

Review Exercise 2

1. A function f is defined by $f : x \mapsto 6x - 9$.
   (i) Find the value of f(3) and of f(−4).
   (ii) Express f^{-1} in similar form.
   (iii) Find the value of $f^{-1}(3)$ and of $f^{-1}(-8)$.

2. A function f is defined by $f : x \mapsto 2px + 3q$. Given that f(2) = 17 and $f^{-1}(5) = -1$, find the value of p and q. Hence,
   (i) evaluate f(4) and f(−5),
   (ii) find $f^{-1}(x)$ and hence evaluate $f^{-1}(10)$ and $f^{-1}(-16)$.

3. Given the function $f : x \mapsto x^2 - 3x + 5$, express each of the following in terms of a.
   (i) f(a)
   (ii) f(2a)
   (iii) f(a + 3)
   (iv) $f(a^2)$
   (v) $f(a^2 + 2)$
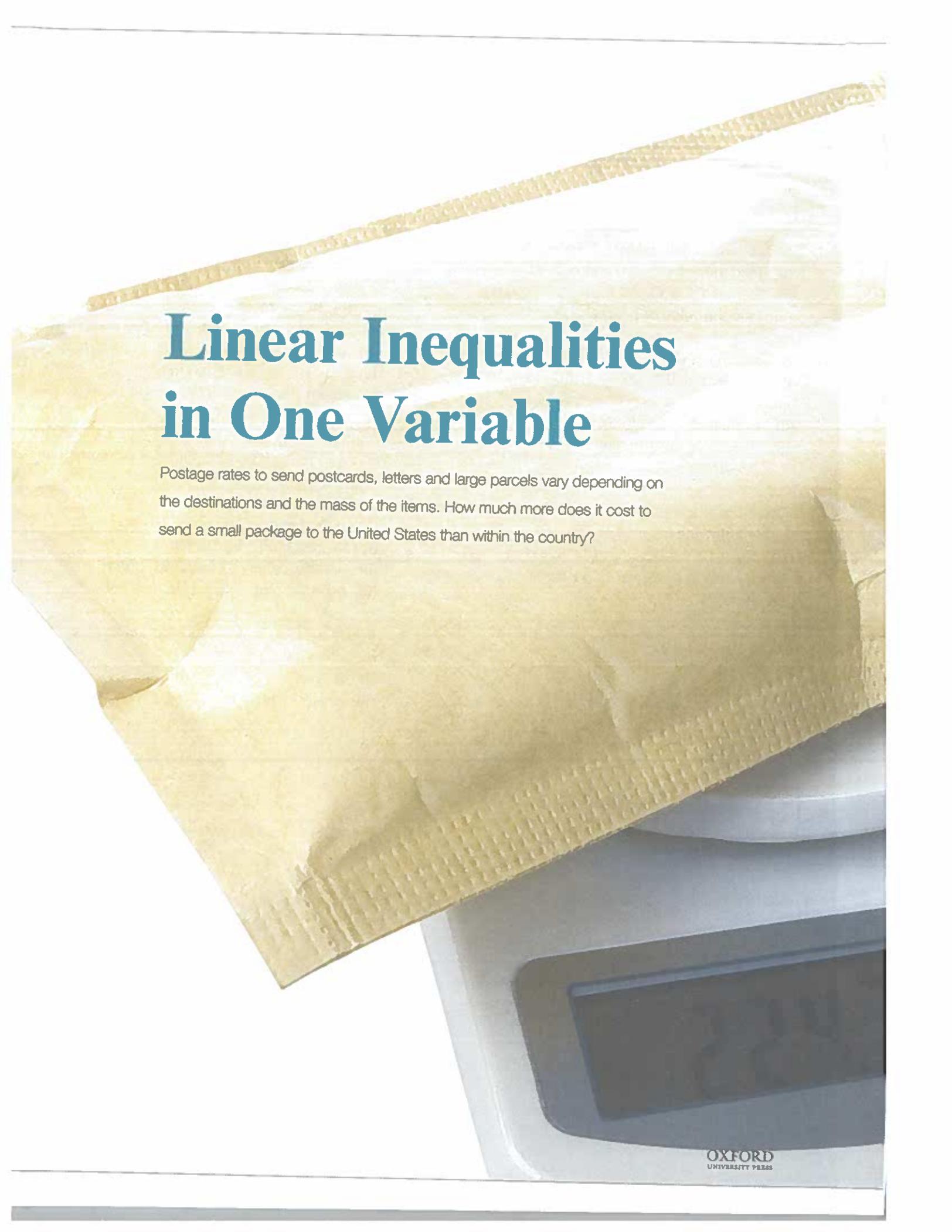
4. Given the function $f(x) = 10x - 7$ and $g(x) = 3 - \frac{1}{4}x$,
   (i) find the values of
      (a) f(2) + g(4),
      (b) f(−1) − g(8),
      (c) 3f(3) − 2g(3).
   (ii) find the value of x for which f(x) = 2g(x).
   (iii) find the value of p for which f(2p) = g(4p).
   (iv) Express $f^{-1}(x)$ and $g^{-1}(x)$ in terms of x.
   (v) Find the value of $f^{-1}(3) - g^{-1}(4)$.

5. A function g is defined by $g : x \mapsto px + q$ and that g(5) = 4 and $g^{-1}(3) = 17$, find the value of p and of q and express g^{-1} in similar form.

6. Given the function $h(x) = x^2 - 7x + 6$,
   (i) express $h(a^2) - h(a)$ in terms of a,
   (ii) find the value of k for which h(k) = 0,
   (iii) express $h(2a) - h(a + 1)$ in terms of a.

7. If $h(x) = 5x^2 + 2x + 1$, find the value of each of the following.
   (i) h(2x)
   (ii) h(x + 1)
   (iii) h(x + 1) − h(x − 1)

Challenge Yourself

1. Find the domain and range of the inverse of the following function.

x	−3	1	3	5
y	2	6	7	2

Hence, determine if the inverse is a function.

Linear Inequalities in One Variable

Postage rates to send postcards, letters and large parcels vary depending on the destinations and the mass of the items. How much more does it cost to send a small package to the United States than within the country?
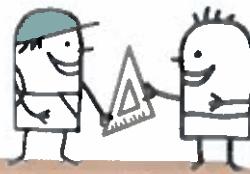
Chapter
Three

LEARNING OBJECTIVES

At the end of this chapter,
you should be able to:

- solve linear inequalities in one variable and
  represent the solution on a number line,
- apply linear inequalities to solve word problems.

3.1 Inequalities
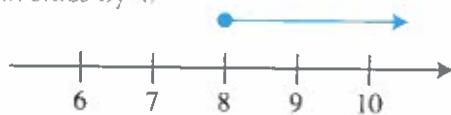
Recap

In Book 1, we have learnt how to solve a simple inequality in the form $ax > b$, $ax \geqslant b$, $ax < b$ and $ax \leqslant b$, where a and b are integers and $aa > 0$.

'To solve an inequality' means to find all the solutions that satisfy the inequality.
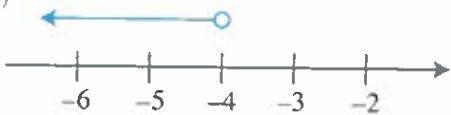
For example, if $4x \geqslant 32$,

then $\qquad x \geqslant 8$. (divide both sides by 4)

If $5y < -20$,

then $y < -4$, (divide both sides by 5)

We will now look at problems involving the multiplication and division of an inequality by a negative number.

Investigation

Properties of Inequalities

In this investigation, we shall explore some properties of inequalities.

1. Consider the inequality $10 > 6$ and complete Table 3.1.

Multiplication by a *negative* number on both sides of the inequality $10 > 6$	$\begin{aligned} \text{LHS} &= 10 \times (-5) \\ &= -50 \\ \text{RHS} &= 6 \times (-5) \\ &= -30 \end{aligned}$	$-50 < -30$	Yes	If $x > y$ and $d < 0$, then dx ____ dy.
Division by a *negative* number on both sides of the inequality $10 > 6$				If $x > y$ and $d < 0$, then $\dfrac{x}{d}$ ____ $\dfrac{y}{d}$.

Table 3.1

2. Do the conclusions which you have drawn from Table 3.1 apply to $10 \geqslant 6$? Can you generalise the conclusions for $x \geqslant y$? What about $x < y$ and $x \leqslant y$?

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From the investigation, we can conclude that:
If we multiply or divide both sides of an inequality by a *negative* number, we will have to *reverse* the inequality sign, i.e.

$$\text{if } x \geq y \text{ and } d < 0, \text{ then } dx \leq dy \text{ and } \frac{x}{d} \leq \frac{y}{d}.$$

Worked Example 1
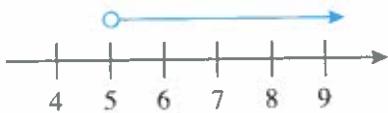
(Solving Linear Inequalities)
Solve each of the following inequalities and illustrate the solutions on a number line.
(a) $-4x < -20$
(b) $-2x \geq 4$

Solution:
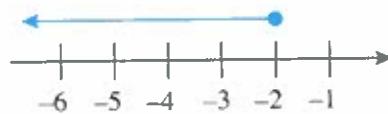
(a)
$$-4x < -20$$
$$-1 \times (-4x) > -1 \times -20 \quad \text{(since } -1 < 0, \text{ change in the inequality sign)}$$
$$4x > 20$$
$$x > \frac{20}{4} \quad \text{(since } 4 > 0, \text{ no change in the inequality sign)}$$
$$\therefore x > 5$$

(b)
$$-2x \geq 4$$
$$-1 \times (-2x) \leq -1 \times 4 \quad \text{(since } -1 < 0, \text{ change in the inequality sign)}$$
$$2x \leq -4$$
$$x \leq \frac{-4}{2} \quad \text{(since } 2 > 0, \text{ no change in the inequality sign)}$$
$$\therefore x \leq -2$$

PRACTISE NOW 1

1. Solve each of the following inequalities and illustrate the solutions on a number line.
   (a) $-6x > -30$
   (b) $-8x \leq 32$
2. Find the smallest integer value of x that satisfies the inequality $-3x < -13$.

In general, to solve an inequality, we

- multiply or divide both sides by a *positive* number *without* having to reverse the inequality signs,

  i.e. if $x \geq y$ and $c > 0$, then $cx \geq cy$ and $\dfrac{x}{c} \geq \dfrac{y}{c}$.

- *reverse* the inequality sign if we multiply or divide both sides by a *negative* number,

  i.e. if $x \geq y$ and $d < 0$, then $dx \leq dy$ and $\dfrac{x}{d} \leq \dfrac{y}{d}$.

In the next section, we will learn how to solve linear inequalities in one variable and represent the solution on a number line.

Inequalities

Investigation

Inequalities

By considering a number line, fill in each blank with '>' or '<'.

1. (a) (i) $6 < 12$ (ii) $6 + 2 \;\square\; 12 + 2$ (iii) $6 - 4 \;\square\; 12 - 4$

   (b) If $6 < 12$ and a is a real number, then $6 + a \;\square\; 12 + a$ and $6 - a \;\square\; 12 - a$.

   (c) If $12 > 6$ and a is a real number, then $12 + a \;\square\; 6 + a$ and $12 - a \;\square\; 6 - a$.

2. (a) (i) $-6 \;\square\; 12$ (ii) $-6 + 2 \;\square\; 12 + 2$ (iii) $-6 - 4 \;\square\; 12 - 4$

   (b) If $-6 < 12$ and a is a real number, then $-6 + a \;\square\; 12 + a$ and $-6 - a \;\square\; 12 - a$.

   (c) If $12 > -6$ and a is a real number, then $12 + a \;\square\; -6 + a$ and $12 - a \;\square\; -6 - a$.

3. (a) (i) $6 \;\square\; -12$ (ii) $6 + 2 \;\square\; -12 + 2$ (iii) $6 - 4 \;\square\; -12 - 4$

   (b) What do you observe about your answers in Question 3(a)?

4. Do the conclusions which you have drawn in Questions 1 to 3 apply to $6 \leq 12$? What about $12 \geq 6$?

From the investigation, we observe that when we add or subtract a positive or a negative number from both sides of an inequality, the inequality sign does not change,

> i.e. if $x > y$, then $x + a > y + a$ and $x - a > y - a$.
>
> If $x \geq y$, then $x + a \geq y + a$ and $x - a \geq y - a$.
>
> Similarly, if $x < y$, then $x + a < y + a$ and $x - a < y - a$.
>
> If $x \leq y$, then $x + a \leq y + a$ and $x - a \leq y - a$.

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One example of a real-life application of inequalities is the speed limit of vehicles travelling on expressways. Can you think of other real-life applications of inequalities? Write down as many as you can.

Worked Example 2

(Solving Linear Inequalities)

Solve each of the following inequalities, illustrating each solution on a number line.

(a) $x + 4 < 3$　　　　　　　(b) $-4y - 5 \geqslant 11$

Solution:

(a) $x + 4 < 3$

$x + 4 - 4 < 3 - 4$　　　(subtract 4 from both sides)

$x < -1$

(b) $-4y - 5 \geqslant 11$
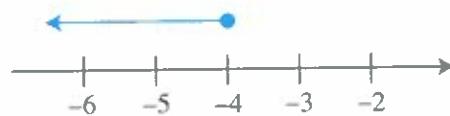
$-4y - 5 + 5 \geqslant 11 + 5$　　　(add 5 to both sides)

$-4y \geqslant 16$

$4y \leqslant -16$　　　(multiply both sides by −1 and change the inequality sign)

$\dfrac{4y}{4} \leqslant \dfrac{-16}{4}$　　　(divide by 4 on both sides)

$y \leqslant -4$

PRACTISE NOW 2

Solve each of the following inequalities, illustrating each solution on a number line.

(a) $x - 3 \geqslant 7$　　　　　　　(b) $-2y + 4 > 3$

Thinking Time

1. Given an equation in the form $ax + b = c$, where a, b and c are constants and $a > 0$, list the steps you would take to find the value of x. Would the steps change if $a < 0$?

2. Given an inequality in the form $ax + b > c$, where a, b and c are constants and $a > 0$, list the steps you would take to find the range of values of x. How would the steps change if $a < 0$?

3. Given an inequality in the form $ax + b \geqslant c$, where a, b and c are constants and $a > 0$, list the steps you would take to find the range of values of x. How would the steps change if $a < 0$?

For the inequalities $ax + b < c$, $ax + b \leqslant c$, $ax + b > c$ and $ax + b \geqslant c$, we say that the corresponding linear equation is $ax + b = c$.

4. How is the solution of each inequality related to that of the corresponding linear equation?

Worked Example 3

(Solving Linear Inequalities)

Solve the inequality $8 - x > 3$ and illustrate the solution on a number line.

(i) If x is a prime number, write down the largest possible value of x that satisfies the inequality.

(ii) Write down the positive integer values of x that satisfy the inequality.

Solution:

$$8 - x > 3$$
$$8 - 8 - x > 3 - 8 \quad \text{(subtract 8 from both sides)}$$
$$-x > -5$$
$$x < 5 \quad \text{(multiply both sides by } -1 \text{ and change the inequality sign)}$$

(i) Largest prime value of x is **3**

(ii) Positive integer values of x are 1, 2, 3 and 4

PRACTISE NOW 3

Solve the inequality $5 - x < -9$ and illustrate the solution on a number line.

(i) If x is a prime number, write down the smallest possible value of x that satisfies the inequality.

(ii) Given that x is a perfect cube, find the smallest possible value of x.

Worked Example 4

Solve each of the following inequalities.

(a) $3x - 2 > 2(1 - x)$ (b) $\dfrac{y}{4} \leqslant \dfrac{y+1}{7}$

Solution:

(a)
$$3x - 2 > 2(1 - x)$$
$$3x - 2 > 2 - 2x \quad \text{(expand the RHS)}$$
$$3x - 2 + 2x > 2 - 2x + 2x \quad \text{(add } 2x \text{ to both sides)}$$
$$5x - 2 > 2$$
$$5x - 2 + 2 > 2 + 2 \quad \text{(add 2 to both sides)}$$
$$5x > 4$$
$$x > \frac{4}{5}$$

(b)
$$\frac{y}{4} \leqslant \frac{y+1}{7}$$
$$4 \times 7 \times \frac{y}{4} \leqslant 4 \times 7 \times \frac{y+1}{7} \quad \text{(multiply by } 4 \times 7 \text{ on both sides)}$$
$$7y \leqslant 4(y + 1)$$
$$7y \leqslant 4y + 4 \quad \text{(expand the RHS)}$$
$$7y - 4y \leqslant 4y + 4 - 4y \quad \text{(subtract } 4y \text{ from both sides)}$$
$$3y \leqslant 4$$
$$y \leqslant \frac{4}{3}$$
$$y \leqslant 1\frac{1}{3}$$

Problem Solving Tip

The LCM of 4 and 7 is 4×7.

PRACTISE NOW 4

1. Solve each of the following inequalities.

   (a) $15x + 1 < 5(3 + x)$ (b) $\dfrac{16y}{3} \geqslant \dfrac{y+1}{2}$

   (c) $\dfrac{1}{2}(z - 4) \leqslant \dfrac{1}{3}(z + 1) + 2$

2. Given that p satisfies the inequality $\dfrac{3}{4}(p - 2) + \dfrac{1}{2} > \dfrac{1}{2}(p - 1)$, find the smallest possible value of p if p is a perfect square.

Exercise 3A

1. Fill in each box with '<', '>', '≤' or '≥'.

   (a) If $x > y$, then $-6x \boxed{} -6y$.

   (b) If $x < y$, then $\dfrac{x}{-30} \boxed{} \dfrac{y}{-30}$.

   (c) If $x \geq y$, then $-4x \boxed{} -4y$.

   (d) If $x \leq y$, then $\dfrac{x}{-10} \boxed{} \dfrac{y}{-10}$.

   (e) $5 + h \boxed{} 7 + h$, where h is a real number

   (f) $5 - k \boxed{} 7 - k$, where k is a real number

2. Solve each of the following inequalities, illustrating each solution on a number line.

   (a) $a + 2 < 3$ (b) $b - 3 \geq 4$

   (c) $-c + 3 > 5$ (d) $4 - d \leq 4$

   (e) $-2e - 1 \leq 2$ (f) $2 + 5f < 0$

   (g) $g - 7 \geq 1 - g$ (h) $5h > 4(h + 1)$

   (i) $8j + 3 < 2(7 - j)$ (j) $4k + 5 \geq 2(-2k)$

   (k) $2(m - 5) \leq 2 - m$ (l) $3(1 - 4n) > 8 - 7n$

3. Solve the inequality $7 + 2x \leq 16$ and illustrate the solution on a number line.

   (i) If x is an integer, write down the largest possible value of x that satisfies the inequality.

   (ii) Given that x is a perfect square, find the largest possible value of x.

4. Solve the inequality $3 - 4x > 3x - 18$ and illustrate the solution on a number line.

   (i) If x is a prime number, write down the possible value(s) of x that satisfies the inequality.

   (ii) Does $x = 0$ satisfy the inequality? Explain your answer.

5. Solve each of the following inequalities, illustrating each solution on a number line.

   (a) $-5x < 25$

   (b) $-12x \geq 138$

   (c) $-y \leq -7$

   (d) $-9y > -35$

   (e) $4(p + 1) < -3(p - 4)$

   (f) $6 - (1 - 2q) \geq 3(5q - 2)$

6. Solve each of the following inequalities.

   (a) $\dfrac{4a}{3} \geq 2$

   (b) $\dfrac{2b - 1}{3} > \dfrac{3b}{5}$

   (c) $\dfrac{c + 4}{4} > \dfrac{c + 1}{3}$

   (d) $\dfrac{2 - d}{2} < \dfrac{3 - d}{4} + \dfrac{1}{2}$

   (e) $\dfrac{1}{4}(e - 2) + \dfrac{2}{3} < \dfrac{1}{6}(e - 4)$

   (f) $\dfrac{f + 1}{2} + \dfrac{3f + 1}{4} \leq \dfrac{3f - 1}{4} + 2$

   (g) $\dfrac{1}{5}(3g + 4) - \dfrac{1}{3}(g + 1) \geq 1 - \dfrac{1}{3}(g + 5)$

   (h) $4\left(\dfrac{h}{3} + \dfrac{3}{4}\right) < 3\left(\dfrac{h}{2} - 5\right)$

7. Given that p satisfies the inequality $\dfrac{1}{6}(2 - p) - 3 \geq \dfrac{p}{10}$, find the largest possible value of p.
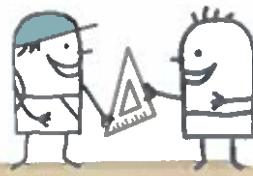
8. Given that $\dfrac{1}{3}(2x - 7) \leq \dfrac{3x + 2}{2}$,

   (i) solve the inequality,

   (ii) find the smallest possible value of x^2.

In this section, we shall take a look at how inequalities are used to solve problems.

Worked Example 5

(Problem Solving involving Inequalities)

Devi scored 66 marks for her first class test and 72 marks for her second class test. What is the minimum mark she must score for her third class test to meet her target of obtaining an average of 75 marks or more for the three tests?

Solution:

Let x be the marks scored by Devi in her third class test.

$$\frac{66 + 72 + x}{3} \geq 75$$

$$3 \times \frac{66 + 72 + x}{3} \geq 3 \times 75 \text{ (multiply by 3 on both sides)}$$

$$66 + 72 + x \geq 225$$

$$138 + x \geq 225$$

$$138 + x - 138 \geq 225 - 138 \text{ (subtract 138 from both sides)}$$

$$x \geq 87$$

$\therefore$ Devi must score at least 87 marks for her third class test.

PRACTISE NOW 5

The minimum mark to obtain a Grade A is 75. Priya managed to achieve an average of Grade A for three of her Science quizzes. What is the minimum mark she scored in her first quiz if she scored 76 and 89 marks in her second and third quiz respectively?

Worked Example 6

An IQ test consists of 20 multiple choice questions. 3 points are awarded for a correct answer and 1 point is deducted for a wrong answer. No points are awarded or deducted for an unanswered question. Raj attempted a total of 19 questions and his total score for the IQ test was above 32. Find the minimum number of correct answers he obtained.

Solution:

Let x and y be the number of correct answers and incorrect answers respectively.

$x + y = 19$ ---------- (1)

$3 \times x + (-1) \times y > 32$

i.e. $\quad 3x - y > 32$ ---------- (2)

From (1),
$y = 19 - x$ ---------- (3)

Substitute (3) into (2):

$3x - (19 - x) > 32$

$3x - 19 + x > 32$

$4x - 19 + 19 > 32 + 19$ (add 19 to both sides)

$\qquad 4x > 51$

$\qquad x > \dfrac{51}{4}$ (divide by 4 on both sides)

$\qquad x > 12.75$

$\therefore$ Raj obtained at least 13 correct answers.

PRACTISE NOW 6
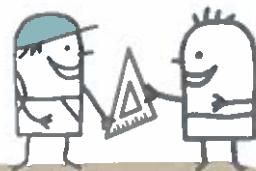
SIMILAR QUESTIONS

Vishal has 12 pieces of $10 and $5 notes in his wallet. If the total value of all the notes is less than $95, what is the maximum number of $10 notes that he has?

Exercise 3B Questions 2, 8, 9

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3.3 Solving Simultaneous Linear Inequalities

Simultaneous Linear Inequalities

To solve linear inequalities simultaneously, we find the solution(s) to each inequality separately, then we consider only the common solutions of the inequalities.

For example, given that $x \geq 5$ and $x \leq 8$, then the range of values of x which satisfies both inequalities is $5 \leq x \leq 8$.

Does $x = 1$ satisfy *both* $3x \leq x + 6$ *and* $2x + 4 < 3x + 6$? Does $x = -3$ satisfy *both* $3x \leq x + 6$ *and* $2x + 4 < 3x + 6$?

Worked Example 7

(Solving Simultaneous Linear Inequalities)
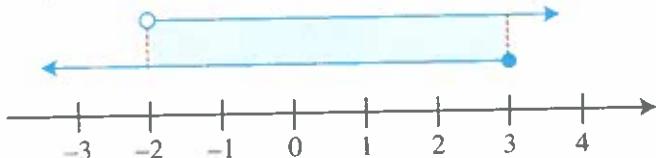
Find the range of values of x for which $3x \leq x + 6$ and $2x + 4 < 3x + 6$.

Solution:

Solving the two linear inequalities separately,

$$3x \leq x + 6 \qquad \text{and} \qquad 2x + 4 < 3x + 6$$
$$3x - x \leq x + 6 - x \qquad\qquad 2x + 4 - 3x < 3x + 6 - 3x$$
$$2x \leq 6 \qquad\qquad -x + 4 < 6$$
$$x \leq \frac{6}{2} \qquad\qquad -x + 4 - 4 < 6 - 4$$
$$x \leq 3 \qquad\qquad -x < 2$$
$$x > -2$$

Representing $x \leq 3$ and $x > -2$ on a number line,

∴ The solutions satisfying both inequalities lie in the overlapping shaded region, i.e. $-2 < x \leq 3$.

PRACTISE NOW 7

Find the range of values of x for which $2x - 3 \leq 7$ and $2x + 1 > -3x - 4$.

(Solving Simultaneous Linear Inequalities)

Solve the inequalities $4x + 14 \leqslant x + 5 < 3x - 1$.

Solution:

Solving the two linear inequalities separately,

$4x + 14 \leqslant x + 5$ and $x + 5 < 3x - 1$

$4x + 14 - x \leqslant x + 5 - x$ $x + 5 - 3x < 3x - 1 - 3x$

$3x + 14 \leqslant 5$ $-2x + 5 < -1$
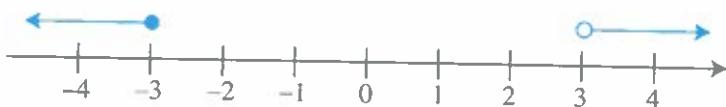
$3x + 14 - 14 \leqslant 5 - 14$ $-2x + 5 - 5 < -1 - 5$

$3x \leqslant -9$ $-2x < -6$

$x \leqslant -3$ $2x > 6$

     $x > 3$

Representing $x \leqslant -3$ and $x > 3$ on a number line,

∴ The simultaneous linear inequalities have **no solution**.

> **ATTENTION**
>
> Since there is no overlapping region on the number line, there is no solution that satisfies both inequalities.

PRACTISE NOW 8

1. Solve the inequalities $8x + 13 \leqslant 4x - 3 < 5x - 11$.

2. Solve the inequalities $\dfrac{y - 2}{3} < \dfrac{2y + 1}{5} \leqslant 3$.

SIMILAR QUESTIONS

Exercise 3B Questions 5(a)–(d), 11, 12, 13(a)–(d), 14(a)–(d), 15(a)–(d), 16–18

Performance Task

The table shows the postage rates for letters and small packages to Malaysia offered by a local company.

Mass (m g)	Postage (cents)
$0 < m \leqslant 20$	45
$20 < m \leqslant 50$	55
$50 < m \leqslant 100$	85
$100 < m \leqslant 200$	185
$200 < m \leqslant 300$	285

Search on the Internet for the postage rates for parcels to Thailand, New Zealand and the United Kingdom, displaying your findings in a table similar to the above.

Exercise 3B

1. On weekends, a movie ticket costs $10.50. Form an inequality and solve it to find the maximum number of tickets Kate can buy with $205.

2. The length and breadth of a rectangle are x cm and y cm respectively. If the rectangle has an area of 24 cm², state the possible pairs of integer values of x and y, where $x > y$.

3. Find the range of values of x which satisfy each of the following pairs of inequalities.
   (a) $x - 4 \le 3$ and $3x \ge -6$
   (b) $2x + 5 < 15$ and $3x - 2 > -6$

4. Find the integer values of x which satisfy each of the following pairs of inequalities.
   (a) $5x - 1 < 4$ and $3x + 5 \ge x + 1$
   (b) $2x - 5 \ge 1$ and $3x - 1 < 26$

5. Solve each of the following pairs of inequalities, illustrating each solution on a number line.
   (a) $-4 \le 2x \le 3x - 2$
   (b) $1 - x < -2 \le 3 - x$
   (c) $3x - 3 < x - 9 < 2x$
   (d) $2x \le x + 6 < 3x + 5$

6. Mr Chua's car consumes petrol at an average rate of 8 litres daily. Before Mr Chua begins his journey, he tops up the petrol in his car to 100 litres. Given that he will next top up the petrol in his car when there are 20 litres left, form an inequality and solve it to find the maximum number of days he can travel before he has to top up the petrol in his car.

7. If the sum of three consecutive integers is less than 75, find the cube of the largest possible integer.

8. In a Math Olympiad quiz, 5 points are awarded for a correct answer and 2 points are deducted for a wrong answer or if a question is left unanswered. Shirley attempted all 30 questions and her total score for the quiz was not more than 66. Find the maximum number of correct answers she obtained.

9. Ethan opened his coin bank to find 50 pieces of $5 and $2 notes. If the total value of all the notes is more than $132, find the minimum number of $5 notes he has.

10. Given that x is a prime number, find the values of x for which $\frac{1}{2}x - 4 > \frac{1}{3}x$ and $\frac{1}{6}x + 1 < \frac{1}{8}x + 3$.

11. An integer x is such that $x + 2 < 5\sqrt{17} < x + 3$. State the value of x.

12. Given that x is a prime number, find the value of x for which $3x - 2 \ge 10 \ge x + 4$.

13. Solve each of the following pairs of inequalities.
   (a) $3 - a \le a - 4 \le 9 - 2a$
   (b) $1 - b < b - 1 < 11 - 2b$
   (c) $3 - c < 2c - 1 < 5 + c$
   (d) $3d - 5 < d + 1 \le 2d + 1$

14. Solve each of the following pairs of inequalities.
   (a) $\frac{a}{4} + 3 \le 4 \le \frac{a}{2} + 6$
   (b) $\frac{b}{3} \ge \frac{b}{2} + 1 \ge b - 1$
   (c) $2(1 - c) > c - 1 \ge \frac{c - 2}{7}$
   (d) $d - 5 < \frac{2d}{5} \le \frac{d}{2} + \frac{1}{5}$

15. Find the integer values of x which satisfy each of the following inequalities.

(a) $3x - 5 < 26 \leqslant 4x - 6$
(b) $3x + 2 < 19 < 5x - 4$
(c) $-4 \leqslant 7 - 3x \leqslant 2$
(d) $-10 < 7 - 2x \leqslant -1$

16. Given that $0 \leqslant x \leqslant 7$ and $1 \leqslant y \leqslant 5$, find

(a) the greatest possible value of $x + y$,
(b) the least possible value of $x - y$,
(c) the largest possible value of xy,
(d) the smallest possible value of $\dfrac{x}{y}$,
(e) the least and greatest possible values of x^2.

17. Given that $-4 \leqslant a \leqslant -1$ and $-6 \leqslant b \leqslant -2$, find
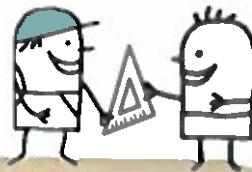
(a) the least possible value of $a + b$,
(b) the greatest possible value of $a - b$,
(c) the smallest possible value of ab,
(d) the largest possible value of $\dfrac{a}{b}$,
(e) the least and greatest possible values of a^2,
(f) the largest value of $b^2 - a$.
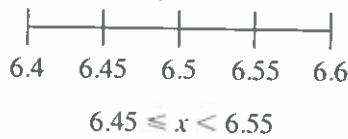
18. State whether each of the following statements is true or false. If your answer is 'false', offer an explanation to support your case.

(a) If $a > b$ and both a and b are negative, then $\dfrac{a}{b} > 1$.
(b) If $a > b$ and both a and b are negative, then $a^3 > b^3$.
(c) If $a > b$ and both a and b are negative, then $\dfrac{b}{a} - \dfrac{a}{b} > 0$.

3.4 Limits of Accuracy

A measurement to the nearest unit refers to an interval of one unit centred on the measurement. E.g. 6.5 is rounded to one decimal place and therefore could represent any measure from 6.45 up to but not including 6.55.

6.4   6.45   6.5   6.55   6.6

$$6.45 \leqslant x < 6.55$$

The bottom end of the range is called the **lower bound** and the top end of the range is the **upper bound**.

PRACTISE NOW

Give the upper and lower bounds for each of the following.

(a) 350 km (to the nearest 50 km)
(b) 1800 ml (to the nearest ml)
(c) 80 s (to the nearest 10 s)

SIMILAR QUESTIONS

Exercise 3C Questions 1–4

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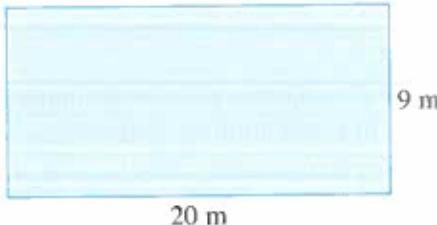
Addition

Finding the upper bound value of an additive sum will require the upper bound limits.

Worked Example 9 (Addition of Limits)

Calculate the upper bound for the perimeter of a rectangle of sides 20 m by 9 m measured to the nearest metre.

9 m

20 m

Solution:

Upper bound limits of 20 m = 20.5

Upper bound limits of 9 m = 9.5

$\therefore$ Upper bound of the perimeter = 2(20.5 + 9.5)

$$= 2(30)$$
$$= 60 \text{ m}$$

PRACTISE NOW 9

Calculate the upper bound for the perimeter of a square of sides 5 m by 5 m measured to the nearest metre.

Subtraction

In situations involving subtraction, different approximations will be required.

Worked Example 10 (Subtraction of Limits)

A 5 m lace needs to be stitched round a place mat. Given that all measurements are corrected to the nearest metre, find the minimum and maximum lengths of lace that will remain if a reel has 20 m of lace.

Solution:

Maximum length of reel = 20.5 m

Minimum length of reel = 19.5 m

Maximum length of lace = 5.5 m

Minimum length of lace = 4.5 m

Maximum length remaining = maximum length of reel − minimum length of lace

$$= 20.5 - 4.5$$
$$= 16 \text{ m}$$

Minimum length remaining = minimum length of reel − maximum length of lace

$$= 19.5 - 5.5$$
$$= 14 \text{ m}$$

Michael wants to knit a scarf using 6 m of wool. Given that all measurements are corrected to the nearest metre, find the minimum and maximum amount of wool that will remain if a reel has 25 m of wool.

::: Multiplication

To get a maximum value in situations involving multiplications of any two given measures, the upper bound value of one measure will be multiplied with the upper bound value of the other measure. In case of the minimum value, the lower bound limits will be multiplied.

Worked Example 11

(Multiplication of Limits)

The dimensions of a room are 30 m by 50 m when corrected to the nearest metre. Express the possible values of its area as an inequality.

Solution:

Maximum value of the length = 50.5 m

Minimum value of the length = 49.5 m

Maximum value of the width = 30.5 m

Minimum value of the width = 29.5 m

Maximum area = $30.5 \times 50.5 = 1540.25$ m²

Minimum area = $29.5 \times 49.5 = 1460.25$ m²

$\therefore 1460.25$ m² $\leq$ area < 1540.25 m²

A garden, in the shape of a rectangle, measures 40 m by 60 m when corrected to the nearest metre. Express the possible values of its area as an inequality.

⠇⠿ Division

In situations involving division of two given measures, we divide the upper bound value of one measure by the lower bound value of the other measure in order to achieve the maximum value, and vice versa to achieve the minimum value.

Worked Example 12 (Division of Limits)

1 litre of paint will cover an area of 10 m². A room has an area of 50 m². Given that all measurements involving area are corrected to the nearest square metre, calculate the maximum and minimum amount of paint needed.

Solution:

For 1 litre of paint,
Upper bound area = 10.5 m²
Lower bound area = 9.5 m²
For the room,
Upper bound area = 50.5 m²
Lower bound area = 49.5 m²

$\therefore$ Maximum amount of paint $= \dfrac{50.5}{9.5} = 5.32$ litres (to 3 s.f.)

Minimum amount of paint $= \dfrac{49.5}{10.5} = 4.71$ litres (to 3 s.f.)

PRACTISE NOW 12

1 litre of water will cover an area of 20 m². A farm has an area of 60 m². Given that all measurements involving area are corrected to the nearest square metre, calculate the maximum and minimum amount of water needed to water the crops.

Exercise 3C
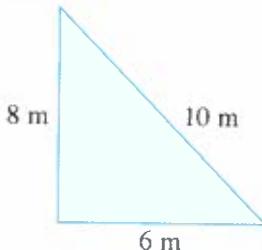
1. Each of the following numbers is correct to two significant figures.

   (i) 4.6   (ii) 0.73
   (iii) 680   (iv) 1000
   (v) 0.035   (vi) 72 000

   (a) Give the upper and lower bounds for each of the following.

   (b) Using x as the number, express the range in which the number lies as an inequality.

2. The capacity of a tank given as 0.6 m³, has been rounded off to the nearest cm³. Calculate the upper and lower bound range of its capacity.

3. The duration of a flash from a camera is timed at 0.004 seconds, accurate to 1/1000 of a second. Calculate the upper and lower bounds for the duration of the flash using inequalities.

4. Give the upper and lower bounds for each of the following.

   (a) 750l (to the nearest 50 l)
   (b) 30 s (to the nearest 10 s)
   (c) 758 kg (to the nearest kg)
   (d) 1500 g (to the nearest g)
   (e) 495 cm (to the nearest 5 cm)

5.

The measurements given are to the nearest metre. Calculate the upper and lower bounds for the perimeter of the above triangle.

6. Of 600 students in a school, 297 are boys. Calculate the maximum number of girls. Number of students is rounded off to the nearest 100.

7. A length of 25.0 cm is cut from a string that measures 50.0 cm. Given that the measurements are corrected to 1 decimal place, express the upper and lower bounds of the length l cm of the string left in the form of an inequality.

8. The radius of the trunk of a tree is measured to the nearest 2 mm. If its radius is 20 cm, find its minimum circumference in centimetres.

9. A car travelling at a speed of 60 km/h (to the nearest 10 km/h) covers a distance in 3 hours (to the nearest hour). Calculate its range of distance in the form of an inequality.

10. The density of a given material is 2.4 g/cm³. If its mass, to the nearest 10 g, is 30 g, find its minimum and maximum volume. Give your answer correct to 3 significant figures.

11. If 1 litre of paint coats an area of 3 m² (to the nearest m²), find the maximum amount of paint is required to coat a wall of area 700 m² (to the nearest m²).

12. A cylindrical water tank has a capacity of 50 l when measured to the nearest 10 l. A bottle has a capacity of 1 l when measured to the nearest litre. Calculate the maximum number of bottles of water that can be drawn from the tank.

Summary

1. **Solving a Linear Inequality**

Case	Adding a number to both sides of the inequality	Subtracting a number from both sides of the inequality
$x > y$	$x + a > y + a$	$x - a > y - a$
$x \geq y$	$x + a \geq y + a$	$x - a \geq y - a$
$x < y$	$x + a < y + a$	$x - a < y - a$
$x \leq y$	$x + a \leq y + a$	$x - a \leq y - a$
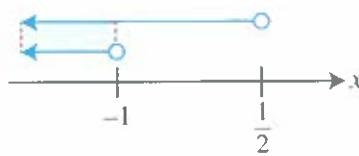
Case	Multiplying a positive number c to both sides of the inequality	Dividing a positive number c from both sides of the inequality	Multiplying a negative number d to both sides of the inequality	Dividing a negative number d from both sides of the inequality
$x > y$	$cx > cy$	$\frac{x}{c} > \frac{y}{c}$	$dx < dy$	$\frac{x}{d} < \frac{y}{d}$
$x \geq y$	$cx \geq cy$	$\frac{x}{c} \geq \frac{y}{c}$	$dx \leq dy$	$\frac{x}{d} \leq \frac{y}{d}$
$x < y$	$cx < cy$	$\frac{x}{c} < \frac{y}{c}$	$dx > dy$	$\frac{x}{d} > \frac{y}{d}$
$x \leq y$	$cx \leq cy$	$\frac{x}{c} \leq \frac{y}{c}$	$dx \geq dy$	$\frac{x}{d} \geq \frac{y}{d}$

When solving a linear inequality of the form $ax + b > cx + d$, apply the above guidelines to reduce the inequality to the form $x > k$ or $x < k$. This is called the **solution** of the inequality.

2. When solving a pair of simultaneous linear inequalities, we only consider the common **solutions** of the inequalities with the aid of a number line.

For example, to solve $2x + 1 < x < 1 - x$, we have

$$2x + 1 < x \quad \text{and} \quad x < 1 - x$$
$$x < -1 \qquad\qquad 2x < 1$$
$$x < \frac{1}{2}$$

The common solution is $x < -1$.

3. The bottom end of the range is called the **lower bound** and top end of the range is the **upper bound**.

Review Exercise 3

1. Solve each of the following inequalities, illustrating each solution on a number line.

   (a) $a - 2 \leqslant 3$ (b) $2b + 1 < 5 - 4b$

   (c) $c \geqslant \frac{1}{2}c - 1$ (d) $\frac{1}{2}d > 1 + \frac{1}{3}d$

   (e) $2(e - 3) \geqslant 1$ (f) $5(f - 4) \leqslant 2f$

   (g) $-3 - g > 2g - 7$ (h) $18 - 3h < 5h - 4$

2. Solve each of the following inequalities.

   (a) $3 + \frac{a}{4} > 5 + \frac{a}{3}$ (b) $\frac{4b}{9} - 5 < 3 - \frac{2b}{3}$

   (c) $\frac{4c}{9} - \frac{3}{4} \geqslant c - \frac{1}{2}$ (d) $\frac{d - 2}{3} < \frac{2d + 3}{5} + \frac{5}{8}$

   (e) $\frac{1}{3}(e + 2) \geqslant \frac{2}{3} + \frac{1}{4}(e - 1)$

   (f) $5 - \frac{2f - 5}{6} \leqslant \frac{f + 3}{2} + \frac{2(f + 1)}{3}$

3. Solve each of the following pairs of inequalities.

   (a) $5 - a \leqslant a - 6 \leqslant 10 - 3a$

   (b) $4 - b < 2b - 1 < 7 + b$

   (c) $4c - 1 < \frac{1}{2} \leqslant 3c + 2$

   (d) $2d + 1 \geqslant d > 3d - 20$

4. Given that $x \leqslant 14\frac{1}{2}$, state the largest possible value of x if x is

   (a) an integer,   (b) a prime number,

   (c) a rational number.

5. Given that $27 - 2x \leqslant 8$, find

   (a) the least value of x,

   (b) the least integer value of x.

6. Find the integer values of x which satisfy each of the following inequalities.

   (a) $5x > 69 - 2x$ and $27 - 2x \geqslant 4$

   (b) $-10 \leqslant x < -4$ and $2 - 5x < 35$

7. Given that $-1 \leqslant x \leqslant 5$ and $2 \leqslant y \leqslant 6$, find the greatest and least values of

   (a) $y - x$,   (b) $\frac{x}{y}$,

   (c) $\frac{x^2}{y}$.

8. Given that $-3 \leqslant x \leqslant 7$ and $4 \leqslant y \leqslant 10$, find

   (a) the smallest possible value of $x - y$,

   (b) the largest possible value of $\frac{x}{y}$,

   (c) the largest possible value of $x^2 - y^2$,

   (d) the smallest possible value of $x^3 + y^3$.

9. The perimeter of a square is at most 81 cm. What is the greatest possible area of the square? Give your answer correct to 4 significant figures.

10. The masses of a sheet of writing paper and an envelope are 3 g and 5 g respectively. It costs 60 cents to send a letter with a mass not exceeding 20 g. Michael has 60 cents worth of stamps. If x is the number of sheets of writing paper, form an inequality in x and find the maximum number of sheets of writing paper that he can use.

11. Lixin is 3 years younger than Rui Feng. If the sum of their ages is at most 50 years, find the maximum possible age of Lixin 5 years ago.

12. Jun Wei has 16 $1 coins and some 20-cent coins in his pocket. Given that the total value of the coins in his pocket is at most $22, find the maximum number of 20-cent coins that he has.

13. In a set of 20 True/False questions, 2 points are awarded for a correct answer and 1 point is deducted for a wrong answer. No points are awarded or deducted for an unanswered question. Amirah answered 15 questions and left the remaining questions unanswered. If her total score is greater than 24, find the maximum number of questions she answered wrongly.

14. Each of the following numbers is correct to the nearest whole number.

   (a) Give the upper and lower bounds for each of the following calculations.

   (b) Using x as the number, express the range in which the number lies as an inequality.

     (i) $570 + 83$ **(ii)** $340 - 26$

     (iii) 17×13 **(iv)** 34×16

     (v) $\dfrac{27}{9}$ **(vi)** $\dfrac{210}{30}$

15. The mass of 20 cabbages is 92 kg correct to 2 significant figures. Calculate the upper and lower bounds for the average mass of one cabbage.

16. The mass of 6 girls given to the nearest 10 kg are 40 kg, 40 kg, 50 kg, 60 kg, 60 kg and 70 kg. Calculate

   (i) the least possible total mass,

   (ii) the greatest possible total mass.

17. Find the maximum volume of a fish tank with dimensions 20 cm by 30 cm by 15 cm measured to the nearest centimetre.

18. A triangle has sides measuring 6 cm, 7 cm and 8 cm. Given that the measurements are to the nearest centimetre, find the least possible perimeter of the triangle.

19. The radius of a circle, rounded to 1 decimal place, measures 9.5 cm. Calculate the upper and lower bounds of

   (a) its circumference,

   (b) its area.

   Give your answers correct to 2 decimal places.

Challenge Yourself

1. It is given that $6 \leqslant x \leqslant 8$ and $0.2 \leqslant y \leqslant 0.5$. If $z = \dfrac{x}{y}$, find the limits in which z must lie.

2. Find the range of values of x for which $\dfrac{3x - 5}{x^2 - 14x + 49} > 0$, $x \neq 7$. Explain your answer.

Indices and Standard Form

The world population is estimated to be about 7 000 000 000 in 2012.

The Bohr radius of a hydrogen atom is estimated to be about 0.000 000 000 053 m.

As it is troublesome to write very large or very small numbers in this manner, they can be represented using standard form which involves indices.

In this chapter, we will learn about indices and standard form.

Chapter
Four
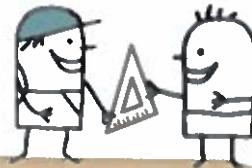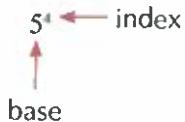
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4.1 Indices

In Book 1, we have learnt how to represent $5 \times 5 \times 5 \times 5$ as 5^4 (read as '5 to the power of 4'). The digit 5 in 5^4 is known as the **base**, and the digit 4 in 5^4 is known as the **index** (plural: indices). 5^4 is called the index notation of $5 \times 5 \times 5 \times 5$.

$$5^4 \longleftarrow \text{index}$$
$$\uparrow$$
$$\text{base}$$

Write $3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3$ in index notation: _____

Investigation

Indices

"Dad, could you please give me an allowance of $10 on the first day of this month, $20 on the second day and $30 on the third day, increasing by $10 each day until the 31ˢᵗ day of this month? Then I will promise not to ask for allowance ever again."   **Son**

"No way, son!"   **Father**

"How about giving me 1 cent on the first day of this month, 2 cents on the second day and 4 cents on the third day, doubling the amount each day until the 31ˢᵗ day of this month? Then I will promise not to ask for allowance ever again."   **Son**

"Deal!"   **Father**

Calculate the amount of allowance the father has to give to his son on the 31ˢᵗ day of the month.

From the investigation, 2^{30} cents may seem to be a small amount, but it is actually equal to 1 073 741 824 cents or about $10.7 million. This shows how 'powerful' the index 30 is!

Comparing Numbers written in Index Form

Work in pairs.

1. Describe in your own words the meaning of 2^{10} and 10^2.

2. Without using a calculator, explain whether 2^{10} or 10^2 is greater.

   Hint: $2^{10} = ()^2 \square\ 10^2$ (Convert 2^{10} to a number with a different base but with index 2)

3. Without using a calculator, explain whether 3^7 or 7^3 is greater.

   Hint: $3^7 = 3(3)^6 = 3()^3 \square\ 7^3$ (Convert 3^7 to a number with a different base but with index 3)
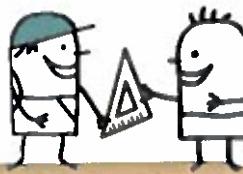
4. If a and b are positive integers such that $b > a$, when will

   (i) $a^b = b^a$, **(ii)** $a^b < b^a$?

   Hint: Use systematic listing, e.g. for $a = 1$, tabulate values for a^b and b^a for integer values of b. Repeat for values of $a = 2, 3, 4, \ldots$

5. In general, if a and b are positive integers such that $b > a$, then $a^b > b^a$ with some exceptions. Based on your work in Question 4, list the values of a and b for which the above statement is not true.

4.2 Laws of Indices

Investigation

Law 1 of Indices

Copy and complete the following.

1. $7^2 \times 7^3 = (7 \times 7) \times ()$

   $= 7 \times 7 \times \ldots \times 7$

   $= 7^{}$

   $= 7^{2+}$

2. $6^4 \times 6^5 = () \times ()$

   $= 6 \times 6 \times \ldots \times 6$

   $= 6^{}$

   $= 6^{4+}$

3. $a^3 \times a^4 = (a \times a \times a) \times ($ [   ] $)$

$\qquad = a \times a \times \dots \times a$

$\qquad = a^{\boxed{}}$

$\qquad = a^{3+\boxed{}}$

4. $a^m \times a^n = (\underbrace{a \times a \times a \times \dots \times a \times a}_{m \text{ times}}) \times (\underbrace{a \times a \times \dots \times a}_{\boxed{} \text{ times}})$

$\qquad = \underbrace{a \times a \times \dots \times a}_{m + \boxed{} \text{ times}}$

$\qquad = a^{m + \boxed{}}$

In general, if a is a real number, and m and n are positive integers, then

> **Law 1 of Indices:** $a^m \times a^n = a^{m+n}$

Worked Example 1

(Law 1 of Indices)

Simplify each of the following, leaving your answer in index notation where appropriate.

(a) $5^8 \times 5^9$ 　　　　　(b) $(-2)^3 \times (-2)$ 　　　　　(c) $6b^3c^5 \times 2bc^4$

Solution:

(a) $5^8 \times 5^9 = 5^{8+9}$ (Law 1)

$\qquad = 5^{17}$

(b) $(-2)^3 \times (-2) = (-2)^{3+1}$ (Law 1)

$\qquad = (-2)^4$

(c) $6b^3c^5 \times 2bc^4 = 12b^{3+1}c^{5+4}$ (Law 1)

$\qquad = 12b^4c^9$

PRACTISE NOW 1

Simplify each of the following, leaving your answer in index notation where appropriate.

(a) $4^7 \times 4^5$

(c) $a^{12} \times a^8$

(b) $(-3)^6 \times (-3)$

(d) $2xy^4 \times 3x^5y^3$

Investigation

Law 2 of Indices

Copy and complete the following. Assume $a \neq 0$.

1. $3^5 \div 3^2 = \dfrac{3 \times 3 \times 3 \times 3 \times 3}{\boxed{}}$

$= 3^{\boxed{}}$

$= 3^{5-\boxed{}}$

2. $\dfrac{10^6}{10^4} = \dfrac{\boxed{}}{\boxed{}}$

$= 10^{\boxed{}}$

$= 10^{6-\boxed{}}$

3. $a^7 \div a^3 = \dfrac{a \times a \times a \times a \times a \times a \times a}{\boxed{}}$

$= a^{\boxed{}}$

$= a^{7-\boxed{}}$

4. $a^m \div a^n = \dfrac{\overbrace{a \times a \times a \times \dots \times a \times a}^{m \text{ times}}}{\underbrace{a \times a \times \dots \times a}_{\boxed{} \text{ times}}}$

$= \underbrace{a \times a \times \dots \times a}_{m - \boxed{} \text{ times}}$

$= a^{m - \boxed{}}$

In general, if a is a real number, and m and n are positive integers such that $m > n$, then

$$\boxed{\textbf{Law 2 of Indices: } a^m \div a^n = a^{m-n}, \text{ if } a \neq 0}$$

Worked Example 2

(Law 2 of Indices)

Simplify each of the following, leaving your answer in index notation where appropriate.

(a) $7^8 \div 7^3$ **(b)** $(-5)^6 \div (-5)$ **(c)** $12b^6 c^3 \div 6b^4 c^2$

Solution:

(a) $7^8 \div 7^3 = 7^{8-3}$ (Law 2)

$= 7^5$

(b) $(-5)^6 \div (-5) = (-5)^{6-1}$ (Law 2)

$= (-5)^5$

(c) $12b^6 c^3 \div 6b^4 c^2 = \dfrac{12b^6 c^3}{6b^4 c^2}$ (Law 2)

$= 2b^{6-4} c^{3-2}$

$= 2b^2 c$

Simplify each of the following, leaving your answer in index notation where appropriate.

(a) $9^7 \div 9^3$

(b) $(-4)^8 \div (-4)$

(c) $a^{10} \div a^6$

(d) $27x^9y^4 \div 9x^6y^3$

Investigation

Law 3 of Indices

Copy and complete the following.

1. $(2^5)^2 = 2^5 \times 2^5$

$\qquad = 2^{5+\,\blacksquare}$ (using Law 1 of Indices)

$\qquad = 2^{5 \times \blacksquare}$

2. $(10^4)^3 = 10^4 \times 10^4 \times 10^4$

$\qquad = 10^{4+\,\blacksquare\,+\,\blacksquare}$ (using Law 1 of Indices)

$\qquad = 10^{4 \times \blacksquare}$

3. $(a^m)^n = \underbrace{(a^m \times a^m \times \ldots \times a^m)}_{n \text{ times}}$

$\underbrace{}_{\blacksquare \text{ times}}$

$\qquad = a^{m+m+\ldots+m}$

$\qquad = a^{m \times \blacksquare}$

In general, if a is a real number, and m and n are positive integers, then

> **Law 3 of Indices:** $(a^m)^n = a^{mn}$

Worked Example 3

(Laws of Indices)

Simplify each of the following, leaving your answer in index notation where appropriate.

(a) $(5^7)^3$

(b) $(h^8)^2$

(c) $(7^p)^4 \times (7^2)^p$

Solution:

(a) $(5^7)^3 = 5^{7 \times 3}$ (Law 3)

$\qquad = 5^{21}$

(b) $(h^8)^2 = h^{8 \times 2}$ (Law 3)

$\qquad = h^{16}$

(c) $(7^p)^4 \times (7^2)^p = 7^{p \times 4} \times 7^{2 \times p}$ (Law 3)

$\qquad = 7^{4p} \times 7^{2p}$

$\qquad = 7^{4p+2p}$ (Law 1)

$\qquad = 7^{6p}$

Chapter 4 Indices and Standard Form

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1. Simplify each of the following, leaving your answer in index notation where appropriate.

(a) $(6^3)^4$ (b) $(k^5)^9$ (c) $(3^q)^6 \times (3^4)^q$

2. Given that $x^8 \times (x^3)^n \div (x^n)^2 = x^{10}$, find the value of n.

Investigation

Law 4 of Indices

Copy and complete the following.

1. $2^3 \times 7^3 = (2 \times 2 \times 2) \times ()$

$\qquad = (2 \times 7) \times (2 \times 7) \times (2 \times)$

$\qquad = (2 \times)^3$

2. $(-3)^2 \times (-4)^2 = (-3) \times (-3) \times () \times ()$

$\qquad = [(-3) \times (-4)] \times [(-3) \times ()]$

$\qquad = [(-3) \times ()]^{}$

3. $a^n \times b^n = \underbrace{(a \times a \times \ldots \times a)}_{n \text{ times}} \times \underbrace{(b \times b \times \ldots \times b)}_{ \text{ times}}$

$\qquad = \underbrace{(a \times b) \times (a \times b) \times \ldots \times (a \times b)}_{ \text{ times}}$

$\qquad = (a \times b)^{}$

In general, if a and b are real numbers, and n is a positive integer, then

Law 4 of Indices: $a^n \times b^n = (a \times b)^n$

Another useful version of Law 4 of Indices can be written as $(a \times b)^n = a^n \times b^n$ or $(ab)^n = a^n b^n$.

Worked Example 4

Simplify each of the following, leaving your answer in index notation where appropriate.

(a) $2^4 \times 7^4$

(b) $(2h^2)^6$

(c) $(xy^2)^3 \times (-3x^2y)^4$

(d) $(4x^2y^3)^3 \div (xy^3)^2$

Solution:

(a) $2^4 \times 7^4 = (2 \times 7)^4$ (Law 4)

$\quad = 14^4$

(b) $(2h^2)^6 = 2^6(h^2)^6$ (Law 4)

$\quad = 64h^{2 \times 6}$ (Law 3)

$\quad = 64h^{12}$

(c) $(xy^2)^3 \times (-3x^2y)^4 = x^3y^{2 \times 3} \times (-3)^4 x^{2 \times 4} y^4$ (Law 4 and Law 3)

$\quad = x^3 y^6 \times 81 x^8 y^4$

$\quad = 81 x^{3+8} y^{6+4}$ (Law 1)

$\quad = 81 x^{11} y^{10}$

(d) $(4x^2y^3)^3 \div (xy^3)^2 = 4^3 x^{2 \times 3} y^{3 \times 3} \div x^2 y^{3 \times 2}$ (Law 4 and Law 3)

$\quad = 64 x^6 y^9 \div x^2 y^6$

$\quad = 64 x^{6-2} y^{9-6}$ (Law 2)

$\quad = 64 x^4 y^3$

PRACTISE NOW 4

SIMILAR QUESTIONS

Simplify each of the following, leaving your answer in index notation where appropriate.

(a) $3^7 \times 8^7$

(b) $(5b^4)^3$

(c) $(-2c^2d^5)^5$

(d) $(m^2n)^4 \times (m^4n^3)^5$

(e) $(-p^7q^5)^2 \div (3p^3q^2)^3$

Class Discussion

Simplification using the Laws of Indices

Work in pairs.

There are two ways of simplifying $(xy^2)^4 \times (3x^2y)^4$. Explain these two ways using the laws of indices.

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Investigation

Law 5 of Indices

Copy and complete the following.

1. $8^3 \div 5^3 = \dfrac{8^3}{5^3}$

$= \dfrac{8 \times 8 \times 8}{\boxed{} \times \boxed{} \times \boxed{}}$

$= \dfrac{8}{\boxed{}} \times \dfrac{8}{\boxed{}} \times \dfrac{8}{\boxed{}}$

$= \left(\dfrac{8}{\boxed{}}\right)^3$

2. $(-12)^4 \div (-7)^4 = \dfrac{(-12)^4}{(-7)^4}$

$= \dfrac{(-12) \times (-12) \times (-12) \times (-12)}{\boxed{} \times \boxed{} \times \boxed{} \times \boxed{}}$

$= \dfrac{(-12)}{\boxed{}} \times \dfrac{(-12)}{\boxed{}} \times \dfrac{(-12)}{\boxed{}} \times \dfrac{(-12)}{\boxed{}}$

$= \left[\dfrac{(-12)}{\boxed{}}\right]^4$

3. Consider $a^n \div b^n$, where $b \neq 0$.

$$a^n \div b^n = \dfrac{\overbrace{a \times a \times \ldots \times a}^{n \text{ times}}}{\underbrace{\boxed{} \times \boxed{} \times \ldots \times \boxed{}}_{n \text{ times}}}$$

$$= \underbrace{\dfrac{a}{\boxed{}} \times \dfrac{a}{\boxed{}} \times \ldots \times \dfrac{a}{\boxed{}}}_{\boxed{} \text{ times}}$$

$$= \left(\dfrac{a}{\boxed{}}\right)^n$$

In general, if a and b are real numbers, and n is a positive integer, then

> **Law 5 of Indices:** $a^n \div b^n = \left(\dfrac{a}{b}\right)^n$, if $b \neq 0$

Another useful version of Law 5 of Indices can be written as $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$ if $b \neq 0$.

Worked Example 5

(Laws of Indices)

Simplify each of the following, leaving your answer in index notation where appropriate.

(a) $10^8 \div 2^8$

(b) $\left(\dfrac{2}{b}\right)^5$

(c) $\left(\dfrac{2p^2}{q^3}\right)^4 \div \dfrac{8p^3}{q^{16}}$

Solution:

(a) $10^8 \div 2^8 = \left(\dfrac{10}{2}\right)^8$ (Law 5)

$\quad\quad\quad = 5^8$

(b) $\left(\dfrac{2}{b}\right)^5 = \dfrac{2^5}{b^5}$ (Law 5)

$\quad\quad\quad = \dfrac{32}{b^5}$

(c) $\left(\dfrac{2p^2}{q^3}\right)^4 \div \dfrac{8p^3}{q^{16}} = \dfrac{2^4 p^{2\times4}}{q^{3\times4}} \times \dfrac{q^{16}}{8p^3}$ (Law 5 and Law 3)

$\quad\quad\quad\quad = \dfrac{16p^8}{q^{12}} \times \dfrac{q^{16}}{8p^3}$

$\quad\quad\quad\quad = 2p^{8-3}q^{16-12}$ (Law 2)

$\quad\quad\quad\quad = 2p^5 q^4$

PRACTISE NOW 5

Simplify each of the following, leaving your answer in index notation where appropriate.

(a) $21^3 \div 7^3$

(b) $(26^5)^3 \div 13^{15}$

(c) $\left(\dfrac{p^2}{q}\right)^3 \div \dfrac{q^7}{p^5}$

(d) $\left(\dfrac{3x^2}{x^3}\right)^3 \div \dfrac{27x^7}{x^{21}}$

Journal Writing

Nora and Farhan were asked to simplify $\left(\dfrac{2x^2}{y}\right)^3$.

Nora wrote this: $\left(\dfrac{2x^2}{y}\right)^3 = \dfrac{2x^6}{y^3}$

Farhan wrote this: $\left(\dfrac{2x^2}{y}\right)^3 = \dfrac{8x^5}{y}$

Both Nora and Farhan obtained the wrong solutions.

(i) Highlight the possible misconceptions that Nora and Farhan might have.

(ii) Provide the correct solution.

Class Discussion

Is $(a + b)^n = a^n + b^n$? Is $(a - b)^n = a^n - b^n$?

Work in pairs.

We have learnt that $(a \times b)^n = a^n \times b^n$ and $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$.

Is it true that $(a + b)^n = a^n + b^n$?

How about $(a - b)^n = a^n - b^n$?

Discuss with your classmates. Provide some counterexamples if necessary.

Exercise 4A

BASIC LEVEL

1. Simplify each of the following, leaving your answer in index notation where appropriate.
   (a) $2^3 \times 2^7$
   (b) $(-4)^6 \times (-4)^5$
   (c) $x^8 \times x^3$
   (d) $(3y^2) \times (8y^7)$

2. Simplify each of the following, leaving your answer in index notation where appropriate.
   (a) $5^8 \div 5^5$
   (b) $(-7)^{11} \div (-7)^4$
   (c) $6x^7 \div x^3$
   (d) $(-15y^9) \div 5y^4$

3. Simplify each of the following, leaving your answer in index notation where appropriate.
   (a) $(9^2)^4$
   (b) $(h^2)^5$
   (c) $3^{14} \times (5^2)^7$
   (d) $2^3 \times 9^3$
   (e) $(2k^6)^3$
   (f) $(-3x^6y^2)^4$

4. Simplify each of the following, leaving your answer in index notation where appropriate.
   (a) $14^{13} \div 7^{13}$
   (b) $(9^5)^4 \div 3^{20}$
   (c) $\left(\dfrac{m}{2}\right)^5$
   (d) $\left(\dfrac{3}{n^2}\right)^3$
   (e) $\left(\dfrac{p^4}{q}\right)^6$
   (f) $\left(-\dfrac{x}{y^2}\right)^4$

INTERMEDIATE LEVEL

5. Simplify each of the following.
   (a) $h^2k \times h^{11}k^9$
   (b) $(-m^7n^3) \times 4m^{11}n^9$
   (c) $11p^6q^7 \times 2p^3q^{10}$
   (d) $h^9k^6 \div h^5k^4$
   (e) $15m^8n^7 \div 3m^2n$
   (f) $(-10x^5y^6) \div (-2xy^3)$

6. Simplify each of the following.
   (a) $(a^2)^3 \times a^5$
   (b) $(b^3)^7 \times (b^4)^5$
   (c) $(c^6)^5 \div (-c^2)$
   (d) $(-3d^3)^2 \div (2d)^3$
   (e) $(e^3)^5 \div (-e^2)^4$
   (f) $(4f^6)^3 \div (-2f^3)^3$

7. Simplify each of the following.
   (a) $(ab^2)^3 \times (2a^2b)^3$
   (b) $c^2d^2 \times (-5c^3d^3)^2$
   (c) $(8e^5f^3)^2 \div (e^3f)^3$
   (d) $16g^8h^7 \div (-2g^3h^2)^3$

8. Simplify each of the following.
   (a) $\left(\dfrac{2a^2}{b}\right) \times \left(\dfrac{a}{b^2}\right)^2$
   (b) $\left(\dfrac{c}{d^2}\right)^3 \times \left(\dfrac{c^3}{2d}\right)^2$
   (c) $\left(\dfrac{3e^3}{f^2}\right)^4 \div \left(\dfrac{27e^9}{f^{11}}\right)$
   (d) $\left(\dfrac{g^2}{h^3}\right)^6 \div \left(\dfrac{-3g^5}{2h^2}\right)^3$
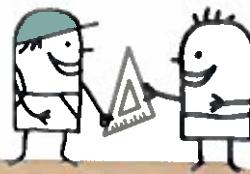
9. Simplify each of the following.

(a) $\dfrac{\left(2x^2y\right)^3}{\left(10xy^3\right)^2} \times \dfrac{\left(5xy^4\right)^3}{4xy}$

(b) $\dfrac{8x^8y^4}{\left(2xy^2\right)^2} \times \dfrac{\left(4x^2y^2\right)^2}{\left(3xy\right)^2}$

(c) $\dfrac{\left(2xy^2\right)^5}{\left(4x^2y\right)^2\left(xy^3\right)}$

(d) $\dfrac{4x^2y^4 \times 8x^4y^2}{\left(4x^2y^2\right)^2}$

10. Given that $\dfrac{\left(2p^3q^4\right)^4}{\left(-3q^5\right)^2} \div \dfrac{\left(4p^2q\right)^2}{9} = \dfrac{p^{a+b}}{q^{a-b}}$, form a pair of simultaneous equations in a and b and hence find the value of a and of b.

4.3 Zero and Negative Indices

In Section 4.2, all the 5 laws of indices apply when the indices are positive integers. In this section, we will consider what happens if the indices are zero or negative integers.

Investigation

Zero Index

3^4 means 3 multiplied by itself 4 times, i.e. $3 \times 3 \times 3 \times 3 = 81$. What does 3^0 mean?

1. Fill in the values of 3^2 and 3^1 in Table 4.1.

Index Form	Value
3^4	81
3^3	27
3^2	
3^1	
3^0	

$\div 3$, $\div 3$, $\div 3$, $\div 3$

Table 4.1

2. What number must you divide 81 (i.e. 3^4) to get 27 (i.e. 3^3)?

3. What number must you divide 27 (i.e. 3^3) to get the value of 3^2?

4. What number must you divide 3^2 to get the value of 3^1?

5. **(a)** By continuing this pattern, what number must you divide 3^1 to get the value of 3^0?

(b) Complete the last row in Table 4.1.

6. Complete Table 4.2.

Index Form	Value
$(-2)^4$	16
$(-2)^3$	-8
$(-2)^2$	
$(-2)^1$	
$(-2)^0$	

$\div(-2)$
$\div(-2)$
$\div(-2)$
$\div(-2)$

Table 4.2

7. Does the pattern work for 0^4, 0^3, 0^2, 0^1 and 0^0? Explain your answer.

Since $5^3 \div 5^3 = \dfrac{5 \times 5 \times 5}{5 \times 5 \times 5}$
$= 1,$

and using the second law of indices,

$5^3 \div 5^3 = 5^{3-3}$
$= 5^0.$

$\therefore 5^0 = 1$

Since $a^4 \div a^4 = \dfrac{a \times a \times a \times a}{a \times a \times a \times a}$
$= 1,$

and using the second law of indices,

$a^4 \div a^4 = a^{4-4}$
$= a^0.$

$\therefore a^0 = 1$

$\therefore$ For the second law of indices to apply for $n = 0$, we define $a^0 = 1$, where a is a real number.

Definition 1: $a^0 = 1$, if $a \neq 0$

Worked Example 6

(Zero Indices)

Evaluate each of the following.

(a) 121^0 (b) $2x^0$ (c) $(2x)^0$

Solution:

(a) $121^0 = 1$

(b) $2x^0 = 2 \times 1$
$= 2$

(c) $(2x)^0 = 1$

PRACTISE NOW 6

1. Evaluate each of the following.
   (a) 2015^0 (b) $(-7)^0$
   (c) $3y^0$ (d) $(3y)^0$

2. Find the value of each of the following.
   (a) $3^0 \times 3^3 \div 3^2$ (b) $3^0 + 3^2$

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∴∴ Negative Indices

Investigation

Negative Indices

Now that we know $3^0 = 1$, we want to find out what 3^{-1} and 3^{-2} are equal to.

1. Continuing the same pattern in the previous investigation, complete Table 4.3.

Index Form	Value
3^2	9
3^1	3
3^0	
3^{-1}	

$\div 3$
$\div 3$
$\div 3$
$\div 3$

Table 4.3

Problem Solving Tip

$$\frac{1}{3} \div 3 = \frac{1}{3} \times \frac{1}{3}$$
$$= \frac{1}{3 \times 3}$$
$$= \frac{1}{3^2}$$

2. Continuing the same pattern in the previous investigation, complete Table 4.4.

Index Form	Value
$(-2)^2$	4
$(-2)^1$	-2
$(-2)^0$	
$(-2)^{-1}$	
$(-2)^{-2}$	

$\div (-2)$
$\div (-2)$
$\div (-2)$
$\div (-2)$

Table 4.4

3. What do you think 0^{-2} is equal to? Explain your answer.

Since $a^4 \div a^7 = \dfrac{a \times a \times a \times a}{a \times a \times a \times a \times a \times a \times a}$

$= \dfrac{1}{a^3}$,

and using the second law of indices,

$a^4 \div a^7 = a^{4-7}$

$= a^{-3}$.

$\therefore a^{-3} = \dfrac{1}{a^3}$

$\therefore$ For the second law of indices to apply for negative indices, we define a^{-n} as $\dfrac{1}{a^n}$, where a is a real number and $a \neq 0$.

> **Definition 2:** $a^{-n} = \dfrac{1}{a^n}$, if $a \neq 0$

Worked Example 7

(Negative Indices)

Evaluate each of the following.

(a) 4^{-3} (b) $(-7)^{-1}$ (c) $\left(\dfrac{3}{5}\right)^{-2}$

Solution:

(a) $4^{-3} = \dfrac{1}{4^3}$

$= \dfrac{1}{64}$

(b) $(-7)^{-1} = \dfrac{1}{(-7)^1}$

$= -\dfrac{1}{7}$

(c) $\left(\dfrac{3}{5}\right)^{-2} = \dfrac{1}{\left(\dfrac{3}{5}\right)^2}$

$= 1 \div \dfrac{3^2}{5^2}$ (Law 3 and Law 5)

$= 1 \times \dfrac{5^2}{3^2}$

$= \dfrac{25}{9}$

$= 2\dfrac{7}{9}$

PRACTISE NOW 7

Evaluate each of the following.

(a) 6^{-2} (b) $(-8)^{-1}$

(c) $\left(\dfrac{4}{5}\right)^{-3}$ (d) $\left(\dfrac{1}{9}\right)^{-1}$

SIMILAR QUESTIONS

Exercise 4B Questions 3(a)–(d)

⠿ Extension of Laws of Indices to Zero and Negative Indices

In Section 4.2, we have learnt that all the 5 laws of indices apply when the indices are positive integers. With the definitions of zero and negative indices, we can now extend all the 5 laws of indices to include *all integer indices*.

Thinking Time

Copy and complete the following.

1. If a and b are real numbers, and m and n are **integers**, then

   Law 1 of Indices: $a^m \times a^n =$ _____ if $a \neq 0$

   Law 2 of Indices: $a^m \div a^n =$ _____ if $a \neq 0$

   Law 3 of Indices: $(a^m)^n =$ _____ if _____

   Law 4 of Indices: $a^n \times b^n =$ _____ if $a, b \neq 0$

   Law 5 of Indices: $a^n \div b^n =$ _____ if _____

2. Notice that some conditions on the bases a and b are now different.

   (i) Why is it necessary for $a \neq 0$ in Law 1?

   (ii) Why is it necessary for $a, b \neq 0$ in Law 4?

3. **(i)** What happens if $m = n$ in Law 2?

   (ii) What happens if $m = 0$ in Law 2?

> **ATTENTION**
>
> Notice that it is no longer necessary for $m > n$ for Law 2 of Indices. Why is this so?

Worked Example 8

(Applications of Laws of Indices, Zero and Negative Indices)

Simplify each of the following, leaving your answer in positive index form.

(a) $a^{-7} \times a^4 \div a^{-3}$

(b) $\dfrac{8b^{-6}c^3}{\left(2b^2c\right)^3}$

(c) $(2d)^0 \div (d^2e^{-4})^{-1}$

(d) $3a + a^{-2} + a^2 \times a - \dfrac{6a^{-1}}{2a^{-4}}$

Solution:

(a) $a^{-7} \times a^4 \div a^{-3} = a^{-7+4-(-3)}$

$\qquad = a^{-3-(-3)}$

$\qquad = a^0$

$\qquad = 1$

(b) $\dfrac{8b^{-6}c^3}{\left(2b^2c\right)^3} = \dfrac{8b^{-6}c^3}{8b^6c^3}$

$\qquad = b^{-6-6}c^{3-3}$

$\qquad = b^{-12}c^0$

$\qquad = \dfrac{1}{b^{12}}$ (since $b^{-12} = \dfrac{1}{b^{12}}$ and $c^0 = 1$)

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(c)
$$(2d)^0 \div (d^2e^{-4})^{-1} = 1 + \frac{1}{d^2e^{-4}}$$
$$= 1 \times d^2e^{-4}$$
$$= \frac{d^2}{e^4}$$

(d)
$$3a + a^{-2} + a^2 \times a - \frac{6a^{-1}}{2a^{-4}}$$
$$= 3a^{1-(-2)} + a^{2+1} - \frac{6a^{-1-(-4)}}{2}$$
$$= 3a^3 + a^3 - 3a^3$$
$$= a^3$$

Simplify each of the following, leaving your answer in positive index form.

(a) $a^{-1} \times a^3 \div a^{-2}$

(b) $(b^{-5}c^2)^{-3}$
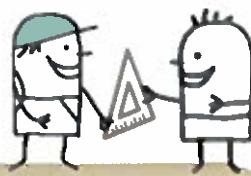
(c) $\dfrac{16d^{-2}e}{(2d^{-1}e)^3}$

(d) $5f^0 \div 3(f^{-2})^2$

(e) $18g^{-6} \div 3(g^{-2})^2$

(f) $6h^2 + 2h^{-2} - h \times h^3 - \dfrac{4}{h^{-4}}$

4.4 Rational Indices

Positive n^{th} Root

In Book 1, we have learnt about the square root and the cube root of a number, e.g.

$\sqrt{9} = 3$ since $3^2 = 3 \times 3 = 9$

and $\sqrt[3]{27} = 3$ since $3^3 = 3 \times 3 \times 3 = 27$.

Since $3^4 = 3 \times 3 \times 3 \times 3 = 81$, then we can define the positive 4th root of 81 to be

$$\sqrt[4]{81} = 3.$$

In general,

> if a is a positive number such that $a = b^n$ for some positive number b, then b is the **positive n^{th} root** of a, and we write $b = \sqrt[n]{a}$.

An expression that involves the **radical sign** $\sqrt{}$ is called a radical expression.

Worked Example 9

(Finding the Positive nth Root)

Evaluate each of the following without the use of a calculator.

(a) $\sqrt[4]{625}$　　　　　　(b) $\sqrt[5]{243}$

Solution:

(a) By prime factorisation, $625 = 5 \times 5 \times 5 \times 5 = 5^4$.

$$\therefore \sqrt[4]{625} = \sqrt[4]{5 \times 5 \times 5 \times 5}$$
$$= 5$$

(b) By prime factorisation, $243 = 3 \times 3 \times 3 \times 3 \times 3 = 3^5$.

$$\therefore \sqrt[5]{243} = \sqrt[5]{3 \times 3 \times 3 \times 3 \times 3}$$
$$= 3$$

PRACTISE NOW 9

Evaluate each of the following without the use of a calculator.

(a) $\sqrt[4]{256}$　　　　　　(b) $\sqrt[5]{1024}$

(c) $\sqrt[3]{\dfrac{8}{27}}$

Rational Indices

So far, all the indices that we have discussed are integers. What happens if the indices are non-integer rational numbers?

For example, what is $3^{\frac{1}{2}}$ equal to?

Let $p = 3^{\frac{1}{2}}$. Then $p^2 = \left(3^{\frac{1}{2}}\right)^2$

$$= 3^{\frac{1}{2} \times 2} \text{ (using Law 3 of Indices)}$$
$$= 3^1$$
$$= 3$$
$$p = \pm\sqrt{3}$$

However, we define $3^{\frac{1}{2}}$ to be equal to the **positive square root** of 3 (and not the positive and negative square roots of 3) because we want $y = a^{\frac{1}{x}}$ to be a function, i.e. for every value of x, there should be exactly one value of y.

Hence, $3^{\frac{1}{2}} = \sqrt{3}$.

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Rational Indices

Work in pairs.

Copy and complete the following.

What is $5^{\frac{1}{3}}$ equal to?

Let $p = 5^{\frac{1}{3}}$. Then $p^3 = (\quad)^3$

$\qquad = 5^{\quad \times 3}$ (using Law 3 of Indices)

$\qquad = 5^1$

$\qquad = 5$

$\qquad \therefore p = \underline{\qquad}$

In this case, there is only one possible value of p.

Hence, $5^{\frac{1}{3}} = \underline{\qquad}$

In general, if n is a **positive integer**, we define

> **Definition 3:** $a^{\frac{1}{n}} = \sqrt[n]{a}$, if $a > 0$

Consider $a^{\frac{1}{n}} = \sqrt[n]{a}$.

1. What happens if $a < 0$?
2. What happens if $a = 0$?

Worked Example **10** (Rational Indices)

Rewrite each of the following in the radical form and hence evaluate the result without the use of a calculator.

(a) $16^{\frac{1}{2}}$

(b) $27^{-\frac{1}{3}}$

Solution:

(a) $16^{\frac{1}{2}} = \sqrt{16}$
$= 4$

(b) $27^{-\frac{1}{3}} = \dfrac{1}{27^{\frac{1}{3}}}$

$= \dfrac{1}{\sqrt[3]{27}}$

$= \dfrac{1}{3}$

PRACTISE NOW 10

Rewrite each of the following in the radical form and hence evaluate the result without the use of a calculator.

(a) $36^{\frac{1}{2}}$

(b) $8^{-\frac{1}{3}}$

(c) $(-125)^{-\frac{1}{3}}$

Investigation

Rational Indices

Copy and complete the following.

What is $5^{\frac{2}{3}}$ equal to?

(a) $5^{\frac{2}{3}} = 5^{2 \times \frac{1}{\square}}$

$= (5^2)^{\frac{1}{\square}}$

$= \sqrt[\square]{5^2}$

(b) $5^{\frac{2}{3}} = 5^{\frac{1}{\square} \times 2}$

$= \left(5^{\frac{1}{\square}}\right)^2$

$= \left(\sqrt[\square]{5}\right)^2$

In general, if m and n are **positive integers**,

$$a^{\frac{m}{n}} = \sqrt[n]{a^m} \text{ or } (\sqrt[n]{a})^m, \text{ if } a > 0$$

Worked Example 11 (Rational Indices)

(a) Evaluate $125^{\frac{2}{3}}$ without the use of a calculator.

(b) Simplify $\dfrac{1}{\sqrt{x^n}}$, expressing your answer in index form.

Solution:

(a) $125^{\frac{2}{3}} = (\sqrt[3]{125})^2$

$= 5^2$

$= 25$

(b) $\dfrac{1}{\sqrt{x^n}} = \dfrac{1}{x^{\frac{n}{2}}}$

$= x^{-\frac{n}{2}}$

PRACTISE NOW 11

1. Evaluate each of the following without the use of a calculator.

    (a) $64^{\frac{2}{3}}$

    (b) $32^{-\frac{3}{5}}$

    (c) $100^{1.5}$

2. Simplify each of the following, expressing your answer in index form.

    (a) $\sqrt[3]{a^n}$

    (b) $\dfrac{1}{\sqrt[5]{x^2}}$

⠿ Extension of Laws of Indices to Rational Indices

Earlier in this section, we have extended all the 5 laws of indices to include all integer indices. With the definition of rational indices, we can now extend all the 5 laws of indices to include *all rational indices*.

Thinking Time

Copy and complete the following.

1. If a and b are real numbers, and m and n are **rational numbers**, then
    Law 1 of Indices: $a^m \times a^n =$ _____ if $a > 0$
    Law 2 of Indices: $a^m \div a^n =$ _____ if $a > 0$
    Law 3 of Indices: $(a^m)^n =$ _____ if _____
    Law 4 of Indices: $a^n \times b^n =$ _____ if $a, b > 0$
    Law 5 of Indices: $a^n \div b^n =$ _____ if _____

2. Notice that some conditions on the bases a and b are now different.
    (i) Why is it necessary for $a > 0$ in Law 1?
    (ii) Why is it necessary for $a, b > 0$ in Law 4?

3. What happens if you do not take care of the conditions?
    The following shows a ridiculous proof that conclude that $1 = -1$.
    Explain what is wrong with the proof.

    $1 = \sqrt{1} = \sqrt{(-1) \times (-1)} = \sqrt{-1} \times \sqrt{-1} = (\sqrt{-1})^2 = (-1)^{\frac{1}{2} \times 2} = (-1)^1 = -1$

Worked Example 12 (Rational Indices)

Simplify each of the following, expressing your answers in positive index form.

(a) $\sqrt[3]{m} \times \sqrt[4]{m^3}$

(b) $\left(m^{\frac{1}{3}}n^{-2}\right)^{\frac{3}{5}}$

(c) $(mn)^{\frac{2}{3}} \div \left(m^{\frac{3}{4}}n^{\frac{1}{3}}\right)^2$

Solution:

(a) $\sqrt[3]{m} \times \sqrt[4]{m^3} = m^{\frac{1}{3}} \times m^{\frac{3}{4}}$

$= m^{\frac{1}{3}+\frac{3}{4}}$

$= m^{\frac{13}{12}}$

(b) $\left(m^{\frac{1}{3}}n^{-2}\right)^{\frac{3}{5}} = m^{\frac{1}{3} \times \frac{3}{5}}n^{\left(-2 \times \frac{3}{5}\right)}$

$= m^{\frac{1}{5}}n^{-\frac{6}{5}}$

$= \dfrac{m^{\frac{1}{5}}}{n^{\frac{6}{5}}}$

(c) $(mn)^{\frac{2}{3}} \div \left(m^{\frac{3}{4}}n^{\frac{1}{3}}\right)^2 = m^{\frac{2}{3}}n^{\frac{2}{3}} \div m^{\left(\frac{3}{4} \times 2\right)}n^{\left(\frac{1}{3} \times 2\right)}$

$= m^{\frac{2}{3}}n^{\frac{2}{3}} \div m^{\frac{3}{2}}n^{\frac{2}{3}}$

$= m^{\frac{2}{3}-\frac{3}{2}}n^{\frac{2}{3}-\frac{2}{3}}$

$= m^{\frac{2}{3}-\frac{3}{2}}n^{0}$

$= m^{-\frac{5}{6}}(1)$

$= \dfrac{1}{m^{\frac{5}{6}}}$

PRACTISE NOW 12

Simplify each of the following, expressing your answers in positive index form.

(a) $\left(m^2\right)^{\frac{5}{6}} \times m^{\frac{1}{3}}$

(b) $\sqrt[5]{m} \div \sqrt[3]{m^2}$

(c) $\left(m^{-3}n^5\right)^{-\frac{1}{3}}$

(d) $\dfrac{m^{-\frac{1}{3}}n^{-\frac{1}{4}}}{\left(m^2n^{-\frac{1}{3}}\right)^{-2}}$

(e) $\left(25m^2n^{-4}\right)^{\frac{1}{2}}\left(m^3n^{-\frac{2}{5}}\right)^2$

(f) $\left(m^2n^{-\frac{1}{7}}\right) \times \sqrt[5]{\left(m^5n^{-5}\right)}$

Equations involving Indices

To solve an equation such as $x^2 = 100$, we take the square root on both sides to obtain $x = \pm 10$. Similarly, to solve the equation $y^3 = 64$, we take the cube root on both sides to obtain $y = 4$.

If we are given the equation $2^x = 32$, how do we find the value of x?

Worked Example 13 (Equations involving Indices)

Solve each of the following equations.

(a) $2^x = 32$

(b) $3^y = \dfrac{1}{9}$

(c) $9^z = 27$

Solution:

(a) $2^x = 32$
$= 2^5$
$x = 5$

(b) $3^y = \dfrac{1}{9}$
$= \dfrac{1}{3^2}$
$= 3^{-2}$
$y = -2$

(c) $9^z = 27$
$(3^2)^z = 3^3$
$3^{2z} = 3^3$
$2z = 3$
$z = \dfrac{3}{2}$
$= 1\dfrac{1}{2}$

PRACTISE NOW 13

Solve each of the following equations.

(a) $5^x = 125$

(b) $7^y = \dfrac{1}{49}$

(c) $8^z = 16$

SIMILAR QUESTIONS

Exercise 4B Questions 8(a)–(d)

Exercise 4B

BASIC LEVEL

1. Evaluate each of the following.

   (a) 17^0

   (b) $\left(-\dfrac{2}{7}\right)^0$

   (c) $4a^0$

   (d) $-8b^0$

   (e) $(72cd^2)^0$

   (f) $7(e^8)^0$

2. Find the value of each of the following.

   (a) $2^0 \times 2^4$

   (b) $7^2 \times 7^0 \div 7$

   (c) $8^0 - 8^2$

   (d) $6^3 + 6^0 - 6$

3. Evaluate each of the following.

   (a) 7^{-3}

   (b) $(-5)^{-1}$

   (c) $\left(\dfrac{3}{4}\right)^{-2}$

   (d) $\left(\dfrac{5}{3}\right)^{-1}$

4. Evaluate each of the following.

   (a) $(7^2)^{-2} \div 7^{-4}$

   (b) $5^0 - 5^{-2}$

   (c) $(2^{15})^0 + \left(\dfrac{3}{5}\right)^{-1}$

   (d) $\left(\dfrac{3}{4}\right)^{-2} \times 3^2 \times 2015^0$

5. Evaluate each of the following without the use of a calculator.

   (a) $\sqrt{196}$

   (b) $\sqrt[3]{125}$

   (c) $\sqrt[5]{\dfrac{1}{32}}$

   (d) $\sqrt[4]{\dfrac{16}{81}}$

6. Rewrite each of the following in the radical form and hence evaluate the result without the use of a calculator.

   (a) $81^{\frac{1}{2}}$

   (b) $(-27)^{\frac{1}{3}}$

   (c) $16^{-\frac{1}{4}}$

   (d) $4^{1.5}$

   (e) $8^{\frac{5}{3}}$

   (f) $(-1000)^{\frac{2}{3}}$

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7. Simplify each of the following, expressing your answer in index form.

(a) $\sqrt[4]{a}$

(b) $\sqrt[3]{b^2}$

(c) $\left(\sqrt[5]{c}\right)^4$

(d) $\dfrac{1}{\sqrt[6]{d}}$

(e) $\dfrac{1}{\sqrt[8]{e^4}}$

(f) $\dfrac{1}{\left(\sqrt[3]{f}\right)^5}$

8. Solve each of the following equations.

(a) $11^a = 1331$

(b) $2^b = \dfrac{1}{128}$

(c) $9^c = 243$

(d) $10^d = 0.01$

9. Simplify each of the following, expressing your answers in positive index form.

(a) $5a^4 \times 3a^2 \div a^{-3}$

(b) $-24b^{-6} \div (3b^{-3})^2$

(c) $(3c)^0 \div (c^{-3}d^5)^{-2}$

(d) $\dfrac{\left(4e^{-6}f^3\right)^2}{8e^{12}f^6}$

(e) $(3g^{-3}h^{-1})^2 \times (-4g^4h^{-2})^2$

(f) $\left(j^2k^{-1}\right)^{-3} \times \left(\dfrac{j^2}{k^3}\right)^{-3}$

(g) $\dfrac{\left(m^5n^3\right) \times \left(m^2\right)^{-2}}{\left(m^{-1}n\right)^2}$

(h) $(5p)^3 - 10p \times 7p^2 + \dfrac{6}{p^{-3}}$

10. Simplify each of the following, expressing your answer in positive index form.
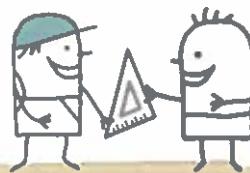
(a) $\sqrt{a} \times \sqrt[3]{a}$

(b) $\sqrt[3]{b^2} \div \sqrt[6]{b}$

(c) $c^{\frac{4}{5}} \times c^{\frac{1}{2}} \div c^{-\frac{2}{5}}$

(d) $d^{\frac{1}{10}} \div d^{-\frac{1}{5}} \times d^{-\frac{3}{2}}$

(e) $\left(e^{-3}f^4\right)^{-\frac{1}{2}}$

(f) $\left(g^{\frac{2}{3}}h^{-\frac{4}{5}}\right)^{\frac{3}{2}}$

11. Simplify each of the following, expressing your answer in positive index form.

(a) $\left(a^{-2}b^3\right)^{\frac{1}{3}} \times \left(a^4b^{-5}\right)^{\frac{1}{2}}$

(b) $\left(c^{-3}d^{\frac{3}{5}}\right)^{-2} \times \left(c^{\frac{4}{5}}d^{-\frac{2}{5}}\right)^5$

(c) $\dfrac{e^{-\frac{1}{3}}f^{-\frac{1}{4}}}{\left(e^2f^{-\frac{1}{3}}\right)^{-2}}$

(d) $\left(\dfrac{g^{-2}h^2}{25}\right)^{-\frac{1}{2}}$

(e) $\left(4j^4k\right)^{\frac{1}{2}} + 2h^3k^{-\frac{1}{2}}$

(f) $\left(m^3n^{-\frac{1}{4}}\right)^4 + \sqrt[5]{32m^4n^{-8}}$

12. Simplify each of the following, expressing your answers in positive index form.

(a) $\left(\dfrac{x^{-4}y^7z^{-6}}{x^3y^{-1}z^3}\right)^3 \times \left(\dfrac{x^5y^2z^{-6}}{x^{-3}y^{-5}z^4}\right)^{-4}$

(b) $\left(\dfrac{x^3y^{-4}z^7}{x^{-5}y^2}\right)^3 \div \left(\dfrac{x^{-4}yz^{-5}}{x^7y^{-3}}\right)^{-2}$

(c) $\dfrac{ab^n}{bc} \times \dfrac{c^nd}{cd} \div \dfrac{b^{n+2}}{c^{n+3}}$

(d) $\dfrac{(a+b)^n}{bc^2} \div \dfrac{(a+b)^{n+3}}{abc}$

4.5 Standard Form

Standard Form

Work in pairs.

Table 4.5 shows some examples of measurements which involve very large or very small numbers.

		Ordinary Notation	Standard Form
(i)	Singapore's population in 2013	5 300 000	5.3×10^6
(ii)	Distance between Earth and the sun	149 600 000 km	1.496×10^8 km
(iii)	Mass of a dust particle	0.000 000 000 753 kg	7.53×10^{-10} kg
(iv)	Mass of an oxygen atom	0.000 000 000 000 000 000 000 000 002 656 kg	2.656×10^{-27} kg
(v)	Number of grains of sand in a bag	29 000	2.9×10
(vi)	Speed of light	300 000 000 m/s	3×10 m/s
(vii)	Wavelength of violet light	0.000 038 cm	3.8×10 cm
(viii)	Mass of a water molecule	0.000 000 000 000 000 000 000 0299	2.99×10 g

Table 4.5

1. The examples in (i)-(ii) involve very large numbers. What do you observe about the powers of 10 in each standard form?
2. The examples in (iii)-(iv) involve very small numbers. What do you observe about the powers of 10 in each standard form?
3. Complete the last column for (v)-(viii) in Table 4.5.

Table 4.6 shows numbers expressed in standard form and numbers not expressed in standard form.

	Standard Form	Not Standard Form
(i)	4.5×10^4	45×10^3
(ii)	2.06×10^8	0.206×10^5
(iii)	3.71×10^{21}	$3.71 \times 10^{21.2}$
(iv)	8.00×10^{-3}	$8.00 \times 10^{-3\frac{1}{2}}$
(v)	9.25×10^{-10}	$92.5 \times 10^{-1.01}$
(vi)	1.0×10^{-16}	10×10^{-17}

Table 4.6

4. For a number in the form $A \times 10^n$ to be considered as standard form, what can you say about A and n? Explain your answer.

The world population, estimated to be about 7 000 000 000 in 2012, can be written as 7.0×10^9. The Bohr radius of a hydrogen atom is 0.000 000 000 053 m, which can be written as 5.3×10^{-11} m.

Both 7.0×10^9 and 5.3×10^{-11} are examples of numbers expressed in standard form.

In general, a number is said to be expressed in **standard form**, or **scientific notation**, when it is written as

$A \times 10^n$, where $1 \leqslant A < 10$ and n is an **integer**.

Worked Example 14 (Standard Form)

Express each of the following numbers in standard form.

(a) 149 600 000   (b) 0.000 038

Solution:

(a) $149\ 600\ 000 = 1.496 \times 10^8$ (move the decimal point 8 places to the left)

(b) $0.000\ 038 = 3.8 \times 10^{-5}$ (move the decimal point 5 places to the right)

PRACTISE NOW 14

1. Express each of the following numbers in standard form.
   (a) 5 300 000
   (b) 600 000 000
   (c) 0.000 048
   (d) 0.000 000 000 21

2. Express each of the following as an integer or a decimal.
   (a) 1.325×10^6
   (b) 4.4×10^{-3}

⠿ Common Prefixes

Have you used an external hard disk with a capacity of 512 *gigabytes*? Have you used a micrometer in the laboratory?

Prefixes are commonly used in our daily lives to denote certain powers of 10. They are related to the special names in the SI system of units. Table 4.7 lists some of the common prefixes and their symbols used for very large and very small numbers.

Power of 10	Name	SI Prefix	Symbol	Numerical Value
10^{12}	trillion	tera-	T	1 000 000 000 000
10^9	billion	giga-	G	1 000 000 000
10^6	million	mega-	M	1 000 000
10^3	thousand	kilo-	k	1000
10^{-3}	thousandth	milli-	m	$0.001 = \dfrac{1}{1000}$
10^{-6}	millionth	micro-	μ	$0.000\ 001 = \dfrac{1}{1\ 000\ 000}$
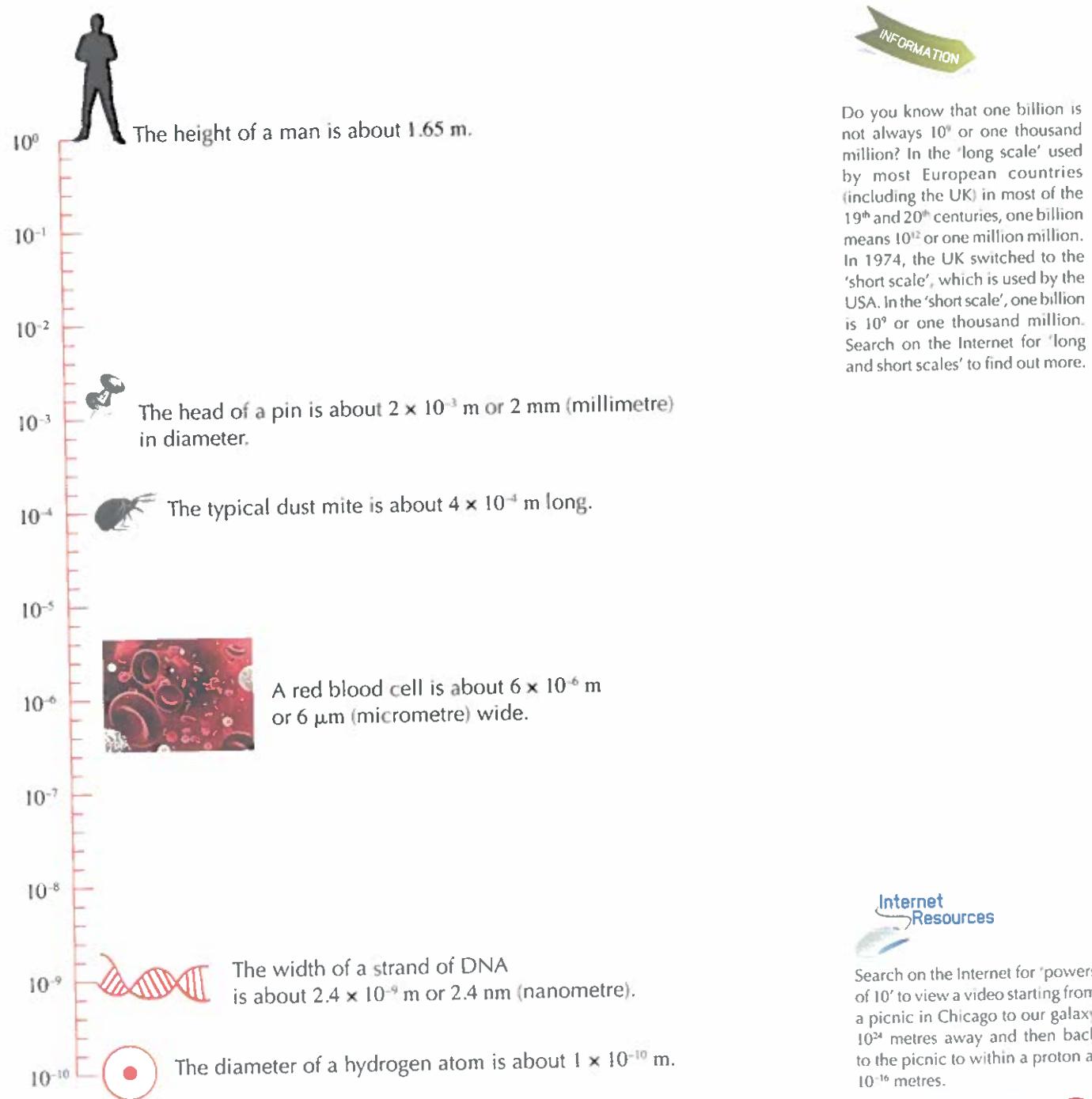
Power of 10	Name	SI Prefix	Symbol	Numerical Value
10^{-9}	billionth	nano-	n	$0.000\,000\,001 = \dfrac{1}{1\,000\,000\,000}$
10^{-12}	trillionth	pico-	p	$0.000\,000\,000\,001 = \dfrac{1}{1\,000\,000\,000\,000}$

Table 4.7

Hence, 8.2×10^9 is read as 8.2 billion. How do you read 3.7×10^{12}?
5.6×10^{-6} is read as 5.6 millionth. How do you read 4.9×10^{-9}?

The figure below shows a range of prefixes used in our daily lives.

10^0 — The height of a man is about 1.65 m.

10^{-1}

10^{-2}

10^{-3} — The head of a pin is about 2×10^{-3} m or 2 mm (millimetre) in diameter.

10^{-4} — The typical dust mite is about 4×10^{-4} m long.

10^{-5}

10^{-6} — A red blood cell is about 6×10^{-6} m or 6 μm (micrometre) wide.

10^{-7}

10^{-8}

10^{-9} — The width of a strand of DNA is about 2.4×10^{-9} m or 2.4 nm (nanometre).

10^{-10} — The diameter of a hydrogen atom is about 1×10^{-10} m.

Worked Example 15

(Common Prefixes in our Daily Lives)

For each of the following, give your answer in standard form.

(a) A male African elephant can weigh as heavy as 7000 kilograms. Express this weight in grams.

(b) The average lifespan of a certain molecule is 0.5 nanoseconds.

Given that 1 nanosecond = 10^{-9} seconds, express this time in seconds.

(c) A steam power plant in Singapore has a capacity of 250 megawatts.

Given that 1 megawatt = 10^6 watts, express this capacity in watts.

Solution:

(a) 7000 kilograms
$= 7000 \times 10^3$ grams
$= 7 \times 10^3 \times 10^3$ grams
$= 7 \times 10^{3+3}$ grams
$= 7 \times 10^6$ grams

(b) 0.5 nanoseconds
$= 0.5 \times 10^{-9}$ seconds
$= 5 \times 10^{-1} \times 10^{-9}$ seconds
$= 5 \times 10^{-1+(-9)}$ seconds
$= 5 \times 10^{-10}$ seconds

(c) 250 megawatts
$= 250 \times 10^6$ watts
$= 2.5 \times 10^2 \times 10^6$ watts
$= 2.5 \times 10^{2+6}$ watts
$= 2.5 \times 10^8$ watts

PRACTISE NOW 15

For each of the following, give your answer in standard form.

(a) The diameter of a human hair is 25.4 micrometres.

Given that 1 micrometre = 10^{-6} metres, express this diameter in metres.

(b) A rain gauge measures rainfall over a period of time. The average annual rainfall in Singapore is 2340 mm. Express this measurement in centimetres.

(c) An external hard drive has a capacity of 4.0 terabytes .

Given that 1 terabyte = 10^{12} bytes, express this capacity in bytes.

Performance Task

If you are buying 1 thumbdrive with a capacity of 1 gigabyte (1 GB), are you getting exactly 1 billion bytes of computer space? Search on the Internet to find out how many bytes 1 GB is actually equal to. Why is it not possible for the manufacturer to produce a thumbdrive with exactly 1 billion bytes?

Hint: How does the number 2 in 2^{30} play a part here?

Search on the Internet to find out why computer storage systems always come in the form of 128 MB, 256 MB, 512 MB and so on. Present your findings to your class.

Worked Example 16

(Applying the Orders of Operation in Standard Form)

With the use of a calculator, find the value of each of the following, giving your answer in standard form, correct to 3 significant figures.

(a) $(7.2 \times 10^3) \times (2.05 \times 10^5)$
(b) $\dfrac{4.5 \times 10^{-3}}{9 \times 10^{-1}}$
(c) $1.35 \times 10^3 + 2.37 \times 10^4$
(d) $\dfrac{2.58 \times 10^{-3} - 4.19 \times 10^{-4}}{3.17 \times 10^2}$

ATTENTION

The buttons on calculators vary with each model. Refer to the manual of your calculator.

Solution:

(a) $(7.2 \times 10^3) \times (2.05 \times 10^5)$

$= \boxed{(}\,\boxed{7}\,\boxed{.}\,\boxed{2}\,\boxed{\times 10^x}\,\boxed{3}\,\boxed{)}\,\boxed{\times}\,\boxed{(}\,\boxed{2}\,\boxed{.}\,\boxed{0}\,\boxed{5}\,\boxed{\times 10^x}\,\boxed{5}\,\boxed{)}\,\boxed{=}$

$= 1\,476\,000\,000$

$= 1.48 \times 10^9$ (to 3 s.f.)

(b) $\dfrac{4.5 \times 10^{-3}}{9 \times 10^{-1}}$

$= \boxed{(}\,\boxed{4}\,\boxed{.}\,\boxed{5}\,\boxed{\times 10^x}\,\boxed{-}\,\boxed{3}\,\boxed{)}\,\boxed{\div}\,\boxed{(}\,\boxed{9}\,\boxed{\times 10^x}\,\boxed{-}\,\boxed{1}\,\boxed{)}\,\boxed{=}$

$= 5 \times 10^{-3}$

(c) $1.35 \times 10^3 + 2.37 \times 10^4$

$= \boxed{1}\,\boxed{.}\,\boxed{3}\,\boxed{5}\,\boxed{\times 10^x}\,\boxed{3}\,\boxed{+}\,\boxed{2}\,\boxed{.}\,\boxed{3}\,\boxed{7}\,\boxed{\times 10^x}\,\boxed{4}\,\boxed{=}$

$= 25\,050$

$= 2.5050 \times 10^4$

$= 2.51 \times 10^4$ (to 3 s.f.)

(d) $\dfrac{2.58 \times 10^{-3} - 4.19 \times 10^{-4}}{3.17 \times 10^2}$

$= \boxed{(}\,\boxed{2}\,\boxed{.}\,\boxed{5}\,\boxed{8}\,\boxed{\times 10^x}\,\boxed{-}\,\boxed{3}\,\boxed{-}\,\boxed{4}\,\boxed{.}\,\boxed{1}\,\boxed{9}\,\boxed{\times 10^x}\,\boxed{-}\,\boxed{4}\,\boxed{)}$

$\boxed{\div}\,\boxed{3}\,\boxed{.}\,\boxed{1}\,\boxed{7}\,\boxed{\times 10^x}\,\boxed{2}\,\boxed{=}$

$= 6.817 \times 10^{-6}$

$= 6.82 \times 10^{-6}$ (to 3 s.f.)

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With the use of a calculator, find the value of each of the following, giving your answer in standard form, correct to 3 significant figures.

(a) $(1.14 \times 10^5) \times (4.56 \times 10^4)$

(b) $(4.2 \times 10^{-4}) \times (2.6 \times 10^2)$

(c) $(2.4 \times 10^8) \div (6 \times 10^4)$

(d) $\dfrac{3.5 \times 10^{-5}}{1.4 \times 10^8}$

(e) $1.14 \times 10^5 + 4.56 \times 10^4$

(f) $4 \times 10^4 - 2.6 \times 10^6$

(g) $\dfrac{2.37 \times 10^{-3} + 3.25 \times 10^{-4}}{4.1 \times 10^5}$

(h) $\dfrac{6.3 \times 10^6}{1.5 \times 10^2 - 3 \times 10^{-1}}$

Worked Example 17 (Applications of Standard Form)

The approximate mass of the moon is 7×10^{19} tonnes while the mass of the earth is approximately 6×10^{24} tonnes. Calculate the number of times that the earth is as heavy as the moon, giving your answer correct to the nearest 1000.

Solution:

Number of times the earth is as heavy as the moon $= \dfrac{6 \times 10^{24}}{7 \times 10^{19}}$

$$= \frac{6}{7} \times 10^{24-19}$$

$$= \frac{6}{7} \times 10^5$$

$$= 86\,000 \text{ (to the nearest 1000)}$$

A Secure Digital (SD) memory card has a capacity of 512 megabytes. Each photograph has a size of 640 kilobytes. Assuming that $1\ \mathbf{MB} = 10^6$ bytes and $1\ \mathbf{kB} = 10^3$ bytes , how many photographs can this memory card store?

Fill in the blanks to convert between ordinary notation and standard form.

1. The distance of the planet Mercury from the sun is about 57 910 000 km (_____ × 10⁷ km) while the distance of Pluto from the sun is about 5 945 900 000 km (_____ × 10⬛ km).

2. The moon travels around the earth at a speed of about 3 683 000 m/h (_____ × 10⬛ km/h).

3. If the diameter of an air molecule is about 0.000 000 0004 m (_____ × 10⬛ m) and there are 500 000 000 000 000 000 000 000 000 (_____ × 10⬛) molecules in a room, then the total volume of the air molecules in the room is _____ × 10⬛ m³.

4. There are about 100 trillion (_____ × 10⬛) cells in the human body. The human heart contains about _____ million (2 × 10⁹) cells and beats about 42 000 000 (_____ × 10⬛) times each year.

Exercise 4C

BASIC LEVEL

1. Express each of the following numbers in standard form.
   (a) 85 300
   (b) 52 700 000
   (c) 0.000 23
   (d) 0.000 000 094

2. Express each of the following as an integer or a decimal.
   (a) 9.6×10^3
   (b) 4×10^5
   (c) 2.8×10^{-4}
   (d) 1×10^{-6}

3. Microwaves are a form of electromagnetic radiation with frequencies between 300 000 000 Hz and 300 GHz. Giving your answer in standard form, express
   (i) 300 000 000 Hz in MHz,
   (ii) 300 GHz in MHz.
   ($1 \text{ MHz} = 10^6 \text{ Hz}$ and $1 \text{ GHz} = 10^9 \text{ Hz}$)

INTERMEDIATE LEVEL

4. (i) A nitrogen atom has an atomic radius of a picometres (pm), where $a = 70$ and $1 \text{ pm} = 10^{-12}$ m. Express this radius in metres. Give your answer in standard form.
   (ii) An oxygen atom has an atomic radius of b nanometres (nm), where $b = 0.074$ and $1 \text{ nm} = 10^{-9}$ m. Express this radius in metres. Give your answer in standard form.
   (iii) Express $a : b$ as a ratio of two integers in its simplest form.

5. The mean distance from the earth to the sun is c megametres (Mm), where $c = 1500$ and $1 \text{ Mm} = 10^6$ m. The mean distance from Pluto to the sun is d terametres (Tm), where $d = 5.91$ and $1 \text{ Tm} = 10^{12}$ m. Express d as a percentage of c. Give your answer in standard form.

6. With the use of a calculator, find the value of each of the following, giving your answer in standard form, correct to 3 significant figures.

(a) $(2.34 \times 10^5) \times (7.12 \times 10^{-4})$

(b) $(5.1 \times 10^{-7}) \times (2.76 \times 10^{-3})$

(c) $(13.4 \times 10^4) \div (4 \times 10^5)$

(d) $\dfrac{3 \times 10^{-4}}{9 \times 10^{-8}}$

(e) $2.54 \times 10^3 + 3.11 \times 10^4$

(f) $6 \times 10^5 - 3.1 \times 10^7$

(g) $\dfrac{4.37 \times 10^{-4} + 2.16 \times 10^{-5}}{3 \times 10^{-3}}$

(h) $\dfrac{2.4 \times 10^{-10}}{7.2 \times 10^{-6} - 3.5 \times 10^{-8}}$

7. With the use of a calculator, find the value of each of the following, giving your answer in standard form, correct to 3 significant figures.

(a) $(1.35 \times 10^{-4})^3$

(b) $6(3.4 \times 10^3)^2$

(c) $\sqrt{1.21 \times 10^8}$

(d) $\sqrt[3]{9.261 \times 10^6}$

(e) $\dfrac{2.3 \times 10^{-2} \times 4.7 \times 10^3}{2 \times 10^3}$

(f) $\dfrac{8 \times 10^2 + 2.5 \times 10^3}{2 \times 10^{-2} - 3.4 \times 10^{-3}}$

8. Given that $P = 7.5 \times 10^3$ and $Q = 5.25 \times 10^4$, express each of the following in standard form.

(a) $2P \times 4Q$

(b) $Q - P$

9. Given that $x = 2 \times 10^{-3}$ and $y = 7 \times 10^{-4}$, evaluate $x + 8y$, giving your answer in standard form.

10. Given that $M = 3.2 \times 10^6$ and $N = 5.0 \times 10^7$, find the value of each of the following, giving your answer in standard form.

(a) MN

(b) $\dfrac{M}{N}$

11. Given that $R = \dfrac{M}{EI}$, find the value of R when $M = 6 \times 10^4$, $E = 4.5 \times 10^8$ and $I = 4 \times 10^2$. Give your answer in standard form.

12. Light travels at a speed of 300 000 000 m/s.

(i) Express this speed in standard form.

(ii) Given that the mean distance from the sun to Jupiter is 778.5 million kilometres, find the time taken, in minutes and seconds, for light to travel from the sun to Jupiter.

13. On a journey from Planet P to Venus, a rocket is travelling at a constant speed. During this journey, the rocket travels past the moon in 4 days. The distance from Planet P to the moon is 4.8×10^5 km.

(i) Find the distance travelled by the rocket in 12 days. Give your answer in standard form.

(ii) Given that the distance between Planet P and Venus is 4.8×10^7 km, find the time taken, in days, for the journey.

ADVANCED LEVEL

14. The table shows the approximate population of the world in the past centuries.

Year	World population
1549	4.20×10^8
1649	5.45×10^8
1749	7.28×10^8
1849	1.17×10^9

Find

(i) the increase in population from 1549 to 1649,

(ii) the number of times that the population in 1849 is as large as that in 1649,

(iii) the number of times that the population of China is as large as that of the world in 1749, given that the population of China in year 2000 is approximately 1.23 billion, where 1 billion $= 10^9$.

1. In index form, $a^n = \underbrace{a \times a \times \ldots \times a \times a}_{n \text{ times}}$, where n is a positive integer.

2. **Laws of Indices**

   Assume that a and b are real numbers and m and n are positive integers.

   > Law 1 of Indices: $a^m \times a^n = a^{m+n}$
   >
   > Law 2 of Indices: $a^m \div a^n = a^{m-n}$, if $a \neq 0$
   >
   > Law 3 of Indices: $(a^m)^n = a^{mn}$
   >
   > Law 4 of Indices: $a^n \times b^n = (a \times b)^n$
   >
   > Law 5 of Indices: $a^n \div b^n = \left(\dfrac{a}{b}\right)^n$, if $b \neq 0$

   If m and n are integers, the conditions for the above 5 laws of indices are different (see page 89).

   If m and n are rational numbers, the conditions for the above 5 laws of indices are again different (see pages 94–95).

3. **Zero Indices**

   If a is a real number, we define $\boxed{a^0 = 1 \text{ if } a \neq 0}$.

4. **Negative Indices**

   If a is a real number, we define $\boxed{a^{-n} = \dfrac{1}{a^n}, \text{ if } a \neq 0}$.

5. **Rational Indices**

   If n is a positive integer, we define $\boxed{a^{\frac{1}{n}} = \sqrt[n]{a} \text{ if } a > 0}$.

   If m and n are positive integers, $\boxed{a^{\frac{m}{n}} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m, \text{ if } a > 0}$.

6. A number is said to be expressed in **standard form**, or **scientific notation**, when it is written as $\boxed{A \times 10^n, \text{ where } 1 \leq A < 10 \text{ and } n \text{ is an } \textbf{integer}}$.

Review Exercise 4

1. Simplify each of the following.
   (a) $(a^3b) \times (a^4b^3)$
   (b) $(6a^5b^4) \div (2a^3b^2)$
   (c) $(-3a^3b^5)^3$
   (d) $\left(\dfrac{2a^2b}{b^3}\right)^3 \div \left(\dfrac{16a^5}{ab^7}\right)$

2. Express each of the following as a power of 5.
   (a) $5^{24} \div 5^8$
   (b) $\dfrac{1}{125}$
   (c) $\sqrt[5]{5}$

3. Evaluate each of the following without the use of a calculator.
   (a) $5^2 \div 5^{-1} \times 5^0$
   (b) $2^{-2} - 3^{-2}$
   (c) $3^{-2} + \left(\dfrac{1}{3}\right)^{-1} - (-3)^0$
   (d) $\left(\dfrac{2}{5}\right)^3 \div \left(\dfrac{9}{2}\right)^{-2}$

4. Evaluate each of the following without the use of a calculator.
   (a) $\sqrt[4]{81}$
   (b) $\sqrt[3]{\dfrac{27}{125}}$
   (c) $16^{1.5}$
   (d) $1024^{-\frac{3}{5}}$

5. Simplify each of the following, expressing your answer in positive index form.
   (a) $\left(\dfrac{3}{x}\right)^{-4}$
   (b) $3 \div x^{-3}$

6. Simplify each of the following, leaving your answer in positive index form.
   (a) $(x^3y^{-2}) \times (x^{-3}y^5)$
   (b) $(5x^2y^3)^0 \div (-2x^{-3}y^5)^{-2}$
   (c) $\left(\dfrac{x^2}{y^{-3}}\right)^4 \div \left(\dfrac{x^5}{y^7}\right)^3$
   (d) $\dfrac{(3x^{-2}y^5)^2 \times (-2x^3y^{-2})^2}{9x^4y^6}$

7. Simplify each of the following, expressing the answers in positive index form.
   (a) $\sqrt[5]{p^3} \times \sqrt[3]{8p}$
   (b) $\left(p^{-3}q^{\frac{3}{5}}\right)^{-\frac{2}{3}} \times \left(p^{\frac{4}{5}}q^{-\frac{2}{3}}\right)^3$
   (c) $\dfrac{p^{\frac{2}{3}}q^{-\frac{2}{5}}}{\left(p^2q^{-\frac{1}{5}}\right)^{-3}}$
   (d) $\left(p^{-\frac{1}{3}}q^2\right)^5 \times \sqrt[3]{27\left(p^{-3}q^2\right)}$

8. (a) Given that $4^{-6} \times 4^x = 1$, find the value of x.
   (b) Find the value of x^3 for which $x^{-3} = 7$.
   (c) If $5^{12} \times 5^{-2} \div 5^x = 25$, find the value of x.

9. Solve each of the following equations.
   (a) $16^a = 8$
   (b) $2015^b = 1$
   (c) $\dfrac{10^c}{10} = 0.01$
   (d) $\dfrac{2^{d-6}}{2} = 2^9$

10. With the use of a calculator, evaluate each of the following, giving your answer in standard form, correct to 3 significant figures.
    (a) $(6.4 \times 10^6) \times (5.1 \times 10^{-3})$
    (b) $(2.17 \times 10^{-5}) \div (7 \times 10^4)$
    (c) $(3.17 \times 10^4) + (2.26 \times 10^5)$
    (d) $(4.15 \times 10^{-3}) - (5.12 \times 10^{-4})$
    (e) $\dfrac{5.1 \times 10^{-6} - 2.34 \times 10^5}{4.87 \times 10^{-3} + 9 \times 10^{-2}}$
    (f) $\dfrac{8.43 \times 10^7 + 6.8 \times 10^8}{(1.01 \times 10^4)^3}$

11. Given that $a = 110\,000\,000$, $b = 12\,100\,000$ and $c = 0.000\,007$, find the value of each of the following, giving your answer in standard form.

(a) $a - b$

(b) $\sqrt[3]{ab}$

(c) $6c^2$

(d) $\dfrac{ac}{b}$

12. The diameter of a circular microorganism is 7 nanometres (nm). Find the

(i) circumference in metres,

(ii) area in square metres,

of the microorganism. Give your answer in standard form. (Take $\pi = 3.142$ and $1\,\text{nm} = 10^{-9}\,\text{m}$.)

13. An astronomical unit (abbreviated as AU, au, a.u., or ua) is a unit of length defined as $149\,597\,870\,700$ metres, which is roughly or exactly the mean distance between the earth and the sun. One astronomical unit (au) is estimated to be 150 gigametres (Gm).

(i) Express this astronomical unit in metres, correct to 4 significant figures. Give your answer in standard form.

(ii) The speed of light is 3×10^8 m/s. How long does light take to travel a distance of 1 au? (Take $1\,\text{Gm} = 10^9\,\text{m}$.)

14. The distance from 2 planets is 240 megametres $(1\,\text{Mm} = 10^6\,\text{m})$.

(i) Express this distance in metres. Give your answer in standard form.

A rocket travels a distance of one metre in 8000 nanoseconds (ns) and $1\,\text{ns} = 10^{-9}\,\text{s}$.

(ii) How long does the rocket take to travel from one planet to the other? Give your answer in seconds.

15. The mass of a hydrogen atom is approximately 1.66×10^{-24} g and that of an oxygen atom is approximately 2.66×10^{-23} g.

(i) Find the mass of a water molecule which consists of two hydrogen and one oxygen atoms. Give your answer in standard form correct to 3 significant figures.

(ii) Find the approximate number of water molecules in a cup of water which has a mass of 280 g. Give your answer in standard form correct to 3 significant figures.

Challenge Yourself

1. Use the numerals 2, 3 and 4 to form a number as large as possible. Show that it is the largest.

2. What is the last digit of 3^{2015}?

3. Find the exact value of $\sqrt{2 + \sqrt{2 + \sqrt{2 + \ldots}}}$.

1. Evaluate each of the following, without the use of a calculator.

   (a) $\left(\dfrac{1}{2}\right)^{-3}$

   (b) $0.04^{-1.5}$

   (c) $\left(1\dfrac{9}{16}\right)^{-\frac{1}{2}}$

   (d) $9^{2.5} + 27^{1\frac{1}{3}}$

   (e) $9^{\frac{1}{2}} - 0.36^{-\frac{1}{2}}$

2. Simplify each of the following, expressing your answer in positive index form.

   (a) $a^5 \div a^{-2}$

   (b) $b^4 \div \sqrt{b} \times b^{-7}$

   (c) $\left(\dfrac{c^{-3}d}{c^2d^{-2}}\right)^{-5}$

3. Solve each of the following equations.

   (a) $7^x = 2^4 \div 4^2$

   (b) $(2y+3)^{\frac{1}{2}} = 5$

4. (i) Solve the inequality $3 - 5p > 17$.

   (ii) Hence, write down the greatest integer value of p which satisfies $17 < 3 - 5p$.

5. When 5 is subtracted from twice a number, the resulting number is less than 12. When 1 is subtracted from three times the number, the resulting number is greater than 12. List the possible integer values of the number.

6. A function f is defined by $f : x \mapsto 10x - 3$, find the inverse function $f^{-1}(x)$.

   Hence evaluate $f^{-1}(9)$, $f^{-1}(-3)$ and $f^{-1}\left(-\dfrac{1}{2}\right)$.

7. The diameter of a circular microorganism is 8.8 nanometres. Find the
   (i) circumference in m,
   (ii) area in m²,
   of the microorganism.
   Give your answers in standard form.
   (Take $\pi = 3.142$, 1 nanometre $= 10^{-9}$ metres.)

8. The perimeter of a square exceeds that of another by 100 cm and the area of the larger square exceeds three times that of the smaller square by 325 cm². Find the length of a side of each of the squares.

9. The distance between P and Q is 330 km. A train, A, travelling from P to Q at an average speed of x km/h takes half an hour less than another train, B, travelling from Q to P at an average speed of $(x - 5)$ km/h. Form an equation in x and find the time taken for each train to travel between P and Q.

10. In the figure, $PQRS$ is a rectangle. The point A lies on PQ and the point B lies on QR such that $\angle SAB = 90°$.

   (i) Given that $PQ = 8$ cm, $QR = 4$ cm, $QB = 3$ cm and $PA = y$ cm, write down expressions in terms of y for AS^2 and AB^2.

   (ii) Form an equation in y and show that it reduces to $y^2 - 8y + 12 = 0$.

   (iii) Solve the equation $y^2 - 8y + 12 = 0$ and hence find the two possible values of the area of $\triangle ABS$.

11. In the figure, the width of the border of the picture is x cm. The picture has an area of 160 cm².

   (i) Form an equation in x and show that it reduces to $x^2 - 18x + 40 = 0$.

   (ii) Solve the equation $x^2 - 18x + 40 = 0$, giving both your answers correct to 2 significant figures.

   (iii) Write down the width of the border, giving your answer correct to 2 significant figures.

A2 Revision Exercise

1. Simplify each of the following, expressing your answer in positive index form.
   (a) $(2ab^2)^3$
   (b) $c^3 \times c^{-2} \div c^0$
   (c) $\left(\dfrac{2}{d}\right)^{-2}$
   (d) $2 \div 4e^{-3}$

2. Given that $5^{-6} \div 5^p = 125^0$, find the value of $\dfrac{1}{2^p}$.

3. Solve each of the following equations.
   (a) $10^{2x+3} = 0.001$
   (b) $y^{-2} = \dfrac{1}{81}$
   (c) $(2z - 1)^{\frac{1}{3}} = 2$

4. Given that $1 \leqslant x \leqslant 3$ and $3 \leqslant y \leqslant 6$, find
   (a) the largest possible value of $x^2 - y$,
   (b) the smallest possible value of $\dfrac{x}{y}$.
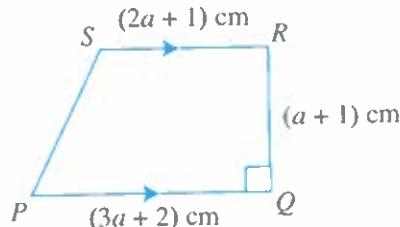
5. Solve the inequality $-3 \leqslant 2q + 7 < 23$, illustrating the solution on a number line.

6. Given that $\dfrac{4x}{5} - \dfrac{3}{10} \leqslant x - 2\dfrac{1}{4}$, find the smallest possible value of x such that
   (a) x is a prime number,
   (b) x is an integer,
   (c) x is a rational number.

7. A function f is defined by $f : x \mapsto \dfrac{2}{3x-2}$ where $x \neq \dfrac{2}{3}$. Find $f^{-1}(x)$ and state the value of x for which f^{-1} is not defined. Hence evaluate $f^{-1}(6)$.

8. The figure shows a trapezium $PQRS$ in which SR is parallel to PQ, $\angle PQR = 90°$, $PQ = (3a + 2)$ cm, $QR = (a + 1)$ cm and $RS = (2a + 1)$ cm.
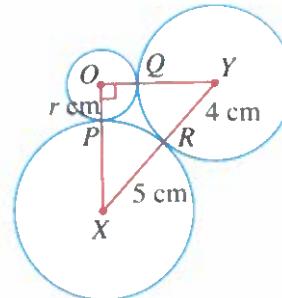
   (i) Find an expression, in terms of a, for the area of the trapezium $PQRS$.
   (ii) Given that the area of the trapezium is 9 cm², form an equation in a and show that it reduces to $5a^2 + 8a - 15 = 0$.

   (iii) Solve the equation $5a^2 + 8a - 15 = 0$, giving both your answers correct to 2 decimal places.
   (iv) Hence, find the length of PQ.

9. A water tank can be filled with water by two pipes in 1 hour and 20 minutes. If the smaller pipe takes 2 hours longer than the larger pipe to fill the tank, find the time taken by each pipe to fill the tank.
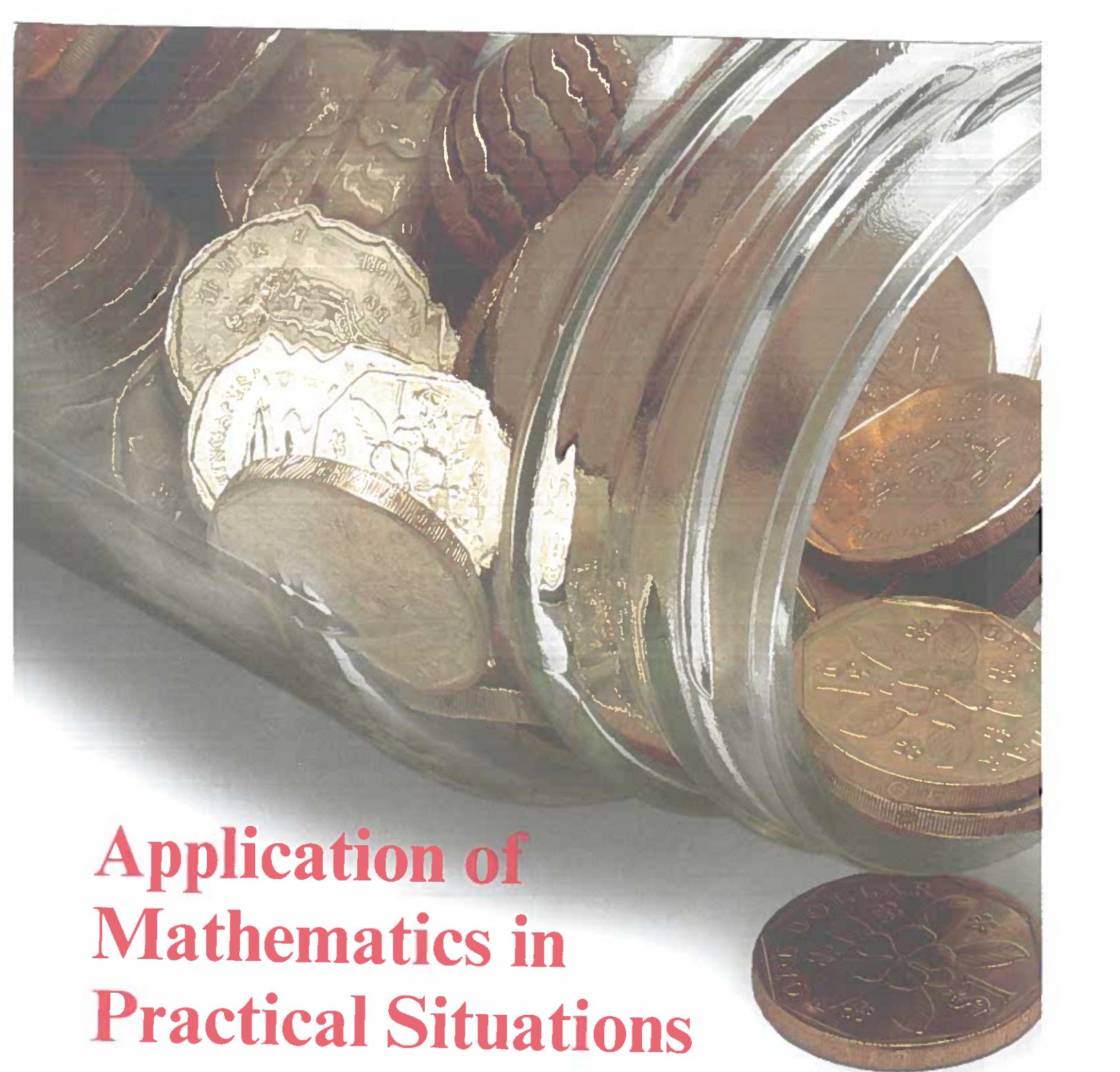
10. The figure shows three circles with centres at O, X and Y. P, Q and R are the points of contact.

   Given that $XR = 5$ cm, $YR = 4$ cm, $OP = r$ cm and $\angle XOY = 90°$,
   (i) write down an equation in r and show that it reduces to $r^2 + 9r - 20 = 0$,
   (ii) find the length of OX and of OY.

11. Michael cycles from A to B, covering a total distance of 50 km. For the first 40 km of his journey, his average speed is x km/h but for the last 10 km, his average speed is 5 km/h less. The total journey takes 2 hours and 40 minutes. Form an equation in x and solve it to find his average speed for the first 40 km of the journey.

Application of Mathematics in Practical Situations

Mathematics can be applied to various practical situations such as computing interest and taxes. Can you think of other applications of Mathematics in practical situations?

Chapter
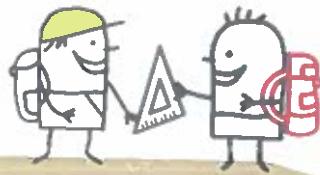
Five

5.1 Profit and Loss

Profit and Loss

A manufacturer produces goods at a certain cost. If the goods are sold at a price *higher* than the cost price, the manufacturer makes a **profit** or **gain**. However, if the manufacturer sells the goods at a price *lower* than the cost price, he suffers a **loss**. We have:

$$\text{Profit} = \text{selling price} - \text{cost price}$$
$$\text{Loss} = \text{cost price} - \text{selling price}$$

We usually express the profit or loss as a percentage of the cost price:

$$\frac{\text{Profit}}{\text{Cost Price}} \times 100\% \quad \text{or} \quad \frac{\text{Loss}}{\text{Cost Price}} \times 100\%$$

However, in the real world, there are cases where the profit or loss is expressed as a percentage of the selling price.

Worked Example 1

(Profit/Loss as a Percentage of Cost Price/Selling Price)

A bag costs $28.

(a) If the bag is sold for $35, express the profit as a percentage of the cost price.

(b) If the bag is sold for $24.50, express the loss as a percentage of the selling price.

Solution:

(a) Profit = selling price – cost price
   = $35 – $28
   = $7

   Required percentage = $\frac{\$7}{\$28} \times 100\%$

   = 25%

(b) Loss = cost price – selling price
   = $28 – $24.50
   = $3.50

   Required percentage = $\frac{\$3.50}{\$24.50} \times 100\%$

   = $14\frac{2}{7}\%$

PRACTISE NOW 1

1. (a) A bicycle which costs $180 is sold for $240. Express the profit as a percentage of the cost price.

   (b) An antique chest which costs $6000 is sold for $5000. Express the loss as a percentage of the selling price.

2. (a) A gold chain which costs $500 is sold at a profit of 27% on the cost price. Find its selling price.

   (b) A car which costs $78 400 is sold at a loss of 6% on the cost price. Find its selling price.

Worked Example 2

(Finding Cost Price)

A bookseller gains 30% on the cost price by selling a book for $65. Calculate the cost price of the book.

Solution:

130% of the cost price = $65

1% of the cost price = $\dfrac{\$65}{130}$

100% of the cost price = $\dfrac{\$65}{130} \times 100$

$= \$50$

The cost price of the book is $50.

PRACTISE NOW 2

1. A shopkeeper gains 35% on the cost price by selling a smartphone for $1282.50. Find the cost price of the smartphone.

2. By selling a book for $16.50, Vishal loses 12% on the cost price. Find the cost price of the book.

Worked Example 3

(Finding Selling Price per Unit of Viable Stock)

Farhan buys 400 kg of bananas at $0.75 per kg. 10% of the bananas are rotten. Find the selling price per kg if he wants to earn a 80% profit on the cost price.

Solution:

Cost price of 400 kg of bananas = $0.75 × 400

$= \$300$

Total selling price of the bananas so as to earn a 80% profit on the cost price

$= \dfrac{180}{100} \times \$300 = \$540$

Mass of bananas that Farhan can sell $= \dfrac{90}{100} \times 400$

$= 360$ kg

Selling price per kg $= \dfrac{\$540}{360}$

$= \$1.50$

PRACTISE NOW 3

A shopkeeper buys 1800 eggs at $1.20 per dozen. 5% of the eggs are rotten. Find the selling price of each egg if he wants to earn a 33% profit on the cost price.

1. Complete the table.
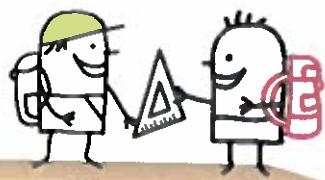
	Cost Price	Selling Price	Profit/ loss	Profit/loss as a percentage of the cost price
(a)	$40	$45		
(b)	$600	$480		
(c)	$88 000			Profit of 4%
(d)	$5680			Loss of 22.5%
(e)		$28.14		Profit of $17\frac{1}{4}$%
(f)		$506.85		Loss of 7%

2. The profit on a refrigerator is 35% of the cost price. If the profit is $280, find
   (i) the cost price,　　(ii) the selling price,
   of the refrigerator.

3. A trader mixes 2 kg of butter which costs $8 per kg with 3 kg of butter which costs $6 per kg. He sells the mixture at $2.55 per 250 g. Express his profit as a percentage of his selling price.

4. A florist buys roses at $18 per dozen. If she sells them at $1.20 each, express her loss as a percentage of her selling price.

5. Shirley buys a fax machine and sells it to Devi at a gain of 25% on the cost price. Devi then sells the fax machine to Amirah at a loss of 25% on the price at which she buys it from Shirley. If Amirah pays $360 for the fax machine, how much did Shirley pay for it?

6. Raj buys 200 boxes of apples at $28 per box. There are 60 apples in each box. 15% of the apples are rotten. Find the selling price per apple if he wants to earn an 80% profit on the cost price.

7. A shopkeeper buys 300 identical articles at a total cost of $1500. He sells 260 articles at a price that is 20% above the cost price. Each of the remaining articles is sold at a price that is 50% of the selling price of each of the 260 articles. Express the shopkeeper's profit as a percentage of his cost price.

5.2 Discount, Taxation and Commission

Discount

Very often, retailers are not able to sell defective merchandise, overstocked items or discontinued models at retail selling prices. To clear the merchandise in stock, the items are usually sold at a lower price, called the **sale price**. The difference between the original selling price, also known as the **marked price**, and the sale price is called the **discount**:

$$\boxed{\text{Discount} = \text{marked price} - \text{sale price}}$$

The discount is often given as a percentage of the marked price:

$$\boxed{\frac{\text{Discount}}{\text{Marked price}} \times 100\%}$$

Worked Example 4

A watch priced at $160 is sold for $100. Find the percentage discount.

Solution:

Discount = marked price − sale price

$$= \$160 - \$100$$
$$= \$60$$

Percentage discount $= \dfrac{\$60}{\$160} \times 100\%$

$$= 37.5\%$$

PRACTISE NOW 4

1. A scarf priced at $100 is sold for $88. Find the percentage discount.
2. The marked price of a washing machine is $600. A discount of 6% is given during a sale. Find the sale price of the washing machine.

Worked Example 5

(Finding Marked Price)
A sculpture is sold for $533 after a discount of 18%.
(i) Calculate the marked price of the sculpture.
(ii) If a 10% discount is given on the marked price of the sculpture before it is sold at a further discount of 8%, would the sale price still be $533? Show your working clearly.

Solution:

(i) 82% of the marked price = $533

1% of the marked price $= \dfrac{\$533}{82}$

100% of the marked price $= \dfrac{\$533}{82} \times 100$

$$= \$650$$

The marked price of the sculpture is $650.

(ii) Sale price of sculpture after a 10% discount $= \dfrac{90}{100} \times \$650$

$$= \$585$$

Sale price of sculpture after a further discount of 8% $= \dfrac{92}{100} \times \$585$

$$= \$538.20$$

No, the sale price would not be $533.

A laptop is sold for $1274 after a discount of 9%.

(i) Find the marked price of the laptop.

(ii) If a 5% discount is given on the marked price of the laptop before it is sold at a further discount of 4%, would the sale price still be $1274? Show your working clearly.

⠿ Goods and Services Tax

The various types of tax collected by the Singapore government go towards the funding of government expenditure such as national defence and education. Examples of taxes include **goods and services tax (GST)** and income tax.

GST is paid in addition to the price of goods and services. GST is usually expressed as a certain percentage of the selling price. GST came into effect in Singapore in 1994, at 3%. It was increased to 4% in 2003, to 5% in 2004 and to 7% in 2007.

GST is also known as value-added tax (VAT) in many other countries.

INFORMATION

The second largest contributor to the Singapore government's tax collection in the financial year from 2009 to 2010 is GST, at $6.9 billion.

Internet
Resources

One of the rationales of GST is for the funding of government expenditures. Search on the Internet to find out other rationales, as well as the advantages and disadvantages of GST. What has the Singapore government done to cushion the effects of GST for lower income groups?

Worked Example 6

(Finding Selling Price inclusive of GST)
A piece of furniture costs $640 before GST. Assuming that GST is at 7%, calculate the total amount of money Raj has to pay for the piece of furniture.

Solution:

GST payable $= \frac{7}{100} \times \$640$

$= \$44.80$

Total amount of money Raj has to pay for the furniture $= \$640 + \$44.80$

$= \$684.80$

1. An article costs $85 before GST. Assuming that GST is at 7%, find the total amount of money a man has to pay for the article.

2. A printer is sold for $642 inclusive of 7% GST. Find the marked price of the printer.

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Investigation

Discount, Service Charge and GST

In this investigation, we shall examine a bill to find out how discount, service charge and GST are calculated in real life.

Consider a restaurant which offers a 20% discount. Kate and Michael receive a bill after a dinner at the restaurant.

1. Kate claims that the GST should be $1.01. Show clearly how Kate calculates the GST and how the restaurant calculates the GST.

2. How would you explain to Kate that the method used by the restaurant in calculating GST is correct?

3. Michael is unhappy the discount is given before service charge and GST. He believes that the total bill will be less if the discount is given after service charge and GST. Kate says that it makes no difference either way. By showing your working clearly, explain why Kate is correct.

4. Michael is not convinced that his argument is wrong. How would you explain to him?

Yummy Restaurant	
Fish and Chips:	$ 8.50
Chicken Chop:	$ 9.50
Subtotal:	$18.00
Discount:	− $ 3.60
Subtotal:	$14.40
Service Charge 10%:	$ 1.44
GST 7%:	$ 1.11
Total:	**$16.95**

From the investigation, we can conclude that:

> Total amount payable = marked price − discount + service charge + GST payable,

where

> Service charge = service charge (in %) × (marked price − discount)

and

> GST payable = GST (in %) × (marked price − discount + service charge).

Another important conclusion is that it does not matter whether the discount (in %) is given before or after service charge and GST. However, in Singapore, discounts (in %) are not given on GST.

Worked Example 7

Mr Wong has dinner with his family at a restaurant which offers a 10% discount. The marked price of the food that they order is $100. Given that there is a service charge of 10% and GST is at 7%, calculate the total amount of money he has to pay.

Solution:

Discount $= \dfrac{10}{100} \times \$100$

$\qquad = \$10$

Service charge $= 10\% \times$ (marked price $-$ discount)

$\qquad\qquad\quad = \dfrac{10}{100} \times (\$100 - \$10)$

$\qquad\qquad\quad = \dfrac{10}{100} \times \$90$

$\qquad\qquad\quad = \$9$

GST payable $= 7\% \times$ (marked price $-$ discount $+$ service charge)

$\qquad\qquad\quad = \dfrac{7}{100} \times (\$100 - \$10 + \$9)$

$\qquad\qquad\quad = \dfrac{7}{100} \times \$99$

$\qquad\qquad\quad = \$6.93$

Total amount payable $=$ marked price $-$ discount $+$ service charge $+$ GST payable

$\qquad\qquad\qquad\qquad = \$100 - \$10 + \$9 + \$6.93$

$\qquad\qquad\qquad\qquad = \$105.93$

PRACTISE NOW 7

SIMILAR QUESTIONS

1. Nora orders a bowl of Tom Yam soup at a restaurant which offers a 15% discount. The marked price of the Tom Yam soup is $6.90. Given that there is a service charge of 10% and GST is at 7%, find the total amount of money she has to pay.

2. Khairul orders one set meal at a restaurant which offers a 20% discount. There is a service charge of 10% and GST is at 7%. Given that he pays a total of $23.54, find the marked price of the set meal.

Investigation

Percentage Point

In this investigation, we shall explore the idea of percentage point and distinguish 'percentage points' from 'percent'. Read the article and answer the questions.

NEWS

GST to Rise to 7% from 1 July 2007

SINGAPORE: On 15 February 2007, the Singapore government released a statement to announce that GST would be increased by two percentage points from 5% to 7% with effect from 1 July 2007. After the increase in GST, government revenues are expected to be raised by $750 million that year and $1.5 billion per year from then onwards.

---

1. Jun Wei says that the increase from 5% to 7% is an increase of 40% and not 2%. How did he arrive at 40%?

2. Do you think Jun Wei is right in saying that the increase is 40%? Explain your answer.

3. Do you notice that the article reported that the increase from 5% to 7% is two percentage points? What does 'two percentage points' mean and how is it different from 'two percent'? Is the distinction between 'two percentage points' and 'two percent' important? Explain your answer.

From the investigation, we note that **percentage point** is defined as the *difference* between two percentages. For example, the increase from 5% to 7% is two percentage points, and not two percent.

Income Tax

Income tax is charged on all incomes derived from Singapore or received in Singapore from sources outside Singapore during the year starting from 1 January and ending on 31 December. The income tax payable is calculated based on the **chargeable income**:

Chargeable income = Total income − Reliefs

The reliefs include personal relief, wife relief, child relief, life insurance premiums, contributions to the Central Provident Fund (CPF) and gifts to charitable organizations in the form of cash, etc. The amount of income tax payable is calculated according to the tax rates. Table 5.1 shows an extract of the tax rates from Explanatory Notes on how to prepare the returns sent to tax payers in 2017 by the Inland Revenue Department.

Chargeable Income	Income Tax Rate (%)	Gross Tax Payable ($)
First $20 000	0	0
Next $10 000	2	200
First $30 000	-	200
Next $10 000	3.50	350
First $40 000	-	550
Next $40 000	7	2 800
First $80 000	-	3 350
Next $40 000	11.5	4 600
First $120 000	-	7 950
Next $40 000	15	6 000
First $160 000	-	13 950
Next $40 000	18	7 200
First $200 000	-	21,150
Next $40 000	19	7 600
First $240 000	-	28 750
Next $40 000	19.5	7 800
First $280 000	-	36 550
Next $40 000	20	8 000
First $320 000	-	44 550
In excess of $320 000	22	

Table 5.1

Worked Example 8

(Finding Tax Payable)

Calculate the tax payable for a chargeable income of $38 500 according to the tax rates given in Table 5.1.

Solution:

On the first $30 000, tax payable = $200
On the next $8500, tax payable = 3.50% of $8500
$$= \$297.50$$

∴ Total tax payable = $497.50

PRACTISE NOW 8

Calculate the tax payable for a chargeable income of $42 000 according to the tax rates given in Table 5.1.

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Worked Example 9

(Income Tax)

In a particular year, Mr Lee earned a gross annual income of $84 000. Of this $84 000, the amount that will not be subjected to income tax is shown in the following table:

Personal relief	$3000
Wife relief	$2000
Child relief	$4000 per child
Central Provident Fund (CPF) contributions	$16 800
Donations	$940

Given that he has 3 children and that for the remaining income that will be taxed, the gross tax payable for the first $40 000 is $550 and the tax rate for the rest is 7%, find his income tax payable.

Solution:

Total reliefs = $3000 + $2000 + 3($4000) + $16 800 + $940

$\qquad$ = $34 740

Taxable income = $84 000 − $34 740

$\qquad$ = $49 260

$$\text{Tax}$$

$49 260 →
- First $40 000 : $550
- Next $9260 at 7% : $648.20

∴ Income tax payable = $550 + $648.20

$\qquad$ = $1198.20

PRACTISE NOW 9

In a particular year, Mr Cheong earned a gross annual income of $284 000. Of this $284 000, the amount that will not be subjected to income tax is shown in the following table:

Personal relief	$3000
Child relief	$4000 per child
CPF contributions	$28 500
Donations	$3500

Given that he has 2 children and that for the remaining income that will be taxed, the gross tax payable for the first $200 000 is $20 750 and the tax rate for the rest is 18%, find his income tax payable.

⠿ Commission

A commission is the payment an agent receives for selling or buying something on behalf of another party. It is usually given as a percentage of the cost price or the selling price.

Worked Example 10 (Finding Commission)

A property agent sells a house for $320 000, of which he receives a commission of 1.5%. Calculate the amount of commission the agent receives.

Solution:

Amount of commission the agent receives $= \frac{1.5}{100} \times \$320\,000$

$$= \$4800$$

PRACTISE NOW 10

1. A property agent sells a house for $528 000, of which he receives a commission of 2%. Find the amount of commission the agent receives.

2. A property agent charges a commission of 3.5% on the selling price of a piece of property. Given that he receives a commission of $25 375, find the selling price of the property.

Exercise 5B

BASIC LEVEL

1. A necklace priced at $580 is sold for $464. Find the percentage discount.

2. The marked price of a folding table at a hypermarket is $45. The hypermarket gives a 12% discount during a sale. Find the sale price of the folding table.

3. During a sale, there is a discount of 7% on a television set. If the discount is $49, find

   (i) the marked price, (ii) the sale price,

   of the television set.

4. A microwave oven costs $270 before GST. Assuming that GST is at 7%, find the total amount of money Ethan has to pay for the microwave oven.

5. An electronic gadget is sold for $1391 inclusive of 7% GST. Find the marked price of the gadget.

6. Using Table 5.1, calculate the income tax payable given the following chargeable incomes.

   (a) $25 400   (b) $38 400

   (c) $72 000   (d) $105 000

7. A property agent charges a commission of 2.5% on the selling price of a house.

   (a) Given that the agent sells a house for $650 000, find the amount of commission he receives.

   (b) On another occasion, he receives a commission of $12 000. Find the selling price of the house.

8. An air conditioner is sold for $700 after a discount of 12.5%.

   (i) Find the marked price of the air conditioner.

   (ii) If a 10% discount is given on the marked price of the air conditioner before it is sold at a further discount of 2.5%, would the sale price still be $700? Show your working clearly.

9. Farhan orders one plate of seafood fried rice at a restaurant which offers a 25% discount. The marked price of the seafood fried rice is $9.50. Given that there is a service charge of 10% and GST is at 7%, find the total amount of money he has to pay.

10. In a particular year, Mr Goh earned a gross annual income of $185 000. Of this $185 000, the amount that will not be subjected to income tax is shown in the following table:

Personal relief	$3000
Child relief	$4000 per child
Parent relief	$5000 per parent
CPF contributions	$18 600

Given that he lives with 1 child and 2 parents, and that for the remaining income that will be taxed, the gross tax payable for the first $120 000 is $7950 and the tax rate for the rest is 15%, find his income tax payable.

11. **Property tax** is a tax on land, houses, flats or buildings, which is computed as follows:

   property tax payable yearly
   = annual value × tax rate,

   where the annual value of a property is the estimated amount of money the owner will get if he or she rents out the property for a year.

   Given that the annual value of a flat is $28 800 and the tax rate is 10% per annum, find the property tax payable for 6 months.

12. Lixin's monthly income consists of a basic salary of $500 and a commission of 4% on her sales for the month. If her income is $1220 for a particular month, find her sales for that month.
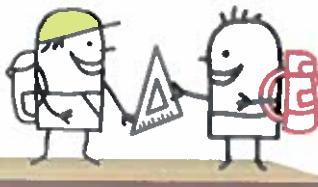
13. Khairul wants to buy a sofa set. He is offered 3 successive discounts of 10%, 20% and 25%, and he can arrange them in any order he wants. Which order will benefit him the most? Explain your answer.

14. Devi orders one bowl of ramen at a Japanese restaurant which offers an 18% discount. There is a service charge of 10% and GST is at 7%. Given that she pays a total of $10.13, find the marked price of the ramen.

15. Mr Neo paid an income tax of $1474 for a particular year. Of his gross annual income, the amount that will not be subjected to income tax is shown in the following table:

Personal relief	$3000
Wife relief	$2000
Child relief	$4000 per child
Parent relief	$5000 per parent
CPF contributions	$15 600
Donations	$200

Given that he lives with 4 children and 2 parents, and that for the remaining income that will be taxed, the gross tax payable for the first $40 000 is $550 and the tax rate for the rest is 7%, find his gross annual income.

5.3 Simple Interest and Compound Interest

In Book 1, we have learnt the concept of rate and how it is applied in the context of speed. There are other applications of rate in real life.

Class Discussion

Body Mass Index

The body mass index (BMI) is a real-life application of rate. The BMI of a person is obtained by dividing his or her mass, in kg, by the square of his or her height, in m, i.e.

$$BMI = \frac{Mass\ (kg)}{(Height\ (m))^2}.$$

ATTENTION

BMI is a rate because it compares two quantities of *different kinds*: 'mass' and 'square of height'.

Table 5.2 shows the different classifications of BMI in Singapore.

Category	BMI range
Emaciation	< 14.9
Underweight	15.0 – 18.4
Normal	18.5 – 22.9
Overweight	23.0 – 27.5
Obese	27.6 – 40.0
Morbidly Obese	> 40.0

Table 5.2

INFORMATION

The unit for BMI is kg/m² but to simplify it, an index is used, i.e. a number without any units.
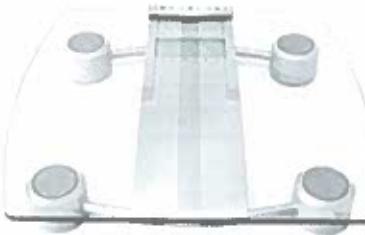
1. Using the formula for BMI given in Table 5.2, calculate your BMI. Which category do you belong to?
2. Find out how medical practitioners in Singapore make use of the BMI to assess patients' health risks.
3. Discuss with your classmates other real-life applications of rates.

Besides BMI, another common real-life application of rate include interest rates.

⠿ Simple Interest

When you deposit money into a bank, you receive interest for allowing the bank to use your money. Similarly, when you borrow money from the bank, you must pay interest for using its money. The interest in both cases is calculated as a percentage (called the **rate**) of the capital (called the **principal**) deposited or borrowed. Interest is sometimes calculated at a fixed yearly rate (called **rate per annum**).

Sometimes, interest rates are calculated on half-yearly or quarterly period, monthly or even daily basis. The amount of interest depends on the length of time the money is deposited or borrowed. If interest is always calculated on the original principal, it is called **simple interest**. When the interest is added to the principal, the sum is called the **amount**.

In general, the formula for finding simple interest is

$$I = \frac{PRT}{100},$$

where I is the amount of **interest**, P is the **principal**, $R\%$ is the **interest rate** per annum (or year) and T is the **number of years**.

Worked Example 11

(Simple Interest)
(a) Jun Wei borrows $1000 from a financial institution that charges simple interest at a rate of 4% per annum. Calculate the amount of interest he has to pay at the end of 3 years.
(b) Huixian invests $4000 in a savings scheme that pays simple interest at a rate of 2% per annum. Calculate the time taken for her investment to grow to $4400.

Solution:

(a) Amount of interest Jun Wei has to pay at the end of 1 year $= \$1000 \times \frac{4}{100}$
$$= \$40$$

Amount of interest Jun Wei has to pay at the end of 3 years $= \$40 \times 3$
$$= \$120$$

(b) Total amount of interest Huixian earns $= \$4400 - \$4000$
$$= \$400$$

Amount of interest Huixian earns per year $= \$4000 \times \frac{2}{100}$
$$= \$80$$

Time taken for her investment to grow to $4400 $= \frac{\$400}{\$80}$
$$= 5 \text{ years}$$

(a) A man takes a loan of $150 000 from a bank to start a business venture. The bank charges him simple interest at a rate of 5.5% per annum. If he plans to repay his loan at the end of 3 years, find the amount of interest he has to pay and hence, the total amount he owes the bank.

(b) Shirley invests $6000 in a savings plan that pays simple interest at a rate of 3% per annum. Find the time taken for her investment to grow to $6720.

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Performance Task

Many adults in Singapore have a credit card and some of them even have multiple credit cards.

(i)   If a cardholder fails to pay a credit card company by the due date, what is the interest rate charged by the company? Are there other related charges that may be incurred?

(ii) A cardholder is allowed to make a minimum monthly payment, instead of the full sum, by the due date. How is the minimum monthly amount usually calculated?

(iii) Given that a man charges $800 to his credit card every month for six months and only pays the minimum amount by the due date each month, find the amount of money he owes the credit card company at the end of the six months.

(iv) Is it advisable to only make the minimum payment every month instead of repaying the full sum owed? Explain your answer.

Present your findings to the class.

Compound Interest

In the previous section, we have learnt that the amount of simple interest one has to pay depends on the amount of money borrowed, i.e. the principal, the interest rate per annum and the loan period in years.

For simple interest, the interest earned every year is the same because it is calculated based on the original principal. Now what happens if the interest is **compounded** yearly? This means that after each year, the interest earned will be counted in the new principal to earn more interest.

Investigation

Simple Interest and Compound Interest

Mr Wong wants to place $1000 in a bank as a fixed deposit for 3 years.

Bank A offers a *simple* interest rate of 2% per annum.

1. Calculate the interest earned and the total amount of money he will have after 3 years.

Bank B offers an interest rate of 2% per annum *compounded* yearly.

2. Copy and complete the following to find the interest earned and the total amount of money he will have at the end of each year.

1st **year:** Principal $P_1 = \$1000$,

Interest $I_1 = \$1000 \times 2\%$

$= \$\underline{\hspace{1cm}}$

Total amount at the end of the 1st year, $A_1 = P_1 + I_1$

$= \$1000 + \$\underline{\hspace{1cm}}$

$= \$1020$

2ⁿᵈ year: Principal $P_2 = A_1 = \$1020$,

$$\text{Interest } I_2 = \$\underline{\hspace{2cm}} \times 2\%$$

$$= \$\underline{\hspace{1.5cm}}$$

Total amount at the end of the 2ⁿᵈ year, $A_2 = P_2 + I_2$

$$= \$1020 + \$\underline{\hspace{1.5cm}}$$

$$= \$\underline{\hspace{1.5cm}}$$

3ʳᵈ year: Principal $P_3 = A_2 = \$\underline{\hspace{1.5cm}}$,

$$\text{Interest } I_3 = \$\underline{\hspace{2cm}} \times 2\%$$

$$= \$\underline{\hspace{1.5cm}}$$

Total amount at the end of the 3ʳᵈ year, $A_3 = P_3 + I_3$

$$= \$1040.40 + \$\underline{\hspace{1.5cm}}$$

$$= \$\underline{\hspace{1.5cm}} \text{ (to the nearest cent)}$$

3. Which bank offers a higher interest and by how much?

From the investigation, finding the compound interest for each year is very tedious. What happens if Mr Wong puts \$1000 in Bank B for 10 years? How do we calculate the compound interest at the end of 10 years?

Hence, there is a need to find a formula to calculate compound interest easily.

From the investigation, we observe the following:

Total amount at the end of the 1ˢᵗ year, $A_1 = P_1 + I_1$

$$= \$1000 + \$1000 \times \frac{2}{100}$$

$$= \$1000\left(1 + \frac{2}{100}\right)$$

Total amount at the end of the 2ⁿᵈ year,

$A_2 = P_2 + I_2$

$$= \$1000\left(1 + \frac{2}{100}\right) + \$1000\left(1 + \frac{2}{100}\right) \times \frac{2}{100}$$

$$= \$1000\left(1 + \frac{2}{100}\right)\left(1 + \frac{2}{100}\right) \quad \text{(Extract common factor } 1000\left(1 + \frac{2}{100}\right)\text{)}$$

$$= \$1000\left(1 + \frac{2}{100}\right)^2$$

Total amount at the end of the 3ʳᵈ year,

$A_3 = P_3 + I_3$

$$= \$1000\left(1 + \frac{2}{100}\right)^2 + \$1000\left(1 + \frac{2}{100}\right)^2 \times \frac{2}{100}$$

$$= \$1000\left(1 + \frac{2}{100}\right)^2\left(1 + \frac{2}{100}\right) \quad \text{(Extract common factor } 1000\left(1 + \frac{2}{100}\right)^2\text{)}$$

$$= \$1000\left(1 + \frac{2}{100}\right)^3$$

By looking at the pattern above, what do you think the total amount of money at the end of 4th year will be? $_____

In general, the formula for finding **compound interest** is

$$A = P\left(1 + \frac{R}{100}\right)^n,$$

where A is the **total amount**, P is the principal, $R\%$ is the **interest rate per annum** (or year) and n is the **number of years**.

Note that if the interest is **compounded monthly**, then $R\%$ is the **interest rate per month** and n is the **number of months**.

Worked Example 12   (Compound Interest)

Find the compound interest on $5000 for 7 years at 3% per annum, compounded annually.

Solution:

$P = \$5000,\ R = 3,\ n = 7$

At the end of 7 years, total amount accumulated is

$$A = P\left(1 + \frac{R}{100}\right)^n$$

$$= \$5000\left(1 + \frac{3}{100}\right)^7$$

$$= \$6149.37 \text{ (to the nearest cent)}$$

$\therefore$ Compound interest $I = A - P$

$$= \$6149.37 - \$5000$$

$$= \$1149.37$$

Problem Solving Tip

Unlike the simple interest formula where the interest I can be found directly, the compound interest formula does not allow you to find I directly.

PRACTISE NOW 12

1. Find the compound interest on $3000 for 4 years at 5% per annum, compounded annually.
2. Find the compound interest on $1500 for 2 years at 2% per annum, compounded **(a)** annually, **(b)** monthly.
3. Mrs Lee places $4000 in a bank as a fixed deposit for 2 years. The bank offers an interest compounded yearly. At the end of 2 years, she receives a total of $4243.60. Find the interest rate.

SIMILAR QUESTIONS

Exercise 5C Questions 9–11, 17–22

Problem Solving Tip

For Question 2, find the interest rate per month first.

Exercise 5C

1. Copy and complete the following table:

	Principal	Interest rate	Time	Simple Interest	Amount
(a)	$12 000	8%	7 years		
(b)	$500	11%		$220	
(c)		9%	4 years	$108	
(d)	$3000		10 years	$1200	
(e)			2 years	$360	$3960
(f)	$1800		18 months	$189	
(g)	$4500		2 years		$5040
(h)		5%		$90	$1290

2. A finance company charges $55 simple interest on a sum of money which is borrowed for five months. Given that the rate of interest is 12% per annum, find the sum of money.

3. A bank charges 2.25% per month simple interest on personal loans. If John borrows $6400 for a period of 2 years 1 month, find the total interest he has to pay.

4. Nora invests $800 that pays simple interest at a rate of 6% per annum and $1200 that pays simple interest at a rate of 7% per annum. What is her total annual interest on these two investments?

5. How long would $1250 have to be deposited at 6% per annum to gain a simple interest of $750?

6. Michael lent Farhan $4800 for 7 months. At the end of this period, Farhan had to pay Michael an interest of $119. What was the simple interest rate per annum?

7. Rui Feng borrows $48 000 from a bank that charges simple interest at a rate of 6% per annum. Find the total amount of money he has to pay the bank at the end of 2 years.

8. A man invests $16 800 in a savings plan that pays simple interest at a rate of 5% per annum. Find the time taken for his investment to grow to $18 900.

9. Kate places $5000 in her bank account. The bank offers an interest of 8% per annum compounded yearly. Find the total interest in her account at the end of 3 years.

10. Find the compound interest on
    (a) $450 for 2 years at 10% per annum compounded yearly,
    (b) $700 for 3 years at 11% per annum compounded yearly,
    (c) $5000 for 2 years at $11\frac{3}{4}$% per annum compounded yearly,
    (d) $1200 for 3 years at 4% per annum compounded yearly,
    (e) $10 000 for 2 years at $7\frac{1}{2}$% per annum compounded yearly

11. Wilson invests $5000 at $5\frac{1}{4}$% per annum compound interest compounded annually. Find the amount at the end of the third year.

12. In a certain year, James put $600 in a bank at the end of March and $400 in the same bank at the end of June. The bank offers 3% per annum simple interest rate. Find the total amount that James receives from the bank at the end of December in that year.

13. A bank increased the rate of simple interest, which it paid to depositors, from 3.5% to 4% annum. Find how much more interest Susan would receive if she deposited $6400 in the bank for 6 months at the new interest rate.

14. Jasmine invested $4000 in a Building Society which paid simple interest at a rate of $7\frac{1}{4}$% per annum to its investors. After 2 years, the rate was increased to 7.6% per annum. Find the amount she had at the end of 7 years.

15. Daniel deposits a certain sum of money in a bank. If the simple interest rate of the bank decreases from $3\frac{3}{4}$% per annum to $3\frac{1}{2}$% per annum, Daniel's interest will increase by $50 in a year. Find the sum of money he deposits.

16. Raj deposits $20 000 in a bank that pays simple interest at a rate of 2.75% per annum. If the interest rate decreases to x% per annum, he will receive $50 less every year. Find the value of x.

17. Rui Feng deposited $15 000 in an account that pays 5.68% compound interest per year. Find the total amount in the account after 6 years if the interest is compounded
    (a) monthly,   (b)   half-yearly.

18. Mr Tan invested $5000 in an endowment fund for 5 years. The fund pays an interest compounded yearly. At the end of 5 years, he received a total of $5800. Find the interest rate.

19. Mr Chua borrows a sum of money from the bank which charges a compound interest of 4.2% per annum, compounded quarterly. Given that Mr Chua had to pay $96.60 in interest payments at the end of the first year, find the original sum of money borrowed, giving your answer correct to the nearest cent.

20. Joan invests $800 at $12\frac{1}{2}$% per annum compound interest compounded monthly. What is the amount of interest at the end of the first year?

21. Kate invests $9000 at 2% per annum compound interest compounded daily. What is this amount at the end of the third day? (1 year = 365 days)

22. Kate has $8000 to invest in either Company A or Company B.

    Company A: 4.9% per annum simple interest
    Company B: 4.8% per annum compound interest, compounded half-yearly

    Kate wishes to invest the money for a period of 4 years.
    (i) Which company should she invest in? Explain your answer.
    (ii) Calculate the difference in interest earned after 4 years.

5.4 Hire Purchase
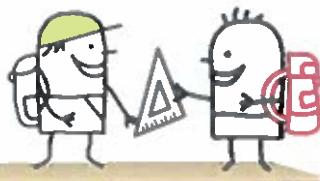
Fig. 5.1 shows an advertisement for the sale of a washing machine.

For Sale: $4500
$1000 deposit
Balance 7%, 2 years

Fig. 5.1

The deposit of $1000 is called the **downpayment**. In this method of purchase, the purchaser can bring home the washing machine after paying the downpayment of $1000. The balance of $3500 will accrue interest at the rate of 7% per annum and the sum will be paid off with monthly payments for 2 years. Each monthly payment is known as an **instalment**.

This method of purchasing goods in which payment of purchase price is spread over a specific period by payment of an initial deposit followed by regular instalments is called **hire purchase**.

Worked Example 13

(Hire Purchase)

A washing machine is priced at $450. A man buys the washing machine on hire purchase according to the following terms: a downpayment of 15% and the remaining to be paid in monthly instalments over 2 years at a simple interest rate of $10\frac{2}{3}\%$ per annum. Calculate

(i) his monthly instalment,

(ii) the total amount the man pays for the washing machine.

Solution:

(i) Downpayment $= \frac{15}{100} \times \$450$

$\qquad\qquad\qquad = \$67.50$

Remaining amount $= \$450 - \$67.50$

$\qquad\qquad\qquad\quad = \$382.50$

Amount of interest the man owes at the end of 1 year $= \$382.50 \times \frac{10\frac{2}{3}}{100}$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = \$40.80$

Amount of interest the man has to pay at the end of 2 years $= \$40.80 \times 2$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = \$81.60$

Total amount to be paid in monthly instalments $= \$382.50 + \$81.60$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad = \$464.10$

Monthly instalment $= \frac{\$464.10}{24}$ (2 years = 24 months)

$\qquad\qquad\qquad\quad = \$19.34$ (to the nearest cent)

(ii) Total amount the man pays for the washing machine $= \$67.50 + \$464.10$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = \$531.60$

PRACTISE NOW 13

An air conditioner is priced at $900. Amirah buys the air conditioner on hire purchase according to the following terms: a downpayment of 20% and the remaining to be paid in monthly instalments over 4 years at a simple interest rate of 10% per annum.
(i) Find her monthly instalment.
(ii) Find the total amount Amirah pays for the air conditioner.
(iii) How much more does she have to pay for buying the air conditioner on hire purchase?

Exercise 5D

1. For each of the following,

   (i) find the additional amount you have to pay by hire purchase,

   (ii) express the additional amount obtained in (i) as a percentage of the cash price.

	Cash Price	Hire Purchase		
		Deposit	Monthly Instalment	Number of instalments
(a)	$360	$50	$40	10
(b)	$900	$150	$75	12
(c)	$25 000	$10 000	$500	36

2. For each of the following,

   (i) find the hire purchase price of the goods,

   (ii) express the amount saved by paying cash as a percentage of the cash price.

Item	Cash Price	Deposit	Monthly Instalment	Number of instalments
(a) VCD player	$200	10%	$9	24
(b) Printer	$350	15%	$18	20
(c) Sofa	$1600	25%	$52	30

3. For each of the following, find

   (i) the monthly instalment,

   (ii) the difference in the hire purchase price and the cash price as a percentage of the cash price.

	Cash Price	Hire Purchase terms
(a)	$800	$100 deposit, simple interest of 8%, 1 year
(b)	$8000	$3200 deposit, simple interest of 10%, 2 years
(c)	$1200	$200 deposit, simple interest of 15%, 1 year

4. Answer the following questions by referring to the advertisement.
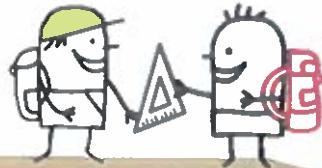
   Laptop
   was $2198
   now $1798
   or $55 monthly × 38
   No deposit

   (a) Find the percentage discount for payment in cash compared to the original price, giving your answer correct to one decimal place.

   (b) What is the difference between the hire purchase price and the original price?

   (c) What is the rate of simple interest charged for hire purchase? Give your answer correct to three significant figures.

5. A computer system is priced at $3200. A man buys the computer system on hire purchase according to the following terms: a downpayment of 15% and the remaining to be paid in monthly instalments over 2 years at a simple interest rate of 9.5% per annum.

   (i) Find his monthly instalment.

   (ii) Find the total amount the man pays for the computer system.

   (iii) How much more does he have to pay for buying the computer system on hire purchase?

6. The price of a sofa is x. A man buys the sofa on hire purchase according to the following terms: a downpayment of 25% and the remaining to be paid in monthly instalments over 30 months at a simple interest rate of 12% per annum. Given that his monthly instalment is $52, find the value of x.

5.5 Money Exchange

Different countries use different forms of currency and their units of money are called by various names. The United Kingdom uses the Sterling pound (£), the United States of America uses the American dollar (US$), Thailand uses the Thai baht (B), Malaysia uses the Malaysia ringgit (RM), Indonesia uses the Indonesia rupiah (Rp), Singapore uses the Singapore dollar (S$) and Pakistan uses the Pakistan rupee (PKR). We can buy or sell foreign currencies at any bank or through a money changer.

Worked Example 14

(Currency Exchange)

(a) The rates of exchange between Australian dollars (A$), Malaysian ringgit (RM) and Singapore dollars (S$) are A$1 = S$1.0318 and RM100 = S$32.368. Convert
   (i) A$543,   (ii) RM4850,
into Singapore dollars, giving your answers correct to the nearest cent.

(b) The rates of exchange between Sterling pounds (£), Japanese yen (¥) and Singapore dollars (S$) are £1 = S$1.7720 and ¥100 = S$1.2484. Convert
   (i) S$6800 into Sterling pounds (£),
   (ii) S$8450 into Japanese yen (¥),
giving your answers correct to the nearest unit of foreign currency.

Solution:

(a) (i) A$1 = S$1.0318

      A$543 = S$1.0318 × 543 = S$560.27 (to the nearest cent)

   (ii) RM100 = S$32.368

      $RM1 = S\$\dfrac{32.368}{100}$

      $RM4850 = S\$\dfrac{32.368}{100} \times 4850 = S\$1569.85$ (to the nearest cent)

(b) (i) S$1.7720 = £1

      $S\$1 = £\dfrac{1}{1.7720}$

      $S\$6800 = £\dfrac{1}{1.7720} \times 6800 = £3837$ (to the nearest £)

   (ii) S$1.2484 = ¥100

      $S\$1 = ¥\dfrac{100}{1.2484}$

      $S\$8450 = ¥\dfrac{100}{1.2484} \times 8450 = ¥676\ 866$ (to the nearest ¥)

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1. (a) The rates of exchange between New Zealand dollars (NZ$), Philippines pesos (P) and Singapore dollars (S$) are NZ$1 = S$0.9451 and P100 = S$2.8542. Convert

    (i) NZ$2360,

    (ii) P25 600,

    into Singapore dollars, giving your answer correct to the nearest cent.

    (b) The rates of exchange between euros (€), Thai baht (baht) and Singapore dollars (S$) are €1 = S$1.7624 and 100 baht = S$4.2136. Convert

    (i) S$5690 into euros,

    (ii) S$7460 into Thai baht,

    giving your answers correct to the nearest unit of foreign currency.

2. A family travels from Hong Kong to Singapore for a holiday. They exchange HK$35 000 to Singapore dollars at an exchange rate of HK$100 = S$16.235. They spend a total of S$3500 in Singapore and convert the remaining Singapore dollars into Hong Kong dollars at the end of the trip at an exchange rate of HK$100 = S$16.242. Find the amount of Hong Kong dollars they receive, giving your answer correct to the nearest dollar.

Journal Writing

Five of your friends and their families are planning a holiday to the United States of America, Australia, New Zealand, Malaysia and Thailand.

1. Find out the exchange rates offered by three money changers on the same day.

2. Compare the rates offered and suggest the outlet which offers the best buying rate and selling rate respectively.
    Hint: You are required to define what is meant by 'buying rate' and 'selling rate'.

3. If an outlet offers the best buying rate, does it mean that it will offer the best selling rate?

Exercise 5E

1. A money exchanger exchanged Thai baht (B) and US dollars (US$) at a rate of 34.77B = US$1.

   (a) Calculate, in Thai baht, the amount received for US$150.

   (b) Calculate, in US$, the amount received for 1617B.

2. The exchange rate between the Sterling pound (£) and the euro (€) during a particular day was £1 to €1.19.

   (a) How many Euros would be equivalent to £320?

   (b) How many pounds would be equivalent to €956?

3. Roger put S$8500 in a 1-year US$ fixed deposit account with a bank at 4% simple interest per annum when the exchange rate was at US$1 = S$1.70.

   (a) How much did he invest in US dollars?

   (b) At the end of one year, he withdrew all his money when the exchange rate was US$1 = S$1.65. Calculate the number of Singapore dollars he made from this investment.

4. The rates of exchange between American dollars (US$), Indonesian rupiah (Rp) and Singapore dollars (S$) are US$1 = S$1.4103 and Rp 100 = S$0.0111. Convert

   (a) US$765,   (b)   Rp 2 560 000,

   into Singapore dollars, giving your answers correct to the nearest cent.

5. The rates of exchange between Australian dollars (A$), Philippines pesos (P) and Singapore dollars (S$) are A$1 = S$1.0373 and P100 = S$2.8250. Convert

   (a) S$350 into Australian dollars,

   (b) S$5670 into Philippine pesos,

   giving your answers correct to the nearest unit of foreign currency.

6. A couple travels from New Zealand to Singapore for a holiday to celebrate their Silver Jubilee wedding anniversary. They exchange NZ$3200 to Singapore dollars at a rate of NZ$100 = S$94.85. They spend a total of S$2560.20 in Singapore and convert the remaining Singapore dollars into New Zealand dollars at the end of the trip at a rate of NZ$100 = S$97.65. Find the amount of New Zealand dollars they receive, giving your answer correct to the nearest cent.

1. **Profit** = selling price – cost price

   Loss = cost price – selling price

   To express the profit or loss as a percentage of the cost price:

   $$\frac{\text{Profit}}{\text{Cost Price}} \times 100\% \quad \text{or} \quad \frac{\text{Loss}}{\text{Cost Price}} \times 100\%$$

2. **Discount = marked price – sale price**

   Percentage discount $= \dfrac{\text{Discount}}{\text{Marked price}} \times 100\%$.

3. In Singapore, **goods and services tax (GST)** is paid in addition to the price of goods and services.

4. **Percentage point** is defined as the *difference* between two percentages.

5. **Chargeable income = Total income – Reliefs**

6. A **commission** is the payment an agent receives for selling or buying something on behalf of another party.

7. **Simple Interest Formula**

   $$I = \frac{PRT}{100},$$

   where I is the amount of interest, P is the principal, $R\%$ is the interest per annum (or year) and T is the number of years.

8. **Compound Interest Formula**

   $$A = P\left(1 + \frac{R}{100}\right)^n,$$

   where A is the total amount, P is the principal, $R\%$ is the interest rate per annum (or year) and n is the number of years.

9. **Hire purchase** is the method of purchasing goods in which payment of purchase price is spread over a specific period by payment of an initial deposit followed by regular instalments.

Review Exercise 5

1. A man buys a dozen cameras for $1800. He sells them at a profit of $36 each. Express his profit as a percentage of his selling price.

2. A shopkeeper sells two items at $48 each. He gains 25% on the cost price for one item and loses 20% on the cost price for the other item. Express his net profit or loss as a percentage of the total cost price of the two items.

3. A car priced at $60 000 is sold for $57 000. Find the percentage discount.

4. The marked price of a textbook is $20. A school is given a 6.5% discount for buying the textbooks in bulk. Find the amount of money the school has to pay for buying 200 textbooks.

5. (a) A retailer buys some bread toasters from a manufacturer at $20 each. In addition to that, he needs to pay a value-added tax of 15%. If he sells a bread toaster to a customer for $26, find the amount of profit he makes.

   (b) The manufacturer later increases the price of each bread toaster by 20%. At the same time, the value-added tax is increased to 25%.

      (i) If the retailer wishes to make the same amount of profit as before, find the price a customer has to pay for a bread toaster.

      (ii) If the retailer sells a bread toaster at a price that is 30% more than the selling price in (b)(i), find the amount of profit he makes on 25 such bread toasters.

6. Nora orders one bowl of noodles at a restaurant which offers a 10% discount. There is a service charge of 10% and GST is at 7%. Given that she pays a total of $7.94, find the marked price of the noodles.

7. In a particular year, Ms Tan earned a gross annual income of $80 000. Of this $80 000, the amount that will not be subjected to income tax is shown in the following table:

Personal relief	$3000
Parent relief	$5000 per parent
CPF contributions	$16 000
Donations	$750

Given that she lives with 2 parents and that for the remaining income that will be taxed, the gross tax payable for the first $40 000 is $550 and the tax rate for the rest is 7%, find her income tax payable.

8. A property agent charges a commission of 5% on the first $50 000 and 2.25% on the remaining selling price of a piece of property. Given that the agent sells a property for $240 000, find the amount of commission he receives.

9. Khairul deposited $600 in a bank at the end of 2010 and another $400 in the same bank at the end of 2011. The bank offers simple interest at a rate of 3% per annum. Find the total amount he has in his bank at the end of 2013.

10. Nora deposited $15 000 in an account that pays 4.12% per annum compound interest. Find the total amount in the account after 3 years if the interest is compounded

    (a) monthly,   (b) half-yearly.

11. A bank pays investors 4% per annum compound interest, compounded half-yearly. Find the original amount Rui Feng invested if he received $5800 as interest at the end of 3 years. Leave your answer correct to the nearest dollar.

12. A printer is priced at $450. A man buys the printer on hire purchase according to the following terms: a downpayment of 15% and the remaining to be paid in monthly instalments over 18 months at a simple interest rate of 12% per annum. Find

   (i) his monthly instalment,

   (ii) the total amount the man pays for the printer.

13. A group of Singaporean students studying in the United Kingdom travels from the United Kingdom to France for a short trip. They exchange a certain amount of Sterling pounds (£) to euros (€) at a rate of £100 = €115. They rent a car in France to travel 1850 km. One litre of petrol allows the car to travel 12 km and each litre of petrol costs €0.80. On top of the amount which they spend on petrol, they spend a total of €3500 in France, including car rental. They convert the remaining euros into sterling pounds at the end of the trip at a rate of £100 = €116.50. Given that they receive £75, find the amount of Sterling pounds that they exchange before going on their trip, giving your answer correct to the nearest pound.

Challenge Yourself

The advertisements of the same television model at three shops are as shown.

Shop A	Shop B	Shop C
$500 before GST	$500 inclusive of GST	$500 (GST absorbed)

The current Goods and Services Tax (GST) rate is 7%.

   (i) Lixin buys the television set from Shop A. How much GST does she pay?

   (ii) Jun Wei buys the television set from Shop B. How much GST does he pay?

Vishal buys the television set from Shop C and is given the following receipt:
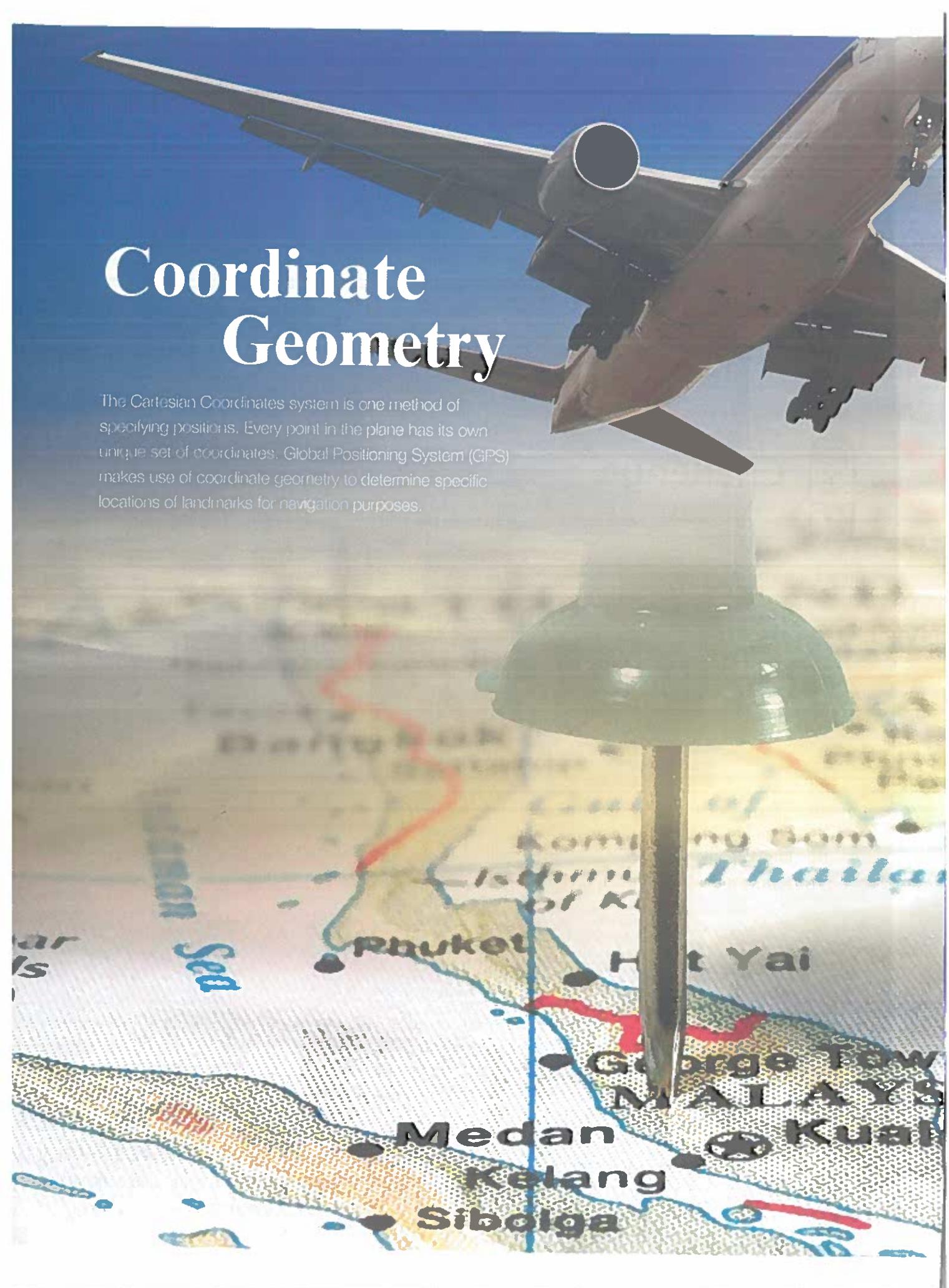
Shop C	
Television Set:	$500.00
Total Incl. GST:	$500.00
Total GST Amount:	$ 32.71

Vishal is surprised the receipt shows that $500 includes GST and he realises that the amount of GST is the same as what his friend Jun Wei pays to Shop B. Thus, he asks the shopkeeper why the advertisement states that his shop is absorbing GST but the GST absorbed is not 7% of $500. The shopkeeper replies that this is how the government calculates GST and he is not complaining about it.

   (iii) If GST were charged on the $500 paid by Vishal, how much GST must the shopkeeper pay to the government? Hence, explain why the shopkeeper says he is not complaining about it.

   (iv) Why does the government calculate the GST for Shop C using the method in the receipt shown above?

   (v) Although Shop B charges GST while Shop C absorbs it, Jun Wei and Vishal each paid $500 inclusive of GST. Does it make a difference whether a shop absorbs GST? Explain your answer.

Coordinate Geometry

The Cartesian Coordinates system is one method of specifying positions. Every point in the plane has its own unique set of coordinates. Global Positioning System (GPS) makes use of coordinate geometry to determine specific locations of landmarks for navigation purposes.

Chapter
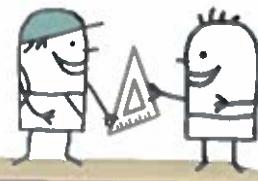
Six

6.1 Gradient of a Straight Line

Recap

In Book 1, we have learnt that a rectangular or **Cartesian** plane consists of two number lines intersecting at right angles at the point O, known as the **origin**. The horizontal and vertical axes are called the **x-axis** and the **y-axis** respectively.
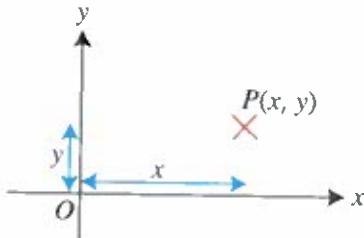
Fig. 6.1

Each point P in the plane is located by an ordered pair (x, y). We say that P has coordinates (x, y).

Gradient of a Straight Line

In Book 2, we have learnt that the gradient of a straight line is the ratio of the vertical change to the horizontal change.
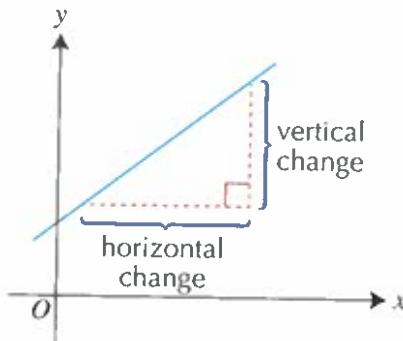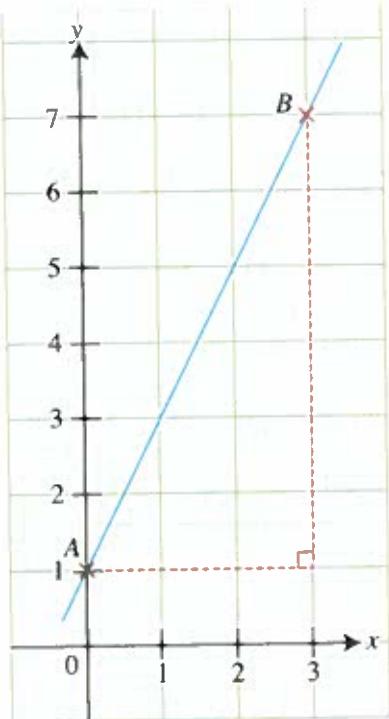
Fig. 6.2
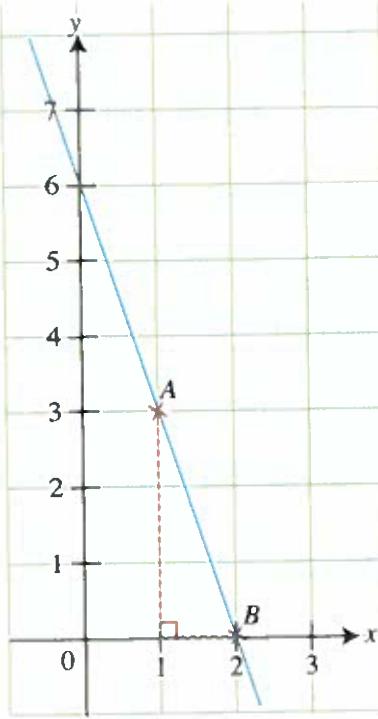
Class Discussion

Finding the Gradient of a Straight Line

1. In Fig. 6.3(a) and (b), *A* and *B* are two points on the line.

(a)   (b)

Fig. 6.3

(i) Find the gradient of each line segment.

(ii) Choose two other points that lie on each of the line segments and calculate the gradient of each line segment. Compare your answers with those obtained in (i). What do you notice? Explain your answer.

2. Given any two points $A(x_1, y_1)$ and $B(x_2, y_2)$, how would you find the gradient of the line passing through *A* and *B*?

3. Using your answer in Question 2, find the gradient of the line passing through each of the following pairs of points.

(a) $(-1, 4)$ and $(3, 7)$ (b) $(-4, -3)$ and $(2, -11)$

(c) $(6, 3)$ and $(-4, 3)$ (d) $(2, -1)$ and $(2, 8)$

Compare your answers with those obtained by your classmates.

From the class discussion, we observe that if $A(x_1, y_1)$ and $B(x_2, y_2)$ are two points on a line, then

$$\boxed{\text{gradient of } AB = \frac{y_2 - y_1}{x_2 - x_1}.}$$
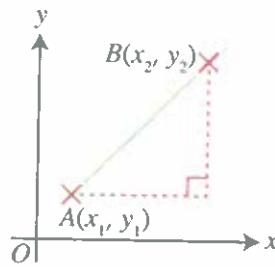
Fig. 6.4

Thinking Time

Instead of writing the gradient of AB as $\frac{y_2 - y_1}{x_2 - x_1}$, we can also write it as $\frac{y_1 - y_2}{x_1 - x_2}$.

Is $\frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1 - y_2}{x_1 - x_2}$? Explain your answer.

Investigation

Gradient of a Straight Line

1. Using a suitable geometry software, draw a line segment with the end-points as $A(-2, 1)$ and $B(0, 5)$.

2. Find the gradient of the line segment AB and record it in Table 6.1. Describe the gradient of the line segment AB using one of the following terms: positive, negative, zero or undefined.

3. Write down the value of $y_2 - y_1$ and of $x_2 - x_1$ in Table 6.1.

4. Repeat Steps 1–3 for each of the following pairs of points.
   (a) $C(7, 5)$ and $D(4, 8)$ (b) $E(-2, 6)$ and $F(-4, 3)$
   (c) $G(1, 1)$ and $H(3, 1)$ (d) $I(-4, 3)$ and $J(-4, 6)$
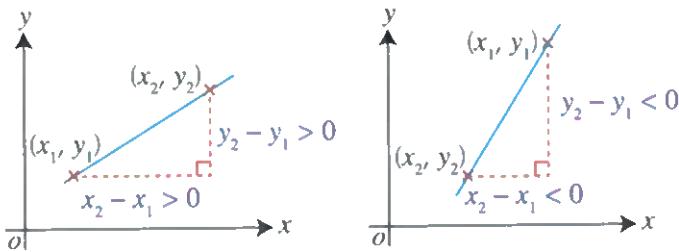
	Coordinates of End-points	Gradient of Line Segment	Sign of Gradient	$y_2 - y_1$	$x_2 - x_1$
(a)	$A(-2, 1)$ and $B(0, 5)$		positive	$5 - 1 =$	$0 - (-2) =$
(b)	$C(7, 5)$ and $D(4, 8)$				
(c)	$E(-2, 6)$ and $F(-4, 3)$				
(d)	$G(1, 1)$ and $H(3, 1)$				
(e)	$I(-4, 3)$ and $J(-4, 6)$				

Table 6.1
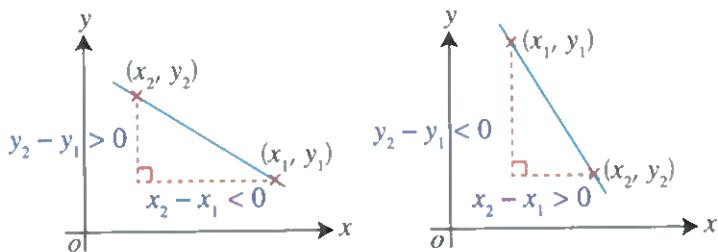
5. **(a)** When $y_2 - y_1 > 0$ and $x_2 - x_1 < 0$, what do you notice about the sign of the gradient?

(b) When $y_2 - y_1 < 0$ and $x_2 - x_1 > 0$, what do you notice about the sign of the gradient?

(c) When the signs of $y_2 - y_1$ and $x_2 - x_1$ are the same, what do you notice about the sign of the gradient?

(d) When $y_2 - y_1 = 0$, what do you notice about the gradient of the line?

(e) When $x_2 - x_1 = 0$, what do you notice about the gradient of the line?

From the investigation, we observe that
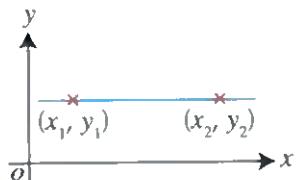
- the gradient of a straight line can be positive, negative, zero or undefined,
- if $y_2 - y_1$ and $x_2 - x_1$ have the same signs, the gradient of the straight line is positive,
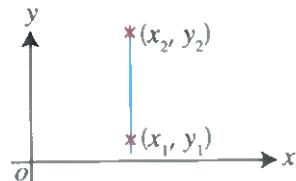
- if $y_2 - y_1$ and $x_2 - x_1$ have opposite signs, the gradient of the straight line is negative,
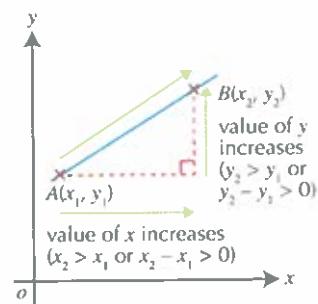
- if $y_2 - y_1 = 0$ or $y_2 = y_1$, the gradient of a horizontal line is zero,

- if $x_2 - x_1 = 0$ or $x_2 = x_1$, the gradient of a vertical line is undefined.

ATTENTION

When the gradient of the line is positive, as the value of x increases (from point A to point B), the value of y also increases.

value of y increases ($y_2 > y_1$ or $y_2 - y_1 > 0$)

value of x increases ($x_2 > x_1$ or $x_2 - x_1 > 0$)

Worked Example 1

Find the gradient of the line passing through each of the following pairs of points.

(a) $A(2, 3)$ and $B(7, 5)$ (b) $P(-2, 8)$ and $Q(1, -1)$

Solution:

(a) Gradient of $AB = \dfrac{y_2 - y_1}{x_2 - x_1}$ (Let $(x_1, y_1) = (2, 3)$ and $(x_2, y_2) = (7, 5)$)

$= \dfrac{5 - 3}{7 - 2}$

$= \dfrac{2}{5}$

Alternatively,

Gradient of $AB = \dfrac{y_1 - y_2}{x_1 - x_2}$

$= \dfrac{3 - 5}{2 - 7}$

$= \dfrac{-2}{-5}$

$= \dfrac{2}{5}$

(b) Gradient of $PQ = \dfrac{y_2 - y_1}{x_2 - x_1}$ (Let $(x_1, y_1) = (-2, 8)$ and $(x_2, y_2) = (1, -1)$)

$= \dfrac{-1 - 8}{1 - (-2)}$

$= \dfrac{-9}{3}$

$= -3$

PRACTISE NOW 1

Find the gradient of the line passing through each of the following pairs of points.

(a) $C(3, 1)$ and $D(6, 3)$

(b) $H(5, -7)$ and $K(0, -2)$

(c) $M(-4, 1)$ and $N(16, 1)$

SIMILAR QUESTIONS

Exercise 6A Questions 1(a)–(f), 2, 9

Worked Example 2
(Using the Gradient to Determine the Coordinates of a Point on the Line)

If the gradient of the line joining the points $(k, 5)$ and $(2, k)$ is -2, find the value of k.

Solution:
Gradient of line, $\dfrac{y_2 - y_1}{x_2 - x_1} = -2$

$\dfrac{k - 5}{2 - k} = -2$ (Let $(x_1, y_1) = (k, 5)$ and $(x_2, y_2) = (2, k)$)

$k - 5 = -2(2 - k)$

$k - 5 = -4 + 2k$

$-1 = k$

$\therefore k = -1$

PRACTISE NOW 2

If the gradient of the line joining the points $(4, -9)$ and $(-3, h)$ is -3, find the value of h.
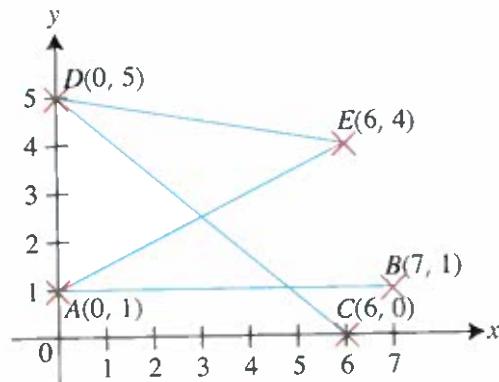
Exercise 6A

BASIC LEVEL

1. Find the gradient of the line passing through each of the following pairs of points.

   (a) $A(0, 0)$ and $B(-2, 1)$

   (b) $C(2, -3)$ and $D(1, 7)$

   (c) $E(-2, 4)$ and $F(-5, 8)$

   (d) $G(-4, 7)$ and $H(1, -8)$

   (e) $I(-2, -5)$ and $J(2, 6)$

   (f) $K(-7, 9)$ and $L(6, 9)$

2. The points $A(0, 1)$, $B(7, 1)$, $C(6, 0)$, $D(0, 5)$ and $E(6, 4)$ are shown in the diagram.

   Find the gradient of each of the line segments AB, AE, DC and DE.

3. If the gradient of the line joining the points $(-3, -7)$ and $(4, p)$ is $\frac{3}{5}$, find the value of p.

4. The coordinates of A and B are $(3k, 8)$ and $(k, -3)$ respectively. Given that the gradient of the line segment AB is 3, find the value of k.

7. The points $P(2, -3)$, $Q(3, -2)$ and $R(8, z)$ are collinear, i.e. they lie on a straight line. Find the value of z.

8. The line joining the points $A(2, t)$ and $B(7, 2t^2 + 7)$ has a gradient of 2. Find the possible values of t.
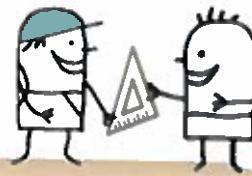
5. The gradient of the line joining the points $(9, a)$ and $(2a, 1)$ is $\frac{2}{a}$, where $a \neq 0$. Find the possible values of a.

6. The points P, Q and R have coordinates $(6, -11)$, $(k, -9)$ and $(2k, -3)$ respectively. If the gradient of PQ is equal to the gradient of PR, find the value of k.

9. The coordinates of the vertices of a square $ABCD$ are $A(0, 6)$, $B(2, 1)$, $C(7, 3)$ and $D(5, 8)$.

  (i) Find the gradient of all 4 sides of $ABCD$.

  (ii) What do you observe about the gradients of the opposite sides of a square?

6.2 Length of a Line Segment

Consider the points $A(1, 1)$ and $B(7, 9)$ as shown in Fig. 6.5.
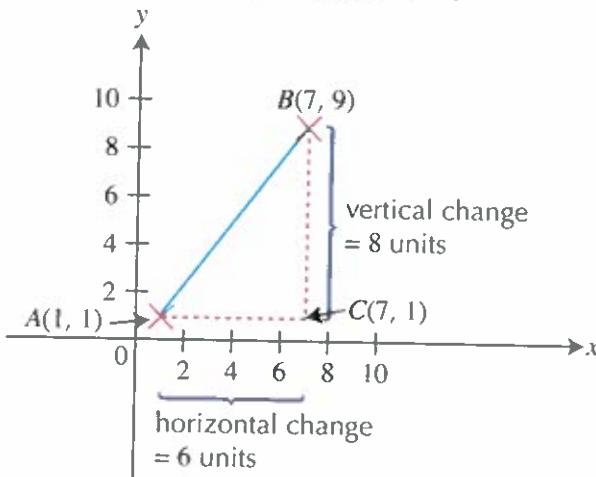
Fig. 6.5

$\triangle ABC$ is formed by drawing AC parallel to the x-axis and BC parallel to the y-axis. The coordinates of the point C are given by $(7, 1)$.

ATTENTION

Recall that a line segment is part of a line with two end-points. A line has no end-points so it does not have a length. However, a line segment has two end-points and so it has length.

ATTENTION

BC is vertical, i.e. x-coordinate of C = x-coordinate of B.
AC is horizontal, i.e. y-coordinate of C = y-coordinate of A.

Hence, $AC = 7 - 1 = 6$

and $BC = 9 - 1 = 8$.

Using Pythagoras' Theorem,

$AB^2 = AC^2 + BC^2$

$= 6^2 + 8^2$

$= 100$

$AB = \sqrt{100}$

$= 10$ units

Consider any two points P and Q with coordinates (x_1, y_1) and (x_2, y_2) respectively. By completing the right-angled $\triangle PQR$, we have the coordinates of R as (x_2, y_1).

Hence, $PR = x_2 - x_1$

and $QR = y_2 - y_1$.

Using Pythagoras' Theorem,

$PQ^2 = PR^2 + QR^2$

$= (x_2 - x_1)^2 + (y_2 - y_1)^2$
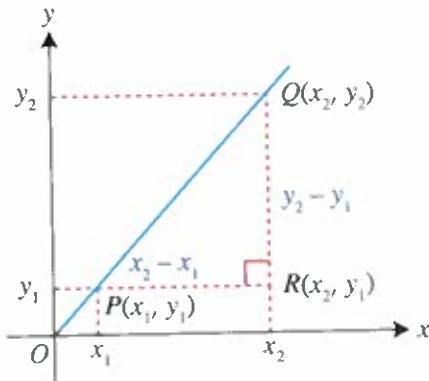
Fig. 6.6

In general, the length of any line segment PQ, where the coordinates of the points P and Q are (x_1, y_1) and (x_2, y_2) respectively, is

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Thinking Time

Instead of writing the length of line segment PQ as $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$,

we can also write it as $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$.

Is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$? Explain your answer.

Worked Example 3

Given that the coordinates of the points A and B are $(-4, 1)$ and $(6, -5)$ respectively, find the length of the line segment AB.

Solution:

Length of line segment $AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$$= \sqrt{[6 - (-4)]^2 + (-5 - 1)^2}$$

$$= \sqrt{10^2 + (-6)^2}$$

$$= \sqrt{100 + 36}$$

$$= \sqrt{136}$$

$$= 11.7 \text{ units (to 3 s.f.)}$$

PRACTISE NOW 3

Find the length of the line segment joining each of the following pairs of points.

(a) $C(6, 2)$ and $D(3, -2)$

(b) $M(-1, 5)$ and $N(6, -4)$

(c) $P(2, 7)$ and $Q(8, 7)$

Worked Example 4

(Using the Length to Determine the Coordinates of a Point on the Line)

Given that the coordinates of the points A and B are $(-3, 2)$ and $(1, -6)$ respectively, find the coordinates of the point C that lies on the y-axis such that $AC = BC$. Hence, find the area of $\triangle ACO$, where O is the origin.

Solution:

Let the coordinates of C be $(0, k)$.

$AC = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$$= \sqrt{[0 - (-3)]^2 + (k - 2)^2}$$

$$= \sqrt{(0 + 3)^2 + (k - 2)^2}$$

$$= \sqrt{9 + (k - 2)^2}$$

Problem Solving Tip

Since C lies on the y-axis, its x-coordinate is 0.

$$BC = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$= \sqrt{(0-1)^2 + [k-(-6)]^2}$$
$$= \sqrt{(-1)^2 + (k+6)^2}$$
$$= \sqrt{1 + (k+6)^2}$$

Since $AC = BC$,

$$\sqrt{9 + (k-2)^2} = \sqrt{1 + (k+6)^2}$$
$$\left(\sqrt{9 + (k-2)^2}\right)^2 = \left(\sqrt{1 + (k+6)^2}\right)^2 \quad \text{(square both sides of the equation)}$$
$$9 + (k-2)^2 = 1 + (k+6)^2$$
$$9 + k^2 - 4k + 4 = 1 + k^2 + 12k + 36$$
$$k^2 - 4k + 13 = k^2 + 12k + 37$$
$$-16k = 24$$
$$16k = -24$$
$$k = -\frac{24}{16}$$
$$= -\frac{3}{2}$$
$$= -1\frac{1}{2}$$
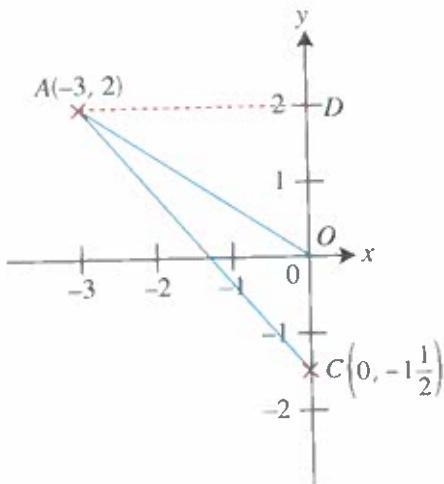
$\therefore$ Coordinates of C are $\left(0,\ -1\frac{1}{2}\right)$

Area of $\triangle ACO = \frac{1}{2} \times$ base $\times$ height
$$= \frac{1}{2} \times OC \times AD$$
$$= \frac{1}{2} \times 1\frac{1}{2} \times 3$$
$$= \frac{9}{4}$$
$$= 2\frac{1}{4} \text{ units}^2$$

PRACTISE NOW 4

Given that the coordinates of the points C and D are $(4, -1)$ and $(-2, 7)$ respectively, find

(a) the coordinates of the point E that lies on the y-axis such that $CE = DE$,

(b) the coordinates of the point F that lies on the x-axis such that $CF = DF$.

Hence, find the area of $\triangle OEF$, where O is the origin.

Worked Example 5

A triangle has vertices $A(0, -5)$, $B(-2, 1)$ and $C(10, 5)$. Show that $\triangle ABC$ is a right-angled triangle and identify the right angle.

Solution:

$$AB^2 = (-2 - 0)^2 + [1 - (-5)]^2$$
$$= (-2)^2 + 6^2$$
$$= 4 + 36$$
$$= 40$$

Problem Solving Tip

Check that the sum of squares of the two shorter sides is equal to the square of the longest side.

$$BC^2 = [10 - (-2)]^2 + (5 - 1)^2$$
$$= 12^2 + 4^2$$
$$= 144 + 16$$
$$= 160$$

$$AC^2 = (10 - 0)^2 + [5 - (-5)]^2$$
$$= 10^2 + 10^2$$
$$= 100 + 100$$
$$= 200$$

Since $AB^2 + BC^2 = 40 + 160$
$$= 200$$
$$= AC^2,$$

the triangle is a right-angled triangle with $\angle ABC = 90°$. (Converse of Pythagoras' Theorem)

PRACTISE NOW 5

SIMILAR QUESTIONS

1. A triangle has vertices $D(6, 1)$, $E(2, 3)$ and $F(-1, -3)$. Show that $\triangle DEF$ is a right-angled triangle and identify the right angle.

2. A triangle has vertices $P(-3, 1)$, $Q(6, 3)$ and $R(1, 8)$. Determine if $\triangle PQR$ is a right-angled triangle.
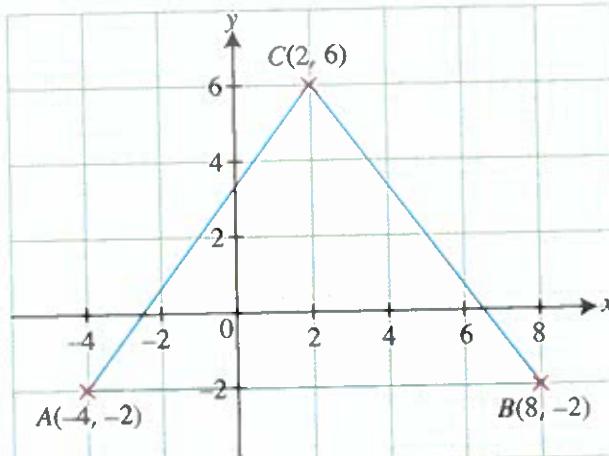
1. Find the length of the line segment joining each of the following pairs of points.

   (a) $A(2, 3)$ and $B(9, 7)$

   (b) $C(3, 6)$ and $D(-5, 9)$

   (c) $E(-1, 4)$ and $F(8, -3)$

   (d) $G(-10, 2)$ and $H(-4, -7)$

2. If the distance between the points $A(p, 0)$ and $B(0, p)$ is 10, find the possible values of p.

3. Given that the coordinates of the points P and Q are $(-2, 6)$ and $(9, 3)$ respectively, find

   (a) the coordinates of the point R that lies on the y-axis such that $PR = QR$,

   (b) the coordinates of the point S that lies on the x-axis such that $PS = QS$.

4. A line segment has two end-points $M(3, 7)$ and $N(11, -6)$. Find the coordinates of the point W that lies on the y-axis such that W is equidistant from M and from N.
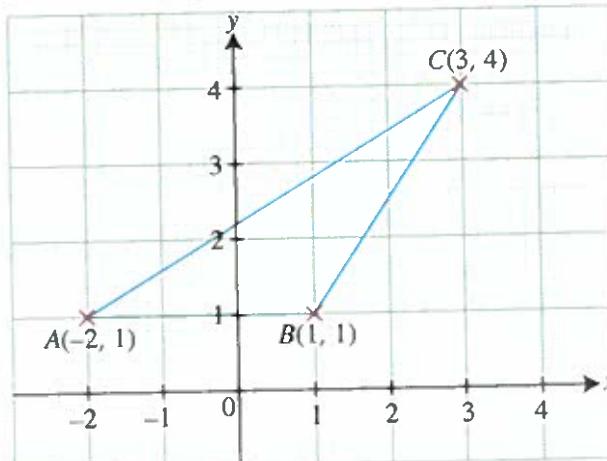
   Hint: The term 'equidistant' means 'same distance'.

5. The vertices of $\triangle ABC$ are $A(-4, -2)$, $B(8, -2)$ and $C(2, 6)$.
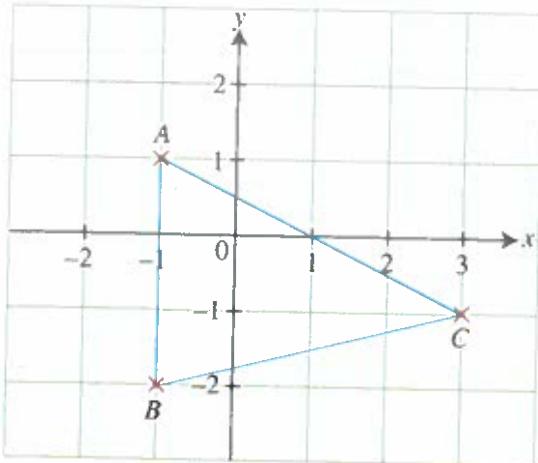
   (i) Find the perimeter and the area of $\triangle ABC$.

   (ii) Hence, find the length of the perpendicular from A to BC.

6. The diagram shows $\triangle ABC$ with vertices $A(-2, 1)$, $B(1, 1)$ and $C(3, 4)$.

   (i) Find the area of $\triangle ABC$.

   (ii) Find the length of AC, giving your answer correct to 2 decimal places.

   (iii) Given that $ABCD$ is a parallelogram, find the coordinates of D.

   (iv) Given that K is the point $(t, 4)$ and the area of $\triangle BCK$ is 12 units², find the possible values of t.

7. The diagram shows $\triangle ABC$ with vertices $A(-1, 1)$, $B(-1, -2)$ and $C(3, -1)$.

(i) Find the lengths of AB, BC and AC.

(ii) Find the area of $\triangle ABC$.

The coordinates of a point E are $(3, k)$ and the area of $\triangle BCE$ is 14 units^2.

(iii) Find the possible values of k.

8. The distance between the points $(1, 2t)$ and $(1 - t, 1)$ is $\sqrt{11 - 9t}$. Find the possible values of t.
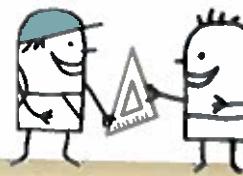
9. **(i)** Show that the points $A(-1, 2)$, $B(5, 2)$ and $C(2, 5)$ are the vertices of an isosceles triangle.

(ii) Find the area of $\triangle ABC$.

10. By showing that the points $P(3, 4)$, $Q(3, 1)$ and $R(8, 4)$ are the vertices of a right-angled triangle, find the length of the perpendicular from P to QR.

ADVANCED LEVEL

11. The vertices of $\triangle PQR$ are $P(1, 3)$, $Q(5, 4)$ and $R(5, 15)$. Find the length of the perpendicular from Q to PR.
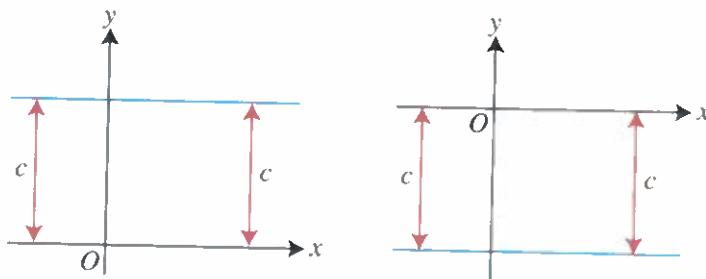
6.3 Equation of a Straight Line

Recap

Horizontal Lines

If a line is parallel to the x-axis and its distance from the x-axis is c, then every point on the line has the same y-coordinate.
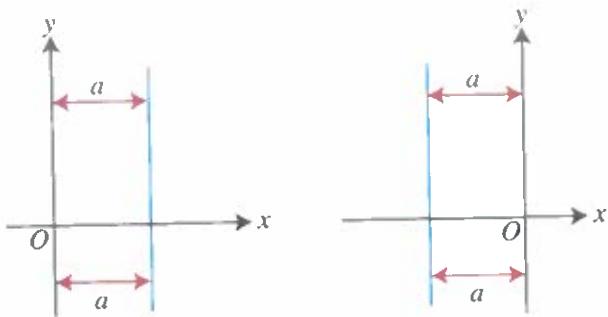
The gradient of a horizontal line is 0.

The equation of a horizontal line is $y = c$ or $y = -c$.

If a line is parallel to the y-axis and its distance from the y-axis is a, then every point on the line has the same x-coordinate.

The gradient of a vertical line is undefined.

The equation of a vertical line is $x = a$ or $x = -a$.

Equation of a Straight Line $y = mx + c$

In Book 2, we have learnt that the equation of a straight line is in the form $y = mx + c$, where the constant m is the **gradient** of the line and the constant c is the **y-intercept**.
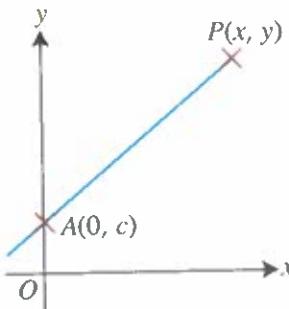
Fig. 6.7

In Fig. 6.7, the straight line passes through the points $A(0, c)$ and $P(x, y)$. If the gradient of the line is m, then

Gradient of line $= m$

$$\frac{y - c}{x - 0} = m$$

$$y - c = mx$$

$$\therefore y = mx + c$$

In general,

> for a straight line passing through the point $(0, c)$ and with gradient m, the equation is
>
> $$y = mx + c.$$

The equation $y = mx + c$ is known as the **gradient-intercept form** of the equation of a straight line. In this equation, m gives the gradient of the straight line, c gives the intercept on the y-axis and $(0, c)$ is the point where the line cuts the y-axis.

Worked Example 6

(Finding the y-intercept given the Gradient and Coordinates of a Point)

Given that $y = 3x + c$ passes through the point $(3, 1)$, find the value of c.

Solution:

Since $(3, 1)$ lies on the line $y = 3x + c$, the coordinates $(3, 1)$ must satisfy the equation,

i.e. $1 = 3(3) + c$ (substitute $x = 3$ and $y = 1$)

$\quad\quad = 9 + c$

$\therefore c = -8$

PRACTISE NOW 6

1. Given that $y = 5x + a$ passes through the point $(-1, 2)$, find the value of a.
2. The point $(6, 8)$ lies on the line $y = -4x + b$. Find the value of b.

Worked Example 7

(Finding the Equation of a Straight Line given the Coordinates of 2 Points)

Find the equation of the straight line passing through each of the following pairs of points.

(a) $A(1, 2)$ and $B(3, 7)$ (b) $C(2, 3)$ and $D(7, 3)$

(c) $E(5, 1)$ and $F(5, 6)$

Solution:

(a) Gradient of $AB = \dfrac{7 - 2}{3 - 1}$

$\quad\quad\quad\quad\quad\quad = \dfrac{5}{2}$

Equation of AB is in the form $y = \dfrac{5}{2}x + c$

Since $(1, 2)$ lies on the line,

$2 = \dfrac{5}{2}(1) + c$

$c = -\dfrac{1}{2}$

$\therefore$ Equation of AB is $y = \dfrac{5}{2}x - \dfrac{1}{2}$

ATTENTION

We can also substitute $(3, 7)$ into the equation of AB to find the value of c.

(b) $C(2, 3)$ and $D(7, 3)$ have the same y-coordinate of value 3.

$\therefore CD$ is a horizontal line with equation $y = 3$.

(c) $E(5, 1)$ and $F(5, 6)$ have the same x-coordinate of value 5.

$\therefore EF$ is a vertical line with equation $x = 5$.

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Find the equation of the straight line passing through each of the following pairs
of points.

(a) $A(-2, 1)$ and $B(5, 3)$

(b) $C(6, 4)$ and $D(-4, 4)$

(c) $E(-3, 5)$ and $F(-3, 8)$

Journal Writing

When we want to find the equation of a straight line, what information do we need?

Consider each of the cases below.

Case 1: Given the gradient m and the y-intercept c
Case 2: Given the gradient m and the coordinates of a point (a, b)
Case 3: Given the coordinates of two points (a, b) and (c, d)

For each case, describe how you would find the equation of the straight line.

Exercise 6C
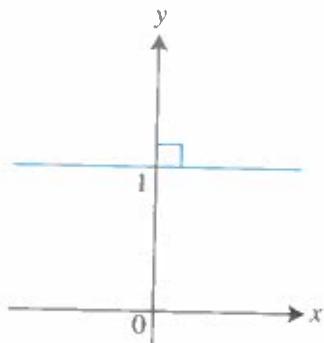
BASIC LEVEL

1. Given that $y = -x + c$ passes through the point $(1, 2)$, find the value of c.

2. The point $(-3, 3)$ lies on the line $y = 4x + k$. Find the value of k.

3. Find the equation of the straight line passing through each of the following pairs of points.

    (a) $A(0, 0)$ and $B(1, -1)$

    (b) $C(1, 3)$ and $D(2, 5)$

    (c) $E(2, 4)$ and $F(-2, 3)$

    (d) $G(-6, -5)$ and $H(4, 4)$

    (e) $I(-2, -4)$ and $J(1, -7)$

    (f) $K(-7, -5)$ and $L(-1, -1)$

    (g) $M(8, 0)$ and $N(-9, 0)$

    (h) $O(0, 0)$ and $P(0, 7)$
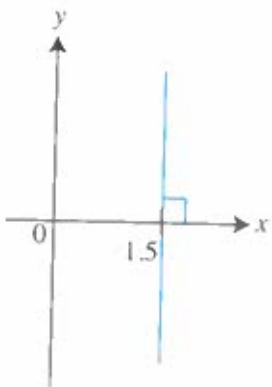
4. Find the equation of each of the straight lines, given the gradient and the coordinates of a point that lies on it.

    (a) $\frac{1}{3}$, $(0, 0)$

    (b) 3, $(1, 1)$

    (c) -3, $(2, -5)$

    (d) $-\frac{1}{2}$, $(5, 7)$

    (e) 0, $(5, 4)$

    (f) a, $(0, a)$

5. Write down the equation of the straight line which passes through the origin and with gradient 2.

6. In each of the following diagrams, find the gradient and the y-intercept of the line where possible. State the equation of each line.

(a)

(b)

(c)

(d)
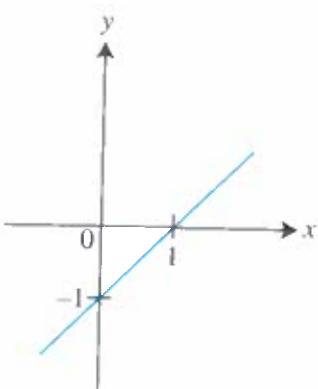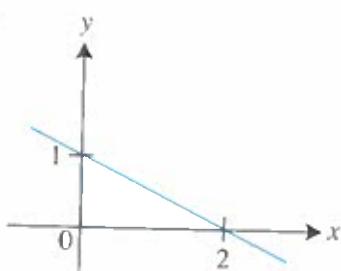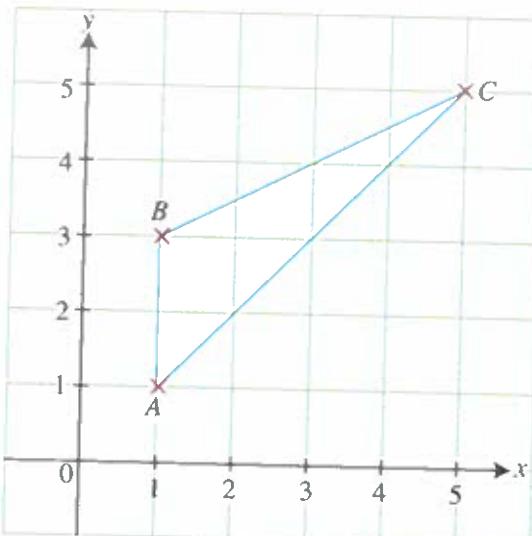
7. The diagram shows $\triangle ABC$ with vertices $A(1, 1)$, $B(1, 3)$ and $C(5, 5)$.

(i) Find the area of $\triangle ABC$.

(ii) Find the gradient of the line passing through B and C.

(iii) Find the equation of the line passing through A and C.

8. The lines $2x - 5 = ky$ and $(k + 1)x = 6y - 3$ have the same gradient. Find the possible values of k.

9. Given the line $\frac{x}{3} + \frac{y}{2} = 1$,

(i) make y the subject of the formula $\frac{x}{3} + \frac{y}{2} = 1$,

(ii) find the gradient of the line,

(iii) find the coordinates of the point at which the line cuts the x-axis.

10. (i) Find the equation of the straight line which passes through the point $(-3, 5)$ and with gradient $-\frac{2}{3}$.

(ii) Given that the line in (i) also passes through the point $(p, 3)$, find the value of p.

11. Find the equation of the straight line passing through the point $(3, -2)$ and having the same gradient as the line $2y = 5x + 7$.

12. (i) Find the equation of the straight line which passes through the point $(3, 1)$ and with gradient 3.

    (ii) Hence, find the coordinates of the point of intersection of the line in (i) with the line $y = x$.

13. The line l has equation $5x + 6y + 30 = 0$. Given that P is the point $(3, -1)$, find

    (i) the coordinates of the point where l crosses the x-axis,

    (ii) the coordinates of the point of intersection of l with the line $x = 2$,

    (iii) the equation of the line passing through P and having the same gradient as l,

    (iv) the equation of the line passing through P and having a gradient of 0.

14. A straight line l passes through the points $A(0, 3)$ and $B(3, 12)$.

    (a) Find

        (i) the gradient of the line l,

        (ii) the equation of the line l.

    (b) The line $x = 3$ is the line of symmetry of $\triangle ABC$. Find the coordinates of C.
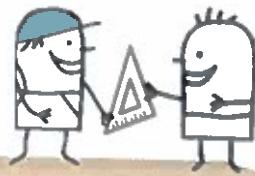
15. If the line $mx = ny + 2$ has the same gradient as the x-axis, find the value of m. State the condition for the line to be parallel to the y-axis instead.

16. The line l has equation $3x + 4y = 24$. It crosses the x-axis at the point A and the y-axis at the point B. Find

    (i) the coordinates of A and of B,

    (ii) the length of the line segment AB,

    (iii) the coordinates of the point C that lies on the line l such that C is equidistant from the coordinate axes,

    (iv) the equation of the line OC, where O is the origin.

17. The coordinates of the points P and Q are $(2, 3)$ and $(9, 5)$ respectively.

    (i) Find the coordinates of the point where the line passing through P and Q intersects the x-axis.

    (ii) Given that $y = 5$ is the line of symmetry of $\triangle PQR$, find the coordinates of R.

    (iii) Calculate the length of PQ.

    (iv) Hence, find the length of the perpendicular from R to PQ.

6.4 Parallel and Perpendicular Lines

Recap

In Book 1, we have learnt that points A, X and B are **collinear** if they lie on the same line.
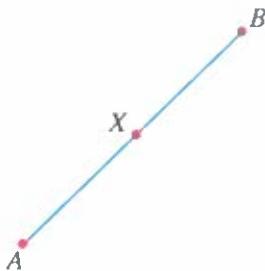
Fig. 6.8

We can see that the line segment AB can be split into two parts namely, AX and XB with X being the common point.

Thinking Time

1. Given three points $A(2, 3)$, $B(3, 5)$ and $C(5, 9)$ which lie on a Cartesian plane, find the gradients of AB, BC and AC. What can you say about the three points?

2. Given that the gradients of AB and BC are equal, what can you say about the points A, B and C?
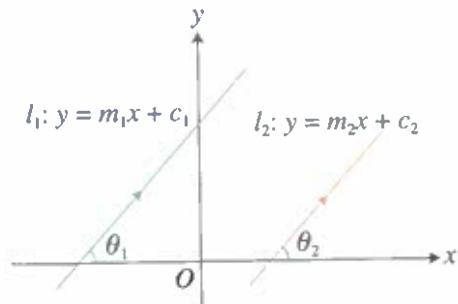
Class Discussion

Parallel and Perpendicular Lines

Work in pairs.
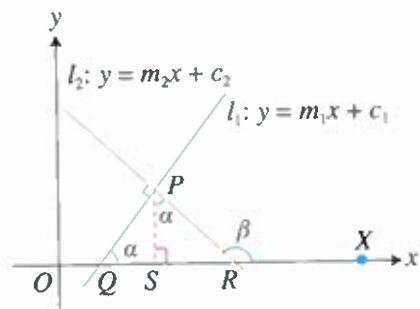
Let us consider the following pair of lines

$$l_1: y = m_1 x + c_1$$
$$l_2: y = m_2 x + c_2.$$

Case 1:

$l_1: y = m_1x + c_1$ $l_2: y = m_2x + c_2$

Let θ_1 and θ_2 be the angles that the lines l_1 and l_2 make with the positive direction of the x-axis respectively.

(i)   If l_1 and l_2 are parallel, what can we say about θ_1 and θ_2? What can we say about the gradients of these 2 lines?

(ii) If $m_1 = m_2$, what can we say about l_1 and l_2?

Case 2:

$l_2: y = m_2x + c_2$ $l_1: y = m_1x + c_1$

Let the lines l_1 and l_2 intersect and form right angles at P. They intersect the x-axis at Q and R respectively. If $\angle PQR = \alpha$, then $\tan \alpha = m_1$; if $\angle PRX = \beta$, then $\tan \beta = m_2$. Let PS be perpendicular to the x-axis, then $\angle SPR = \alpha$.

(i)   Consider $\triangle PQS$. Since $m_1 = \tan \alpha$, write down an expression for m_1 in terms of the lengths.

(ii) Consider $\triangle PRS$. Since $m_2 = \tan \beta$, write down an expression for m_2 in terms of the lengths.

Hence, show that $m_2 = -\dfrac{1}{m_1}$.

(iii) If l_1 and l_2 are perpendicular, what can we say about m_1 and m_2?

(iv) If $m_1m_2 = -1$, what can we say about l_1 and l_2?

From the Class Discussion, we observe that for two lines l_1 and l_2,

> (i)   If l_1 is **parallel** to l_2, then their gradients are the same.
>
> (ii) If l_1 is **perpendicular** to l_2, then the product of their gradients is −1.

Worked Example 8 (Parallel Lines and Collinear Points)

The coordinates of 4 points are $O(0, 0)$, $A(2, k)$, $B(2k, 9)$ and $C(3k, 2k + 7)$. Find the value(s) of k if

(a) the points O, A and B are collinear,

(b) OA is parallel to BC.

Solution:

(a) If O, A and B are collinear, they lie on the same straight line,

i.e. gradient of OA = gradient of OB.

$$\frac{k - 0}{2 - 0} = \frac{9 - 0}{2k - 0}$$

$$\frac{k}{2} = \frac{9}{2k}$$

$$2k^2 = 18$$

$$k^2 = 9$$

$$k = \pm 3$$

(b) OA is parallel to BC,

i.e. gradient of OA = gradient of BC.

$$\frac{k - 0}{2 - 0} = \frac{2k + 7 - 9}{3k - 2k}$$

$$\frac{k}{2} = \frac{2k - 2}{k} \quad \text{(simplify both expressions before cross-multiplying)}$$

$$k^2 = 4k - 4$$

$$k^2 - 4k + 4 = 0$$

$$(k - 2)^2 = 0$$

$$k = 2$$

PRACTISE NOW 8

The coordinates of 4 points are $A(0, 9)$, $B(k + 1, k + 4)$, $C(2k, k + 3)$ and $D(2k + 2, k + 6)$. Find the value(s) of k if

(a) the points A, B and C are collinear,

(b) AB is parallel to CD.

Worked Example 9

(Perpendicular Lines)
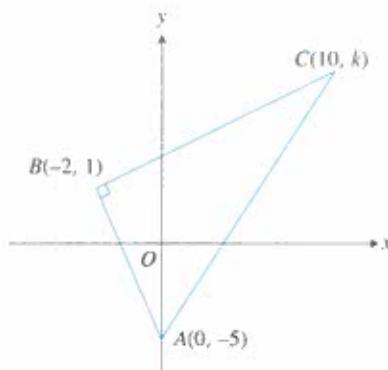
The vertices of $\triangle ABC$ are $A(0, -5)$, $B(-2, 1)$ and $C(10, k)$. Find the value of k if $\angle ABC = 90°$.

Solution:

Since $\angle ABC = 90°$, gradient of $AB \times$ gradient of $BC = -1$.

$$\frac{1 - (-5)}{-2 - 0} \times \frac{k - 1}{10 - (-2)} = -1$$

i.e.
$$(-3)\left(\frac{k-1}{12}\right) = -1$$

$$k - 1 = 4$$

$$k = 5$$

Class Discussion

Perpendicular Lines

Can you use Pythagoras' Theorem to verify your answer to Worked Example 9?

PRACTISE NOW 9

The vertices of $\triangle ABC$ are $A(0, 1)$, $B(-1, -2)$ and $C(2, k)$. Find the value of k if $\angle ABC = 90°$.

Worked Example 10

The figure shows a quadrilateral $ABCD$ in which the points A and D lie on the y-axis and x-axis respectively. C is the point $(16, 11)$ and the equation of AB is $y = 2x + 4$. If AD and BC are perpendicular to AB, find

(i) the coordinates of A and of D,
(ii) the equation of BC,
(iii) the length of CD.

Solution:

(i) At A, $x = 0$, i.e. $y = 2(0) + 4 = 4$.
$\therefore A(0, 4)$
Let D be the point $(k, 0)$.
Gradient of $AB = 2 \Rightarrow$ Gradient of $AD = -\dfrac{1}{2}$
i.e. $\dfrac{4 - 0}{0 - k} = -\dfrac{1}{2}$
$k = 8$
$\therefore D(8, 0)$

(ii) Gradient of BC = Gradient of $AD = -\dfrac{1}{2}$
Equation of BC is $\quad y - 11 = -\dfrac{1}{2}(x - 16)$
$2y + x = 38$

(iii) Length of $CD = \sqrt{(16 - 8)^2 + (11 - 0)^2}$
$= \sqrt{185}$
$= 13.6$ units
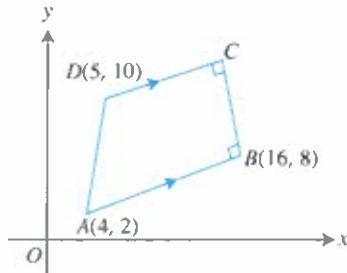
> **RECALL**
>
> Length of a line segment
> $= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

PRACTISE NOW 10

The figure shows a trapezium $ABCD$ in which AB is parallel to DC and BC is perpendicular to both AB and DC. The coordinates of A, B and D are $(4, 2)$, $(16, 8)$ and $(5, 10)$ respectively. Find the equation of DC and of BC.

> **SIMILAR QUESTIONS**
>
> Exercise 6D Questions 5, 11, 14–16

Exercise 6D

1. Find the gradient of a line paraallel to

   (a) $y = 8x + 5$,   (b) $y = -\frac{1}{2}x - 7$.

2. Find the gradient of a line perpendicular to

   (a) $y = 3x + 2$,   (b) $y = -\frac{3}{4} + 6$.

3. The coordinates of 4 points are $O(0, 0)$, $A(2, 3k)$, $B(4k, 6)$ and $C(10k, 7)$. Find the value(s) of k if
   (a) the points O, A and B are collinear,
   (b) OA is parallel to BC.

4. The coordinates of 3 points are $A(1, 1)$, $B(-1, 4)$ and $C(6, k)$. Find the value of k if AB is perpendicular to BC.
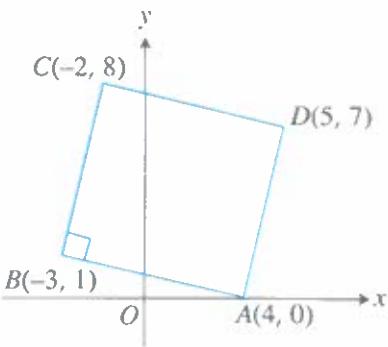
5. Find the equation of the line passing through the point
   (a) $(-2, 5)$ and parallel to the line $3y + 7x = 29$,
   (b) $(-1, -6)$ and perpendicular to the line $42x - 7y = 5$,
   (c) $(4, 8)$ and parallel to the line $3x + y = 17$,
   (d) $(2, -3)$ and perpendicular to the line $y + 2x = 13$.

6. The vertices of $\triangle ABC$ are $A(-1, -3)$, $B(2, 3)$ and $C(k + 5, k)$. Find the value of k if AB is perpendicular to BC.

7. Show that $P(-1, 3)$, $Q(6, 8)$ and $R(11, 1)$ are the vertices of an isosceles triangle. Determine whether $\angle PQR$ is a right angle, showing your working clearly.

8. Given that A is the point $(0, 4)$ and B is $(6, 6)$, find
   (i) the point C on the x-axis such that $AB = BC$,
   (ii) the point D on the y-axis such that $\angle ABD = 90°$.

9. A point P is equidistant from $R(-2, 4)$ and $S(6, -4)$ and its x-coordinate is twice its y-coordinate.
   (i) Find the coordinates of P.
   (ii) Hence, determine whether P, R and S are collinear, showing your working clearly.

10. The coordinates of 3 points are $A(-1, -6)$, $B(3, -12)$ and $C(k, 6)$. Find the value of k if
    (a) A, B and C are collinear,
    (b) AB is perpendicular to AC.

11. The coordinates of three points are $A(-1, -3)$, $B(2, 3)$ and $C(6, k)$. If AB is perpendicular to BC, find
    (i) the value of k,   (ii) the gradient of AC.
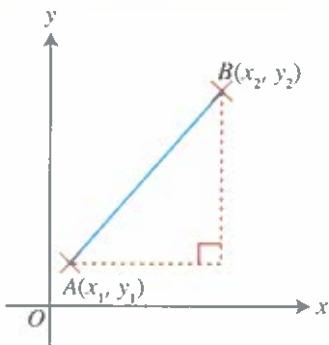
12. Determine whether the 4 points $(2, 1)$, $(-1, -5)$, $(1, 5)$ and $(-2, -1)$ are the vertices of a parallelogram, showing your working clearly.

13. Determine whether the 4 points $(5, 8)$, $(7, 5)$, $(3, 5)$ and $(5, 2)$ are the vertices of a rhombus, showing your working clearly.

14. The coordinates of three points are $A(-1, -6)$, $B(3, 12)$ and $C(k, 6)$. Find the value(s) of k if
    (a) A, B and C are collinear,
    (b) AB is perpendicular to AC,
    (c) BC is perpendicular to AC.

15. The coordinates of the vertices of a triangle ABC are $A(1, 2)$, $B(6, 7)$ and $C(7, 2)$. Find the equations of the perpendicular bisectors of
    (a) AB,   (b) BC.

16. The coordinates of the vertices of a square are $A(4, 0)$, $B(-3, 1)$, $C(-2, 8)$ and $D(5, 7)$.

(i) Find the gradient of the line segment joining the points $A(4, 0)$ and $C(-2, 8)$.

(ii) Given that B is the point $(-3, 1)$, explain why $AB = BC$ and that $\angle ABC = 90°$, showing your working clearly.

(iii) Hence, or otherwise, calculate the area of the square.

Summary
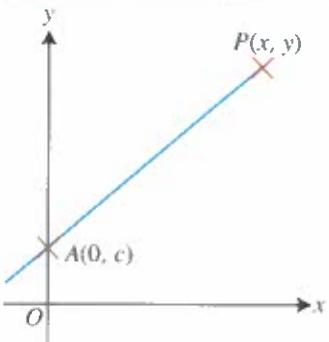
1.	**Gradient** If $A(x_1, y_1)$ and $B(x_2, y_2)$ are two points on a line, then gradient of AB is $\dfrac{y_2 - y_1}{x_2 - x_1}$. **Length** The length of any line segment AB, where the coordinates of the points A and B are (x_1, y_1) and (x_2, y_2) respectively is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.
2.	**Equation of an oblique line** The equation of a straight line passing through the point $(0, c)$ and with gradient m is $y = mx + c$.
3.	**Equation of a horizontal line** The equation of a straight line that is parallel to the x-axis and which passes through the point (a, b) is $y = b$. It has a gradient of **0**. **Equation of a vertical line** The equation of a straight line that is parallel to the y-axis and which passes through the point (a, b) is $x = a$. Its gradient is **undefined**.
4.	For two lines l_1 and l_2, (i) If l_1 is **parallel** to l_2, then their gradients are the same. (ii) If l_1 is **perpendicular** to l_2, then the product of their gradients is -1.
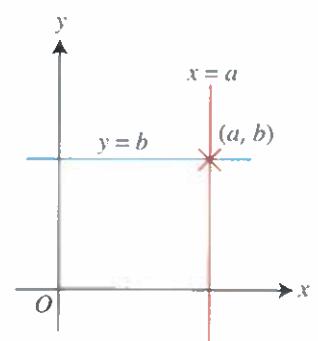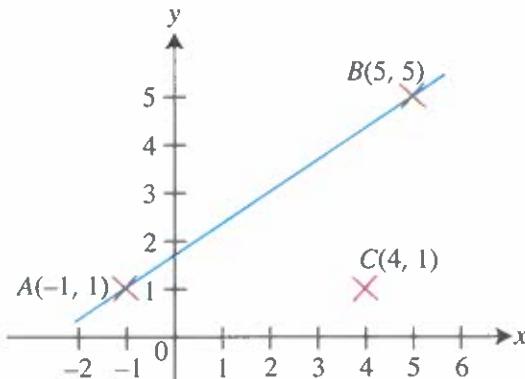
Review Exercise 6

1. A straight line has a gradient of 2 and passes through the point $(0, -3)$.
   (i) Write down the equation of the straight line.
   (ii) Given that the line also passes through the point $(4, k)$, find the value of k.

2. The equation of a straight line is $6x + 2y = 7$.
   (i) Find the gradient of the line.

   Another line with equation $y = mx + c$ has the same gradient as $6x + 2y = 7$ and passes through the point $(3, 5)$.

   (ii) Find the value of c.

3. The coordinates of the points A and B are $(1, 5)$ and $(2, -3)$ respectively. Find the equation of the line passing through the origin and having the same gradient as AB.

4. The line l has equation $3x - 4y = 24$. It intersects the x-axis at A and the y-axis at B. Given that M is the point $(4, -3)$, find
   (i) the gradient of l,
   (ii) the length of AB,
   (iii) the equation of the line passing through B and having the same gradient as OM, where O is the origin.

5. The coordinates of the points A and B are $(0, 6)$ and $(8, 0)$ respectively.
   (i) Find the equation of the line passing through A and B.

   Given that the line $y = x + 1$ cuts the line AB at the point M, find

   (ii) the coordinates of M,
   (iii) the equation of the line which passes through M and is parallel to the x-axis,
   (iv) the equation of the line which passes through M and is parallel to the y-axis.
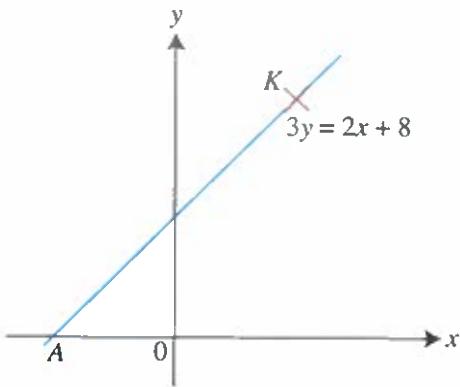
6. The diagram shows the line l passing through the points $A(-1, 1)$ and $B(5, 5)$.

   Given that C is the point $(4, 1)$, find
   (i) the gradient of l,
   (ii) the equation of l,
   (iii) the area of $\triangle ABC$,
   (iv) the length of BC, giving your answer correct to 2 decimal places.
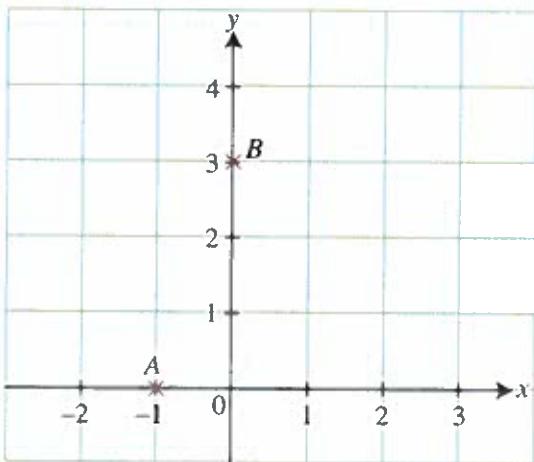
7. The diagram shows the graph of $3y = 2x + 8$. The line cuts the x-axis at A and the point K lies on the line such that its distance from the x-axis is twice that from the y-axis.

Find

(i) the coordinates of the point A,

(ii) the coordinates of the point K,

(iii) the area of $\triangle AKH$, where H is the point $(4, 0)$.
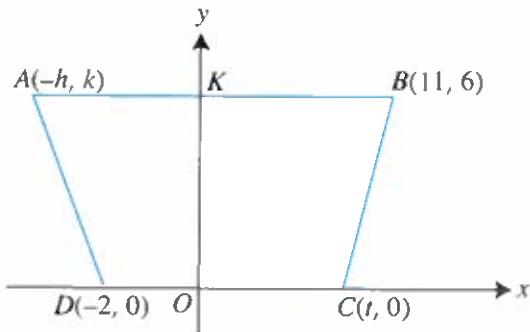
8. The diagram shows the points $A(-1, 0)$ and $B(0, 3)$.

(i) Find the equation of the line passing through A and B.

(ii) Given that the length of AB is $\sqrt{h}$ units, find the value of h.

The point $(-5, k)$ lies on BA produced.

(iii) Find the value of k.

(iv) Given that $y = x + 1$ is the line of symmetry of $\triangle ABC$, find the coordinates of C.
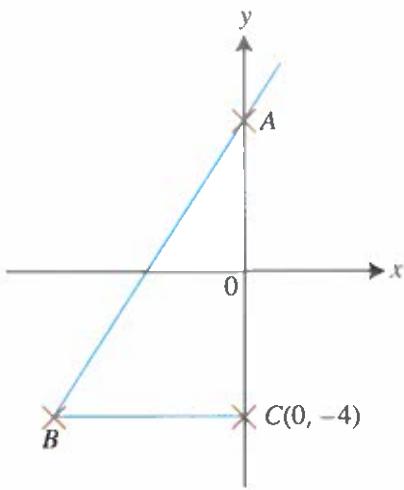
9. The equation of a straight line l_1 is $5y + 12x - 60 = 0$. It intersects the x-axis at P and the y-axis at Q.

(i) Write down the coordinates of P and of Q.

(ii) Find the length of PQ.

Another line l_2 has the same gradient as l_1 and passes through the point $(0, -2)$.

(iii) Find the equation of the line l_2.

(iv) Given that the y-axis is the line of symmetry of $\triangle PQR$, find the coordinates of R.

10. The diagram shows a trapezium $ABCD$ in which AB is parallel to DC and the area of $ABCD$ is 84 units^2.
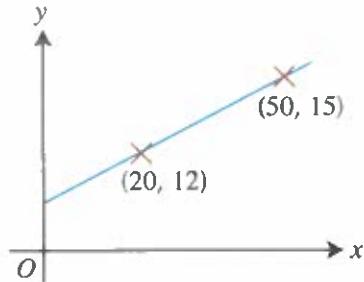
Given that the vertices are $A(-h, k)$, $B(11, 6)$, $C(t, 0)$ and $D(-2, 0)$, find an expression for t in terms of h.

11. In the diagram, C is the point $(0, -4)$ and A is a point on the y-axis. The line AB meets the horizontal line through C at B.

(i) Write down the equation of the line passing through B and C.

(ii) Given that the equation of the line passing through A and B is $2y - 5x = 4$, find the area of $\triangle ABC$.

(iii) Given that the length of AB is $\sqrt{l}$ units, find the value of l.

(iv) Find the length of the perpendicular from C to AB.

(v) Given that the coordinates of D are $(2.4, 2)$, show that $ABCD$ is a parallelogram.

12. A spring is suspended freely. When a mass of 20 g is attached to the spring, it has a length of 12 cm. When a mass of 50 g is attached to the spring, it has a length of 15 cm. The graph below shows how the length, y cm, of the spring, varies with the mass, x g, attached to it.
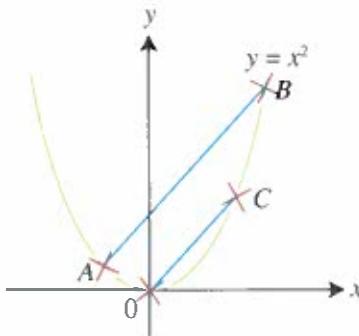
(i) Find an expression for y in terms of x.

(ii) State what the value of the y-intercept represents.

13. Find the equation of the passing through the point $\left(\dfrac{10}{11}, \dfrac{3}{11} \right)$ and perpendicular to the line $3x + 5y = 13$.

14. Find the equation of the line passing through the point $(2, 3)$ and parallel to the line $y = 2x - 13$.

Challenge Yourself

The diagram shows two points A and B on the graph of $y = x^2$. A line, drawn from the origin O and parallel to AB, intersects the graph at C. The coordinates of A, B and C are (a, a^2), (b, b^2) and (c, c^2) respectively. Show that $a + b = c$.

Graphs of Functions and Graphical Solution

A newspaper article states that the growth in the number of members of a social network increased exponentially in its first year of operation. What is meant by an exponential increase? How can you represent this information on a graph?

Chapter
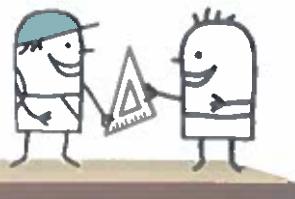
Seven

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7.1 Graphs of Cubic Functions

In Book 2, we have learnt how to draw the graphs of $y = ax^2 + bx + c$, where a, b and c are constants and $a \neq 0$. In Chapter 1 of this book, we have learnt how to sketch graphs of the form $y = (x - h)(x - k)$ and $y = -(x - h)(x - k)$, where h and k are constants, and $y = (x - p)^2 + q$ and $y = -(x - p)^2 + q$, where p and q are constants.

In this section, we will learn how to draw the graphs of cubic functions.

In general, cubic functions are of the form $y = ax^3 + bx^2 + cx + d$, where a, b, c and d are constants and $a \neq 0$.
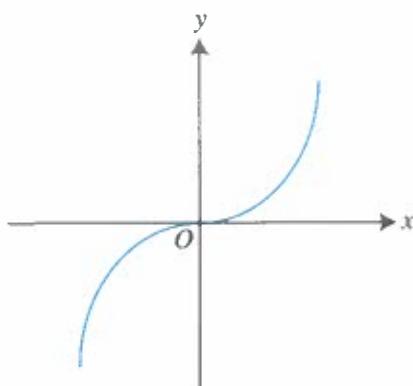
Investigation
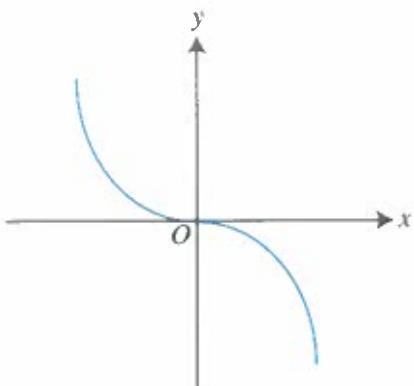
Graphs of Cubic Functions

1. Using a graphing software, draw each of the following graphs.

   (a) $y = x^3$ (b) $y = 2x^3$

   (c) $y = 5x^3$ (d) $y = -x^3$

   (e) $y = -2x^3$ (f) $y = -5x^3$

2. For the graph of $y = ax^3$, where a is a constant, how does the value of a affect the shape of the graph?

3. Using a graphing software, draw each of the following graphs.

   (a) $y = x^3 - x^2 + 1$ (b) $y = x^3 + 4x^2 - 3$

   (c) $y = x^3 + x$ (d) $y = -x^3 + x^2 - 2$

   (e) $y = -x^3 + 2x^2 + 1$ (f) $y = -x^3 - 0.5x - 1$

4. For each of the graphs in Question 3, how does the coefficient of x^3 affect the shape of the graph?

From the investigation, we observe that for the graph of a cubic function of the form $y = ax^3$, the graph takes the shape in Fig. 7.1 (a) and (b) for positive and negative values of the coefficient of x^3.
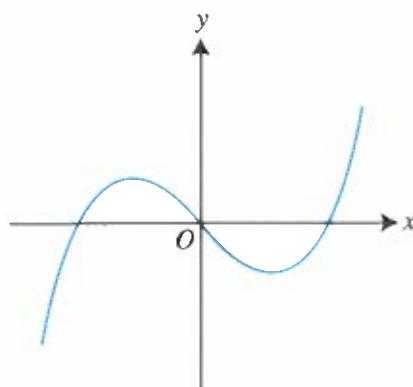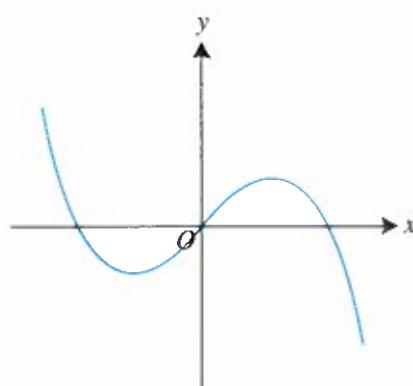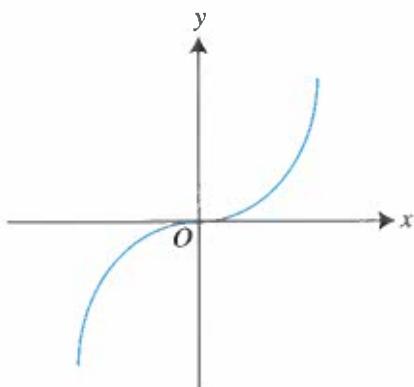
(a) $y = ax^3$, where $a > 0$ **(b)** $y = ax^3$, where $a < 0$

Fig. 7.1

In general, the graph of a cubic function, i.e. a function of the form $y = ax^3 + bx^2 + cx + d$ would take the shape in Fig. 7.2 (but may not pass through the origin if $d \neq 0$).

(a) $y = ax^3 + bx^2 + cx + d$, where $a > 0$

(b) $y = ax^3 + bx^2 + cx + d$, where $a < 0$

Fig. 7.2

Worked Example 1

Using a scale of 2 cm to represent 1 unit on the x-axis and 2 cm to represent 10 units on the y-axis, draw the graphs of $y = x^3 + 3$ and $y = -x^3 - 3$ for $-3 \leqslant x \leqslant 3$.

For each graph, find

(i) the value of y when $x = 1.5$,

(ii) the value of x when $y = 20$.
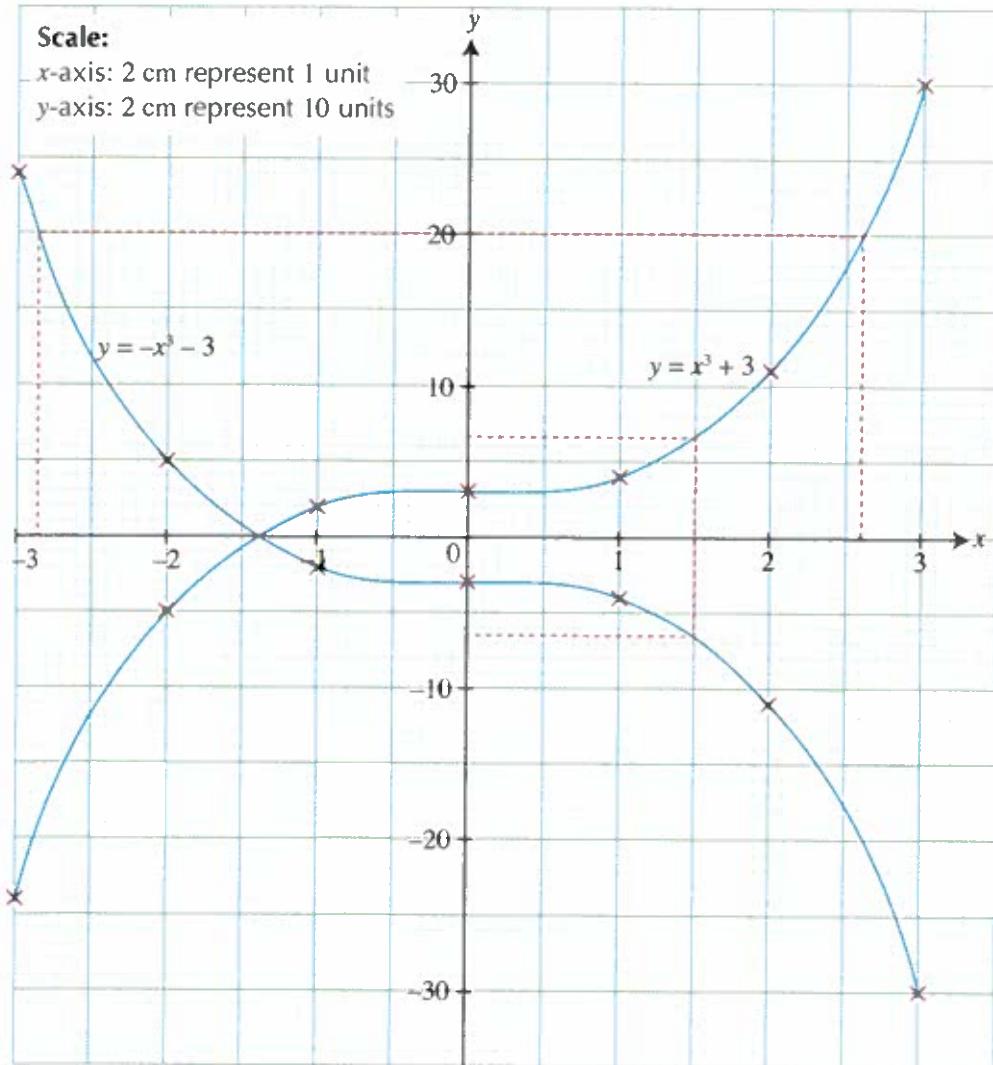
Solution:

x	-3	-2	-1	0	1	2	3
$y = x^3 + 3$	-24	-5	2	3	4	11	30
$y = -x^3 - 3$	24	5	-2	-3	-4	-11	-30

Scale:
x-axis: 2 cm represent 1 unit
y-axis: 2 cm represent 10 units

(i) Consider $y = x^3 + 3$.

From the graph, when $x = 1.5$, $y = 6.5$.

Consider $y = -x^3 - 3$.

From the graph, when $x = 1.5$, $y = -6.5$.

(ii) Consider $y = x^3 + 3$.

From the graph, when $y = 20$, $x = 2.6$.

Consider $y = -x^3 - 3$.

From the graph, when $y = 20$, $x = -2.85$.
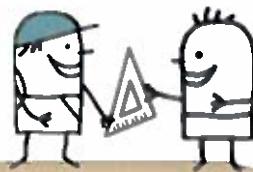
Using a scale of 2 cm to represent 1 unit on the x-axis and 2 cm to represent 10 units on the y-axis, draw the graphs of $y = x^3 + 2$ and $y = -x^3 - 2$ for $-3 \leqslant x \leqslant 3$.

For each graph, find

(i) the value of y when $x = 2.5$,

(ii) the value of x when $y = 15$.

7.2 Graphs of Reciprocal Functions

Graph of $y = \dfrac{a}{x}$

Investigation
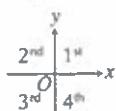
Graphs of $y = \dfrac{a}{x}$

Using a graphing software, draw the graph of $y = \dfrac{a}{x}$ for $a = 1$, $a = 5$, $a = -1$ and $a = -3$.

1. **(i)** For $a > 0$, which quadrants do the graphs lie in?

   (ii) For $a < 0$, which quadrants do the graphs lie in?

2. What can you say about the rotational symmetry of each of the graphs?

3. Do the graphs intersect the x-axis and the y-axis? Explain your answer.

INFORMATION

The four quadrants on the Cartesian plane are labelled 1st, 2nd, 3rd and 4th as follows:
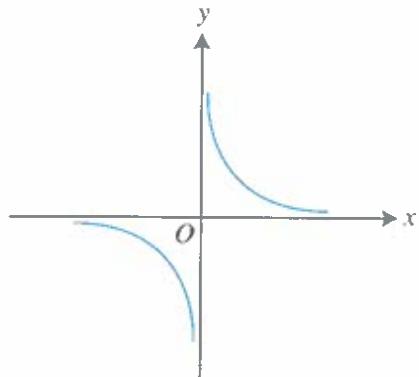
RECALL
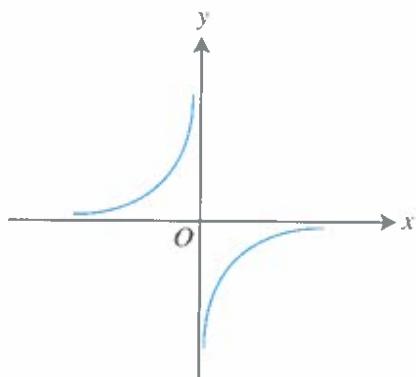
The order of rotational symmetry about a particular point is the number of distinct ways in which a figure can map onto itself by rotation in 360°.

From the investigation, we observe that for the graph of $y = \dfrac{a}{x}$,

- when $x = 0$, the function $y = \dfrac{a}{x}$ is not defined, i.e. there is a break when $x = 0$,
- there is rotational symmetry of order 2 about the origin, i.e. it maps onto itself twice by rotation in $360°$,
- if $a > 0$, the graph would take the shape in Fig. 7.3(i),
  if $a < 0$, the graph would take the shape in Fig. 7.3(ii).

(i) $y = \dfrac{a}{x}$, where $a > 0$ (ii) $y = \dfrac{a}{x}$, where $a < 0$

Fig. 7.3

Consider the graph in Fig. 7.3(i) $y = \dfrac{a}{x}$, where $a > 0$. The graph consists of two parts that lie in the 1st and 3rd quadrants. In the 1st quadrant, we observe that:

- as x increases, y decreases;
- as x approaches zero, y becomes very large;

  e.g. for $a = 1$, $y = \dfrac{1}{x}$, if $x = 0.000\ 001$, $y = \dfrac{1}{0.000\ 001} = 1\ 000\ 000$;

- as x becomes very large, y approaches zero;

  e.g. for $a = 1$, $y = \dfrac{1}{x}$, if $x = 1\ 000\ 000$, $y = \dfrac{1}{1\ 000\ 000} = 0.000\ 001$;

- the curve gets very close to the x-axis and y-axis but never touches them.

Can you describe the part of the graph that is in the 3rd quadrant?

Can you describe the graph of $y = \dfrac{a}{x}$, where $a < 0$?

Thinking Time

What are the equations of the lines of symmetry of the graph $y = \dfrac{a}{x}$ when
(a) $a > 0$?   (b) $a < 0$?

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Worked Example 2

(Drawing the Graph of $y = \dfrac{a}{x}$)

Using a scale of 1 cm to represent 1 unit on the x-axis and 1 cm to represent 5 units on the y-axis, draw the graph of $y = \dfrac{6}{x}$ for $-5 \leqslant x \leqslant 5$, $x \neq 0$. Find

(i) the value of y when $x = 1.4$,

(ii) the value of x when $y = -8$.
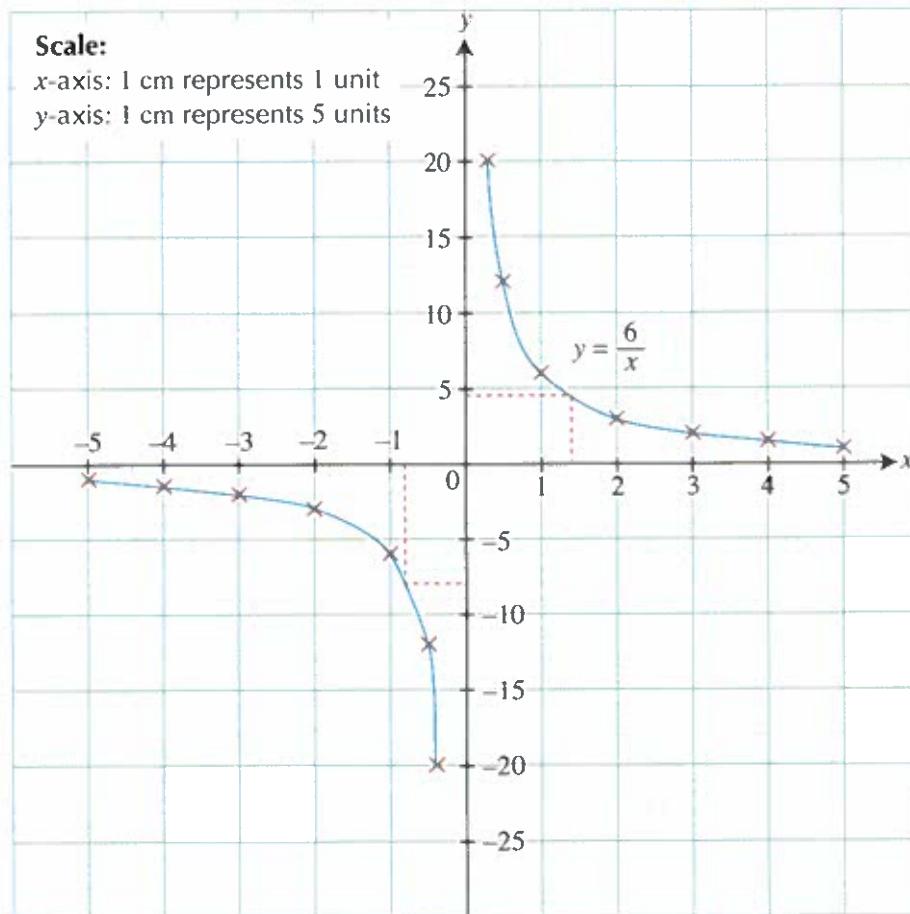
Solution:

x	-5	-4	-3	-2	-1	-0.5	-0.3	0.3	0.5	1	2	3	4	5
y	-1.2	-1.5	-2	-3	-6	-12	-20	20	12	6	3	2	1.5	1.2

Scale:
x-axis: 1 cm represents 1 unit
y-axis: 1 cm represents 5 units

$y = \dfrac{6}{x}$

(i) From the graph, when $x = 1.4$, $y = 4.5$.

(ii) From the graph, when $y = -8$, $x = -0.8$.

ATTENTION

For **(i)**, although the answer is 4.2857... by calculation, the answer obtained from the graph can only be accurate up to half of a small square grid, which is 0.5. Similarly, for **(ii)**, although $x = -0.75$ by calculation, the answer obtained from the graph is accurate to half of a small square grid, i.e. 0.1.

PRACTISE NOW 2

Using a scale of 1 cm to represent 1 unit on both axes, draw the graph of $y = \dfrac{3}{x}$ for $-5 \leqslant x \leqslant 5$, $x \neq 0$. Find

(i) the value of y when $x = 2.5$,

(ii) the value of x when $y = -1.2$.

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Graph of $y = \dfrac{a}{x^2}$

Investigation
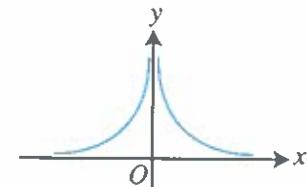
Graphs of $y = \dfrac{a}{x^2}$

Using a graphing software, draw the graph of $y = \dfrac{a}{x^2}$ for $a = 2$, $a = 4$, $a = -1$ and $a = -3$.
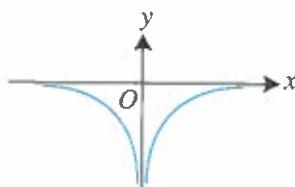
1. **(i)** For $a > 0$, which quadrants do the graphs lie in?

   (ii) For $a < 0$, which quadrants do the graphs lie in?

2. What can you say about the line symmetry of each of the graphs?

3. Do the graphs intersect the x-axis and the y-axis? Explain your answer.

From the investigation, we observe that for the graph of $y = \dfrac{a}{x^2}$,

- when $x = 0$, the function $y = \dfrac{a}{x^2}$ is not defined, i.e. there is a break when $x = 0$,

- if $a > 0$, the values of y are always positive, i.e. the graph lies entirely above the x-axis;

  if $a < 0$, the values of y are always negative, i.e. the graph lies entirely below the x-axis.

- the graph is symmetrical about the y-axis, i.e. the y-axis is the line of symmetry,

- if $a > 0$, the graph would take the shape in Fig. 7.4(i),

  if $a < 0$, the graph would take the shape in Fig. 7.4(ii).

(i) $y = \dfrac{a}{x^2}$, where $a > 0$ **(ii)** $y = \dfrac{a}{x^2}$, where $a < 0$

Fig. 7.4

Consider the graph in Fig. 7.4(i) $y = \dfrac{a}{x^2}$, where $a > 0$. The graph consists of two parts that lie in the 1st and 2nd quadrants. In the 1st quadrant, we observe that:

- as x increases, y decreases;

- as x approaches zero, y becomes very large;

  e.g. for $a = 1$, $y = \dfrac{1}{x^2}$, if $x = 0.001$, $y = \dfrac{1}{0.001^2} = 1\,000\,000$;

- as x becomes very large, y approaches zero;

  e.g. for $a = 1$, $y = \dfrac{1}{x^2}$, if $x = 1000$, $y = \dfrac{1}{1000^2} = 0.000\,001$;

- the curve gets very close to the x-axis and y-axis but never touches them.

Can you describe the part of the graph that is in the 2nd quadrant?

Can you describe the graph of $y = \dfrac{a}{x^2}$, where $a < 0$?

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Worked Example 3
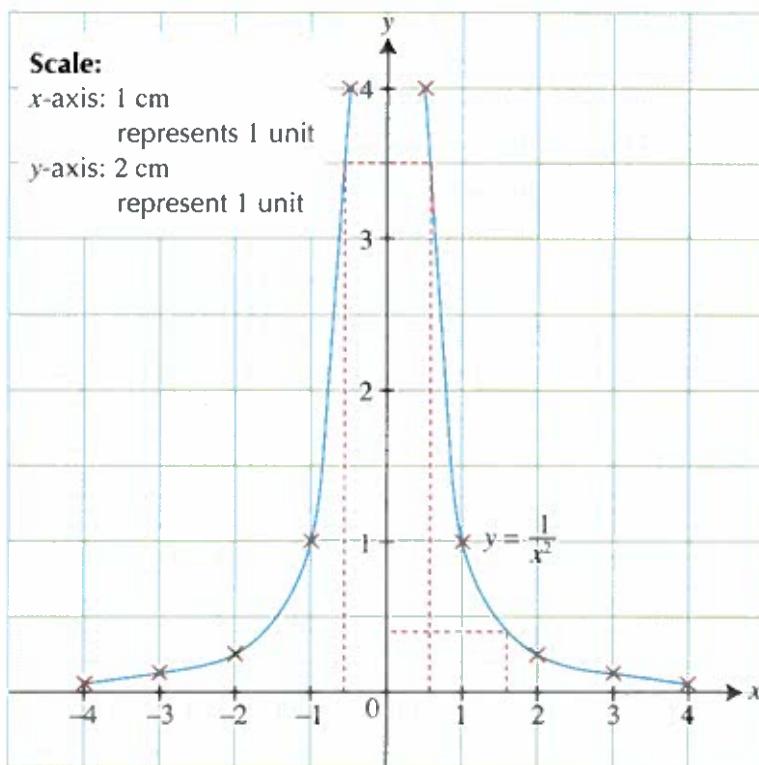
(Drawing the Graph of $y = \dfrac{a}{x^2}$)

Using 1 cm to represent 1 unit on the x-axis and 2 cm to represent 1 unit on the y-axis, draw the graph of

$y = \dfrac{1}{x^2}$ for $-4 \leqslant x \leqslant 4$, $x \neq 0$. Find

(i) the value of y when $x = 1.6$,

(ii) the values of x when $y = 3.5$.

Solution:

x	-4	-3	-2	-1	-0.5	0.5	1	2	3	4
y	0.06	0.11	0.25	1	4	4	1	0.25	0.11	0.06

Scale:
x-axis: 1 cm represents 1 unit
y-axis: 2 cm represent 1 unit

$y = \dfrac{1}{x^2}$

(i) From the graph, when $x = 1.6$, $y = 0.4$.

(ii) From the graph, when $y = 3.5$, $x = 0.6$ or $x = -0.6$.

ATTENTION

The accuracy of the answer can only be accurate up to half of a small square grid.

PRACTISE NOW 3

Using 1 cm to represent 1 unit on the x-axis and 2 cm to represent 1 unit on the y-axis, draw the graph of $y = -\dfrac{2}{x^2}$ for $-4 \leqslant x \leqslant 4$, $x \neq 0$. Find

(i) the value of y when $x = 1.5$,

(ii) the values of x when $y = -3.2$.

SIMILAR QUESTIONS

Exercise 7A Questions 4, 9, 10, 13

Exercise 7A

1. The table below shows some values of x and the corresponding values of y, where $y = x^3$.

x	-3	-2	-1	0	1	2	3
y		-8			1		27

(a) Copy and complete the table.

(b) Using a scale of 2 cm to represent 1 unit, draw a horizontal x-axis for $-3 \leqslant x \leqslant 3$.

Using a scale of 2 cm to represent 5 units, draw a vertical y-axis for $-27 \leqslant y \leqslant 27$.

On your axes, plot the points given in the table and join them with a smooth curve.

(c) Use your graph to find

(i) the value of y when $x = 1.5$,

(ii) the value of x when $y = 12$.

2. The table below shows some values of x and the corresponding values of y, correct to 1 decimal place, where $y = 2x^3 + 3$.

x	-2	-1.5	-1	-0.5	0	0.5	1	1.5	2
y	-13	-3.8	1	p	3	3.3	5	9.8	19

(a) Find the value of p.

(b) Using a scale of 4 cm to represent 1 unit, draw a horizontal x-axis for $-2 \leqslant x \leqslant 2$.

Using a scale of 1 cm to represent 5 units, draw a vertical y-axis for $-15 \leqslant y \leqslant 20$.

On your axes, plot the points given in the table and join them with a smooth curve.

(c) Use your graph to find

(i) the value of y when $x = -1.2$,

(ii) the value of x when $y = 14$.

3. The table below shows some values of x and the corresponding values of y, where $y = \dfrac{4}{x}$.

x	$\dfrac{1}{4}$	$\dfrac{1}{2}$	1	2	3	4	5
y	16			2		1	

(a) Copy and complete the table.

(b) Using a scale of 2 cm to represent 1 unit on the x-axis and 1 cm to represent 1 unit on the y-axis, draw the graph of $y = \dfrac{4}{x}$ for $\dfrac{1}{4} \leqslant x \leqslant 5$.

(c) Use your graph to find

(i) the value of y when $x = 3.6$,

(ii) the value of x when $y = 1.5$.

4. The table below shows some values of x and the corresponding values of y, correct to 1 decimal place, where $y = \dfrac{10}{x^2}$.

x	1	2	3	4	5
y	10	2.5	a	0.6	b

(a) Find the value of a and of b.

(b) Using a scale of 2 cm to represent 1 unit on both axes, draw the graph of $y = \dfrac{10}{x^2}$ for $1 \leqslant x \leqslant 5$.

(c) Use your graph to find

(i) the value of y when $x = 2.8$,

(ii) the value of x when $y = 4.4$.

5. Using a scale of 2 cm to represent 1 unit on the x-axis and 2 cm to represent 5 units on the y-axis, draw the graph of $y = 3x - x^3$ for $-3 \leqslant x \leqslant 3$. Use your graph to find

(i) the value of y when $x = 1.4$,

(ii) the values of x when $y = -6.6$.

6. Using a suitable scale, draw the graph of $y = x^3 - 6x^2 + 13x$ for $0 \leqslant x \leqslant 5$. Use your graph to find

   (a) the value(s) of y when

       (i) $x = 1.5$,   (ii) $x = 3.5$,   (iii) $x = 4.45$.

   (b) the value of x when

       (i) $y = 7$,   (ii) $y = 15$,   (iii) $y = 22$.

7. Using a scale of 4 cm to represent 1 unit on both axes, draw the graph of $y = -\dfrac{2}{x} - 1$ for $\dfrac{1}{2} \leqslant x \leqslant 4$. Use your graph to find

   (i) the value of y when $x = 2.5$,

   (ii) the value of x when $y = -1.6$.

8. The table below shows some values of x and the corresponding values of y, correct to 1 decimal place, where $y = x - \dfrac{3}{x}$.

x	0.5	1	2	3	4	5	6
y	−5.5	−2	0.5	h	3.3	4.4	k

   (a) Find the value of h and of k.

   (b) Using a scale of 2 cm to represent 1 unit on both axes, draw the graph of $y = x - \dfrac{3}{x}$ for $0.5 \leqslant x \leqslant 6$.

   (c) Use your graph to find

       (i) the value of y when $x = 1.6$,

       (ii) the value of x when $y = -2.5$.

9. Using a scale of 2 cm to represent 1 unit on the x-axis and 4 cm to represent 1 unit on the y-axis, draw the graph of $y = 2 - \dfrac{3}{x^2}$ for $1 \leqslant x \leqslant 6$. Use your graph to find

   (i) the value of y when $x = 1.5$,

   (ii) the value of x when $y = 1.5$.

10. Using a scale of 2 cm to represent 1 unit on both axes, draw the graph of $y = x + \dfrac{2}{x^2}$ for $1 \leqslant x \leqslant 6$. Use your graph to find

    (i) the value of y when $x = 5.4$,

    (ii) the values of x when $y = 3$.

11. Using a suitable scale, draw the graph of $y = x^3 - 2x - 1$ for $-3 \leqslant x \leqslant 3$.

    (a) Use your graph to find the x-coordinates of the points of intersection of the curve with the x-axis.

    (b) On the same axes, draw the straight line $y = x$ for $-3 \leqslant x \leqslant 3$.

        (i) Write down the x-coordinates of the points at which the line $y = x$ meets the curve $y = x^3 - 2x - 1$.

        (ii) Hence, state the solutions of the equation $x^3 - 2x - 1 = x$. Explain your answer.

12. The variables x and y are connected by the equation $y = x + \dfrac{1}{2x} - 1$.

    The table below shows some values of x and the corresponding values of y, correct to 1 decimal place.

x	0.1	0.5	0.8	1	1.5	2	2.5	3	3.5	4
y	4.1	0.5	0.4	0.5	0.8	1.3	p	2.2	2.6	3.1

    (a) Calculate the value of p.

    (b) Using a scale of 4 cm to represent 1 unit on both axes, draw the graph of $y = x + \dfrac{1}{2x} - 1$ for $0.1 \leqslant x \leqslant 4$.

    (c) Use your graph to find the values of x in the range $0.1 \leqslant x \leqslant 4$ for which $x + \dfrac{1}{2x} = 1$.

13. Using a scale of 2 cm to represent 1 unit on the x-axis and 1 cm to represent 1 unit on the y-axis, draw the graph $y = \dfrac{1}{4}x^2 + \dfrac{8}{x} - 9$ for $0.5 \leqslant x \leqslant 7$.

    (a) Use your graph to find the minimum value of y in the given range.
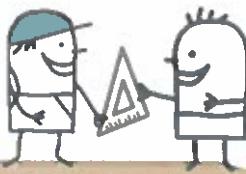
    (b) By drawing suitable straight lines to the graph, solve each of the following equations, giving your answers correct to 1 decimal place.

        (i) $\dfrac{1}{4}x^2 + \dfrac{8}{x} = 6$

        (ii) $\dfrac{1}{4}x^2 + \dfrac{8}{x} = x + 4$

        (iii) $\dfrac{1}{4}x^2 + 2x = 15 - \dfrac{8}{x}$

7.3 Graphs of Exponential Functions

Graphs of $y = a^x$ and $y = ka^x$

Investigation

Graphs of $y = a^x$ and $y = ka^x$

1. Using a graphing software, draw each of the following graphs.

   (a) $y = 2^x$ (b) $y = 3^x$

   (c) $y = 4^x$ (d) $y = 5^x$

2. For each of the graphs in Question 1, answer the following questions.

   (a) Write down the coordinates of the point where the graph intersects the y-axis.

   (b) As x increases, what happens to the value of y?

   (c) Does the graph intersect the x-axis?

3. How does the value of a affect the shape of the graph of $y = a^x$?

4. Using a graphing software, draw each of the following graphs.

   (a) $y = 2^x$ (b) $y = 3(2^x)$

   (c) $y = 5(2^x)$ (d) $y = -2^x$

   (e) $y = -4(2^x)$

5. For each of the graphs in Question 4, answer the following questions.

   (a) Write down the coordinates of the point where the graph intersects the y-axis.

   (b) As x increases, what happens to the value of y?

   (c) Does the graph intersect the x-axis?

6. How does the value of k affect the shape of the graph of $y = ka^x$?

From the investigation, we observe that for the graph of $y = a^x$,
- the values of y are always positive, i.e. the graph lies entirely above the x-axis,
- the graph intersects the y-axis at $(0, 1)$.
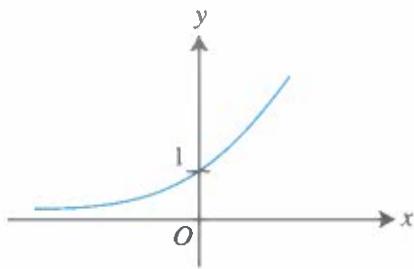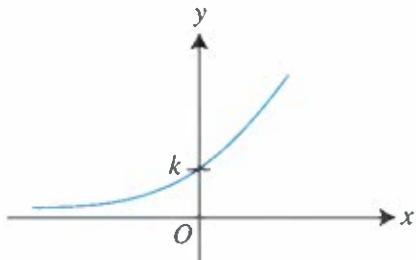
Fig. 7.5
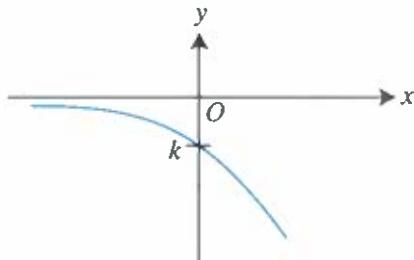
As the positive value of x increases and tends to the right of the graph, the value of y increases very rapidly and approaches infinity. When x is negative and tends to the left of the graph, y becomes smaller as x becomes smaller. The curve gets very close to the x-axis but never touches it.

For the graph of $y = ka^x$,

- if $k > 0$, the values of y are always positive, i.e. the graph lies entirely above the x-axis (see Fig. 7.6(i)),

  if $k < 0$, the values of y are always negative, i.e. the graph lies entirely below the x-axis (see Fig. 7.6(ii)),

- the graph intersects the y-axis at $(0, k)$.

(i) $y = ka^x$, where $k > 0$
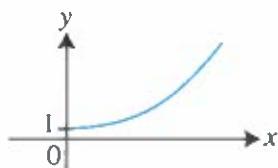
(ii) $y = ka^x$, where $k < 0$

Fig. 7.6

Journal Writing

A newspaper article states that the growth in the number of members of a social network increased **exponentially** in its first year of operation and can be represented by the equation $y = 28^x$, where x is the number of months and y is the number of members.

(i) Describe how the number of members of the social network changes with time.
(ii) Search on the Internet for more real-life applications of exponential graphs.
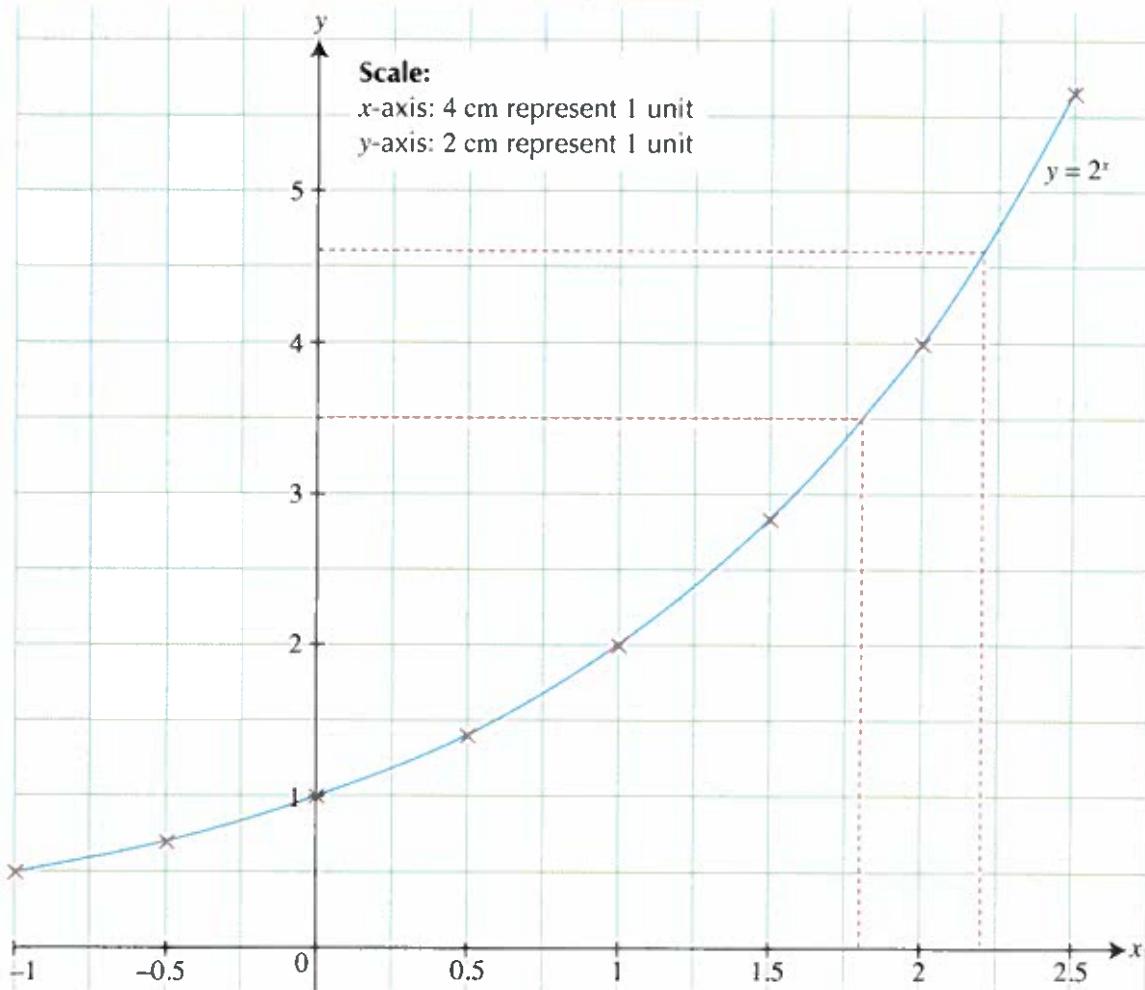
(Graph of y = aˣ)

Using a scale of 4 cm to represent 1 unit on the *x*-axis and 2 cm to represent 1 unit on the *y*-axis, draw the graph of $y = 2^x$ for $-1 \leqslant x \leqslant 2.5$. Use your graph to find

(i) the value of *y* when *x* = 1.8,
(ii) the value of *x* when *y* = 4.6.

Solution:

x	−1	−0.5	0	0.5	1	1.5	2	2.5
y	0.5	0.71	1	1.41	2	2.83	4	5.66

Scale:
x-axis: 4 cm represent 1 unit
y-axis: 2 cm represent 1 unit

$y = 2^x$

(i) From the graph, when *x* = 1.8, *y* = 3.5.

(ii) From the graph, when *y* = 4.6, *x* = 2.2.

PRACTISE NOW 4

Using a scale of 4 cm to represent 1 unit on the *x*-axis and 2 cm to represent 1 unit on the *y*-axis, draw the graph of $y = 3^x$ for $-2 \leqslant x \leqslant 2$. Use your graph to find

(i) the value of *y* when *x* = −1,
(ii) the value of *x* when *y* = 0.7.

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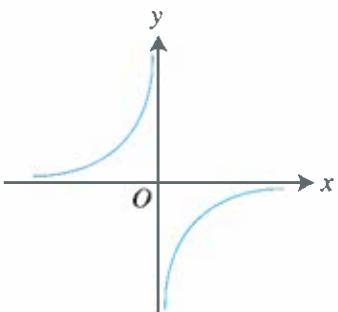
Matching Graphs of Power Functions with the Corresponding Functions
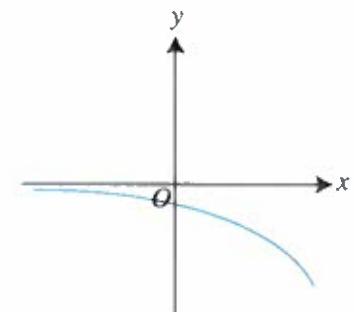
Work in pairs.

Match the graphs with their respective functions and justify your answers. If your classmate does not obtain the correct answer, explain to him what he has done wrong.
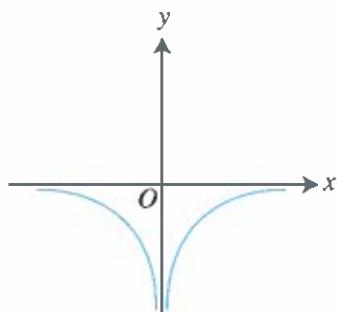
A: $y = 2x^3$	**B**: $y = -\dfrac{6}{x}$	**C**: $y = \dfrac{5}{2x^2}$	**D**: $y = 5^x$
E: $y = -\dfrac{3}{x^2}$	**F**: $y = -2(6^x)$	**G**: $y = \dfrac{1}{2x}$	**H**: $y = -3x^3$
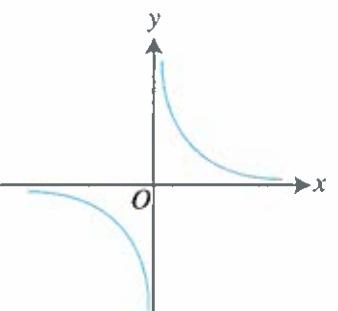
Graph 1
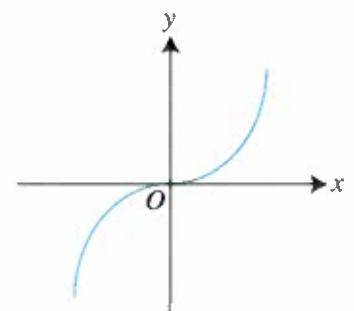
Graph 2
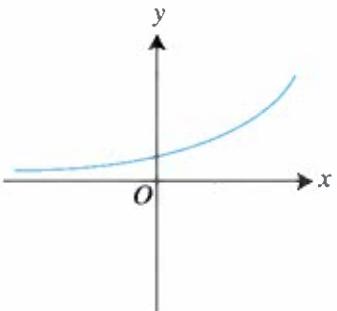
Graph 3
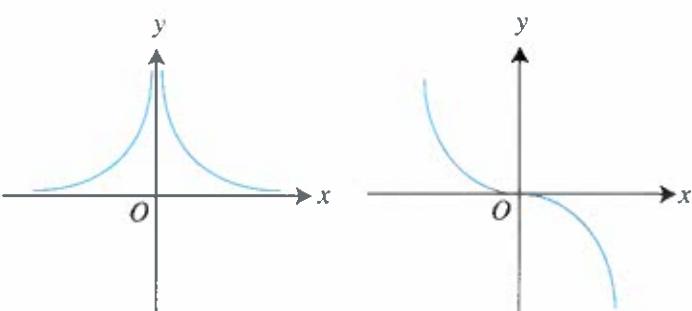
Graph 4

Graph 5
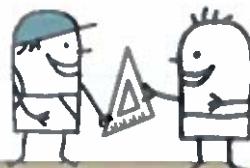
Graph 6

Graph 7

Graph 8
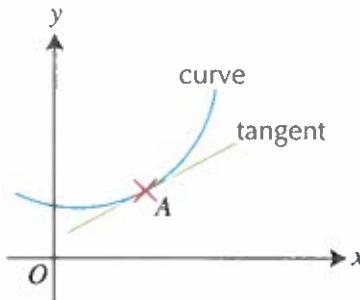
SIMILAR
QUESTIONS

Exercise 7B Question 6

7.4 Gradient of a Curve

When a straight line touches a curve at a single point A, the line is called the **tangent** to the curve at the point A.

When a line l_1 touches the curve at P, l_1 is called the tangent to the curve at P. Similarly, when a line l_2 touches the curve at Q, l_2 is called the tangent to the curve at Q.
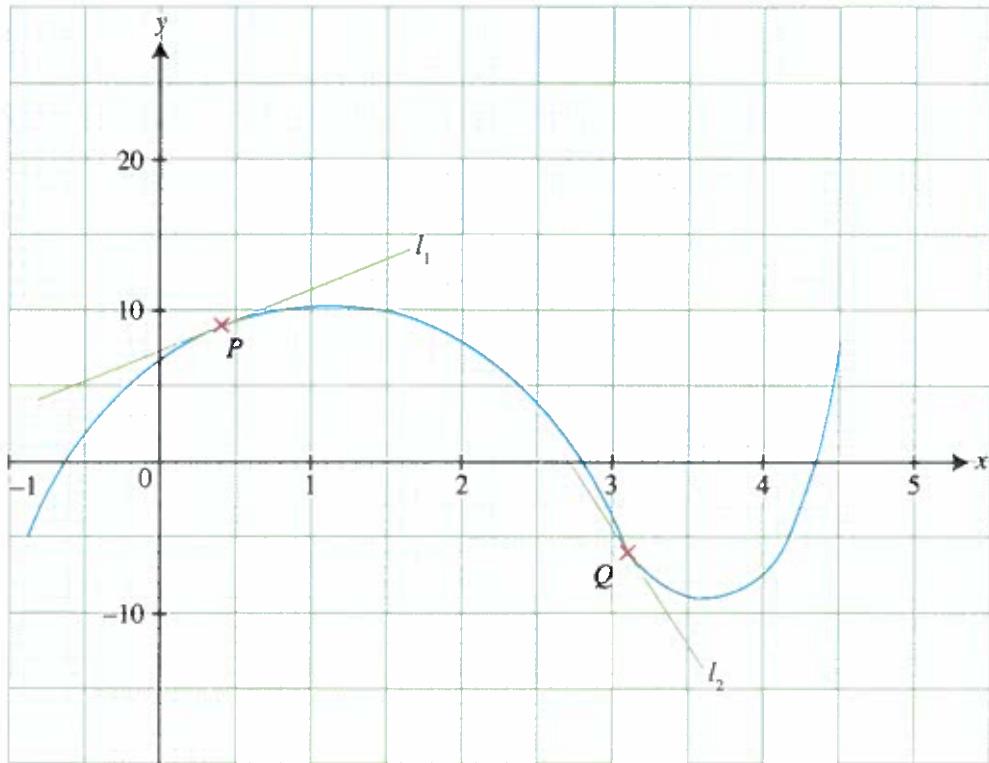
Fig. 7.7

The **gradient of the curve at a point** is defined as the **gradient of the tangent to the curve at that point**. Hence, the gradient of the curve at P in Fig. 7.7 is equal to the gradient of the line l_1 and the gradient of the curve at Q is equal to the gradient of the line l_2.

(Gradient of a Curve)

The variables x and y are connected by the equation $y = \frac{1}{2}(5x - x^2)$.

The table below shows some values of x and the corresponding values of y.

x	$-\frac{1}{2}$	0	1	2	$2\frac{1}{2}$	3	4	5
y	a	0	2	3	b	3	2	0

(a) Find the value of a and of b.

(b) Using a scale of 2 cm to represent 1 unit on both axes, draw the graph of $y = \frac{1}{2}(5x - x^2)$ for $-\frac{1}{2} \leqslant x \leqslant 5$.

(c) By drawing a tangent, find the gradient of the curve at the point (1, 2).

(d) The gradient of the curve at the point (h, k) is zero.

    (i) Draw the tangent at the point (h, k).

    (ii) Hence, find the value of h and of k.

Solution:

(a) When $x = -\frac{1}{2}$,

$$y = \frac{1}{2}\left[5\left(-\frac{1}{2}\right) - \left(-\frac{1}{2}\right)^2\right]$$

$$= -1\frac{3}{8}$$

$$\therefore a = -1\frac{3}{8}$$

$$= -1.375$$

When $x = 2\frac{1}{2}$,

$$y = \frac{1}{2}\left[5\left(2\frac{1}{2}\right) - \left(2\frac{1}{2}\right)^2\right]$$

$$= 3\frac{1}{8}$$

$$\therefore b = 3\frac{1}{8}$$

$$= 3.125$$

Problem Solving Tip

Give the value of a and of b in decimals for easy plotting of points.
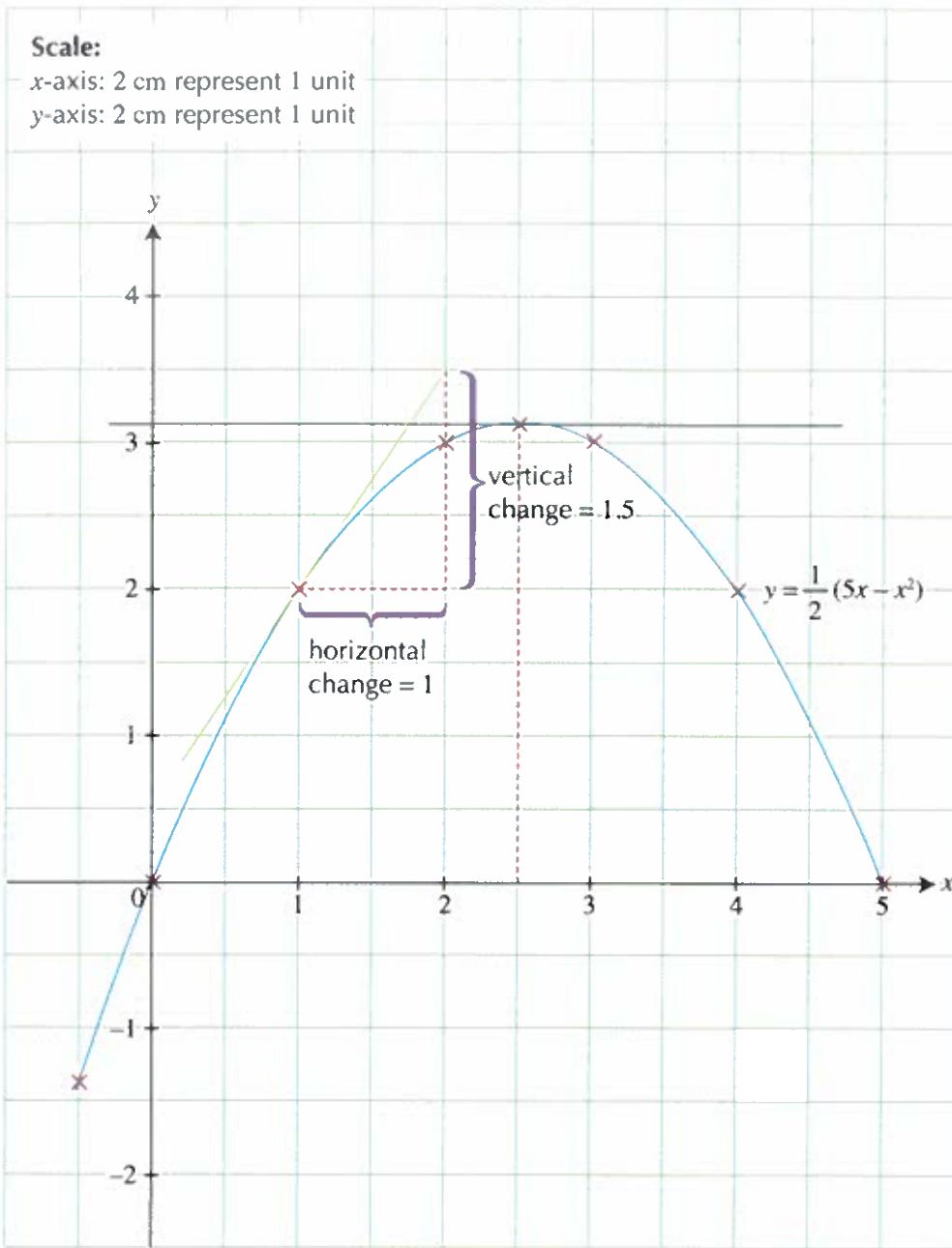
(b)

Scale:
x-axis: 2 cm represent 1 unit
y-axis: 2 cm represent 1 unit

The graph shows the curve $y = \frac{1}{2}(5x - x^2)$ with a tangent drawn at the point $(1, 2)$. The labels indicate:
- vertical change = 1.5
- horizontal change = 1

(c) A tangent is drawn to the curve at the point $(1, 2)$.

From the graph,

$$\text{Gradient} = \frac{\text{vertical change}}{\text{horizontal change}}$$

$$= \frac{1.5}{1}$$

$$= 1.5$$

(d) A line parallel to the *x*-axis at the maximum point of the curve has a gradient equal to zero. From the graph and table, $h = 2.5$, $k = 3\frac{1}{8}$.

The variables x and y are connected by the equation $y = x^2 - 4x$.

The table below shows some values of x and the corresponding values of y.

x	−1	0	1	2	3	4	5
y	a	0	−3	−4	b	0	5

(a) Find the value of a and of b.

(b) Using a scale of 2 cm to represent 1 unit on both axes, draw the graph of $y = x^2 - 4x$ for $-1 \le x \le 5$.

(c) By drawing a tangent, find the gradient of the curve at the point where $x = 2.8$.

(d) The gradient of the curve at the point (h, k) is zero.

   (i) Draw the tangent at the point (h, k).

   (ii) Hence, find the value of h and of k.

Exercise 7B

BASIC LEVEL

1. The table below shows some values of x and the corresponding values of y, where $y = 4^x$.

x	−1	−0.5	0	0.5	1	1.5	2	2.5
y	0.25		1	2	4			

  (a) Copy and complete the table.

  (b) Using a scale of 4 cm to represent 1 unit, draw a horizontal x-axis for $-1 \le x \le 2.5$.

     Using a scale of 1 cm to represent 2 units, draw a vertical y-axis for $0 \le y \le 32$.

     On your axes, plot the points given in the table and join them with a smooth curve.

  (c) Use your graph to find

    (i) the value of y when $x = 1.8$,

    (ii) the value of x when $y = 0.4$.

2. The variables x and y are connected by the equation $y = 3(2^x)$.

The table below shows some values of x and the corresponding values of y correct to 1 decimal place.

x	−1	−0.5	0	0.5	1	1.5	2	2.5
y	1.5	2.1			6		12	17.0

  (a) Copy and complete the table.

  (b) Using a scale of 4 cm to represent 1 unit on the x-axis and 1 cm to represent 1 unit on the y-axis, draw the graph of $y = 3(2^x)$ for $-1 \le x \le 2.5$.

  (c) Use your graph to find

    (i) the values of y when $x = 0.7$ and $x = 2.3$,

    (ii) the values of x when $y = 2.5$ and $y = 7.4$.

3. Using a scale of 4 cm to represent 1 unit on the x-axis and 1 cm to represent 2 units on the y-axis, draw the graph of $y = -2(3^x)$ for $-2 \leqslant x \leqslant 2$. Use your graph to find
   (i) the value of y when $x = 1.2$,
   (ii) the value of x when $y = -6.7$.

4. The table below shows some values of x and the corresponding values of y, correct to 1 decimal place, where $y = 2 + 2^x$.
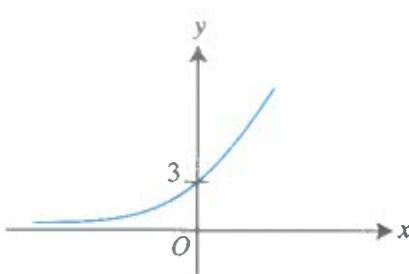
x	-1	-0.5	0	1	1.5	2	2.5	3
y	a	2.7	3	4	4.8	6	b	10

   (a) Find the value of a and of b.
   (b) Using a scale of 4 cm to represent 1 unit on the x-axis and 2 cm to represent 1 unit on the y-axis, draw the graph of $y = 2 + 2^x$ for $-1 \leqslant x \leqslant 3$.
   (c) Use your graph to find
      (i) the values of y when $x = -0.7$ and $x = 2.7$,
      (ii) the values of x when $y = 5.3$ and $y = 7.5$.

5. Using a scale of 4 cm to represent 1 unit on the x-axis and 1 cm to represent 1 unit on the y-axis, draw the graph of $y = 3^x$ for $-2 \leqslant x \leqslant 2$.
   (a) Use your graph to find the value of x when $y = 5.8$.
   (b) On the same axes, draw the graph of $y = \frac{1}{2}x - \frac{1}{x}$, $x \neq 0$.
      (i) Write down the coordinates of the point at which the graph of $y = \frac{1}{2}x - \frac{1}{x}$ meets the curve $y = 3^x$.
      (ii) Hence, state the solution of the equation $3^x + \frac{1}{x} - \frac{1}{2}x = 0$.

6. The sketch represents the graph of $y = ka^x$, where $a > 0$.

   Write down the value of k.

7. The table below shows some values of x and the corresponding values of y, where $y = (x + 2)(4 - x)$.

x	-2	-1	0	1	2	3	4
y	0		8	9			0
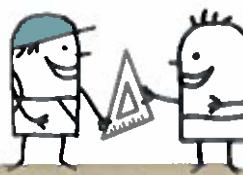
   (a) Copy and complete the table.
   (b) Using a scale of 2 cm to represent 1 unit on both axes, draw the graph of $y = (x + 2)(4 - x)$ for $-2 \leqslant x \leqslant 4$.
   (c) By drawing a tangent, find the gradient of the curve at the point where $x = -1$.
   (d) The gradient of the curve at the point (h, k) is zero.
      (i) Draw the tangent at the point (h, k).
      (ii) Hence, find the value of h and of k.

8. (a) Using a scale of 2 cm to represent 1 unit on the x-axis and 2 cm to represent 5 units on the y-axis, draw the graph of $y = 12 + 10x - 3x^2$ for $-2 \leqslant x \leqslant 5$.

   (b) Find the gradient of the curve when $x = 4$.

   (c) Find the gradient of the curve at the point where the curve intersects the y-axis.

10. (a) Using suitable scale, draw the graph of $y = 1 + \dfrac{1}{x}$ for $0.5 \leqslant x \leqslant 3$.

    (b) On the same axes, draw the line $y = -x$.

    (c) Hence, find the coordinates on the graph of $y = 1 + \dfrac{1}{x}$ at which the gradient of the curve is -1.

9. Using a scale of 4 cm to represent 1 unit on the x-axis and 1 cm to represent 2 units on the y-axis, draw the graph of $y = 2^x + \dfrac{1}{x^2}$ for $-2 \leqslant x \leqslant 3$.

   (a) (i) On the same axes, draw the line $y = 1 - x$.

   (ii) Hence, solve the equation
   $$2^x + \frac{1}{x^2} - 1 + x = 0.$$

   (b) Explain why the graph of $y = 2^x + \dfrac{1}{x^2}$ will not lie below the x-axis for all real values of x.

7.5 Applications of Graphs in Real-World Contexts

In this section, we will apply our knowledge of coordinate geometry and graphs to analyse and interpret graphs in various real-world contexts, including distance-time and speed-time graphs. Important features of the graphs such as intercepts, gradients, variables and scale of the x- and y-axes will provide information to help us analyse the graphs.

For example, the gradient of a line segment in a distance-time graph gives the speed. If the graph is a curve, the gradient of the curve at a point will give the speed at that instant.

Linear Distance-Time Graphs

Work in pairs.

Fig. 5.8 shows the graph of a cyclist's journey between 0800 and 1200. The graph can be divided into 4 sections – 0800 to 0900, 0900 to 0930, 0930 to 1030 and 1030 to 1200.
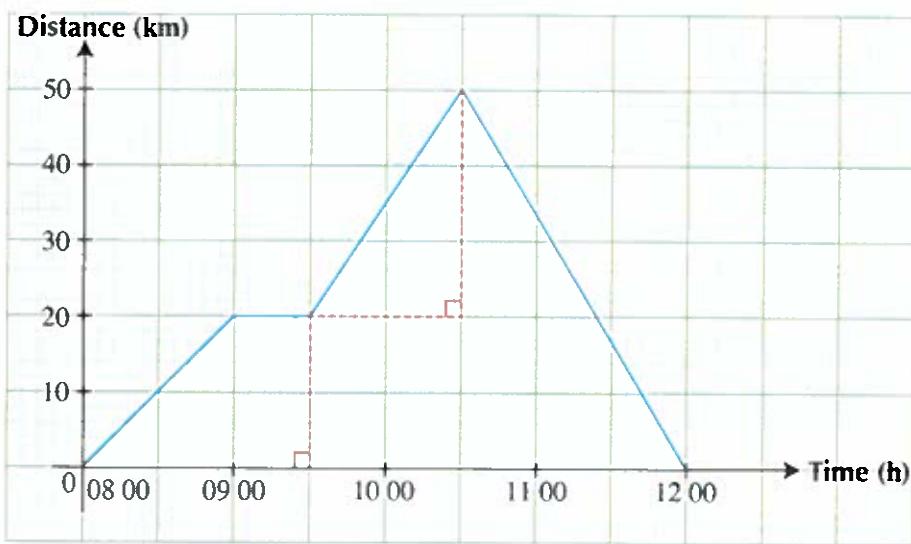
Fig. 7.8

Since the gradient of the graph from 0800 to 0900 = $\dfrac{20 \text{ km}}{1 \text{ h}}$

$$= 20 \text{ km/h,}$$

the cyclist travels at a constant speed of 20 km/h in the first hour.

1. Consider the section of the graph from 0900 to 0930. Since the graph is a horizontal line, what is its gradient? State clearly what this gradient represents.

2. Find the gradient of the section of the graph from 0930 to 1030. What does this gradient tell you about the motion of the cyclist?

3. Find the gradient of the section of the graph from 1030 to 1200. What does the negative gradient represent? Describe briefly the motion of the cyclist.

4. Explain why the average speed of the cyclist cannot be calculated by using $\dfrac{20 + 0 + 30 + 50}{4}$ km/h. Hence, find the **average speed** of the cyclist for the whole journey.

RECALL
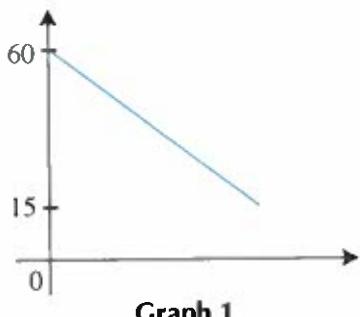
The average speed of an object is defined as the total distance travelled by the object per unit time.
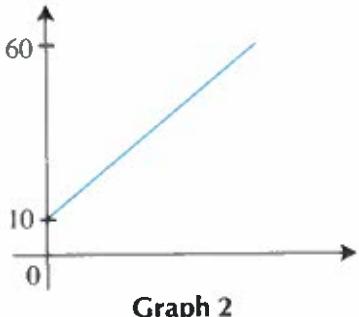
Match the scenarios with their respective graphs and justify your answers.
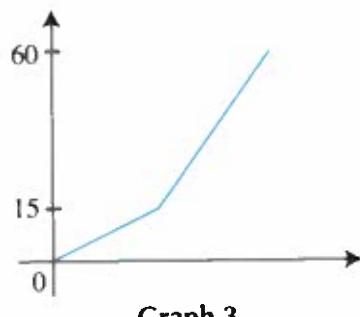
A: A few years ago, the exchange rate between Singapore dollars and Hong Kong dollars was S\$1 = HK\$6.	**B:** The height of water in a uniform cylindrical container increased at a constant rate from 10 cm to 60 cm.	**C:** Mr Neo was driving at a constant speed of 60 km/h when he suddenly applied the brakes and came to a stop.
D: The battery level in a smartphone decreased non-uniformly from 60% to 15%.	**E:** The temperature of a substance in a freezer decreased uniformly from 60 °C to 15 °C in 20 minutes.	**F:** A plant grew slowly at a constant rate to a height of 15 cm when it was kept indoors for 4 weeks, then grew more quickly at a constant rate to a height of 60 cm when it was placed outdoors for the next 4 weeks.
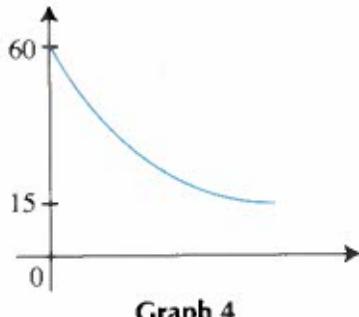
Graph 1
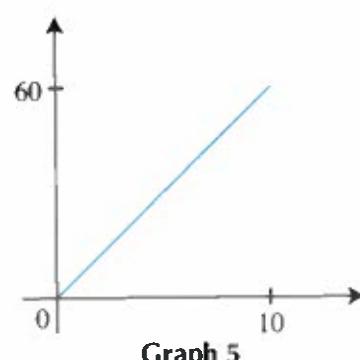
Graph 2
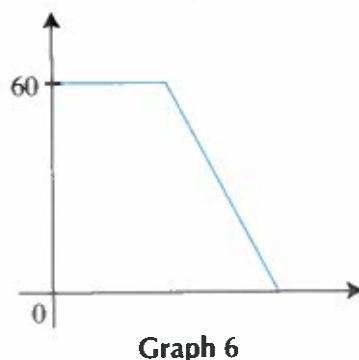
Graph 3

Graph 4

Graph 5

Graph 6

Worked Example 6

(Distance-Time Curve)

A train started from station *A* and travelled to station *B* 8 km from *A*.
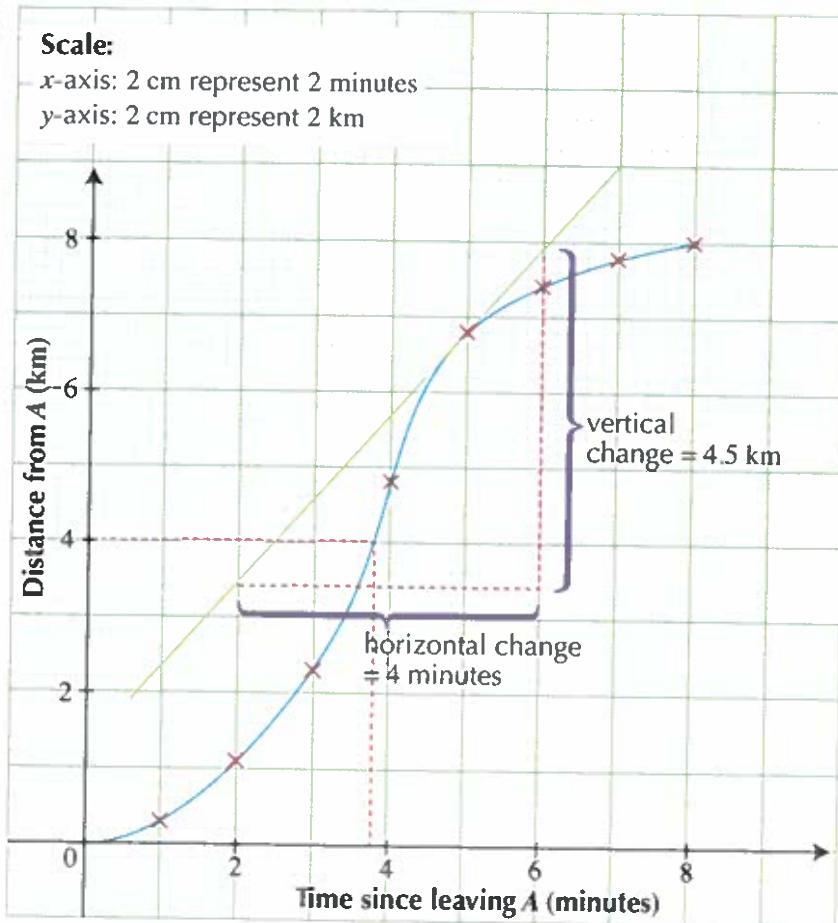
The table below shows the readings of the time, in minutes, since leaving station *A* and the corresponding distance, in km, from *A*.

Time (in minutes)	1	2	3	4	5	6	7	8
Distance (in km)	0.3	1.1	2.3	4.8	6.8	7.4	7.8	8.0

(a) Using a scale of 2 cm to represent 2 minutes on the horizontal axis and 2 cm to represent 2 km on the vertical axis, plot the points given in the table and join them with a smooth curve.

(b) Use your graph to estimate the time taken to travel the first 4 km of the journey.

(c) By drawing a tangent, find the approximate speed of the train 5 minutes after it has left station *A*.

(d) By considering the gradient of the graph, compare and describe briefly the motion of the train during the first 4 minutes and the last 4 minutes of the journey.

Solution:

(a)

Scale:
x-axis: 2 cm represent 2 minutes
y-axis: 2 cm represent 2 km

vertical change = 4.5 km

horizontal change ≑ 4 minutes

Distance from *A* (km)

Time since leaving *A* (minutes)

(b) From the graph, the train takes approximately 3.8 minutes to travel the first 4 km.

(c) The gradient of the tangent at the point 5 minutes after it left station A gives the speed at that particular point. It is called the instantaneous speed. A tangent is drawn to the curve at the point 5 minutes after it has left station A.

From the graph,

$$\text{Gradient} = \frac{\text{vertical change}}{\text{horizontal change}}$$

$$= \frac{4.5 \text{ km}}{4 \text{ minutes}}$$

$$= \frac{4.5 \text{ km}}{\frac{4}{60} \text{ h}}$$

$$= 67.5 \text{ km/h}$$
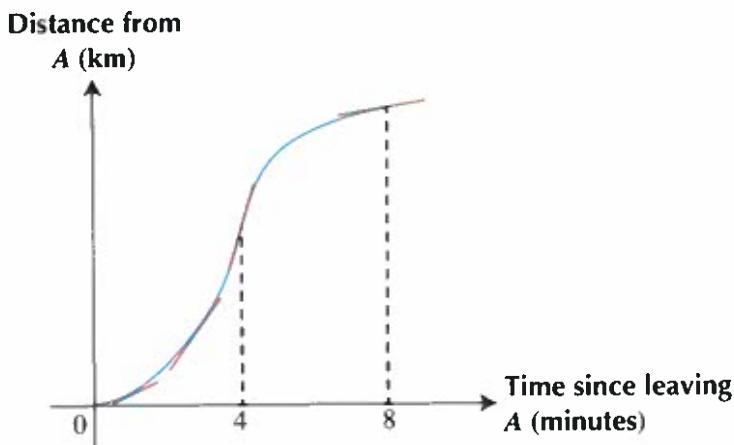
∴ The speed of the train 5 minutes after it has left station A is approximately 67.5 km/h.

(d) During the first 4 minutes, the speed of the train increases as the gradient of the curve increases.

During the last 4 minutes, the speed of the train decreases as the gradient of the curve decreases.

A train started from station P and travelled to station Q, 7.6 km from P.

The table below shows the readings of the time, in minutes, since leaving station P and the corresponding distance, in km, from P.

Time (in minutes)	1	2	3	4	5	6	7	8
Distance (in km)	0.2	0.8	2.6	5.0	6.5	7.2	7.5	7.6

(a) Using a scale of 2 cm to represent 2 minutes on the horizontal axis and 2 cm to represent 2 km on the vertical axis, plot the points given in the table and join them with a smooth curve.

(b) Use your graph to estimate the time taken to travel the first 4 km of the journey.

(c) By drawing a tangent, find the approximate speed of the train 6 minutes after it has left station P.

(d) By considering the gradient of the graph, compare and describe briefly the motion of the train during the first 4 minutes and the last 4 minutes of the journey.

Thinking Time

Fig. 7.9 shows three containers, each of which is being filled with liquid at a constant rate from a tap to a height of a cm. The containers are initially empty.

The graph of Fig. 7.9(a) shows the height (h cm) of the liquid as the container is being filled in t seconds.

(a)

(b)

(c)
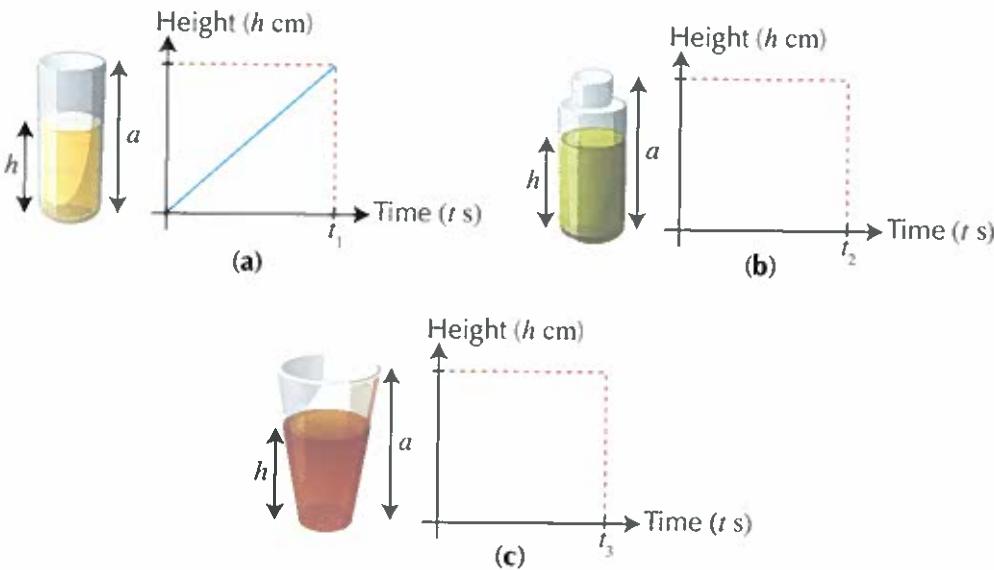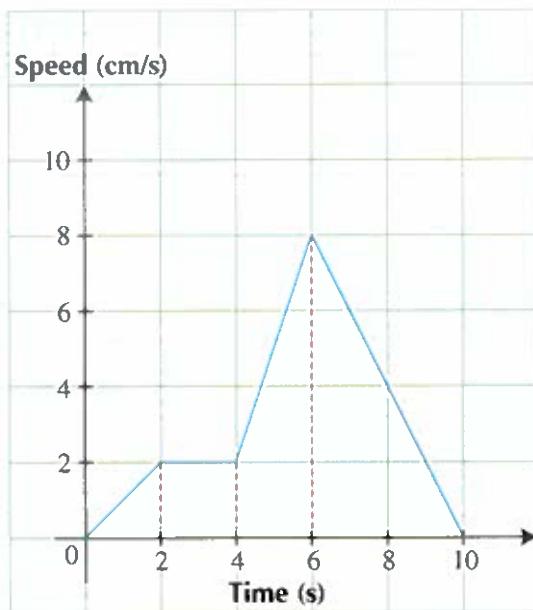
Fig. 7.9

1. Explain clearly why the graph of Fig. 7.9(a) is a straight line.

2. Complete the graphs in Fig. 7.9(b) and (c). Explain the shape of each graph obtained.

Worked Example 7

The graph shows the speed of an object over a period of 10 seconds.

Speed (cm/s) vs **Time (s)**

(i) Find the acceleration in the first 2 seconds,

(ii) Given that the distance travelled is given by the area under the speed-time graph, find the average speed during the whole journey.

(iii) Find the deceleration in the last 2 seconds.

Solution:

(i) Acceleration $= \dfrac{2 \text{ cm/s}}{2 \text{ s}}$

$= 1 \text{ cm/s}^2$

(ii) Total distance = area under graph
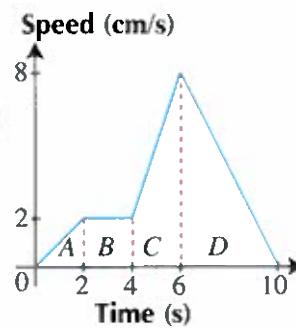
$= \text{Area of } (A + B + C + D)$

$= \left(\dfrac{1}{2} \times 2 \times 2\right) + (2 \times 2) + \dfrac{1}{2}(2+8) \times 2 + \left(\dfrac{1}{2} \times 8 \times 4\right)$

$= 2 + 4 + 10 + 16$

$= 32 \text{ cm}$

Average speed $= \dfrac{\text{Total distance}}{\text{Total time}}$

$= \dfrac{32}{10}$

$= 3.2 \text{ cm/s}$

Speed (cm/s) vs **Time (s)** with regions $A \mid B \mid C \mid D$

> **ATTENTION**
>
> For (i), the gradient can be found by $\dfrac{\text{vertical change}}{\text{horizontal change}}$ or by using the formula $\dfrac{y_2 - y_1}{x_2 - x_1}$.
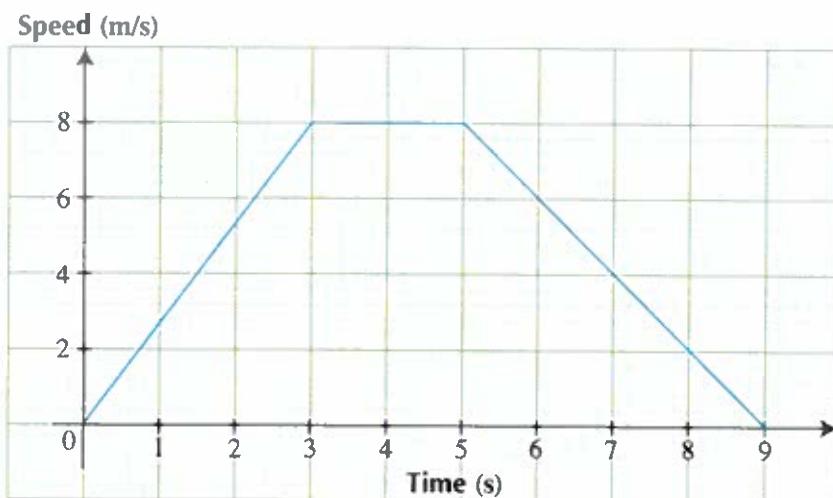
> **ATTENTION**
>
> Acceleration of an object = rate of change of speed = gradient of a speed-time graph
>
> If the gradient is negative, we say that the object is decelerating. The unit for acceleration is always of speed per unit time, i.e. if the unit of speed is cm/s, then the unit of acceleration is cm/s².

(iii) Deceleration in the last 2 seconds = deceleration in the last 4 seconds

$= \dfrac{8 - 0}{10 - 6}$

$= 2 \text{ cm/s}^2$

The graph shows the speed of an object over a period of 9 seconds.

Speed (m/s)

Time (s)

(i) Find the acceleration in the first 3 seconds.

(ii) Given that the distance travelled is given by the area under the speed-time graph, find the average speed during the whole journey.

(iii) Find the deceleration in the last 3 seconds.

Worked Example 8 (Speed-Time Graph)

A particle moves along a straight line from A to B so that, t seconds after leaving A, its speed, v m/s, is given by $v = 3t^2 - 15t + 20$.

The table below shows some values of t and the corresponding values of v.

t	0	1	1.5	2	2.5	3	4	5
v	20	8	a	2	b	2	8	20

(a) Find the value of a and of b.

(b) Using a scale of 2 cm to represent 1 second on the horizontal axis and 2 cm to represent 5 m/s on the vertical axis, draw the graph of $v = 3t^2 - 15t + 20$ for $0 \leqslant t \leqslant 5$.

(c) Use your graph to estimate
  (i) the value of t when the speed is 10 m/s,
  (ii) the time at which the acceleration is zero,
  (iii) the gradient at $t = 4$, and explain what this value represents,
  (iv) the time interval when the speed is less than 15 m/s.

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Solution:

(a) When $t = 1.5$,

$$v = 3(1.5)^2 - 15(1.5) + 20$$
$$= 4.25$$
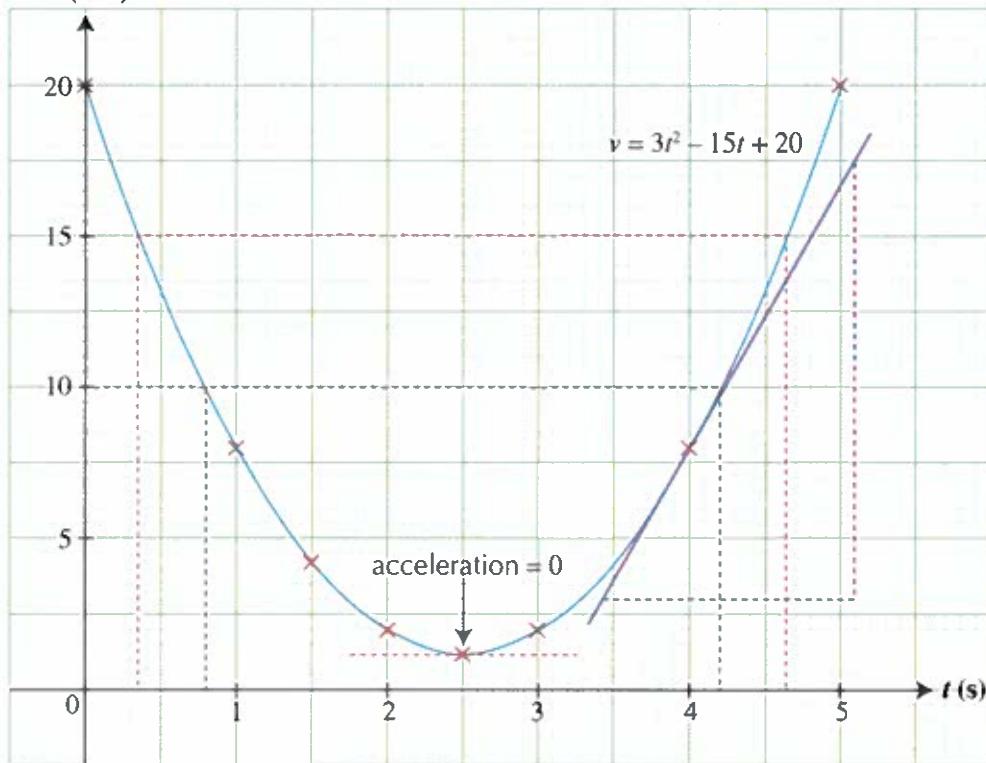
$$\therefore a = 4.25$$

When $t = 2.5$,

$$v = 3(2.5)^2 - 15(2.5) + 20$$
$$= 1.25$$

$$\therefore b = 1.25$$

(b)

(c) **(i)** From the graph, when $v = 10$,

$t = 0.8$ or $t = 4.2$.

(ii) The acceleration is zero when the gradient of the curve is zero.

From the graph, the acceleration is zero at $t = 2.5$.

(iii) A tangent is drawn to the curve at the point $t = 4$.

From the graph,

$$\text{Gradient} = \frac{\text{vertical change}}{\text{horizontal change}}$$
$$= \frac{14.5}{1.65}$$
$$\approx 9$$

$\therefore$ The acceleration of the particle at $t = 4$ is approximately 9 m/s².

(iv) From the graph, when $v < 15$, $0.35 < t < 4.65$.

In **(iii)**, the unit for acceleration is $\frac{m/s}{s}$, i.e. m/s².

A particle moves along a straight line from P to Q so that, t seconds after leaving P, its speed, v m/s, is given by $v = 2t^2 - 9t + 12$.

The table below shows some values of t and the corresponding values of v.

t	0	1	2	3	4	5
v	12	5	a	3	8	b

(a) Find the value of a and of b.

(b) Using a scale of 2 cm to represent 1 second on the horizontal axis and 2 cm to represent 5 m/s on the vertical axis, draw the graph of $v = 2t^2 - 9t + 12$ for $0 \leqslant t \leqslant 5$.

(c) Use your graph to estimate

  (i) the values of t when the speed is 7 m/s,

  (ii) the time at which the acceleration is zero,

  (iii) the gradient at $t = 4.5$, and explain what this value represents,

  (iv) the time interval when the speed is less than 10 m/s.
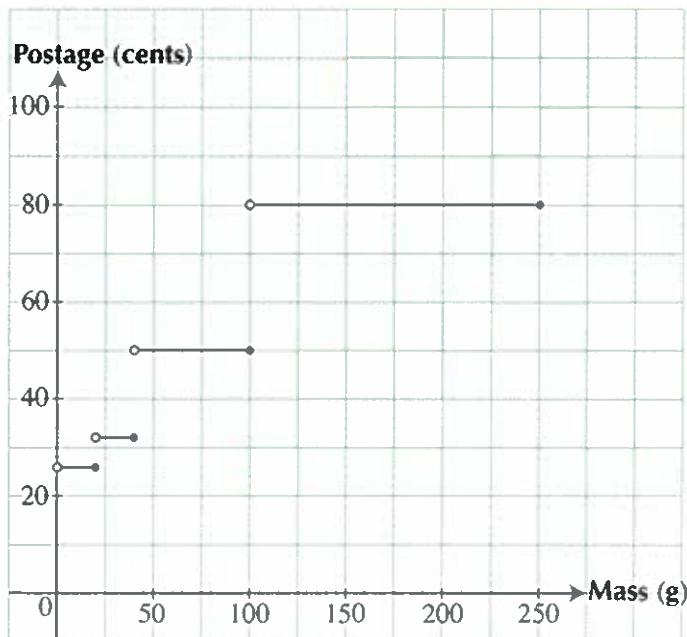
⠿ Other Graphs

Graphs can also be used in other real-world contexts such as in the calculation of postage rates, parking charges and labour costs.

Worked Example 9

(Graphs involving Rates)

The step-function graph below shows the local postage rates for letters, postcards and small packages offered by Company A.

Problem Solving Tip

When the mass m is such that $0 < m \leqslant 20$, the postage is 26 cents. Similarly, for $20 < m \leqslant 40$, the postage is 32 cents.

ATTENTION

An empty node ○ indicates that the point is excluded from the graph and a shaded node ● indicates that the point is included.
For example, when the mass is 100 g, the postage is 50 cents, not 80 cents.

(a) Write down the postage to mail a letter with a mass of 80 g.

Company B offers the following postage rates: 60 cents for the first 50 g and 0.2 cents for each subsequent gram.

(b) Given that Huixian wishes to post a letter with a mass of 150 g, insert the graph corresponding to the rates offered by Company B and use your graph to determine which company offers a lower postage.
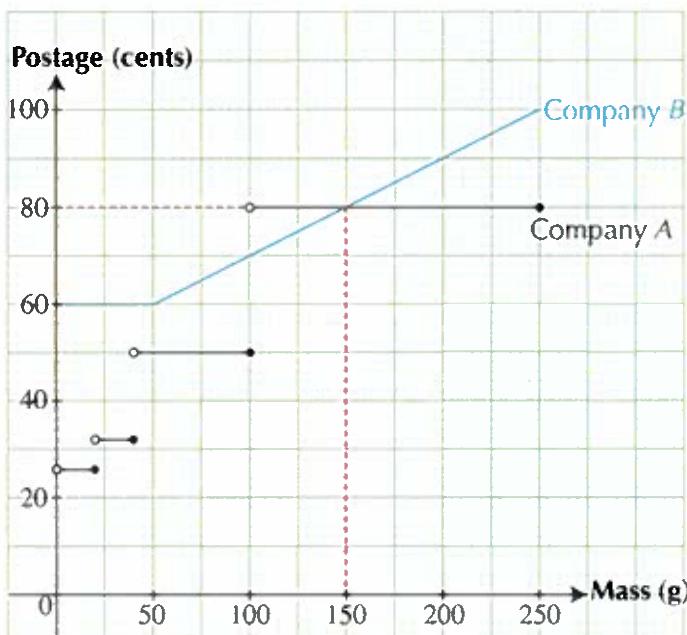
Solution:

(a) From the graph, the postage is 50 cents.

(b) For the first 50 g, the postage is 60 cents.

When the mass is 100 g, the postage is $60 + 0.2 \times (100 - 50) = 70$ cents.
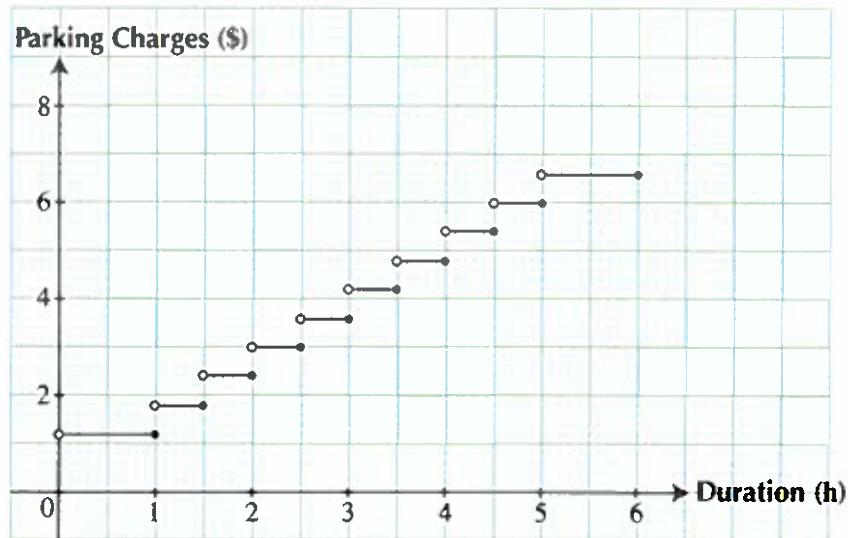
When the mass is 250 g, the postage is $60 + 0.2 \times (250 - 50) = 100$ cents.

From the graph, for a mass of 150 g, both companies charge the same postage of 80 cents.

The step-function graph below shows the parking charges for the first 6 hours at Carpark *X*.

Carpark *Y* has the following charges:
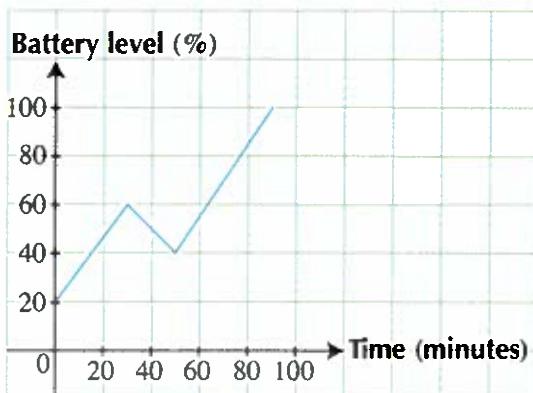
> Free for the first 12 minutes
> 2.5 cents per minute thereafter

Insert the graph corresponding to the rates offered by Carpark *Y* and use your graph to determine which carpark Mr Wong should choose if he has to park for 2 hours.

Worked Example 10

(Other Types of Graphs)

The graph below shows the battery level of a smartphone. It had an initial level of 20%, increasing to 60% in half an hour while connected to the power supply. Farhan then removed the smartphone from the power supply to watch a 20-minute long video clip, before connecting the smartphone to the power supply again.

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(a) Find the battery level of the smartphone when Farhan was exactly halfway through the video clip.

(b) Find the rate of increase in the battery level of the smartphone when it was connected to the power supply again.

Solution:

(a) From the graph, the battery level was 50%.

(b) To find the rate of increase in the battery level, we need to calculate the gradient of the line from the 50th minute to the 90th minute.

$$\text{Gradient} = \frac{100 - 40}{90 - 50}$$
$$= 1.5\%/\text{minute}$$

∴ The rate of increase in the battery level is 1.5%/minute.

The graph below shows the heart rate, in beats per minute, of an adult who is at the park. He rests at the bench for the first 10 minutes, after which he begins to brisk walk for 10 minutes. He then slows down for 5 minutes, before brisk walking again for a further 5 minutes. He then jogs at a constant speed for 10 minutes, before gradually slowing down.
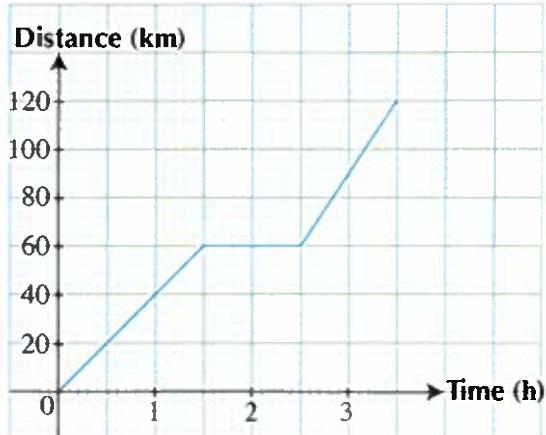
(a) Write down his resting heart rate.

(b) Find the rate of increase in his heart rate as he brisk walks for the first time.

(c) Find the rate of decrease in his heart rate as he slows down in the last 20 minutes.

Exercise 7C
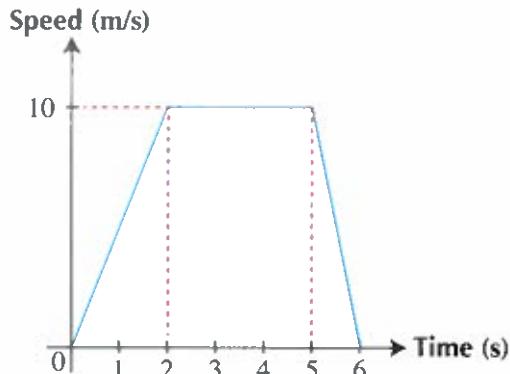
1. A cyclist set out at 0900 for a destination 40 km away. He cycled at a constant speed of 15 km/h until 1030. Then he rested for half an hour before completing his journey at a constant speed of 20 km/h.
   (i) Draw the distance-time graph to represent the journey.
   (ii) Hence, find the time at which the cyclist reached his destination, giving your answer to the nearest minute.

2. Raj starts a 30-km journey at 0900. He maintains a constant speed of 20 km/h for the first 45 minutes and then stops for a rest. He then continues his journey at a constant speed of 30 km/h, finally arriving at his destination at 1120.
   (i) Find the distance travelled in the first 45 minutes.
   (ii) Draw the distance-time graph to represent the journey.
   (iii) Hence, state the duration of his stop, giving your answer in minutes.

3. The figure shows the distance-time graph of a car.
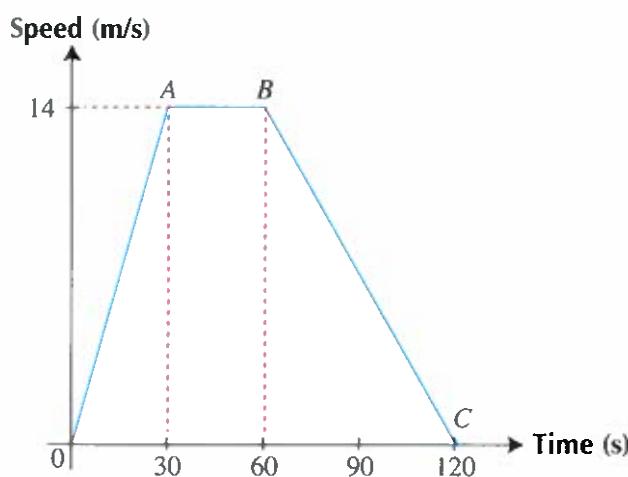
   (i) Find the duration during which the car is not moving.
   (ii) Find the average speed of the car in the first 2 hours of the journey.
   (iii) Find the average speed of the car for the whole journey.
   (iv) Draw the speed-time graph of the car for the whole journey.

4. The graph shows the speed-time graph of a car.

   (i) Find the acceleration in the first 2 seconds.
   (ii) Given that the distance travelled is given by the area under the speed-time graph, find the average speed during the whole journey.

5. The graph shows the speed, v m/s, of a car after t seconds.

   (i) State what the gradient of OA represents.
   (ii) Find the speed of the car when $t = 15$.

6. A lift moves from ground level to a height of 60 metres in 10 seconds, stops for 10 seconds and then descends to the ground in 10 seconds. The table shows the height, h m, of the lift on the upward and downward journeys, t seconds after leaving ground level.

t (in seconds)	0	2	4	6	8	10
h (in m)	0	3	16	44	57	60

t (in seconds)	20	22	24	26	28	30
h (in m)	60	57	44	16	3	0

(i) Using a scale of 2 cm to represent 5 seconds, draw a horizontal t-axis for $0 \leqslant t \leqslant 30$.

Using a scale of 1 cm to represent 5 metres, draw a vertical h-axis for $0 \leqslant h \leqslant 60$.

On your axes, plot the points given in the table and join them with a smooth curve.

(ii) Find the gradient of the graph at $t = 8$ and explain briefly what this gradient represents.

A construction worker, waiting at the 40-metre level, starts to walk down at $t = 5$.

(iii) Assuming that he descends at a steady speed of 0.8 m/s, use your graph to find the time when the worker and the lift are at the same height.

7. A company which manufactures automated vehicles is putting them on a test run. One of the vehicles starts from a point X and travels to a point Y, 3 km away. The table shows the distance, d km, of the vehicle from X, t minutes after leaving X.

Time (in minutes)	0	1	2	3	4	5	6
Distance (in km)	0	0.2	0.7	1.8	2.5	2.9	3.0

(a) Using a scale of 2 cm to represent 1 minute on the horizontal axis and 4 cm to represent 1 km on the vertical axis, plot the points given in the table and join them with a smooth curve.

(b) Use your graph to find

(i) the approximate time taken to travel the first 1 km,

(ii) the gradient of the graph when $t = 1\frac{1}{2}$ and explain briefly what this value represents.

(iii) the time taken to travel the last 1 km.

8. Ethan and Michael start moving towards each other at the same time. The initial distance between them is 32 km.

(a) Given that Ethan is cycling at a constant speed of 20 km/h and Michael is walking at a constant speed of 7 km/h, draw a distance-time graph to illustrate this information.

(b) Use your graph to find

(i) how long it will take for them to pass each other,
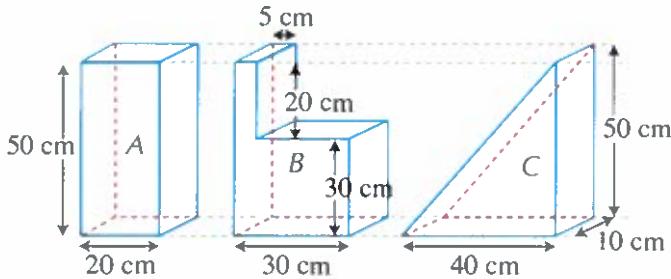
(ii) the times when they will be 5 km apart.

9. At 0900, Shirley travels to meet Kate, who stays 20 km away. Shirley travels at a uniform speed of 18 km/h for half an hour. She rests for 20 minutes and then continues her journey at a uniform speed of 8 km/h.
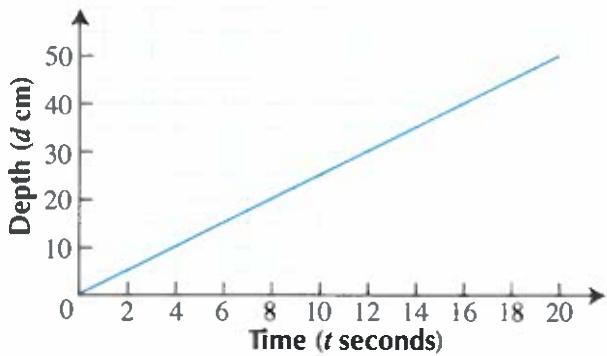
At 0900, Kate sets off from home on the same road to meet Shirley and travels at a uniform speed of 7 km/h.

(a) Draw the distance-time graph for the above information.

(b) Use your graph to find

   (i) the time at which Shirley and Kate meet,

   (ii) the distance away from Kate's home when they meet.

10. The figure shows three containers, each with a height of 50 cm and a width of 10 cm. The other dimensions are as shown. The containers are initially empty and it takes 20 seconds to fill each container at a constant rate.
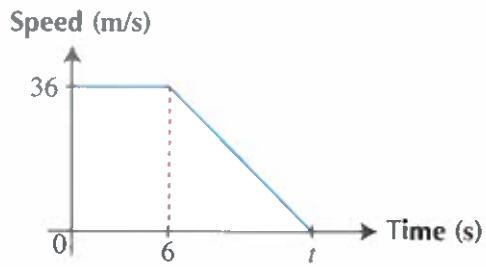
The diagram below shows the relationship between the depth, d cm, of the liquid and the time, t seconds, taken to fill container A.
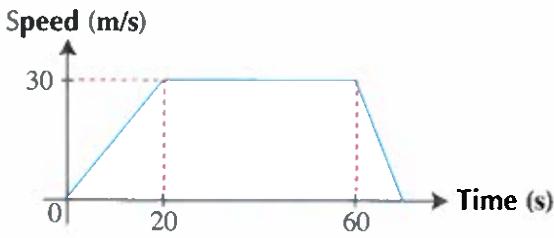
On the same diagram, sketch the graph of the depth of the liquid against time for container B and container C.

11. The diagram shows the speed-time graph of an object which travels at a constant speed of 36 m/s and then slows down at a rate of 12 m/s², coming to rest at time t seconds.

(i) Find the value of t.

(ii) Given that the distance travelled when the object is slowing down is 54 m, find the average speed for the whole journey.

12. The diagram shows the speed-time graph of a train.

(i) Find the acceleration of the train during the first 20 seconds.

(ii) Given that the train decelerates at a rate of 0.75 m/s², find the time taken for the whole journey.

13. A particle moves along a straight line from A to B so that, t minutes after leaving A, its speed, v m/min, is given by $v = t^2 - 7t + 16$.

t (minutes)	0	1	2	3	4	5	6
v (m/min)	16	10	6	a	4	6	b

(a) Find the value of a and of b.

(b) Using a scale of 2 cm to represent 1 minute on the horizontal axis and 1 cm to represent 1 m/min on the vertical axis, draw the graph of $v = t^2 - 7t + 16$ for $0 \leqslant t \leqslant 6$.

(c) Use your graph to estimate

   (i) the value(s) of t when the speed is 7 m/min,

   (ii) the time at which the speed is a minimum,

   (iii) the gradient at $t = 2$, and explain what this value represents,

   (iv) the time interval when the speed is not more than 5 m/min.

14. The speed of a body, v m/s, after time t seconds is given in the table.

t (s)	0	2	4	6	8	10	12
v (m/s)	0	2	7	12	19	28	42

(i) Using a scale of 1 cm to represent 1 second on the horizontal axis and 1 cm to represent 5 m/s on the vertical axis, plot the graph of v against t for $0 \leqslant t \leqslant 12$.

(ii) Use your graph to estimate the speed of the body when $t = 5$ and when $t = 11$.

(iii) By drawing two tangents, find the acceleration of the body when $t = 4$ and when $t = 10$.

15. Object P moves along a straight line from A to B so that, t hours after leaving A, its speed, v km/h, is given by $v = 3t^2 - 17t + 30$.

t (h)	0	1	2	3	4	5
v (km/h)	30	16	h	6	k	20
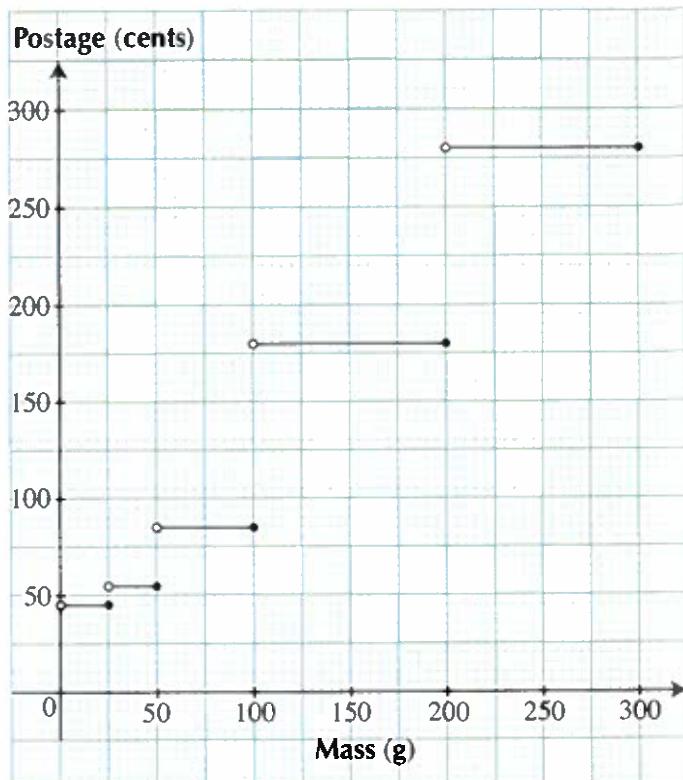
(a) Find the value of h and of k.

(b) Using a scale of 2 cm to represent 1 hour on the horizontal axis and 2 cm to represent 5 km/h on the vertical axis, draw the graph of $v = 3t^2 - 17t + 30$ for $0 \leqslant t \leqslant 5$.

(c) Use your graph to estimate

    (i) the time at which the speed is a minimum,

    (ii) the gradient at $t = 4.5$, and explain what this value represents,

    (iii) the time interval when the speed does not exceed 10 km/h.

Object Q moves along a straight line from A to B with a constant speed of 24 km/h.

(d) Use your graph to determine the value of t at which both objects have the same speed.

16. A taxi starts from rest and accelerates at a uniform rate for 45 seconds to reach a speed of 30 m/s. It then travels at this constant speed. Sketch the speed-time graph and use it to find the speed after 10 seconds.

17. The step-function graph below shows the postage rates to Malaysia for letters and small packages offered by Company A.
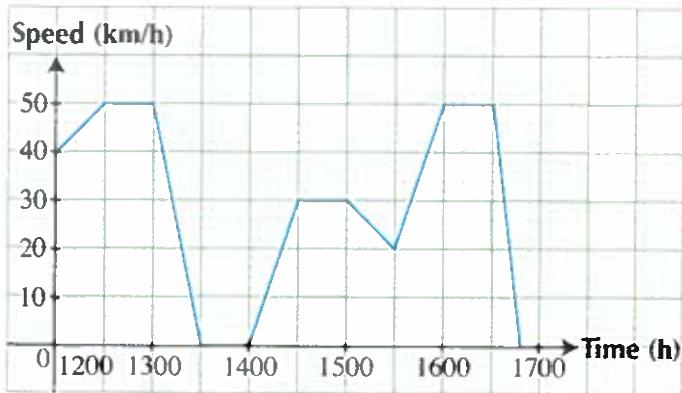
(a) Write down the postage to mail a letter with a mass of 50 g to Malaysia.

Company B offers the following postage rates to Malaysia: $1 for the first 80 g and 1 cent for each subsequent gram.

(b) Given that Devi wishes to post a small package with a mass of 220 g to Malaysia, determine which company offers a lower postage. Show your working to support your answer.
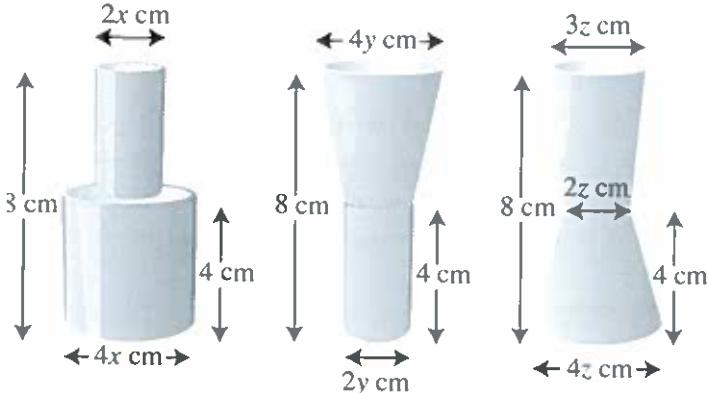
18. The graph below shows the speed of a coach as it ferried passengers from Blue Town to Summer City via the highway one afternoon. The coach made only one stop during the journey.

Speed (km/h)

(a) Write down the duration of the stop.

(b) Find the initial acceleration of the coach.

(c) Given that the distance travelled is given by the area under the speed-time graph, explain why the distance between Blue Town and Summer City is less than 250 km.

(d) Determine the time when the coach reached Summer City.

19. The diagram below shows three containers of a fixed volume and varying cross-sectional areas, each with a height of 8 cm. The containers are initially empty. A tap is used to fill each of the containers at a constant rate.

Given that it takes 60 seconds to fill each of the containers, sketch the graph of the height (h cm) of the water level against time (t s) for each of the containers.
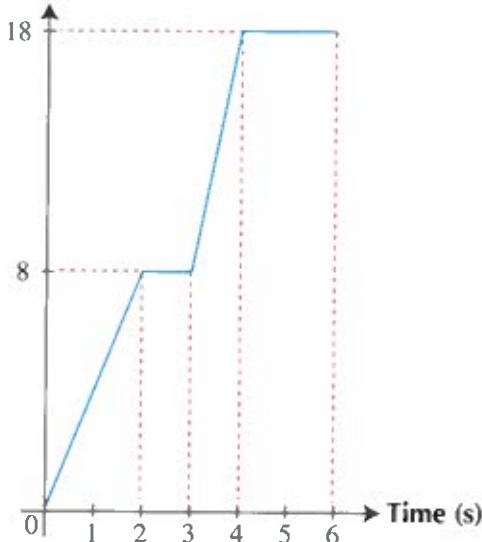
20. A toy car starts from a point A and moves towards a point B, which it reaches after 7 seconds. The speed, v cm/s, after t seconds, is given in the table.

t (s)	0	1	2	3	4	5	6	7
v (cm/s)	0	4.5	8	10.5	12	12.5	12	10.5

(a) Using a scale of 2 cm to represent 1 second on the horizontal axis and 2 cm to represent 2 cm/s on the vertical axis, plot the graph of v against t for $0 \leqslant t \leqslant 7$.

(b) Use your graph to estimate

   (i) the acceleration of the body when $t = 2$ and when $t = 6$,

   (ii) the time interval when the speed is greater than 11 m/s.

(c) Given that this motion can be modelled by the equation $v = at^2 + bt + c$, where a, b and c are constants, find the values of a, b and c.

21. The graph shows the distance-time graph of a body during a period of 6 seconds.

Distance (m)

Sketch the speed-time graph for the same journey.

22. The speed of an object, v m/s, at time t seconds, is given by $v = 6 + 2t$.
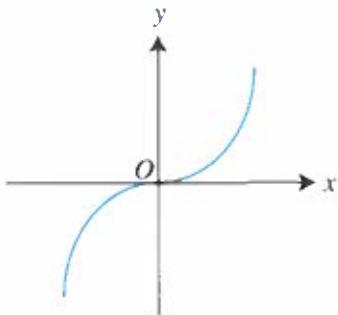
(a) Sketch the speed-time graph for the motion.

(b) Find the speed when $t = 3$.

(c) Sketch the acceleration-time graph for the motion.
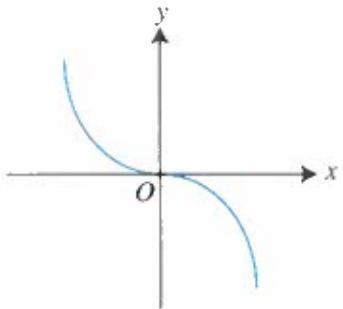
1. **Graphs of Power Functions** $y = ax^n$
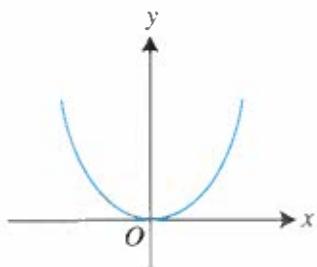
(a) $n = 3, a > 0, y = ax^3$
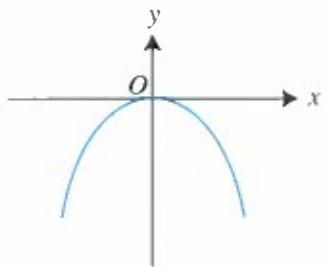
$n = 3, a < 0, y = ax^3$
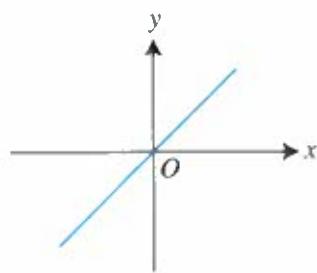
(b) $n = 2, a > 0, y = ax^2$
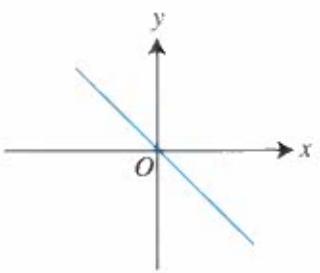
$n = 2, a < 0, y = ax^2$
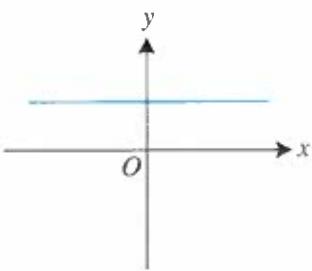
(c) $n = 1, a > 0, y = ax$
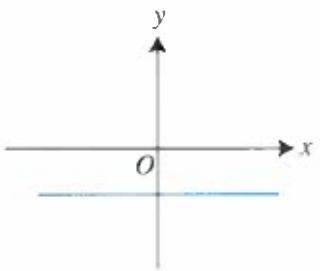
$n = 1, a < 0, y = ax$
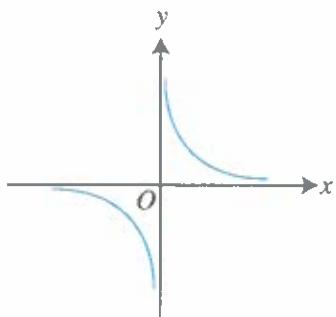
(d) $n = 0, a > 0, y = a$
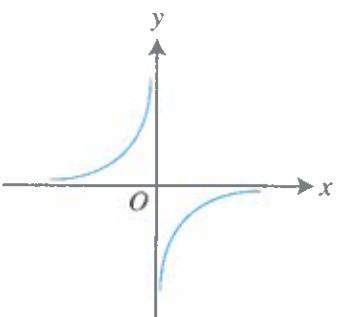
$n = 0, a < 0, y = a$
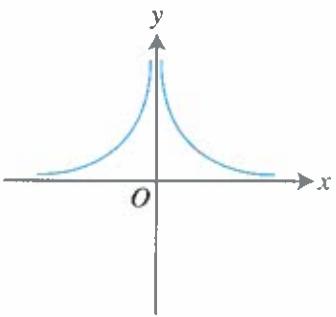
(e) $n = -1, a > 0, y = \dfrac{a}{x}$
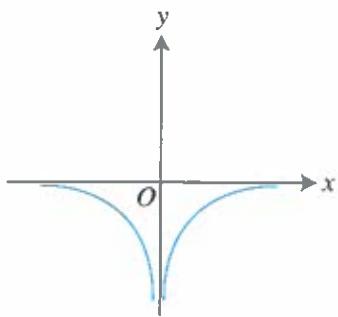
$n = -1, a < 0, y = \dfrac{a}{x}$

(f) $n = -2, a > 0, y = \dfrac{a}{x^2}$
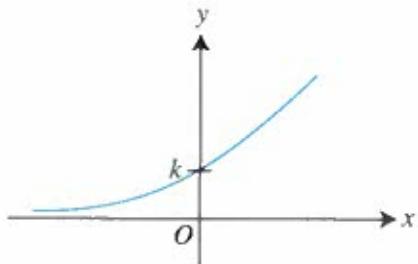
$n = -2, a < 0, y = \dfrac{a}{x^2}$
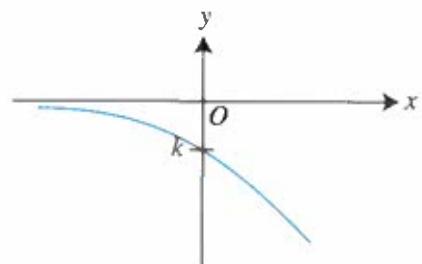
2. Graph of $y = ka^x$, where $a > 0$

$k > 0, y = ka^x$

$k < 0, y = ka^x$

3. Gradient of a Curve

The gradient of a curve at a point can be obtained by drawing a tangent to the curve at that point and finding the gradient of the tangent.

1. The table below shows some values of x and the corresponding values of y, where $y = x^3 - 3x - 10$.

x	-3	-2	-1	0	1	2	3	4
y	-28			-10			8	42

(a) Copy and complete the table.

(b) Using a scale of 1 cm to represent 1 unit, draw a horizontal x-axis for $-3 \leqslant x \leqslant 4$.

Using a scale of 1 cm to represent 5 units, draw a vertical y-axis for $-28 \leqslant y \leqslant 42$.

On your axes, plot the points given in the table and join them with a smooth curve.

(c) Use your graph to find

    (i) the value of y when $x = 1.8$,

    (ii) the value of x when $y = 10$.

2. The variables x and y are connected by the equation $y = x(x-2)(x+2)$.

The table below shows some values of x and the corresponding values of y.

x	-3	-2	-1	0	1	2	3
y	-15	0		0	-3		
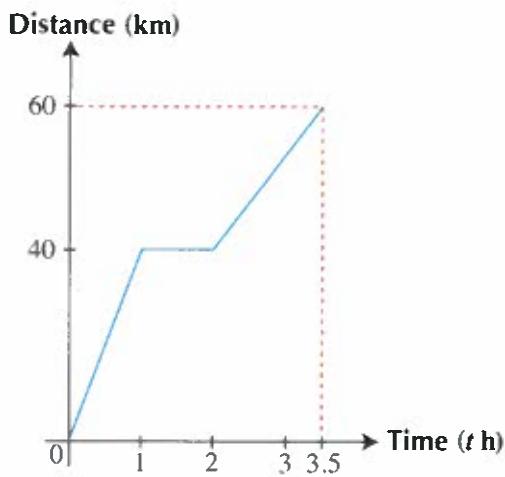
(a) Copy and complete the table.

(b) Using a scale of 2 cm to represent 1 unit on the x-axis and 2 cm to represent 5 units on the y-axis, draw the graph of $y = x(x-2)(x+2)$ for $-3 \leqslant x \leqslant 3$.

(c) Use your graph to find

    (i) the value of y when $x = 1.4$,

    (ii) the value of x when $y = 4.5$,

    (iii) the solutions to the equation $x(x-2)(x+2) = 0$.

3. Using a scale of 4 cm to represent 1 unit on the x-axis and 2 cm to represent 1 unit on the y-axis, draw the graph of $y = 1 - 2x - \dfrac{1}{x}$ for $-4 \leqslant x \leqslant -0.25$.

(a) Use your graph to find

    (i) the value of y when $x = -0.75$,

    (ii) the values of x when $y = 4.5$.

(b) Write down the coordinates of the point on the curve where the tangent to the curve is a horizontal line.

4. Using a suitable scale, draw the graph of $y = 3^x - 2$ for $-1.5 \leqslant x \leqslant 2$. Using your graph,

    (i) solve the equation $3^x = 2$,

    (ii) find the coordinates of the point on the graph of $y = 3^x - 2$ where the gradient of the tangent is 2.
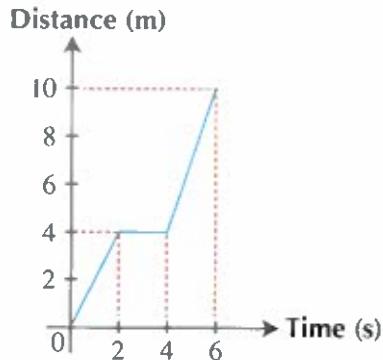
5. Using a scale of 4 cm to represent 1 unit on both axes, draw the graph of $y = x - 2 + \dfrac{3}{x}$ for $0.5 \leqslant x \leqslant 4$.

    (i) State the minimum value of y and the corresponding value of x.

    (ii) Find the range of values of x for which $y < 2.2$.

    (iii) By drawing a tangent, find the gradient of the graph at the point where $x = 3$.

    (iv) Using your graph, find the value of x for which $2x + \dfrac{3}{x} = 8$.

6. A coach travelled from the airport to the hotel in 20 minutes at a constant speed of 45 km/h. After stopping for half an hour, it travelled back to the airport at a constant speed of 60 km/h.

   (i) Draw the distance-time graph to represent the journey.

   (ii) Hence, find the average speed of the coach for the whole journey.

7. The graph shows the distance-time graph of a heavy goods vehicle.

   Distance (km)

   (a) Find

   (i) the time interval during which the vehicle stopped to unload goods,

   (ii) the speed when $t = 3$,

   (iii) the maximum speed during the journey,

   (iv) the average speed for the whole journey.

   (b) Sketch the speed-time graph for the motion.
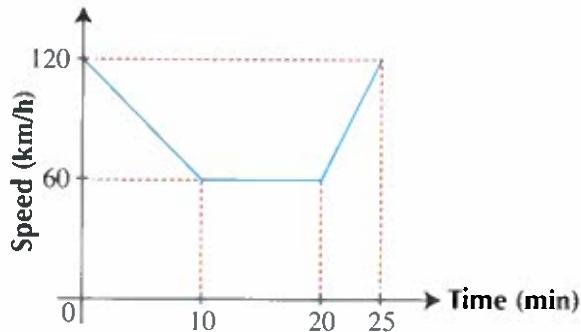
8. The diagram shows the speed-time graph of a particle over a period of 6 seconds.

   Distance (m)

   Calculate

   (i) its acceleration during the first two seconds,

   (ii) the greatest acceleration,

   (iii) the total distance moved,

   (iv) the average speed of the particle.

9. The graph illustrates the speed of a car during a period of 25 minutes.

   Find

   (i) the total distance travelled over the 25 minutes,

   (ii) the average speed during the 25 minutes.

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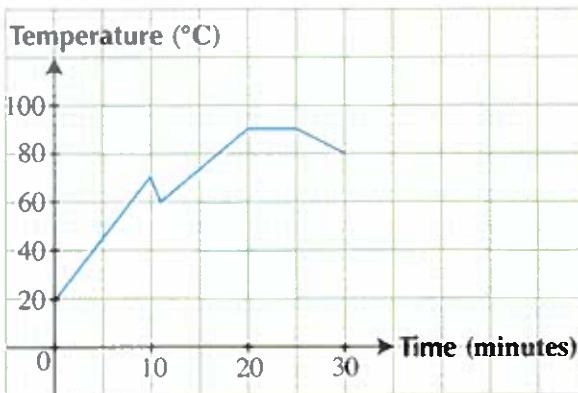
10. The intensity of illumination, I units, at a point on a screen a distance of D cm from the light source is modelled by the equation $I = \dfrac{k}{D^2}$, where k is a constant.

(i) Using the data in the table, find the value of k.

Distance (D cm)	10		20	25		
Intensity (I units)		1250	800		294	200

(ii) Complete the table and use a suitable scale to draw the graph of I against D.

(iii) From the graph, find the intensity of illumination when the light source is 30 cm from the screen.

(iv) What can you say about the relationship between I and D^2?
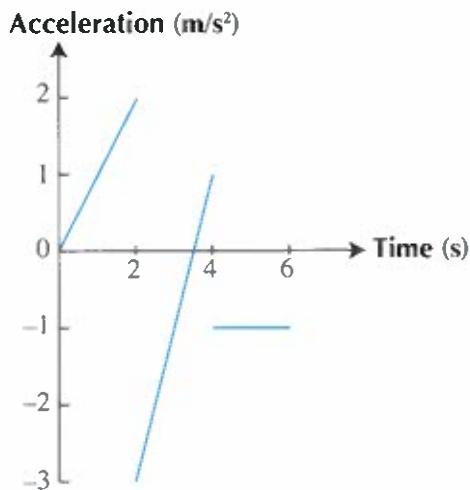
11. The graph below shows the temperature of soup as it was being heated in an electric cooker. Ten minutes later, Mrs Wong added some fresh vegetables from the refrigerator, continued to heat the soup until it reached a temperature of 90 °C and switched off the cooker after five minutes.

(a) Find the rate of increase in the temperature of the soup in the first ten minutes.

(b) Suggest why there is a drop in the temperature between the 10th and the 11th minute.

(c) Find the rate of decrease in the temperature of the soup after the cooker was switched off.

Challenge Yourself

The graph shows an acceleration-time graph of an object travelling in a straight line.

Sketch a possible speed-time graph for the motion of the object. Explain your answer.
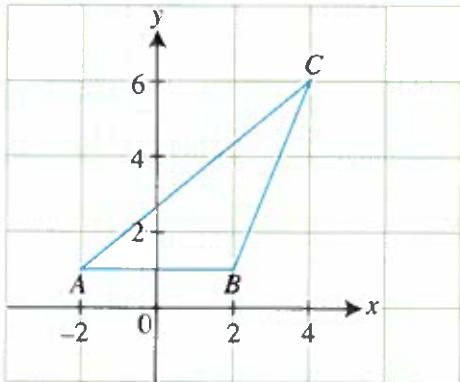
B1 Revision Exercise

1. The cost of manufacturing an article is $160. The manufacturer sells it to a dealer at a gain of 45% and the dealer sells it to a purchaser for $266.80. Express the dealer's profit as a percentage of the price at which he buys the article from the manufacturer.

2. A trader mixes Coffee A which costs $9 per kg with Coffee B which costs $13 per kg, in the ratio 3 : 1. The mixture is then sold at $1.25 per 100 g. Express his profit as a percentage of his cost price.

3. Find the equation of the line joining the points $A(5, 7)$ and $B(8, 12)$.
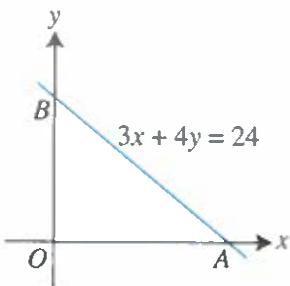
4. The figure shows the points $A(-2, 1)$, $B(2, 1)$ and $C(4, 6)$.

Find
(i) the area of $\triangle ABC$,
(ii) the coordinates of the point D such that $ABCD$ is a parallelogram,
(iii) the coordinates of the point P such that $ACPB$ is a parallelogram.
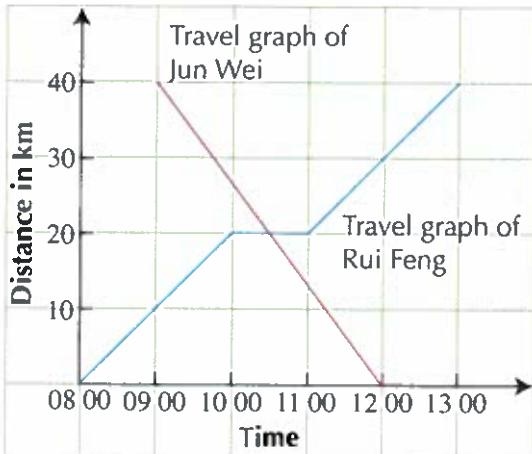
5. The line $3x + 4y = 24$ cuts the x-axis and the y-axis at A and B respectively.

Find
(i) the coordinates of A and of B,
(ii) the area of $\triangle ABC$, where C is the point $(-5, 0)$,
(iii) the gradient of BC,
(iv) the length of the perpendicular from C to AB.

6. The travel graphs below show the journeys of Rui Feng and Jun Wei. Rui Feng starts from the train station at 0800 and travels towards the airport, 40 km away. Jun Wei starts from the airport at 0900 and travels towards the train station.
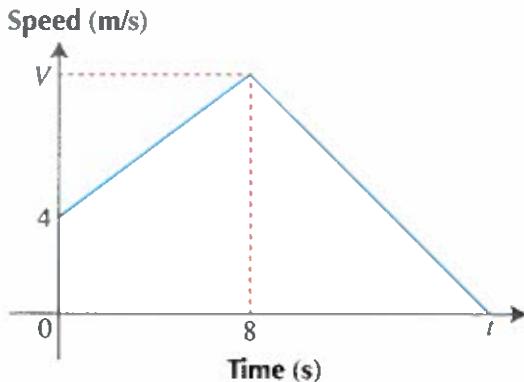
From the graph, find
(i) Jun Wei's average speed for the whole journey,
(ii) the time when Rui Feng and Jun Wei meet and how far they are from the airport when they meet,
(iii) the time interval during which Rui Feng took a rest,
(iv) the distance between Rui Feng from the airport when Jun Wei reaches the train station.

7. The diagram shows the speed-time graph of a particle over a period of t seconds.

Speed (m/s)

Time (s)

(i) Given that the acceleration of the particle during the first 8 seconds of its motion is 1.5 m/s^2, find the value of V.

(ii) Given that the retardation of the particle is 1 m/s^2, find the value of t.

8. The variables x and y are connected by the equation $y = \dfrac{12}{x} + x - 6$.
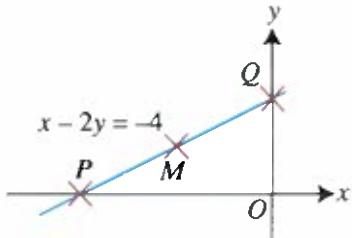
The table below shows some values of x and the corresponding values of y.

x	1	1.5	2	3	4	5	6	7	8
y	7	h	2	1	1	1.4	2	k	3.5

(a) Find the value of h and of k.

(b) Using a scale of 2 cm to represent 1 unit on each axis, draw the graph of $y = \dfrac{12}{x} + x - 6$ for $1 \leqslant x \leqslant 8$.

(c) Use your graph to find
   (i) the value of y when $x = 2.3$,
   (ii) the minimum value of y.

(d) By drawing a tangent, find the gradient of the curve at the point where $x = 5$.

(e) Use your graph to obtain one solution of the equation $x^2 + 12 = 10x$.
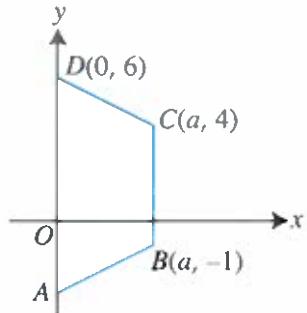
1. By selling an antique vase for $408, a storekeeper suffers a loss of 4% on the cost price.

   (i) Find the cost price of the vase.

   (ii) If the vase is sold for $510, express the storekeeper's gain or loss as a percentage of the cost price.

2. Farhan's monthly income consists of a basic salary of $520 and a commission of 25% on his sales for the month. If he sold $5264 worth of goods in a particular month, find his total income for that month.

3. Find the equation of the line passing through the point (2, –5) and parallel to the line $5x + 7y = 46$.

4. The line $x - 2y = -4$ cuts the x-axis and the y-axis at P and Q respectively. M is a point on PQ such that it is equidistant from the coordinate axes.

   Find

   (i) the coordinates of P and of Q,

   (ii) the coordinates of M,

   (iii) the area of $\triangle PMO$.

5. $ABCD$ is a trapezium in which $DC = \sqrt{13}$ units. The coordinates of B, C and D are $(a, -1)$, $(a, 4)$ and $(0, 6)$ respectively and the gradient of AB is $\frac{2}{3}$.
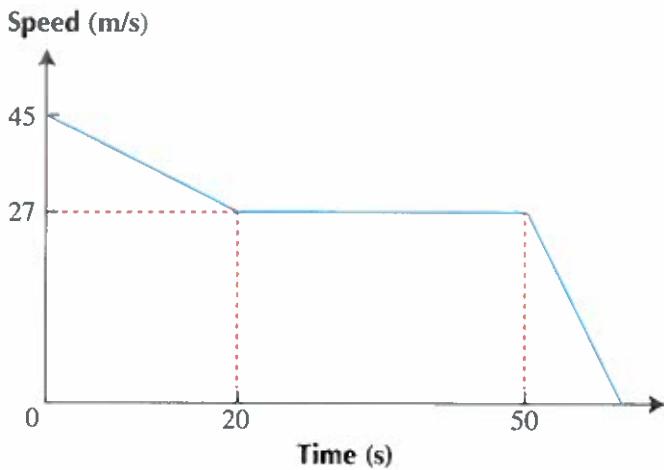
   Find

   (i) the coordinates of the point C,

   (ii) the coordinates of the point A,

   (iii) the area of the trapezium $ABCD$,

   (iv) the length of AB,

   (v) the equation of AB.

6. A coach leaves Watertown for Sandcity 120 km away at 1100 and travels at a uniform speed of 50 km/h. An hour later, a car travelling at a uniform speed of 80 km/h leaves Sandcity for Watertown by the same route.

   (a) Draw the distance-time graph to represent the journey.

   (b) Use your graph to find

       (i) the time when the car meets the coach and the distance from Watertown at this instant,

       (ii) the distance between the coach and the car at 1300.

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7. The diagram shows the speed-time graph of a car which decelerates uniformly from 45 m/s to 27 m/s in 30 seconds. It then travels at a constant speed of 27 m/s for 30 seconds.

Speed (m/s)

Time (s)

(i) Given that the car begins to decelerate uniformly at 0.6 m/s² until it comes to rest, find the total time taken for the journey.
Give your answer in minutes and seconds.

(ii) Sketch the acceleration-time graph for the motion.

8. A string of length 15 cm is used to form a rectangle. Given that the area of the rectangle is y cm² and that one side of the rectangle is x cm long, show that $y = \frac{1}{2}x(15 - 2x)$.

The table below shows some values of x and the corresponding values of y.

x	0.5	1	2	3	4	5	6	6.5
y	3.5	6.5	11	13.5	14	12.5	9	p

(i) Find the value of p.

(ii) Using a scale of 2 cm to represent 1 unit, draw a horizontal x-axis for $0 \leqslant x \leqslant 7$.
Using a scale of 1 cm to represent 1 unit, draw a vertical y-axis for $0 \leqslant y \leqslant 16$.
On your axes, plot the points given in the table and join them with a smooth curve.

(iii) Use your graph to find the solutions of the equation $x(15 - 2x) = 23$.

(iv) By drawing a tangent, find the gradient of the curve at the point $(5, 12.5)$.

(v) Use your graph to find the maximum value of y and the corresponding value of x.

(vi) Hence, write down the dimensions of the rectangle when the area is a maximum. What can you say about your answer?

Further Trigonometry

Many measurements in this world are difficult or impossible to obtain directly. How do you measure the height of the Eiffel Tower or of Mount Everest? With the help of trigonometry, these measurements can be easily obtained.

Chapter
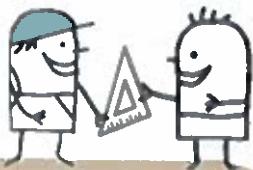
Eight

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8.1 Sine and Cosine of Obtuse Angles

Recap

In Book 2, we have learnt that the trigonometric ratios of an acute angle A are defined based on a right-angled triangle as follows:

$$\sin A = \frac{opp}{hyp}$$

$$\cos A = \frac{adj}{hyp}$$

$$\tan A = \frac{opp}{adj}$$
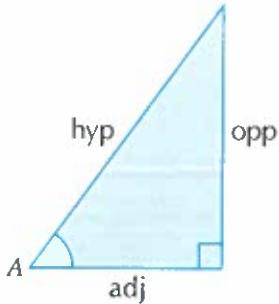
Fig. 8.1

What happens if A is an **obtuse** angle as shown below?
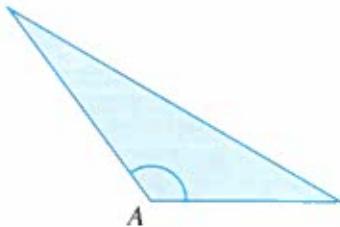
Fig. 8.2

In order to find the sides and angles of an obtuse-angled triangle, we will need to extend the definitions of trigonometric ratios. In this chapter, we will learn the trigonometric ratios of the sine and cosine of obtuse angles.

⠿ Sine and Cosine of Obtuse Angles

Fig. 8.3(**a**) shows a circle with centre O and radius r units.

$P(x, y)$ is a point on the circle and $\triangle OPQ$ is a right-angled triangle. $\angle A$ is an acute angle.
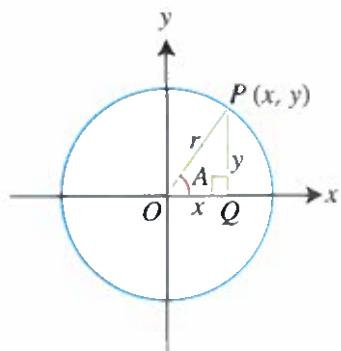
$$\therefore \sin A = \frac{\text{opp}}{\text{hyp}} = \frac{y}{r} \text{ and } \cos A = \frac{\text{adj}}{\text{hyp}} = \frac{x}{r}$$

In other words, we have extended the definition of the sine and cosine of an angle A in terms of the coordinates of a point $P(x, y)$:
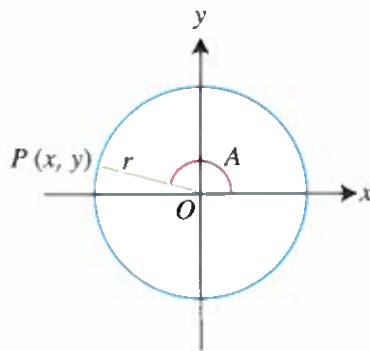
$$\sin A = \frac{y}{r} \text{ and } \cos A = \frac{x}{r}$$

If A is an acute angle, then x, y and r are positive.

In other words, $\sin A$ and $\cos A$ are positive if A is acute.

(a) $\angle A$ is acute   **(b)** $\angle A$ is obtuse

Fig. 8.3

Fig. 8.3(**b**) shows a circle with centre O and radius r units.

$P(x, y)$ is a point on the circle and $\angle A$ is an **obtuse** angle.

Using the extended definitions,
$$\sin A = \frac{y}{r} \text{ and } \cos A = \frac{x}{r}.$$

However, x is now negative, but y and r are still positive.

In other words, if A is an obtuse angle, then $\sin A$ is still positive but $\cos A$ is negative.

Use your calculator to find the value of $\sin 150°$ and of $\cos 150°$.

Which trigonometric ratio is positive and which one is negative?

Investigation

Relationship between Trigonometric Ratios of Acute and Obtuse Angles

Use your calculator to evaluate the sine and cosine of each of the following pairs of angles, leaving your answers correct to 3 significant figures where necessary.

	A	$180° - A$	$\sin A$	$\sin (180° - A)$	$\cos A$	$\cos (180° - A)$
(a)	30°	150°				
(b)	76°	104°				
(c)	111°	69°				
(d)	167°	13°				

Table 8.1

1. What do you notice about $\sin A$ and $\sin (180° - A)$?

2. What do you notice about $\cos A$ and $\cos (180° - A)$?

In general, if A is acute or obtuse,

$$\sin A = \sin (180° - A)$$
$$\cos A = -\cos (180° - A)$$

To prove the above relationship, consider Fig. 8.4 which shows a circle with centre O and radius r units. $P(x, y)$ is a point on the circle where $x < 0$ and $y > 0$.

$\angle A$ is an obtuse angle.
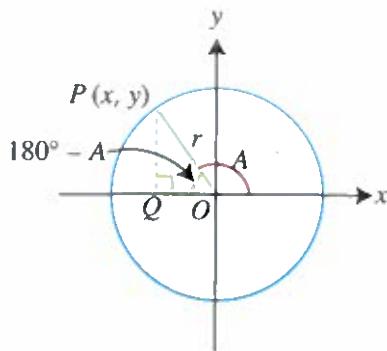
Fig. 8.4

Using the extended definitions,

$$\sin A = \frac{y}{r} \text{ and } \cos A = \frac{x}{r} . \text{ --- (1)}$$

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In the right-angled $\triangle OPQ$,

$OQ = -x$ (so that OQ will be positive since $x < 0$)

and $PQ = y$ (since $y > 0$).

Since $180° - A$ is an acute angle, we can use the definitions for acute angles:

$$\sin(180° - A) = \frac{\text{opp}}{\text{hyp}} = \frac{PQ}{OP} = \frac{y}{r}$$

and $\cos(180° - A) = \frac{\text{adj}}{\text{hyp}} = \frac{OQ}{OP} = \frac{-x}{r}$ --- (2)

Comparing (1) and (2),

$$\sin A = \frac{y}{r} = \sin(180° - A)$$

but $\cos A = \frac{x}{r} = -\cos(180° - A)$

Worked Example 1

(Relationship between Trigonometric Ratios of Acute and Obtuse Angles)

Given that $\sin 55° = 0.819$ and $\cos 136° = -0.719$ when corrected to 3 significant figures, find the value of each of the following without the use of a calculator.

(a) $\sin 125°$ (b) $\cos 44°$

Solution:

(a) $\sin 125° = \sin(180° - 125°)$

$\qquad\quad = \sin 55°$

$\qquad\quad = 0.819$

(b) $\cos 44° = -\cos(180° - 44°)$

$\qquad\quad = -\cos 136°$

$\qquad\quad = -(-0.719)$

$\qquad\quad = 0.719$

PRACTISE NOW 1

1. Given that $\sin 84° = 0.995$ and $\cos 129° = -0.629$ when corrected to 3 significant figures, find the value of each of the following without the use of a calculator.

   (a) $\sin 96°$ (b) $\cos 51°$

2. Given that $\sin 172° = 0.139$ and $\cos 40° = 0.766$ when corrected to 3 significant figures, find the value of $\sin 8° - \cos 140°$ without the use of a calculator.
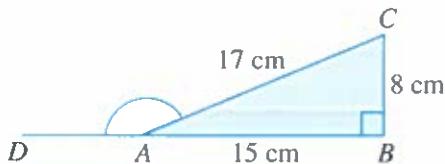
Worked Example 2

(Relationship between Trigonometric Ratios of Acute and Obtuse Angles)

In the figure, DAB is a straight line, $\angle ABC = 90°$, $AB = 15$ cm, $BC = 8$ cm and $AC = 17$ cm.

Find the value of each of the following.

(a) $\sin \angle DAC$ (b) $\cos \angle DAC$

(c) $\tan \angle ACB$

Solution:

(a) $\sin \angle DAC = \sin (180° - \angle DAC)$

$\qquad = \sin \angle BAC$

$\qquad = \dfrac{\text{opp}}{\text{hyp}}$ (since $\sin A = \sin (180° - A)$)

$\qquad = \dfrac{BC}{AC}$

$\qquad = \dfrac{8}{17}$

(b) $\cos \angle DAC = -\cos (180° - \angle DAC)$

$\qquad = -\cos \angle BAC$

$\qquad = -\dfrac{\text{adj}}{\text{hyp}}$ (since $\cos A = -\cos (180° - A)$)

$\qquad = -\dfrac{AB}{AC}$

$\qquad = -\dfrac{15}{17}$

(c) $\tan \angle ACB = \dfrac{\text{opp}}{\text{adj}}$

$\qquad = \dfrac{AB}{BC}$

$\qquad = \dfrac{15}{8}$

$\qquad = 1\dfrac{7}{8}$
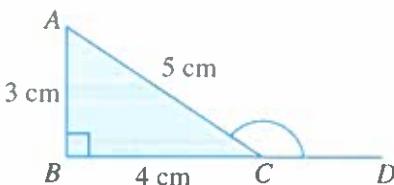
1. In the figure, BCD is a straight line, $\angle ABC = 90°$, $AB = 3$ cm, $BC = 4$ cm and $AC = 5$ cm.
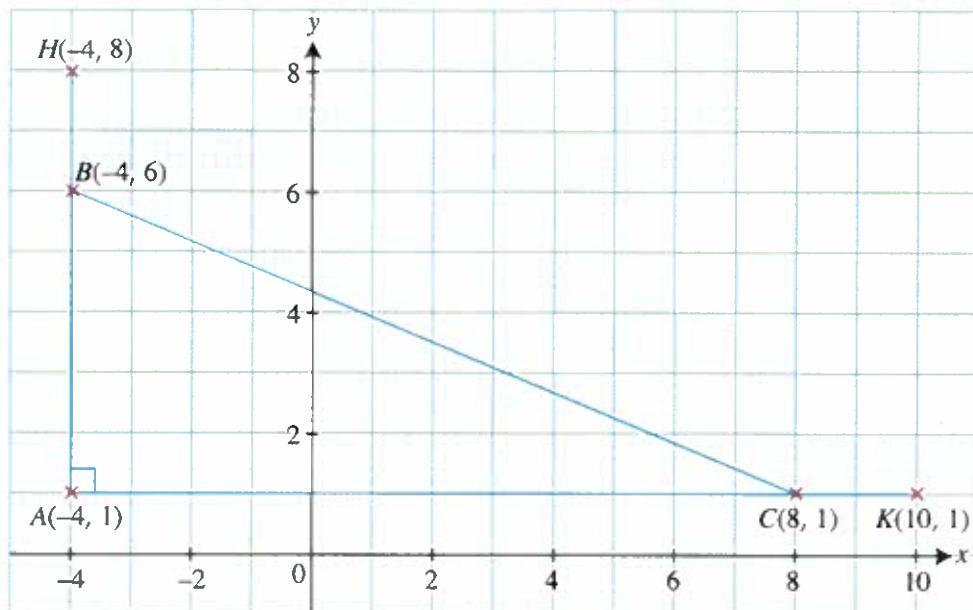
Find the value of each of the following.

(a) $\sin \angle ACD$ (b) $\cos \angle ACD$

(c) $\tan \angle BAC$

2. The figure shows $\triangle ABC$ with vertices $A(-4, 1)$, $B(-4, 6)$ and $C(8, 1)$. $H(-4, 8)$ and $K(10, 1)$ lie on AB produced and AC produced respectively.

ATTENTION

The scale for the x-axis and the y-axis must be the same for the trigonometric ratios of the angles to be correct. If the scale used is different, the lengths of the sides of the triangle would not match the distance between two points represented by coordinates.

(a) Find the length of BC.

(b) State the value of each of the following.

     (i) $\sin \angle HBC$ (ii) $\cos \angle BCK$

     (iii) $\tan \angle ABC$

Worked Example 3

(Solving Simple Trigonometric Equations)

Given that $0° \leqslant x \leqslant 180°$, find the possible values of x for each of the following equations.

(a) $\sin x = 0.45$ (b) $\cos x = -0.834$

Solution:

(a) Since $\sin x$ is positive, x can either be an acute angle or an obtuse angle.

$\sin x = 0.45$

$\quad x = \sin^{-1} 0.45 = 26.7°$ (to 1 d.p.)

$\qquad$ or $180° - 26.7° = 153.3°$ (to 1 d.p.)

$\therefore x = 26.7°$ or $153.3°$

(b) Since $\cos x$ is negative, x is an obtuse angle.

$\cos x = -0.834$

$\quad x = \cos^{-1} (-0.834)$

$\qquad = 146.5°$ (to 1 d.p.)

PRACTISE NOW 3

Given that $0° \leqslant x \leqslant 180°$, find the possible values of x for each of the following equations.

(a) $\sin x = 0.415$ (b) $\cos x = -0.234$ (c) $\cos x = 0.104$
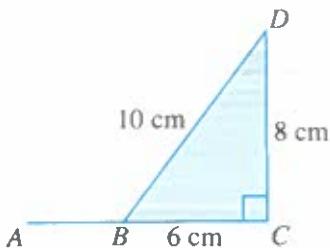
Exercise 8A

BASIC LEVEL

1. Express each of the following as a trigonometric ratio of the acute angle.

   (a) $\sin 110°$ (b) $\sin 176°$

   (c) $\sin 98°$ (d) $\cos 99°$

   (e) $\cos 107°$ (f) $\cos 175°$

2. Given that $\sin 32° = 0.530$ and $\cos 145° = -0.819$ when corrected to 3 significant figures, find the value of each of the following without the use of a calculator.

   (a) $\sin 148°$ (b) $\cos 35°$

3. Given that $\sin 45° = \cos 45° = 0.707$ when corrected to 3 significant figures, find the value of each of the following without the use of a calculator.

   (a) $2 \cos 45° + 3 \sin 135°$

   (b) $3 \cos 135° + 4 \sin 135°$

   (c) $\cos 135° - 2 \sin 45°$
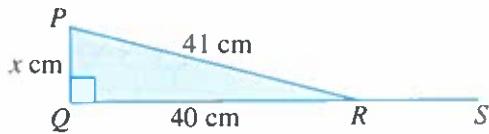
4. In the figure, ABC is a straight line, $\angle BCD = 90°$, $BC = 6$ cm, $CD = 8$ cm and $BD = 10$ cm.

Find the value of each of the following.

(a) $\sin \angle ABD$ **(b)** $\cos \angle DBA$

(c) $\tan \angle CBD$

5. In the figure, QRS is a straight line, $\angle PQR = 90°$, $PQ = x$ cm, $QR = 40$ cm and $PR = 41$ cm.

(a) Find the value of x.

(b) Find the value of each of the following.

    (i) $\sin \angle PRS$ **(ii)** $\cos \angle PRS$

    (iii) $\tan \angle PRQ$

6. Find an acute angle whose sine is

(a) 0.52,   **(b)** 0.75,

(c) 0.875,   **(d)** 0.3456.

7. Find an obtuse angle whose sine is

(a) 0.52,   **(b)** 0.75,

(c) 0.875,   **(d)** 0.3456.
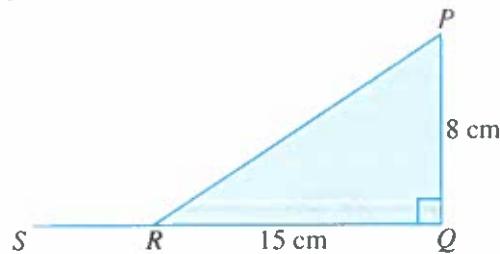
8. Find an acute angle whose cosine is

(a) 0.67,   **(b)** 0.756,

(c) 0.5,   **(d)** 0.985.

9. Given that $0° \leqslant x \leqslant 180°$, find the possible values of x for each of the following equations.

(a) $\sin x = 0.753$ **(b)** $\sin x = 0.952$

(c) $\sin x = 0.4714$ **(d)** $\cos x = -0.238$

(e) $\cos x = -0.783$ **(f)** $\cos x = 0.524$
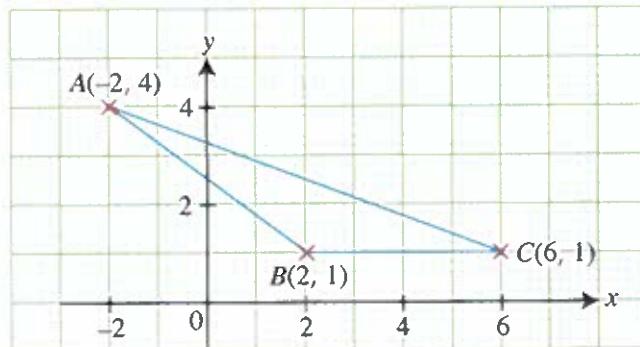
10. In the figure, SRQ is a straight line, $\angle PQR = 90°$, $PQ = 8$ cm and $QR = 15$ cm.

Find the value of each of the following.

(a) $\sin \angle PRS$ **(b)** $\cos \angle SRP$

(c) $\tan \angle PRQ$
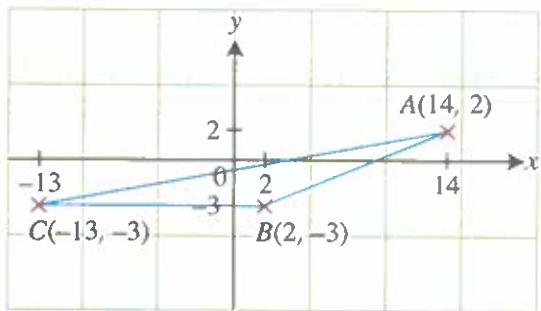
11. The figure shows $\triangle ABC$ with vertices $A(-2, 4)$, $B(2, 1)$ and $C(6, 1)$.

Find the value of each of the following.

(a) $\sin \angle ABC$ **(b)** $\cos \angle ABC$

(c) $\tan \angle ACB$

12. The figure shows $\triangle ABC$ with vertices $A(14, 2)$, $B(2, -3)$ and $C(-13, -3)$.

Find the value of each of the following.

(a) $\sin \angle ABC$ (b) $\cos \angle ABC$

(c) $\tan \angle ACB$
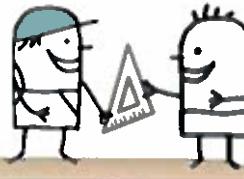
13. Given that $\sin x° = \sin 27°$, where $0° < x < 180°$, write down the possible values of x.

14. Given that $0° < x < 180°$, find the possible values of x for each of the following equations.

(a) $\sin (x + 10°) = 0.47$ (b) $\cos (x - 10°) = -0.56$

8.2 Area of Triangle

In primary school, we have learnt that the area of a triangle is given by the formula:

$$\text{Area of triangle} = \frac{1}{2} \times \text{base} \times \text{height or } \frac{1}{2}bh$$
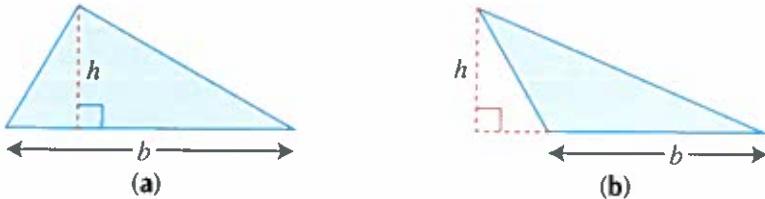
Fig. 8.5

What happens if the height of a triangle is not given?

In real life, if a farmer has a triangular field with a given base, it is not easy to find the height of the triangle.

For example, if he is to start measuring from the point B in Fig. 8.6(a), when he reaches D on AC, $\angle BDC$ may not be a right angle so BD may not be the height.

Fig. 8.6

INFORMATION
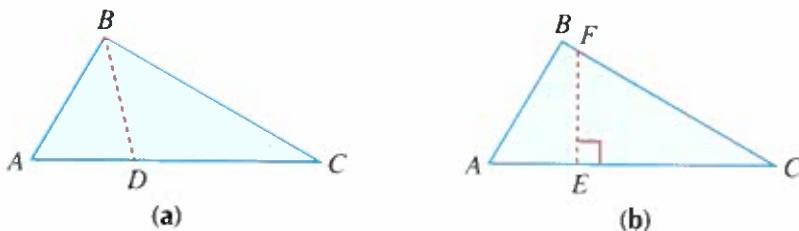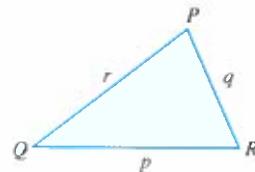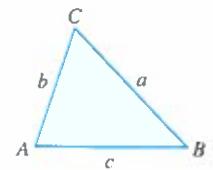
In this chapter, we shall use small letters to denote the lengths of the sides facing the angles, which are correspondingly denoted by capital letters.

Hence, we label two triangles as follows:

If the farmer is to choose a point E on AC in Fig. 8.6**(b)** and he walks in the direction perpendicular to AC, he may not end up at B.

In other words, there is a need to find another formula for the area of a triangle.

Fig. 8.7 shows two triangles.
In Fig. 8.7**(a)**, $\angle C$ is acute, while in Fig. 8.7**(b)**, $\angle C$ is obtuse.
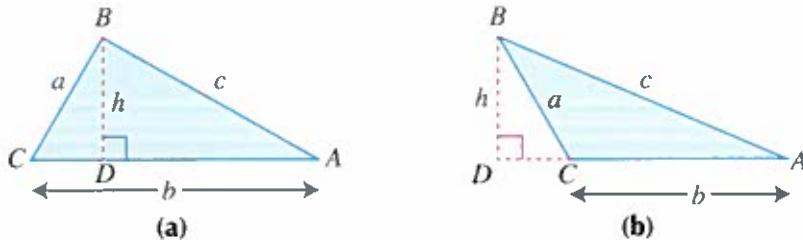
Fig. 8.7

In Fig. 8.7**(a)**, consider $\triangle BCD$.

Then $\sin C = \dfrac{\text{opp}}{\text{hyp}}$

$\qquad = \dfrac{h}{a}$

$\qquad h = a \sin C$

$\therefore$ Area of $\triangle ABC = \dfrac{1}{2} bh$

$\qquad\qquad\qquad = \dfrac{1}{2} b(a \sin C)$

$\qquad\qquad\qquad = \dfrac{1}{2} ab \sin C$

In Fig. 8.7**(b)**, consider $\triangle BCD$.

Then $\sin C = \dfrac{\text{opp}}{\text{hyp}}$

$\qquad = \dfrac{h}{a}$

$\therefore \sin \angle ACB = \sin (180° - \angle ACB)$

$\qquad\qquad\quad = \sin \angle BCD$

$\qquad\qquad\quad = \dfrac{h}{a}$

i.e. $h = a \sin \angle ACB$

$\qquad = a \sin C$

$\therefore$ Area of $\triangle ABC = \dfrac{1}{2} bh$

$\qquad\qquad\qquad = \dfrac{1}{2} ab \sin C$

By considering $\sin A$ and $\sin B$ in a similar way, we can show that:

Area of $\triangle ABC = \dfrac{1}{2} bc \sin A$ and $\dfrac{1}{2} ac \sin B$ respectively.

In general,

$$\boxed{\text{Area of } \triangle ABC = \frac{1}{2} ab \sin C = \frac{1}{2} bc \sin A = \frac{1}{2} ac \sin B}$$

ATTENTION
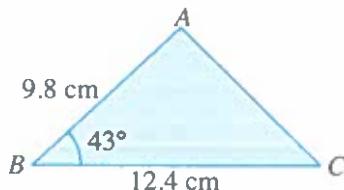
In the formula $\frac{1}{2} ab \sin C$, notice that the angle C is in between the two sides a and b, i.e. C is called the **included** angle.

Worked Example 4

(Finding the Area of a Triangle)

Find the area of $\triangle ABC$, given that $AB = 9.8$ cm, $BC = 12.4$ cm and $\angle ABC = 43°$.

Solution:

We have $a = 12.4$, $c = 9.8$ and $B = 43°$.

$$\text{Area of } \triangle ABC = \frac{1}{2}ac \sin B$$

$$= \frac{1}{2} \times 12.4 \times 9.8 \times \sin 43°$$

$$= 41.4 \text{ cm}^2$$

PRACTISE NOW 4
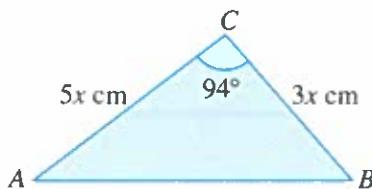
Find the area of $\triangle ABC$, given that $BC = 31.8$ m, $AC = 24.8$ m and $\angle ACB = 49°$.

Worked Example 5

(Problem involving Area of a Triangle)

In $\triangle ABC$, $CA = 5x$ cm, $CB = 3x$ cm and $\angle ACB = 94°$.

Given that the area of $\triangle ABC$ is 145 cm², find the value of x.

Solution:
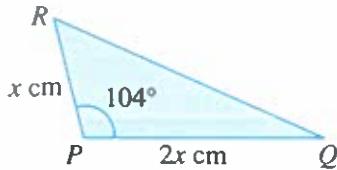
We have $a = 3x$, $b = 5x$ and $C = 94°$.

$$\text{Area of } \triangle ABC = \frac{1}{2}ab \sin C$$

$$145 = \frac{1}{2} \times 3x \times 5x \times \sin 94°$$

$$= 7.5x^2 \sin 94°$$

$$x^2 = \frac{145}{7.5 \sin 94°}$$

$$x = \sqrt{\frac{145}{7.5 \sin 94°}} \quad \text{(since } x \text{ is positive)}$$

$$= 4.40 \text{ (to 3 s.f.)}$$

1. In $\triangle PQR$, $PQ = 2x$ cm, $PR = x$ cm and $\angle QPR = 104°$.

Given that the area of $\triangle PQR$ is 12.5 cm², find the value of x.
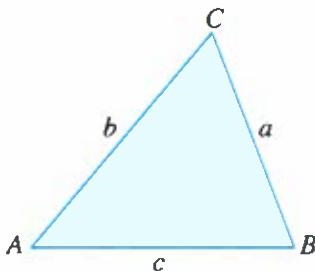
2. In $\triangle XYZ$, $XY = 5$ cm, $YZ = 6$ cm and the area of $\triangle XYZ$ is 12 cm². Find $\angle XYZ$.

Thinking Time

In real life, a 'triangular' field is not always exactly a triangle. Therefore, a small error in measuring the included angle may result in a large error in the area of the triangle if we use the formula $\frac{1}{2}ab\sin C$. However, the error is usually not so large if we can use the lengths of the three sides of the triangle to find its area.

Heron of Alexandria (around AD 75) established a formula for finding the area of a triangle using the lengths of its sides only. The area of $\triangle ABC$ is given by $\sqrt{s(s-a)(s-b)(s-c)}$, where $s = \dfrac{a+b+c}{2}$ is half of the perimeter.

Verify that the above formula is correct for each of the following cases:

(a) $a = 6$ cm, $b = 8$ cm and $c = 10$ cm

(b) $a = 8$ cm, $b = 9$ cm and $c = 10$ cm

(c) $a = 5$ cm, $b = 3$ cm and $c = 7$ cm
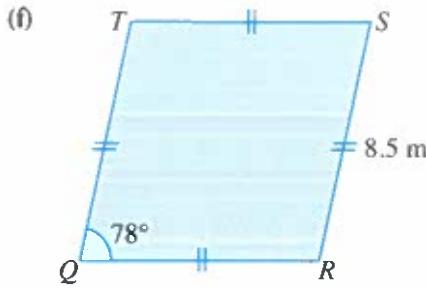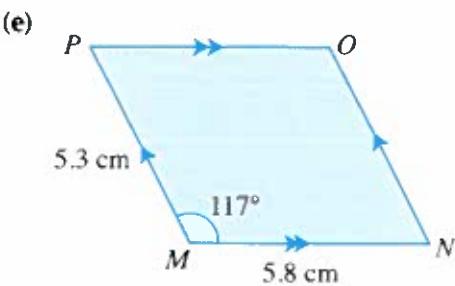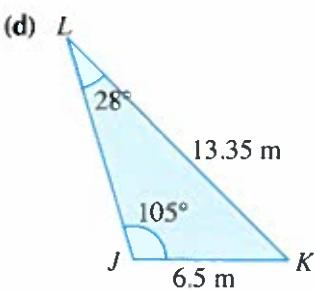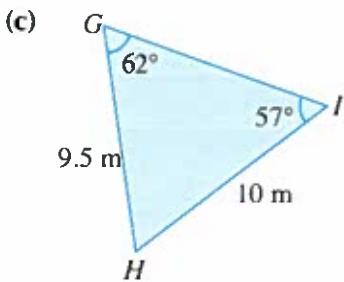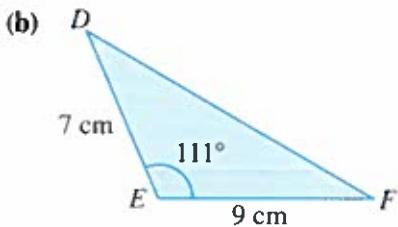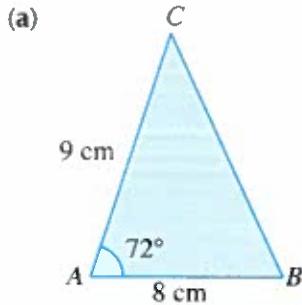
Can you find a proof for this formula?
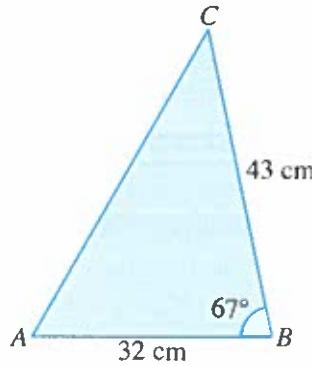
BASIC LEVEL

1. Find the area of each of the following figures.

(a)

(b)

(c)

(d)

(e)

(f)
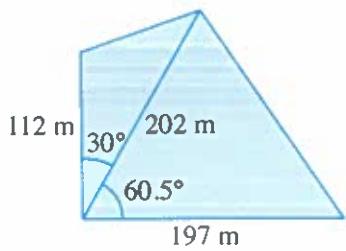
2. Find the area of $\triangle ABC$, given that $AB = 22$ cm, $AC = 15$ cm and $\angle BAC = 45°$.

3. In $\triangle PQR$, $\angle P = 72°$, $q = 152$ cm and $r = 125$ cm. Find the area of $\triangle PQR$.

4. In $\triangle ABC$, $AB = 32$ cm, $BC = 43$ cm and $\angle ABC = 67°$.
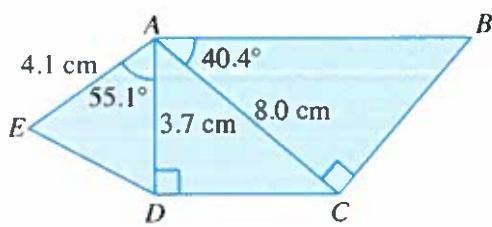
(i) Find the area of $\triangle ABC$.

(ii) Hence, find the perpendicular distance from A to BC.

5. The figure shows the plan of two neighbouring estates in the form of 2 triangles.

Calculate the area of the two estates.
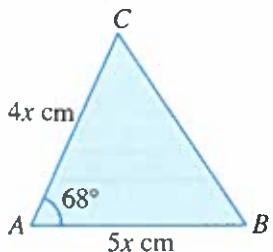
6. In the figure, $\angle ADC = \angle ACB = 90°$, $\angle EAD = 55.1°$, $\angle CAB = 40.4°$, $AE = 4.1$ cm, $AD = 3.7$ cm and $AC = 8.0$ cm.

Find

(i) $\angle ACD$, **(ii)** the length of AB,

(iii) the area of $\triangle AED$.
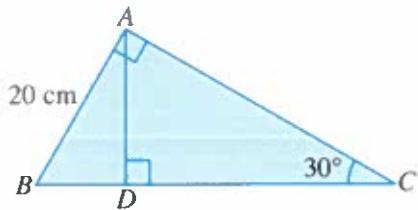
7. In $\triangle ABC$, $AB = 5x$ cm, $AC = 4x$ cm and $\angle BAC = 68°$.

Given that the area of $\triangle ABC$ is 97 cm², find the value of x.

8. In the figure, $AB = 20$ cm, $\angle BAC = 90°$, $\angle ACB = 30°$ and AD is perpendicular to BC.
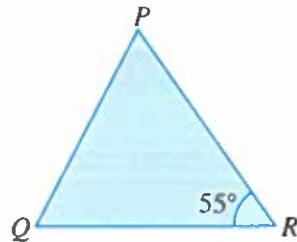
Find

(i) $\angle BAD$, **(ii)** BD,

(iii) the area of $\triangle ABC$.

9. The diagonals of a parallelogram have lengths 15.6 cm and 17.2 cm. They intersect at an angle of 120°. Find the area of the parallelogram.

10. In $\triangle PQR$, $\angle PRQ = 55°$, $3QR = 4PR$ and the area of $\triangle PQR$ is 158 cm².

Find the length of QR.
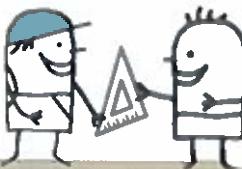
11. Given that the area of a rhombus is 40 cm² and that each side has a length of 15 cm, find the angles of the rhombus.

12. In quadrilateral $ABCD$, $AB = 3.2$ cm, $BC = 5.1$ cm, $\angle CBD = 34.4°$ and the length of the diagonal BD is 7.5 cm. Given further that the area of $\triangle ABD$ is 11.62 cm² and $\angle ABD$ is obtuse, find

(i) the area of $\triangle BCD$,

(ii) $\angle ABD$.

8.3 Sine Rule

Investigation

Sine Rule

Go to http://www.shinglee.com.sg/StudentResources/ and open the geometry software template 'Sine Rule'.

Fig. 8.8 shows a triangle *ABC* and a table of values.

Sine Rule

Click and move the points A, B and C so that you will get different types of triangles, e.g. acute-angled, right-angled, obtuse-angled, equilateral, isosceles and scalene

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Show How to do animation

Right-angled triangle   Isosceles Triangle   Equilateral Triangle

What do you notice about these values?

A	B	C	a	b	c	$\frac{a}{\sin(A)}$	$\frac{b}{\sin(B)}$	$\frac{c}{\sin(C)}$
69.7°	76.4°	33.9°	8.06 cm	8.36 cm	4.80 cm	8.60 cm	8.60 cm	8.60 cm

Fig. 8.8

1. The labelling of the sides of the triangle with reference to the vertices is important. Copy and complete the following.

    (a) The length of the side of the triangle opposite vertex *A* is labelled *a*.

    (b) The length of the side of the triangle opposite vertex *B* is labelled _____.

    (c) The length of the side of the triangle opposite vertex *C* is labelled _____.

2. Click and drag each of the vertices A, B and C to get different types of triangles. To obtain special triangles such as a right-angled triangle, an isosceles triangle and an equilateral triangle, click on the respective buttons in the template. For each triangle, copy and complete Table 8.2.

No.	$\angle A$	$\angle B$	$\angle C$	a	b	c	$\dfrac{a}{\sin A}$	$\dfrac{b}{\sin B}$	$\dfrac{c}{\sin C}$
1.									
2.									
3.									
4.									
5.									
6.									

Table 8.2
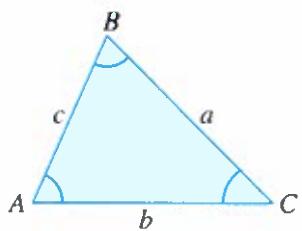
3. What do you notice about the last 3 columns in Table 8.2?

4. Click on the button 'Show how to do animation' in the template and it will show you how to add 10 more entries to the table as the points A, B and C move automatically. What do you notice about the last 3 columns of the table in the template?

5. Hence, write down a formula relating the quantities in the last 3 columns of the table. This is called the **Sine Rule**. Notice that for each fraction, *the side must be opposite the angle.*

6. Do you think the relationship $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$ is also true? Explain your answer.

7. Copy and complete the following:

The lengths of the sides of a triangle are p_____ to the sine of the angles opposite the sides.

RECALL

In a triangle,
- the largest angle is opposite the longest side,
- the smallest angle is opposite the shortest side.

From the investigation, we can conclude that:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \quad \text{or} \quad \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \, ,$$

where A, B and C are the three interior angles of $\triangle ABC$ opposite the sides whose lengths are a, b and c respectively.

This is called the **Sine Rule**.

We can prove the sine rule using the formula for the area of a triangle obtained from the previous section, as shown on the next page.

For any triangle ABC,

$$\frac{1}{2}bc \sin A = \frac{1}{2}ac \sin B = \frac{1}{2}ab \sin C.$$

Dividing each side by $\frac{1}{2}abc$,
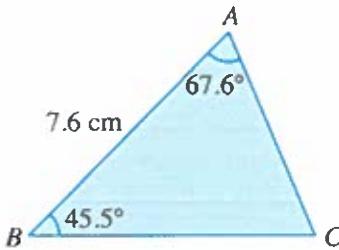
$$\frac{\frac{1}{2}bc \sin A}{\frac{1}{2}abc} = \frac{\frac{1}{2}ac \sin B}{\frac{1}{2}abc} = \frac{\frac{1}{2}ab \sin C}{\frac{1}{2}abc}.$$

$$\therefore \quad \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Worked Example 6

(Using Sine Rule when given 2 angles and 1 side)

In $\triangle ABC$, $\angle A = 67.6°$, $\angle B = 45.5°$ and $AB = 7.6$ cm. Find

(i) $\angle C$,   (ii) the length of BC,

(iii) the length of AC.

Solution:

(i) $\angle C = 180° - 67.6° - 45.5°$ ($\angle$ sum of a $\triangle$)

$\quad\quad = 66.9°$

(ii) Using sine rule,

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$

$$\frac{a}{\sin 67.6°} = \frac{7.6}{\sin 66.9°}$$

$$a = \frac{7.6 \sin 67.6°}{\sin 66.9°}$$

$$= 7.64 \text{ cm (to 3 s.f.)}$$

$\therefore BC = 7.64$ cm
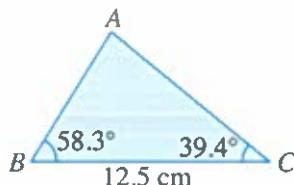
(iii) Using sine rule,

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\frac{b}{\sin 45.5°} = \frac{7.6}{\sin 66.9°}$$

$$b = \frac{7.6 \sin 45.5°}{\sin 66.9°}$$

$$= 5.89 \text{ cm (to 3 s.f.)}$$

$\therefore AC = 5.89$ cm

In $\triangle ABC$, $\angle B = 58.3°$, $\angle C = 39.4°$ and $BC = 12.5$ cm. Find
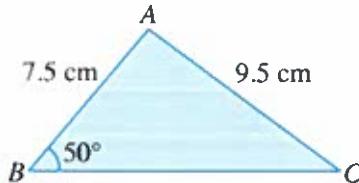
(i) $\angle A$, (ii) the length of AB, (iii) the length of AC.

Worked Example 7

(Using Sine Rule when given 2 sides and 1 non-included angle)

In $\triangle ABC$, $\angle B = 50°$, $AB = 7.5$ cm and $AC = 9.5$ cm. Find

(i) $\angle C$, (ii) $\angle A$,

(iii) the length of BC.

Solution:

(i) Using sine rule,

$$\frac{\sin C}{c} = \frac{\sin B}{b}$$

$$\frac{\sin C}{7.5} = \frac{\sin 50°}{9.5}$$

$$\sin C = \frac{7.5 \sin 50°}{9.5}$$

$$= 0.6048 \text{ (to 4 s.f.)}$$

$\angle C = \sin^{-1} 0.6048 = 37.21°$ (to 2 d.p.)

or $180° - 37.21° = 142.78°$ (to 2 d.p.)

Since $c < b$, then $\angle C < \angle B$, hence $\angle C$ cannot be $142.78°$.

$\therefore \angle C = 37.2°$ (to 1 d.p.)

(ii) $\angle A = 180° - 50° - 37.21°$ ($\angle$ sum of a $\triangle$)

$\qquad = 92.8°$ (to 1 d.p.)

(iii) Using sine rule,

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\frac{a}{\sin 92.79°} = \frac{9.5}{\sin 50°}$$
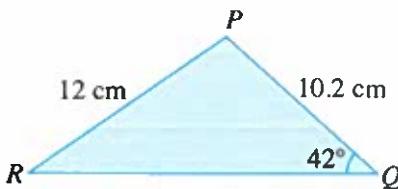
$$a = \frac{9.5 \sin 92.79°}{\sin 50°}$$

$$= 12.4 \text{ cm (to 3 s.f.)}$$
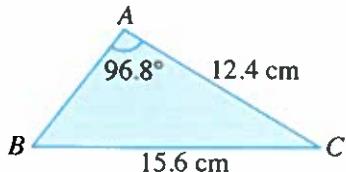
$\therefore BC = 12.4$ cm

PRACTISE NOW 7

1. In $\triangle PQR$, $\angle Q = 42°$, $PR = 12$ cm and $PQ = 10.2$ cm. Find

(i) $\angle R$,   **(ii)** $\angle P$,   **(iii)** the length of QR.

2. In $\triangle ABC$, $\angle BAC = 96.8°$, $AC = 12.4$ cm and $BC = 15.6$ cm. Find

(i) $\angle ABC$,   **(ii)** $\angle BCA$,   **(iii)** the length of AB.

Worked Example 8

(Ambiguous Case of Sine Rule)

In $\triangle ABC$, $\angle ABC = 55°$, $AB = 16.3$ cm and $AC = 14.3$ cm.
Find $\angle ACB$, $\angle BAC$ and the length of BC.

Solution:

Using sine rule,

$$\frac{\sin \angle ACB}{16.3} = \frac{\sin 55°}{14.3}$$

$$\sin \angle ACB = \frac{16.3 \sin 55°}{14.3}$$

$$= 0.9337 \text{ (to 4 s.f.)}$$
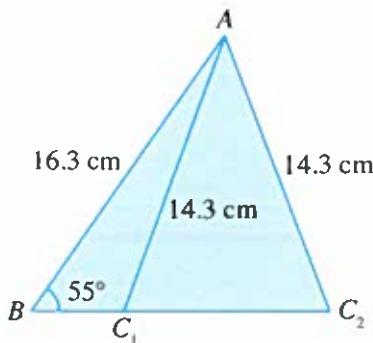
$\angle ACB = \sin^{-1} 0.9337 = 69.02°$ (to 2 d.p.)

or $180° - 69.02° = 110.98°$ (to 2 d.p.) (Since $c > b$, then $\angle C > \angle B$, i.e. $\angle ACB > 55°$,
                                                                    hence both answers are possible.)

When $\angle ACB = 69.02°$, $\angle BAC = 180° - 55° - 69.02°$

$$= 55.98°$$

When $\angle ACB = 110.98°$, $\angle BAC = 180° - 55° - 110.98°$

$$= 14.02°$$

Notice that it is possible to construct two different triangles from the information above.

A set of information that will give two sets of solutions is said to be *ambiguous*.
Hence, for ambiguous cases, two sets of solutions will be obtained.

Case 1: when $\angle ACB = 69.02°$ and $\angle BAC = 55.98°$

Using sine rule,

$$\frac{a}{\sin 55.98°} = \frac{16.3}{\sin 69.02°}$$

$$a = \frac{16.3 \sin 55.98°}{\sin 69.02°}$$

$$= 14.5 \text{ cm (to 3 s.f.)}$$

$\therefore \angle ACB = 69.0°$, $\angle BAC = 56.0°$ and $BC = 14.5$ cm

Case 2: when $\angle ACB = 110.98°$ and $\angle BAC = 14.02°$

Using sine rule,
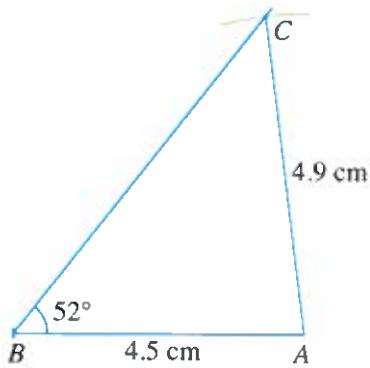
$$\frac{a}{\sin 14.02°} = \frac{16.3}{\sin 110.98°}$$

$$a = \frac{16.3 \sin 14.02°}{\sin 110.98°}$$

$$= 4.23 \text{ cm (to 3 s.f.)}$$

$\therefore \angle ACB = 111.0°$, $\angle BAC = 14.0°$ and $BC = 4.23$ cm

---

If sides b, c and $\angle B$ (i.e. 2 sides and a non-included angle) are given, take note of each of the following cases.

Case 1: $b > c$
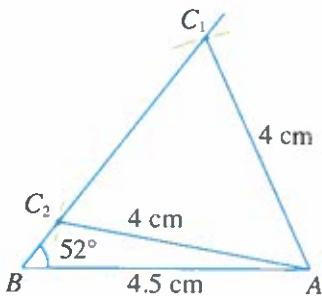
For example, construct $\triangle ABC$ such that $\angle ABC = 52°$, $AC = 4.9$ cm and $AB = 4.5$ cm.
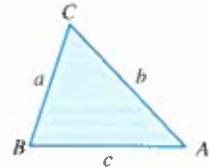
When $b > c$, one triangle can be constructed.
Therefore, there is only one value of $\angle ACB$.

Case 2: $b < c$

For example, construct $\triangle ABC$ such that $\angle ABC = 52°$, $AC = 4$ cm and $AB = 4.5$ cm.
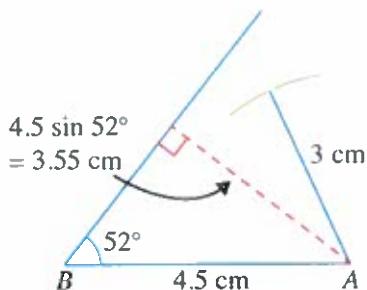
When $b < c$, two triangles can be constructed.
Therefore, there are two possible values of $\angle ACB$.

RECALL

In $\triangle ABC$, a is the side opposite to $\angle A$, b is the side opposite to $\angle B$ and c is the side opposite to $\angle C$.

Case 3: $b < c \sin B$, where $c \sin B$ is the perpendicular distance from A to the line through B

For example, construct $\triangle ABC$ such that $\angle ABC = 52°$, $AB = 4.5$ cm and $AC = 3$ cm.

When $b < c \sin B$, no triangle can be constructed.

In $\triangle ABC$, $\angle ABC = 46°$, $AB = 9.8$ cm and $AC = 7.1$ cm. Find $\angle ACB$, $\angle BAC$ and the length of BC.

Journal Writing

To solve a triangle means to find all the unknown sides and/or angles. From the worked examples, state the given conditions when sine rule can be used to solve a triangle.
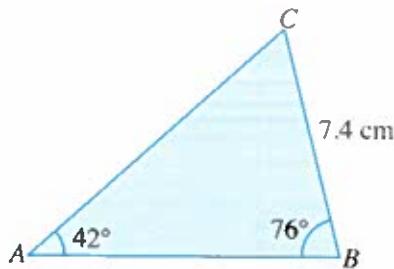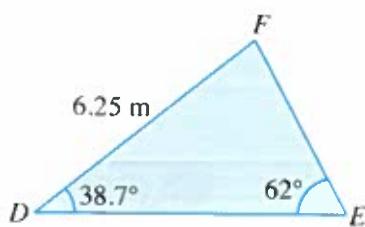
1. For each of the following triangles, find the unknown angles and sides.
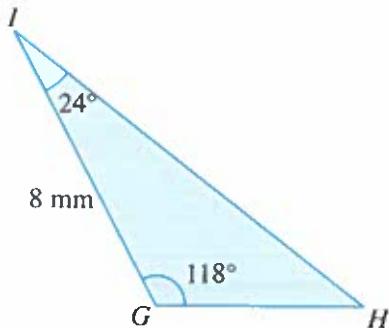
    (a)

    (b)
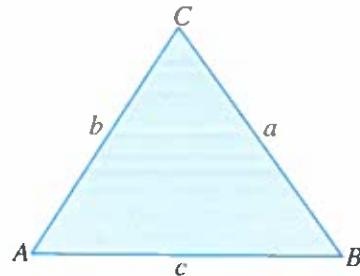
    (c)
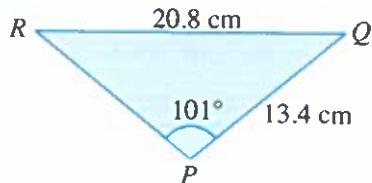
2. In $\triangle PQR$, $QR = 7$ cm, $\angle PQR = 47°$ and $\angle PRQ = 97°$. Find the length of PQ.

3. In $\triangle PQR$, $\angle P = 75°$, $\angle Q = 60°$ and $q = 14$ cm. Find the length of the longest side.

4. For each of the following triangles ABC, find the unknown angles and sides.

    (a) $\angle A = 92.0°$, $b = 6.93$ cm and $a = 15.3$ cm

    (b) $\angle B = 98.0°$, $a = 14.5$ m and $b = 17.4$ m

    (c) $\angle C = 35.0°$, $b = 8.7$ cm and $c = 9.5$ cm
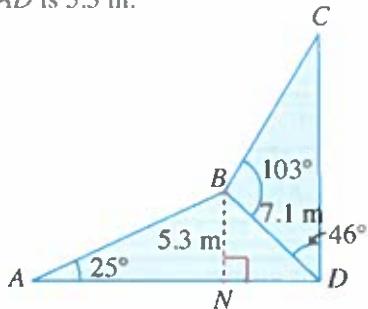
5. In $\triangle PQR$, $\angle P = 101°$, $PQ = 13.4$ cm and $QR = 20.8$ cm. Find

    (i) $\angle R$,　　　　　　　**(ii)** $\angle Q$,

    (iii) the length of PR.

6. In $\triangle ABC$, $\angle ABC = 91°$, $BC = 7.4$ cm and $AC = 11.6$ cm. Find

    (i) $\angle BAC$,　　　　　　**(ii)** $\angle ACB$,

    (iii) the length of AB.
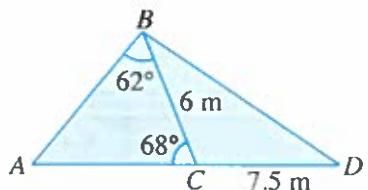
7. The figure shows a metal framework in which AD is horizontal with $BD = 7.1$ m, $\angle BAD = 25°$, $\angle BDC = 46°$, $\angle DBC = 103°$ and the height of B above AD is 5.3 m.

Find

(i) the length of the metal bar AB,

(ii) the angle that BD makes with BN,

(iii) the length of the metal bar CD.
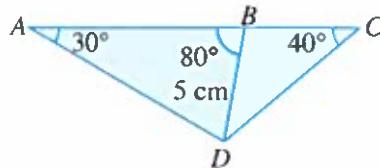
8. In the figure, A, C and D are three points along a straight road where $\angle ABC = 62°$, $\angle ACB = 68°$, $BC = 6$ m and $CD = 7.5$ m.

Find

(i) the distance AC,

(ii) the area of the region enclosed by AB, BD and DA.
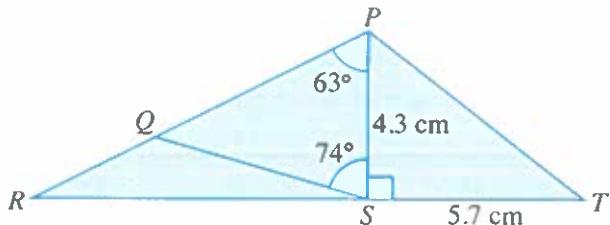
9. An experiment is carried out to determine the extension of springs. Springs are attached to a horizontal bar at A, B and C and are joined to a mass D.

Given that $\angle ACD = 40°$, $\angle CAD = 30°$, $\angle ABD = 80°$ and $BD = 5$ cm, find
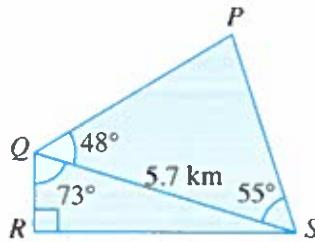
(i) the distance between A and B,

(ii) the distance between B and C,

(iii) the vertical distance between the mass and the horizontal bar.

10. In the figure, RST is a straight line, $\angle PST = 90°$, $\angle SPR = 63°$, $\angle PSQ = 74°$, $PS = 4.3$ cm and $ST = 5.7$ cm.
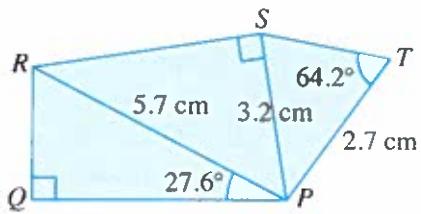
(i) Determine if QS is parallel to PT.

(ii) Find the length of PR.

(iii) Find the length of QS.

11. In the figure, $PQRS$ is a nature reserve. A 5.7 km long walkway connects Q to S. It is given that $\angle QRS = 90°$, $\angle SQR = 73°$, $\angle PQS = 48°$ and $\angle PSQ = 55°$.

Find the area of the nature reserve.

12. In the figure, $\angle PQR = \angle PSR = 90°$, $\angle QPR = 27.6°$, $\angle PTS = 64.2°$, $PR = 5.7$ cm, $PS = 3.2$ cm and $PT = 2.7$ cm.

Find
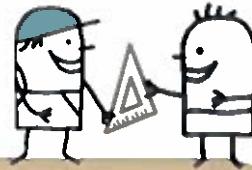
(i) the length of QR,

(ii) $\angle SPR$,

(iii) $\angle PST$.

13. For the data of each of the following triangles, determine whether it is an ambiguous case. Explain your answer.

(a) $\triangle ABC$, $\angle A = 92°$, $b = 7.5$ cm, $a = 8.5$ cm

(b) $\triangle DEF$, $\angle D = 47°$, $d = 75$ m, $e = 80$ m

(c) $\triangle GHI$, $g = 37$ mm, $h = 37$ mm, $\angle G = 58°$

(d) $\triangle JKL$, $j = 19$ cm, $k = 15$ cm, $\angle K = 39°$

(e) $\triangle MNO$, $n = 80$ m, $o = 67$ m, $\angle O = 43°$

(f) $\triangle PQR$, $p = 19$ mm, $q = 25$ mm, $\angle Q = 52°$

14. Determine whether it is possible to construct each of the following triangles with the given conditions.

(a) $\triangle ABC$, $AB = 6$ cm, $BC = 8$ cm, $\angle ABC = 90°$ and $\angle ACB = 35°$

(b) $\triangle PQR$, $PQ = 6$ cm, $PR = 5$ cm, $\angle PQR = 30°$ and $\angle PRQ = 36.9°$

(c) $\triangle LMN$, $LM = 6.9$ cm and $LN = 7.8$ cm, $\angle LMN = 42°$ and $\angle LNM = 57°$

(d) $\triangle GHK$, $GH = 6.4$ cm and $GK = 12.8$ cm, $\angle GHK = 90°$ and $\angle HGK = 60°$

15. In $\triangle ABC$, $\angle BAC = 58°$, $BC = 14.0$ cm and $AC = 15.4$ cm. Find $\angle ABC$, $\angle ACB$ and the length of AB.

16. On a map whose scale is 8 cm to 1 km, an undeveloped plot of land is shown as a quadrilateral $ABCD$. The length of the diagonal AC is 7 cm, $\angle BAC = 55°$, $\angle BCA = 77°$, $\angle DAC = 90°$ and $\angle DCA = 40°$. Find

(i) the length, in cm, of the side AB on the map,

(ii) the length, in km, which is represented by AD,

(iii) the area, in km², which is represented by $\triangle ADC$.

17. In $\triangle ABC$, $\angle A = 35°$, $BC = 5$ cm and $\sin B = \dfrac{4}{3}\sin A$.

(i) Calculate two possible values of $\angle B$.

(ii) Find the length of AC.

8.4 Cosine Rule

From the journal writing in Section 8.3, we have observed that the Sine Rule can be used to solve a triangle if the following are given:

(1) Two angles and the length of one side (see Worked Example 6); or

(2) The lengths of two sides and one non-included angle (see Worked Example 7).

What happens if the lengths of two sides and an included angle are given (see Fig. 8.9)? Can you try to use sine rule to solve the triangle?
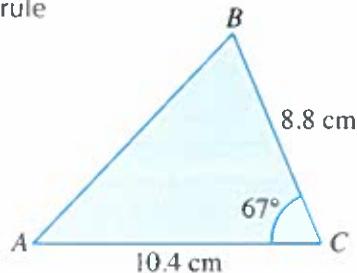
Fig. 8.9

Investigation

Cosine Rule

Go to http://www.shinglee.com.sg/StudentResources/ and open the geometry software template 'Cosine Rule'.

Fig 8.10 shows a triangle ABC and a table of values.

Cosine Rule

Click and move the points A, B and C so that you will get different types of triangles, e.g. acute-angled, right-angled, obtuse-angled, equilateral, isosceles and scalene

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Show How to do animation

Right-angled Triangle | Isosceles Triangle | Equilateral Triangle

Q1. What do you notice about the values in the first two columns, middle two columns and last two columns of the table below? This is called the cosine rule.

What do you notice about these values?

What do you notice about these values?

What do you notice about these values?

a^2	$(b^2+c^2)-2 \cdot b \cdot c \cos(A)$	b^2	$(a^2+c^2)-2 \cdot a \cdot c \cos(B)$	c^2	$(a^2+b^2)-2 \cdot a \cdot b \cos(C)$
58.16 cm²	58.16 cm²	64.55 cm²	64.55 cm²	19.12 cm²	19.12 cm²

Angle A = 68.7° a = 7.63 cm
Angle B = 79.0° b = 8.03 cm
Angle C = 32.3° c = 4.37 cm

Fig 8.10

1. The labelling of the sides of the triangle with reference to the vertices is important. Copy and complete the following.

   (a) The length of the side of the triangle opposite vertex A is labelled a.

   (b) The length of the side of the triangle opposite vertex B is labelled _____.

   (c) The length of the side of the triangle opposite vertex C is labelled _____.

2. Click and drag each of the vertices A, B and C to get different types of triangles. To obtain special triangles such as a right-angled triangle, an isosceles triangle and an equilateral triangle, click on the respective buttons in the template. For each triangle, copy and complete Table 8.3.

No.	$\angle A$	$\angle B$	$\angle C$	a^2	$b^2 + c^2 - 2bc \cos A$	b^2	$a^2 + c^2 - 2ac \cos B$	c^2	$a^2 + b^2 - 2ab \cos C$
1.									
2.									
3.									
4.									
5.									
6.									

Table 8.3

3. What do you notice about the last 6 columns in Table 8.3?

4. Click on the button 'Show how to do animation' in the template and it will show you how to add 10 more entries to the table as the points A, B and C move automatically. What do you notice about the last 6 columns of the table in the template?

5. Hence, write down a formula relating the quantities in the last 6 columns of the table. This is called the **Cosine Rule**. Notice that for each fraction, the *side by itself must be opposite the angle*.

6. For each of the formulae in Question 5, make the angle the subject of the formula.

From the investigation, we can conclude that:

$$a^2 = b^2 + c^2 - 2bc \cos A$$
$$b^2 = a^2 + c^2 - 2ac \cos B \qquad \text{or} \qquad$$
$$c^2 = a^2 + b^2 - 2ab \cos C$$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$
$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$$
$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

where A, B and C are the three interior angles of $\triangle ABC$ opposite the sides whose lengths are a, b and c respectively.

This is called the **Cosine Rule**.

We can prove the cosine rule as follows. Without loss of generality, we will just prove $a^2 = b^2 + c^2 - 2bc \cos A$.

There are three cases: $\angle A$ is an acute angle, $\angle A$ is an obtuse angle and $\angle A$ is a right angle. We will show the case when A is acute.
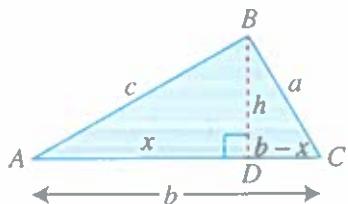
Fig. 8.11

In $\triangle BCD$,

$$a^2 = h^2 + (b - x)^2 \text{ (Pythagoras' Theorem)}$$
$$= h^2 + b^2 - 2bx + x^2$$
$$= b^2 + (h^2 + x^2) - 2bx \quad \text{--- (1)}$$

In $\triangle BAD$,

$$c^2 = h^2 + x^2 \text{ (Pythagoras' Theorem) --- (2)}$$

and $\cos A = \dfrac{x}{c}$,

i.e. $x = c \cos A$ --- (3)

Substituting (2) and (3) into (1),

$a^2 = b^2 + (h^2 + x^2) - 2bx$
$\quad = b^2 + c^2 - 2bc \cos A$ (proven)
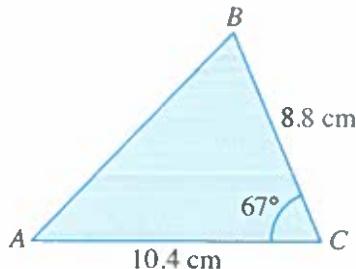
Thinking Time

1. Prove the Cosine Rule for Case 2 where $\angle A$ is an obtuse angle.

2. Let us consider Case 3 where $\angle A$ is a right angle.

    (a) What happens to the formula for the Cosine Rule $a^2 = b^2 + c^2 - 2bc \cos A$ if $A = 90°$?

    (b) Is this formula always true if $A = 90°$? Explain your answer.

3. Copy and complete the following:

    _____ Theorem is a special case of the Cosine Rule.

Worked Example 9

In $\triangle ABC$, $BC = 8.8$ cm, $AC = 10.4$ cm and $\angle ACB = 67°$. Find

(i) the length of AB,   (ii) $\angle ABC$,

(iii) $\angle BAC$.

Solution:

(i) Using cosine rule,

$$c^2 = a^2 + b^2 - 2ab \cos C$$

$$AB^2 = 8.8^2 + 10.4^2 - 2 \times 8.8 \times 10.4 \times \cos 67°$$

$$= 114.1 \text{ (to 4 s.f.)}$$

$$\therefore AB = \sqrt{114.1}$$

$$= 10.7 \text{ cm (to 3 s.f.)}$$

(ii) Using sine rule,

$$\frac{\sin \angle ABC}{AC} = \frac{\sin \angle ACB}{AB}$$

$$\frac{\sin \angle ABC}{10.4} = \frac{\sin 67°}{10.68}$$

$$\sin \angle ABC = \frac{10.4 \sin 67°}{10.68}$$

$$= 0.8963 \text{ (to 4 s.f.)}$$

$$\angle ABC = \sin^{-1} 0.8963 = 63.68° \text{ (to 2 d.p.)}$$

or $180° - 63.68° = 116.32°$ (to 2 d.p.)

Since $AC < AB$, then $\angle B < \angle C$, hence $\angle B$ cannot be $116.32°$.

$$\therefore \angle ABC = 63.7° \text{ (to 1 d.p.)}$$

ATTENTION

In order for the final answer to be accurate to three significant figures, any intermediate working must be correct to four significant figures, i.e. $AB \approx 10.68$ cm.
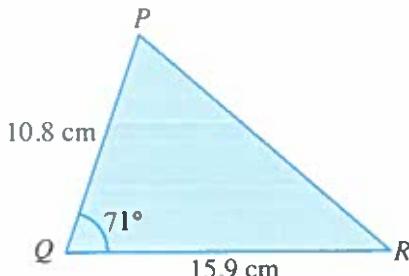
(iii) $\angle BAC = 180° - 63.68° - 67°$ ($\angle$ sum of a $\triangle$)

$$= 49.3° \text{ (to 1 d.p.)}$$

In $\triangle PQR$, $PQ = 10.8$ cm, $QR = 15.9$ cm and $\angle PQR = 71°$. Find
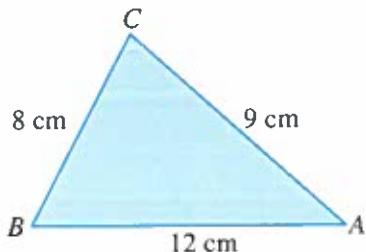
(i) the length of PR,   (ii) $\angle QPR$,   (iii) $\angle PRQ$.

Worked Example 10

(Using Cosine Rule when given 3 sides)

In $\triangle ABC$, $AB = 12$ cm, $BC = 8$ cm and $AC = 9$ cm.
Find the size of the smallest angle.

Solution:

The smallest angle is the angle opposite the shortest side, i.e. $\angle BAC$.
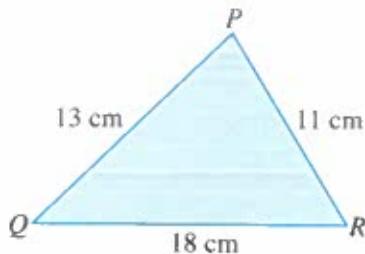
Using cosine rule,

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$= \frac{9^2 + 12^2 - 8^2}{2 \times 9 \times 12}$$

$$= \frac{161}{216}$$

$$\angle A = \cos^{-1} \frac{161}{216} = 41.8° \text{ (to 1 d.p.)}$$

$\therefore$ The smallest angle is $41.8°$.

In $\triangle PQR$, $PQ = 13$ cm, $QR = 18$ cm and $PR = 11$ cm. Find the size of the largest angle.
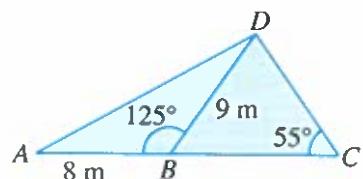
Exercise 8D

BASIC LEVEL

1. In $\triangle ABC$, $a = 5$ cm, $b = 7$ cm and $\angle C = 60°$. Find c.

2. In $\triangle GHI$, $g = 9$ cm, $i = 7$ cm and $\angle H = 30°$. Find h.

3. In $\triangle MNO$, $m = 4.2$ cm, $n = 5.8$ cm and $\angle O = 141.4°$. Find o.

4. In $\triangle XYZ$, $x = 7$ m, $y = 8$ m and $z = 9$ m. Find the unknown angles.

5. In $\triangle ABC$, $AB = 6.7$ cm, $BC = 3.8$ cm and $AC = 5.3$ cm. Find the size of the smallest angle.

6. In $\triangle PQR$, $PQ = 7.8$ cm, $QR = 9.1$ cm and $PR = 4.9$ cm. Find the size of the largest angle.

INTERMEDIATE LEVEL
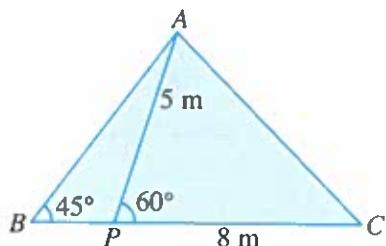
7. In the figure, the point B lies on AC such that $AB = 8$ m, $BD = 9$ m, $\angle ABD = 125°$ and $\angle BCD = 55°$.

Find

(i) the length of CD, 　(ii) the length of AD.

8. The figure shows the cross section of the roof of an old cottage. It is given that $AP = 5$ m, $PC = 8$ m, $\angle APC = 60°$ and $\angle ABC = 45°$.
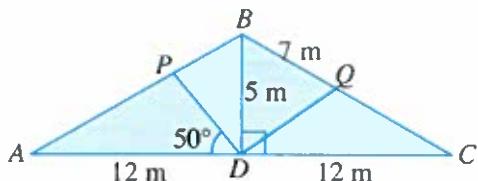
Find

(i) the length of AB, 　(ii) the length of AC.

9. In $\triangle ABC$, $BC = 4$ cm. M is the midpoint of BC such that $AM = 4$ cm and $\angle AMB = 120°$. Find

   (i)   the length of AC,

   (ii) the length of AB,

   (iii) $\angle ACB$.
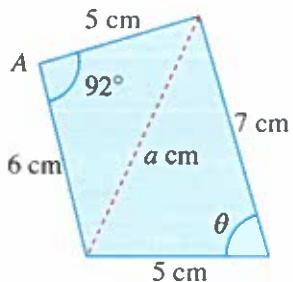
10. The figure shows the supports of the roof of a building in which $BD = 5$ m, $AD = CD = 12$ m, $BQ = 7$ m and $\angle PDA = 50°$.

   Find

   (i) $\angle BAD$,

   (ii) the length of the support PD,

   (iii) the length of the support DQ.
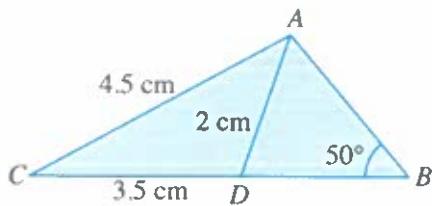
11. The figure shows a quadrilateral with the dimensions as shown.

   Find

   (i)   the value of a,

   (ii) θ.

12. In the figure, D is a point on CB such that $AD = 2$ cm, $AC = 4.5$ cm, $CD = 3.5$ cm and $\angle ABD = 50°$.

   Find

   (i) $\angle ADB$,

   (ii) the shortest distance from A to CB,

   (iii) the length of BD.
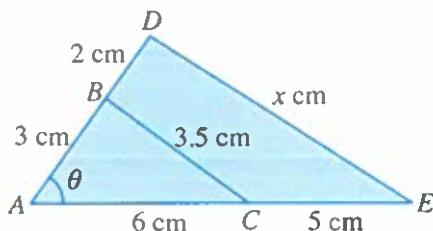
13. On a map whose scale is 2 cm to 5 km, a farm is shown as a triangle XYZ. Given that $XY = 9$ cm, $YZ = 12$ cm and $XZ = 8$ cm, find

   (i)   the length, in km, which is represented by XZ,

   (ii) $\angle YXZ$,

   (iii) the area, in km^2, which is represented by $\triangle XYZ$.

14. In a trapezium $ABCD$, AB is parallel to DC, $AB = 4.5$ cm, $BC = 5$ cm, $CD = 7.5$ cm and $AD = 6$ cm. The point X lies on CD such that BX is parallel to AD. Find $\angle BCX$ and the length of BD.

15. The figure shows two triangles ABC and ADE.
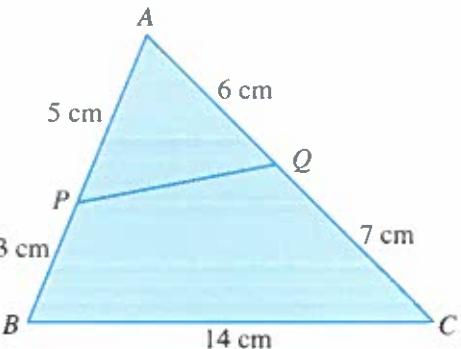
(i) Determine if $\triangle ADE$ is an enlargement of $\triangle ABC$.

(ii) Find the value of $\cos\theta$.

(iii) Hence, find the value of x.

16. In $\triangle ABC$, $AB = 8$ cm, $BC = 5$ cm and $CA = 6$ cm. BC is produced to R so that $CR = 3$ cm.

(i) Express $\cos\angle BCA$ in the form $\dfrac{p}{q}$, where p and q are integers.

(ii) Hence, find the length of AR.

17. In the figure, the point P lies on AB such that $AP = 5$ cm and $PB = 3$ cm. The point Q lies on AC such that $AQ = 6$ cm and $QC = 7$ cm.

Find the length of PQ.

Summary
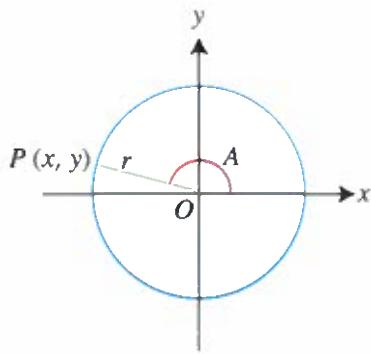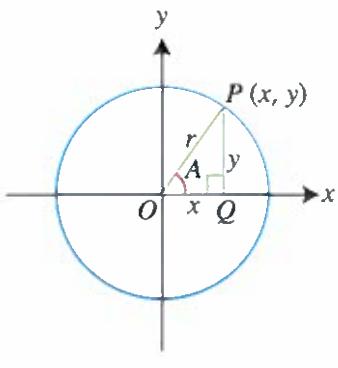
1. For any angle A, the sine and cosine of an angle A are defined as follows:

$$\sin A = \frac{y}{r} \text{ and } \cos A = \frac{x}{r},$$

where (x, y) are the coordinates of a point P on a circle with centre O and radius r as shown.
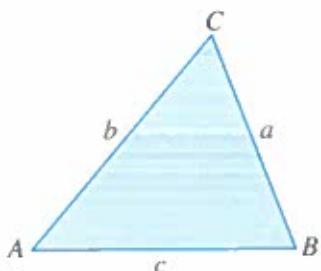
(a)

(b)

2. $\sin A = \sin(180° - A)$

    $\cos A = -\cos(180° - A)$

3. Area of $\triangle ABC = \dfrac{1}{2}ab \sin C$

4. **Sine Rule**

    In any $\triangle ABC$,

    $$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

    or

    $$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

    The Sine Rule can be used to solve a triangle (i.e. find the unknown sides and angles) if the following are given:

    *   two angles and the length of one side; or
    *   the lengths of two sides and one **non-included** angle

5. **Cosine Rule**

    In any $\triangle ABC$,

    $$a^2 = b^2 + c^2 - 2bc \cos A \qquad \cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

    $$b^2 = a^2 + c^2 - 2ac \cos B \qquad \text{or} \qquad \cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

    $$c^2 = a^2 + b^2 - 2ab \cos C \qquad \cos C = \frac{a^2 + b^2 - c^2}{2ab}.$$

    The Cosine Rule can be used to solve a triangle if the following are given:
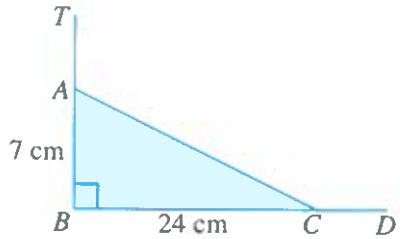
    *   the lengths of all three sides; or
    *   the lengths of two sides and an **included** angle

Review Exercise 8
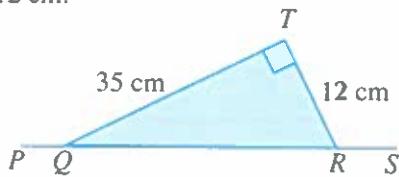
1. In the figure, A is a point on TB such that $AB = 7$ cm and C is a point on BD such that $BC = 24$ cm.

Given that $\angle TBD = 90°$, find
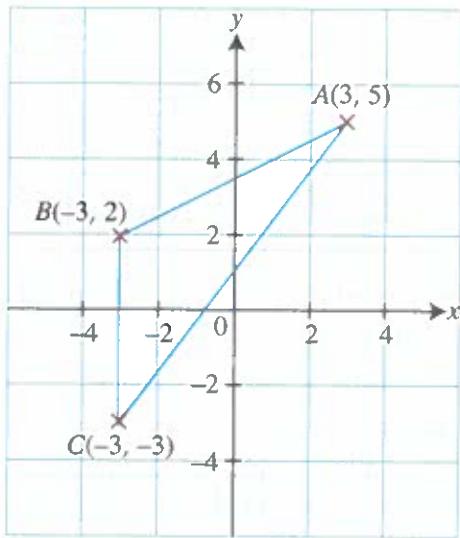
(a) AC,

(b) the value of each of the following.

(i) $\tan \angle ACB$

(ii) $\cos \angle ACD$

(iii) $\sin \angle TAC$

2. In the figure, $PQRS$ is a straight line and $\angle QTR = 90°$. It is also given that $QT = 35$ cm and $TR = 12$ cm.

(a) Find the length of QR.

(b) Express each of the following as a fraction in its simplest form.

(i) $\sin \angle PQT$

(ii) $\cos \angle PQT$

(iii) $\tan \angle TQR + \tan \angle TRQ$
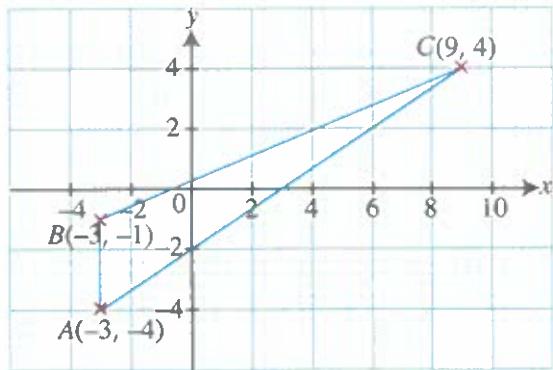
3. The figure shows $\triangle ABC$ with vertices $A(3, 5)$, $B(-3, 2)$ and $C(-3, -3)$.

Find the value of each of the following.

(i) $\cos \angle ABC$

(ii) $\sin \angle ABC$

(iii) $\tan \angle ACB$

4. The figure shows the points $A(-3, -4)$, $B(-3, -1)$ and $C(9, 4)$.
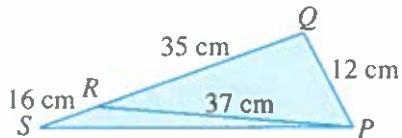
Find

(i) the length of BC and of AC,

(ii) the value of $\sin \angle ABC$ and of $\cos \angle ABC$,

(iii) the area of $\triangle ABC$,

(iv) the length of the perpendicular from B to AC.

5. Given that $0° < x < 180°$, find the possible values of x for each of the following equations.
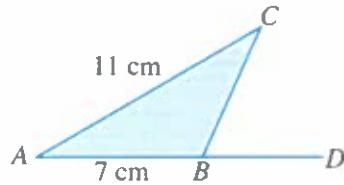
(a) $\sin x = 0.419$ (b) $\cos x = 0.932$

(c) $\tan x = 0.503$ (d) $\cos x = -0.318$

6. In $\triangle PQR$, $PQ = 12$ cm, $QR = 35$ cm and $PR = 37$ cm.

(a) Explain why $\angle PQR$ is a right angle.

(b) QR is produced to S such that $RS = 16$ cm. Find

(i) the value of $\cos \angle PRS$ and of $\sin \angle PRS$,

(ii) $\angle RPS$.
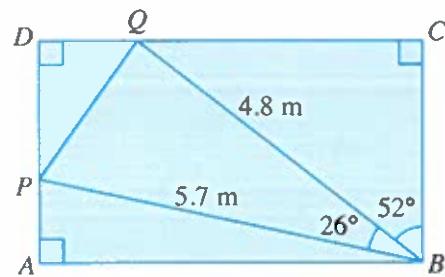
7. In the figure, ABD is a straight line, $AB = 7$ cm, $AC = 11$ cm and $\sin \angle CBD = \dfrac{3}{4}$.

Find

(i) $\sin \angle ACB$, giving your answer as a fraction in its simplest form,

(ii) $\angle BAC$,

(iii) the area of $\triangle ABC$,
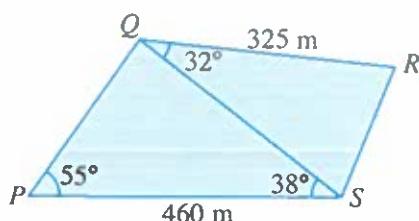
(iv) the length of BC.

8. In the figure, $ABCD$ is a rectangular hoarding and PQ, BP and BQ are three pieces of wood nailed at the back to support the hoarding.

Given that $BP = 5.7$ m, $BQ = 4.8$ m, $\angle PBQ = 26°$ and $\angle CBQ = 52°$, find
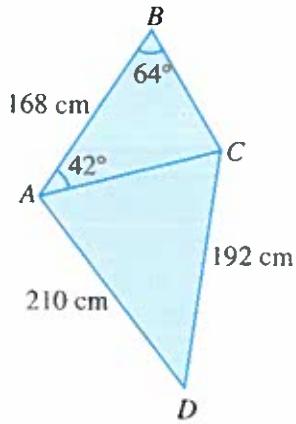
(i) the width of the hoarding, BC,

(ii) the length of AP,

(iii) the area enclosed by the three pieces of wood, $\triangle PBQ$,

(iv) the length of PQ,

(v) $\angle BPQ$.

9. The figure shows a park *PQRS* where *PS* = 460 m, *QR* = 325 m, ∠*PSQ* = 38°, ∠*QPS* = 55° and ∠*RQS* = 32°.
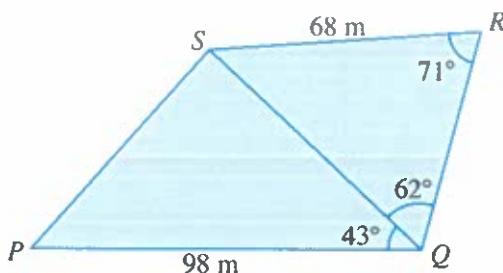
(i) Find the length of *QS*.

(ii) Find the length of *RS*.

(iii) What is the shortest distance between *Q* and *PS*?

(iv) Find the area occupied by the park.

10. Four children are standing in the field at the points *A*, *B*, *C* and *D*, playing a game of 'Catch Me'. It is given that *AB* = 168 cm, *AD* = 210 cm, *CD* = 192 cm, ∠*BAC* = 42° and ∠*ABC* = 64°.
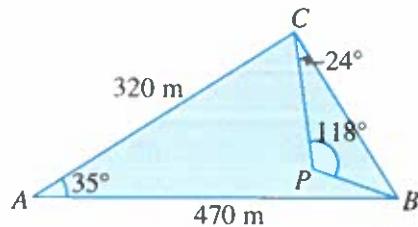
(i) Given that Huixian, who is standing at *A*, runs towards Lixin who is standing at *C*, find the distance *AC* that Huixian has to run to reach Lixin.

(ii) Find ∠*ADC*.

(iii) Calculate the area of *ABCD*, giving your answer in m².

11. The figure shows a quadrilateral *PQRS*. Given that *PQ* = 98 m, *RS* = 68 m, ∠*SQR* = 62°, ∠*PQS* = 43° and ∠*SRQ* = 71°, find
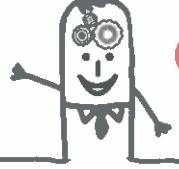
(i) the area of △*SQR*,

(ii) the length of *PS*.

12. The figure shows a triangular park *ABC* where *AB* = 470 m, *AC* = 320 m and ∠*BAC* = 35°. *P* is a lamp post inside the park such that ∠*BPC* = 118° and ∠*PCB* = 24°.
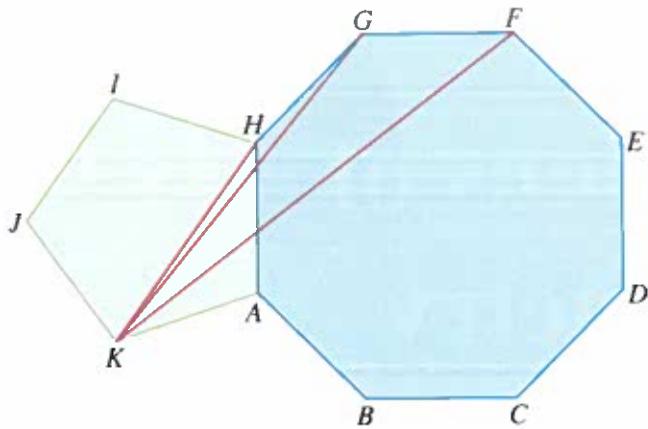
Find

(i) the area of the park,

(ii) the distance between *B* and *C*,

(iii) the distance between *P* and *C*.

In the figure, *ABCDEFGH* is a regular octagon with sides 8 cm and *AHIJK* is a regular pentagon. Find

(a) the length of *HK* and of *GK*,

(b) the area of △*FGK*.

Applications of Trigonometry

Trigonometry can be applied to improve navigation. If we have a fixed reference point, we are able to indicate the direction of an object from this point. This is known as the bearing of an object with respect to a fixed point.

Chapter
Nine
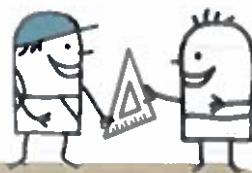
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9.1 Angles of Elevation and Depression

Recap

In Book 2, we have learnt that trigonometry can be used to find the heights of buildings and mountains. We have also learnt that a clinometer may be used to obtain the angle of elevation of the top of an object, as shown in Fig. 9.1.
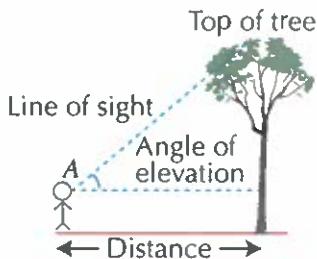
Fig. 9.1

In this section, we will learn about angles of elevation and depression and how they may be used to solve simple problems.

Angles of Elevation and Depression

Fig. 9.2 shows Khairul standing in front of a vertical wall BC. A is the point where his eyes are and AD is an imaginary horizontal line from his eyes to the wall.
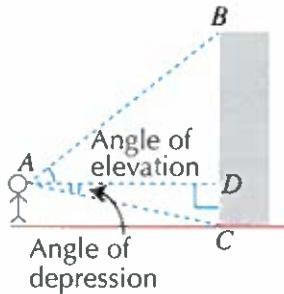
Fig. 9.2
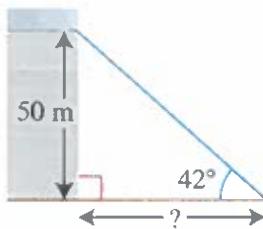
When Khairul looks at the top of the wall, B, the angle between the horizontal AD and the line of sight AB, i.e. $\angle BAD$, is called the **angle of elevation**.

When Khairul looks at the bottom of the wall, C, the angle between the horizontal AD and the line of sight AC, i.e. $\angle CAD$, is called the **angle of depression**.
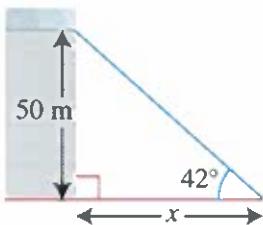
Worked Example 1

A window of a building is 50 m above the ground. Given that the angle of elevation of the window from a point on the ground is 42°, find the distance of the point on the ground from the foot of the building.

Solution:

Let x m be the distance of the point on the ground from the foot of the building.

$$\tan 42° = \frac{50}{x}$$

$$x \tan 42° = 50$$
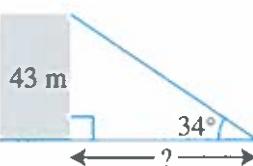
$$x = \frac{50}{\tan 42°}$$

$$= 55.5 \text{ (to 3 s.f.)}$$
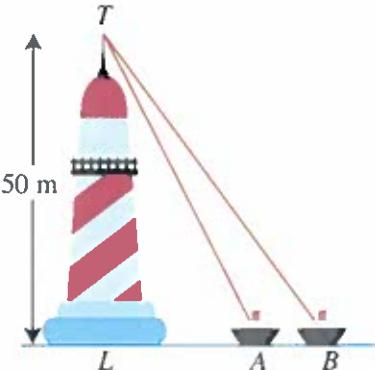
∴ The distance of the point on the ground from the foot of the building is 55.5 m.

PRACTISE NOW 1

1. The angle of elevation of the top of an office tower of height 43 m from a point on level ground is 34°. Find the distance of the point on the ground from the foot of the tower.

2. A lighthouse TL has a height of 50 m. The angles of elevation of the top of the lighthouse T from boat A and boat B are 48° and 38° respectively. Find the distance between boats A and B.
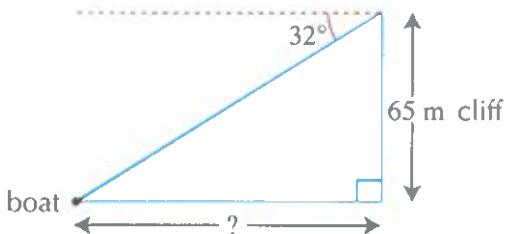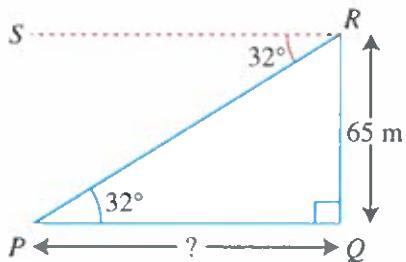
Worked Example 2

A cliff is 65 m high. Given that the angle of depression of a boat from the top of the cliff is 32°, find the distance between the boat and the base of the cliff.

Solution:

Method 1:

$\angle RPQ = 32°$ (alt. $\angle$s, $SR \parallel PQ$)

$\tan 32° = \dfrac{65}{PQ}$

$\therefore PQ = \dfrac{65}{\tan 32°}$

$\qquad = 104$ m (to 3 s.f.)

Method 2:

$\angle PRQ + 32° = 90°$ ($\angle QRS$ is a right angle.)

$\therefore \angle PRQ = 90° - 32°$

$\qquad\quad = 58°$
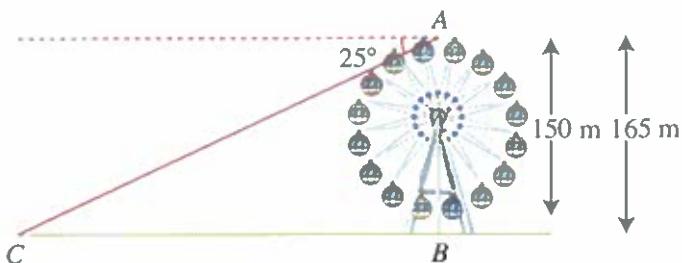
$\tan 58° = \dfrac{PQ}{65}$

$\therefore PQ = 65 \tan 58°$

$\qquad = 104$ m (to 3 s.f.)
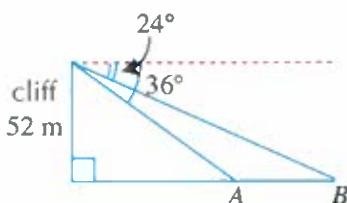
$\therefore$ The distance between the boat and the base of the cliff is 104 m.

1. The Singapore Flyer is an iconic giant observation wheel built on top of a terminal building. The diameter of the wheel is approximately 150 m and the highest point of the wheel is about 165 m above the ground. From the point A at the top of wheel, Nora observes that the angle of depression of a sports car C on the ground is 25°.

Find

(i) the distance of the sports car from a point B which is on ground level directly below A,

(ii) the angle of depression of the sports car from the centre of the wheel, W.

2. From the top of a cliff 52 m high, the angles of depression of two ships A and B due east of it are 36° and 24° respectively.

Calculate the distance between the two ships.
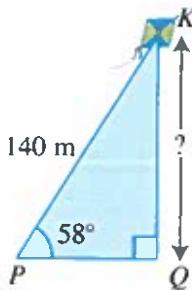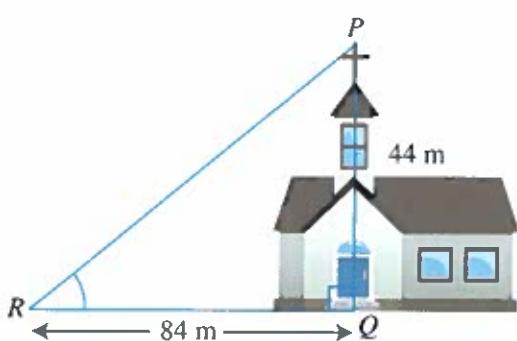
1. Lixin, standing at P, is flying a kite attached to a string of length 140 m. The angle of elevation of the kite K from her hand is 58°. Assuming that the string is taut, find the height of the kite above her hand.
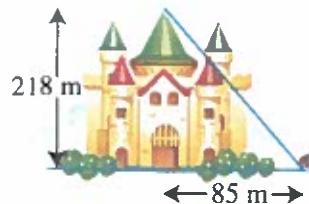
2. Two buildings on level ground are 120 m and 85 m tall respectively. Given that the angle of elevation of the top of the taller building from the top of the shorter building is 33.9°, find the distance between the two buildings.

3. At a certain time in a day, a church spire PQ, 44 m high, casts a shadow RQ, 84 m long. Find the angle of elevation of the top of the spire from the point R.

4. A building is 41 m high. Given that the angle of depression of a fire hydrant from the top of the building is 33°, find the distance between the fire hydrant and the foot of the building.
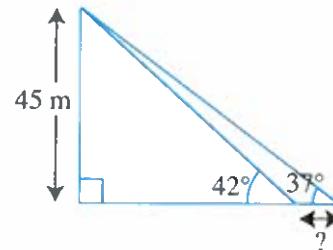
5. A boat is 65.7 m away from the base of the cliff. Given that the angle of depression of the boat from the top of the cliff is 28.9°, find the height of the cliff.

6. A castle has a height of 218 m. Given that an unusual bird is 85 m away from the foot of the castle, find the angle of depression of the bird from the top of the castle.
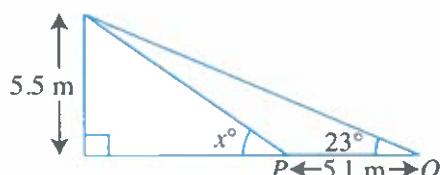
7. A clock tower has a height of 45 m. The angles of elevation of the top of the clock tower from two points on the ground are 42° and 37° respectively.

Find the distance between the two points.
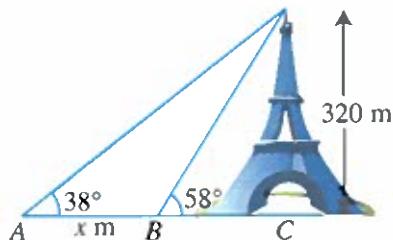
8. A castle stands on top of a mountain. At a point on level ground which is 55 m away from the foot of the mountain, the angles of elevation of the top of the castle and the top of the mountain are 60° and 45° respectively. Find the height of the castle.

9. An overhead bridge has a height of 5.5 m. The angles of elevation of the top of the bridge from two points P and Q on the ground are $x°$ and 23° respectively.

Given that the distance between P and Q is 5.1 m, find the value of x.

10. The Eiffel Tower in Paris has a height of 320 m. When Kate stands at the point A, the angle of elevation of the top of the tower is 38°. Kate walks x metres to a point B and observes that the angle of elevation of the top of the tower is now 58°.
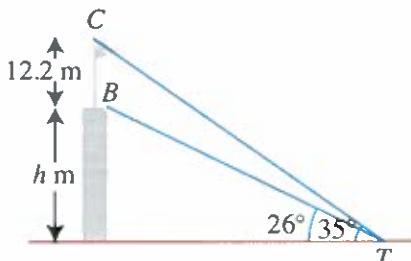
Find the value of x.

11. From the top of a cliff 88 m high, the angles of depression of two boats due west of it are 23° and 18° respectively. Calculate the distance between the two boats.

12. A satellite dish stands at the top of a cliff. From the top of the satellite dish, the angle of depression of a ship which is 80 m away from the base of the cliff is 37°. From the foot of the satellite dish, the angle of depression of the same ship is 32°. Find the height of the satellite dish.

13. A flagpole of height 12.2 m is placed on top of a building of height h metres. From a point T on level ground, the angle of elevation of the base of the flagpole B is 26° and the angle of elevation of the top of the flagpole C is 35°.

Find the value of h.

14. A tower with a height of 27 m stands at the top of a cliff. From the top of the tower, the angle of depression of a guard house is 56°. From the foot of the tower, the angle of depression of the same guard house is 49°. Find

(i) the distance between the base of the cliff and the guard house,

(ii) the height of the cliff.

9.2 Bearings
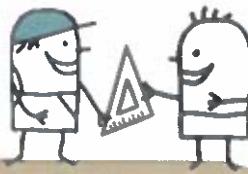
Fig. 9.3 shows the positions of four points A, B, C and D relative to an origin O. N, E, S and W represent the directions north, east, south and west from O respectively.
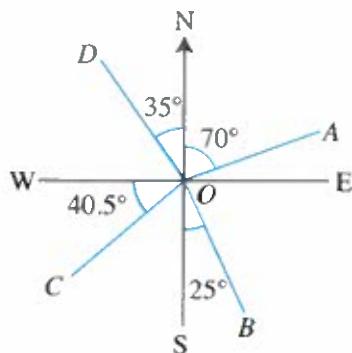
Fig. 9.3

The bearing of A from O is an angle measured from the north, at O, in a clockwise direction and is always written as a three-digit number. Hence, the bearing of A from O is 070°.

The bearing of B from O is equal to $180° - 25°$. Hence, the bearing of B from O is 155°.

The bearing of C from O is equal to $270° - 40.5°$. Hence, the bearing of C from O is 229.5°.

What is the bearing of D from O?
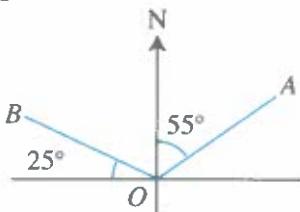
The bearing of N is taken as 000° or 360°. The bearing of E from O is 090°. Similarly, the bearing of S from O is 180°. What is the bearing of W from O?

When reading compass bearings, directions are usually measured from either the north or the south. For example, 070° is written as N70°E and 210° is written as S30°W. When reading true bearings, directions are given in terms of the angles measured clockwise from the north.

Worked Example 3

(Finding the Bearing)
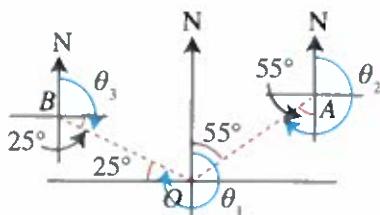
Find the bearing of

(a) A from O,   (b) B from O,

(c) O from A,   (d) O from B.

Solution:

(a) Bearing of A from O is 055°

(b) The bearing of B from O is given by the reflex angle θ_1, which is (270° + 25°).
∴ Bearing of B from O is 295°

(c) The bearing of O from A is given by the reflex angle θ_2, which is (180° + 55°).
∴ Bearing of O from A is 235°

(d) The bearing of O from B is given by the obtuse angle θ_3, which is (90° + 25°).
∴ Bearing of O from B is 115°

> **Problem Solving Tip**
>
> For **(c)**, follow the steps as shown.
> **Step 1:** Draw the north line from A.
> **Step 2:** Draw the angle clockwise from the north line to OA.
> **Step 3:** Find the angle θ_2.

PRACTISE NOW 3
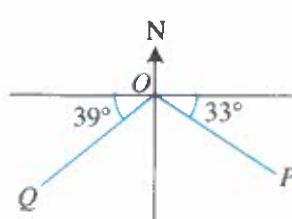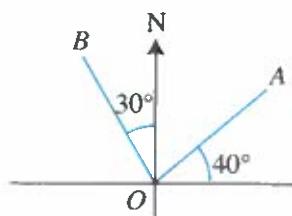
1. Find the bearing of
   (a) A from O,   (b) B from O,
   (c) O from A,   (d) O from B.

2. Find the bearing of
   (a) P from O,   (b) Q from O,
   (c) O from P,   (d) O from Q.

SIMILAR QUESTIONS
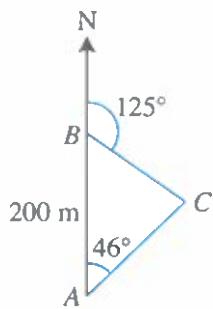
Exercise 9B Questions 1–3, 6–8

Worked Example 4

(Problem involving Bearings)

Three points A, B and C are on level ground such that B is due north of A, the bearing of C from A is 046° and the bearing of C from B is 125°. Given that the distance between A and B is 200 m, find the distance of C from A.

Solution:

Since the bearing of C from B is 125°,

$\angle ABC = 180° - 125° = 55°$

i.e. $\angle ACB = 180° - 46° - 55°$ ($\angle$ sum of a $\triangle$)

$\qquad = 79°$

Using sine rule,

$\dfrac{AC}{\sin 55°} = \dfrac{200}{\sin 79°}$
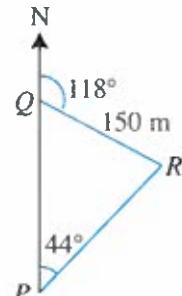
$AC = \dfrac{200 \sin 55°}{\sin 79°}$

$\qquad = 167$ m (to 3 s.f.)

PRACTISE NOW 4

SIMILAR QUESTIONS
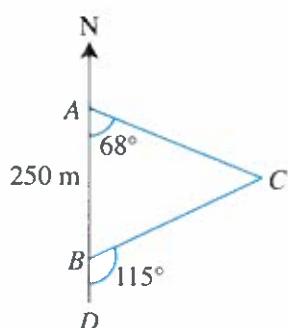
Exercise 9B Questions 4, 5, 9, 14

1. Three points P, Q and R are on level ground such that P is due south of Q, the bearing of R from Q is 118° and the bearing of R from P is 044°. Given that the distance between Q and R is 150 m, find the distance of P from Q.
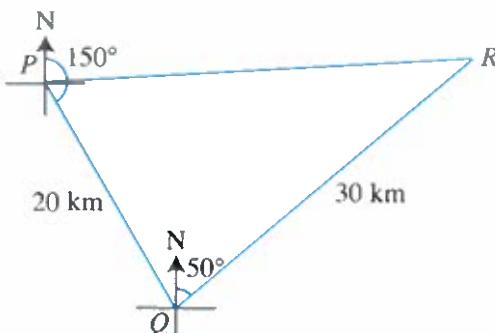
2. In the figure, A, B and C are points of an amusement park on level ground, with A due north of B. Given that $\angle BAC = 68°$, $AB = 250$ m and $\angle CBD = 115°$, find

   (i) the bearing of B from C,

   (ii) the length of AC and of BC.

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Worked Example 5

A boat sailed 20 km from a point P to an island Q, on a bearing of 150°. It then sailed another 30 km on a bearing of 50° to a lighthouse R. Find the distance and the bearing of the lighthouse from P.

Solution:

$\angle PQR = 30° + 50° = 80°$

Using cosine rule,

$PR^2 = 20^2 + 30^2 - 2 \times 20 \times 30 \times \cos 80°$

$\quad = 1092$ (to 4 s.f.)

$PR = \sqrt{1092}$

$\quad = 33.0$ km (to 3 s.f.)

∴ The lighthouse is 33.0 km away from P.

Using sine rule,

$$\frac{\sin \angle QPR}{30} = \frac{\sin 80°}{33.05}$$

$$\sin \angle QPR = \frac{30 \sin 80°}{33.05}$$

$$= 0.8940 \text{ (to 4 s.f.)}$$

$$\angle QPR = \sin^{-1} 0.8940$$

$$= 63.39° \text{ (to 2 d.p.)}$$

$150° - 63.39° = 86.61°$ (to 2 d.p.)

∴ Bearing of the lighthouse from P is 086.6°

> **Problem Solving Tip**
>
> To find distance PR using cosine rule, we need to find $\angle PQR$ first.
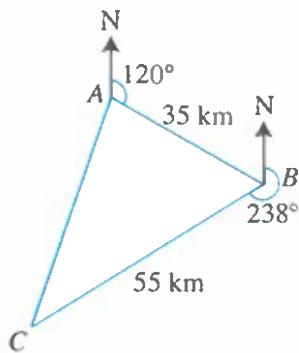
> **Problem Solving Tip**
>
> To find the bearing of R from P, we need to find $\angle QPR$ first.

The figure shows three towns on level ground. Given that the bearing of B from A is 120°, the bearing of C from B is 238°, $AB = 35$ km and $BC = 55$ km, find

(i) the distance between towns A and C,

(ii) the bearing of town C from town A.
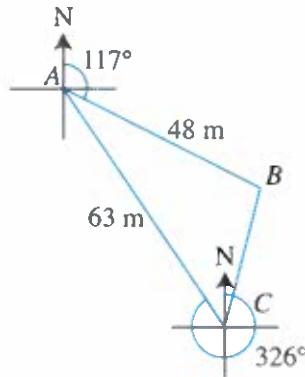
Worked Example 6

(Problem involving Bearings)

The figure shows three points on level ground. The bearing of B from A is 117°, the bearing of A from C is 326°, $AB = 48$ m and $AC = 63$ m. Calculate

(i) the length of BC,

(ii) the bearing of B from C,

(iii) the shortest distance from B to AC.
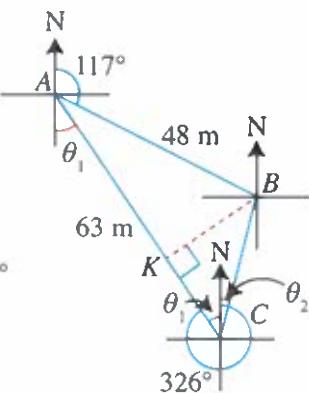
Solution:

(i) $\theta_1 = 360° - 326°$ ($\angle$s at a point)

$= 34°$

i.e. $\angle BAC = 180° - 117° - 34°$ (adj. $\angle$s on a str. line)

$= 29°$

Using cosine rule,

$$BC^2 = 48^2 + 63^2 - 2 \times 48 \times 63 \times \cos 29°$$

$$= 983.3 \text{ (to 4 s.f.)}$$

$$BC = \sqrt{983.3}$$

$$= 31.4 \text{ km (to 3 s.f.)}$$

Problem Solving Tip

Since we are given two sides of the triangle (AB and AC), in order to find the third side (BC), we first have to find $\angle BAC$. From the bearings given in the question, how can we find $\angle BAC$?

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(ii) Using sine rule,

$$\frac{\sin \angle ACB}{48} = \frac{\sin 29°}{31.36}$$

$$\sin \angle ACB = \frac{48 \sin 29°}{31.36}$$

$$= 0.7421 \text{ (to 4 s.f.)}$$

$$\angle ACB = \sin^{-1} 0.7421$$

$$= 47.91° \text{ (to 2 d.p.)}$$

$$\theta_2 = 47.91° - \theta_1$$

$$= 13.9° \text{ (to 1 d.p.)}$$

∴ Bearing of B from C is $013.9°$

(iii) The shortest distance from B to AC is BK, where BK is perpendicular to AC.
In $\triangle ABK$,

$$\sin \angle BAC = \frac{BK}{48}$$

$$BK = 48 \sin 29°$$

$$= 23.3 \text{ m (to 3 s.f.)}$$
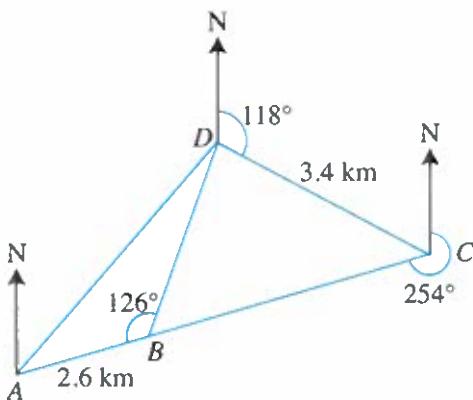
PRACTISE NOW 6

A, B, C and D are four points on level ground. The bearing of C from D is $118°$ and the bearing of A from C is $254°$.

Given that $AB = 2.6$ km, $CD = 3.4$ km, $\angle ABD = 126°$ and ABC is a straight line, find
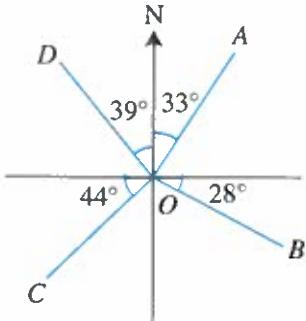
(i) the bearing of B from D,

(ii) the distance between B and D,

(iii) the distance between A and D,

(iv) the shortest distance from B to AD.

BASIC LEVEL
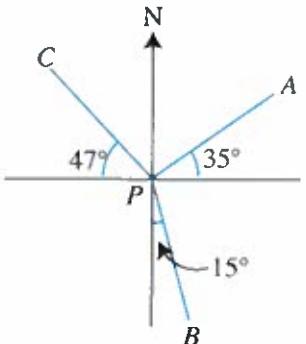
1. The figure shows the positions of O, A, B, C and D.

Find the bearing of

(a) A from O,

(b) B from O,

(c) C from O,

(d) D from O.
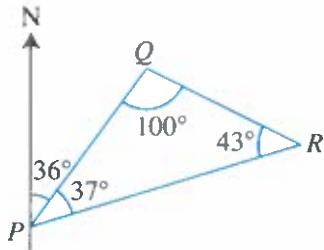
2. The figure shows the positions of P, A, B and C.

Find the bearing of

(a) A from P,

(b) B from P,

(c) C from P,

(d) P from A,

(e) P from B,

(f) P from C.

3. The figure shows the positions of P, Q and R.

Find the bearing of

(a) Q from P,

(b) P from Q,

(c) R from P,

(d) P from R,

(e) Q from R,

(f) R from Q.

4. A point Q is 24 km from P and on a bearing of 072° from P. From Q, Vishal walks at a bearing of 320° to a point R, located directly north of P. Find

(a) the distance between P and R,

(b) the distance between Q and R.
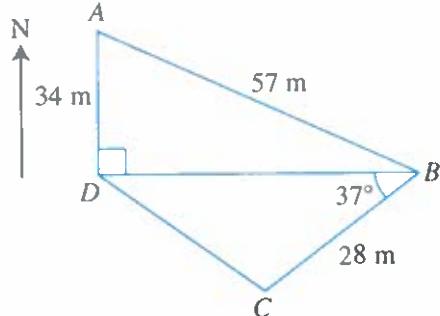
5. A petrol kiosk P is 12 km due north of another petrol kiosk Q. The bearing of a police station R from P is 135° and that from Q is 120°. Find the distance between P and R.

INTERMEDIATE LEVEL

6. A, B, C and D are the four corners of a rectangular plot marked out on level ground. Given that the bearing of B from A is 040° and that the bearing of C from A is 090°, find the bearing of

(a) B from C,

(b) A from C,

(c) D from C.

7. P, Q and R are three points on level ground. Given that the bearing of R from P is 135°, $\angle PQR = 55°$ and $\angle PRQ = 48°$, find the bearing of

(a) P from R,   (b) Q from R,

(c) P from Q.

8. A, B and C are three points on level ground. Given that the bearing of B from A is 122°, $\angle CAB = 32°$ and $\angle ABC = 86°$, find the possible bearing(s) of C from B.

9. A bus stop is 280 m due north of a taxi stand. Nora walks from the taxi stand in the direction 050°. Calculate how far she has to walk before she is

(a) equidistant from the bus stop and the taxi stand,

(b) as close as possible to the bus stop,

(c) due east of the bus stop.

10. A helicopter flies 30 km from a point P to another point Q on a bearing of 128°. It then flies another 25 km to a point R on a bearing of 295°. Find the distance between P and R.

11. In the figure, P, Q and R are points on level ground in a park. R is 600 m from Q and on a bearing of 305° from Q. Q is 950 m from P and on a bearing of 078° from P.

Find

(i) the distance between P and R,

(ii) the bearing of R from P.

12. The figure shows four points on level ground. A is due north of D, B is due east of D and $\angle DBC = 37°$.

Given that $AD = 34$ m, $AB = 57$ m and $BC = 28$ m, find

(i) the bearing of B from A,

(ii) the shortest distance from C to BD,

(iii) the bearing of D from C.

13. A, B and C are three points on level ground. The bearing of B from A is 068° and the bearing of C from A is 144°.

Given that $AB = 370$ m and $AC = 510$ m, find

(i) the distance between B and C,

(ii) $\angle ACB$,

(iii) the bearing of C from B,

(iv) the shortest distance from A to BC.

14. Two cruise ships P and Q leave the port at the same time. P sails at 10 km/h on a bearing of 030° and Q sails at 12 km/h on a bearing of 300°. Find their distance apart and the bearing of P from Q after 2 hours.

15. P, Q and R represent three ports. Q is 35 km from P and on a bearing of 032° from P. R is 65 km from P and on a bearing of 108° from P.
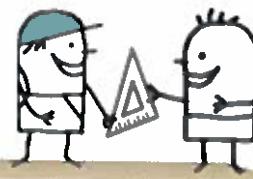
(a) Find

    (i) the distance between Q and R,

    (ii) the bearing of R from Q.

A ship sets sail at 0930 from P directly to R at an average speed of 30 km/h and reaches a point S due south of Q.

(b) Find the time when it reaches S.

9.3 Three-Dimensional Problems

A plane is a flat surface like the floor or the surface of a whiteboard. It has two dimensions (2D) – length and breadth.

A solid has three dimensions (3D) – length, breadth and height/thickness/depth.

Investigation

Visualising 3D Solids

1. Look at your school desk or table. It has a rectangular top (ignore the rounded corners, if any). Fig. 9.4 shows a photo of a school desk viewed from the top. Measure the angles of the two corners of the rectangular top, $\angle ABC$ and $\angle BCD$. Do you get 90° for both angles?
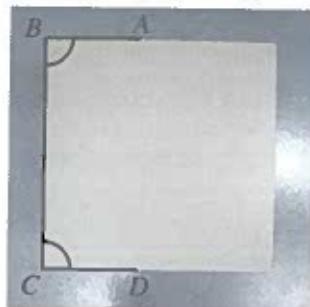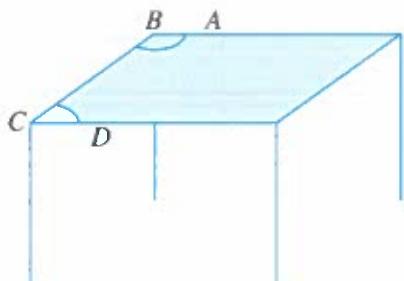
Fig. 9.4

2. Fig. 9.5(a) shows a photo of the same desk viewed from the side. Measure $\angle ABC$ and $\angle BCD$ again. Do you get 90°, smaller than 90°, or larger than 90°?

The following figures show the top view and the front view of a structure.

Top view   **Front view**

Notice that there are no hidden lines. Draw the side view of the structure.

(a)   (b)
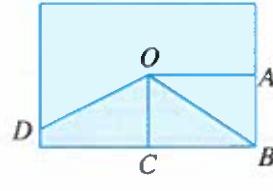
Fig. 9.5

In other words, drawing a 3D solid or object on a flat surface **may make a right angle look smaller or larger than 90°.**
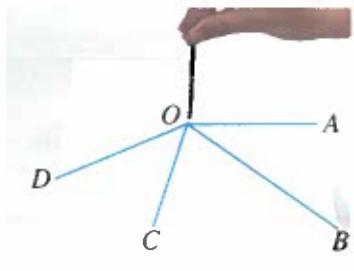
3. Fig. 9.6(a) shows a plane with a few lines drawn on it.

Place a pencil perpendicular to the plane in Fig. 9.6(a), as shown in Fig. 9.6(b).
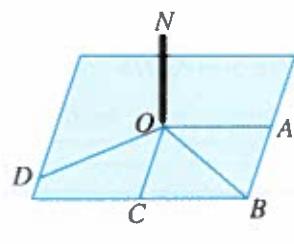
Use a set square to check whether the pencil is perpendicular to every line on the plane in Fig. 9.6(a).

(a)   (b)   (c)

Fig. 9.6

Your pencil is called a **normal** to the plane since it is perpendicular to every line on the plane.

4. In Fig. 9.6(c), $\angle NOA$ looks like it is a 90° angle, but $\angle NOB$ does not look like a 90° angle. Is $\angle NOB = 90°$? Explain your answer.

5. Fig. 9.7 shows a cuboid. Dotted lines represent lines that are hidden, i.e. you cannot see them from the front.
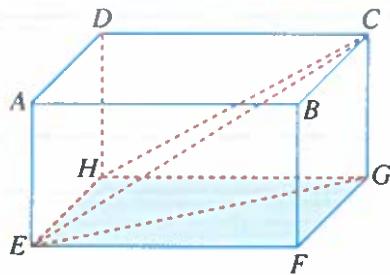
Fig. 9.7

There are two methods to determine whether a triangle in the above cuboid is a right-angled triangle.

Method 1: Find a rectangle

To determine whether $\triangle EFG$ and $\triangle CGH$ are right-angled triangles:

(a) $\triangle EFG$ lies on the plane $EFGH$. Is the plane $EFGH$ a rectangle? Explain your answer.

(b) Thus, is $\angle EFG = 90°$ and $\triangle EFG$ a right-angled triangle? Explain your answer.

(c) Using the same method as above, determine whether $\triangle CGH$ is a right-angled triangle by identifying the appropriate rectangle and the right angle of the triangle.

Method 2: Find a normal to a plane

To determine whether $\triangle CGE$ and $\triangle CHE$ are right-angled triangles:

(a) $EFGH$ is a horizontal plane. Is the vertical line CG a normal to the plane $EFGH$? Explain your answer.

(b) Is the line GE a line on the plane $EFGH$?

(c) Thus, is $\angle CGE = 90°$ and $\triangle CGE$ a right-angled triangle? Explain your answer.

(d) Using the same method as above, determine whether $\triangle CHE$ is a right-angled triangle by shading the appropriate plane, and identifying the corresponding normal and the right angle of $\triangle CHE$.

Problem Solving Tip
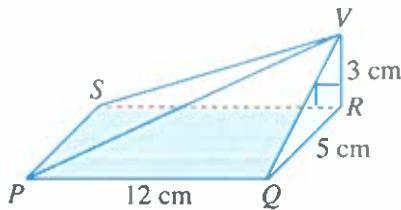
A normal to a plane is perpendicular to every line on the plane.

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Worked Example 7

(Three-dimensional Problem)

The figure shows a pyramid with a rectangular base $PQRS$ and vertex V vertically above R.
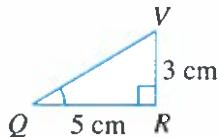
Given that $PQ = 12$ cm, $QR = 5$ cm and $VR = 3$ cm, find

(i) $\angle VQR$, (ii) $\angle VPR$.

Solution:

(i) In $\triangle VQR$,

$$\tan \angle VQR = \frac{3}{5}$$

$$\angle VQR = \tan^{-1} \frac{3}{5}$$

$$= 31.0° \text{ (to 1 d.p.)}$$

> **Problem Solving Tip**
>
> The basic technique used in solving a three-dimensional problem is to reduce it to a problem in a plane.
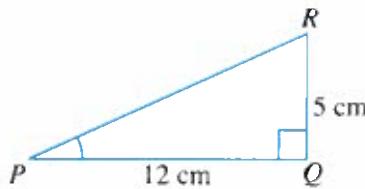
(ii) In $\triangle PQR$, $\angle PQR = 90°$.
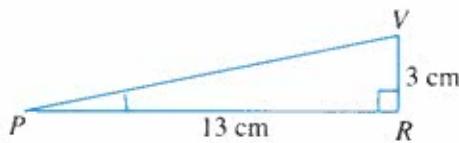
Using Pythagoras' Theorem,

$$PR^2 = PQ^2 + QR^2$$

$$= 12^2 + 5^2$$

$$= 144 + 25$$

$$= 169$$

$$PR = \sqrt{169}$$

$$= 13 \text{ cm}$$

> **JustForFun**
>
> Two shapes are as shown.
>
> (a)
>
> (b)
>
> Cut each shape along the dotted lines and use the pieces to form two squares.

In $\triangle VRP$, $\angle VRP = 90°$ (VR is the normal to the plane $PQRS$).

$$\tan \angle VPR = \frac{3}{13}$$

$$\angle VPR = \tan^{-1} \frac{3}{13}$$

$$= 13.0° \text{ (to 1 d.p.)}$$

1. The figure shows a wedge with a horizontal base *ABCD* and a vertical face *PQCB*. *APQD* is a rectangular sloping surface and $\triangle ABP$ and $\triangle DCQ$ are right-angled triangles in the vertical plane.
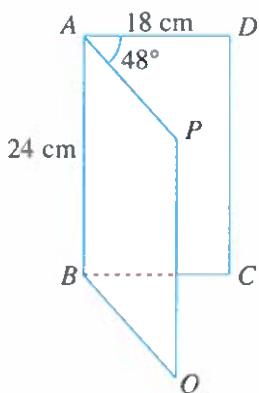
Given that $CQ = BP = 8$ cm, $DC = AB = 15$ cm and $AD = BC = PQ = 24$ cm, find

(i) $\angle BAC$,   (ii) $\angle AQC$,   (iii) $\angle CDQ$.

2. The figure shows a photo frame which can be opened about *AB*. *ABCD* and *ABQP* are rectangles. The frame is opened through 48° as shown.

Given that $AB = 24$ cm and $AP = AD = 18$ cm, find
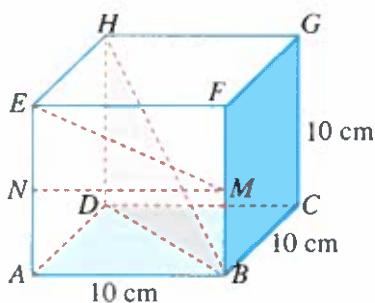
(i) the length of the straight line *CQ*,

(ii) $\angle CAQ$.

Worked Example 8

(Three-dimensional Problem)

The figure shows a cube of length 10 cm. *M* and *N* are the midpoints of *BF* and *AE* respectively.

Find
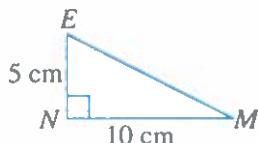
(i) $\angle MEN$,   (ii) $\angle EMN$,

(iii) $\angle MEF$,   (iv) $\angle HBD$.

Solution:

(i) In $\triangle MEN$,

$$\tan \angle MEN = \frac{10}{5}$$
$$= 2$$
$$\angle MEN = \tan^{-1} 2$$
$$= 63.4° \text{ (to 1 d.p.)}$$

5 cm, E, N, M, 10 cm

(ii) In $\triangle MEN$,

$$\tan \angle EMN = \frac{5}{10}$$
$$= \frac{1}{2}$$
$$\angle EMN = \tan^{-1} \frac{1}{2}$$
$$= 26.6° \text{ (to 1 d.p.)}$$

Problem Solving Tip
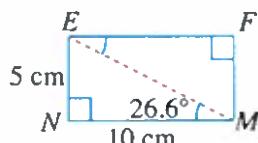
In **(ii)**, an alternative method to solve for $\angle EMN$ is to use the sum of angles in a triangle, i.e. $\angle EMN = 180° - \angle ENM - \angle MEN$.

(iii) $\angle MEF = \angle EMN$ (alt. $\angle$s, $EF \parallel NM$)
$$\therefore \angle MEF = 26.6°$$

E, F, 5 cm, N, 26.6°, M, 10 cm

JustForFun

Is it possible to construct each of the following three-dimensional objects?

(iv) In $\triangle BCD$, $\angle BCD = 90°$.

Using Pythagoras' Theorem,
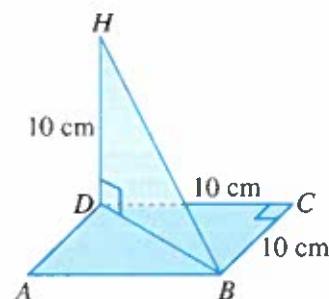
$$DB^2 = BC^2 + DC^2$$
$$= 10^2 + 10^2$$
$$= 100 + 100$$
$$= 200$$
$$DB = \sqrt{200}$$
$$= 14.14 \text{ cm (to 4 s.f.)}$$

H, 10 cm, 10 cm, D, C, 10 cm, A, B

(a)

In $\triangle HBD$, $\angle HDB = 90°$.
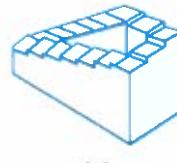
$$\tan \angle HBD = \frac{10}{14.14}$$
$$= 0.7072 \text{ (to 4 s.f.)}$$
$$\angle HBD = \tan^{-1} 0.7072$$
$$= 35.3° \text{ (to 1 d.p.)}$$

(b)

(c)

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Applications of Trigonometry   Chapter 9   280
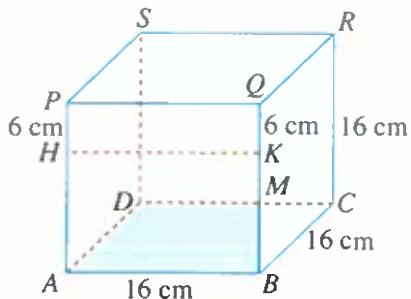
1. The figure shows a cube of length 16 cm.
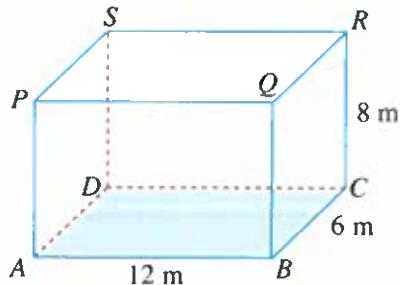
Given that $PH = QK = 6$ cm, find
(i) $\angle BCK$, (ii) $\angle SBD$, (iii) $\angle BDK$.

2. The figure shows a cuboid where $AB = 12$ m, $BC = 6$ m and $CR = 8$ m.
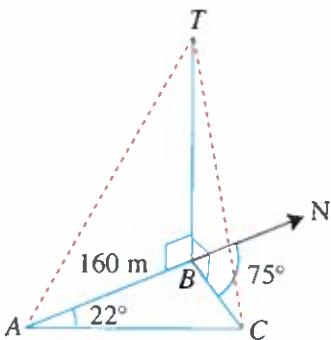
Find
(i) $\angle ABP$, (ii) $\angle BCQ$, (iii) $\angle CAR$.

Worked Example 9

(Angle of Elevation in a Three-dimensional Problem)

Three points A, B and C are on level ground. B is due north of A, the bearing of C from A is 022° and the bearing of C from B is 075°.

(i) Given that A and B are 160 m apart, find the distance between B and C.

A vertical mast BT stands at B such that $\tan \angle TAB = \dfrac{3}{16}$.

(ii) Find the angle of elevation of T from C.

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Solution:
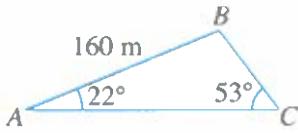
(i) $\angle BCA = 75° - 22°$ (ext. $\angle$ of a $\triangle$ = sum of int. opp. $\angle$s)

$\quad\quad = 53°$

Using sine rule,

$$\frac{BC}{\sin 22°} = \frac{160}{\sin 53°}$$

$$BC = \frac{160 \sin 22°}{\sin 53°}$$

$$= 75.0 \text{ m (to 3 s.f.)}$$
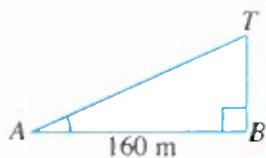
(ii) In $\triangle TAB$,

$$\tan \angle TAB = \frac{3}{16}$$

$$\frac{BT}{160} = \frac{3}{16}$$

$$BT = \frac{3}{16} \times 160$$

$$= 30 \text{ m}$$
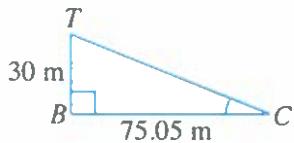
In $\triangle TCB$,

$$\tan \angle TCB = \frac{30}{75.05}$$

$$\angle TCB = \tan^{-1} \frac{30}{75.05}$$

$$= 21.8° \text{ (to 1 d.p.)}$$
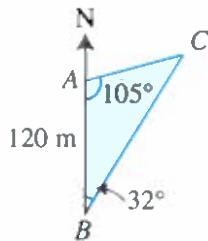
$\therefore$ The angle of elevation of T from C is $21.8°$.

PRACTISE NOW 9

Three points A, B and C are on level ground. B is due south of A, the bearing of C from B is $032°$ and $\angle CAB = 105°$.

(i) Given that A and B are 120 m apart, find the distance between B and C.

A vertical mast CT of height 25 m stands at C.
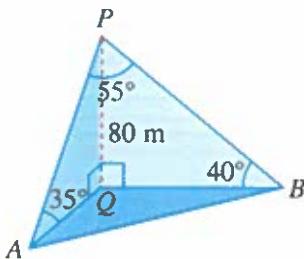
(ii) Find the angle of elevation of T from B.

Worked Example 10

(Angle of Depression in a Three-dimensional Problem)

Priya is in a cable car P at a height of 80 m above the ground. She observes a statue at A and fountain at B. Given that the angles of depression of the statue and the fountain are 35° and 40° respectively and that $\angle APB = 55°$, find the distance between A and B.

Solution:

In $\triangle APQ$, $\angle AQP = 90°$.

$$\sin 35° = \frac{80}{AP}$$

$$AP = \frac{80}{\sin 35°}$$

$$= 139.5 \text{ m (to 4 s.f.)}$$

In $\triangle BPQ$, $\angle BQP = 90°$.

$$\sin 40° = \frac{80}{BP}$$

$$BP = \frac{80}{\sin 40°}$$

$$= 124.5 \text{ m (to 4 s.f.)}$$

Using cosine rule,

$$AB^2 = AP^2 + BP^2 - 2 \times AP \times BP \cos 55°$$

$$= 139.5^2 + 124.5^2 - 2 \times 139.5 \times 124.5 \cos 55°$$

$$= 15\,040 \text{ (to 4 s.f.)}$$

$$AB = \sqrt{15\,040}$$

$$= 123 \text{ m (to 3 s.f.)}$$

> **Problem Solving Tip**
>
> In $\triangle ABP$, in order to find AB, we first have to find AP and BP. Since $\triangle APQ$ and $\triangle BPQ$ are right-angled triangles, how can we find AP and BP?

PRACTISE NOW 10

Rui Feng is on the top T of an observation tower OT. The height of the tower is 54 m. He observes a car that has broken down at A, causing a traffic jam to B. Given that the angles of depression of A and B are 42° and 38° respectively and that $\angle ATB = 48°$, find the distance between A and B.
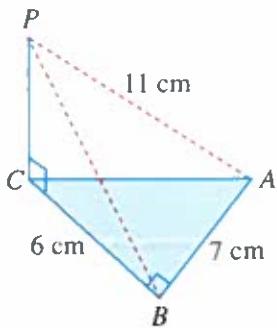
> **SIMILAR QUESTIONS**
>
> Exercise 9C Questions 12, 13, 17

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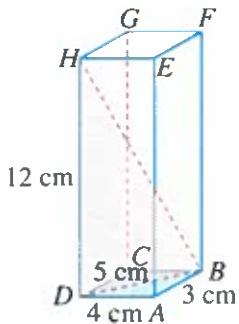
1. The figure shows $\triangle ABC$, right-angled at B and lying in a horizontal plane. P is a point vertically above C.

Given that $AB = 7$ cm, $BC = 6$ cm and $AP = 11$ cm, find

(i) AC,

(ii) PC,

(iii) $\angle PAC$,

(iv) the angle of elevation of P from B.
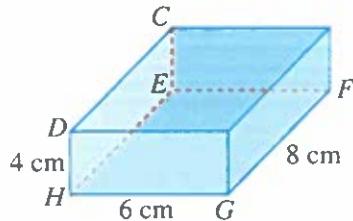
2. The figure shows a rectangular box in which $AB = 3$ cm, $AD = 4$ cm, $BD = 5$ cm and $DH = 12$ cm.

Find

(i) the length of BH, (ii) $\angle BDC$,

(iii) $\angle HBD$.
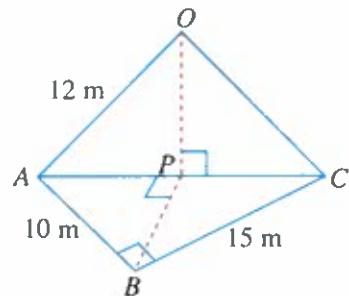
3. A rectangular block of sugar has a horizontal base $EFGH$. The corners C and D are vertically above E and H respectively. It is given that $DH = 4$ cm, $GH = 6$ cm and $FG = 8$ cm.

Find

(i) $\angle DGH$,

(ii) HF,

(iii) the angle of elevation of D from F.
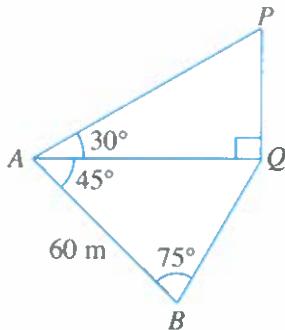
4. The figure shows three points A, B and C on horizontal ground where $\angle ABC$ is a right angle. AOC represents a vertical triangular wall with P as the foot of the perpendicular from O to AC.

Given that $\angle APB = 90°$, $AB = 10$ m, $BC = 15$ m and $OA = 12$ m, find

(i) $\angle BAC$,

(ii) the length of AP,

(iii) the length of OP,

(iv) the angle of elevation of O from B.
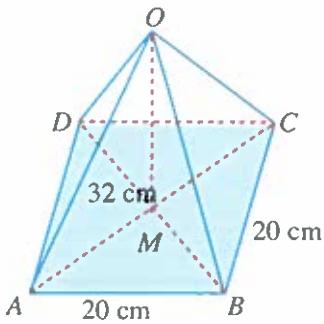
5. In the figure, the angle of elevation of the top of a vertical tower PQ from a point A is $30°$.

Given that Q, the foot of the tower, is on the same horizontal plane as A and B, and that $AB = 60$ m, $\angle BAQ = 45°$ and $\angle ABQ = 75°$, find the height of the tower.
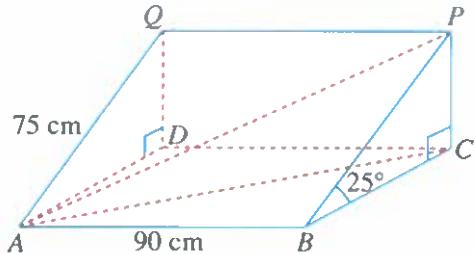
6. $OABCD$ is a pyramid. The square base $ABCD$ has sides of length 20 cm and lies in a horizontal plane. M is the point of intersection of the diagonals of the base and O is vertically above M.

Given that $OA = 32$ cm, find

(i) the length of AM,

(ii) the height of the pyramid,

(iii) $\angle OAM$.
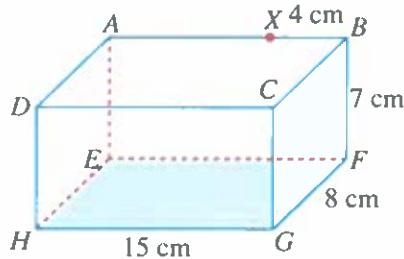
7. In the figure, $ABPQ$ is the rectangular sloping surface of a desk with $ABCD$ lying in a horizontal plane. Q and P lie vertically above D and C.

Given that $AB = PQ = 90$ cm, $AQ = BP = 75$ cm and $\angle PBC = \angle QAD = 25°$, find

(i) AC,   **(ii)** $\angle PAC$,

(iii) $\angle CAB$.
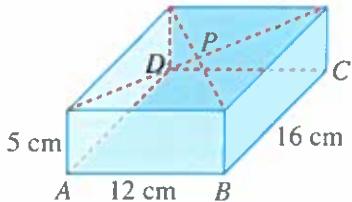
8. In the figure, a cuboid has a horizontal base $EFGH$ where $HG = 15$ cm, $GF = 8$ cm and $BF = 7$ cm. X is a point on AB such that $XB = 4$ cm.

Find

(i) $\angle CEG$,   **(ii)** $\angle GXF$.

9. P is the centre of the upper face of the rectangular block with $ABCD$ as its base.
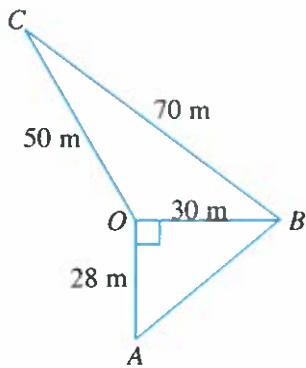
Find

(i) $\angle PAC$,   **(ii)** $\angle PAB$.

10. P, Q and R are three points on level ground with Q due east of P and R due south of P. A vertical mast PT stands at P and the angle of elevation of the top T from Q is 3.5°. Given that $PQ = 1000$ m and $PR = 750$ m, find

   (i) the bearing of Q from R,

   (ii) the height of the mast,

   (iii) the angle of elevation of T from R.

11. The figure shows four points O, A, B and C which lie on level ground in a campsite. O is due north of A and B is due east of O.
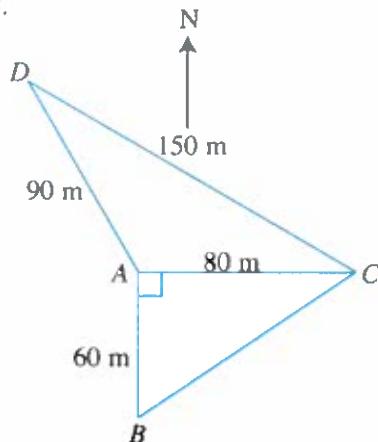
   (a) Given that $OA = 28$ m, $OB = 30$ m, $OC = 50$ m and $BC = 70$ m, find

   (i) the bearing of A from B,

   (ii) $\angle COB$,

   (iii) the bearing of C from O.

   A vertical flag pole stands at the point B such that the angles of elevation from O, A and C are measured.

   (b) Given that the greatest of these 3 angles of elevation is 29°, calculate the height of the flag pole.

12. A, B, C and D are four points on horizontal ground. B is due south of A and the bearing of C from A is 090°.
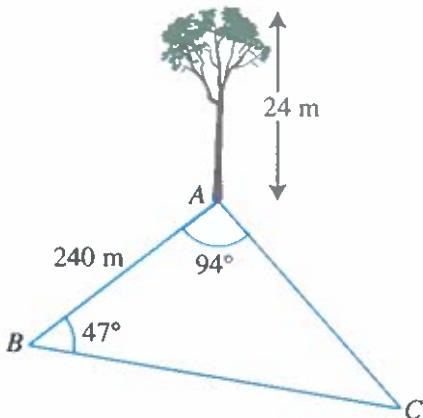
   (a) Given that $AB = 60$ m, $AC = 80$ m and $CD = 150$ m, find

   (i) the bearing of C from B,

   (ii) the bearing of D from A.

   A vertical mast stands at A and the angle of depression of C from the top of the mast is 8.6°.

   (b) Find the height of the mast and the angle of depression of D from the top of the mast.

13. A tree of height 24 m stands vertically at A on the ground of an island. Two boats are at B and C such that $\angle BAC = 94°$, $\angle ABC = 47°$ and $AB = 240$ m.

(a) Find
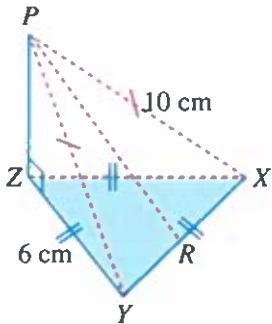
   (i) the distance between B and C,

   (ii) the area of $\triangle ABC$,

   (iii) the shortest distance from A to BC.

The boat at B sails in a straight line towards C.

(b) Find the greatest angle of depression of the boat from the top of the tree.
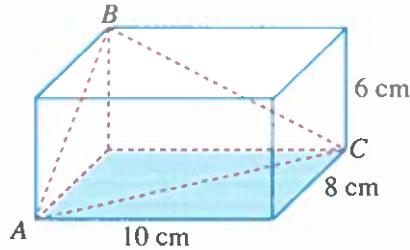
14. In the figure, XYZ is an equilateral triangle with sides of length of 6 cm lying in a horizontal plane. P lies vertically above Z, R is the midpoint of XY and $PX = PY = 10$ cm.

Find

(i) $\angle PYZ$,

(ii) $\angle PRZ$.
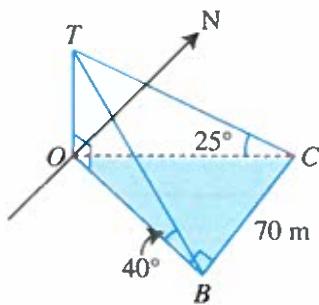
15. The figure shows a block of wood in the shape of a cuboid with dimensions 10 cm by 8 cm by 6 cm. Huixian cuts the block into two pieces such that the cutting tool passes through the points A, B and C as shown.

Given that the triangular surface ACB on one piece of the block is to be coated with varnish, find
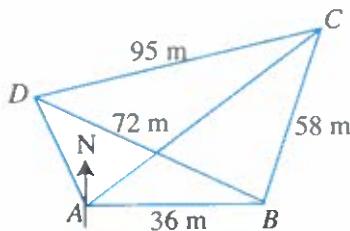
(i) $\angle ABC$,

(ii) the area of the surface that is to be coated with varnish.

16. Amirah stands at a point B, due east of a vertical tower OT, and observes that the angle of elevation of the top of the tower T is $40°$. She walks 70 m due north and finds that the angle of elevation of T from her new position at C is $25°$.

Find the height of the tower and hence distance OB.

17. The figure shows a plot of land *ABCD* such that *B* is due east of *A* and the bearing of *C* from *A* is 048°.

(i) Given that *AB* = 36 m, *BC* = 58 m, *BD* = 72 m and *CD* = 95 m, find the bearing of *C* from *B*.

A vertical control tower of height 35 m stands at *B*. Shirley cycles from *C* to *D* and reaches a point *P* where the angle of depression of *P* from the top of the tower is the greatest.

(ii) Find the angle of depression of *P* from the top of the tower.

Summary

1. Angles of Elevation and Depression

Angle of Elevation

When a person looks at the top of the wall, *B*, the angle between the horizontal *AD* and the line of sight *AB*, i.e. ∠*BAD*, is called the angle of elevation.

Angle of Depression

When a person looks at the bottom of the wall, *C*, the angle between the horizontal *AD* and the line of sight *AC*, i.e. ∠*CAD*, is called the angle of depression.

2. Bearings

The bearing of a point A from another point O is an angle measured from the north, at O, in a clockwise direction and it always written as a three-digit number.

The bearing of A from O is $050°$.

The bearing of O from A is $230°$.
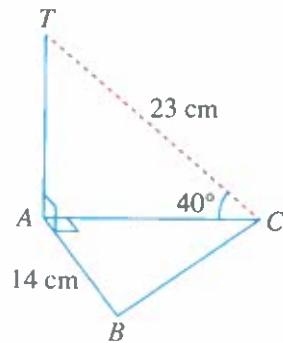
3. The basic technique used in solving a three-dimensional problem is to reduce it to a problem in a plane.

Review Exercise 9
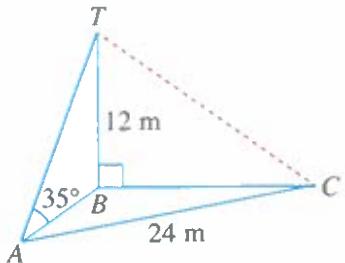
1. Two points A and B, 35 m apart on level ground, are due east of the foot of a castle. The angles of elevation of the top of the castle from A and B are $47°$ and $29°$ respectively. Find the height of the castle.

2. The angle of depression of a rabbit from the top of a cliff is $24°$. After the rabbit hops a distance of 80 m horizontally towards the base of the cliff, the angle of depression of the rabbit from the top of the cliff becomes $32°$. Find the height of the cliff.

3. The captain of a ship observes that the angle of elevation of a lighthouse is $12°$. When he sails a further distance of 200 m away from the lighthouse, the angle of elevation becomes $10°$. Find the height of the lighthouse.

4. ABC is a triangle lying on a horizontal plane with $\angle BAC = 90°$ and $AB = 14$ cm. T is a point vertically above A, $TC = 23$ cm and the angle of elevation of T from C is $40°$.
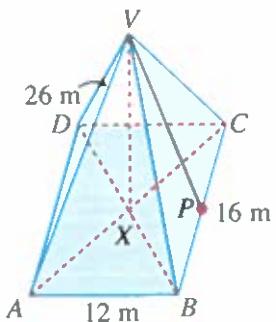
Find

(i) the height of AT,

(ii) the angle of elevation of T from B,

(iii) the length of BC.

5. $ABCT$ is a triangular pyramid with $\triangle ABC$ as its base and BT as its height. It is given that $AC = 24$ m, $BT = 12$ m, $\angle TAB = 35°$ and $\angle ABC = 90°$, find
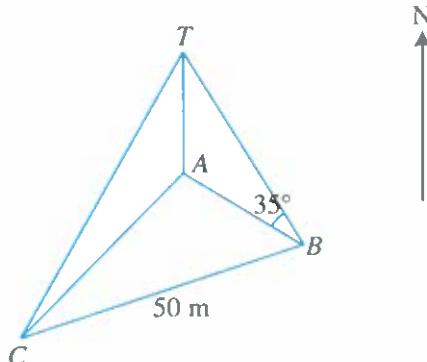
(i) the length of AB,

(ii) the length of BC,

(iii) the angle of depression of C from T.

6. The figure shows a roof in the shape of a right pyramid on a horizontal rectangular base $ABCD$, where $AB = 12$ m, $BC = 16$ m and $VA = 26$ m.
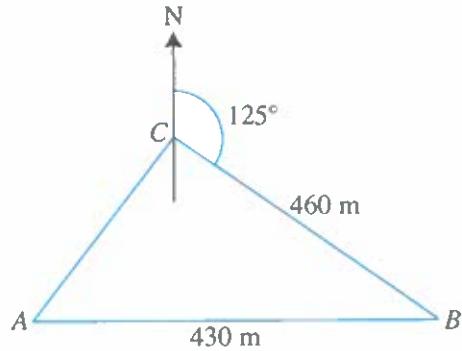
(i) Given that X is the midpoint of AC, find the height, VX, of the roof.

(ii) Find $\angle AVC$.

A passer-by notices a crack along VP, where P is the midpoint of BC.

(iii) Find the length of the crack.

7. Three points A, B and C lie on a horizontal ground. T is the top of a vertical tower standing on A. The bearings of B and C from A are 135° and 225° respectively and the bearing of C from B is 250°. If the distance between B and C is 50 m and the angle of elevation of T from B is 35°, calculate the height of the tower and the angle of elevation of T from C.

8. In the figure, A, B and C are three points on a horizontal field. A is due west of B, the bearing of B from C is 125°, $AB = 430$ m and $BC = 460$ m.
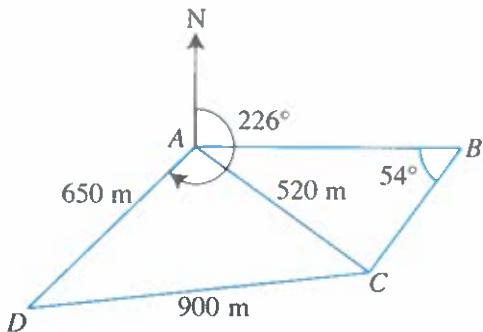
(a) Find

(i) the distance between A and C,

(ii) $\angle ACB$,

(iii) the bearing of C from A,

(iv) the area of $\triangle ABC$.

At a certain instant, a hot air balloon is at a point which is directly above C.

(b) Given that the angle of elevation of the hot air balloon from B is 5.2°, find the angle of elevation of the hot air balloon from A.

9. The figure shows the positions of four points A, B, C and D on level ground. B is due east of A, the bearing of D from A is 226°, $\angle ABC = 54°$, $AC = 520$ m, $AD = 650$ m and $CD = 900$ m.
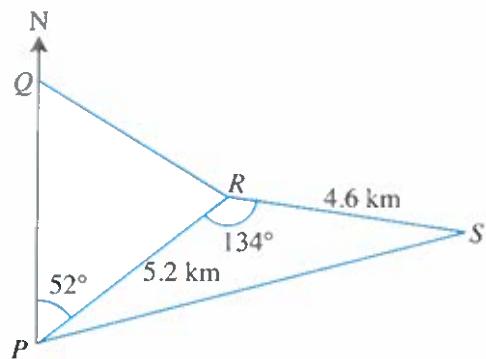
(a) Find

    (i) $\angle CAD$,

    (ii) the bearing of C from A,

    (iii) the distance between B and C,

    (iv) the shortest distance from A to CD.

A vertical tower of height 80 m stands at A. Devi walks from C to D and reaches a point P where the angle of elevation of the top of the tower from P is the greatest.

(b) Find the angle of elevation of the top of the tower from P.

10. The figure shows the position of a post office P and three train stations Q, R and S. Q is due north of P, the bearing of R from P is 052°, $\angle PRS = 134°$, $PR = 5.2$ km and $RS = 4.6$ km.

(i) Calculate the distance between P and S.

(ii) Find the bearing of S from R.

(iii) Given that the bearing of R from Q is 122°, find the distance between P and Q.

An office building of height 75 m stands at R.

(iv) Given that Huixian walks in a straight line along PS, calculate the greatest angle of elevation of the top of the building as she walks along PS.
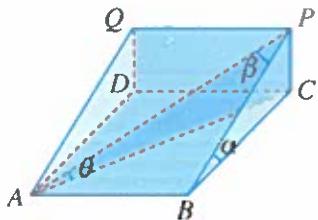
A policeman is standing at a point due north of P such that he is equidistant from both P and R.

(v) Find the distance between the policeman and P.
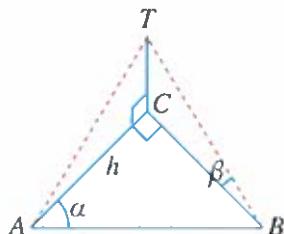
1. In the figure, *ABCD* is horizontal and *ABPQ* is a rectangular plane.

Given that $\angle PCA = \angle PCB = \angle QDC = 90°$, $\angle PBC = \alpha$, $\angle APB = \beta$ and $\angle PAC = \theta$, express $\sin \theta$ in terms of the trigonometric ratios of α and β only.

2. In the figure, $\triangle ABC$ is on horizontal ground and *CT* is vertical.

Given that $\angle ACB = 90°$, $\angle BAC = \alpha$, $\angle TBC = \beta$ and $AC = h$ metres, express the length of *TB* in terms of h and the trigonometric ratios of α and β.

Arc Length, Area of Sector and Radian Measure

The cross section of train tunnels may be in the shape of a major segment of a circle. What is meant by a 'major segment of a circle'? How do we find the cross-sectional area of such a train tunnel?

Chapter
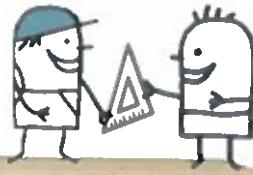
Ten

10.1 Length of Arc
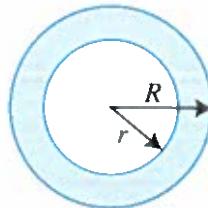
⠿ Recap

We have learnt that the circumference of a circle can be found using the formula $2\pi r$ or πd and the area of a circle can be found using the formula πr^2, where r is the radius and d is the diameter of the circle.

Worked Example 1

(Problem involving Circles)

Find the area of the washer shown, given that its external diameter is 6.4 cm and the diameter of the hole is 3.6 cm.

Solution:

External radius of ring, $R = \dfrac{6.4}{2}$

$= 3.2$ cm

Internal radius of ring, $r = \dfrac{3.6}{2}$

$= 1.8$ cm

∴ The area of the washer $= \pi R^2 - \pi r^2$

$= \pi(3.2)^2 - \pi(1.8)^2$

$= \pi(3.2^2 - 1.8^2)$

$= 7\pi$

$= 22.0$ cm^2 (to 3 s.f.)

PRACTISE NOW 1

Find the area of the ring whose external and internal diameters are 40 mm and 33 mm.

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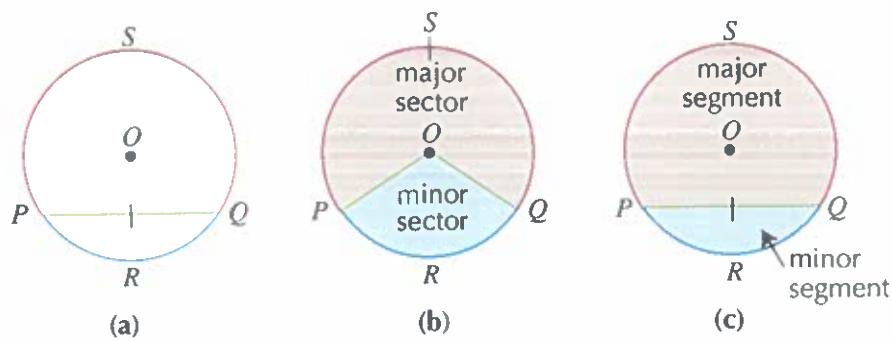
Fig. 10.1

Fig. 10.1 **(a)** shows a circle with centre O. The line PQ is called a **chord**. PRQ is part of the circumference which is called an **arc**. The arc PRQ is called the **minor arc** and the arc PSQ is called the **major arc**.

The part of a circle enclosed by any two radii of a circle and an arc is called a **sector**. In Fig. 10.1 **(b)**, the region enclosed by the radii OP, OQ and the minor arc PRQ is called a **minor sector** of the circle. The region enclosed by the radii OP, OQ and the major arc PSQ is called a **major sector** of the circle.

In Fig. 10.1 **(c)**, the chord PQ divides the circle into two **segments**. The region enclosed by the chord PQ and the *minor arc PRQ* is called a **minor segment**. The region enclosed by the chord PQ and the *major arc PSQ* is called the **major segment**.
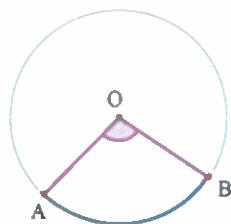
Investigation

Arc Length

Go to http://www.shinglee.com.sg/StudentResources/ and open the geometry template Arc Length as shown.

1. The template shows a circle with $\angle AOB$ subtended by the (blue) arc AB at the centre O.

Arc Length

Show How to do animation

Length of Blue Arc AB	Circumference	(Length of Blue Arc AB) / Circumference	Angle AOB	(Angle AOB) / 360°
4.16 cm	14.93 cm	0.279	100.39°	0.279

Fig. 10.2

2. Click and drag point *B* to change the size of the (blue) arc *AB*. To change the radius of the circle, move point *A*. Copy and complete Table 10.1.
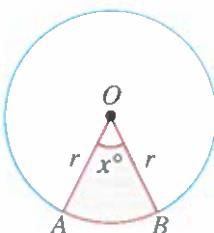
No.	Length of Blue Arc *AB*	Circumference of Circle	Length of Blue Arc *AB* / Circumference of Circle	$\angle AOB$	$\dfrac{\angle AOB}{360°}$
(a)					
(b)					
(c)					
(d)					
(e)					

Table 10.1

3. What do you notice about the fourth column and the last column of Table 10.1?

4. Click on the button 'Show how to do animation' in the template and it will show you how to add 10 more entries to the table as the points *A* and *B* move automatically. What do you notice about the fourth column and the last column of the table in the template?

5. Hence, write down a formula for finding the length of an arc of a circle.

From the investigation, we observe that

$$\text{arc length} = \frac{x°}{360°} \times \text{circumference}$$
$$= \frac{x°}{360°} \times 2\Box r,$$
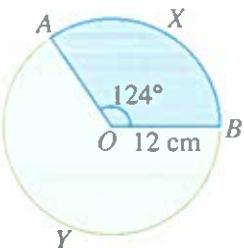
where $x°$ is the angle subtended by the arc at the centre of the circle of radius r.

Chapter 10 Arc Length, Area of Sector and Radian Measure

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Worked Example 2

(Finding the Arc Length)

In the figure, O is the centre of a circle of radius 12 cm and $\angle AOB = 124°$.

Find

(i) the length of the minor arc AXB,

(ii) the perimeter of the major sector $OAYB$.

Solution:

(i) Length of minor arc $AXC = \dfrac{124°}{360°} \times 2\square \times 12$

$= 26.0$ cm (to 3 s.f.)

(ii) Perimeter of major sector = length of arc $AYB + OA + OB$

$= \dfrac{360° - 124°}{360°} \times 2\square \times 12 + 12 + 12$

$= 73.4$ cm (to 3 s.f.)

Problem Solving Tip
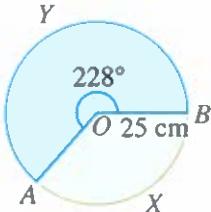
Alternatively,
Length of arc AYB
$= 2\pi r -$ length of arc AXB.

PRACTISE NOW 2

1. In the figure, O is the centre of a circle of radius 25 cm and reflex $\angle AOB = 228°$.
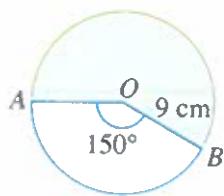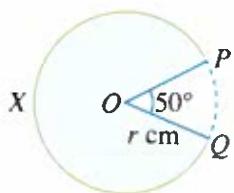
Find

(i) the length of the major arc AYB,

(ii) the perimeter of the minor sector $OAXB$.

2. In the figure, O is the centre of a circle of radius 9 cm and $\angle AOB = 150°$.

Find the perimeter of the shaded region, giving your answer in the form $a + b\pi$, where a and b are rational numbers.

3. The figure shows the design of a logo in which a sector has been removed from the circle, centre O and radius r cm.
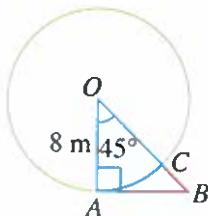
Given that the length of the major arc PXQ is 36 cm and $\angle POQ = 50°$, find the value of r.

Worked Example 3

(Finding the Arc Length)

In the figure, O is the centre of a circle of radius 8 m. The points A and C lie on the circumference of the circle and OCB is a straight line.

Given that $\angle AOB = 45°$ and OA is perpendicular to AB, find the perimeter of the shaded region ABC.

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Solution:

$\angle ABC = 180° - 90° - 45°$

$\qquad = 45°$

i.e. $\triangle AOB$ is isosceles and $AB = 8$ m.

Length of arc $AC = \dfrac{45°}{360°} \times 2\pi \times 8$

$\qquad\qquad\qquad = 2\pi$ m

Using Pythagoras' Theorem,

$\qquad OB^2 = 8^2 + 8^2$

$\qquad OB = \sqrt{8^2 + 8^2}$

$\qquad\quad = \sqrt{128}$

$\qquad\quad = 11.31$ m (to 4 s.f.)

$BC = OB - OC$

$\qquad = 11.31 - 8$

$\qquad = 3.31$ m

$\therefore$ Perimeter of shaded region $ABC = AB + BC +$ length of arc AC

$\qquad\qquad\qquad\qquad\qquad = 8 + 3.31 + 2\pi$

$\qquad\qquad\qquad\qquad\qquad = 17.6$ m (to 3 s.f.)
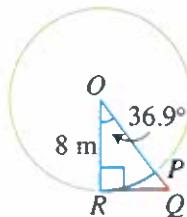
PRACTISE NOW 3

1. In the figure, O is the centre of a circle of radius 8 m. The points P and R lie on the circumference of the circle and OPQ is a straight line.

Given that $\angle QOR = 36.9°$ and OR is perpendicular to QR, find the perimeter of the shaded region PQR.

2. The figure shows a sector of a circle of radius 10 cm. Given that the angle at the centre of the circle is 80°, find the perimeter of the sector.

Worked Example 4

(Finding the Arc Length)

In the figure, AQB is the minor arc of a circle with centre O and radius 17 cm. ARB is a semicircle with AB as its diameter and P as its centre. $\triangle OPB$ is a right-angled triangle with $\angle OPB = 90°$ and $\angle POB = 59°$.

Find

(i) the length of AB,

(ii) the perimeter of the shaded region.

Solution:

(i) In $\triangle OPB$,

$$\sin \angle POB = \frac{PB}{OB}$$

$$\sin 59° = \frac{PB}{17}$$

$$PB = 17 \sin 59°$$

$$= 14.57 \text{ cm (to 4 s.f.)}$$

> **ATTENTION**
>
> In $\triangle OPB$, PB is the side opposite $\angle POB$ and OB is the hypotenuse.

$$\therefore AB = 2PB$$

$$= 2(14.57)$$

$$= 29.1 \text{ cm (to 3 s.f.)}$$

(ii) $\angle AOB = 2\angle POB$

$$= 2(59°)$$

$$= 118°$$
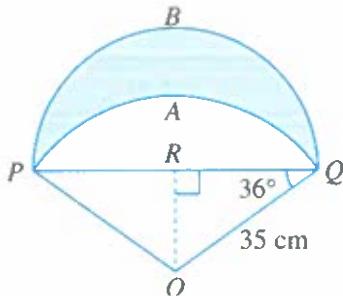
Length of arc $AQB = \dfrac{118°}{360°} \times 2\pi \times 17$

$$= 35.01 \text{ cm (to 4 s.f.)}$$

Length of arc $ARB = \dfrac{1}{2} \times 2\pi \times 14.57$

$$= 45.77 \text{ cm (to 4 s.f.)}$$

$\therefore$ Perimeter of shaded region $= 35.01 + 45.77$

$$= 80.8 \text{ cm (to 3 s.f.)}$$

In the figure, PAQ is the minor arc of a circle with centre O and radius 35 cm. PBQ is a semicircle with PQ as its diameter and R as its centre. $\triangle ORQ$ is a right-angled triangle with $\angle ORQ = 90°$ and $\angle RQO = 36°$.
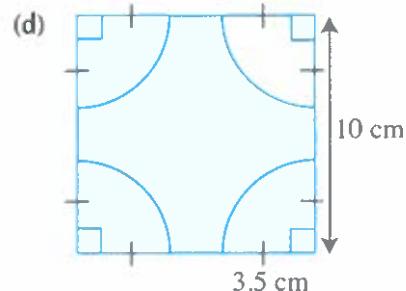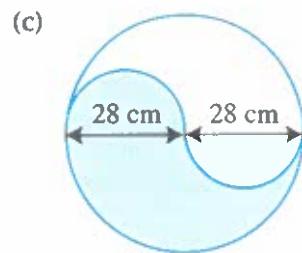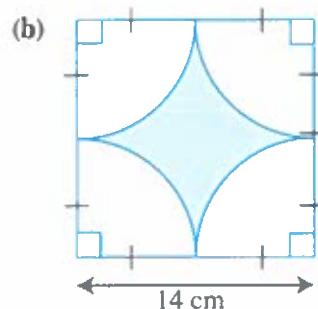
Find

(i) the length of PQ,

(ii) the perimeter of the shaded region.

Exercise 10A
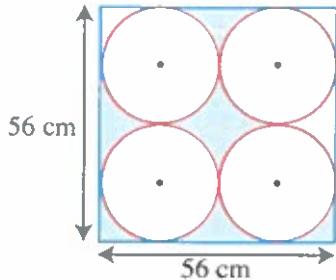
BASIC LEVEL

1. Find the diameters of the circles given the following areas.

   (a) 616 mm²

   (b) $779\dfrac{5}{8}$ m²

   (c) 3850 cm²

2. Find the areas of the following rings whose external and internal diameters are

   (a) 15 cm and 13 cm,

   (b) 1.2 m and 0.9 m

3. Find **(i)** the area and **(ii)** the perimeter of the shaded region in each of the following figures given that all arcs are either quadrants or semicircles.

(a)

(b)

(c)

(d)
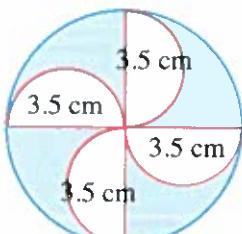
4. Find the area of the shaded region in the following figures.

(a)

56 cm

56 cm

(b)

3.5 cm

3.5 cm

3.5 cm

3.5 cm

(c)

14 cm   14 cm   14 cm

(d)

48 cm

14 cm
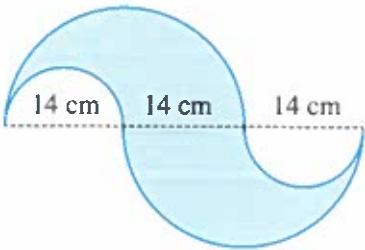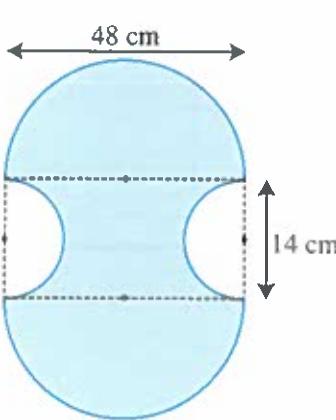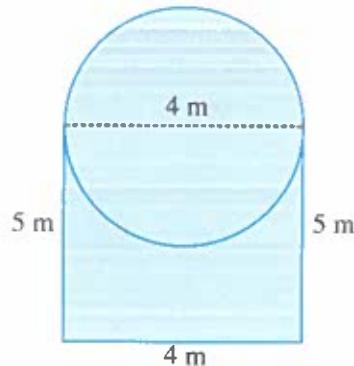
5. The radius of a circular pond is 12 m. Grass grows around the pond, two metres away from its edge. Find, in terms of π,

(a) the circumference of the circle forming the outer edge of the grass,

(b) the area of the land between the pond and the outer edge of the grass
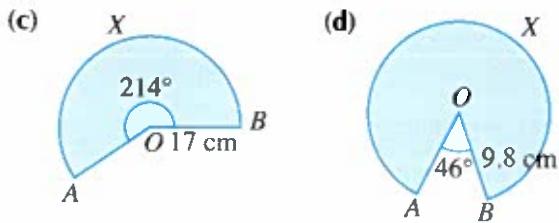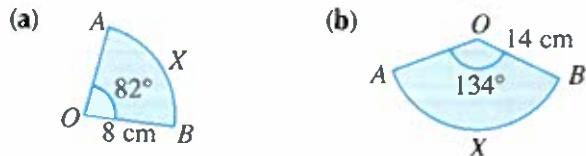
6. John decides to paint this figure on his driveway in yellow with black lines.

4 m

5 m   5 m

4 m
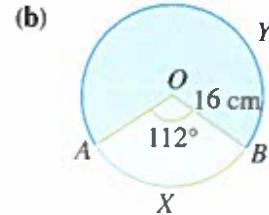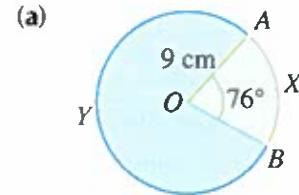
Find

(a) the area of the shaded region that he will paint,

(b) the total length that he will paint with black paint.
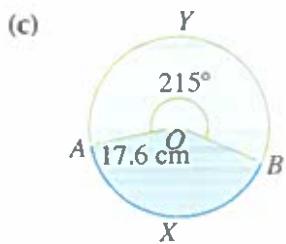
7. Find the length of each of the following arcs AXB.

(a)

$82°$

8 cm

(b)

14 cm

$134°$

(c)

$214°$

O 17 cm

(d)

$46°$ 9.8 cm
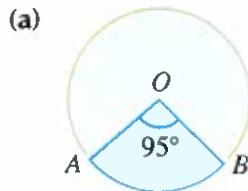
8. For each of the following circles, find

(i) the length of the minor arc AXB,

(ii) the perimeter of the major sector $OAYB$.

(a)

9 cm

$76°$

(b)

16 cm

$112°$

(c)

215°

A 17.6 cm O B

Y

X
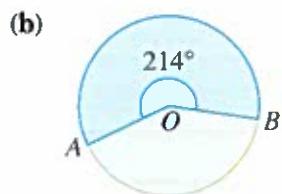
9. Find the radius of each of the following circles.

(a)

O

A 95° B

Length of minor arc = 26.53 cm

(b)

214°

A O B

Length of major arc = 104.6 cm
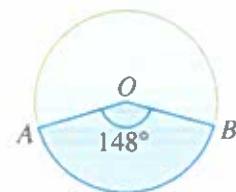
10. The radius of a circle is 14 m. Find the angle at the centre of the circle subtended by an arc of length

(a) 12 m,   (b) 19.5 m,

(c) 64.2 m,   (d) 84.6 m,
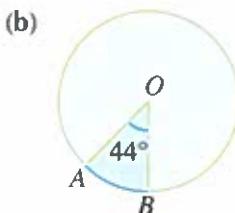
giving your answers correct to the nearest degree.

11. The hour hand of a large clock mounted on a clock tower travels through an angle of 45°. If the hour hand is 1.5 m long, how far does the tip of the hour hand travel?

12. A piece of wire 32 cm long is bent to form a sector of a circle of radius 6 cm. Find the angle subtended by the wire at the centre of the circle.

13. Find the radius of each of the following circles.

(a)

O

A 148° B
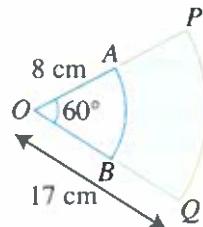
Perimeter of minor sector = 77.91 cm

(b)

O

44°

A

B
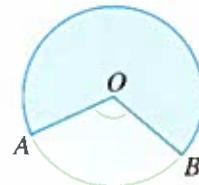
Perimeter of major sector = 278.1 cm

14. The figure shows two sectors OAB and OPQ with O as the common centre. The lengths of OA and OQ are 8 cm and 17 cm respectively.

P

8 cm A

O 60°

B

17 cm Q

Given that $\angle AOB = \angle POQ = 60°$, find the perimeter of the shaded region, giving your answer in the form $a + b\pi$, where a and b are rational numbers.

15. In the figure, the length of the minor arc is $\frac{7}{24}$ of the circumference of the circle.

O

A

B
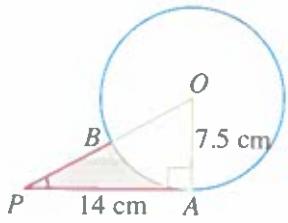
(i) Find $\angle AOB$.

(ii) Given that the diameter of the circle is 14 cm, find the length of the minor arc.

16. In the figure, O is the centre of a circle of radius 7.5 cm. The points A and B lie on the circumference of the circle and OBP is a straight line.

Given that $PA = 14$ cm and OA is perpendicular to AP, find

(i) $\angle POA$,

(ii) the perimeter of the shaded region PBA.
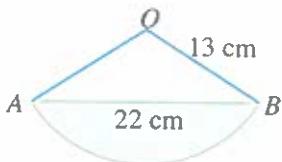
17. The figure shows a circle with centre O and radius 26 cm. Triangles OPR and OQR are congruent and $\angle OPR = \angle OQR = 90°$.

Given that $\angle POQ = 138°$, find

(i) the length of QR,

(ii) the perimeter of the shaded region.
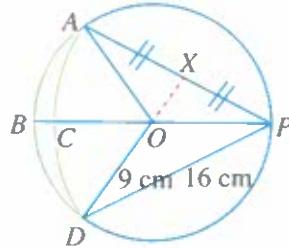
18. The figure shows a sector of a circle with centre O and radius 13 cm.

Given that the length of the chord $AB = 22$ cm,

(i) show that $\angle AOB$ is approximately 115.6°,

(ii) find the perimeter of the shaded region.

19. In the figure, ABD is the minor arc of a circle with centre O and radius 9 cm. ACD is the minor arc of a circle with centre P and radius 16 cm. The point X lies on AP such that $AX = XP$.
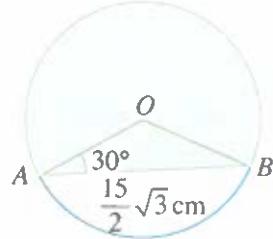
Find

(i) $\angle APB$,

(ii) $\angle AOB$,

(iii) the perimeter of the shaded region.
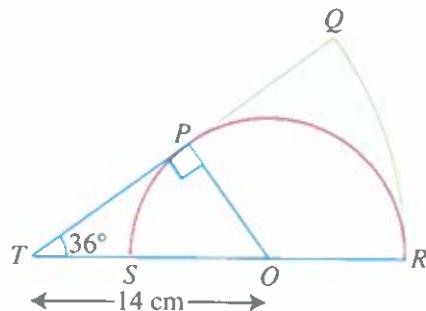
20. In the figure, O is the centre of a circle. The points A and B lie on the circumference of the circle such that $AB = \dfrac{15}{2}\sqrt{3}$ cm and $\angle OAB = 30°$.
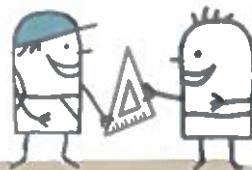
Find the perimeter of the shaded region.

21. The figure shows a semicircle with centre O and diameter SR. QR is an arc of another circle with centre T and T lies on RS produced.

Given that $TO = 14$ cm, $\angle OTP = 36°$ and OP is perpendicular to PT, find the perimeter of the shaded region.

10.2 Area of Sector

We have revised the formula to finding the area of a circle in the previous section. In this section, we will look at expressing the area of a sector as a fraction of the area of a circle.

Investigation

Area of Sector

Go to http://www.shinglee.com.sg/StudentResources/ and open the geometry template Sector Area as shown below.

1. The template shows a circle with centre O and a (yellow) sector OAB.

Sector Area
© Shinglee Pte Ltd, 2015
Show How to do animation

Area of Shaded Sector OAB	Area of Circle	$\dfrac{(\text{Area of Shaded Sector OAB})}{(\text{Area of Circle})}$	Angle AOB	$\dfrac{(\text{Angle AOB})}{360^\circ}$
4.95 cm²	17.75 cm²	0.279	100.45°	0.279

Fig. 10.3

2. Click and drag point B to change the size of the (yellow) sector OAB. To change the radius of the circle, move point A. Copy and complete Table 10.2.
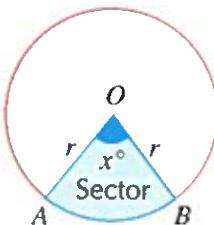
No.	Area of Shaded Sector OAB	Area of Circle	$\dfrac{\text{Area of Sector } OAB}{\text{Area of Circle}}$	$\angle AOB$	$\dfrac{\angle AOB}{360^\circ}$
(a)					
(b)					
(c)					
(d)					
(e)					

Table 10.2

3. What do you notice about the third last column and the last column of Table 10.2?

4. Click on the button 'Show how to do animation' in the template and it will show you how to add 10 more entries to the table as the points A and B move automatically. What do you notice about the fourth column and the last column of the table in the template?

5. Hence, write down a formula for finding the area of a sector of a circle.

From the investigation, we observe that

$$\text{area of a sector} = \frac{x°}{360°} \times \text{area of the circle}$$
$$= \frac{x°}{360°} \times \Box r^2,$$
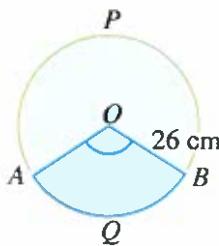
where $x°$ is the angle of the sector subtended at the centre of the circle of radius r.

Worked Example 5

(Finding the Area of a Sector)

In the figure, O is the centre of a circle of radius 26 cm. The length of minor arc AQB is 52 cm.

(i) Show that $\angle AOB$ is approximately 114.6°.

(ii) Hence, find the area of the major sector $OAPB$.

Solution:

(i) Since the length of minor arc AQB is 52 cm,

$$\frac{\angle AOB}{360°} \times 2\Box \times 26 = 52 \quad \left(\text{use arc length} = \frac{x°}{360°} \times 2\Box r\right)$$

$$\frac{\angle AOB}{360°} = \frac{52}{52\Box}$$

$$= \frac{1}{\Box}$$

$$\angle AOB = \frac{1}{\Box} \times 360°$$

$$= 114.6° \text{ (to 1 d.p.)}$$

(ii) Reflex $\angle AOB = 360° - 114.59°$ ($\angle$s at a point)

$$= 245.4°$$

Area of major sector $OAPB = \dfrac{245.4°}{360°} \times \pi \times 26^2$
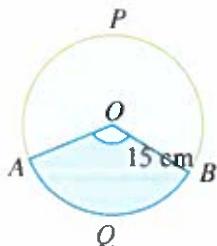
$$= 1450 \text{ cm}^2 \text{ (to 3 s.f.)}$$

In the figure, O is the centre of a circle of radius 15 cm. The length of minor arc AQB is 33 cm.
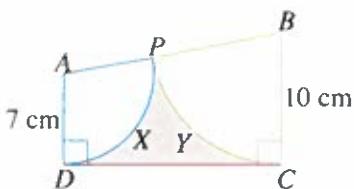
(i) Show that $\angle AOB$ is approximately $126.1°$.

(ii) Hence, find the area of the major sector $OAPB$.

Worked Example 6

(Finding the Area of a Sector)

In the figure, $ABCD$ is a trapezium in which AD is parallel to BC, $AD = 7$ cm, $BC = 10$ cm and $\angle ADC = \angle BCD = 90°$. PXD is an arc of a circle centre A and PYC is an arc of a circle centre B.
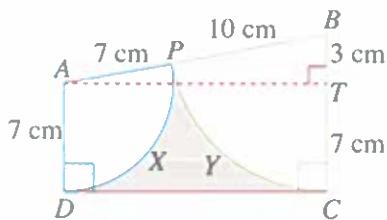
(i) Show that $\angle BAD$ is approximately $100.2°$.

(ii) Hence, find the area of the shaded region.

Solution:

(i) Draw a line AT such that T lies on BC and AT is perpendicular to BC.

$BT = BC - TC$

$\quad = 10 - 7$

$\quad = 3$ cm

$AB = 7 + 10 \ (AB = AP + PB)$

$\quad = 17$ cm

In $\triangle ATB$,

$\sin \angle BAT = \dfrac{3}{17}$

$\quad \angle BAT = \sin^{-1} \dfrac{3}{17}$

$\qquad = 10.16°$ (to 2 d.p.)

$\angle BAD = 90° + 10.16°$

$\quad = 100.16°$

$\quad = 100.2°$ (to 1 d.p.)

ATTENTION

In $\triangle ATB$, BT is the side opposite $\angle BAT$ and AB is the hypotenuse.

(ii) Using Pythagoras' Theorem,

$AT^2 = 17^2 - 3^2$

$AT = \sqrt{17^2 - 3^2}$

$\quad = \sqrt{280}$

$\quad = 16.73$ cm (to 4 s.f.)

$\angle ABT = 180° - 90° - 10.16°$

$\quad = 79.84°$ (to 2 d.p.)
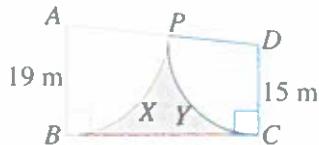
Area of shaded region

= Area of trapezium $ABCD$ – area of sector APD – area of sector BCP

$= \dfrac{1}{2}(7 + 10)(16.73) - \dfrac{100.2°}{360°} \times \square \times 7^2 - \dfrac{79.84°}{360°} \times \square \times 10^2$

$= 29.7$ cm^2 (to 3 s.f.)

In the figure, *ABCD* is a trapezium in which *AB* is parallel to *DC*, *AB* = 19 m, *DC* = 15 m and ∠*ABC* = ∠*DCB* = 90°. *PXB* is an arc of a circle centre *A* and *PYC* is an arc of a circle centre *D*.

(i) Show that ∠*ADC* is approximately 96.8°.

(ii) Hence, find the area of the shaded region.

Exercise 10B

BASIC LEVEL
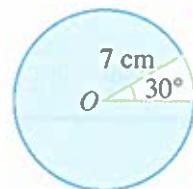
1. Copy and complete the table for sectors of a circle.

	Radius	Angle at centre	Arc length	Area	Perimeter
(a)	7 cm	72°			
(b)	35 mm				136 mm
(c)		270°		1848 mm²	
(d)		150°	220 cm		
(e)	14 m		55 m		
(f)		75°		154 cm²	
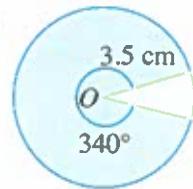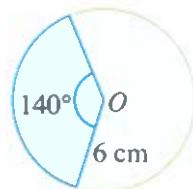
2. For each of the following circles, find
   (i) the perimeter,
   (ii) the area,
   of the minor sector.
   (a)
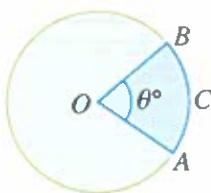
   (b)

   (c)
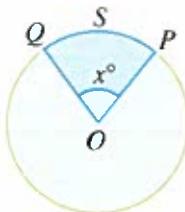
3. The figure shows a circle with centre O and $\angle AOB = \theta°$. The circumference of the circle is 88 cm.
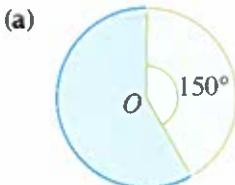
Find the length of arc ACB and the area of sector $OACB$ for each of the following values of θ.

(a) 60   (b) 99

(c) 126   (d) 216

4. The figure shows a circle with centre O and $\angle POQ = x°$. The area of the circle is 3850 cm².
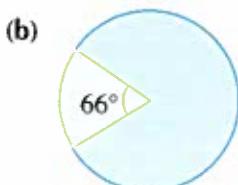
Find the area of sector $OPSQ$ and the length of arc PSQ for each of the following values of x.

(a) 36   (b) 84

(c) 108   (d) 198

5. Find the radius of each of the following circles.

(a)

Area of minor sector = 114 cm²

(b)
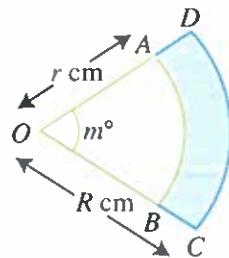
Area of major sector = 369 cm²

6. The diameter of a circle is 18 cm. Find the angle subtended by the arc of a sector with each of the following areas.

(a) 42.6 cm²   (b) 117.2 cm²

(c) 214.5 cm²   (d) 18.9 cm²
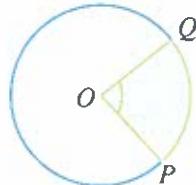
7. The figure shows two sectors OAB and ODC with O as the common centre.

Given that $OA = r$ cm, $OC = R$ cm and $\angle AOB = m°$, find the perimeter and the area of the shaded region $ABCD$ for each of the following cases.

(i) $r = 10$, $R = 20$, $m = 45$

(ii) $r = 5$, $R = 8$, $m = 120$

(iii) $r = 35$, $R = 49$, $m = 160$
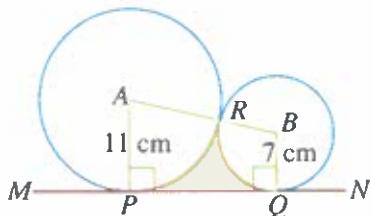
8. In the figure, the area of the shaded sector POQ is $\frac{5}{18}$ of the area of the whole circle.

(i) Find $\angle POQ$.

(ii) Given that the area of the shaded sector is 385 cm², find the diameter of the circle.
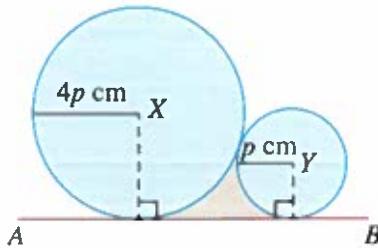
9. During an Art lesson, students are required to make a shape in the form of a sector of a circle of radius 12 cm. If the perimeter of the shape is 38 cm, find the area of the paper used in making the shape.

10. The figure shows two circular discs of radii 11 cm and 7 cm touching each other at R and lying on a straight line $MPQN$.

(i) Show that $\angle PAB$ is approximately $77.2°$.

(ii) Hence, find the area of the shaded region.
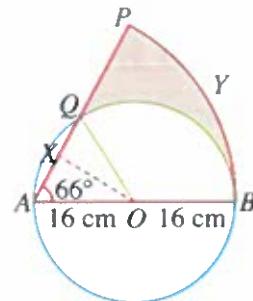
11. Two circular discs of radii $4p$ cm and p cm touch each other externally and lie on a straight line AB as shown.

Find an expression, in terms of p, for the area enclosed by the two discs and the line AB.
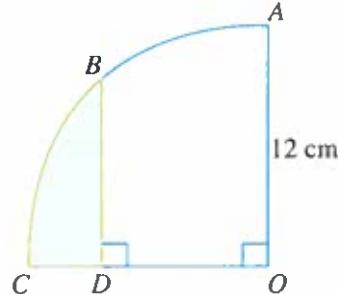
12. In the figure, O is the centre of a circle with radius 16 cm. PYB is the minor arc of a circle centre A and radius 32 cm. OX divides $\triangle OAQ$ into two congruent triangles.

Given that $\angle OAQ = 66°$, find

(i) $\angle BOQ$,

(ii) the length of AQ,

(iii) the perimeter of the shaded region,

(iv) the area of the shaded region.
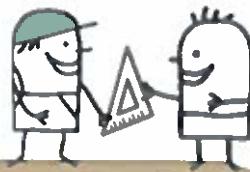
13. The figure shows a quadrant of a circle of radius 12 cm.

Given that B is the midpoint of the arc AC, find

(i) the length of BD,

(ii) the perimeter of the shaded region,

(iii) the area of the shaded region.

10.3 Radian Measure

⠿ Conversion Between Degrees and Radians

So far we have been using the measurement of 360° to denote the angle for one complete revolution. However, this value is arbitrary and in some branches of mathematics, angular measurement cannot be conveniently done in degrees. Hence, a new unit, called the **radian**, is introduced to describe the magnitude of an angle.

Consider a circle with centre O and radius r units (see Fig. 10.4(a)).

Suppose the arc AB has a length of r units.

Then the measure of the angle θ, subtended by the arc AB at the centre O, is defined to be one **radian**.

In other words, a radian is the size of the angle subtended at the centre of a circle by an arc equal in length to the radius of the circle.

(a)   (b)
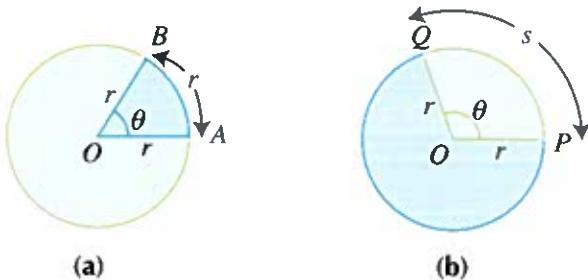
Fig. 10.4

In general, the angle θ, in radians, subtended at the centre of a circle with radius r units by an arc PQ of length s units, is equal to the ratio of the arc length to the radius:

$$\theta \text{ (in radians)} = \frac{s}{r}$$

The abbreviation of 'radian' is 'rad'.

For example, if the radius of the circle is 2 cm and arc $PQ = 2$ cm, then $\theta = 1$ rad; if the radius of the circle is 2 cm and arc $PQ = 4$ cm, then $\theta = 2$ rad.

Investigation

Visualise the Size of an Angle of 1 radian

For this activity, you will require a sheet of paper, a pair of compasses and a piece of string (5 cm long).

1. On a sheet of paper, use a rule and a pair of compasses to draw a circle, centre O and with radius 5 cm.

2. Mark a point A on the circumference, as shown in Fig. 10.5.

3. Using a piece of string of length 5 cm, mark a point B on the circumference such that the length of arc AB is 5 cm.

4. Consider the minor sector AOB. Notice that the radius is equal to the arc length. What is the value of θ in radians?

$\theta \text{ (in radians)} = \frac{s}{r}$
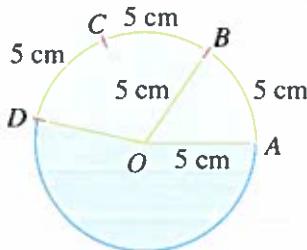
Fig. 10.5

5. Repeat Step 3 to mark out arcs $BC, CD, DE, \ldots$ of length 5 cm each around the circle.

6. **(a)** Estimate the size of the angle, in radians, subtended at the centre by
   (i) a semicircle,   **(ii)** a circle.
   (b) Using this estimate, what is the approximate size of $\angle AOB$ (1 radian) in degrees?
   (c) Hence, what is an angle of 1 radian approximately equal to in degrees?

7. **(a)** Using the formula for the circumference of a circle, find the exact value of the angle at the centre of the circle in radians.
   (b) Hence, find the exact value of an angle of 1 radian in degrees.

In general, for a circle of radius r units,

if arc $PQ = r$, then $\theta = 1$ rad;

if arc $PQ = 2r$, then $\theta = 2$ rad;

if arc $PQ = 2\pi r$, then $\theta = 2\pi$ rad.

When arc $PQ = 2\pi r$, it means that the arm OP has made one complete revolution, i.e. OM has moved through an angle of 360°, i.e. 2π rad = 360°.

$$\boxed{\pi \text{ rad} = 180°}$$
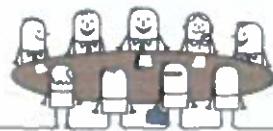
$1 \text{ rad} = \dfrac{180°}{\pi} \approx 57.3°$

$1° = \dfrac{\pi}{180°} \approx 0.017\ 46 \text{ rad}$

The table gives a conversion table for some special angles.

Angle (°)	Angle (rad)
30	$\frac{\pi}{6}$
45	$\frac{\pi}{4}$
60	$\frac{\pi}{3}$
90	$\frac{\pi}{2}$
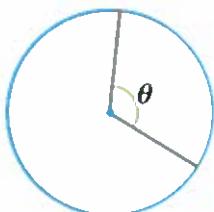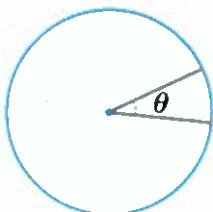
Estimate the Size of Angles in Radians

Work in pairs.
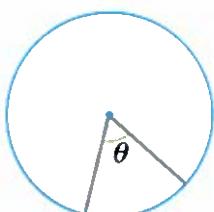
Look at the figures below.
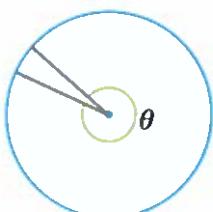
(a)
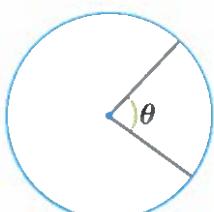
(b)
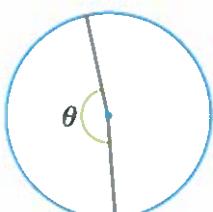
(c)

(d)

(e)

(f)

Identify which figure shows an angle θ of

(i) 1 radian,   (ii) 2 radians,   (iii) 3 radians,

(iv) 6 radians,   (v) 0.5 radians,   (vi) 1.5 radians.

Worked Example 7

(Converting from Radians to Degrees)

Convert each of the following angles from radians to degrees.

(a) $\dfrac{\pi}{12}$ (b) 2.8

Solution:

(a) Since π radians = 180°,

$$\dfrac{\pi}{12} \text{ rad} = \dfrac{180°}{12}$$
$$= 15°$$

(b) Since π radians = 180°,

$$1 \text{ radian} = \dfrac{180°}{\pi}.$$

$$2.8 \text{ radians} = 2.8 \times \dfrac{180°}{\pi}$$
$$= 160.4° \text{ (to 1 d.p.)}$$

JustForFun

Gottfried Wilhelm von Leibnitz (1646-1716) was a German genius who won recognition in law, philosophy, religion, literature, metaphysics and mathematics. He developed a method of calculating π without reference to a circle. He proved that $\dfrac{\pi}{4}$ could be determined to any desired degree of accuracy by the formula:

$$\dfrac{\pi}{4} = 1 - \dfrac{1}{3} + \dfrac{1}{5} - \dfrac{1}{7} + \dfrac{1}{9} - \dfrac{1}{11} + \dots$$

PRACTISE NOW 7

SIMILAR QUESTIONS

Exercise 10C Questions 1(a)–(d)

Convert each of the following angles from radians to degrees.

(a) $\dfrac{\pi}{15}$ (b) $\dfrac{3\pi}{2}$

(c) 3.04   (d) 8

Worked Example 8

(Converting from Degrees to Radians)

Convert each of the following angles from degrees to radians.

(a) 27°   (b) 315°

Solution:

(a) Since 180° = π radians,

$$27° = \dfrac{\pi}{180°} \times 27°$$
$$= 0.471 \text{ radians (to 3 s.f.)}$$

(b) Since 180° = π radians,

$$315° = \dfrac{\pi}{180°} \times 315°$$
$$= \dfrac{7\pi}{4}$$
$$= 5.50 \text{ radians (to 3 s.f.)}$$

PRACTISE NOW 8

SIMILAR QUESTIONS

Exercise 10C Questions 2(a)–(d), 3(a)–(d)

Convert each of the following angles from degrees to radians.

(a) 36°   (b) 288°   (c) 197.5°   (d) 400°

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Arc Length, Area of Sector and Radian Measure   Chapter 10   316

Use of Calculators

Worked Example 9

(Use of a Calculator)

Find the value of each of the following.

(a) sin 1.2   (b) cos 0.879

(c) tan 1.012

Solution:

(a) To find the value of sin 1.2, first set the calculator to the 'radian' mode.
Press $\boxed{\sin}\boxed{1}\boxed{.}\boxed{2}\boxed{=}$ and the display shows 0.932 039 086,
i.e. sin 1.2 = 0.932 (to 3 s.f.)

(b) To find the value of cos 0.879, press $\boxed{\cos}\boxed{0}\boxed{.}\boxed{8}\boxed{7}\boxed{9}\boxed{=}$ and the display shows 0.637 921 564,

i.e. cos 0.879 = 0.638 (to 3 s.f.)

(c) Press $\boxed{\tan}\boxed{1}\boxed{.}\boxed{0}\boxed{1}\boxed{2}\boxed{=}$ and the display shows 1.599 298 86,
i.e. tan 1.012 = 1.60 (to 3 s.f.)

PRACTISE NOW 9

Find the value of each of the following.

(a) sin 0.65   (b) cos 0.235   (c) tan 1.23

Worked Example 10

(Use of a Calculator)

For each of the following, find the value of x in the range $0 < x < \frac{\pi}{2}$.

(a) sin x = 0.45   (b) cos x = 0.605

(c) tan x = 2.4

Solution:

(a) For sin x = 0.45, press $\boxed{\sin^{-1}}\boxed{0}\boxed{.}\boxed{4}\boxed{5}\boxed{=}$ to get 0.466 765 339,
i.e. when sin x = 0.45,

x = 0.467 radians (to 3 s.f.)

(b) For $\cos x = 0.605$, press $\boxed{\cos^{-1}}\,\boxed{0}\,\boxed{.}\,\boxed{6}\,\boxed{0}\,\boxed{5}\,\boxed{=}$ to get 0.921 030 459,

i.e. when $\cos x = 0.605$,

$x = 0.921$ radians (to 3 s.f.)

(c) For $\tan x = 2.4$, press $\boxed{\tan^{-1}}\,\boxed{2}\,\boxed{.}\,\boxed{4}\,\boxed{=}$ to get 1.176 005 207,

i.e. when $\tan x = 2.4$,

$x = 1.18$ radians (to 3 s.f.)

PRACTISE NOW 10

For each of the following, find the value of x in the range $0 < x < \dfrac{\square}{2}$.

(a) $\sin x = 0.87$ **(b)** $\cos x = 0.347$ **(c)** $\tan x = 0.88$
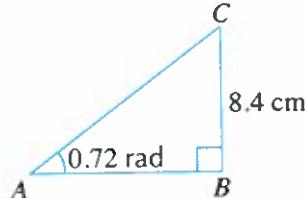
Worked Example 11

(Solving a Triangle in Radian Mode)

In $\triangle ABC$, $\angle ABC = \dfrac{\square}{2}$. Given that $BC = 8.4$ cm and $\angle BAC = 0.72$ radians, calculate the length of

(i) AB, **(ii)** AC.

Solution:

(a) $\tan \angle CAB = \dfrac{\text{opp}}{\text{adj}} = \dfrac{BC}{AB}$

$\tan 0.72 = \dfrac{8.4}{AB}$

$\therefore AB = \dfrac{8.4}{\tan 0.72}$

$= 9.58$ cm (to 3 s.f.)

(b) $\sin \angle CAB = \dfrac{\text{opp}}{\text{hyp}} = \dfrac{BC}{AC}$

$\sin 0.72 = \dfrac{8.4}{AC}$

$\therefore AC = \dfrac{8.4}{\sin 0.72}$

$= 12.7$ cm (to 3 s.f.)

Problem Solving Tip

$\angle ABC = \dfrac{\square}{2} = \dfrac{180°}{2} = 90°$

i.e. $\triangle ABC$ is a right-angled triangle.

ATTENTION
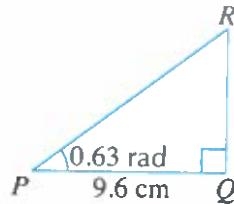
We can also use Pythagoras' Theorem to find the length of AC, i.e. $AC^2 = AB^2 + BC^2$

In $\triangle PQR$, $\angle PQR = \frac{\square}{2}$. Given that $PQ = 9.6$ cm and $\angle QPR = 0.63$ radians, calculate the length of

(i) QR, (ii) PR.
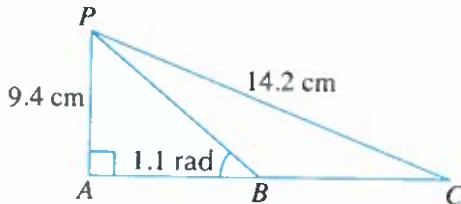
Worked Example 12

(Solving a Triangle in Radian Mode)

In $\triangle PAC$, $PA = 9.4$ cm, $PC = 14.2$ cm and $\angle PAC = \frac{\square}{2}$.

B lies on AC such that $\angle PBA = 1.1$ radians.

Find

(i) $\angle ACP$ in radians,

(ii) the length of AB,

(iii) the length of BC.

Solution:

(i) In $\triangle PAC$,

$\sin \angle ACP = \dfrac{9.4}{14.2}$

$\therefore \angle ACP = \sin^{-1}\left(\dfrac{9.4}{14.2}\right)$

$\qquad = 0.723$ radians (to 3 s.f.)

(ii) In $\triangle PAB$,

$\tan 1.1 = \dfrac{9.4}{AB}$

$AB = \dfrac{9.4}{\tan 1.1}$

$\qquad = 4.78$ cm (to 3 s.f.)

(iii) Using Pythagoras' Theorem,

$$PC^2 = PA^2 + AC^2$$
$$14.2^2 = 9.4^2 + AC^2$$
$$AC^2 = 14.2^2 - 9.4^2$$
$$= 113.28$$
$$AC = \sqrt{113.28}$$
$$= 10.64 \text{ cm (to 4 s.f.)}$$

$$\therefore BC = AC - AB$$
$$= 10.64 - 4.78$$
$$= 5.86 \text{ cm (to 3 s.f.)}$$
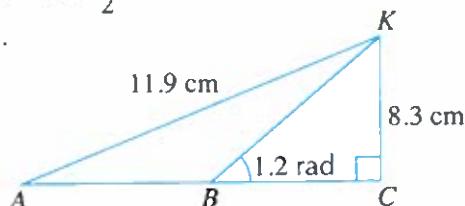
In $\triangle ACK$, $KC = 8.3$ cm, $AK = 11.9$ cm and $\angle ACK = \dfrac{\square}{2}$.
B lies on AC such that $\angle KBC = 1.2$ radians.

Find
(i) $\angle KAC$ in radians,
(ii) the length of BC,
(iii) the length of AB.

Exercise 10C

1. Convert each of the following angles from radians to degrees, giving your answer correct to 1 decimal place.

   (a) $\dfrac{5\square}{6}$

   (b) $\dfrac{\square}{7}$

   (c) 3.2

   (d) 2.56

2. Convert each of the following angles from degrees to radians.

   (a) 37.4°

   (b) 78.9°

   (c) 142°

   (d) 308°

3. Convert each of the following angles from degrees to radians, leaving your answer in terms of π.

   (a) 15°

   (b) 18°

   (c) 75°

   (d) 225°

4. Find the value of each of the following, where all the angles are in radians.

   (a) $\sin 0.8$

   (b) $\cos 0.543$

   (c) $\tan 1.5$

   (d) $\sin \dfrac{\square}{8}$
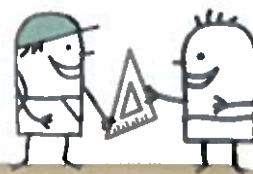
   (e) $\cos 0.45\pi$

   (f) $\tan \dfrac{2\square}{5}$

5. For each of the following, find the value of x in the range $0 < x < \frac{\pi}{2}$.

(a) $\sin x = 0.74$ **(b)** $\cos x = 0.17$

(c) $\tan x = 0.48$ **(d)** $\sin x = 0.147$

(e) $\cos x = 0.769$ **(f)** $\tan x = 1.256$

6. In $\triangle ABC$, $\angle ABC = \frac{\pi}{2}$. Given that $AB = 8.7$ cm and $\angle CAB = 0.93$ radians, calculate the length of

(i) BC,   **(ii)** AC.

7. A point C on level ground is 12.7 m away from the foot B of a tree AB.

Given that $\angle BAC = 1.08$ radians, find

(i) the height of the tree,

(ii) the distance between A and C.

8. The figure shows the cross section of a piece of cake PQR, in which $PR = 16.8$ cm, $\angle QPR = 0.98$ radians and $\angle PQR = \frac{\pi}{2}$. The edges PQ and QR are to be sprinkled with chocolate rice and icing sugar respectively.

Find the length of the cake which is

(i) to be sprinkled with chocolate rice,

(ii) to be sprinkled with icing sugar.

9. At Large Splash, there is a swimming pool with 2 water slides SP and SQ.

Given that the points P, Q and R lie on the surface of the swimming pool such that $SP = 13.9$ m, $SR = 7.4$ m and $\angle QSR = 0.85$ radians, find

(i) $\angle PSQ$, the angle between the slides,

(ii) the length of slide SQ,

(iii) PQ, the distance between the bottom of both slides.

10. Two points P and Q, 11.9 m apart on level ground, are due east of the foot A of a tree TA.

Given that $\angle TPA = 0.72$ radians and $\angle TQA = 1.3$ radians, find the height of the tree.

10.4 Arc Length and Area of Sector using Radian Measure

From the investigation in Section 10.1 on the formula for arc length, we have discovered that

$$\frac{\text{Length of arc } AB}{\text{Circumference}} = \frac{\angle AOB}{360°}$$

i.e. Length of arc $AB = \dfrac{x°}{360°} \times$ Circumference

However, if $\angle AOB = \theta$ is measured in radians, then

$$\frac{\text{Length of arc } AB}{\text{Circumference}} = \frac{\theta}{2\pi}$$

i.e. Length of arc $AB = \dfrac{\theta}{2\pi} \times$ Circumference

$$= \frac{\theta}{2\pi} \times 2\pi r$$

$$= r\theta$$

In general,
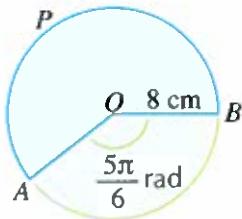
$$\boxed{s = r\theta,}$$

where s is the length of an arc of a circle of radius r units and θ, in radians, is the angle subtended by the arc at the centre of the circle.

Worked Example 13

(Finding Arc Length using Radian Measure)

In the figure, O is the centre of a circle of radius 8 cm and $\angle AOB = \dfrac{5\pi}{6}$ radians.

Find the length of the major arc APB.

Solution:

Reflex $\angle AOB = 2\pi - \dfrac{5\pi}{6}$ $(360° = 2\pi)$

$\qquad\qquad = \dfrac{7\pi}{6}$ radians

Length of major arc $APB = 8 \times \dfrac{7\pi}{6}$ (using $s = r\theta$, θ in radians)

$\qquad\qquad\qquad\qquad = \dfrac{28\pi}{3}$

$\qquad\qquad\qquad\qquad = 29.3$ cm (to 3 s.f.)
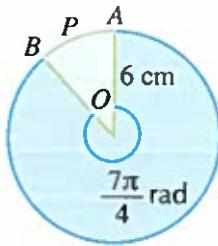
PRACTISE NOW 13

In the figure, O is the centre of a circle of radius 6 cm and reflex $\angle AOB = \dfrac{7\pi}{4}$ radians. Find the length of the minor arc APB.

From the investigation in Section 8.2 on the formula for area of a sector, we have discovered that

$$\frac{\text{Area of sector } OAB}{\text{Area of circle}} = \frac{\angle AOB}{360°}$$

$$\text{i.e. Area of sector } OAB = \frac{x°}{360°} \times \text{Area of circle}$$

However, if $\angle AOB = \theta$ is measured in radians, then

$$\frac{\text{Area of sector } OAB}{\text{Area of circle}} = \frac{\theta}{2\pi}$$

$$\text{i.e. Area of sector } OAB = \frac{\theta}{2\pi} \times \text{Area of circle}$$

$$= \frac{\theta}{2\pi} \times \pi r^2$$

$$= \frac{1}{2} r^2 \theta$$

In general,

$$\boxed{A = \frac{1}{2} r^2 \theta,}$$
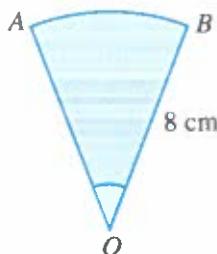
where A is the area of a sector of a circle of radius r units and θ, in radians, is the angle subtended by the arc at the centre of the circle.

Worked Example 14 (Finding Area of a Sector using Radian Measure)

The figure shows a sector OAB of radius 8 cm and a perimeter of 23 cm. Find

(i) $\angle AOB$ in radians,

(ii) the area of the sector OAB.

Solution:

(i) Given that the perimeter = 23 cm,

$$r + r + s = 23$$
$$8 + 8 + s = 23$$
$$16 + s = 23$$
$$s = 23 - 16$$
$$= 7 \text{ cm}$$

Using $s = r\theta$,

$$7 = 8\theta$$
$$\theta = \frac{7}{8} \text{ radians}$$

$$\therefore \angle AOB = \frac{7}{8} \text{ radians}$$

(ii) Area of sector OAB
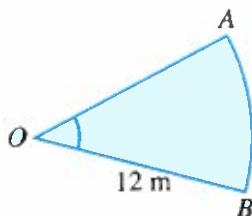
$$= \frac{1}{2} \times 8^2 \times \frac{7}{8}$$
$$= 28 \text{ cm}^2$$

PRACTISE NOW 14

1. The figure shows a sector OAB of radius 12 m and a perimeter of 33 m.
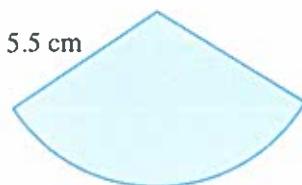
Find

(i) $\angle AOB$ in radians,

(ii) the area of the sector OAB.

2. A circle of radius 5.5 cm has a sector with an area of 30.25 cm².
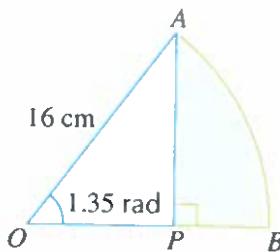
Calculate the perimeter of this sector.

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In the figure, AB is an arc of a circle centre O and radius 16 cm. P is a point on OB such that AP is perpendicular to OB and $\angle AOB = 1.35$ radians.

(i) Show that OP is approximately 3.50 cm.
(ii) Find the length of AP.
(iii) Find the area of the shaded region.

Solution:

(i) In $\triangle AOP$,

$$\cos 1.35 = \frac{OP}{16}$$
$$\therefore OP = 16 \cos 1.35$$
$$= 3.50 \text{ cm (to 3 s.f.)}$$

(ii) In $\triangle AOP$,

$$\sin 1.35 = \frac{AP}{16}$$
$$\therefore AP = 16 \sin 1.35$$
$$= 15.6 \text{ cm (to 3 s.f.)}$$

(iii) Area of shaded region = area of sector OAB – area of $\triangle AOP$
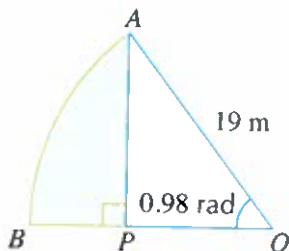
$$= \frac{1}{2} \times 16^2 \times 1.35 - \frac{1}{2} \times 3.504 \times 15.61$$
$$= 145 \text{ cm}^2$$

PRACTISE NOW 15

In the figure, AB is an arc of a circle centre O and radius 19 m. P is a point on OB such that AP is perpendicular to OB and $\angle AOB = 0.98$ radians.
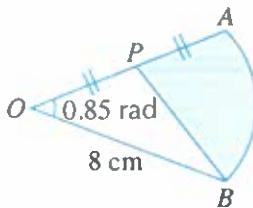
(i) Show that OP is approximately 10.6 m.
(ii) Find the length of AP.
(iii) Find the area of the shaded region.

Worked Example 16

(Problem involving Area of a Triangle)

In the figure, OAB is a sector of a circle with centre O and radius 8 cm.

P is the midpoint of OA and $\angle AOB = 0.85$ radians. Find

(i) the length of the arc AB,

(ii) the area of the shaded region.

Solution:

(i) Length of arc $AB = r\theta$

$$= 0.85 \times 8$$
$$= 6.8 \text{ cm}$$

(ii) Area of sector $OAB = \dfrac{1}{2}r^2\theta$

$$= \dfrac{1}{2} \times 8^2 \times 0.85$$
$$= 27.2 \text{ cm}^2$$

Area of $\triangle OPB = \dfrac{1}{2} \times OP \times OB \times \sin \angle POB$ (using formula Area $= \dfrac{1}{2}ab\sin C$)

$$= \dfrac{1}{2} \times 4 \times 8 \times \sin 0.85 \quad (OP = \dfrac{1}{2}OB)$$
$$= 12.02 \text{ cm}^2 \text{ (to 4 s.f.)}$$

$\therefore$ Area of shaded region = Area of sector OAB – area of $\triangle OPB$

$$= 27.2 - 12.02$$
$$= 15.2 \text{ cm}^2 \text{ (to 3 s.f.)}$$

PRACTISE NOW 16

In the figure, OAB is a sector of a circle with centre O and radius 12 cm. K is a point on OB such that $KB = 5.5$ cm and $\angle AOB = 1.08$ radians. Find

(i) the length of the arc AB,

(ii) the area of the shaded region.

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Worked Example 17

(Finding Area of Segment using Radian Measure)

The figure shows the major segment ABC representing the cross section of a train tunnel, centre O and radius 3.6 m, such that $\angle AOC = 2.2$ radians.

(i) Show that AC is approximately 6.42 m.

(ii) Find the length of the major arc ABC, giving your answer in terms of π.

(iii) Find the area of the cross section of the tunnel, i.e. area of major segment ABC.

Solution:

(i) Using cosine rule,

$$AC^2 = OA^2 + OC^2 - 2(OA)(OC) \cos \angle AOC$$
$$= 3.6^2 + 3.6^2 - 2(3.6)(3.6) \cos 2.2$$
$$= 41.17 \text{ (to 4 s.f.)}$$
$$AC = 6.42 \text{ m (to 3 s.f.)}$$

(ii) Reflex $\angle AOC = (2\pi - 2.2)$ radians

$\therefore$ Length of major arc $ABC = 3.6 \times (2\pi - 2.2)$
$$= (7.2\pi - 7.92) \text{ m}$$

RECALL

Area of triangle $= \dfrac{1}{2} ab \sin C$

(iii) Area of major segment ABC = Area of major sector $OABC$ + area of $\triangle OAC$
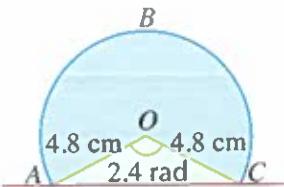
$$= \frac{1}{2} \times 3.6^2 \times (2\pi - 2.2) + \frac{1}{2} \times 3.6^2 \times \sin 2.2$$
$$= 31.7 \text{ m}^2 \text{ (to 3 s.f.)}$$

$\therefore$ Area of the cross section of the tunnel $= 31.7$ m^2

The figure shows the major segment *ABC* representing the cross section of a tunnel in a children's toy set, centre *O* and radius 4.8 cm, such that $\angle AOC = 2.4$ radians.
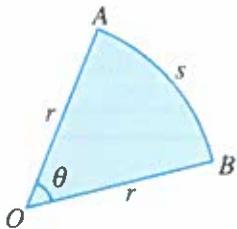
(i)   Show that *AC* is approximately 8.95 cm.

(ii) Find the length of the major arc *ABC*, giving your answer in terms of π.

(iii) Find the area of the cross section of the tunnel, i.e. area of major segment *ABC*.

Exercise 10D
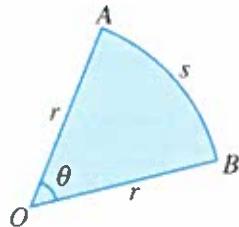
BASIC LEVEL

1. For each of the following sectors, find the arc length.
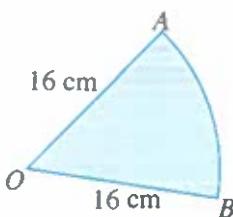
(a) $r = 6$ cm, $\theta = 1.6$ rad

(b) $r = 14$ cm, $\theta = 0.25$ rad

(c) $r = 25$ m, $\theta = 1.75$ rad

(d) $r = 12$ mm, $\theta = \dfrac{3}{4}$ rad

2. For each of the following sectors, find the area.

(a) $r = 8$ cm, $\theta = 2.2$ rad

(b) $r = 17$ cm, $\theta = 0.46$ rad

(c) $r = 33$ m, $\theta = \dfrac{1}{5}$ rad

(d) $r = 94$ mm, $\theta = 0.6$ rad

3. The figure shows a sector OAB of radius 16 cm and a perimeter of 50 cm.
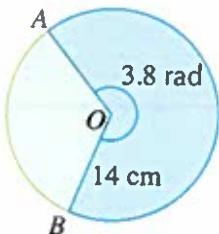
Find

(i) $\angle AOB$ in radians,

(ii) the area of the sector OAB.

4. Copy and complete the table for sectors of a circle.

	Radius	Angle at centre	Arc length	Area
(a)	4 cm	1.25 rad		
(b)	6 cm		9 cm	
(c)		0.8 rad	9.6 m	
(d)		1.2 rad		60 m²
(e)	8 mm			64 mm²
(f)			6 mm	27 mm²

5. OAB is a major sector of a circle centre O and radius 14 cm.
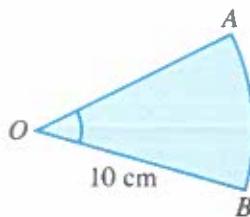
Given that reflex $\angle AOB = 3.8$ radians, find

(i) the perimeter of major sector OAB,

(ii) the area of the minor sector OAB.
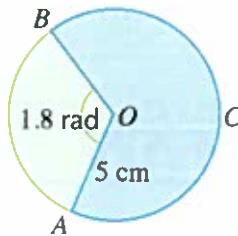
6. Given that a sector OAB of radius 10 cm has an area of 60 cm², find the perimeter of the sector.

7. Given that a sector of radius 18 m has an area of 729 m², find the perimeter of the sector.
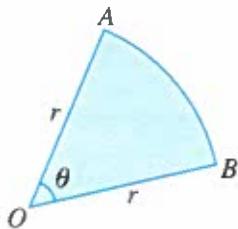
8. A, B and C are points on a circle with centre O and radius 5 cm, with obtuse $\angle AOB = 1.8$ radians.
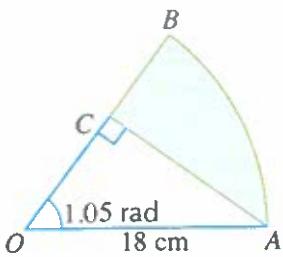
(i) Find the area of minor sector AOB.

(ii) Write down an expression, in terms of π, for the size of reflex $\angle AOB$.

(iii) Find the length of major arc ACB, giving your answer in the form $a + b\pi$, where a and b are rational numbers.

9. Given that the perimeter of the sector AOB is 18 cm and that the area is 8 cm²,

(i) form a pair of simultaneous equations involving r and θ,

(ii) find the value of r and of θ.
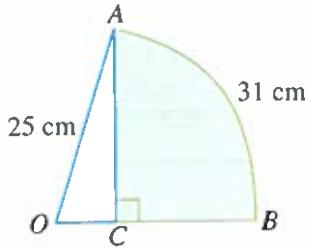
10. In the figure, AB is an arc of a circle centre O and radius 18 cm. C is a point on OB such that AC is perpendicular to OB and $\angle AOB = 1.05$ radians.

Find
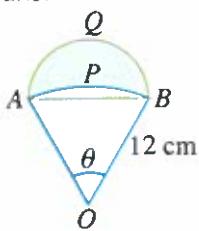
(i) the length of AC,

(ii) the area of the shaded region.

11. In the figure, OAB is a sector of a circle centre O and radius 25 cm. AC is perpendicular to OB and the length of minor arc AB is 31 cm.
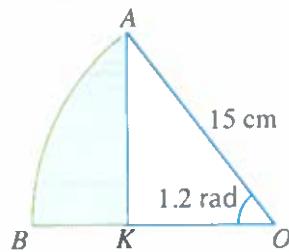
Find the area of the shaded region.

12. In the figure, APB is an arc of a circle centre O and radius 12 cm. ABQ is a semicircle with AB as diameter. $\triangle AOB$ is an equilateral triangle and $\angle AOB = \theta$ radians.

(i) Write down the value of θ.

(ii) Find the length of the arc APB.

(iii) Find the area of the shaded region.
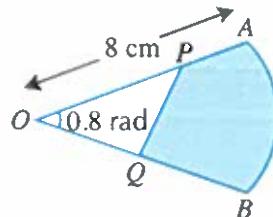
13. In the figure, OAB is a sector of a circle with centre O and radius 15 cm. K is a point on OB such that $2OK = 3BK$ and $\angle AOB = 1.2$ radians.

Find

(i) the length of the arc AB,

(ii) the area of the shaded region.
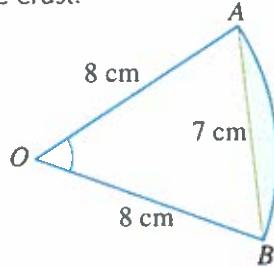
14. The figure shows a sector of a circle OAB, with centre O and radius 8 cm. The point P lies on OA such that $\dfrac{OP}{PA} = \dfrac{3}{2}$ and the point Q lies on OB such that $OQ : QB = 3 : 4$.

Given that $\angle AOB = 0.8$ radians, find

(i) the length of the arc AB,

(ii) the area of the shaded region.
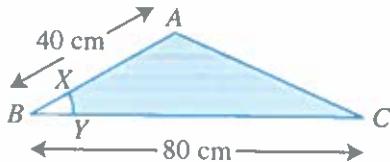
15. The figure shows a slice of an apple pie in the shape of a sector of a circle with centre O and radius 8 cm. A cut is made along AB to remove some of the crust.

Given that $AB = 7$ cm, find

(i) the maximum number of slices that can be obtained from one full pie,

(ii) the area of the shaded segment that has been removed.
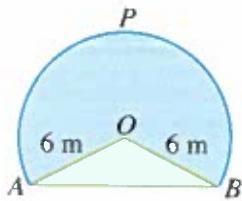
16. In $\triangle ABC$, $AB = 40$ cm, $BC = 80$ cm and $\angle ABC = \dfrac{\pi}{6}$. XY is an arc of a circle centre B and radius 10 cm.

Find

(i) the area of $\triangle ABC$,

(ii) the area of the shaded region.
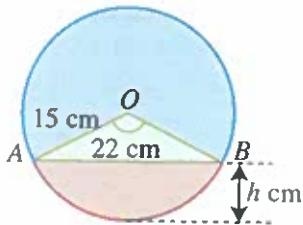
17. The figure shows the major segment APB centre O and radius 6 m.

Given that the length of the major arc APB is 24 m, find

(i) reflex $\angle AOB$ in radians,

(ii) the length of AB,

(iii) the area of the major segment APB.
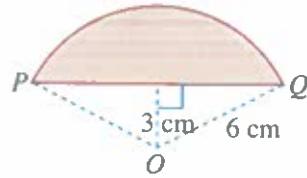
18. The figure shows the cross section of a water pipe of radius 15 cm. Water flowing in the pipe has a height of h cm and a horizontal width of 22 cm.

Find

(i) the value of h,

(ii) the area of the cross section which contains water.

19. The figure shows a minor segment of a circle centre O and radius 6 cm. PQ is a chord of the circle such that its distance from O is 3 cm.
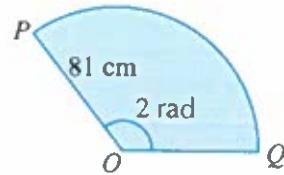
Find

(i) obtuse $\angle POQ$ in radians,

(ii) the area of the shaded segment.
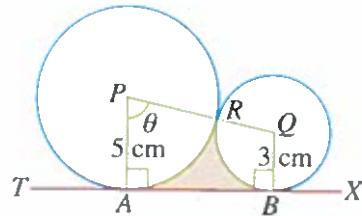
20. An arc PQ of a circle, centre O, subtends an angle of 2 radians at the centre.

This sector is folded to form a right circular cone so that the arc PQ becomes the circumference of the base. Find the height of the cone.
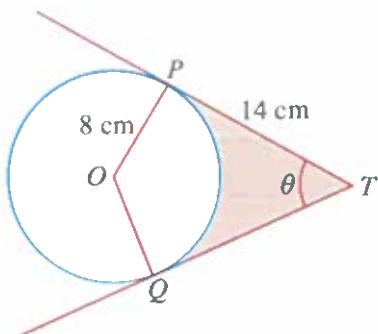
21. Two circles with centres P and Q touch each other externally at R. $TABX$ is a common tangent to the two circles where $\angle BAP = \angle ABQ = \dfrac{\pi}{2}$ radians.

Given that the radii of the circles are 5 cm and 3 cm respectively and $\angle APQ = \theta$ radians, find
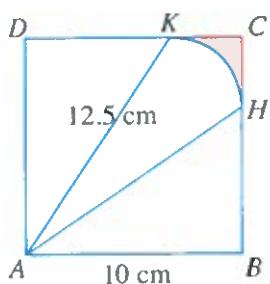
(i) the value of θ,

(ii) the area of the shaded region.

22. In the figure, O is the centre of the circle whose radius is 8 cm. TP and TQ are tangents to the circle such that $\angle OPT = \angle OQT = \frac{\pi}{2}$ radians.

Given that $TP = 14$ cm and $\angle PTQ = \theta$ radians, find

(i) the value of θ,

(ii) the area of the shaded region.
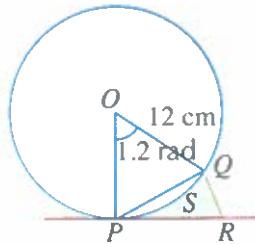
23. In the figure, $ABCD$ is a square of side 10 cm.

Given that HK is an arc of a circle centre A and radius AK where $AK = 12.5$ cm, find

(i) $\angle HAK$ in radians,

(ii) the area of the shaded region HCK.
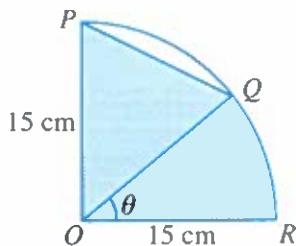
24. The figure shows a circle centre O and radius 12 cm.

Given that $\angle POQ = 1.2$ radians and PR is a tangent to the circle where $\angle OPR = \frac{\pi}{2}$ radians and the length of PR is equal to that of the arc PSQ, find

(i) the length of the arc PSQ,

(ii) the area of the segment PSQ,

(iii) the length of the chord PQ,

(iv) the area of the shaded region $PSQR$.

25. The figure shows a quadrant of a circle centre O and radius 15 cm. Q is a point on the arc such that the lengths of arcs PQ and QR are in the ratio $4 : 3$.

Given that $\angle QOR = \theta$ radians, find

(i) the value of θ in terms of π,

(ii) the area of the shaded region.
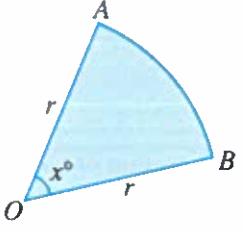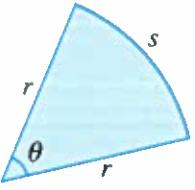
1. A radian is the size of the angle subtended at the centre of a circle by an arc equal in length to the radius of the circle.

$$\pi \text{ radians} = 180°$$
$$1 \text{ radian} = \frac{180°}{\pi}$$
$$1° = \frac{\pi}{180°} \text{ radians}$$
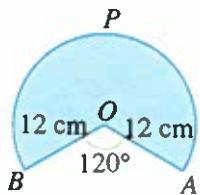
2. To find the arc length and area of a sector:

For a sector subtending an angle $x°$ at the centre of a circle of radius r:	For a sector of a circle of radius r enclosed by two radii that subtend an angle of θ radians at the centre:
Arc length $AB = \dfrac{x°}{360°} \times 2\pi r$ Area of sector $OAB = \dfrac{x°}{360°} \times \pi r^2$	Arc length, $s = r\theta$ Area of sector, $A = \dfrac{1}{2}r^2\theta$

Review Exercise 10
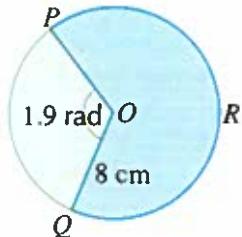
1. In the figure, **BPA** is the major arc of a circle centre O and radius 12 cm.

Given that $\angle BOA = 120°$, find
(i) the length of arc **BPA**,
(ii) the area of the sector **OBPA**.
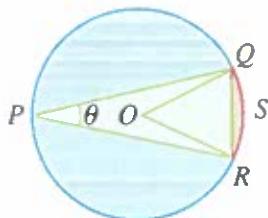
2. The figure shows a circle with centre O, radius 8 cm and $\angle POQ = 1.9$ radians.

Find
(i) the area of the minor sector **POQ**,
(ii) reflex $\angle POQ$ in terms of π,
(iii) the length of the major arc **PRQ** in terms of π.

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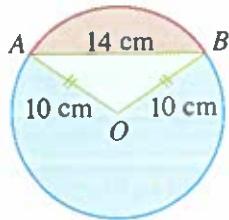
3. In the figure, O is the centre of a circle of radius 8 cm and $\triangle PQR$ is an isosceles triangle.

Given that $\angle QPR = \theta$ radians and $QR = 7$ cm, find

(i) the value of θ,

(ii) the area of the minor segment QSR.
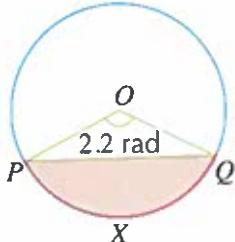
4. A log cake baked for a Christmas party has a circular cross section with centre O and radius 10 cm. The top section of the cake is sliced off along AB and is to be replaced with cake toppings.

Given that $AB = 14$ cm, find
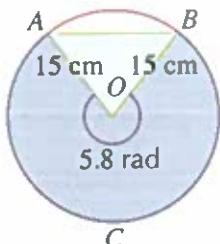
(i) $\angle AOB$ in radians,

(ii) the cross-sectional area of the cake that is sliced off.

5. The figure shows the cross section of a drainage pipe. The shaded minor segment PQX of the circle centre O, represents the part of the pipe which contains mud.
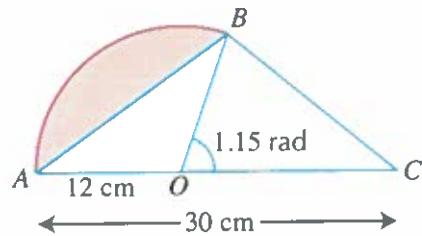
Given that $\angle POQ = 2.2$ radians, express the area of the segment as a percentage of the area of the circle, giving your answer correct to the nearest integer.

6. A dart board is in the shape of a circle with centre O and radius 15 cm. $\triangle OAB$ is shaded green such that $OA = OB = 15$ cm. The major segment $OACB$ is shaded purple such that reflex $\angle AOB = 5.8$ radians.

(i) Show that the area of $\triangle OAB$ is approximately 52.3 cm².

(ii) Given that the dart lands on the board, find the probability that it lands on the unshaded area, giving your answer in standard form.
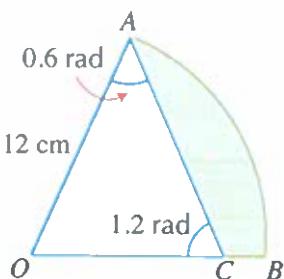
7. In the figure, OAB is a sector of a circle with centre O and radius 12 cm. AOC is a straight line such that $AC = 30$ cm and $\angle BOC = 1.15$ radians.

Calculate

(i) the length of the arc AB,

(ii) the area of the shaded segment,

(iii) the area of $\triangle BOC$,

(iv) the length of BC.
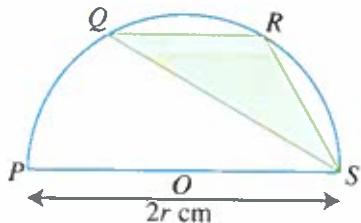
8. In the figure, *OAB* is a sector of a circle with centre *O* and radius 12 cm. *C* is a point on radius *OB* such that $\angle OAC = 0.6$ radians and $\angle OCA = 1.2$ radians.

A
0.6 rad
12 cm
1.2 rad
O   C   B
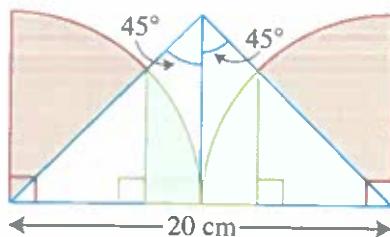
Find

(i) the length of *BC* and of *AC*,

(ii) the area of the shaded region.
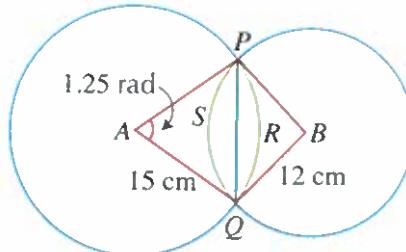
9. In the figure, *PQRS* is a semicircle with centre at *O*. Given that $PS = 2r$ cm and the lengths of arcs *PQ*, *QR* and *RS* are equal, calculate the area of the shaded region, giving your answer in terms of *r*.

Q   R

P   O   S
        2r cm

10. Find the total area of the shaded region in the figure below, giving your answer in terms of ϖ.

45°   45°

20 cm

11. The points *P*, *R* and *Q* lie on the circumference of the circle with centre *A* and radius 15 cm such that $\angle PAQ = 1.25$ radians.

P
1.25 rad
A   S   R   B
15 cm   12 cm
Q

(a) Find

(i) the length of the chord *PQ*,

(ii) the area of the minor sector *APQ*,

(iii) the area of $\triangle PAQ$.

A second circle with centre *B* and radius 12 cm passes through the points *P*, *S* and *Q*.
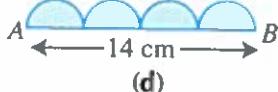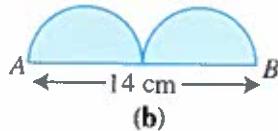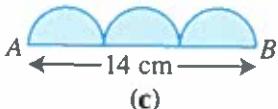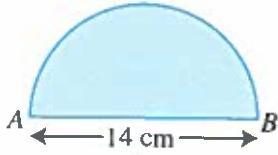
(b) Calculate the area of the shaded region.

Challenge Yourself
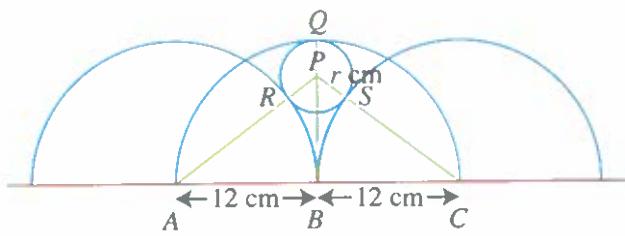
1. The figures below show various semicircles.

In each figure, all the semicircles are of the same size and $AB = 14$ cm.

A ←— 14 cm —→ B
(a)

A ←— 14 cm —→ B
(b)

A ←— 14 cm —→ B
(c)

A ←— 14 cm —→ B
(d)

Find the perimeter of each figure, leaving your answer in terms of π. What do you observe? Can you generalise your observations?
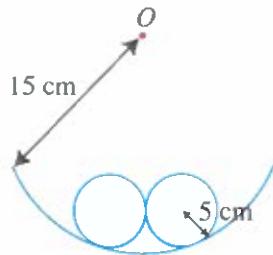
2. The figure shows three semicircles each of radius 12 cm with centres at A, B and C in a straight line. A fourth circle with a centre at P and radius r cm is drawn to touch the other three semicircles.

Given that BPQ is a straight line, find
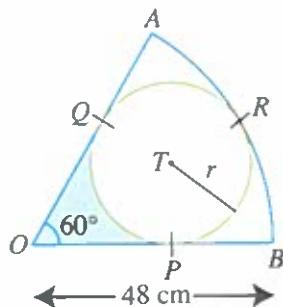
(i) the value of r,

(ii) $\angle PAC$ in radians,

(iii) the area of the shaded region.

3. The figure shows two identical circles of radius 5 cm touching each other externally. The two circles also touch a larger circle, centre O and radius 15 cm internally.

(i) Show that the perimeter of the shaded region is $\dfrac{35\pi}{3}$ cm.

(ii) Find the area of the shaded region.

4. In the figure, the circle PQR with centre T and with radius r is enclosed in the sector OAB with centre at O and radius 48 cm.
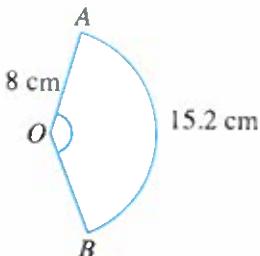
Given that $\angle AOB = 60°$ and $\angle OPT = \angle OQT = \dfrac{\pi}{2}$ radians, find

(i) the value of r,

(ii) the area of the shaded region.

C1 Revision Exercise

1. Given that $0 \leqslant x \leqslant 2\pi$, find the values of x for which $\sin x = 0.345$.
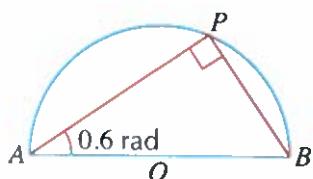
2. In the figure, OAB is a sector of a circle of radius 8 cm and the length of arc $AB = 15.2$ cm.

   Find
   (i) $\angle AOB$ in radians,
   (ii) the area of the sector.
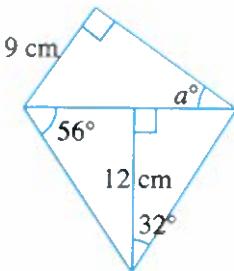
3. The figure shows a semicircle $AOBP$ where O is the centre and $\triangle APB$ is right-angled at P.
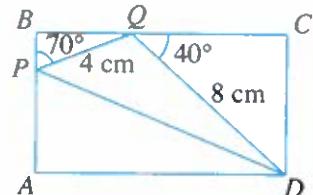
   Given that $AB = 16$ cm and $\angle PAB = 0.6$ radians, find
   (i) the length of AP,
   (ii) the area of $\triangle PAB$,
   (iii) the area of the shaded region.

4. Find the value of a in the figure.
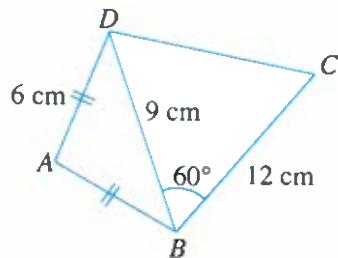
5. In the figure, $ABCD$ is a rectangle. The points P and Q lie on AB and BC respectively such that $PQ = 4$ cm, $QD = 8$ cm, $\angle BPQ = 70°$ and $\angle CQD = 40°$.

   Find
   (i) the length of CD,
   (ii) $\angle APD$,
   (iii) the length of AD.

6. The figure shows a quadrilateral $ABCD$ in which $AB = AD = 6$ cm, $BD = 9$ cm, $BC = 12$ cm and $\angle CBD = 60°$.
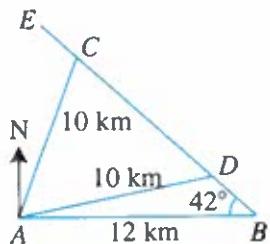
   Find
   (i) $\angle BAD$,
   (ii) the area of $\triangle ABD$,
   (iii) the area of the quadrilateral $ABCD$,
   (iv) $\angle ABC$,
   (v) the length of DC.
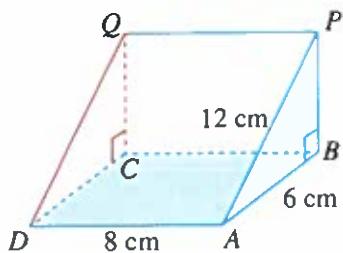
7. The figure shows a point B which lies 12 km due east of A. A straight road BE makes an angle of $42°$ with AB. C and D are two points on the road such that $AD = AC = 10$ km.

Find
   (i)   the bearing of C from A,
   (ii) the bearing of D from A,
   (iii) the distance between C and D.

8. The diagram shows a right triangular prism with $\angle ABP = 90°$ and $ABCD$ lying on a horizontal table.
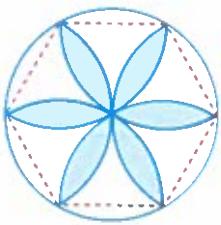
Given that $AB = 6$ cm, $AD = 8$ cm and $AP = 12$ cm, find
   (i) $\angle PAB$,
   (ii) the length of PB,
   (iii) $\angle PDB$.

9. From the top T of an observation tower, which is 20 m high, the angle of depression of a ship, A, which is due east of T, is $25.4°$. Another ship, B, which is due west of T, finds the angle of elevation of T to be $54.7°$. Calculate the distance between A and B.
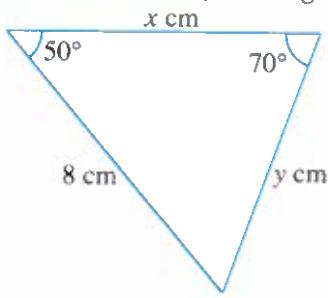
C2 Revision Exercise

1. Given that $0 \leqslant x \leqslant 3$, find the value of x for which $\cos x = -0.5$.

2. Given that $\sin 40° = 0.643$ and $\cos 15° = 0.966$ when corrected to 3 significant figures, find the value of each of the following without the use of a calculator.
   (a) $\sin 140°$ (b) $\cos 165°$

3. In the figure, the radius of the circle and of each arc is 4 cm.
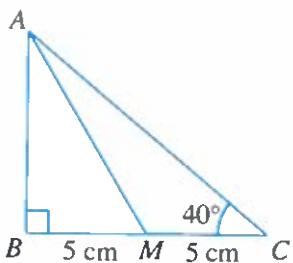
   Find the area of the shaded region.

4. Find the value of x and of y in the given triangle.
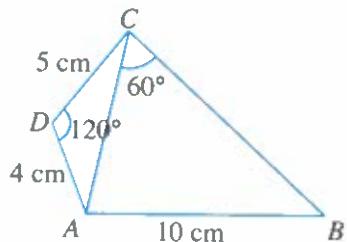
5. In the figure, $\triangle ABC$ is right-angled at B and M is the midpoint of BC.

   Given that $BM = MC = 5$ cm and $\angle ACB = 40°$, find
   (i) the length of AM,
   (ii) the length of AC,
   (iii) $\angle AMB$,
   (iv) $\angle CAM$.
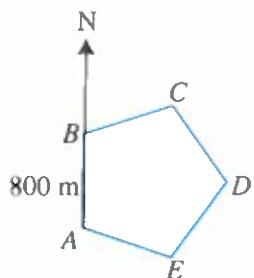
6. The figure shows a quadrilateral $ABCD$ in which $AD = 4$ cm, $DC = 5$ cm, $AB = 10$ cm, $\angle ADC = 120°$ and $\angle ACB = 60°$.

   Find
   (i) the length of AC,
   (ii) $\angle DAC$,
   (iii) $\angle ABC$
   (iv) the length of BD,
   (v) the area of the quadrilateral $ABCD$.
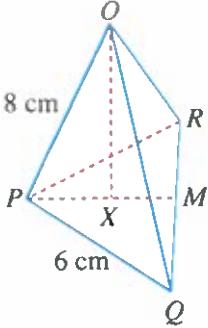
7. During a junior sailing competition, the course of the race is in the form of a regular pentagon $ABCDE$, where B is due north of A and $AB = 800$ m.
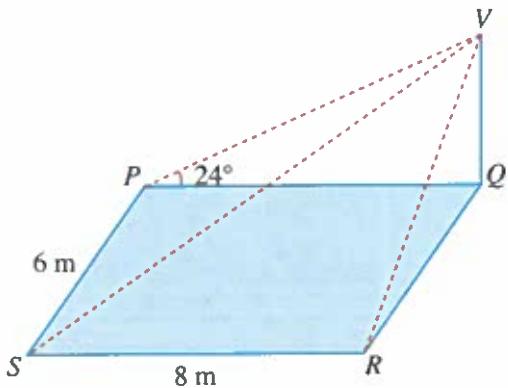
   (a) Find the bearing of
       (i) C from B,
       (ii) E from A,
       (iii) D from E.
   (b) Calculate the area of $ABCDE$, giving your answer in standard form correct to 4 significant figures.

8. *OPQR* is a triangular pyramid. *M* is the midpoint of *QR*, *PQ* = *QR* = *PR* = 6 cm and *OP* = *OQ* = *OR* = 8 cm.
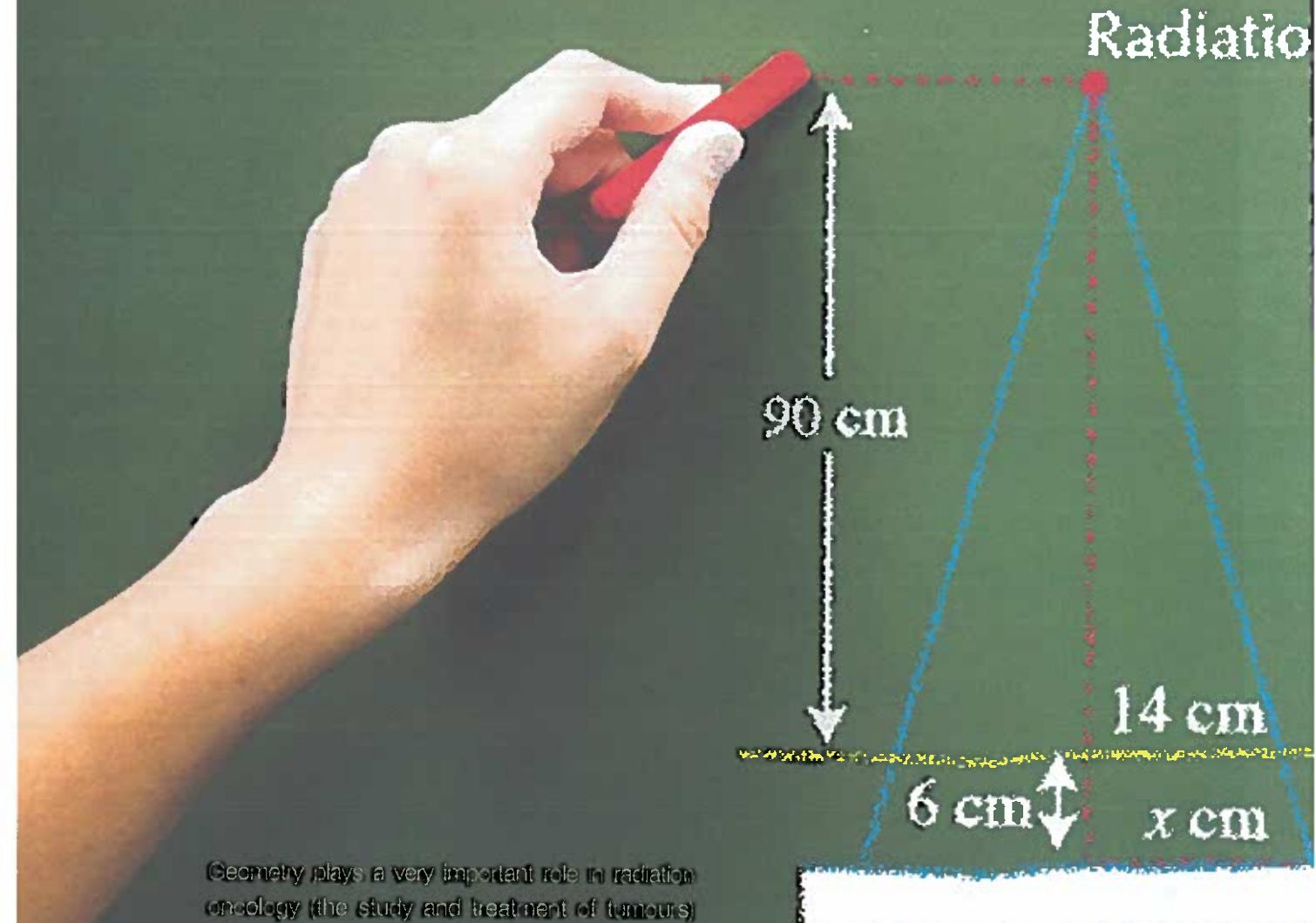
(a) Find
  (i) the height *OX* of the pyramid,
  (ii) *OPX*.
(b) Given that *PM* : *XM* = 2 : 1, find ∠*OMX*.

9. A vertical pole *VQ* stands at one corner of a horizontal rectangular field.

Given that *SR* = 8 m, *PS* = 6 m and the angle of elevation of *V* from *P* is 24°, find
(i) the distance between *V* and *Q*,
(ii) the distance between *V* and *S*,
(iii) the angle between *VS* and *VQ*,
(iv) the angle of elevation of *V* from *R*.

Congruence and Similarity Tests

90 cm

Radiatio

14 cm

6 cm   x cm
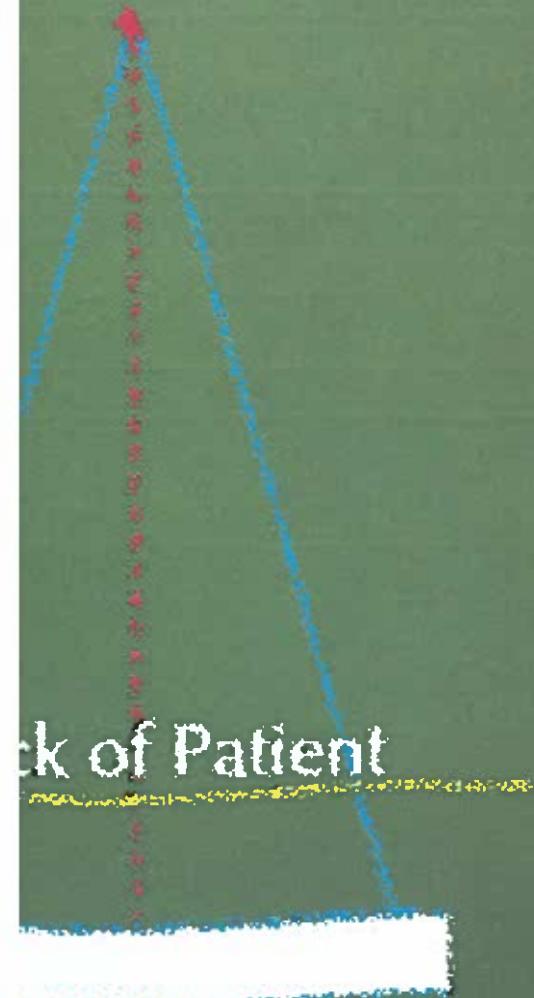
Spinal

Geometry plays a very important role in radiation oncology (the study and treatment of tumours) when determining the safe level of radiation to be administered to spinal cords of cancer patients. The picture shows how far apart two beams of radiation must be placed so that they will not overlap at the spinal cord, or else a double dose of radiation will endanger the patient.

Chapter
Eleven

urce S

ck of Patient

d
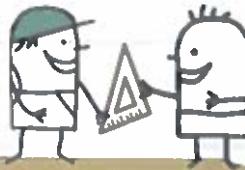
At the end of this chapter, you should be able to:
- apply the four congruence tests to determine whether two or more triangles are congruent.
- apply the three similarity tests to determine whether two or more triangles are similar.
- solve problems involving congruent and/or similar triangles.

11.1 Congruence Tests

Recap

In Book 2, we have learnt that congruent figures have exactly the same shape and size; and they can be mapped onto one another under translation, rotation and reflection.

For congruent triangles, this would mean that all the corresponding lengths are equal and all the corresponding angles are equal.

For example, $\triangle ABC$ is congruent to $\triangle XYZ$ (and we write $\triangle ABC \equiv \triangle XYZ$) if and only if

$AB = XY$

$BC = YZ$

$AC = XZ$

$B\hat{A}C = Y\hat{X}Z$
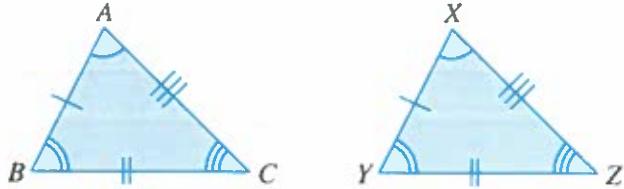
$A\hat{B}C = X\hat{Y}Z$

$A\hat{C}B = X\hat{Z}Y$

Fig. 11.1

Consider $\triangle EFG$, where

$EF = AB$

$FG = BC$

$EG = AC$

$F\hat{E}G = B\hat{A}C$

$E\hat{F}G = A\hat{B}C$

$E\hat{G}F = A\hat{C}B$
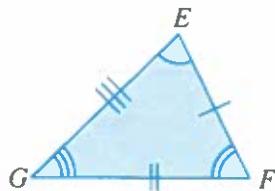
Fig. 11.2

$\triangle EFG$ is still congruent to $\triangle ABC$ even though $\triangle EFG$ is laterally inverted, because $\triangle EFG$ can be mapped onto $\triangle ABC$ by a reflection (and a translation if necessary).

In this section, we will investigate whether we need all the 6 conditions (i.e. 3 pairs of corresponding sides are equal and 3 pairs of corresponding angles are equal) to prove that two triangles are congruent.

⠿ SSS Congruence Test
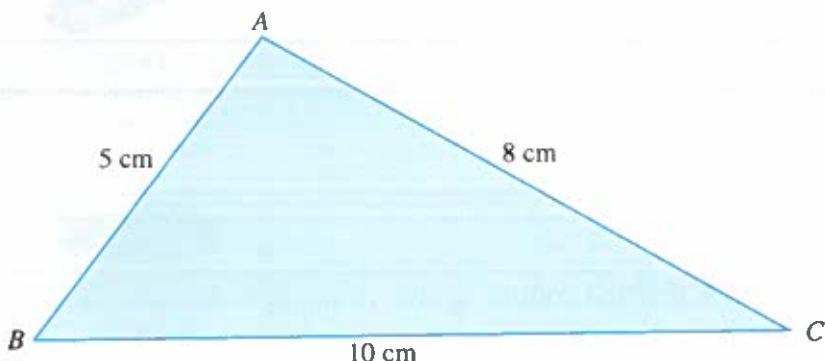
⬭ Investigation

SSS Congruence Test

1. Cut 3 satay sticks so that the lengths of the sticks are 5 cm, 8 cm and 10 cm.

2. Try to form a triangle using the 3 satay sticks in as many ways as possible, such that the lengths of the triangle correspond to the lengths of the 3 sticks.

   (If you do not have satay sticks, you can try to construct the triangle using the three given lengths in as many ways as possible.)

3. Do you get the following triangle?

   If you get a different triangle, flip it over and see if it fits onto this triangle.

   Is it possible to get other triangles?

Fig. 11.3

> **ATTENTION**
>
> If the sum of the lengths of the two shorter sticks is less than or equal to the length of the longest stick, then you cannot form a triangle.

4. Try using 3 other satay sticks of different lengths, such that the sum of the lengths of the two shorter sticks is greater than the length of the longest stick, and see if you always get a unique triangle (regardless of reflection) no matter how you try to form a triangle.

5. What can you conclude from this investigation?
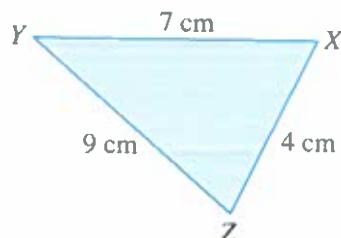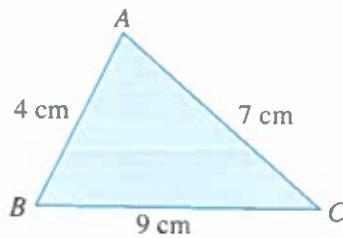
From the investigation, we observe the following:

> **SSS Congruence Test**: If the 3 sides of a triangle are equal to the 3 corresponding sides of another triangle, then the two triangles are congruent.

Worked Example 1

(Proving that Two Triangles are Congruent using the SSS Congruence Test)

Prove that the following two triangles are congruent.

Solution:

$A \leftrightarrow X$

$B \leftrightarrow Z$

$C \leftrightarrow Y$

$AB = XZ = 4$ cm

$BC = ZY = 9$ cm

$AC = XY = 7$ cm

$\therefore \triangle ABC \equiv \triangle XZY$ (SSS)

PRACTISE NOW 1

1. Copy and complete the proof to show that the following two triangles are congruent.

Solution:

$A \leftrightarrow \underline{\quad}$

$B \leftrightarrow \underline{\quad}$

$C \leftrightarrow \underline{\quad}$

$AB = \underline{\quad} = 5$ m

$BC = \underline{\quad} = \underline{\quad}$ m

$AC = \underline{\quad}$ (given)

$\therefore \triangle ABC \equiv \triangle \underline{\quad}$ ($\underline{\quad}$)

2. The diagram shows a kite *WXYZ*.

Identify the two congruent triangles in the kite and prove that they are congruent.

⠿ SAS Congruence Test

Investigation

SAS Congruence Test

Part 1

1. Try to construct △*XYZ* such that *XY* = 3 cm, *YZ* = 6 cm and $X\hat{Y}Z$ = 50° in as many ways as possible.
   Compare the triangle you have drawn with those drawn by your classmates.

2. Do you get the following triangles (not drawn to scale)?
   Both triangles are actually congruent to each other: you can map both triangles together by a reflection.
   Is it possible to get other triangles?
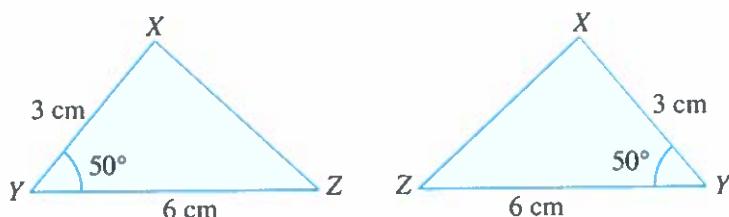
Fig. 11.4

3. Try to construct △*XYZ* for other dimensions, where *XY* and *YZ* have a fixed length and $X\hat{Y}Z$ is a fixed angle, in as many ways as possible and see if you always get a unique triangle (regardless of reflection).

4. Notice that the given $X\hat{Y}Z$ is between the two given sides *XY* and *YZ*: $X\hat{Y}Z$ is called the **included angle**.

5. What can you conclude from part 1 of this investigation?

Part 2

6. Try to construct △*ABC* such that *AB* = 5 cm, *AC* = 3 cm and *AB̂C* = 30° in as many ways as possible.

7. Fig. 11.5 shows one possible △*ABC* that satisfies the given dimensions.
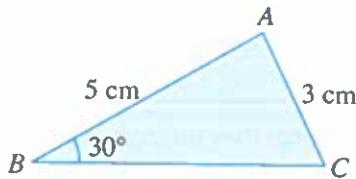
Fig. 11.5

Is it possible to construct a different △*ABC* (excluding a laterally inverted triangle)?

8. Notice that the given *AB̂C* is not in between the two given sides *AB* and *AC*, i.e. *AB̂C* is *not the included angle*.

9. What can you conclude from part 2 of this investigation?

From the investigation, we observe the following:

> **SAS Congruence Test:** If 2 sides and the included angle of a triangle are equal to the 2 corresponding sides and the corresponding included angle of another triangle, then the two triangles are congruent.

If the given angle is not the included angle, then *SSA may **not** be a congruence test*.

For example, in the above investigation, there are two ways to construct △*ABC* as shown in Fig. 11.6.
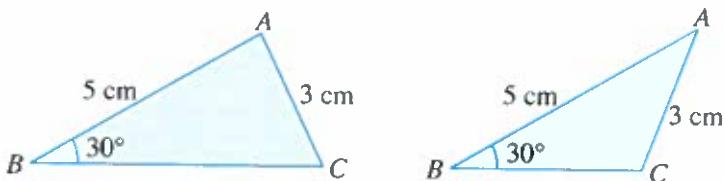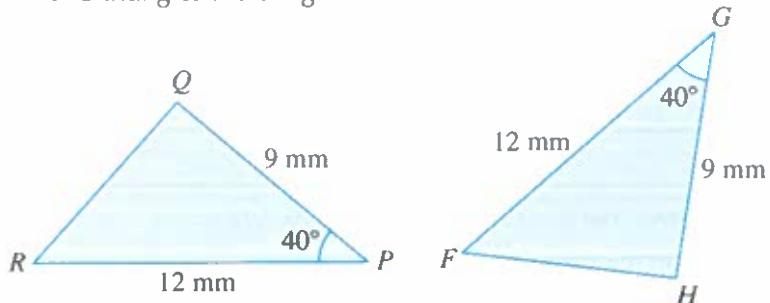
Fig. 11.6

Since the two triangles are not congruent, then SSA is not a congruence test in general, although there are exceptions (see RHS Congruence Test later in this section).

Worked Example 2

(Proving that Two Triangles are Congruent using the SAS Congruence Test)

Copy and complete the proof to show that the following two triangles are congruent.

Solution:

$P \leftrightarrow$ ____

$Q \leftrightarrow$ ____

$R \leftrightarrow$ ____

$PQ =$ ____ $= 9$ mm

$Q\hat{P}R =$ ____ $=$ ____°

$PR =$ ____ $=$ ____ mm

$\therefore \triangle PQR \equiv \triangle$ ____ (SAS)
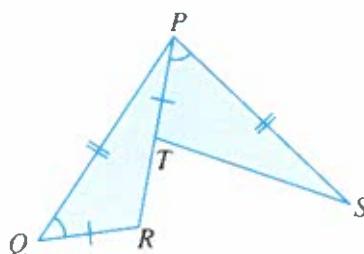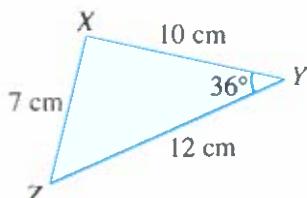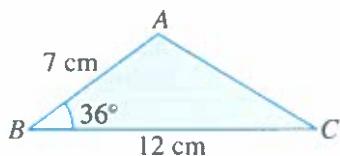
> **Problem Solving Tip**
>
> It is easier to match the vertices with the given angle first. Then match the vertex opposite either the 9 mm or 12 mm side for both triangles.

PRACTISE NOW 2

1. Prove that the following two triangles, where $P\hat{Q}R = S\hat{P}T$, are congruent.

2. Determine whether the following triangles are congruent.
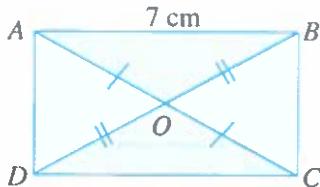
Worked Example 3

(Proving that Two Triangles are Congruent)

In the diagram, AOC and BOD are straight lines, $OA = OC$, $OB = OD$ and $AB = 7$ cm.

(i) Prove that $\triangle AOB$ is congruent to $\triangle COD$.

(ii) Find the length of CD.

Solution:

(i) $A \leftrightarrow C$

$O \leftrightarrow O$

$B \leftrightarrow D$

$OA = OC$ (given)

$A\hat{O}B = C\hat{O}D$ (vert. opp. $\angle$s)

$OB = OD$ (given)

$\therefore \triangle AOB \equiv \triangle COD$ (SAS)

(ii) Since $\triangle AOB \equiv \triangle COD$, then all the corresponding sides are equal.

$\therefore CD = AB$

$\quad = 7$ cm
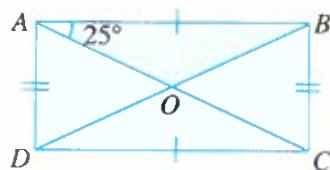
PRACTISE NOW 3

1. In the diagram, $ABCD$ is a rectangle, the two diagonals AC and BD intersect at O and $C\hat{A}B = 25°$.
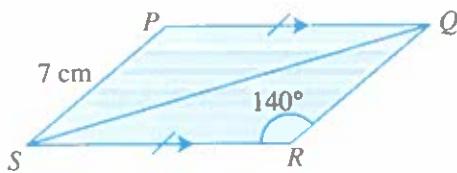
(i) Prove that $\triangle AOB$ is congruent to $\triangle COD$.

(ii) Find $B\hat{D}C$.

2. In the diagram, PQ is equal and parallel to SR, $PS = 7$ cm and $Q\hat{R}S = 140°$.

(i) Identify two congruent triangles and prove that they are congruent.

(ii) Find the length of QR and $Q\hat{P}S$.

⠿ AAS Congruence Test

Investigation

AAS Congruence Test

Part 1

1. Try to construct $\triangle PQR$ such that $PQ = 7$ cm, $Q\hat{P}R = 40°$ and $P\hat{Q}R = 60°$ in as many ways as possible.

   Compare the triangles you have drawn with those drawn by your classmates.
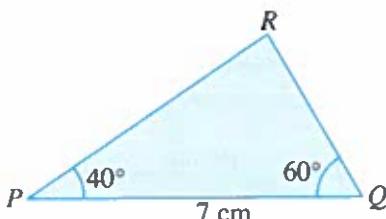
Fig. 11.7

2. Do you always get a unique triangle (regardless of reflection) as shown above (not drawn to scale)?

3. What can you conclude from part 1 of this investigation?

Part 2

4. Try to construct $\triangle ABC$ such that $AB = 6$ cm, $B\hat{A}C = 50°$ and $A\hat{C}B = 70°$ in as many ways as possible.

Fig. 11.8

Problem Solving Tip
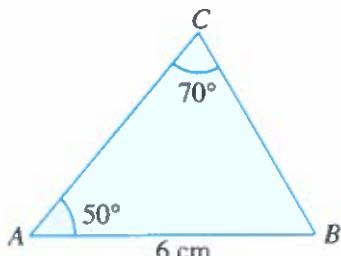
An easier way to construct the vertex C is to find $A\hat{B}C$ first, and then construct $A\hat{B}C$.

5. Do you always get a unique triangle (regardless of reflection)?
6. What can you conclude from part 2 of this investigation?
7. Does it matter whether the given side of the triangle is between the 2 given angles (as in part 1) or if it is not between the 2 given angles (as in part 2)? Explain your answer.

   Hint: See Problem Solving Tip for Question 4 in this investigation.

From the investigation, we observe the following:
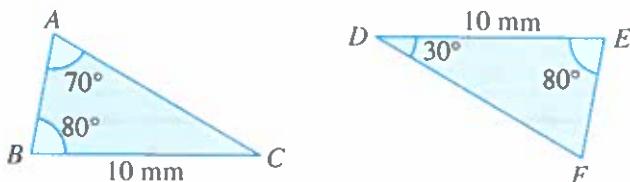
> **AAS Congruence Test:** If 2 angles and 1 side of a triangle are equal to the 2 corresponding angles and the corresponding side of another triangle, then the two triangles are congruent.

Since it does not matter whether or not the given side is between the two given angles, it does not matter whether we write AAS Congruence Test or ASA Congruence Test (unlike SAS Congruence Test where the given angle must be the included angle).

Worked Example 4

(Proving that Two Triangles are Congruent using the AAS Congruence Test)

Copy and complete the proof to show that the following two triangles are congruent.

Solution:

In $\triangle DEF$, $E\hat{F}D = 180° - 80° - 30°$ ($\angle$ sum of a $\triangle$)
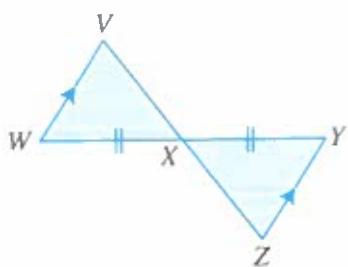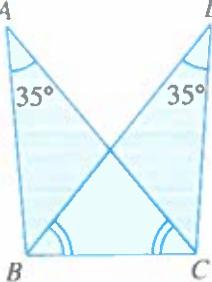
$\qquad\qquad = 70°$

$A \leftrightarrow \underline{\quad}$

$B \leftrightarrow \underline{\quad}$

$C \leftrightarrow \underline{\quad}$

$A\hat{B}C = \underline{\quad\quad} = 80°$

$B\hat{A}C = \underline{\quad\quad} = \underline{\quad}°$

$BC = \underline{\quad\quad} = \underline{\quad}$ mm

$\therefore \triangle ABC \equiv \triangle\underline{\quad\quad}$ (AAS)

In each part, prove that the following two triangles are congruent.

(a)

(b)

RHS Congruence Test

Investigation

RHS Congruence Test

1. Try to construct $\triangle DEF$ such that $D\hat{E}F = 90°$, $DE = 3$ cm and $DF = 5$ cm in as many ways as possible.
   Compare the triangles you have drawn with those drawn by your classmates.
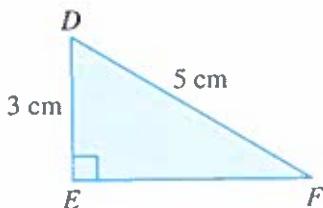
Fig. 11.9

2. Do you always get a unique triangle (regardless of reflection) as shown above (not drawn to scale)?

3. What can you conclude from the above investigation?

4. Do you notice that the above congruence test is a special case of SSA where the given angle is a right angle?

From the investigation, we observe the following:

RHS Congruence Test: If the hypotenuse (H) and 1 side (S) of a right-angled (R) triangle are equal to the hypotenuse and 1 side of another right-angled triangle, then the two right-angled triangles are congruent.

In general, SSA is not a congruence test, but there are exceptions, one of which is the RHS Congruence Test.

INFORMATION
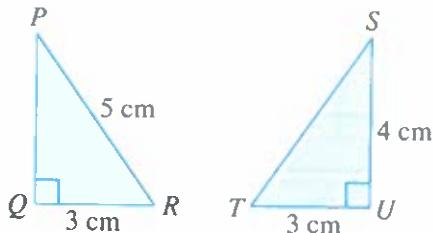
The 4 congruence tests covered so far are not the only congruence tests. There are more which are not included in the syllabus. An example is SSA, which is a congruence test if the given angle is obtuse.

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(Proving that Two Triangles are Congruent using the RHS Congruence Test)

Copy and complete the proof to show that the following two triangles are congruent.

Solution:

By Pythagoras' Theorem,

$$ST = \sqrt{TU^2 + SU^2}$$

$$= \sqrt{3^2 + 4^2}$$

$$= \sqrt{25}$$

$$= 5 \text{ cm}$$

Problem Solving Tip

Sometimes there is a need to find the other unknown side in either triangle.

$P \leftrightarrow \underline{\quad}$

$Q \leftrightarrow \underline{\quad}$

$R \leftrightarrow \underline{\quad}$

$P\hat{Q}R = \underline{\quad\quad} = \underline{\quad}°$

$PR = \underline{\quad\quad} = \underline{\quad} \text{ cm}$

$QR = \underline{\quad\quad} = \underline{\quad} \text{ cm}$
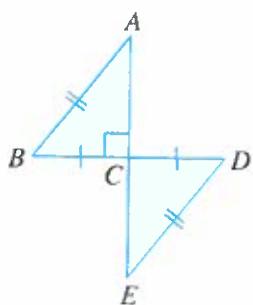
$\therefore \triangle PQR \equiv \triangle\underline{\quad}$ (RHS)
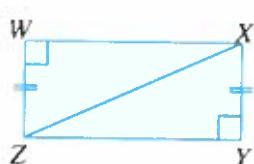
PRACTISE NOW 5

In each part, prove that the following two triangles are congruent.

(a)

(b)

SIMILAR QUESTIONS

Exercise 11A Questions 1(d), 2(d), 3(d), 4(g),(h)
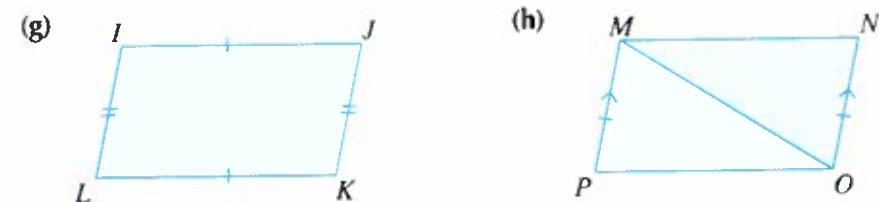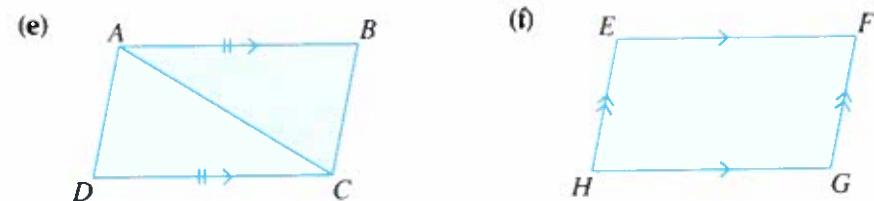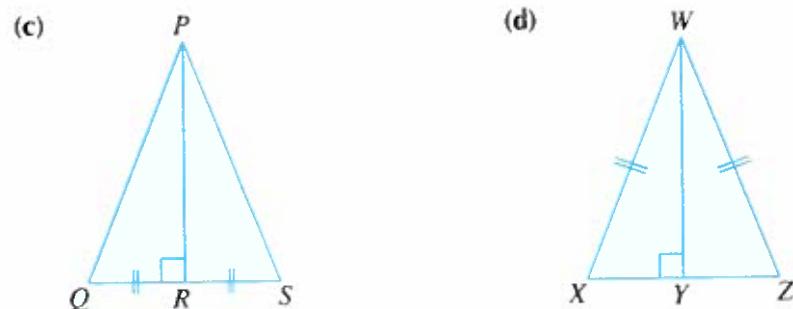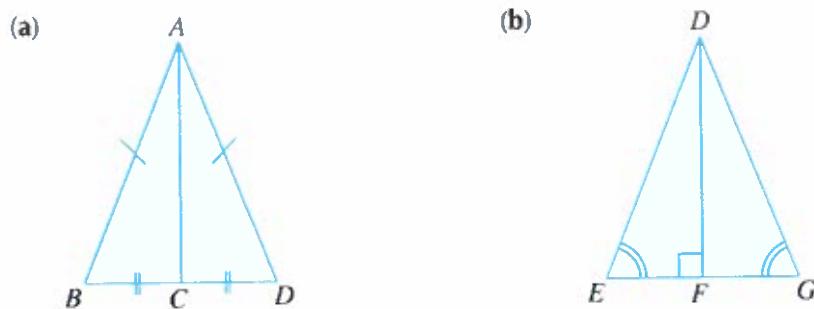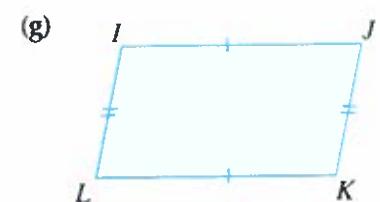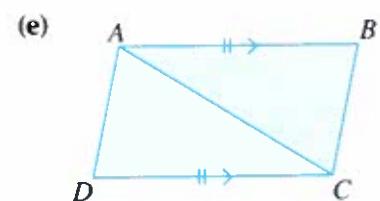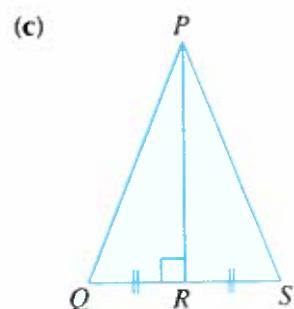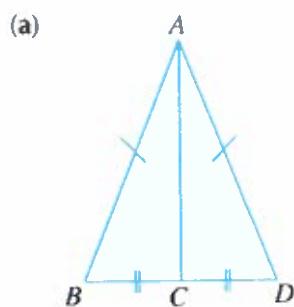
Consolidation for Congruence Tests

Work in pairs.

In each diagram, identify a pair of congruent triangles and prove that they are congruent.

(a)

(b)

(c)

(d)

(e)

(f)

(g)

(h)

SIMILAR
QUESTIONS

Exercise 11A Questions 7, 8
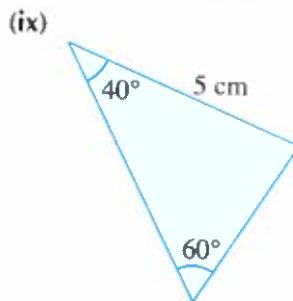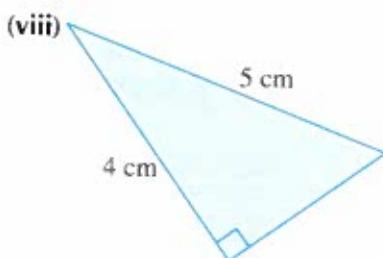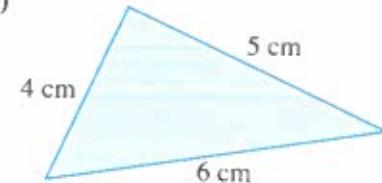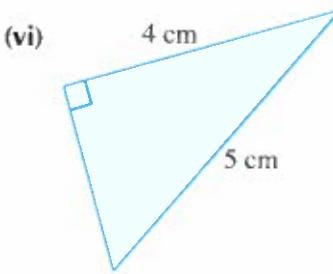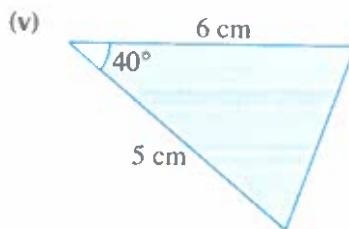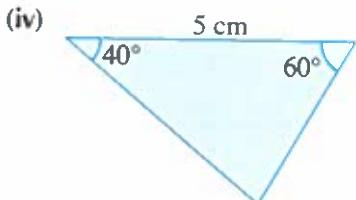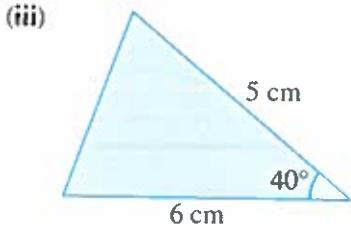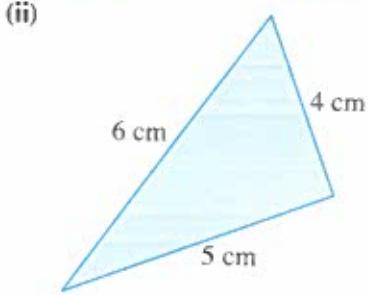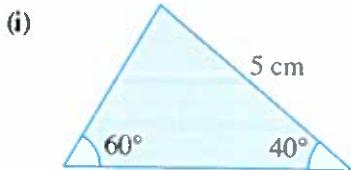
BASIC LEVEL
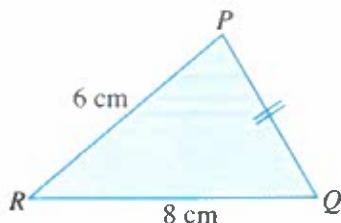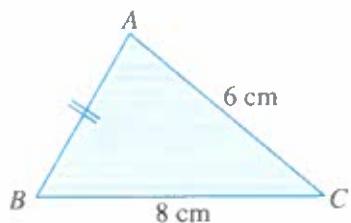
1. Identify a pair of congruent triangles from the following triangles (not drawn to scale), based on each of the following congruence tests:

    (a) SSS Congruence Test,

    (b) SAS Congruence Test,

    (c) AAS Congruence Test,

    (d) RHS Congruence Test.

    (i)

    5 cm
    60°   40°

    (ii)

    4 cm
    6 cm
    5 cm

    (iii)

    5 cm
    40°
    6 cm

    (iv)

    5 cm
    40°   60°

    (v)

    6 cm
    40°
    5 cm

    (vi)

    4 cm
    5 cm

    (vii)

    5 cm
    4 cm
    6 cm

    (viii)

    5 cm
    4 cm

    (ix)

    40°   5 cm
    60°

2. Copy and complete the proof to show that each of the following pairs of triangles are congruent.

(a)

$A \leftrightarrow$ ___

$B \leftrightarrow$ ___

$C \leftrightarrow$ ___

$AB =$ ___ (given)

$BC =$ ___ $= 8$ cm

$AC =$ ___ $=$ ___ cm

∴ $\triangle ABC \equiv \triangle$ ___ (___)

(b)

$D \leftrightarrow$ ___

$E \leftrightarrow$ ___

$F \leftrightarrow$ ___

$DE =$ ___ $= 3$ m

$D\hat{E}F =$ ___ $= 70°$

$EF =$ ___ $=$ ___ m

∴ $\triangle DEF \equiv \triangle$ ___ (___)

(c)

$L \leftrightarrow$ ___

$M \leftrightarrow$ ___

$N \leftrightarrow$ ___

$L\hat{M}N =$ ___ $= 30°$

$L\hat{N}M =$ ___ $=$ ___ °

$MN =$ ___ $=$ ___ cm

∴ $\triangle$ ___ $\equiv \triangle UVW$ (___)

(d)

$G \leftrightarrow$ ___

$H \leftrightarrow$ ___

$I \leftrightarrow$ ___

$G\hat{H}I =$ ___ $=$ ___ °

$GI =$ ___ $= 13$ mm

$HI =$ ___ $=$ ___ mm

∴ $\triangle$ ___ $\equiv \triangle STU$ (___)

3. Determine whether each of the following pairs of triangles are congruent.

(a)

(b)

(c)

(d)

4. In each diagram, identify a pair of congruent triangles and prove that they are congruent.

(a)

(b)

(c)

(d)

(e)

(f)

(g)

(h)

5. In the diagram, *RTV* and *STU* are straight lines, $RT = VT$, $ST = UT$, $RS = 4$ cm and $S\hat{R}T = 80°$.
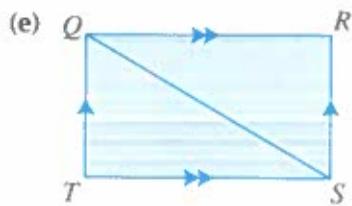
(i) Prove that $\triangle RST$ is congruent to $\triangle VUT$.

(ii) Find the length of *UV*.

(iii) Find $U\hat{V}T$.

(iv) Hence, other than $RS = UV$, what can you conclude about the lines *RS* and *UV*?

6. In the diagram, $GH = JI$ and $GI = JH$.

(i) Identify a triangle that is congruent to $\triangle GHI$ and prove that they are congruent.

(ii) If $H\hat{J}I = 60°$ and $G\hat{I}H = 40°$, find $G\hat{H}I$.
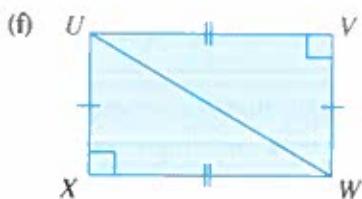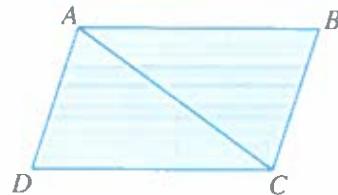
7. In each diagram, identify a pair of congruent triangles and prove that they are congruent.

(a)

(b)

(c)

(d)

(e)

(f)
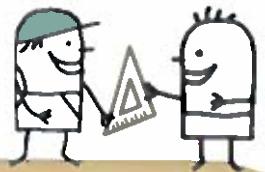
8. The diagram shows a parallelogram *ABCD*.

Use three different ways to prove that △*ABC* and △*CDA* are congruent.

11.2 Similarity Tests

⠿ Recap

In Book 2, we have learnt that similar figures have exactly the same shape but *not* necessarily the same size (i.e. congruence is a special case of similarity); and they can be mapped onto one another under enlargement.

For similar triangles, this would mean that all the corresponding angles are equal, and all the corresponding sides are proportional (i.e. all the ratios of the corresponding sides are equal).

For example, △*ABC* is similar to △*XYZ* if and only if

$B\hat{A}C = Y\hat{X}Z$
$A\hat{B}C = X\hat{Y}Z$
$A\hat{C}B = X\hat{Z}Y$
$\dfrac{AB}{XY} = \dfrac{BC}{YZ} = \dfrac{AC}{XZ}$

Fig. 11.10

ATTENTION

The vertices of the 2 triangles must match:
$A \leftrightarrow X$
$B \leftrightarrow Y$
$C \leftrightarrow Z$

In this section, we will investigate whether we need all the 6 conditions (i.e. 3 pairs of corresponding sides are equal and 3 ratios of corresponding sides are equal) to prove that two triangles are similar.

Chapter 11   Congruence and Similarity Tests

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⠿ AA Similarity Test

Investigation

AA Similarity Test

1. Construct $\triangle ABC$ and $\triangle XYZ$ of different sizes such that $B\hat{A}C = Y\hat{X}Z = 30°$ and $A\hat{B}C = X\hat{Y}Z = 50°$.

Fig. 11.11

2. Find $A\hat{C}B$ and $X\hat{Z}Y$ by using angle sum of triangle $= 180°$.
   Is $A\hat{C}B = X\hat{Z}Y$?

3. Measure the lengths of all the sides of the two triangles that you have constructed (not the ones shown in Fig. 11.11) and calculate the ratios $\frac{AB}{XY}$, $\frac{BC}{YZ}$ and $\frac{AC}{XZ}$ correct to 2 significant figures. Is $\frac{AB}{XY} = \frac{BC}{YZ} = \frac{AC}{XZ}$?

4. From what you have learnt in Book 2 which you have recapped at the beginning of Section 11.2, since all the corresponding angles of $\triangle ABC$ and $\triangle XYZ$ are equal and all the 3 ratios of their corresponding sides are equal, are the two triangles similar?

5. It is very important to note that the above **given conditions** are **2 pairs of corresponding angles equal** (i.e. $B\hat{A}C = Y\hat{X}Z$ and $A\hat{B}C = X\hat{Y}Z$). Are these given conditions enough to prove that the two triangles are similar?

From the investigation, we observe the following:

> **AA Similarity Test:** If 2 angles of a triangle are equal to the 2 corresponding angles of another triangle, then the two triangles are similar.

Thinking Time
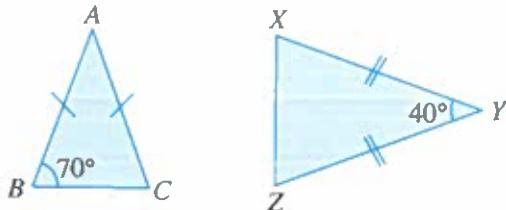
1. Why is the AAA Similarity Test not necessary?

2. Do two congruent triangles satisfy the AA Similarity Test? Is congruence a special case of similarity?

Worked Example 6

(Proving that Two Triangles are Similar using the AA Similarity Test)

Copy and complete the proof to show that the following two triangles are similar.

Solution:

$A\hat{C}B = A\hat{B}C$ (base ∠s of isos. Δ)

$\qquad = 70°$

$Y\hat{X}Z = Y\hat{Z}X$

$\qquad = \dfrac{180° - 40°}{2}$ (base ∠s of isos. Δ)

$\qquad = 70°$

$A \leftrightarrow Y$

$B \leftrightarrow \underline{\quad}$

$C \leftrightarrow \underline{\quad}$

$A\hat{B}C = \underline{\quad} = 70°$

$A\hat{C}B = \underline{\quad} = \underline{\quad}°$

$\therefore \triangle ABC$ is similar to $\triangle\underline{\quad}$ (2 pairs of corr. ∠s equal).

> **Problem Solving Tip**
>
> Sometimes there is a need to find the other unknown angles in both triangles.

PRACTISE NOW 6

1. In each part, determine whether the following two triangles are similar. Explain or prove your answers.

   (a)

(b)

(c)

(d)
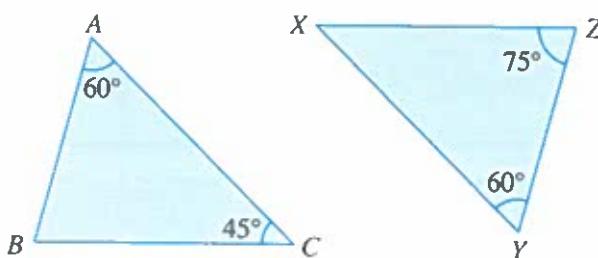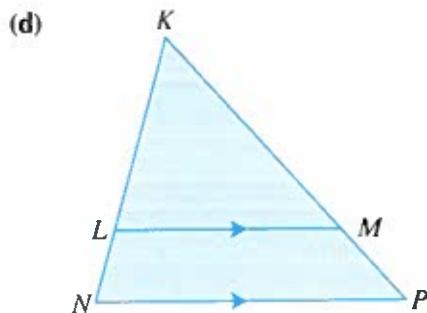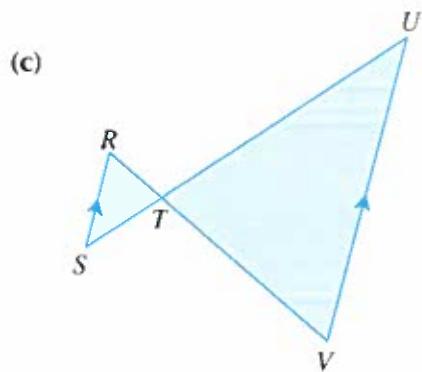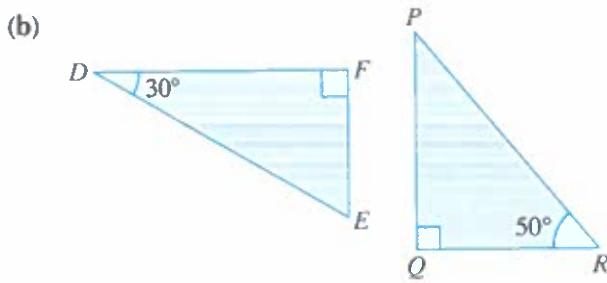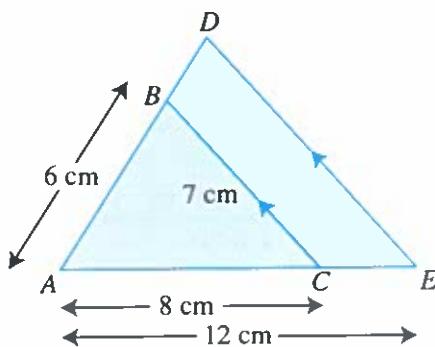
2. In the figure, BC is parallel to DE, $AB = 6$ cm, $AC = 8$ cm, $BC = 7$ cm and $AE = 12$ cm.

(i)   Show that $\triangle ABC$ is similar to $\triangle ADE$.

(ii) Find the length of DE and of BD.

(iii) What can you conclude about the ratios $\dfrac{AB}{BD}$ and $\dfrac{AC}{CE}$ for such a figure?

⸬ SSS Similarity Test

Investigation

SSS Similarity Test

1. Construct $\triangle DEF$ such that $DE = 2$ cm, $EF = 3$ cm and $DF = 4$ cm.
2. Construct $\triangle PQR$ such that $PQ = 4$ cm, $QR = 6$ cm and $PR = 8$ cm.
   Fig. 11.12 shows the two triangles (not drawn to scale).
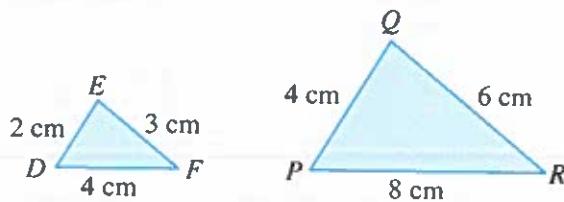
Fig. 11.12

3. Calculate the ratios $\dfrac{DE}{PQ}$, $\dfrac{EF}{QR}$ and $\dfrac{DF}{PR}$.
   Is $\dfrac{DE}{PQ} = \dfrac{EF}{QR} = \dfrac{DF}{PR}$?

4. Measure all the angles of the two triangles which you have constructed (not the ones shown in Fig. 11.12).
   Are $E\hat{D}F = Q\hat{P}R$, $D\hat{E}F = P\hat{Q}R$ and $D\hat{F}E = P\hat{R}Q$?

5. From what you have learnt in Book 2 which you have recapped at the beginning of Section 11.2, since all the corresponding angles of $\triangle DEF$ and $\triangle PQR$ are equal and all the 3 ratios of their corresponding sides are equal, are the two triangles similar?

6. It is very important to note that the above **given conditions** are **3 ratios of corresponding sides equal** (i.e. $\dfrac{DE}{PQ} = \dfrac{EF}{QR} = \dfrac{DF}{PR}$). Are these given conditions enough to prove that the two triangles are similar?

From the investigation, we observe the following:

> **SSS Similarity Test:** If the 3 ratios of the corresponding sides of two triangles are equal, then the two triangles are similar.
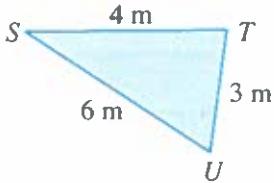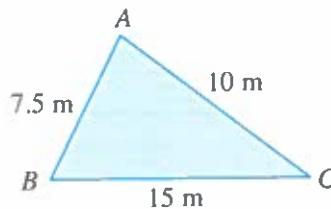
What are the similarities and the differences between the SSS Congruence Test and the SSS Similarity Test?

Worked Example 7

(Proving that Two Triangles are Similar using the SSS Similarity Test)

Copy and complete the proof to show that the following two triangles are similar.

Solution:

$A \leftrightarrow T$

$B \leftrightarrow$ ___

$C \leftrightarrow$ ___

$\dfrac{AB}{TU} = \dfrac{7.5}{3} = 2.5$

$\dfrac{AC}{TS} = \dfrac{\quad}{\quad} = \underline{\quad}$

$\dfrac{BC}{\quad} = \dfrac{\quad}{6} = \underline{\quad}$
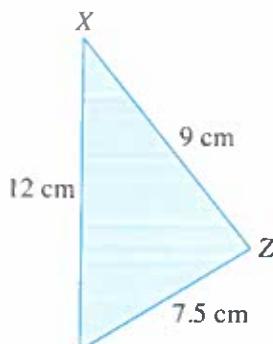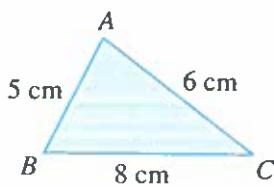
$\therefore \triangle ABC$ is similar to $\triangle$____ (3 ratios of corr. sides equal).
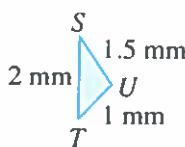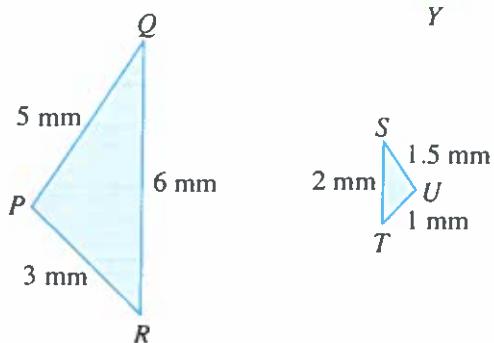
> **Problem Solving Tip**
>
> Match the vertex opposite the longest side for both triangles first, i.e. $A \leftrightarrow T$. Then do the same for the vertex opposite the shortest side and so on.

In each part, determine whether the following two triangles are similar. Explain or prove your answers.

(a)

(b)

SAS Similarity Test

Investigation

SAS Similarity Test

1. Construct $\triangle ABC$ such that $A\hat{B}C = 70°$, $AB = 3$ cm and $BC = 5$ cm.
2. Construct $\triangle PQR$ such that $P\hat{Q}R = 70°$, $PQ = 4.5$ cm and $QR = 7.5$ cm.

   Fig. 11.13 shows the two triangles (not drawn to scale).
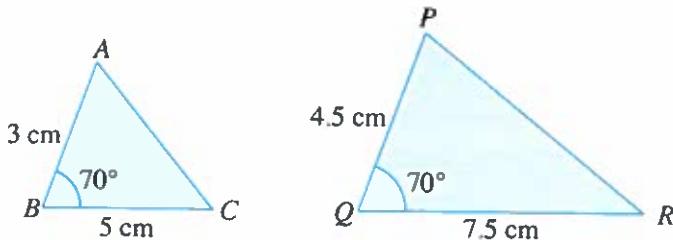
Fig. 11.13

3. Calculate the ratios $\dfrac{PQ}{AB}$ and $\dfrac{QR}{BC}$.

   Is $\dfrac{PQ}{AB} = \dfrac{QR}{BC}$?

4. Measure the length of AC and of PR in the two triangles which you have constructed (not the ones shown in Fig. 11.13). Then calculate $\dfrac{PR}{AC}$.

   Is $\dfrac{PR}{AC} = \dfrac{PQ}{AB} = \dfrac{QR}{BC}$?

5. Measure all the other unknown angles in the two triangles which you have constructed.

   Are $B\hat{A}C = Q\hat{P}R$ and $A\hat{C}B = P\hat{R}Q$?

6. From what you have learnt in Book 2 which you have recapped at the beginning of Section 11.2, since all the corresponding angles of $\triangle ABC$ and $\triangle PQR$ are equal and all the 3 ratios of their corresponding sides are equal, are the two triangles similar?

7. It is very important to note that the above **given conditions** are **2 ratios of corresponding sides equal** and **the pair of *included* angles equal** (i.e. $\dfrac{PQ}{AB} = \dfrac{QR}{BC}$ and $P\hat{Q}R = A\hat{B}C$). Are these given conditions enough to prove that the two triangles are similar?

From the investigation, we observe the following:

> **SAS Similarity Test:** If 2 ratios of the corresponding sides of two triangles are equal and the pair of included angles are also equal, then the two triangles are similar.

Thinking Time

1. What are the similarities and the differences between the SAS Congruence Test and the SAS Similarity Test?
2. Unlike the need for the AAS Congruence Test, why is the AAS Similarity Test not necessary?
3. Is there a RHS Similarity Test? Investigate and explain your answer.

(Proving that Two Triangles are Similar using the SAS Similarity Test)

Copy and complete the proof to show that the following two triangles are similar.

Solution:

$D \leftrightarrow$ ___

$E \leftrightarrow$ ___

$F \leftrightarrow$ ___

$E\hat{D}F =$ ___ (common angle)

$\dfrac{DE}{DG} = \dfrac{1}{2}$

$\dfrac{DF}{DH} =$ ___

$\therefore \dfrac{DE}{DG} = \dfrac{DF}{DH}$

$\therefore \triangle DEF$ is similar to $\triangle$ ___ (2 ratios of corr. sides and included $\angle$ equal).

ATTENTION
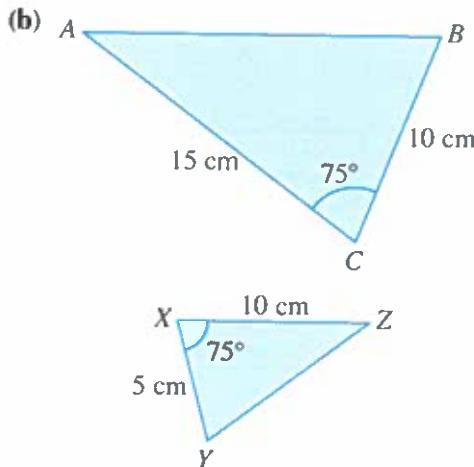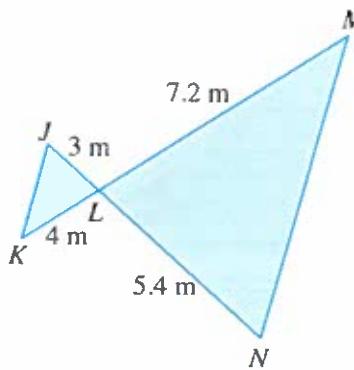
The common angle is the included angle between the given corresponding sides.

PRACTISE NOW 8

In each part, determine whether the following two triangles are similar. Explain or prove your answers.

(a)

(b)
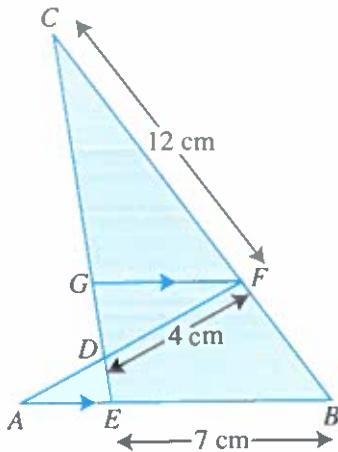
SIMILAR QUESTIONS

Exercise 11B Questions 1(c), 2(c), 3(c), 4(c),(d)

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Worked Example 9

In the diagram, GF is parallel to AB, $AB = CF = 12$ cm, $BF = FG$, $DF = 4$ cm and $EB = 7$ cm.

(i) Show that $\triangle ABF$ is congruent to $\triangle CFG$.

(ii) Show that $\triangle CDF$ is similar to $\triangle ADE$.

(iii) Find the length of DE.

Solution:

(i) $A \leftrightarrow C$

$B \leftrightarrow F$

$F \leftrightarrow G$

$AB = CF$ (given)

$A\hat{B}F = C\hat{F}G$ (corr. $\angle$s, $AB \parallel GF$)

$BF = FG$ (given)

$\therefore \triangle ABF \equiv \triangle CFG$ (SAS)

> **Problem Solving Tip**
>
> In (i), since $BF = FG$ (given), then the vertices opposite these sides must match, i.e. $A \leftrightarrow C$; similarly for $AB = CF$. Since only 2 pairs of equal sides are given, we may consider using the SAS Congruence Test.

(ii) $C \leftrightarrow A$

$D \leftrightarrow D$

$F \leftrightarrow E$

$C\hat{D}F = A\hat{D}E$ (vert. opp. $\angle$s)

Since $\triangle CFG$ and $\triangle ABF$ are congruent,

then $F\hat{C}G = B\hat{A}F$,

i.e. $F\hat{C}D = E\hat{A}D$.

$\therefore \triangle CDF$ is similar to $\triangle ADE$ (2 pairs of corr. $\angle$s equal).

> **Problem Solving Tip**
>
> In (ii), since the lengths of $\triangle ADE$ are not given, we may consider using the AA Similarity Test.

(iii) Since $AB = CF = 12$ cm, then $AE = 12 - 7 = 5$ cm.

Since $\triangle ADE$ and $\triangle CDF$ are similar, then

$$\frac{DE}{DF} = \frac{AE}{CF}$$

i.e. $\dfrac{DE}{4} = \dfrac{5}{12}$

$\therefore \quad DE = \dfrac{5}{12} \times 4$
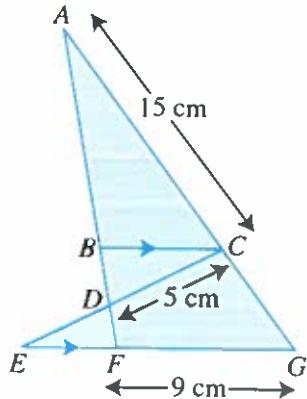
$\qquad\quad = 1\dfrac{2}{3}$ cm

PRACTISE NOW 9
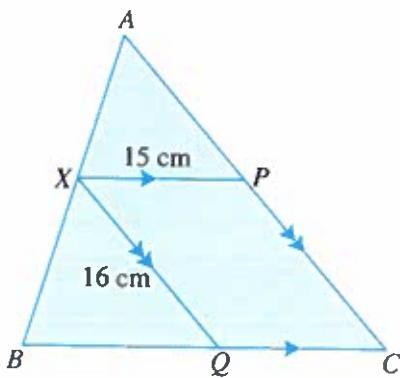
1. In the diagram, BC is parallel to EG, $AC = EG = 15$ cm, $BC = CG$, $CD = 5$ cm and $FG = 9$ cm.

(i) Show that $\triangle ABC$ and $\triangle ECG$ are congruent.

(ii) Show that $\triangle ACD$ and $\triangle EFD$ are similar.

(iii) Find the length of DF.

2. The side AB of $\triangle ABC$ is divided at X in the ratio $3 : 4$. P and Q are points on CA and CB such that XP and XQ are parallel to BC and AC respectively.

Given that $XP = 15$ cm and $XQ = 16$ cm, find the length of BQ and of AC.
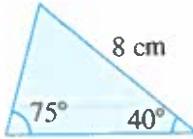
BASIC LEVEL
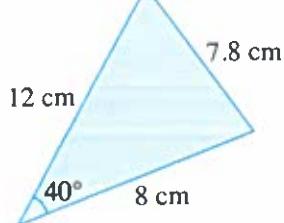
1. Identify a pair of similar triangles from the following triangles (not drawn to scale), based on in each of the following similarity tests:
   (a) AA Similarity Test,
   (b) SSS Similarity Test,
   (c) SAS Similarity Test.

   (i)

   8 cm
   75° 40°

   (ii)

   7.8 cm
   12 cm
   40° 8 cm

   (iii)

   75°
   40°
   12 cm

   (iv)

   15.4 cm
   18 cm
   24 cm 40°

   (v)

   40° 85°
   12 cm

   (vi)

   4 cm
   3.9 cm
   6 cm

   (vii)

   55°
   8 cm
   40°

   (viii)

   4.5 cm
   6 cm
   40°

2. Copy and complete the proof to show that each of the following pairs of triangles are similar.

   (a)

   A
   70°
   60°
   B   C

   S
   70°
   T   50° U

   $S\hat{T}U = 180° - 70° - 50°$ ($\angle$ sum of a $\triangle$)

   $= \underline{\quad}°$

   $A \leftrightarrow \underline{\quad}$

   $B \leftrightarrow \underline{\quad}$

   $C \leftrightarrow \underline{\quad}$

   $B\hat{A}C = \underline{\quad} = 70°$

   $A\hat{B}C = \underline{\quad} = \underline{\quad}°$

   $\therefore \triangle ABC$ is similar to $\triangle\underline{\quad}$

   (_____ pairs of corr. _____ equal).

(b)

$X \leftrightarrow$ ___

$Y \leftrightarrow$ ___

$Z \leftrightarrow$ ___

$\dfrac{XY}{NM} = \dfrac{24}{} =$ ___

$\dfrac{XZ}{} = \dfrac{}{} =$ ___

$\dfrac{YZ}{} = \dfrac{}{} =$ ___

$\therefore \triangle XYZ$ and $\triangle$ _____ are similar

(_____ ratios of corr. _____ equal).

(c)

$D \leftrightarrow$ ___

$E \leftrightarrow$ ___

$F \leftrightarrow$ ___

$D\hat{E}F =$ ___ $=$ ___ $°$

$\dfrac{DE}{GI} = \dfrac{}{} =$ ___

$\dfrac{EF}{} = \dfrac{}{} =$ ___

$\therefore \dfrac{DE}{GI} = \dfrac{EF}{}$

$\therefore \triangle DEF$ is similar to $\triangle$ _____

(_____ ratios of corr. _____ and _____ angle equal).

3. Determine whether each of the following pairs of triangles are similar.
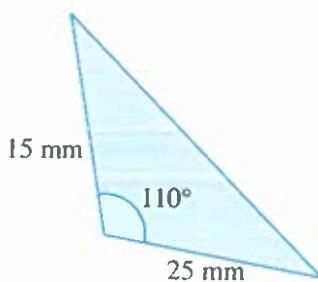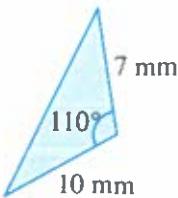
(a)

(b)
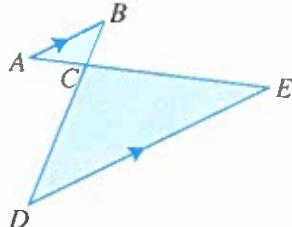
(c)
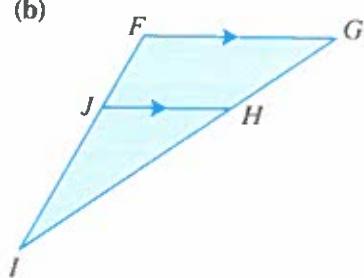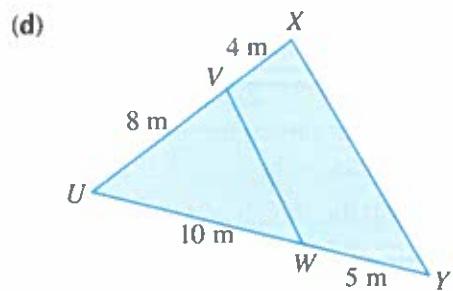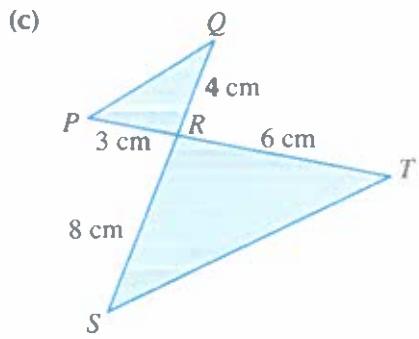
4. In each diagram, identify a pair of similar triangles and prove that they are similar.

(a)

(b)

(c)

(d)

(d)
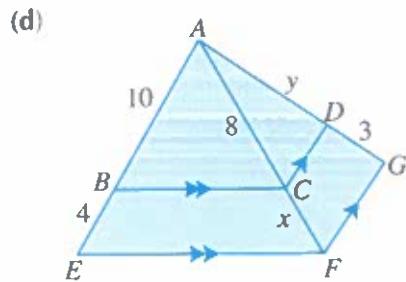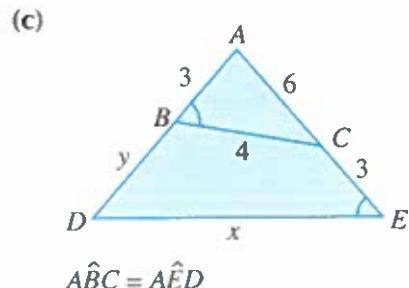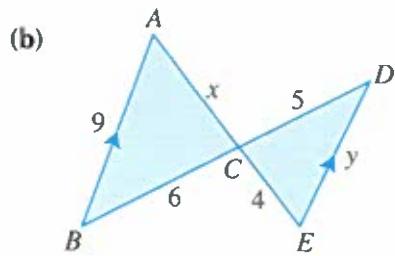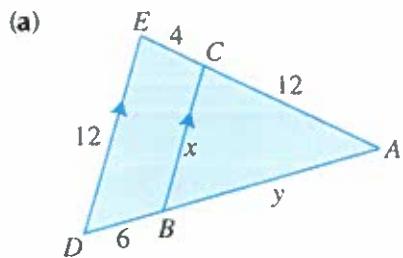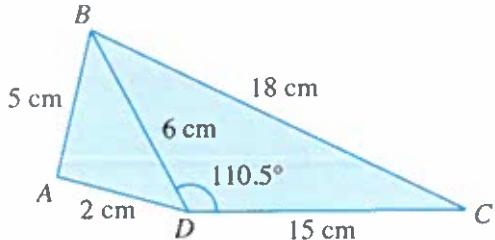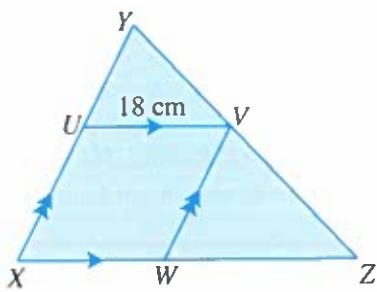
5. In each of the following figures, identify the similar triangles and find the value of x and of y. All lengths are given in cm.

(a)

(b)

(c)

$A\hat{B}C = A\hat{E}D$

6. In the diagram, $AB = 5$ cm, $AD = 2$ cm, $BD = 6$ cm, $CD = 15$ cm, $BC = 18$ cm and $B\hat{D}C = 110.5°$.
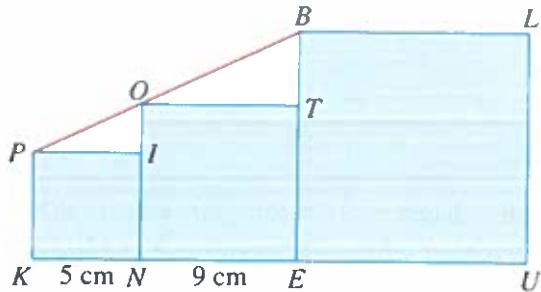
(i) Prove that $\triangle ABD$ and $\triangle DCB$ are similar.

(ii) Find $D\hat{A}B$.

7. In the diagram, U divides YX in the ratio $2 : 3$. $UVWX$ is a rhombus in which $UV = 18$ cm.
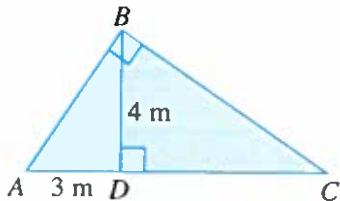
Find the length of XY and of WZ.

8. In the diagram, the lengths of the sides of the squares *PINK* and *NOTE* are 5 cm and 9 cm respectively. Given that *POB* is a straight line, find the length of the side of the square *BLUE*.
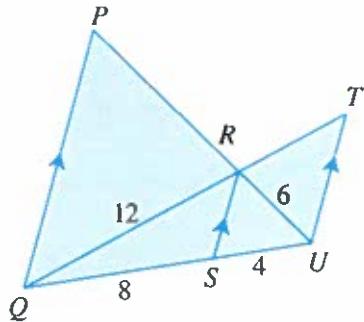
9. In the diagram, $A\hat{B}C = B\hat{D}C = 90°$, $AD = 3$ m and $BD = 4$ m.

(i) Identify 3 similar triangles and prove that they are similar.

(ii) Find the length of *BC* and of *CD*.
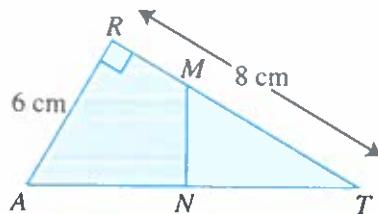
10. In the diagram, *PQ*, *RS* and *TU* are parallel lines, $QS = 8$ cm, $SU = 4$ cm, $QR = 12$ cm and $RU = 6$ cm.
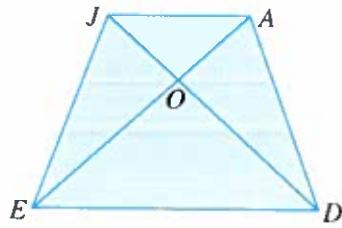
(i) Find the length of *RT* and of *PR*.

(ii) Calculate the ratio $PQ : TU$.

11. *RAT* is a right-angled triangle with $A\hat{R}T = 90°$, $RA = 6$ cm and $RT = 8$ cm. If the triangle is folded along the line *MN*, vertex *A* coincides with the vertex *T*.

(i) Prove that *MN* is perpendicular to *AT*.

(ii) Name a pair of similar triangles and prove that they are similar.

(iii) Hence, find the length of *MN*.
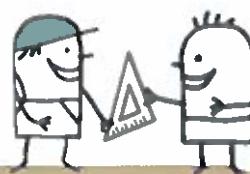
12. In the quadrilateral *JADE*, $AO = JO$ and $EO = DO$.

Explain why $\triangle AJD$ is congruent to $\triangle JAE$.

11.3 Applications of Congruent and Similar Triangles
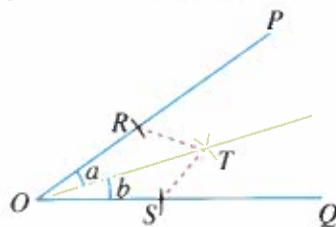
In this section, we will apply the concepts of congruent and similar triangles to solve problems in mathematics and in real life.

Worked Example 10

(Application of Congruent Triangles)

In Book 1, we have learnt how to construct the bisector of a given angle as shown in the diagram. Prove that OT is the angle bisector of $P\hat{O}Q$.

RECALL

Draw arcs OR and OS, where $OR = OS$.

Draw arcs RT and ST, where $RT = ST$.

Prove that OT is the angle bisector of $P\hat{O}Q$, i.e. $\angle a = \angle b$.

Solution:

$R \leftrightarrow S$

$O \leftrightarrow O$

$T \leftrightarrow T$

$OR = OS$

$RT = ST$

$OT = OT$ (common side)

$\therefore \triangle ROT$ and $\triangle SOT$ are congruent (SSS Congruence Test).

$\therefore a = b$, i.e. OT is the angle bisector of $P\hat{O}Q$.
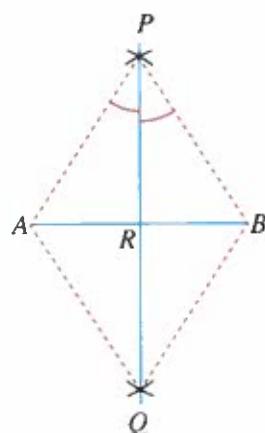
PRACTISE NOW 10

In Book 1, we have learnt how to construct the perpendicular bisector of a given line segment as shown in the diagram. Prove that PQ is the perpendicular bisector of AB.

SIMILAR QUESTIONS

Exercise 11C Questions 1, 2, 4, 5, 7

RECALL
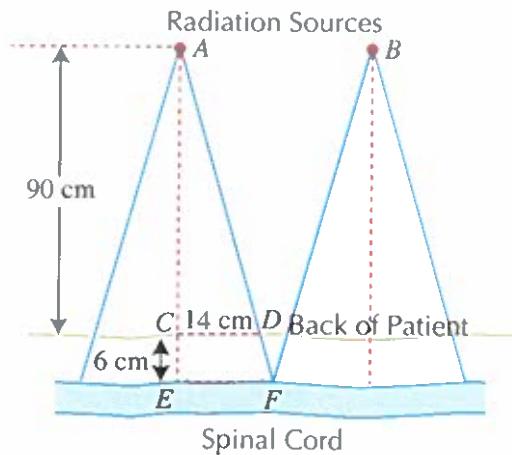
Draw arcs AP, AQ, BP and BQ, where $AP = AQ = BP = BQ$.

Prove that PQ is the perpendicular bisector of AB.

(Application of Similar Triangles)

The chapter opener mentions the importance of geometry in radiation oncology (the study and treatment of tumours). The diagram below shows how far apart two beams of radiation must be placed so that they will not overlap at the spinal cord, or else a double dose of radiation will endanger the patient.

Radiation Sources

Find the distance between the two radiation sources A and B.

Solution:

$A \leftrightarrow A$

$C \leftrightarrow E$

$D \leftrightarrow F$

$$\frac{EF}{CD} = \frac{AE}{AC}$$

$$\frac{EF}{14} = \frac{90+6}{90}$$
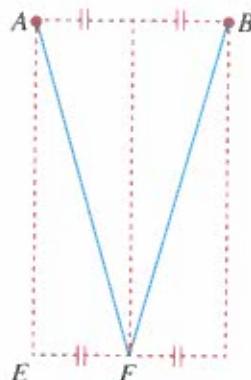
$$EF = \frac{96}{90} \times 14$$

$$= 14.93 \text{ cm (to 4 s.f.)}$$
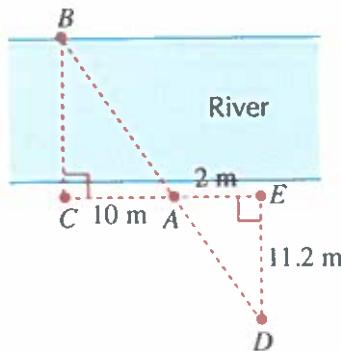
$\therefore AB = 2 \times EF$

$$= 2 \times 14.93$$
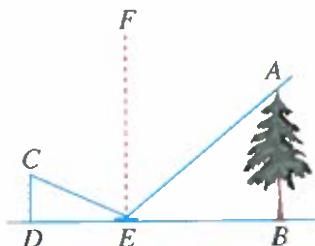
$$= 29.9 \text{ cm (to 3 s.f.)}$$

1. To estimate the width of a river, a man selects an object B at the far bank and stands at C such that BC is perpendicular to the river bank (see diagram). He then walks perpendicular to BC for 10 metres until he reaches A, where he plants a vertical pole. He continues to walk another 2 metres until point E, before he walks along ED such that $A\hat{E}D = 90°$. He stops walking when he reaches D, where DAB forms a straight line. He measures and finds that the length of DE is 11.2 m. Find the length of BC.

2. To determine the height AB of a tree, Michael places a mirror on the ground at E. From E, he walks backwards to a point D, where he is just able to see the top of the tree in the mirror. CD, FE and AB are perpendicular to the line DEB.

Problem Solving Tip

According to the Law of Reflection in Physics, the angle of incidence, $A\hat{E}F$, is equal to the angle of reflection, $C\hat{E}F$.

Given that $BE = 18$ m, $ED = 2.1$ m and that his eyes at C are 1.4 m above the ground, find the height of the tree.
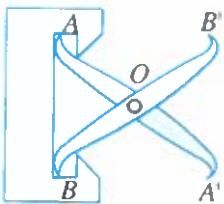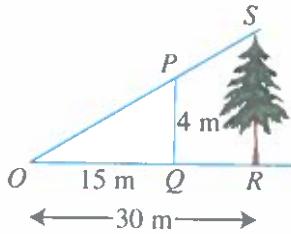
1. The diagram illustrates how the length *AB* (which cannot be measured directly) of a pond is measured. Choose a point *C* and measure the length of *AC* and the length of *BC*. Produce *AC* and *BC* to *A'* and *B'* respectively, so that *CA'* = *AC* and *CB'* = *BC*. By measuring the length of *B'A'*, we will be able to find the length of *AB*. Why is this so?

2. To measure the width of the internal trough, *AB*, of a machine tool which cannot be measured directly, we make use of a device as shown in the diagram. The device is made up of two parts, *AA'* and *BB'*, hinged halfway at *O*. By measuring the distance between *A'* and *B'*, we will be able to obtain the length of *AB*. Why is this so?
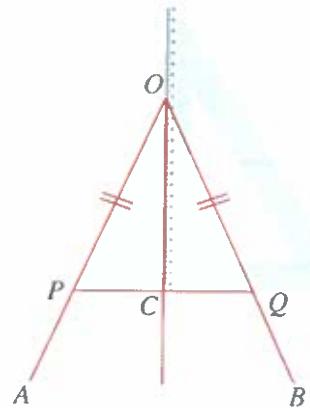
3. The figure shows a tree *SR* and a pole *PQ* casting shadows of lengths 30 m and 15 m respectively. If the height of the pole is 4 m, find the height of the tree.
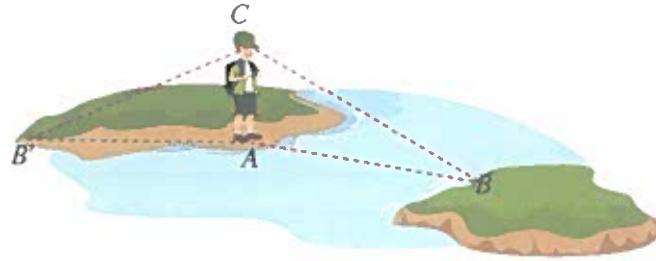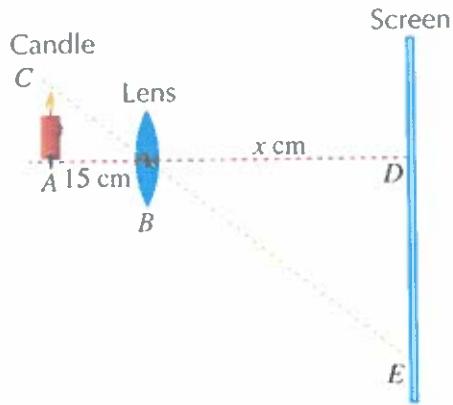
4. In the diagram, *P* lies on *OA* and *Q* lies on *OB* such that *OP* = *OQ*. Place a set square with one side along *PQ* and another side passing through *O*, as shown in the diagram. Explain why *OC* is the angle bisector of *AÔB*.

5. The diagram shows Ethan standing at a point *A* along a river bank. He looks directly across to the opposite bank, adjusting his cap so that his line of vision *CB* passes through the lowest point at the rim of his cap and falls on the point *B*. He then turns around without moving his head. His new line of vision *CB'* through the lowest point at the rim of his cap now falls on a point *B'* on the same side of the river. State which measurement he can make in order to find the width *AB* of the river. Explain your answer.
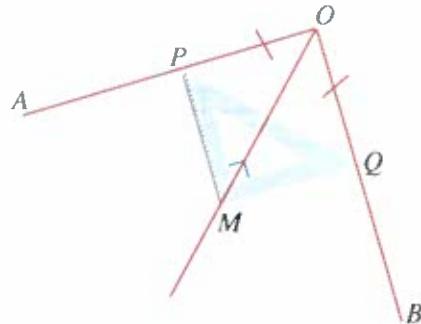
6. A candle is placed 15 cm from a lens and a screen is placed at a distance of x cm from the lens as shown in the diagram. The image of the candle, DE, captured on the screen, is inverted and is 3 times the length of the candle. Find the value of x.

7. Using a set square, we can bisect a given angle. In the diagram, P and Q are marked along the arms, OA and OB of $A\hat{O}B$ respectively, such that $OP = OQ$. Move a 90°-45°-45° set square away from O until the 45° edges coincide with P and Q as shown in the diagram. Explain why OM is the angle bisector of $A\hat{O}B$.

Summary

1. The following shows 4 congruence tests and 3 similarity tests.

Congruence Tests		Similarity Tests
SSS Congruence Test	↔	SSS Similarity Test
SAS Congruence Test	↔	SAS Similarity Test
AAS Congruence Test	↔	AA Similarity Test
		(Since AA is enough, there is no need to use the AAS Similarity Test.)
RHS Congruence Test		
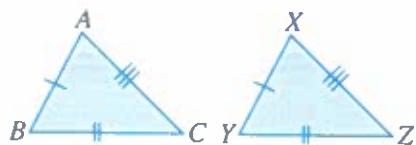
2. In general, SSA is not a congruence test.
   An exception is the RHS congruence test.

3. Congruence Tests
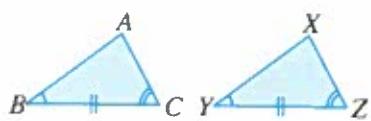
| SSS Congruence Test	SAS Congruence Test
3 corresponding sides are equal, i.e. $AB = XY$, $BC = YZ$, $AC = XZ$	2 corresponding sides and the included angle are equal, i.e. $AB = XY$, $BC = YZ$, $A\hat{B}C = X\hat{Y}Z$
AAS Congruence Test	**RHS Congruence Test**
1 corresponding side and 2 corresponding angles are equal, i.e. $BC = YZ$, $A\hat{B}C = X\hat{Y}Z$, $A\hat{C}B = X\hat{Z}Y$	1 side and the hypotenuse of the right-angled triangle are equal, i.e. $BC = YZ$, $AC = XZ$, $A\hat{B}C = X\hat{Y}Z = 90°$
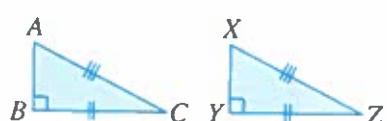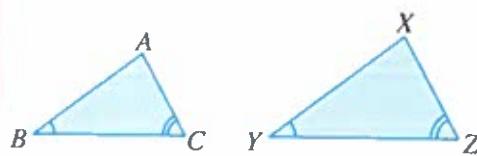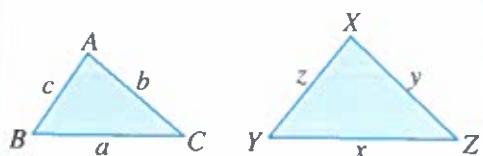
4. Similarity Tests
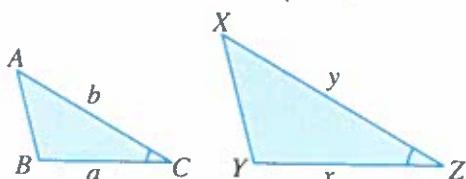
AA Similarity Test	SSS Similarity Test
2 corresponding angles are equal, i.e. $A\hat{B}C = X\hat{Y}Z$, $A\hat{C}B = X\hat{Z}Y$	The 3 ratios of the corresponding sides are equal, i.e. $\dfrac{a}{x} = \dfrac{b}{y} = \dfrac{c}{z}$ or $\dfrac{x}{a} = \dfrac{y}{b} = \dfrac{z}{c}$

SAS Similarity Test

2 ratios of the corresponding sides and the included angle are equal,

i.e. $\dfrac{a}{x} = \dfrac{b}{y}$ or $\dfrac{x}{a} = \dfrac{y}{b}$, $A\hat{C}B = X\hat{Z}Y$

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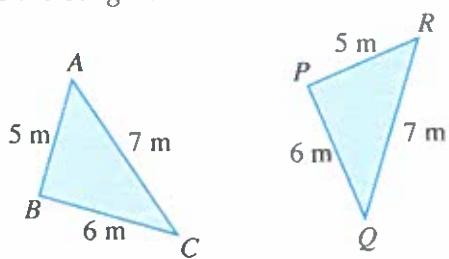
Review Exercise 11
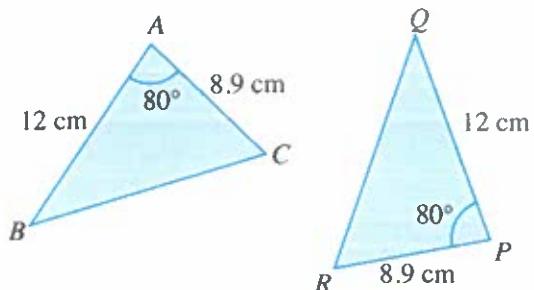
1. Determine whether each of the following pairs of triangles are congruent. If they are congruent, state the congruence test.
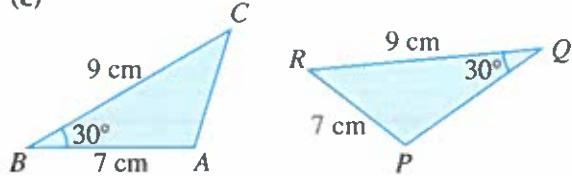
(a)
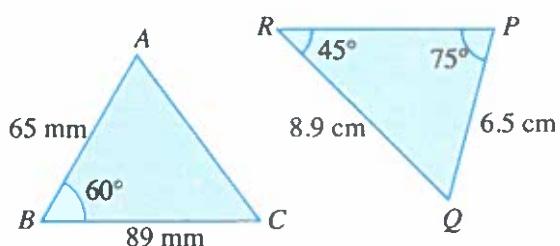
(b)
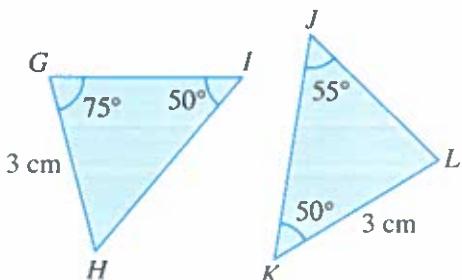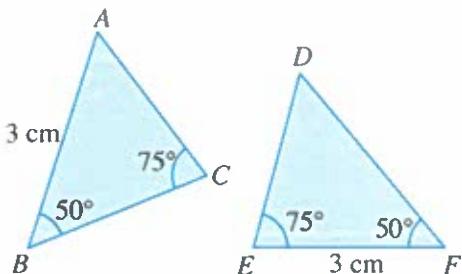
(c)

(d)
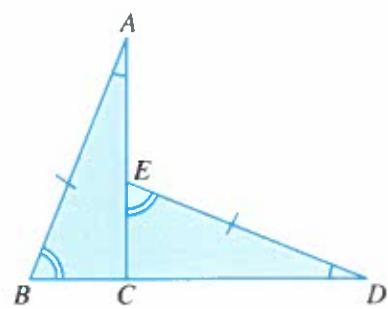
2. Identify a pair of congruent triangles and state the congruence test.
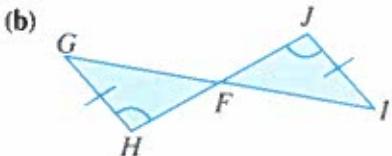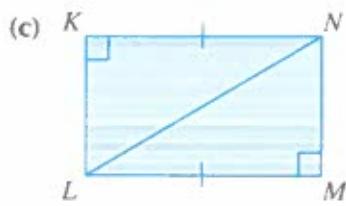
3. Which of the following pairs of triangles are congruent? If they are congruent, state the congruence test and name the other three pairs of equal measurements.

(a)

(b)

(c)

(d)

(e)

(f)
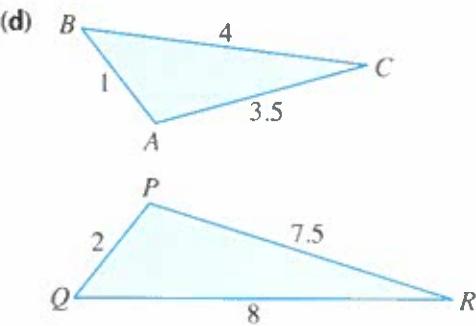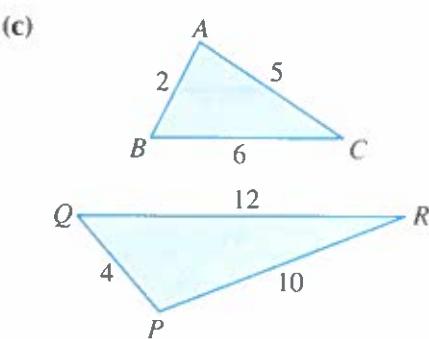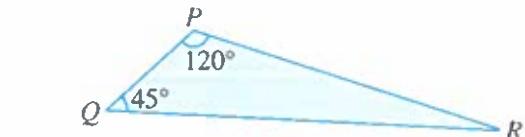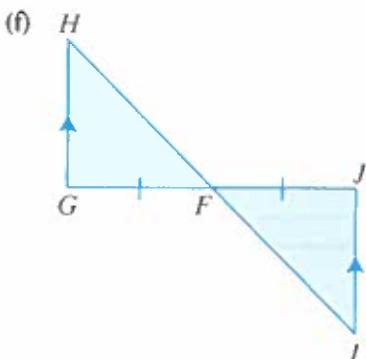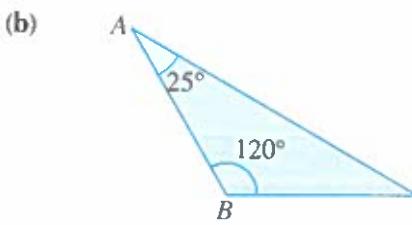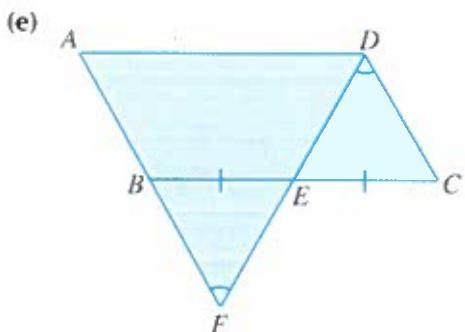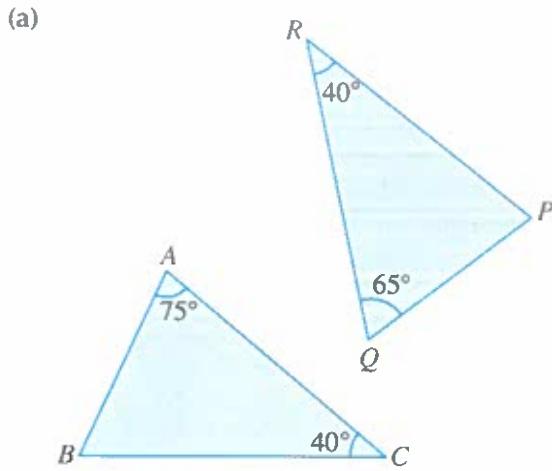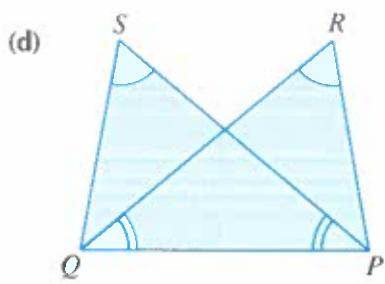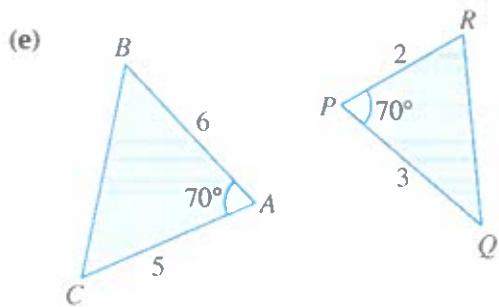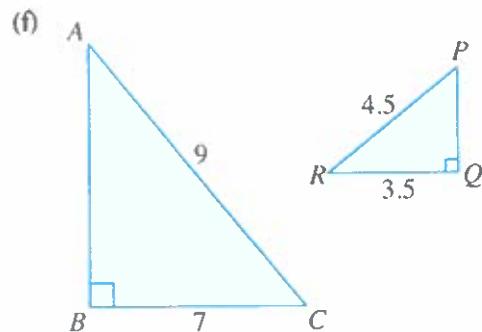
4. Which of the following pairs of triangles are similar? If they are similar, state the reason for similarity. All lengths are given in cm.

(a)

(b)

(c)

(d)

(e)

(f)

5. In the diagram, AOB and COD are straight lines, $AO = BO$ and $CO = DO$.

(i) Identify a pair of congruent triangles and state the congruence test.

(ii) Write down two pairs of equal angles.

6. In the diagram, PQ is equal and parallel to RS, $PR = 5$ cm and $Q\hat{S}R = 50°$.
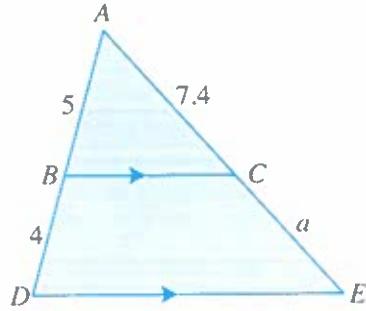
(i) Identify a pair of congruent triangles and state the congruence test.

(ii) Find the length of QS and $Q\hat{P}R$.

7. In the diagram, P and Q are points along the arms OA and OB of $A\hat{O}B$ respectively such that $OP = OQ$. A set square is used to construct perpendiculars to OA and OB at P and Q respectively. The perpendiculars meet at C. Explain why OC is the angle bisector of $A\hat{O}B$.
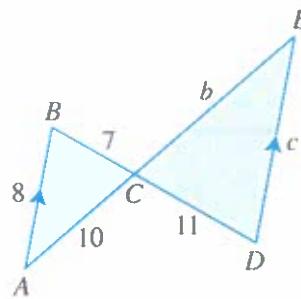
8. In each of the following figures, find the value of each of the unknowns. All lengths given are in cm.
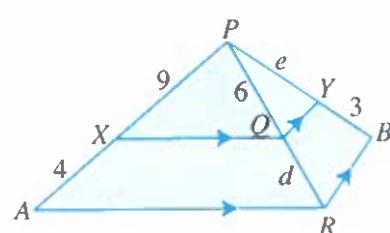
(a)
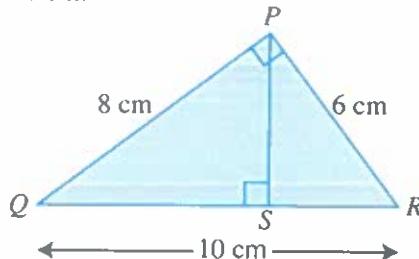
(b)
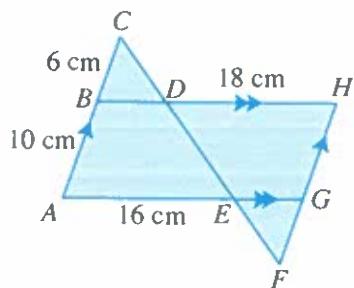
(c)

9. In the figure, the angle QPR is a right angle, PS is perpendicular to QR, $PQ = 8$ cm, $PR = 6$ cm and $QR = 10$ cm.
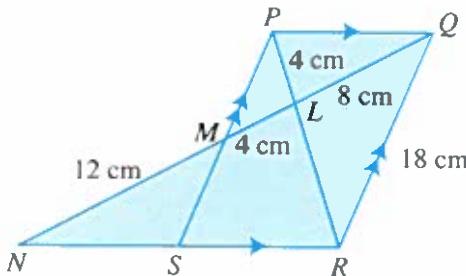
(i) Name a triangle similar to $\triangle PQS$.

(ii) Calculate the length of QS.

**10.** In the diagram, ABC, $CDEF$, FGH, BDH and AEG are straight lines. BH is parallel to AG, AC is parallel to FH, $AB = 10$ cm, $BC = 6$ cm, $AE = 16$ cm and $DH = 18$ cm.
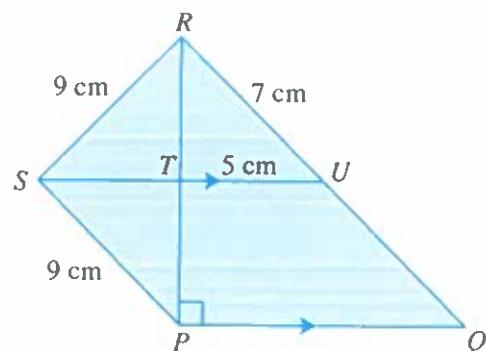
(a) (i) Identify two triangles similar to $\triangle BCD$.

(ii) Calculate the length of BD.

(b) Find the length of EG and of FH.

(c) Prove that $\triangle ACE$ and $\triangle HFD$ are similar.

11. In the diagram, PLR and $QLMN$ are straight lines, PQ is parallel to NR, SP is parallel to RQ, $QL = 8$ cm, $LM = PL = 4$ cm, $MN = 12$ cm and $QR = 18$ cm.
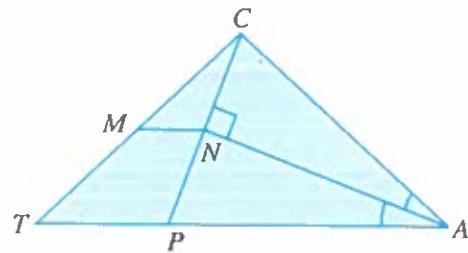
(a) (i) Name a triangle similar to $\triangle PLQ$.

(ii) Calculate the length of LR.

(b) (i) Name the triangle similar to $\triangle NQR$.

(ii) Calculate the length of MS.

(c) Name three other pairs of similar triangles.
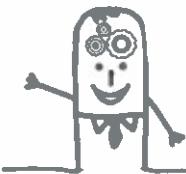
12. In the diagram, STU, RTP and RUQ are straight lines, SU is parallel to PQ, $R\hat{P}Q = 90°$, $SR = SP = 9$ cm, $TU = 5$ cm and $RU = 7$ cm.

(i) Identify two triangles which are congruent.

(ii) Find the length of UQ and of PQ.

13. In $\triangle CAT$, M is the midpoint of CT, $C\hat{A}N = P\hat{A}N$ and CP is a straight line that is perpendicular to NA.
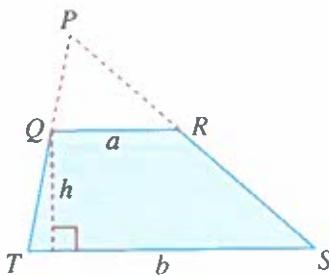
(i) Explain why $\triangle CAN$ is congruent to $\triangle PAN$.

(ii) Hence, or otherwise, explain why $MTAN$ is a trapezium.
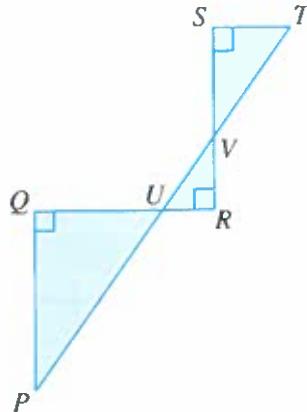
1. The figure shows a trapezium $QRST$, where $QR = a$ units and $ST = b$ units. The height of the trapezium is h units.
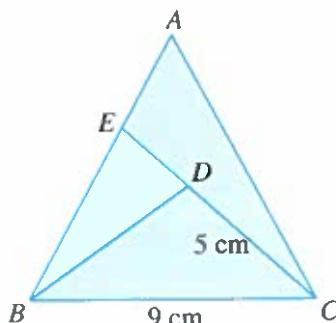
   By using similar triangles, show that the area of the trapezium is given by $\frac{1}{2}(a+b)h$.

2. In the diagram, $P\hat{Q}R = Q\hat{R}S = R\hat{S}T = 90°$, $PQ = QR = RS = 5$ cm and $ST = 1$ cm.

   Given that PUT and RS intersect at V, find the length of QU.

3. In the diagram, $AB = AC$, $CB = CE$, $BD = BE$, $BC = 9$ cm and $CD = 5$ cm.

   Find the length of AC.

Area and Volume of Similar Figures and Solids

The scale drawing of a structure is a representation of the actual size of the structure. This drawing shows all the dimensions which are necessary to build the structure. From the scale drawing, we are able to make use of the relationship between the lengths, the areas and the volumes of similar objects to calculate the actual surface area and volume of the structure.
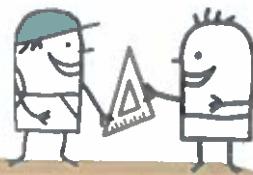
Chapter Twelve

12.1 Area of Similar Figures

In Book 2, we have learnt about similar triangles. In this chapter, we will learn how to find the area of similar figures and the volume of similar solids.

Investigation

Areas of Similar Figures

1. Table 12.1 shows three squares. Are they similar? Explain your answer.

Square			
Length of Square	1 cm	2 cm	3 cm
Area of Square			

Table 12.1

2. Complete Table 12.1 to find the area of each square.

3. **(a)** The length of the second square is double that of the first square. What is the relationship between their areas?

   (b) The length of the third square is three times that of the first square. What is the relationship between their areas?

4. Let the length and the area of a square be l_1 and A_1 respectively. Let the length and the area of a second square be l_2 and A_2 respectively. Note that the two squares are **similar**.

   Express the following ratio of areas in terms of l_1 and l_2.

$$\frac{A_2}{A_1} =$$

5. Is the formula in Question 4 always true?

   Let us investigate what happens if we have similar triangles instead.

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Table 12.2 shows three **similar** triangles.

Triangle			
Length of Corresponding Side of Triangle	1 unit	2 units	3 units
Area of Triangle	1 square unit		

Table 12.2

6. **(a)** The length of a side of the second triangle is double that of the corresponding side of the first triangle.
   What is the relationship between their areas?
   (b) The length of a side of the third triangle is three times that of the corresponding side of the first triangle.
   What is the relationship between their areas?

7. Let the length and the area of a triangle be l_1 and A_1 respectively.
   Let the length and the area of a second *similar* triangle be l_2 and A_2 respectively.
   Express the following ratio of areas in terms of l_1 and l_2.

$$\frac{A_2}{A_1} =$$

In general, the ratio of the areas of two **similar** figures is the square of the ratio of their corresponding lengths, i.e.

$$\frac{A_2}{A_1} = \left(\frac{l_2}{l_1}\right)^2$$

where A_1 and l_1 are the area and the length of the first figure respectively,
and A_2 and l_2 are the area and the length of the second similar figure respectively.
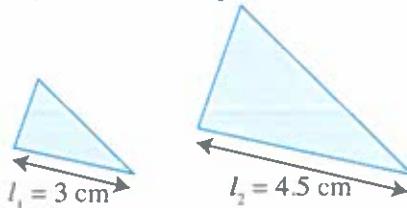
Worked Example 1
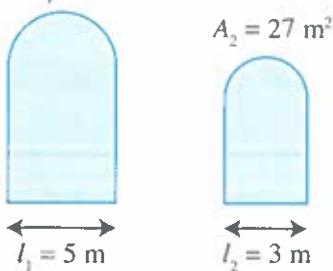
Find the unknown area of each of the following pairs of similar figures.

(a)

$A_1 = 4$ cm^2 $A_2 = ?$

$l_1 = 3$ cm $l_2 = 4.5$ cm

(b) $A_1 = ?$

$A_2 = 27$ m^2

$l_1 = 5$ m $l_2 = 3$ m

Solution:

(a) $\dfrac{A_2}{A_1} = \left(\dfrac{l_2}{l_1}\right)^2$

$\dfrac{A_2}{4} = \left(\dfrac{4.5}{3}\right)^2$

$\quad = \dfrac{9}{4}$

$\therefore A_2 = \dfrac{9}{4} \times 4$

$\quad = 9$ cm^2

(b) $\dfrac{A_1}{A_2} = \left(\dfrac{l_1}{l_2}\right)^2$

$\dfrac{A_1}{27} = \left(\dfrac{5}{3}\right)^2$

$\quad = \dfrac{25}{9}$

$\therefore A_1 = \dfrac{25}{9} \times 27$

$\quad = 75$ m^2

INFORMATION

To simplify $\left(\dfrac{4.5}{3}\right)^2$ without using a calculator, we have

$\left(\dfrac{4.5}{3}\right)^2 = \left(\dfrac{4.5 \times 2}{3 \times 2}\right)^2$

$= \left(\dfrac{9}{6}\right)^2$

$= \left(\dfrac{3}{2}\right)^2$
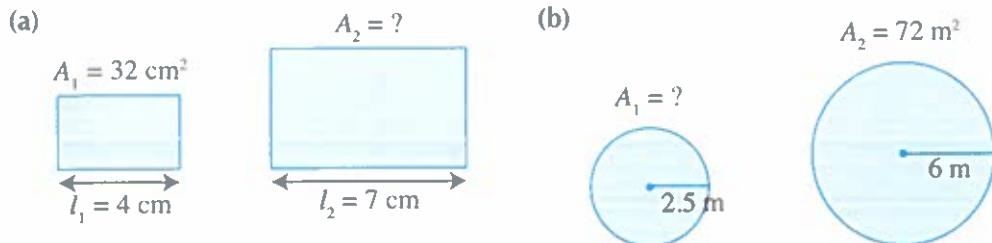
$= \dfrac{9}{4}$

Problem Solving Tip

For (b), write the unknown A_1 first. It will help in subsequent algebraic manipulations.

Find the unknown area of each of the following pairs of similar figures.

(a)

$A_1 = 32$ cm^2

$A_2 = ?$

$l_1 = 4$ cm

$l_2 = 7$ cm

(b)

$A_1 = ?$

$A_2 = 72$ m^2

2.5 m

6 m
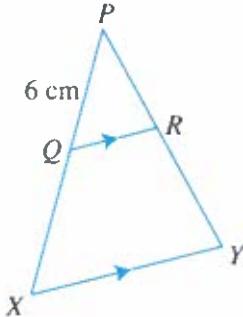
Worked Example 2

(Finding the Length of Similar Figures)

In the figure, QR is parallel to XY, $PQ = 6$ cm and the areas of $\triangle PQR$ and $\triangle PXY$ are 9 cm^2 and 64 cm^2 respectively. Find the length of QX.

P

6 cm

Q — R

X — Y

Solution:

Since QR is parallel to XY, $\triangle PQR$ and $\triangle PXY$ are similar.

$$\left(\frac{PX}{PQ}\right)^2 = \frac{\text{Area of } \triangle PXY}{\text{Area of } \triangle PQR}$$

$$\left(\frac{PX}{6}\right)^2 = \frac{64}{9}$$

$$\frac{PX^2}{36} = \frac{64}{9}$$

$$PX^2 = \frac{64}{9} \times 36$$

$$= 256$$

$$PX = \sqrt{256}$$

$$= 16 \text{ cm } (-16 \text{ is rejected because } PX > 0)$$

$$QX = PX - PQ$$

$$= 16 - 6$$

$$= 10 \text{ cm}$$

ATTENTION
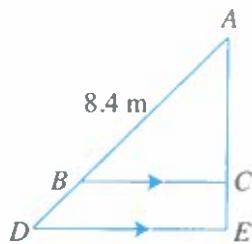
In $\triangle PQR$ and $\triangle PXY$,
$\angle P$ is a common angle,
$\angle PQR = \angle PXY$ (corr. $\angle$s) and
$\angle PRQ = \angle PYX$ (corr. $\angle$s).

In the figure, BC is parallel to DE, $AB = 8.4$ m and the areas of $\triangle ABC$ and $\triangle ADE$ are 49 m² and 100 m² respectively. Find the length of BD.
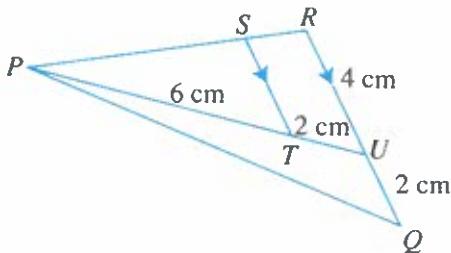
Worked Example 3
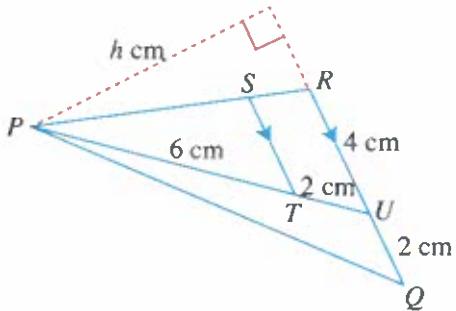
(Problem involving Similar Figures)

In the figure, ST is parallel to RU, $RU = 4$ cm, $UQ = 2$ cm, $PT = 6$ cm and $TU = 2$ cm. Given that the area of $\triangle PUR$ is 12 cm², find the area of

(i) $\triangle PQU$,   (ii) $\triangle PTS$.

Solution:

(i) Notice that $\triangle PUR$ and $\triangle PQU$ have a common height corresponding to the bases RU and UQ respectively.

Let the common height be h cm.

$$\frac{\text{Area of } \triangle PQU}{\text{Area of } \triangle PUR} = \frac{\frac{1}{2} \times UQ \times h}{\frac{1}{2} \times RU \times h}$$

$$= \frac{UQ}{RU}$$

$$\frac{\text{Area of } \triangle PQU}{12} = \frac{2}{4}$$

$$\text{Area of } \triangle PQU = \frac{2}{4} \times 12$$

$$= 6 \text{ cm}^2$$

(ii) Since ST is parallel to RU, $\triangle PST$ and $\triangle PRU$ are similar.

$$\frac{\text{Area of } \triangle PTS}{\text{Area of } \triangle PUR} = \left(\frac{PT}{PU}\right)^2$$

$$\frac{\text{Area of } \triangle PTS}{12} = \left(\frac{6}{8}\right)^2$$

$$= \frac{9}{16}$$

$$\text{Area of } \triangle PTS = \frac{9}{16} \times 12$$

$$= 6.75 \text{ cm}^2$$
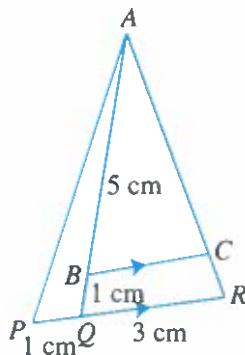
In $\triangle PST$ and $\triangle PRU$,
$\angle P$ is a common angle,
$\angle PST = \angle PRU$ (corr. $\angle$s) and
$\angle PTS = \angle PUR$ (corr. $\angle$s).

PRACTISE NOW 3

In the figure, BC is parallel to QR, $AB = 5$ cm, $BQ = 1$ cm, $PQ = 1$ cm and $QR = 3$ cm. Given that the area of $\triangle AQR$ is 21 cm², find the area of

(i) $\triangle APQ$,   **(ii)** $\triangle ABC$.
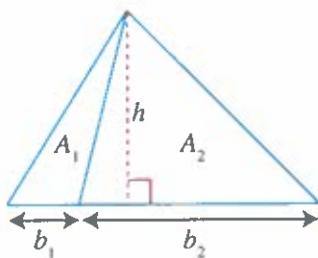
From Worked Example 3, we can conclude that the ratio of the areas of two triangles having a *common height* h is equal to the ratio of the lengths of the bases of the two triangles, i.e.

$$\frac{A_1}{A_2} = \frac{b_1}{b_2},$$
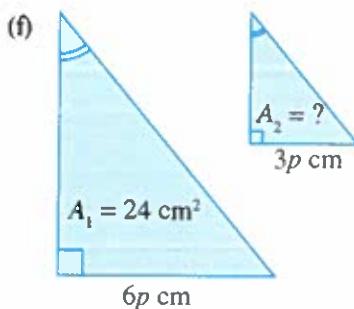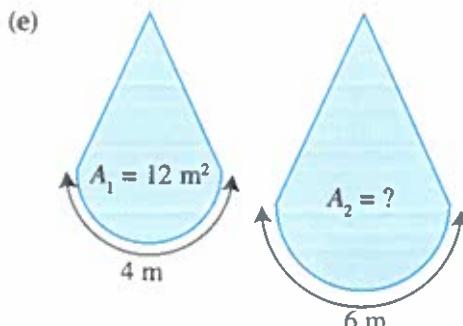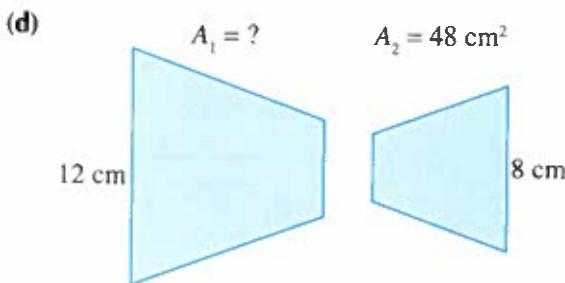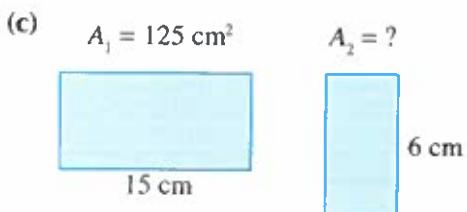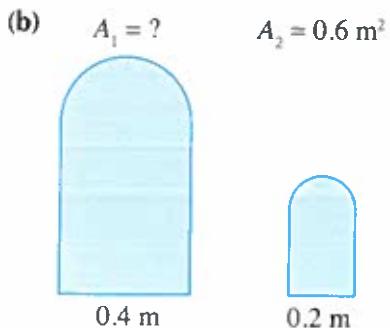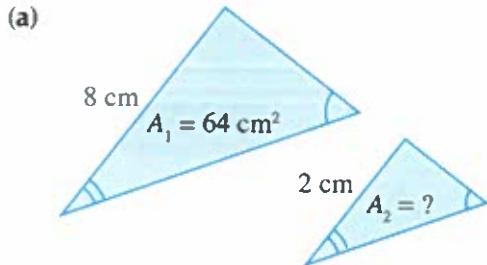
where A_1 and b_1 are the area and the length of the base of the first triangle respectively, and A_2 and b_2 are the area and the length of the base of the second triangle respectively.
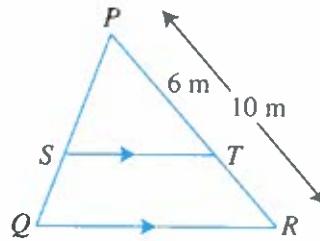
BASIC LEVEL

1. Find the unknown area of each of the following pairs of similar figures.

(a)

8 cm $A_1 = 64$ cm^2

2 cm $A_2 = ?$

(b) $A_1 = ?$ $A_2 = 0.6$ m^2

0.4 m   0.2 m

(c) $A_1 = 125$ cm^2 $A_2 = ?$

15 cm   6 cm

(d) $A_1 = ?$ $A_2 = 48$ cm^2

12 cm   8 cm

(e)

$A_1 = 12$ m^2 $A_2 = ?$

4 m   6 m

(f)

$A_2 = ?$ 3p cm

$A_1 = 24$ cm^2

6p cm

2. Find the ratio of the areas of two circles whose radii are 4 cm and 7 cm.

3. A triangular plot of land *PQR* is such that *PT* = 6 m and *PR* = 10 m. *ST* and *QR* are two water pipes that are parallel to each other. The area of Δ*PST* is 24 m^2.

P
6 m 10 m
S → T
Q → R
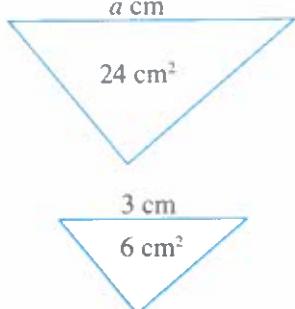
Find the area of the land occupied by
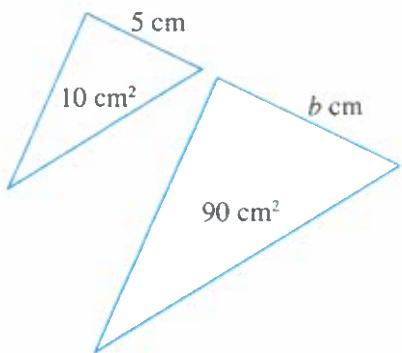
(i) Δ*PQR*,   (ii) *SQRT*.

4. Find the unknown value in each of the following pairs of similar figures.

(a)

a cm

24 cm²

3 cm

6 cm²

(b)

5 cm

10 cm²

b cm

90 cm²

(c)

240 m²

15 m²

5 m

c m

(d) 27 cm²

6 cm

12 cm²

d cm
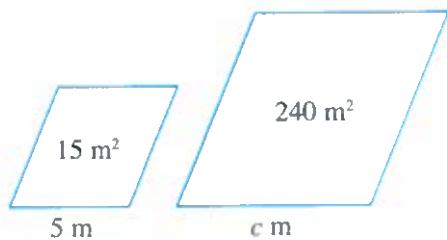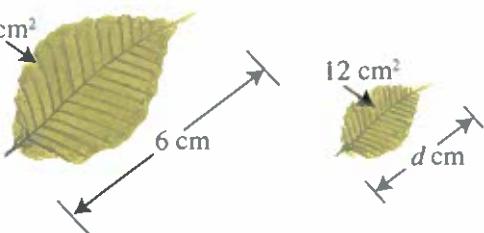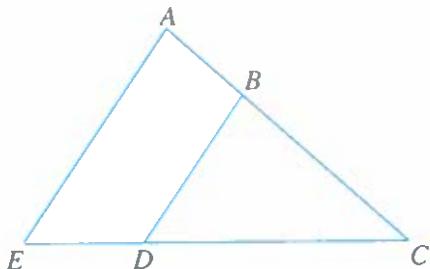
5. The perimeters of two similar regular hexagons are 10 m and 8 m. Given that the area of the larger hexagon is 200 m², find the area of the smaller hexagon.
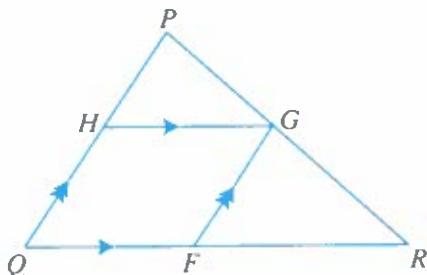
6. In the figure, $\triangle CAE$ is an enlargement of $\triangle CBD$ with a scale factor of $\frac{4}{3}$.

Given that the area of $\triangle CBD$ is 9 cm², find the area of $ABDE$.
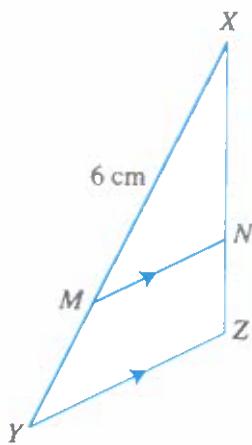
7. In a scale drawing of a house, the width, 150 cm, of a door is represented by a line 30 mm long. Find the actual land area, in square metres, occupied by the house if the corresponding area on the plan is 3250 cm².

8. In the figure, HG is parallel to QR, GF is parallel to PQ and $QF : FR = p : q$.
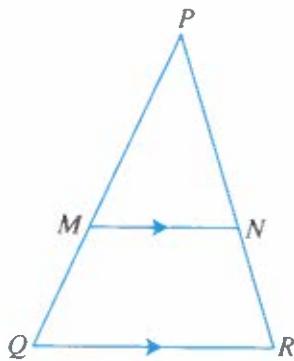
Find the ratio of the area of $\triangle PHG$ to that of $\triangle PQR$ in terms of p and q.

9. Two solid cones are geometrically similar and the height of one cone is 1.5 times that of the other. Given that the height of the smaller cone is 12 cm and its surface area is 124 cm², find

(i) the height, **(ii)** the surface area, of the larger cone.

10. In the figure, $\triangle XYZ$ is an enlargement of $\triangle XMN$.
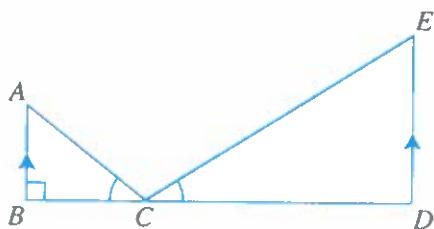
Given that $XM = 6$ cm and that the areas of $\triangle XMN$ and $MYZN$ are 14 cm² and 22 cm² respectively, find the length of MY.

11. In the figure, MN is parallel to QR.
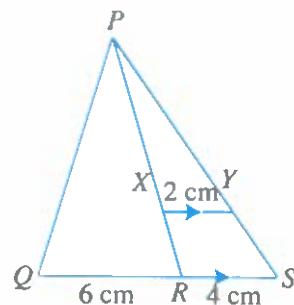
If the areas of $\triangle PMN$ and trapezium $MQRN$ are in the ratio $9 : 16$, find the ratio $MN : QR$.

12. In the figure, BCD is a straight line and BA is parallel to DE. $\angle ABC = 90°$ and $\angle ACB = \angle ECD$. The areas of $\triangle ABC$ and $\triangle CDE$ are 25 cm² and 64 cm² respectively.

Given further that CD is 4.5 cm longer than BC, find the length of BC.

13. In the figure, XY is parallel to RS, $XY = 2$ cm, $QR = 6$ cm and $RS = 4$ cm.
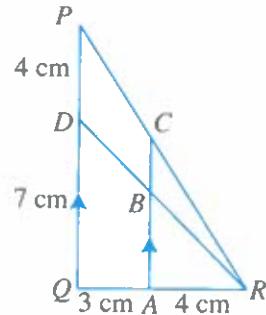
Given that the area of $\triangle PXY$ is 10 cm², find the area of

(i) $\triangle PRS$, (ii) $\triangle PQR$.
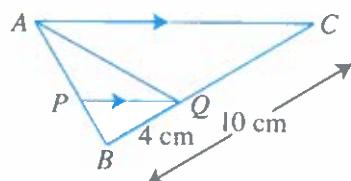
14. In the figure, AC is parallel to QP, $AR = 4$ cm, $QA = 3$ cm, $DQ = 7$ cm and $PD = 4$ cm.

Find

(i) the length of BC,

(ii) the ratio of the area of $\triangle ARB$ to that of $\triangle BRC$,

(iii) the ratio of the area of $\triangle BRC$ to $ABDQ$.

15. In the figure, PQ is parallel to AC.
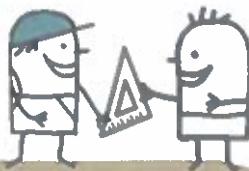
Given that $BQ = 4$ cm, $BC = 10$ cm and the area of $\triangle BPQ$ is 8 cm², find the area of

(i) $\triangle ABC$, (ii) $\triangle PQC$,

(iii) $\triangle AQC$.

12.2 Volume of Similar Solids

In Section 12.1, we have learnt how to find the area of similar figures. In this section, we will learn how to find the volume of similar solids.

Investigation

Volume and Mass of Similar Solids

1. Table 12.3 shows three cubes. Are they similar? Explain your answer.

Cube			
Length of Cube	1 cm	2 cm	3 cm
Volume of Cube			

Table 12.3

2. Complete Table 12.3 to find the volume of each cube.
3. (a) The length of the second cube is double that of the first cube. What is the relationship between their volumes?
   (b) The length of the third cube is three times that of the first cube. What is the relationship between their volumes?
4. Let the length and the volume of a cube be l_1 and V_1 respectively. Let the length and the volume of a second cube be l_2 and V_2 respectively. Note that the two cubes are **similar**. Express the following ratio of volumes in terms of l_1 and l_2.

$$\frac{V_2}{V_1} =$$

5. Is the formula in Question 4 always true?

   Let us investigate what happens if we have similar cylinders instead.

   Fig. 12.1 shows two **similar** cylinders: they have exactly the same shape, i.e. the ratio of the corresponding lengths is a constant k, called the **scale factor** of the solids.

   For example,

   $$\frac{h_2}{h_1} = k$$

   where h_1 and h_2 are the heights of the first and second cylinders respectively.
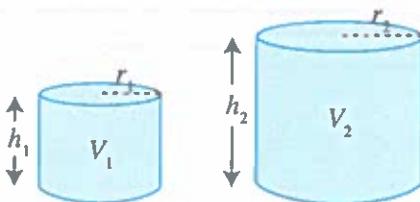
   Fig. 12.1

   The radii of the circular cross sections of the two cylinders are r_1 and r_2 respectively. What is the value of $\frac{r_2}{r_1}$?

6. The volume of the first cylinder is $V_1 = \pi r_1^2 h_1$.

   (a) Find the volume of the second cylinder, V_2, in terms of r_1 and h_1.

   (b) Hence, find the volume of the second cylinder, V_2, in terms of V_1.

7. Express the following ratio of volumes in terms of k, then in terms of h_1 and h_2, and then in terms of r_1 and r_2.

   $$\frac{V_2}{V_1} =$$

8. If two similar solids are made of the same material, what is the relationship between their masses m_1 and m_2?

   Let the density of the material be d. Then

   $$d = \frac{m_1}{V_1} = \frac{m_2}{V_2},$$

   where V_1 and V_2 are the volumes of the solids respectively.

   Express the following in terms of V_1 and V_2.

   $$\frac{m_2}{m_1} =$$

   In general, the ratio of the volumes of two similar solids is the cube of the ratio of their corresponding lengths, and the ratio of their masses is equal to the ratio of their volumes, i.e.
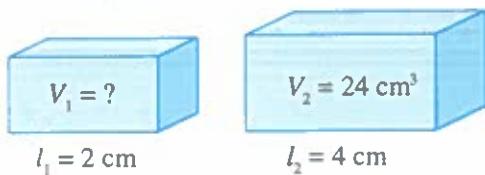
   $$\frac{V_2}{V_1} = \left(\frac{l_2}{l_1}\right)^3 \text{ and } \frac{m_2}{m_1} = \frac{V_2}{V_1},$$

   where V_1, l_1 and m_1 are the volume, length and mass of the first solid respectively, and V_2, l_2 and m_2 are the volume, length and mass of the second similar solid respectively.

Worked Example 4

(Finding the Volume of Similar Solids)

The figure shows two toy blocks which take the shape of a pair of similar cuboids.

$V_1 = ?$

$V_2 = 24$ cm^3

$l_1 = 2$ cm $\qquad l_2 = 4$ cm

Find the volume, V_1, of the smaller block.

Solution:

$$\frac{V_1}{V_2} = \left(\frac{l_1}{l_2}\right)^3$$

$$\frac{V_1}{24} = \left(\frac{2}{4}\right)^3$$
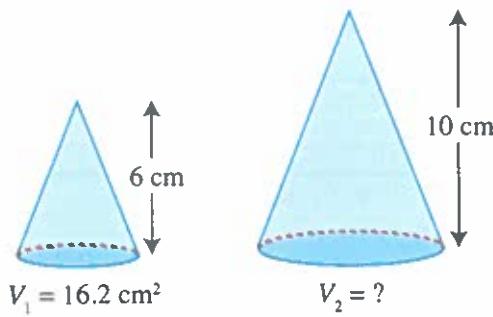
$$= \left(\frac{1}{2}\right)^3$$

$$= \frac{1}{8}$$

$$\therefore V_1 = \frac{1}{8} \times 24$$

$$= 3 \text{ cm}^3$$
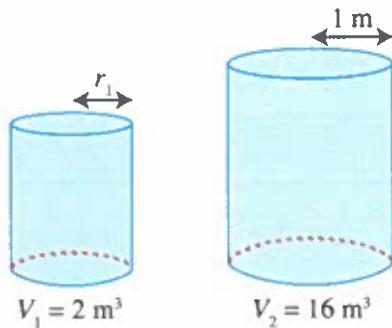
PRACTISE NOW 4

1. The figure shows two chocolate hats which take the shape of a pair of similar cones.

   6 cm

   10 cm

   $V_1 = 16.2$ cm^2 $\qquad V_2 = ?$

   Find the volume, V_2, of the larger chocolate hat.

2. Find the unknown radius, r_1, for the following pair of similar cylinders.

   1 m

   r_1

   $V_1 = 2$ m^3 $\qquad V_2 = 16$ m^3

Worked Example 5

Two solid spheres of diameters 4 m and 5 m are made of the same material. Given that the smaller sphere has a mass of 120 kg, find the mass of the larger sphere.

Solution:

Let m_1, V_1 and l_1 be the mass, volume and diameter of the smaller sphere respectively, and m_2, V_2 and l_2 be the mass, volume and diameter of the larger sphere respectively.

$$\frac{m_2}{m_1} = \frac{V_2}{V_1}$$

$$= \left(\frac{l_2}{l_1}\right)^3$$

$$\frac{m_2}{120} = \left(\frac{5}{4}\right)^3$$

$$m_2 = \left(\frac{5}{4}\right)^3 \times 120$$

$$= 234\frac{3}{8}$$

∴ The mass of the larger sphere is $234\frac{3}{8}$ kg.

PRACTISE NOW 5

1. Two similar solid triangular prisms have heights 5 cm and 8 cm. Given that the smaller prism has a mass of 80 g, find the mass of the larger prism, giving your answer correct to the nearest integer.

2. The figure shows a statue with a height of 20 cm and a mass of 3 kg. Michael wishes to make a similar statue with a height of 2 m using the same material.
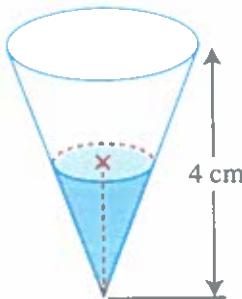
Find the mass of the statue made by Michael.

Worked Example 6

The figure shows an inverted conical container of height 4 cm. It contains a volume of water which is equal to one-eighth of its full capacity.

Find

(i) the depth of the water,

(ii) the ratio of the area of the top surface of the water to the area of the top surface of the container.

Solution:

(i) Let V_1 and h_1 be the volume and height of the smaller cone respectively, and V_2 and h_2 be the volume and height of the larger cone respectively.

$$\frac{V_1}{V_2} = \left(\frac{h_1}{h_2}\right)^3$$

$$\frac{1}{8} = \left(\frac{h_1}{4}\right)^3 \quad \text{(Since } V_1 = \frac{1}{8}V_2 \text{, then } \frac{V_1}{V_2} = \frac{1}{8} \text{.)}$$

$$\left(\frac{h_1}{4}\right)^3 = \frac{1}{8}$$

$$\frac{h_1}{4} = \sqrt[3]{\frac{1}{8}}$$

$$= \frac{1}{2}$$

$$h_1 = \frac{1}{2} \times 4$$

$$= 2$$
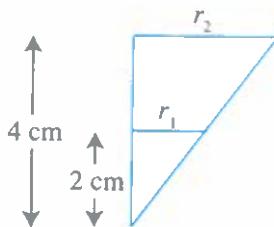
∴ The depth of the water is 2 cm.

(ii) The top surface of the water and that of the container are circles.

Let r_1 and r_2 be the radii of the smaller circle and the larger circle respectively.

Using similar triangles,

$$\frac{r_1}{r_2} = \frac{h_1}{h_2}$$

$$= \frac{2}{4}$$

$$= \frac{1}{2}$$

4 cm
2 cm
r_2
r_1

Let A_1 and A_2 be the areas of the smaller circle and larger circle respectively.

$$\frac{A_1}{A_2} = \left(\frac{r_1}{r_2}\right)^2$$
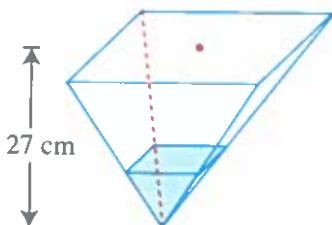
$$= \left(\frac{1}{2}\right)^2$$

$$= \frac{1}{4}$$

∴ The ratio of the area of the top surface of the water to the area of the top surface of the container is $1 : 4$.

PRACTISE NOW 6

The figure shows a container in the shape of an inverted right pyramid of height 27 cm. It contains a volume of vegetable oil which is equal to one-sixth of its full capacity.

27 cm
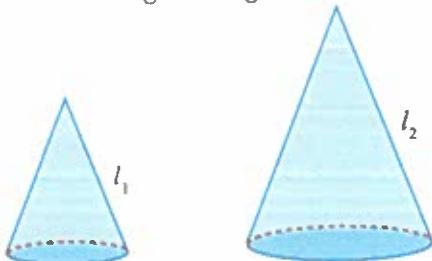
Find

(i) the depth of the vegetable oil,

(ii) the ratio of the area of the top surface of the vegetable oil to the area of the top surface of the container, giving your answer in the form $1 : n$.

Thinking Time

1. Two similar cones with slant heights are given as shown.

Express the ratio of the total surface area of the smaller cone to that of the larger cone in terms of l_1 and l_2. Explain your answer.

2. The similarity ratio formulae, $\dfrac{A_2}{A_1} = \left(\dfrac{l_2}{l_1}\right)^2$ and $\dfrac{m_2}{m_1} = \dfrac{V_2}{V_1} = \left(\dfrac{l_2}{l_1}\right)^3$, have some real-life implications.

For example, why is it not possible for a human to be a giant with a height of about 20 m?

Journal Writing

Search on the Internet to find out the locations of 5 different Merlions in Singapore which are recognised by the Singapore Tourism Board.

(a) Find the ratio of the heights of these 5 Merlions.

(b) Hence, find the ratio of
   (i) the total surface area of these Merlions,
   (ii) the volume of material used to construct these Merlions.

State any assumptions that you have made.
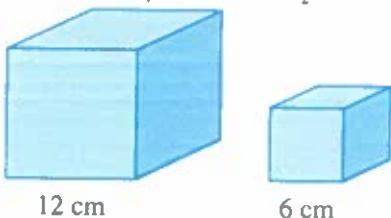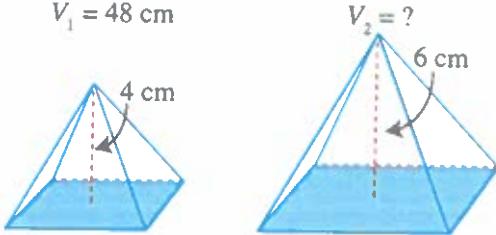
BASIC LEVEL

1. Find the unknown volume of each of the following pairs of similar solids.

(a)

$V_1 = ?$ $V_2 = 72 \text{ cm}^3$

12 cm   6 cm

(b) $V_1 = 48 \text{ cm}$ $V_2 = ?$

6 cm

4 cm

(c) $V_1 = ?$ $V_2 = 12 \text{ cm}^3$

2.5 cm

7.5 cm

(d)

$V_1 = ?$ $V_2 = 2464 \text{ m}^3$

4 m   16 m

(e)

$V_1 = 3.2 \text{ m}^3$

$V_2 = ?$

64 cm

32 cm
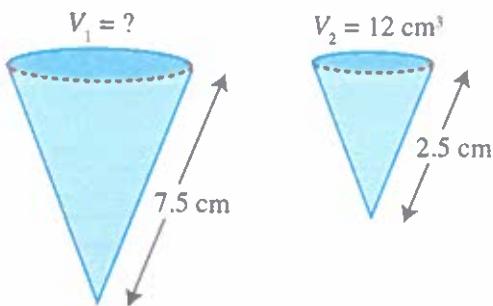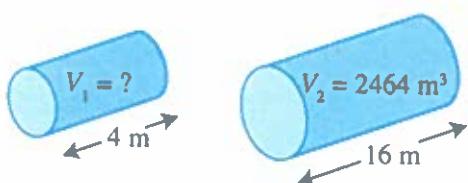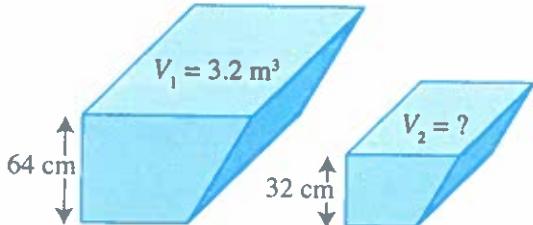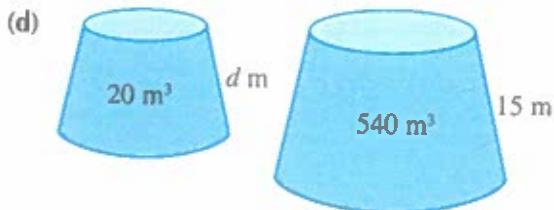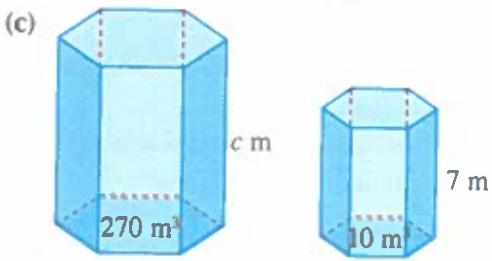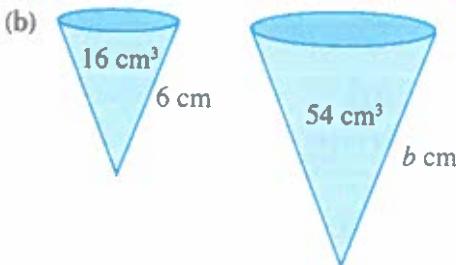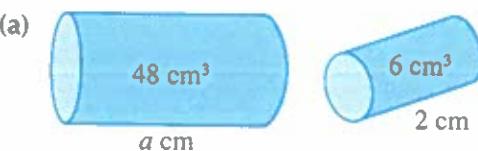
2. Find the ratio of the volumes of
   (a) two similar solid cylinders of circumferences 10 cm and 8 cm,
   (b) two similar solid cones of heights 9 cm and 12 cm,
   (c) two solid spheres of radii 4 cm and 6 cm.

3. In a restaurant, a Junior glass has a height of 6 cm and a Senior glass has a height of 9 cm. Given that the capacity of a Senior glass is 540 cm³, find the capacity of the Junior glass.

4. Find the unknown value in each of the following pairs of similar solids.

(a)

48 cm³   6 cm³

2 cm

a cm

(b)

16 cm³

6 cm   54 cm³

b cm

(c)

c m

270 m³   7 m

10 m

(d)

20 m³ d m

540 m³   15 m
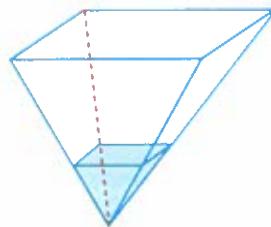
5. The areas of the bases of two similar cones are in the ratio $9 : 16$.
   (i) Find the ratio of the heights of the cones.
   (ii) Given that the volume of the larger cone is 448 cm³, find the volume of the smaller cone.

6. The masses of two spheres of the same material are 640 kg and 270 kg. Find the ratio of their diameters.

7. A certain brand of chilli flakes comes in similar bottles of two sizes – 'mini' and 'ordinary'. The 'mini' bottle has a mass of 280 g and a height of 15 cm. Given that the 'ordinary' bottle has a mass of 750 g, find its height.

8. Two similar solid candy canes have heights 4 cm and 7 cm.
   (i) Find the ratio of the total surface areas of the candy canes.
   (ii) Given that the smaller candy cane has a mass of 10 g, find the mass of the larger candy cane.
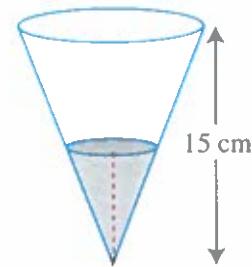
9. The volume of one sphere is 4 times that of another sphere. Given that the radius of the smaller sphere is 3 cm, find the radius of the larger sphere.

10. The mass of a glass figurine of height 6 cm is 500 g. Find the mass of a similar glass figurine if it has a height of 4 cm.

11. A train is 10 m long and has a mass of 72 tonnes. A similar model, made of the same material, is 40 cm long. (1 tonne = 1000 kg)
    (i) Find the mass of the model.
    (ii) Given that the tank of the model train contains 0.85 litres of water when it is full, find the capacity of the tank of the train, giving your answer correct to the nearest integer.

12. The masses of two similar plastic boxes are 8.58 kg and 4.29 kg. Given that the first box has a base area of 12.94 m², find the base area of the second box.

13. The figure shows a container in the shape of an inverted right pyramid which contains some water. The area of the top surface of the container is 63 cm² and the area of the top surface of the water is 28 cm².

Find
   (i) the depth of the water if its volume is 336 cm³,
   (ii) the ratio of the depth of the water to the height of the container,
   (iii) the capacity of the container.

14. The figure shows an inverted conical container of height 15 cm found in a laboratory. It contains a volume of mercury which is equal to $\frac{8}{27}$ of its full capacity.
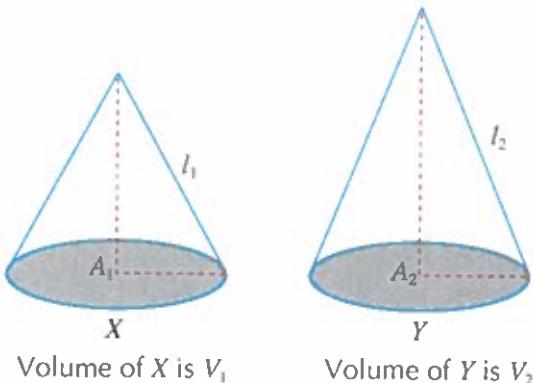
Find
   (i) the depth of the mercury,
   (ii) the area of the mercury that is exposed to the air if the area of the top surface of the container is 45 cm²,
   (iii) the capacity of the container.

15. A clay model has a mass of x^2 kg and a height of 30 cm. A similar clay model has a mass of $(x + 0.3)$ kg and a height of 20 cm. Find the value of x.

Volume of X is V_1 Volume of Y is V_2

If X and Y are two similar solids, then
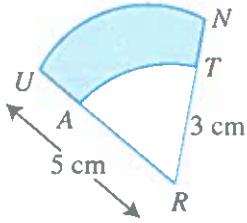
- Ratio of their corresponding lengths is $\dfrac{l_2}{l_1}$

- Ratio of their areas, $\dfrac{A_2}{A_1} = \left(\dfrac{l_2}{l_1}\right)^2$

- Ratio of their volumes, $\dfrac{V_2}{V_1} = \left(\dfrac{l_2}{l_1}\right)^3$

Review Exercise 12

1. Find the ratio of the areas of two similar triangles if the lengths of their corresponding sides are

   (a) 3 cm and 5 cm,   (b) 4.5 m and 9 m,

   (c) 2 mm and 3 mm.

2. Find the ratio of the areas of two similar triangles if their perimeters are 294 cm and 336 cm.

3. (i) If the length of each side of a regular pentagon is doubled, what will happen to its area?

   (ii) Given that the length of each side of a pentagon, whose area is 25 cm², is doubled, find the area of the enlarged pentagon.

   (iii) If the length of each side of a regular n-sided polygon is tripled, what will happen to its area?
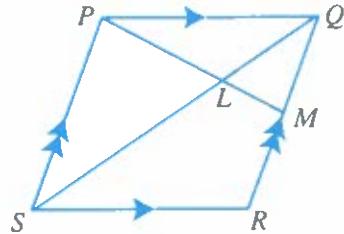
4. The volumes of two similar cylinders are in the ratio 8 : 27. Find the ratio of the base radii of the cylinders.

5. The volumes of two similar water jugs are in the ratio 27 : 64. Find
   (i) the ratio of the heights of the jugs,
   (ii) the ratio of the total surface areas of the jugs.

6. A model of a marble statue of height 3.2 m is made of the same material as the statue. Given that the height of the model is 40 cm and its mass is 12 kg, find the mass of the statue in tonnes.
   (1 tonne = 1000 kg)

7. In the figure, the radii of the sectors RAT and RUN are 3 cm and 5 cm respectively.
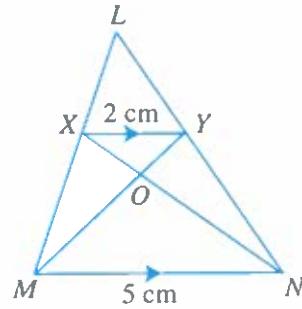
   Find the ratio of the area of the shaded region to that of the area of sector RUN.

8. Two artificial ponds are similar in every aspect. The perimeter of the surface of the larger pond is three times that of the smaller pond.
   (i) Write down the ratio of their surface areas.
   (ii) Given that the larger pond contains 10 800 litres of water, find the amount of water contained in the smaller pond.

9. In the figure, $PQRS$ is a parallelogram. PLM and SLQ are straight lines and M is the midpoint of QR.
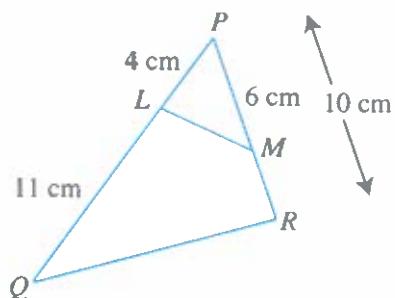
   (a) Identify a triangle similar to $\triangle QLM$.
   (b) Write down the numerical value of
       (i) $\dfrac{\text{area of } \triangle QLM}{\text{area of } \triangle SLP}$,
       (ii) $\dfrac{LS}{QS}$,
       (iii) $\dfrac{\text{area of } \triangle PLS}{\text{area of } \triangle PQS}$.

10. In the figure, XY is parallel to MN where $XY = 2$ cm and $MN = 5$ cm. XN and YM meet at O.
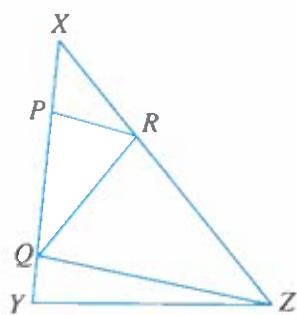
   (a) Find the value of
       $\dfrac{\text{area of quadrilateral } XMNY}{\text{area of } \triangle LMN}$.
   (b) (i) Identify a triangle similar to $\triangle XOY$.
       (ii) Write down the value of $\dfrac{YO}{OM}$.
       (iii) Show that the area of $\triangle XOY$: area of $\triangle XOM$: area of $\triangle MON = 4 : 10 : 25$.
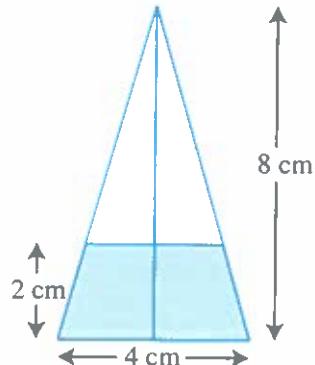
11. In the figure, PLQ and PMR are straight lines, $PL = 4$ cm, $LQ = 11$ cm, $PM = 6$ cm and $PR = 10$ cm.

(i) Show that $\triangle PQR$ and $\triangle PML$ are similar.

(ii) Write down the value of $\dfrac{\text{area of } \triangle PQR}{\text{area of } \triangle PML}$.

(iii) Given that the area of $\triangle PML$ is 6 cm², find the area of the quadrilateral $LMRQ$.

12. In the figure, $XP = \dfrac{1}{2}PQ$, $QY = \dfrac{1}{3}PQ$ and $XR = \dfrac{2}{5}XZ$.
Find the ratio of the area of $\triangle QRZ$ to that of $\triangle QYZ$.
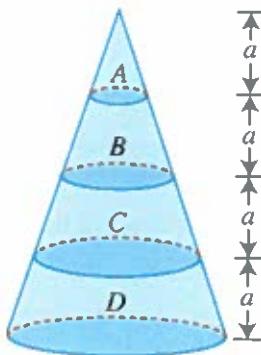
13. The figure shows a vertical section through the axis of a solid paper weight made in the shape of a right circular cone. Its height is 8 cm and the diameter of its base is 4 cm. The shaded portion is made of lead 2 cm thick while the unshaded portion is made of wood. The density of lead and wood are 11.3 g/cm³ and 0.9 g/cm³ respectively.

Find
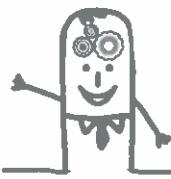
(i) the total volume of the paper weight,

(ii) the volume of the wooden portion,

(iii) the total mass of the paper weight.

14. A right circular cone is divided into 4 portions, A, B, C and D, by planes parallel to the base. The height of each portion is a units.

Find
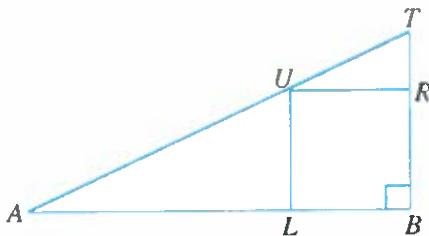
(i) the ratio of the volume of A to that of B,

(ii) the ratio of the volume of B to that of C,

(iii) the ratio of the sum of the volumes of A, B and C to that of D.
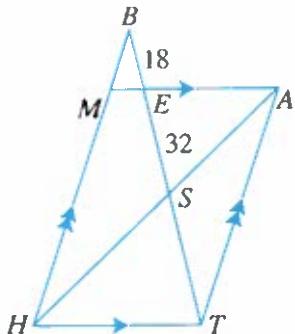
1. In the figure, *BAT* is a right-angled triangle with $AB = 2BT$. A square *BLUR* is inscribed in $\triangle BAT$ such that *L*, *U* and *R* lie on the sides of $\triangle BAT$.

   Hint: Let $BT = x$ and $BR = y$.
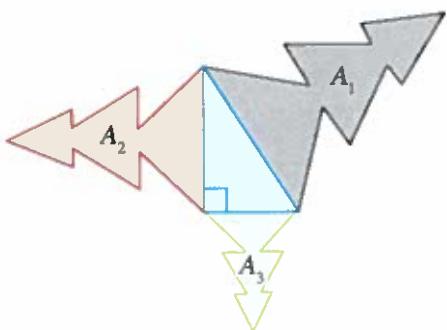
   Find the ratio of the area of the square *BLUR* to that of $\triangle BAT$.

2. In the figure, *MATH* is a parallelogram. *BMH* and *BEST* are straight lines, $BE = 18$ cm and $ES = 32$ cm.

   (i) Explain why $\triangle BME$ and $\triangle BHT$ are similar.
   (ii) Name 2 other pairs of similar triangles.
   (iii) Find the length of *ST*.
   (iv) Find the ratio of the areas of $\triangle BME$ and $\triangle BHT$.

3. The diagram shows a right-angled triangle. The three shaded figures are similar. Their areas are A_1, A_2 and A_3 as indicated. Prove that $A_1 = A_2 + A_3$. This is called the Generalised Pythagoras' Theorem.

Geometrical Properties of Circles

Structures such as bridges and archways are sometimes in the shape of an arc of a circle. The balcony of an apartment may also take the shape of an arc. If we are given an arc of a circle, are we able to determine the centre of the circle?

Chapter
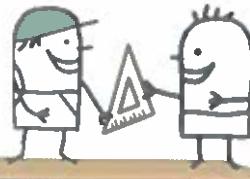
Thirteen

LEARNING OBJECTIVES

At the end of this chapter, you should be able to:

- apply the symmetry properties of circles:
  - equal chords are equidistant from the centre,
  - the perpendicular bisector of a chord passes through the centre,
  - tangents from an external point are equal in length,
  - the line joining an external point to the centre of the circle bisects the angle between the tangents,
- apply the angle properties of circles:
  - the angle in a semicircle is a right angle,
  - the angle between the tangent and radius of a circle is a right angle,
  - the angle at the centre is twice the angle at the circumference,
  - angles in the same segment are equal,
  - angles in opposite segments are supplementary,
  - angles in alternate segments are equal.

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13.1 Symmetric Properties of Circles

In this section, we will learn four symmetric properties of circles – two of them on chords and the other two on tangents.

Perpendicular Bisector of a Chord

Investigation

Circle Symmetric Property 1

Go to http://www.shinglee.com.sg/StudentResources/ and open the geometry template Circle Symmetric Property 1 as shown below.

There are three conditions:

Condition A: The line *l* (or *OM*) passes through the centre *O* of the circle.

Condition B: The line *l* (or *OM*) is perpendicular to the chord *AB*.

Condition C: The line *l* (or *OM*) **bisects** the chord *AB*.

Note that the chord *AB* must *not* be the diameter of the circle.

In this investigation, you will learn that **any two** of the above three conditions will imply the third one.

> **INFORMATION**
>
> 'Bisect' means 'cut into two equal parts.'

Part 1

1. The template shows a circle with centre *O* and the line *OM* perpendicular to the chord *AB*. Which two of the above three conditions are given?

Circle Symmetric Property 1a

Given Condition A: The line *l* (or *OM*) passes through the centre *O* of the circle.
Given Condition B: The line *l* (or *OM*) is **perpendicular** to the chord *AB*.
Move the points A, B and R.

Fig. 13.1

2. Click and drag point A or B to change the chord.

Click and drag point R to change the size of the circle.

(a) What do you notice about the length of AM and of MB?

(b) What do you call the point M?

Part 2

3. Click on the 'Next' button. The next page of the template shows a circle with centre O, and the line OM bisecting the chord AB.

Which two of the three conditions on the previous page are given?

Circle Symmetric Property 1b

Given Condition A: The line l (or OM) passes through the centre O of the circle.
Given Condition C: The line l (or OM) bisects the chord AB.
Move the points A, B and R.

Fig. 13.2

4. Click and drag point A or B to change the chord.

Click and drag point R to change the size of the circle.

What do you notice about the size of $\angle AMO$ and $\angle BMO$?

5. Use a sheet of paper to draw and cut out a circle. To find the centre of a circle, fold the circle into two equal halves, and then fold again into two equal halves as shown in Fig. 13.3(**a**).
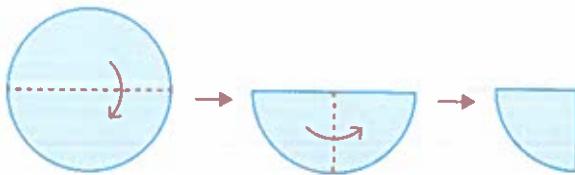
Fig. 13.3(a)

Open up the paper as shown in Fig. 13.3(**b**), where the dotted lines indicate the lines obtained from the above paper folding.

Fig. 13.3(b)

Mark the centre of the circle as *O* in Fig. 13.3(**b**). Why is this the centre of the circle?

> INFORMATION
>
> Can you think of other ways of folding to obtain the centre of the circle?

6. Using the same circle as in Question 5, fold along a chord *AB* that is not a diameter of the circle and then fold it into two equal halves as shown in Fig. 13.3(**c**).
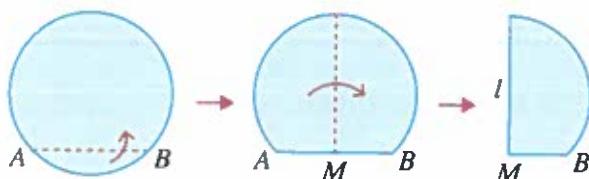
Fig. 13.3(c)

Open up the paper as shown in Fig. 13.3(**d**), where the dotted lines indicate the lines obtained from the above paper folding.
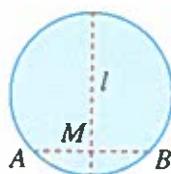
Fig. 13.3(d)

As the paper is folded into two equal halves, the line *l* bisects the chord *AB* and ∠*AMB*. Since ∠*AMB* = 180°, *l* is perpendicular to the chord *AB*.

(a) Which two of the three conditions on page 411 are satisfied?

(b) Does the line *l* pass through the centre *O* of the circle that you have marked in Question 5?

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From the investigation, there are three parts to **Circle Symmetric Property 1** (any two of the three conditions will imply the third one):

(i) If a line *l* **passes through the centre** of a circle and is **perpendicular to a chord** *AB* (which is not the diameter) of the circle, then the line *l* **bisects the chord** *AB*.

(ii) If a line *l* **passes through the centre** of a circle and **bisects a chord** *AB* (which is not the diameter) of the circle, then the line *l* is **perpendicular to the chord** *AB*.

(iii) If a line *l* **bisects the chord** *AB* of a circle and is **perpendicular to the chord** *AB* (i.e. *l* is the perpendicular bisector of the chord *AB*), then the line *l* **passes through the centre** of the circle. (This is also true if the chord is a diameter of the circle.)
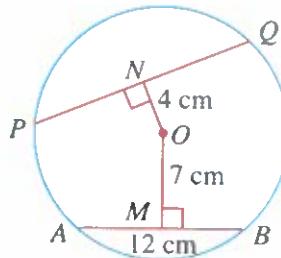
Thinking Time

1. If a line *l* passes through the centre of a circle and is perpendicular to a chord *AB* (which is not the diameter) of the circle, by using congruent triangles, prove that the line *l* bisects the chord *AB*.

2. If a line *l* passes through the centre of a circle and bisects a chord *AB* (which is not the diameter) of the circle, by using congruent triangles, prove that the line *l* is perpendicular to the chord *AB*.

Worked Example 1

In the figure, AB and PQ are chords of the circle with centre O. The point M lies on AB such that OM is perpendicular to AB and the point N lies on PQ such that ON is perpendicular to PQ.

Given that $AB = 12$ cm, $OM = 7$ cm and $ON = 4$ cm, find the length of the chord PQ, giving your answer correct to 2 decimal places.

Solution:

OM bisects AB (perpendicular bisector of chord).

$\therefore AM = MB$

$\quad = \dfrac{12}{2}$

$\quad = 6$ cm

Consider $\triangle OMA$.

$OA^2 = AM^2 + OM^2$ (Pythagoras' Theorem)

$\quad = 6^2 + 7^2$

$\quad = 85$

Since $OP = OA$ (radii of circle),
then $OP^2 = OA^2 = 85$.

Consider $\triangle ONP$.

$OP^2 = ON^2 + PN^2$ (Pythagoras' Theorem)

$OA^2 = 4^2 + PN^2$ (since $OA = OP$, radii of circle)

$\quad 85 = 16 + PN^2$

$\therefore PN^2 = 85 - 16$

$\qquad = 69$

i.e. $PN = \sqrt{69}$ (since length $PN > 0$)
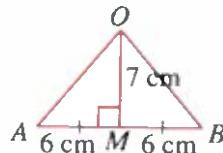
ON bisects PQ. (perpendicular bisector of chord)

$\therefore PQ = 2 \times PN$

$\quad = 2 \times \sqrt{69}$

$\quad = 16.61$ cm (to 2 d.p.)

Problem Solving Tip
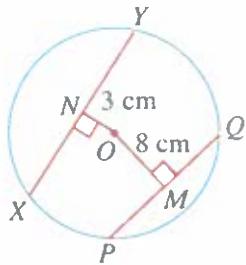
To find PQ, we first try to relate PQ to the information given in the question, i.e. OM, ON, the perpendicular distance of chords from the centre and the length of chord AB.

Since $OP = OA$ (radii of circle) and we have two right-angled triangles OMA and ONP, can we make use of Pythagoras' Theorem to find PN?
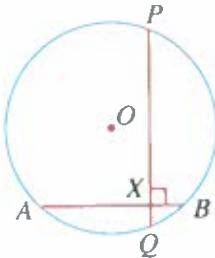
1. In the figure, PQ and XY are chords of the circle with centre O. The point M lies on PQ such that OM is perpendicular to PQ and the point N lies on XY such that ON is perpendicular to PQ. Given that $XY = 26$ cm, $OM = 8$ cm and $ON = 3$ cm, find the length of the chord PQ, giving your answer correct to 2 decimal places.

2. The figure shows a circle with centre O and radius 7 cm. The chords AB and PQ have lengths 11 cm and 13 cm respectively, and intersect at right angles at X. Find the length of OX.
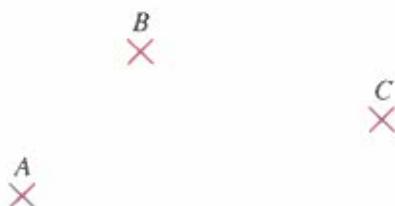
Class Discussion

Application of Circle Symmetric Property 1
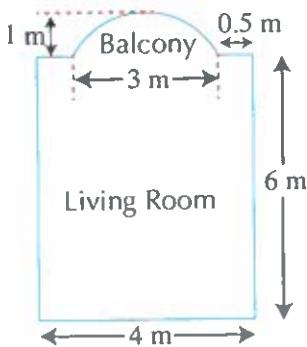
Work in pairs.

1. Construct a circle that passes through the three given points A, B and C.

B
✕

C
✕

A
✕

2. The diagram shows the plan of a living room with a balcony (not drawn to scale). The living room is rectangular (6 m by 4 m) and the balcony is an arc of a circle (see dimensions in diagram). Using a scale of 2 cm to represent 1 m, draw an accurate scale drawing of the living room with the balcony.

   Hint: Use the method in Question 1 to draw the arc of the balcony.

∷ Equal Chords

Investigation

Circle Symmetric Property 2

Go to http://www.shinglee.com.sg/StudentResources/ and open the geometry template Circle Symmetric Property 2 as shown below.

Part 1

1. The template shows a circle with centre O and two equal chords.

Circle Symmetric Property 2a

Given Conditions: Equal chords, i.e. the chords AB and PQ have equal length.
Move the point A or B to change the length of both chords.
Move the point P until it coincides with A.

Next

Fig. 13.3

2. Click and drag point A or B to change the lengths of both chords.

   Click and drag point R to change the size of the circle.

   Click and drag point P until it coincides with the point A.

   What do you notice about the distance of both chords from the centre O?

3. Copy and complete the following sentence.

   In general, equal chords of a circle are _____ from the centre of the circle.

Part 2

4. Click on the 'Next' button. The next page of the template shows two chords of a circle that are equidistant from its centre O.

Circle Symmetric Property 2b

Given Conditions: Chords that are equidistant from centre of circle.
Move the point M to change distance of chord from centre O of circle.
Move the point P until both chords coincide.

Previous

Fig. 13.4

5. Click and drag point M to change the distance of both chords from the centre O.

   Click and drag point R to change the size of the circle.

   Click and drag point P until both chords coincide.

   What do you notice about the lengths of both chords?

6. Copy and complete the following sentence.

   In general, chords that are equidistant from the centre of a circle are _____ in length.

From the investigation, there are two parts to **Circle Symmetric Property 2**:

(i) Equal chords of a circle are equidistant from the centre of the circle.

(ii) Chords that are equidistant from the centre of a circle are equal (in length).

Worked Example 2

The lengths of two parallel chords of a circle of radius 12 cm are 8 cm and 14 cm respectively. Find the distance between the chords.

Solution:

There are two possible cases about the positions of the two chords AB and XY (equal chords). Let $AB = 8$ cm and $XY = 14$ cm.

Case 1: The chords are on opposite sides of the centre O.
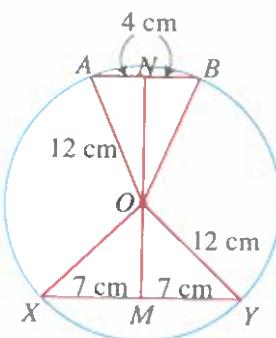
In $\triangle AON$,

$ON^2 = 12^2 - 4^2$ (Pythagoras' Theorem)

$\quad = 128$

$ON = \sqrt{128}$

$\quad = 11.31$ cm (to 4 s.f.)

In $\triangle YOM$,

$OM^2 = 12^2 - 7^2$ (Pythagoras' Theorem)

$\quad = 95$

$OM = \sqrt{95}$

$\quad = 9.747$ cm (to 4 s.f.)

Distance between the chords = MN

$\qquad\qquad = NO + OM$

$\qquad\qquad = 11.31 + 9.747$

$\qquad\qquad = 21.1$ cm (to 3 s.f.)
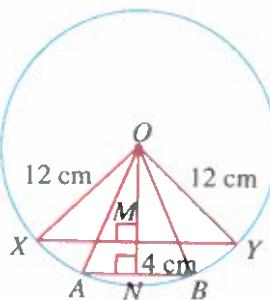
Case 2: The chords are on the same side of the centre O.

Distance between the chords = MN

$\qquad\qquad = ON - OM$

$\qquad\qquad = 11.31 - 9.747$

$\qquad\qquad = 1.56$ cm (to 3 s.f.)

∴ The distance between the chords can either be 21.1 cm or 1.56 cm.

PRACTISE NOW 2

The lengths of two parallel chords of a circle of radius 20 cm are 10 cm and 30 cm respectively. Find the distance between the chords.

BASIC LEVEL

1. Given that O is the centre of each of the following circles, find the values of the unknowns.

   (a)

   13 cm O 5 cm $b°$ a cm a cm

   (b)

   26 m c m O 17 m $d°$ 16 m

   (c)

   10 mm $f°$ e mm O 5 mm   12 mm
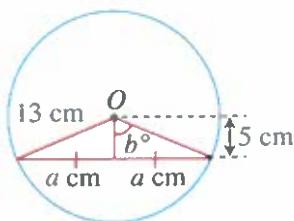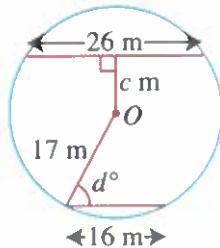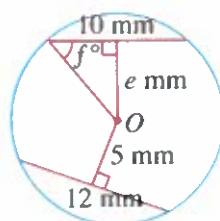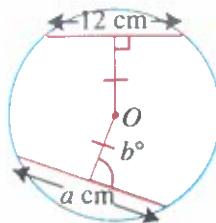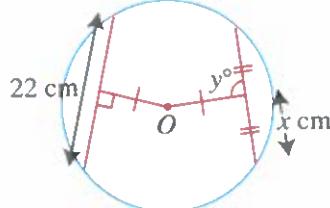
2. AB is a chord of a circle, centre O and with radius 17 cm. Given that $AB = 16$ cm, find the perpendicular distance from O to AB.

3. A chord of length 24 m is at a distance of 5 m from the centre of a circle. Find the radius of the circle.

4. A chord of a circle of radius 8.5 cm is 5 cm from the centre. Find the length of the chord.

5. Given that O is the centre of each of the following circles, find the values of the unknowns.

   (a)

   12 cm O $b°$ a cm

   (b)

   22 cm O $y°$ x cm
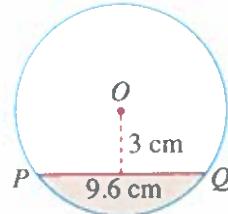
INTERMEDIATE LEVEL
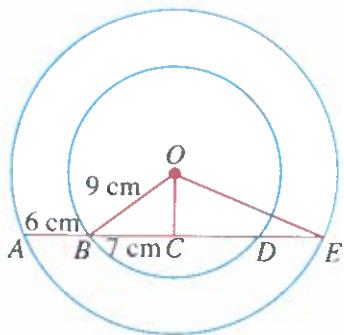
6. The figure shows the cross section of a circular water pipe. The shaded region shows the water flowing through the pipe.

   O 3 cm P 9.6 cm Q

   Given that $PQ = 9.6$ cm and that the surface of the water is 3 cm below the centre O of the circle, find the cross-sectional area of the water pipe.

7. The perpendicular bisector of a chord XY cuts XY at N and the circle at P. Given that $XY = 16$ cm and $NP = 2$ cm, calculate the radius of the circle.

8. The figure shows two concentric circles with centre O. The points A and E lie on the circumference of the larger circle while the points B and D lie on the circumference of the smaller circle.

Given that $ABCDE$ is a straight line, $OB = 9$ cm, $AB = 6$ cm, $BC = 7$ cm and $AC = CE$, find

(i) the length of OC,

(ii) the length of OE.

9. The lengths of two parallel chords of a circle of radius 5 cm are 6 cm and 8 cm respectively. Find the distance between the chords.

10. Two parallel chords PQ and MN are 3 cm apart on the same side of a circle where $PQ = 7$ cm and $MN = 14$ cm. Calculate the radius of the circle.

11. The radius of a circle is 17 cm. A chord XY lies 9 cm from the centre and divides the circle into two segments. Find the perimeter of the minor segment.

Radius of a Circle and Tangent to a Circle

A straight line cutting a circle at two distinct points is called a **secant**.

In Fig. 13.5(**a**), AB is a secant.

(a)

(b)
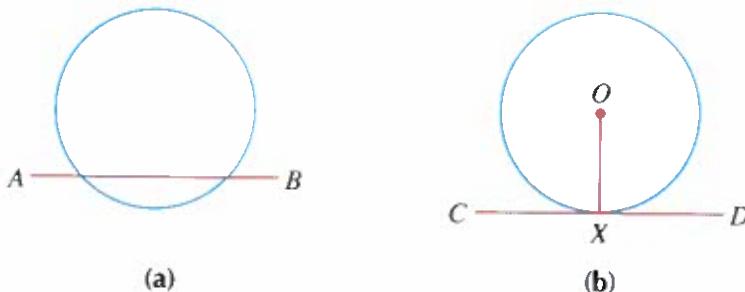
Fig. 13.5

If a straight line and a circle have only one point of contact, then that line is called a **tangent**.

In Fig. 13.5(**b**), CD is a tangent and X is the point of contact.

Investigation

Circle Symmetric Property 3

Go to http://www.shinglee.com.sg/StudentResources/ and open the geometry template
Circle Symmetric Property 3 as shown below.

1. The template shows a circle with centre O and radius OP, which is perpendicular
   to the chord at A.

Circle Symmetric Property 3

From Symmetric Property 1a, 1b or 1c, OP is the **perpendicular bisector** of the chord.

Click here to show or hide Secant

Move the point A until it reaches P. You can also move the points P and R.

Fig. 13.6

2. Click on the button 'Click here to show or hide Secant'. It will reveal a secant
   that coincides with the chord, i.e. the secant is also perpendicular to the
   radius OP. Unlike a chord which is a *line segment* with two end points,
   a secant is a *line* that cuts the circle at two different points.

3. Click and drag point P to move the radius OP and the secant around the circle.

   Click and drag point R to change the size of the circle.

   Click and drag point A until it coincides with with the point P.

   (a) What do you notice about the secant? What has it become?

   (b) What is the angle between the tangent at the point of contact P and the radius
       of the circle?

4. Copy and complete the following sentence.

   In general, the tangent at the point of contact is _____ to the radius
   of the circle.
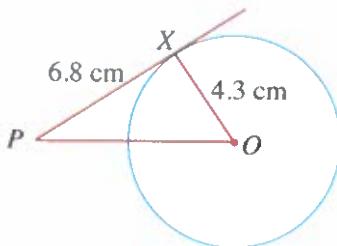
From the investigation, **Circle Symmetric Property 3** states that:

> The tangent at the point of contact is perpendicular to the radius of the circle.

Worked Example 3

(Application of Circle Symmetric Property 3)

In the figure, PX is a tangent to the circle, centre O.

Given that $PX = 6.8$ cm and $OX = 4.3$ cm, find

(i) $\angle OPX$,

(ii) the length of OP,

(iii) the area of $\triangle OPX$.

Solution:

(i) $\angle OXP = 90°$ (tangent $\perp$ radius)

In $\triangle OPX$,

$$\tan \angle OPX = \frac{OX}{PX}$$
$$= \frac{4.3}{6.8}$$
$$\angle OPX = \tan^{-1} \frac{4.3}{6.8}$$
$$= 32.3° \text{ (to 1 d.p.)}$$

(ii) In $\triangle OPX$,

$$OP^2 = 6.8^2 + 4.3^2 \text{ (Pythagoras' Theorem)}$$
$$= 64.73$$
$$OP = \sqrt{64.73}$$
$$= 8.05 \text{ cm (to 3 s.f.)}$$

ATTENTION

For **(ii)**, trigonometric ratios may be used to find the length of OP, i.e. $\sin 32.31° = \frac{4.3}{OP}$ before solving for OP.
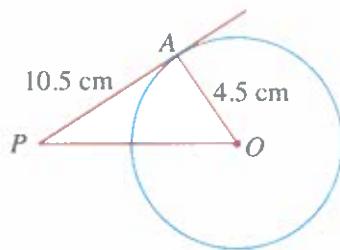
(iii) Area of $\triangle OPX = \frac{1}{2} \times PX \times OX$ (use $\frac{1}{2} \times$ base $\times$ height)

$$= \frac{1}{2} \times 6.8 \times 4.3$$
$$= 14.62 \text{ cm}^2$$

OXFORD
UNIVERSITY PRESS
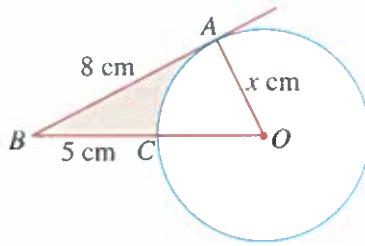
1. In the figure, *PA* is a tangent to the circle, centre *O*.

Given that *PA* = 10.5 cm and *OA* = 4.5 cm, find

(i) ∠*OPA*,

(ii) the length of *OP*,

(iii) the area of △*OPA*.

2. In the figure, *AB* is a tangent to the circle, centre *O*.

Given that *AB* = 8 cm, *BC* = 5 cm and *OA* = *x* cm, find

(i) the value of *x*,

(ii) ∠*AOB*,

(iii) the area bounded by *AB*, *BC* and the minor arc *AC*.

Investigation

Circle Symmetric Property 4

Go to http://www.shinglee.com.sg/StudentResources/ and open the geometry template Circle Symmetric Property 4 as shown below.

1. The template shows a circle with centre O and two tangents from an external point P touching the circle at A and B respectively.

Circle Symmetric Property 4

Given Conditions: Tangents from external point P.
Move the points P and R. But point P must be an external point outside the circle.

AP = 7.16 cm
BP = 7.16 cm

Angle OPA = 23.47°
Angle OPB = 23.47°

Angle AOP = 66.53°
Angle BOP = 66.53°

Fig. 13.7

2. Click and drag point P to change the position of the external point, but P must remain outside the circle.

    Click and drag point R to change the size of the circle.

    (a) What do you notice about the length of AP and of BP?

    (b) What do you notice about $\angle OPA$ and $\angle OPB$?

3. Copy and complete the following sentences.

    In general,

    (a) tangents from an external point are _____ (in length);

    (b) the line from the centre of a circle to an external point _____ the angle between the two tangents.

4. Prove the two results in Question 3.

    Hint: For Question 3, how are $\triangle OAP$ and $\triangle OBP$ related?
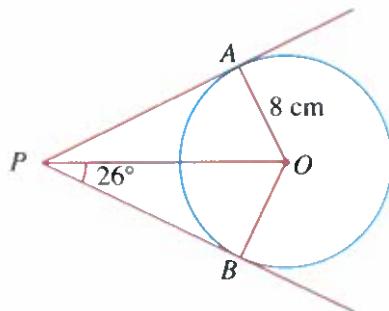
From the investigation, there are two parts to **Circle Symmetric Property 4**:

(i) Tangents from an external point are equal (in length).

(ii) The line from the centre of a circle to an external point bisects the angle between the two tangents from the external point.

Worked Example 4

(Application of Circle Symmetric Property 4)

In the figure, *PA* and *PB* are tangents to the circle with centre *O*.

Given that $OA = 8$ cm and $\angle OPB = 26°$, find

(i) $\angle AOB$,

(ii) the length of *AP*,

(iii) the area of the quadrilateral *APBO*.

Solution:

(i) $\angle OBP = \angle OAP = 90°$ (tangent $\perp$ radius)

$\angle OPA = \angle OPB = 26°$ (symmetric properties of tangents to circle)

$\angle AOB = 360° - \angle OAP - \angle OBP - \angle APB$ ($\angle$sum of a quadrilateral)

$\qquad = 360° - 90° - 90° - (26° + 26°)$

$\qquad = 128°$

(ii) In $\triangle OAP$,

$$\tan \angle APO = \frac{OA}{AP}$$

$$\tan 26° = \frac{8}{AP}$$

$$AP = \frac{8}{\tan 26°}$$

$$= 16.4 \text{ cm (to 3 s.f.)}$$

> **ATTENTION**
>
> In Worked Example 4, $PA = PB$ (equal tangents).

(iii) Area of $\triangle OAP = \dfrac{1}{2} \times AP \times OA$ (use $\dfrac{1}{2} \times$ base $\times$ height)

$$= \dfrac{1}{2} \times 16.40 \times 8$$

$$= 65.61 \text{ cm}^2$$
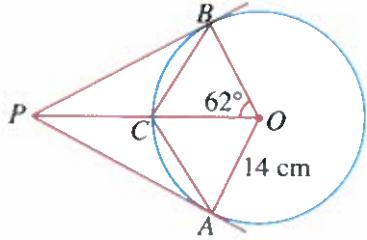
Area of quadrilateral $APBO = 2 \times$ area of $\triangle OAP$

$$= 2 \times 65.61$$

$$= 131 \text{ cm}^2 \text{ (to 3 s.f.)}$$

PRACTISE NOW 4

1. In the figure, *PA* and *PB* are tangents to the circle with centre *O*.
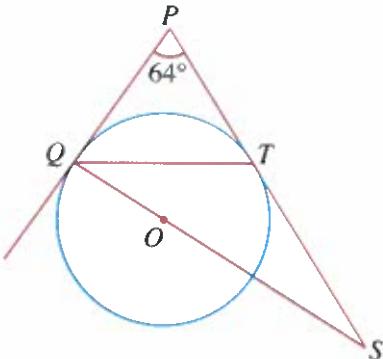
Given that $OA = 14$ cm and $\angle BOP = 62°$, find

(i) $\angle OPB$,

(ii) $\angle OAC$,

(iii) the length of *BP*,

(iv) the area of the quadrilateral *APBO*.

2. In the figure, *PQ* and *PT* are tangents to the circle, centre *O*, at the points *Q* and *T* respectively. *PT* produced meets *QO* produced at *S*.
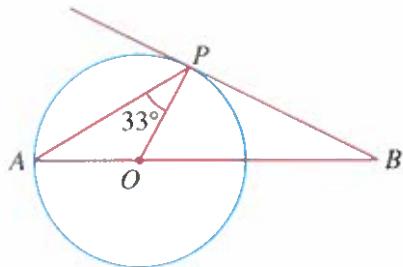
Given that $\angle QPT = 64°$, find $\angle SQT$.
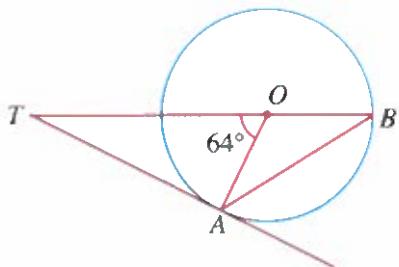
BASIC LEVEL

1. In the figure, *BP* is a tangent to the circle with centre *O*.

Given that ∠*APO* = 33°, find ∠*PBA*.
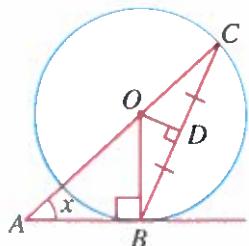
2. In the figure, *O* is the centre of the circle passing through the points *A* and *B*. *TA* is a tangent to the circle at *A* and *TOB* is a straight line.

Given that ∠*AOT* = 64°, find
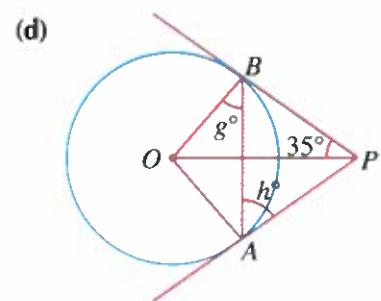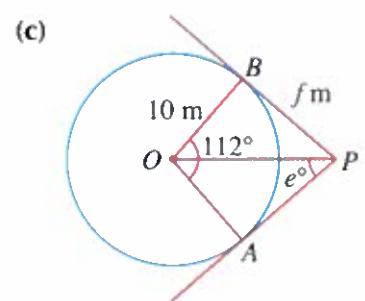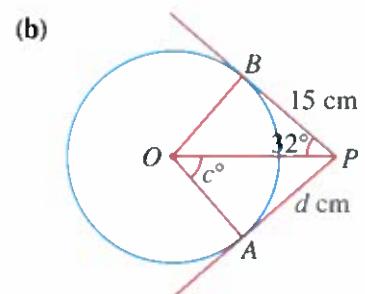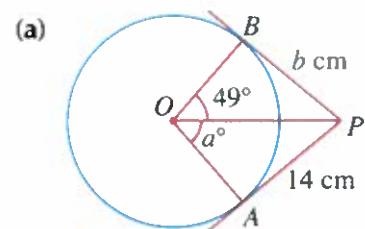
(i) ∠*ATB*, **(ii)** ∠*TAB*.

3. In the figure, *AB* is a tangent to the circle with centre *O*. *D* is the midpoint of the chord *BC*.

Given that ∠*BAC* = *x*, find ∠*COD* in terms of *x*.
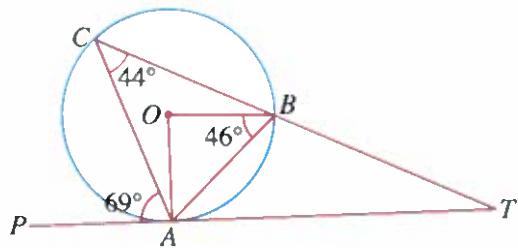
4. Given that *PA* and *PB* are tangents to each of the following circles with centre *O*, find the values of the unknowns.

(a)

(b)

(c)

(d)

(e)

(f)
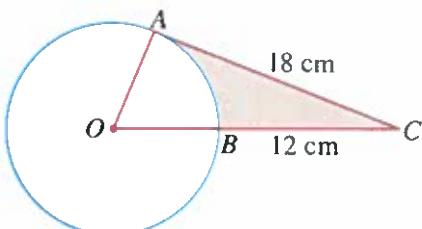
5. In the figure, *PAT* is a tangent to the circle, centre *O*, at *A*. *C* is a point on the circle such that *TBC* is a straight line and $\angle ACB = 44°$.

Given that $\angle OBA = 46°$ and $\angle PAC = 69°$, find

(i) $\angle BAT$,   (ii) $\angle PTC$.

6. The figure shows a circle, centre *O*. *AC* is a tangent to the circle at *A* and *OBC* is a straight line.

Given that $AC = 18$ cm and $BC = 12$ cm, find
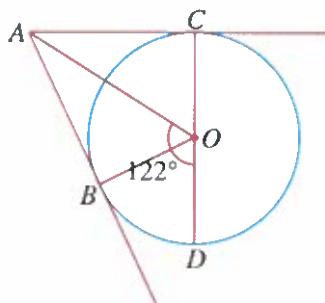
(i) the radius of the circle,

(ii) $\angle AOB$,

(iii) the area of the shaded region.

7. *PQ* is a chord of a circle with centre *O*. Given that $\angle POQ = 84°$, find the obtuse angle between *PQ* and the tangent at *P*.

8. The tangent from a point *P* touches a circle at *N*. Given that the radius of the circle is 5.6 cm and that *P* is 10.6 cm away from the centre, find the length of the tangent *PN*.

9. A point *T* is 9.1 m away from the centre of a circle. The tangent from *T* to the point of tangency is 8.4 m. Find the diameter of the circle.

10. In the figure, *AB* and *AC* are tangents to the circle at *B* and *C* respectively. *O* is the centre of the circle and $\angle AOD = 122°$.
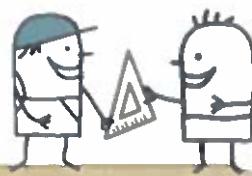
Find $\angle BAC$.

11. The tangents from a point *T* touch a circle, centre *O*, at the points *A* and *B*. Given that $\angle AOT = 51°$, find $\angle BAT$.

12. Two concentric circles have radii 12 cm and 25.5 cm respectively. A tangent to the inner circle cuts the outer circle at the points *H* and *K*. Find the length of *HK*.

13.2 Angle Properties of Circles

In this section, we will learn the angle properties of circles.

Angles at Centre and Angles at Circumference

Fig. 13.8(**a**) shows a circle with centre O.

$\angle AOB$ is an angle **subtended at the centre** of the circle by the (blue) *minor* arc AXB.

$\angle APB$ is an angle **subtended at the circumference** of the circle by the same *minor* arc AXB.

Fig. 13.8(**b**) shows another circle with centre O.

$\angle AOB$ is an angle subtended at the centre of the circle by the (red) *major* arc AYB.

$\angle AQB$ is an angle subtended at the circumference of the circle by the same *major* arc AYB.

(a)	(b)	(c)
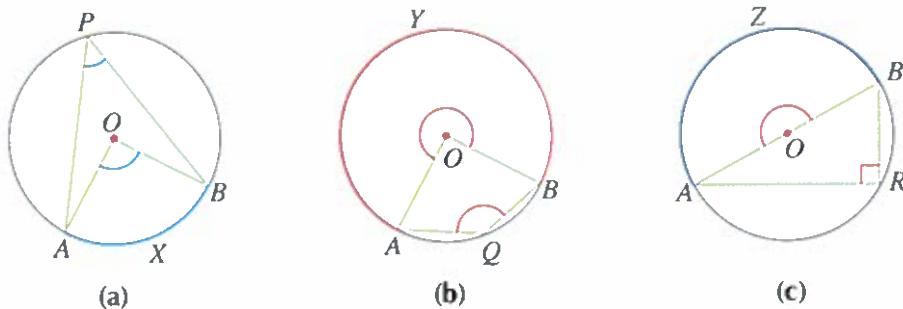
Fig. 13.8

One way to recognise which angle is subtended by which arc is to look at the shape of the arc. For example, the shape of the blue arc indicating $\angle APB$ in Fig. 13.8(**a**) is the same shape as that of the blue minor arc AXB which subtends the angle; and the shape of the red arc indicating $\angle AOB$ in Fig. 13.8(**b**) is the same shape as that of the red major arc AYB which subtends the angle.

Consider Fig. 13.8(**c**). Can you identify the angle subtended at the centre of the circle and the angle subtended at the circumference by the semicircle AZB?
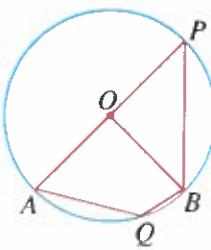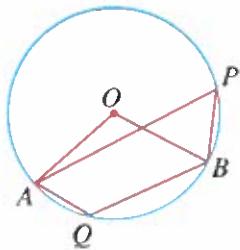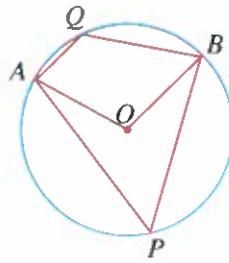
Identifying Angles at the Centre and at the Circumference

Work in pairs to identify each of the following by using a different coloured pencil or pen to draw the angle in each circle.

(a) Angle at centre subtended by the minor arc *AQB*

(b) Angle at circumference subtended by the minor arc *AQB*

(c) Angle at centre subtended by the major arc *APB*

(d) Angle at circumference subtended by the major arc *APB*

Investigation

Circle Angle Property 1

Go to http://www.shinglee.com.sg/StudentResources/ and open the geometry template Circle Angle Property 1 as shown below.

1. The template shows a circle with centre O. $\angle AOB$ is an angle at the centre while $\angle APB$ is an angle at the circumference subtended by the same (minor or major) arc AB.

<div>

Circle Angle Property 1: Angle at Centre

Given Condition: Angle at centre $\angle AOB$ and angle at circumference $\angle APB$ subtended by same (minor or major) arc AB

Set angle at centre = 60 degrees   Set angle at centre = 90 degrees   Set angle at centre = 120 degrees

Set angle at centre = 145 degrees   Set angle at centre = 250 degrees   Set angle at centre = 320 degrees

You can also move the points R and P.

Angle at circumference $\angle$ APB = 33.20° (yellow angle)
Angle at centre $\angle$ AOB = 66.4° (pink angle)

</div>

Fig. 13.9

2. Click on the action buttons in the template to set $\angle AOB$ to the values below. You can also move the point R to change the size of the circle, and the point P to change $\angle APB$. Copy and complete Table 13.1 below.

$\angle AOB$	60°	90°	120°	145°	250°	320°
$\angle APB$						
$\dfrac{\angle AOB}{\angle APB}$						

Table 13.1

3. What is the relationship between $\angle AOB$ and $\angle APB$?

4. Copy and complete the following sentence.

In general, an angle at the centre of a circle is _____ that of any angle at the circumference subtended by the same arc.

From the investigation, **Circle Angle Property 1** states that:

> An angle at the centre of a circle is **twice** that of any angle at the
> circumference subtended by the same arc.

To prove Circle Angle Property 1, we have to consider 4 cases.
The following proof applies to Case 1 and Case 2.

Case 1 and Case 2:

Fig. 13.10 shows Case 1 (angles subtended by minor arc AB) and Case 2 (angles
subtended by major arc AB). The proofs for both cases are actually the same.

Join P to O and produce PO to cut the circle at X.

Let $\angle APX = a$ and $\angle BPX = b$.
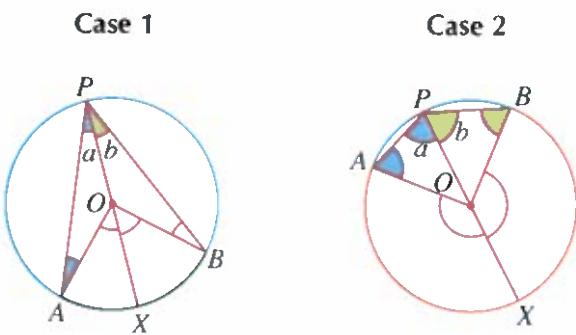
Fig. 13.10

Since $OA = OP$ (radii),
$\triangle AOP$ is an isosceles triangle.

Then $\angle OAP = \angle OPA$ (base $\angle$s of isos. $\triangle$)
$\qquad\qquad = a$
$\therefore \angle AOX = \angle OAP + \angle OPA$ (ext. $\angle$ of $\triangle$)
$\qquad\qquad = 2a$

Similarly, $\triangle BOP$ is an isosceles triangle,
$\angle OBP = \angle OPB = b$ and $\angle BOX = 2b$.
$\therefore \angle AOB = \angle AOX + \angle BOX$
$\qquad\qquad = 2a + 2b$
$\qquad\qquad = 2(a + b)$
$\qquad\qquad = 2 \times (\angle APX + \angle BPX)$
$\qquad\qquad = 2 \times \angle APB \text{ (proven)}$
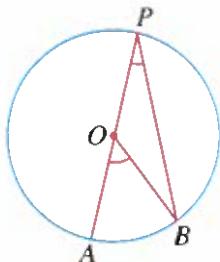
The Thinking Time on the next page considers the next 2 cases.

Fig. 11.11 shows Case 3 and Case 4 (two special cases of angles subtended by minor arc *AB*). In each case, prove Circle Angle Property 1.

Case 3   Case 4
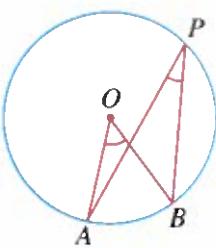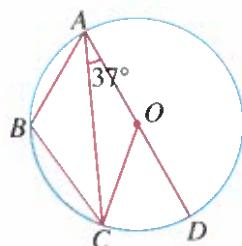
Fig. 13.11

Worked Example 5

(Application of Circle Angle Property 1)

A, *B*, *C* and *D* are four points on a circle with centre *O*. Given that *AOD* is a diameter of the circle and $\angle CAD = 37°$, find

(i) $\angle COD$,

(ii) $\angle ABC$.

Solution:

(i) $\angle COD = 2 \times \angle CAD$ ($\angle$ at centre $= 2 \angle$ at $\odot^{ce}$)

$\qquad = 2 \times 37°$

$\qquad = 74°$
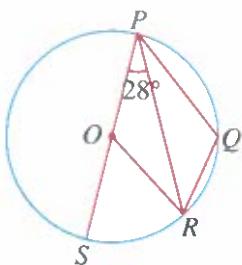
(ii) Reflex $\angle AOC = 180° + 74°$

$\qquad\qquad\qquad = 254°$

$\therefore \angle ABC = \dfrac{254°}{2}$ ($\angle$ at centre $= 2 \angle$ at $\odot^{ce}$)

$\qquad\qquad = 127°$
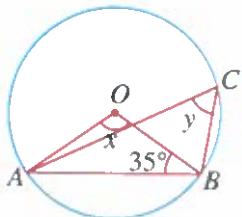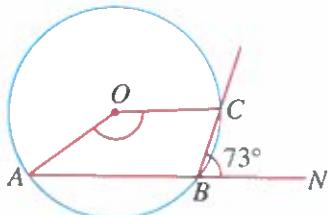
1. P, Q, R and S are four points on a circle with centre O.
    Given that POS is a diameter of the circle and $\angle OPR = 28°$, find

(i) $\angle SOR$,   (ii) $\angle PQR$.

2. Given that O is the centre of the circle and $\angle ABO = 35°$, find the angles marked x and y.

3. In the figure, O is the centre of the circle and A and B lie on the circumference such that ABN is a straight line.

Given that C lies on the circumference such that $\angle NBC = 73°$, find the obtuse angle AOC.

⋮⋮ Angle in a Semicircle

Investigation

Circle Angle Property 2

Go to http://www.shinglee.com.sg/StudentResources/ and open the geometry template Circle Angle Property 2 as shown below.

1. The template shows a circle with centre O. $\angle AOB$ is an angle at the centre while $\angle APB$ is an angle at the circumference.

Circle Angle Property 2: Angle in Semicircle

Given Condition: Angle in semicircle

[Set angle at centre = 180 degrees]

You can also move the points R and P.

Angle at circumference $\angle APB = 43.20°$ (yellow angle)
Angle at centre $\angle AOB = 86.4°$ (pink angle)

Fig. 13.12

2. Click on the action button in the template to set $\angle AOB = 180°$. You can also move the point R to change the size of the circle, and the point P to change $\angle APB$.

   (a) What is $\angle APB$ equal to?

   (b) What is the special name given to the sector APB when $\angle AOB = 180°$?

3. Copy and complete the following sentence.

   In general, an angle in a semicircle is always equal to _____.

4. Prove the angle property in Question 3.
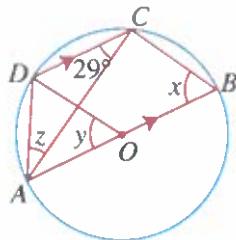
From the investigation, **Circle Angle Property 2** states that:

> An angle in a semicircle is always equal to 90°.

Worked Example 6

(Application of Circle Angle Property 2)

A, B, C and D are four points on a circle with centre O.
Given that AOB is a diameter of the circle, DC is parallel
to AB and $\angle DCA = 29°$, find the angles marked x, y and z.

Solution:

$\angle ACB = 90°$ (rt. $\angle$ in a semicircle)

$\angle CAB = 29°$ (alt. $\angle$s, $AB \parallel DC$)

$\therefore x = 180° - 90° - 29°$ ($\angle$ sum of a $\triangle$)

$\quad = 61°$

$y = 2 \times \angle ACD$ ($\angle$ at centre $= 2 \angle$ at $\odot^{ce}$)

$\quad = 2 \times 29°$

$\quad = 58°$

$\angle ADO = \angle DAO$ (base $\angle$ of isos. $\triangle$)

$\quad\quad = \dfrac{1}{2}(180° - 58°)$

$\quad\quad = 61°$

$\therefore z = \angle ADO - \angle CAB$

$\quad = 61° - 29°$

$\quad = 32°$
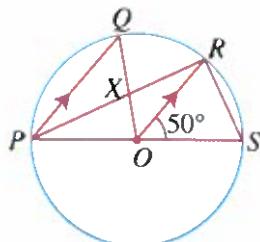
PRACTISE NOW 6

P, Q, R and S are four points on a circle with centre O.
Given that POS is a diameter of the circle, PQ is parallel
to OR and $\angle ROS = 50°$, find

(i) $\angle OPR$,

(ii) $\angle QOR$,

(iii) $\angle PXQ$.
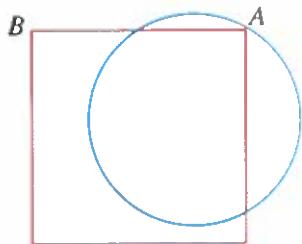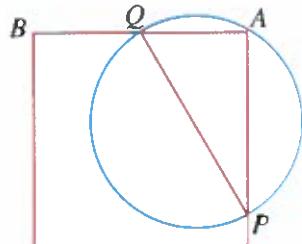
Do you know how the centre of a circle can be determined? Follow the instructions given and discover the answer yourself.
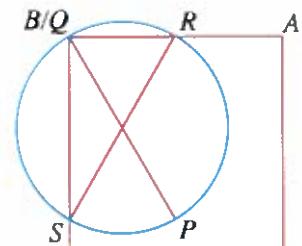
1. Place a rectangular sheet of paper under a circle such that one of its corners touches the circle, say at the point A.

2. Join the two points, P and Q, as shown.

3. Move the same sheet of paper such that another of its corners touches the circle, say at the point B. Join the two points R and S as shown.

The result would show that the point of intersection of PQ and RS gives the centre of the circle. Explain why this is true.

⬚⬚ Angles in Same or Opposite Segments

Fig. 13.13(a) shows a circle with a chord AB that divides the circle into two segments.

The larger segment APQB is called the major segment (shaded blue) while the smaller segment AXYB is called the minor segment (shaded green). ∠APB and ∠AQB are angles subtended at the circumference of the circle by the same minor arc AB. Since ∠APB and ∠AQB lie in the same (major) segment, they are called **angles in the same segment**. ∠AXB and ∠AYB are angles subtended at the circumference of the circle by the same major arc AB. Since ∠AXB and ∠AYB lie in the same (minor) segment, they are called angles in the same segment.

(a)   (b)
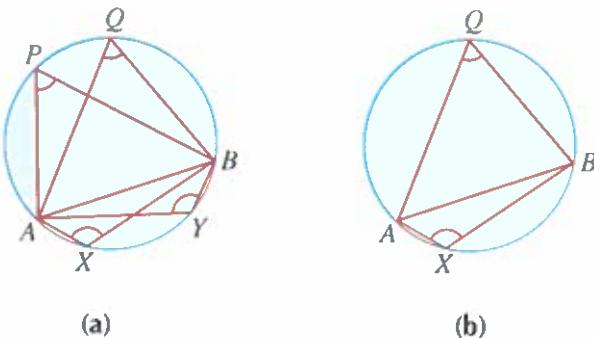
Fig. 13.13

Fig. 13.13(b) shows a circle with a chord AB that divides the circle into two segments. The segment AQB and the segment AXB are called **opposite segments** (*not* different segments). ∠AQB and ∠AXB are angles subtended at the circumference of the circle by the minor arc AB and by the major arc AB respectively. Since ∠AQB and ∠AXB lie in opposite segments, they are called **angles in opposite segments**.

ATTENTION

Opposite segments must be formed by the *same chord*.
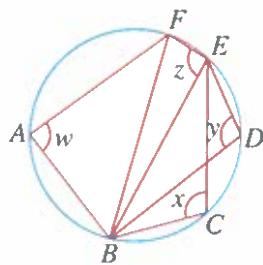
Class Discussion

Angles in Same or Opposite Segments

Work in pairs.

The figure on the right shows a circle with four angles labelled *w*, *x*, *y* and *z*. Work in pairs to identify which pairs of the four angles are in the same segment and which pairs of the four angles are in opposite segments. For each case, specify the chord that forms the segment(s). In particular, are ∠*w* and ∠*y* angles in opposite segments? Explain your answer.

Investigation

Circle Angle Property 3

Go to http://www.shinglee.com.sg/StudentResources/ and open the geometry template Circle Angle Property 3 as shown below.

1. The template shows a circle with centre O. $\angle APB$ and $\angle AQB$ are angles in the same (minor or major) segment.

Circle Angle Property 3: Angles in Same Segment

Given Condition: Angles in **same (minor or major) segment**

Move the points A. B. R. P and Q. Make sure you explore the case when both angles are in a **minor segment** also.

Why do you think the relationship is always true? [Show Hint]

Angle at circumference $\angle APB = 59.31°$ (yellow angle)

Angle at circumference $\angle AQB = 59.31°$ (pink angle)

Fig. 13.14

2. Click and drag point A or B to change the size of $\angle APB$ and of $\angle AQB$.
   Click and drag point R to change the size of the circle.
   Click and drag point P or Q to change the position of $\angle APB$ and of $\angle AQB$.
   What do you notice about $\angle APB$ and $\angle AQB$?

3. Copy and complete the following sentence.
   In general, angles in the same segment are _____.

4. Prove the angle property in Question 3. You can also click on the button 'Show Hint' in the template.

> **ATTENTION**
>
> To adjust $\angle APB$ and $\angle AQB$ until they are in the same minor segment, click and drag point A or B until arc $APQB$ is a minor arc.

From the investigation, Circle Angle Property 3 states that:
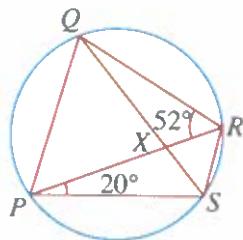
> Angles in the same segment are equal.

Worked Example 7

In the figure, P, Q, R and S are points on the circumference of a circle. Given that PR and QS intersect at the point X, $\angle RPS = 20°$ and $\angle PRQ = 52°$, find

(i) $\angle SQR$,

(ii) $\angle QSP$,

(iii) $\angle PXQ$.

Solution:

(i) $\angle SQR = \angle RPS$ ($\angle$s in same segment)

$\qquad = 20°$

(ii) $\angle QSP = \angle PRQ$ ($\angle$s in same segment)

$\qquad = 52°$

(iii) $\angle PXQ = \angle SQR + \angle PRQ$ (ext. $\angle$ = sum of int. opp. $\angle$s)

$\qquad = 20° + 52°$
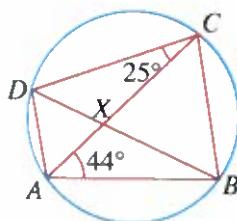
$\qquad = 72°$

PRACTISE NOW 7
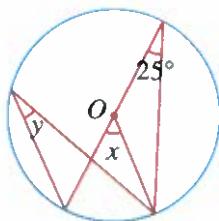
1. In the figure, A, B, C and D are points on the circumference of a circle. Given that AC and BD intersect at the point X, $\angle BAC = 44°$ and $\angle ACD = 25°$, find

    (i) $\angle CDX$,   (ii) $\angle ABX$,

    (iii) $\angle CXB$.
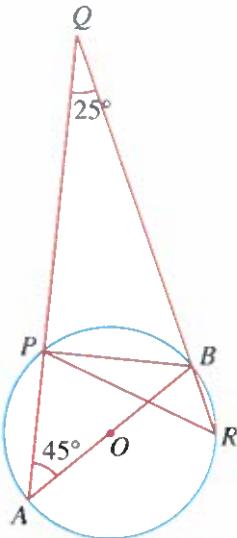
2. Given that O is the centre of the circle, find the angles marked x and y.

3. In the figure, AB is a diameter of the circle with centre O. APQ and RBQ are straight lines. Find $\angle BPR$.

Investigation

Circle Angle Property 4

Go to http://www.shinglee.com.sg/StudentResources/ and open the geometry template Circle Angle Property 4 as shown below.

1. The template shows a circle with centre O. $\angle APB$ and $\angle AQB$ are angles in opposite (minor or major) segments.

Circle Angle Property 4: Angles in Opposite Segments

Given Condition: Angles in **opposite segments**

Move the points A, B, R, P and Q.

Why do you think the relationship is always true? [Show Hint]

Angle at circumference $\angle APB = 59.85°$ (yellow angle)
Angle at circumference $\angle AQB = 120.15°$ (pink angle)

Fig. 13.15

2. Click and drag the point A or B to change the size of $\angle APB$ and of $\angle AQB$.

   Click and drag the point R to change the size of the circle.

   Click and drag the point P or Q to change the position of $\angle APB$ and of $\angle AQB$.

   What do you notice about $\angle APB$ and $\angle AQB$?

3. Copy and complete the following sentence.

   In general, angles in opposite segments are **supplementary**, i.e. they add up to _____.

4. Prove the angle property in Question 3. You can also click on the button 'Show Hint' in the template.
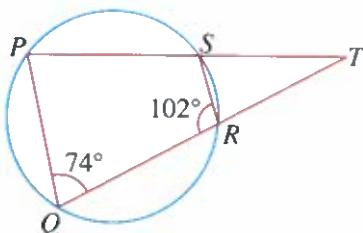
From the investigation, **Circle Angle Property 4** states that:

> Angles in opposite segments are *supplementary*, i.e. they add up to 180°.

Worked Example 8

(Application of Circle Angle Property 4)

In the figure, P, Q, R and S are points on the circumference of the circle. PST and QRT are straight lines, $\angle PQR = 74°$ and $\angle QRS = 102°$.

Find

(i) $\angle QPS$,

(ii) $\angle RTS$,

(iii) $\angle RST$.

Solution:
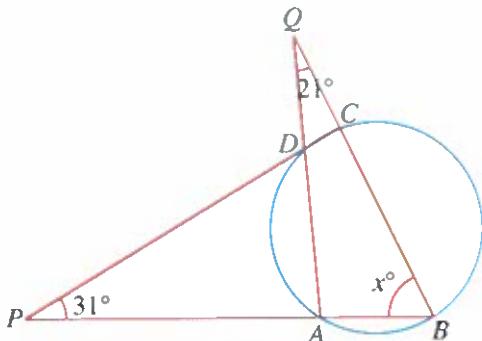
(i) $\angle QPS = 180° - 102°$ ($\angle$s in opp. segments)

$\quad\quad\quad = 78°$

(ii) $\angle RTS = 180° - 74° - 78°$ ($\angle$ sum of a $\triangle$)

$\quad\quad\quad = 28°$

(iii) $\angle RST = 102° - 28°$ (ext. $\angle$ = sum of int. opp. $\angle$s)

$\quad\quad\quad = 74°$

1. In the figure, A, B, C and D are points on the circumference of a circle. PAB, QCB, PDC and QDA are straight lines.
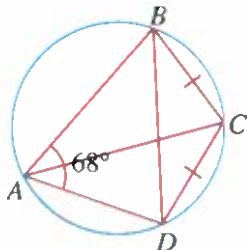
Given that $\angle BPC = 31°$, $\angle AQB = 21°$ and $\angle PBQ = x°$, find

(i) $\angle BAD$ in terms of x, (ii) $\angle BCD$ in terms of x,

(iii) the value of x, (iv) $\angle PAD$.

2. In the figure, A, B, C and D are points on the circumference of the circle and $BC = CD$.

Given that $\angle BAD = 68°$, find $\angle BAC$.

⠿ Angles in Alternate Segments

In Fig. 13.16, TA is the tangent to the circle at A and AQ is a chord at the point of contact. $\angle TAQ$ is the angle between the tangent TA and the chord AQ at A. $\angle QPA$ is the angle subtended by chord AQ in the **alternate segment**. We will now learn a theorem that states how they are related.
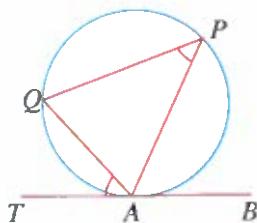
Fig. 13.16

Investigation

Circle Angle Property 5

Go to http://www.shinglee.com.sg/StudentResources/ and open the geometry software template Tangent-Chord Theorem as shown below:

1. The template shows a circle with centre O. The pink angle, $\angle x$, is the angle between the tangent and the chord at the point of contact A. The yellow angle, $\angle y$, is an angle subtended by the chord in the alternate segment with reference to $\angle x$. Notice that the two angles are on different sides of the chord.

Tangent-Chord Theorem (or Alternate Segment Theorem)

Given Conditions: Angles in alternate segment

Move the points A, B, R and P.

Angle between tangent and chord $\angle x =$ (pink angle)

        Angle in alternate segment $\angle y =$ (yellow angle)

Show Hint for Proof

Fig. 13.16

2. Click and drag the point A or B to change the size of $\angle x$ and of $\angle y$.
   Click and drag the point R to change the size of the circle.
   Click and drag the point P to change the position of $\angle x$ and of $\angle y$.
   What do you notice about $\angle x$ and $\angle y$?

3. Copy and complete the following sentence.
   In general, angles in alternate segments are _____.

4. Prove the angle property in Question 1. You can also click on the button 'Show Hint' in the template.

This is also called the **Tangent-Chord Theorem** (or **Alternate Segment Theorem**).
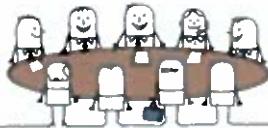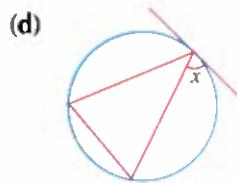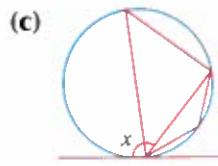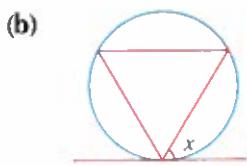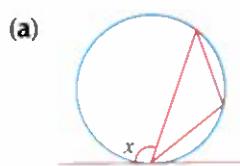
From the investigation, **Circle Angle Property 5** states that:

> The angle between a tangent and a chord at the point of contact is equal to the angle subtended by the chord in the alternate segment.

Thinking Time

Identify the angle that is equal to $\angle x$ in each of the following circles and label it as $\angle y$.

(a)

(b)

(c)

(d)
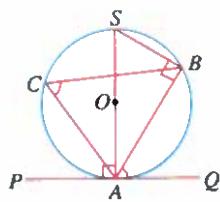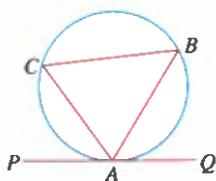
Class Discussion

Proof of Tangent-Chord Theorem

Work in pairs.

Consider the circle in Fig. 13.8(a), where A, B and C lie on the circumference of the circle and PQ is a tangent to the circle at A.

To prove that $\angle BAQ = \angle ACB$, in Fig. 13.18(b), draw a diameter AOS and join SB.

Fig 13.18(a)

Fig 13.18(b)

Copy and complete the following.

$\angle ABS = \underline{\quad}$ (rt. $\angle$ in a semicircle)

$\angle SAQ = \underline{\quad}$ (tangent $\perp$ radius)

$\angle ASB = \angle ACB$ (\underline{\hspace{3cm}})

$\angle ASB + \angle SAB = 90°$ (\underline{\hspace{3cm}})

$\angle SAB + \angle BAQ = 90°$ (\underline{\hspace{3cm}})
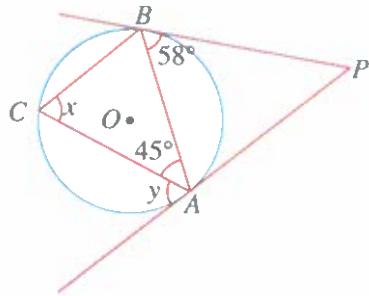
$\therefore \angle ASB = \angle BAQ$

$\angle BAQ = \angle ACB$ (since $\angle ASB = \underline{\quad}$)

Worked Example 9

(Tangent-Chord Theorem)

Given that *PA* and *PB* are tangents to a circle with centre *O*, $\angle ABP = 58°$ and $\angle BAC = 45°$, find the angles marked *x* and *y*.

Solution:

$x = 58°$ ($\angle$s in alt. segment)

$\angle PAB = \angle PBA$

$\qquad = 58°$ (base $\angle$s of isos. $\triangle PAB$)

$y = 180° - 58° - 45°$ (adj. $\angle$s on a str. line)

$\quad = 77°$
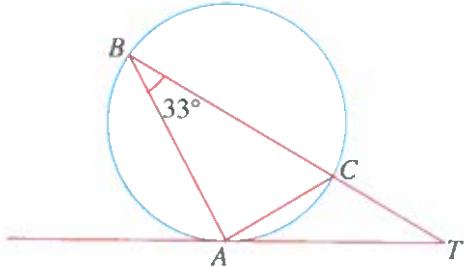
$\therefore x = 58°$, $y = 77°$

PRACTISE NOW 9
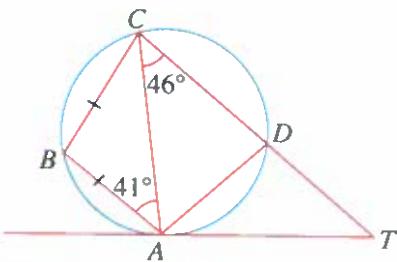
1. *BC* is a diameter of the circle and *TA* is the tangent to the circle at *A*.

Given that $\angle ABC = 33°$, find $\angle ATC$.

2. *TA* is the tangent to the circle at *A*, $AB = BC$, $\angle BAC = 41°$ and $\angle ACT = 46°$.
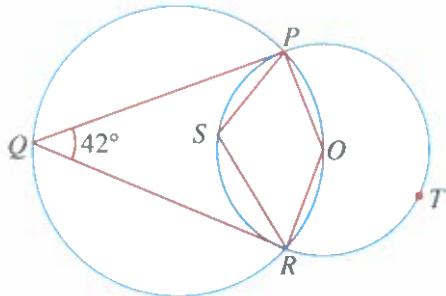
Find

(i) $\angle CAD$,

(ii) $\angle ATC$.

(Application of Circle Properties)

In the figure, O is the centre of the smaller circle passing through the points P, S, R and T. The points P, Q, R and O lie on the larger circle.

Given that $\angle PQR = 42°$, find $\angle PSR$.

Solution:

$\angle POR = 180° - 42°$ ($\angle$s in opp. segments)

$\qquad = 138°$

Reflex $\angle POR = 360° - 138°$ ($\angle$s at a point)

$\qquad = 222°$
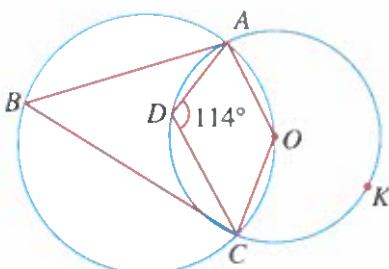
$\angle PSR = \dfrac{222°}{2}$ ($\angle$ at centre = 2 $\angle$ at $\odot^{ce}$)

$\qquad = 111°$

In the figure, O is the centre of the smaller circle passing through the points A, D, C and K. The points A, B, C and O lie on the larger circle.
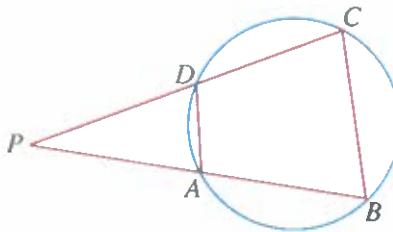
Given that $\angle ADC = 114°$, find $\angle ABC$.

Worked Example 11

(Application of Circle Properties)

In the figure, A, B, C and D are points on the circle. PAB and PDC are straight lines.

(i) Show that △PAD is similar to △PCB.

(ii) Given also that PA = 12 cm, AD = 7 cm and PC = 28 cm, find the length of BC.
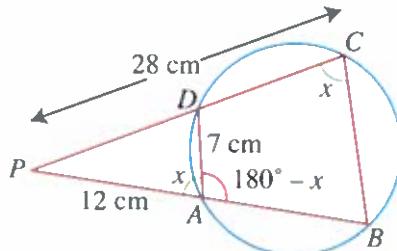
Solution:

(i) Let $\angle BCD = x$.

Then $\angle BAD = 180° - x$ ($\angle$s in opp. segments)

i.e. $\angle PAD = 180° - (180° - x)$

$\qquad\qquad = x$

In △PAD and △PCB,
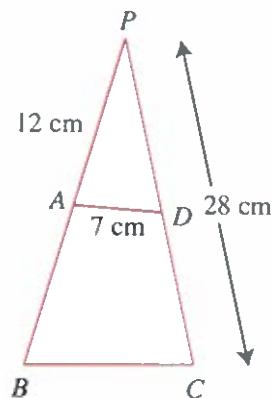
$\qquad$ $\angle P$ is a common angle.

$\qquad$ $\angle PAD = \angle PCB$

∴ △PAD is similar to △PCB. (2 pairs of corr. ∠s equal)

(ii) Using similar triangles,
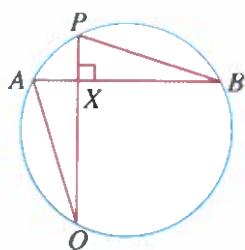
$$\frac{BC}{DA} = \frac{PC}{PA}$$

$$\frac{BC}{7} = \frac{28}{12}$$

$$BC = \frac{28}{12} \times 7$$

$$= 16\frac{1}{3} \text{ cm}$$

In the figure, A, P, B and Q are points on the circle. The chords AB and PQ intersect at right angles at X.

(i) Show that △AXQ is similar to △PXB.

(ii) Given also that AX = 5 cm, QX = 10.5 cm and PX = 3.4 cm, find the length of BX.
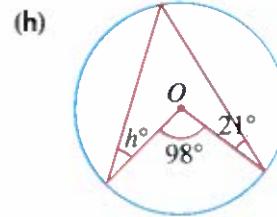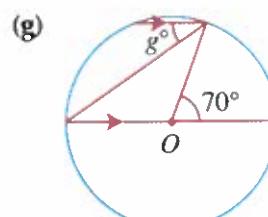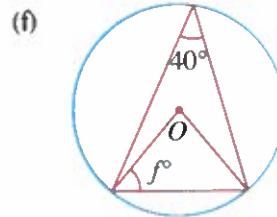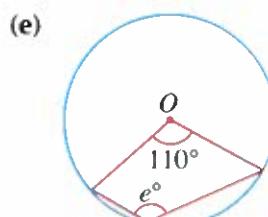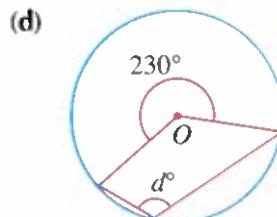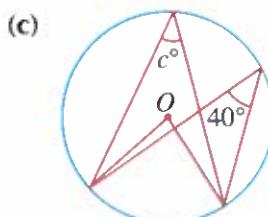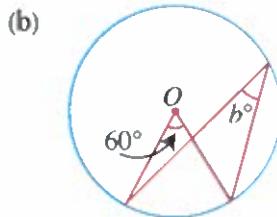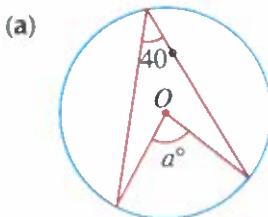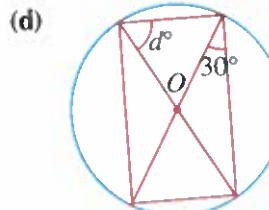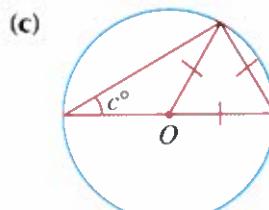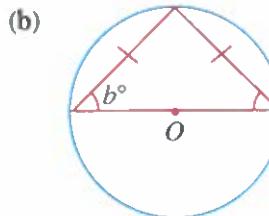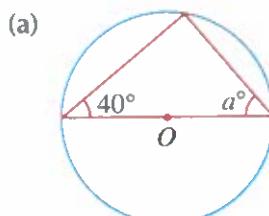
1. Given that O is the centre of each of the following circles, find the value of each of the unknowns.

(a)

(b)

(c)

(d)

(e)

(f)

(g)

(h)

2. Given that O is the centre of each of the following circles, find the value of each of the unknowns.

(a)

(b)

(c)

(d)

3. Find the value of each of the unknowns.

(a)

(b)
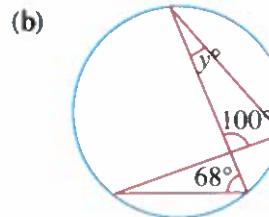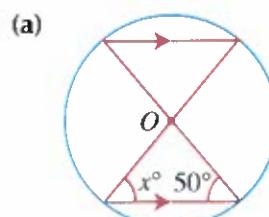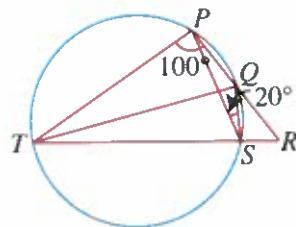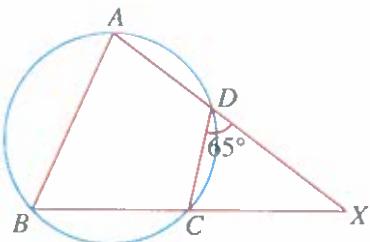
4. In the figure, $\angle TPQ = 100°$ and $\angle PSQ = 20°$.
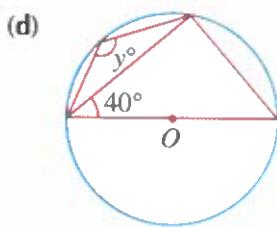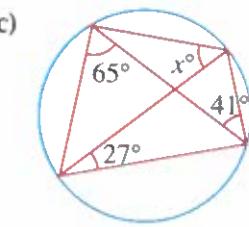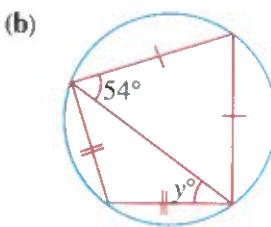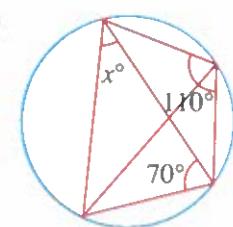
Find $\angle PQT$.

5. In the figure, A, B, C and D are points on the circle such that AD produced meets BC produced at X.
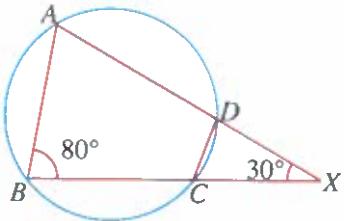
Given that $\angle CDX = 65°$, find $\angle ABC$.

6. Find the values of the unknowns.

(a)

(b)

(c)

(d)
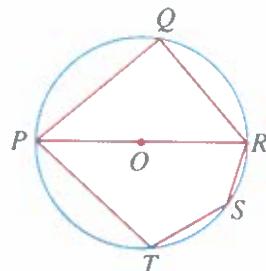
7. In the figure, A, B, C and D are points on the circle such that AD produced meets BC produced at X.

Given that $\angle ABC = 80°$ and $\angle AXB = 30°$, find
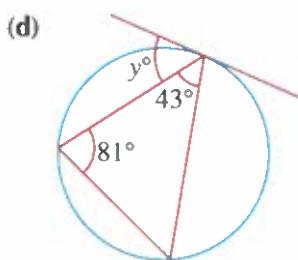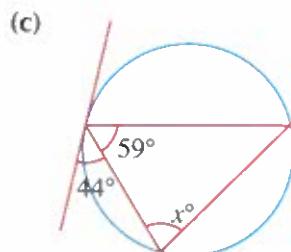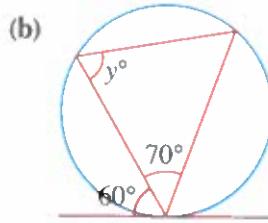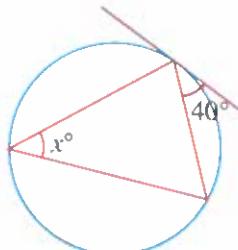
(i) $\angle BAD$, (ii) $\angle XCD$.

8. In the figure, O is the centre of the circle.
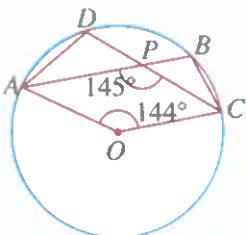
Find the sum of $\angle PQR$, $\angle PRS$ and $\angle PTS$.

9. Find the value of each of the unknowns.

(a)

(b)

(c)

(d)
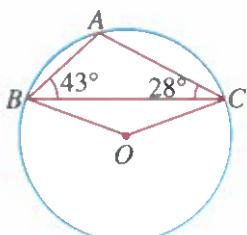
10. In the figure, O is the centre of the circle, $\angle AOC = 144°$ and $\angle APC = 145°$.

Find $\angle BAD$.
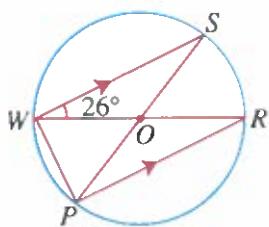
11. In the figure, O is the centre of the circle, $\angle ABC = 43°$ and $\angle ACB = 28°$.

Find

(i) $\angle OBA$,

(ii) $\angle OCA$.
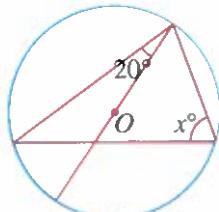
12. In the figure, O is the centre of the circle, $\angle SWR = 26°$. WS is parallel to PR.
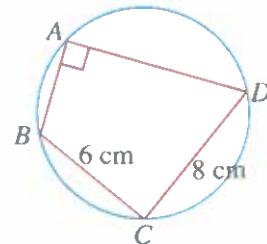
Find

(i) $\angle PWR$,

(ii) $\angle SPW$.

13. Given that O is the centre of the circle, find the value of x.
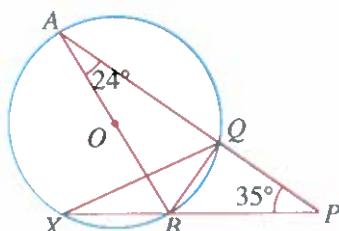
14. In the figure, $\angle BAD = 90°$, $BC = 6$ cm and $CD = 8$ cm.
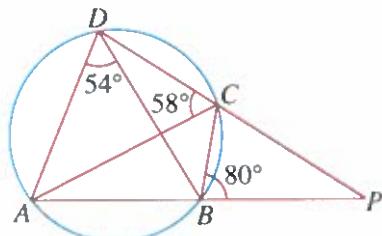
Find the area of the circle.

15. In the figure, A, Q, B and X are points on the circle. AB is a diameter of the circle.
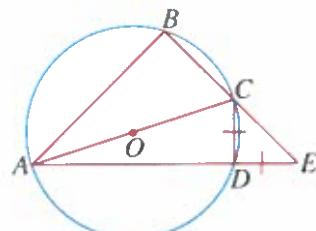
Given that $\angle BAP = 24°$ and $\angle BPA = 35°$, find $\angle BQX$.

16. In the figure, $\angle ADB = 54°$, $\angle ACD = 58°$ and $\angle CBP = 80°$.
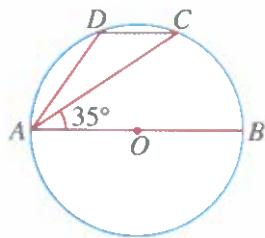
Find $\angle APD$.

17. In the figure, O is the centre of the circle.

Given that $CD = DE$, find $\angle BAD$.

18. In the figure, AB is a diameter of the circle.
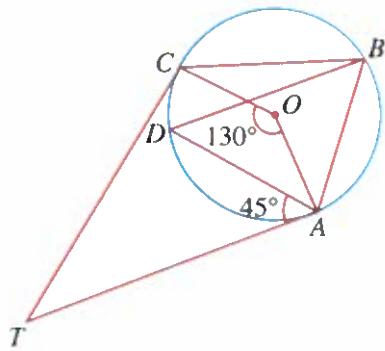
Given that $\angle CAB = 35°$, find $\angle ADC$.
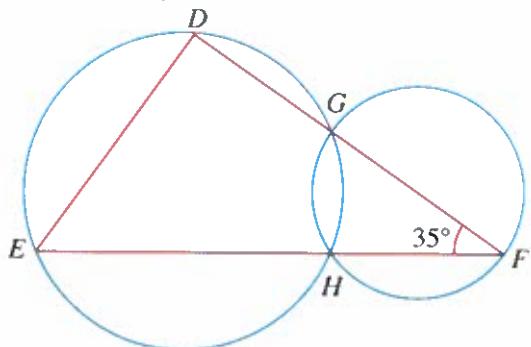
19. In the figure, A, B, C and D are points on the circle. The tangents from a point T touch the circle at the points A and C.

Given that $\angle TAD = 45°$ and $\angle AOC = 127°$, find

(i) $\angle CBD$, **(ii)** $\angle ATC$.
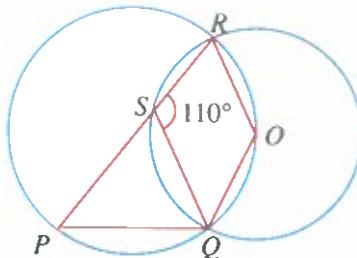
20. In the figure, two circles intersect at the points G and H. GF is a diameter of the circle GHF and $\angle GFH = 35°$. ED is a chord in the larger circle and EHF is a straight line.
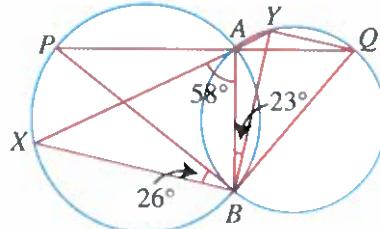
Find

(i) $\angle EDG$, **(ii)** $\angle DEF$.

21. In the figure, two circles intersect at the points Q and R. O is the centre of the circle SQR, $\angle RSQ = 110°$ and PSR is a chord in the larger circle.

Find $\angle QPS$.
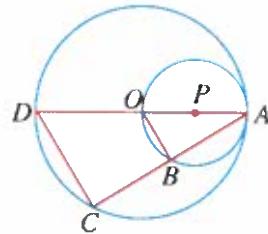
22. In the figure, points P, A, B and X lie on the larger circle and Q, B, A and Y lie on the smaller circle. PAQ and XAY are straight lines, $\angle BAX = 58°$, $\angle PBX = 26°$ and $\angle ABY = 23°$.
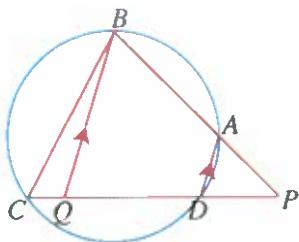
Find

(i) $\angle AQB$, **(ii)** $\angle AYQ$.

23. In the figure, O is the centre of the larger circle passing through the points A, C and D with DOA as a diameter. P is the centre of the smaller circle through points O, B and A, with OPA as a diameter.

(a) Show that $\triangle ABO$ is similar to $\triangle ACD$.

(b) Given also that $AP = 4$ cm and $OB = 4.5$ cm, find the length of

(i) OC, **(ii)** CD.
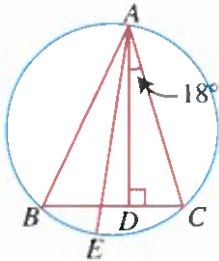
24. In the figure, A, B, C and D are points on the circle. PAB and $PDQC$ are straight lines. QB is parallel to DA.

(i) Show that $\triangle PAD$ is similar to $\triangle PBQ$.

(ii) Name another triangle that is similar to $\triangle PAD$. Explain your answer.
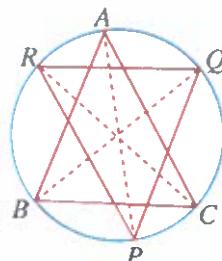
25. In the figure, A, B, E and C are points on the circle. AE is the diameter of the circle and AD is the height of $\triangle ABC$.

Given that $\angle CAD = 18°$, find $\angle BAE$.
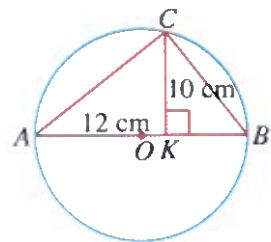
26. In the figure, A, Q, C, P, B and R are points on the circle. AP, BQ and CR are the angle bisectors of $\angle A$, $\angle B$ and $\angle C$ respectively.

Given that $\angle A = 50°$, $\angle B = 70°$ and $\angle C = 60°$, find $\angle P$, $\angle Q$ and $\angle R$.

27. In the figure, AOB is a diameter of the circle, centre O. C is a point on the circumference such that CK is perpendicular to AB.

(i) Show that $\triangle ACK$ is similar to $\triangle CBK$.

(ii) Given also that $AK = 12$ cm and $CK = 10$ cm, find the radius of the circle.
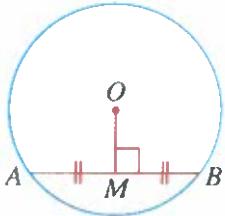
Symmetric Properties of Circle

Property 1:
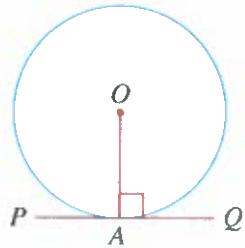
Perpendicular bisector of chord

The perpendicular bisector of a chord of a circle passes through the centre of the circle, i.e. $AM = MB \Leftrightarrow OM \perp AB$

Property 2:

Equal chords

Chords that are equidistant from the centre of a circle are equal in length, i.e. $PQ = AB \Leftrightarrow OM = ON$

Property 3:

Tangent perpendicular to radius

The tangent at the point of contact is perpendicular to the radius of a circle, i.e. $PQ \perp OA$

Property 4:

Equal tangents
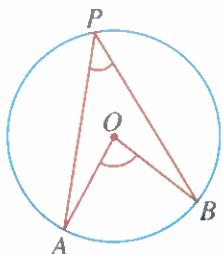
Tangents from an external point are equal in length.

The line from the centre of a circle to an external point bisects the angle between the two tangents from the external point, i.e. $PA = PB$.

Property 1:

Angle at centre = 2 × Angle at circumference

An angle at the centre of a circle is twice that of any angle at the circumference subtended by the same arc,

i.e. $\angle AOB = 2 \times \angle APB$

Property 2:
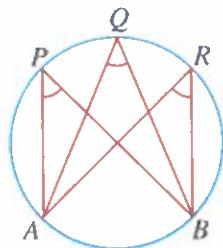
Right angle in semicircle

An angle in a semicircle is always equal to 90°, i.e. AOB is a diameter $\Leftrightarrow \angle APB = 90°$

Property 3:
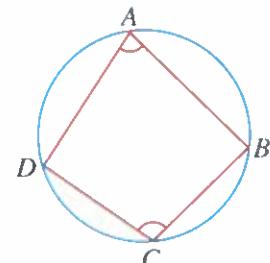
Angles in same segment are equal

Angles in the same segment are equal,

i.e. $\angle APB = \angle AQB = \angle ARB$

Property 4:
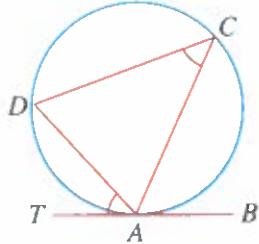
Angles in opposite segments are supplementary

Angles in opposite segments are supplementary,

i.e. $\angle DAB + \angle DCB = 180°$

Property 5:

Angles in alternate segments are equal

The angle between a tangent and a chord at the point of contact is equal to the angle subtended by the chord in the alternate segment,

i.e. $\angle DAT = \angle ACD$
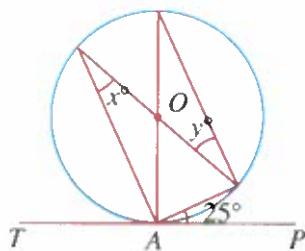
Review Exercise 13
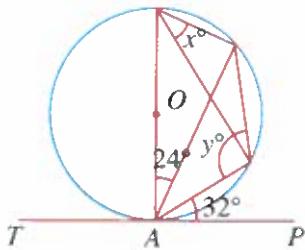
1. Given that *PAT* is a tangent to each of the following circles with centre *O*, find the values of the unknowns.
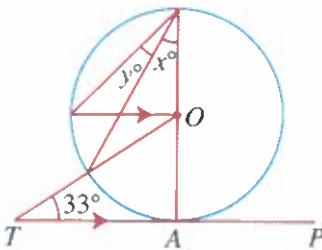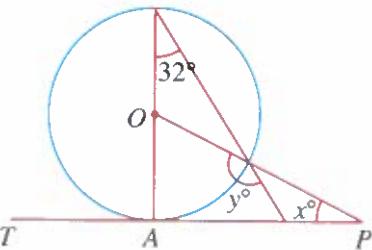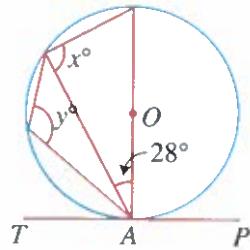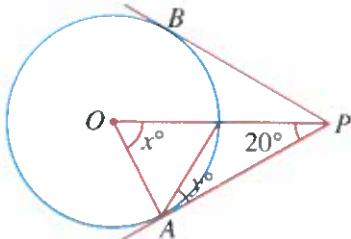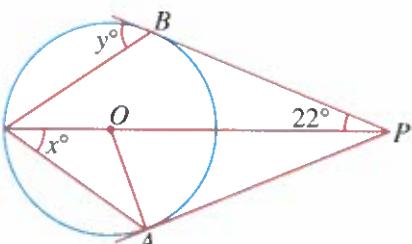
(a)

(b)

(c)

(d)

(e)

(f)

2. Given that *PA* and *PB* are tangents to each of the following circles with centre *O*, find the values of the unknowns.

(a)

(b)

(c)

(d)

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(e)

(f)

3. Given that O is the centre of each of the following circles, find the value of each of the unknowns.

(a)

(b)

(c)

(d)

(e)

(f)
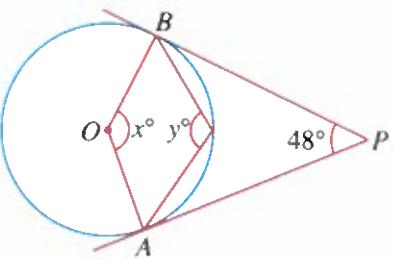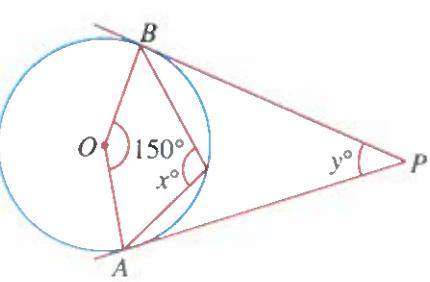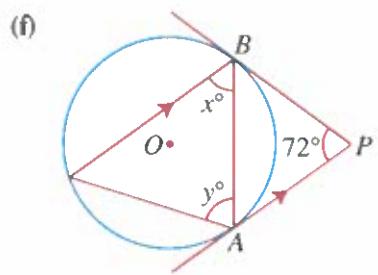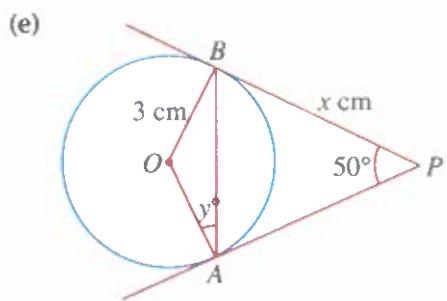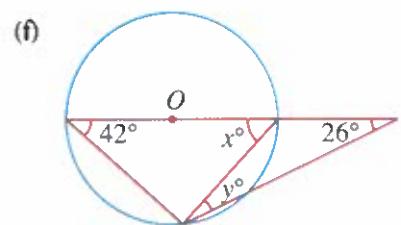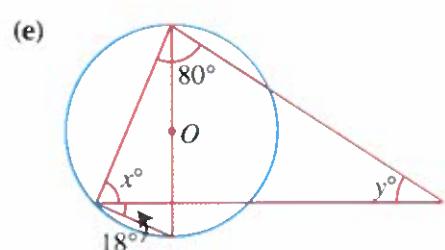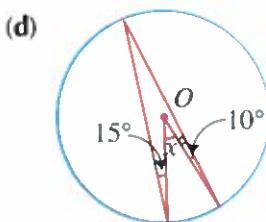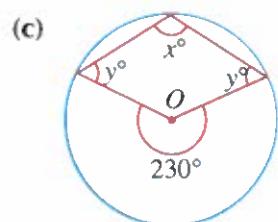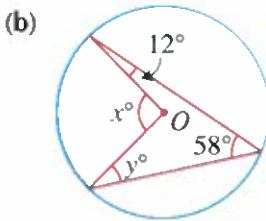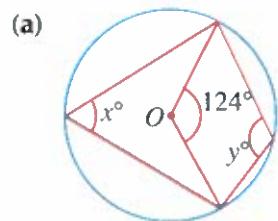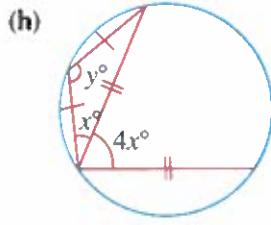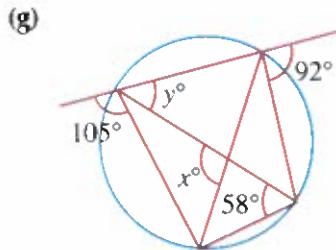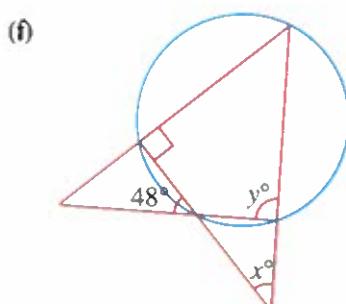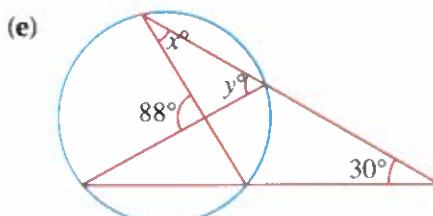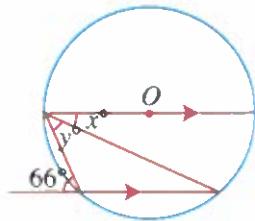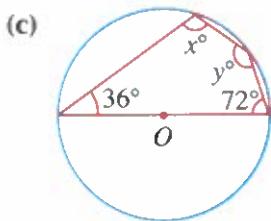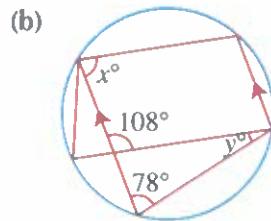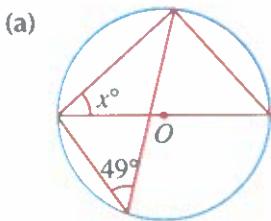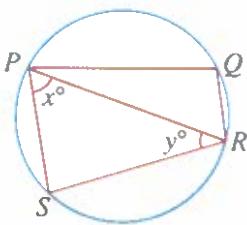
4. Given that O is the centre of each of the following circles, find the values of the unknowns.

(a)

(b)

(c)

(d)

(e)

(f)

(g)

(h)
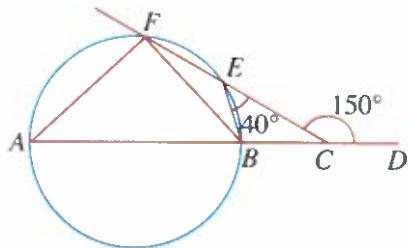
5. In the figure, P, Q, R and S are points on the circle.
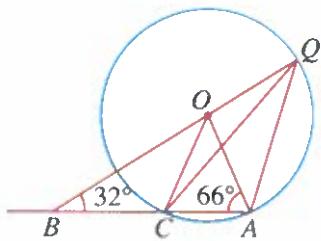
Express $\angle PQR$ in terms of x and y.

6. In the figure, A, B, E and F are points on the circle. AB is the diameter of the circle. $ABCD$ and CEF are straight lines, $\angle BEC = 40°$ and $\angle FCD = 150°$.

Find $\angle EBF$.
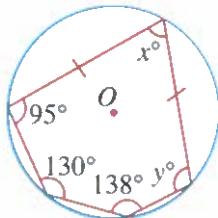
7. In the figure, Q, A and C are points on the circle, centre O. BOQ and BCA are straight lines, $\angle OAC = 66°$ and $\angle OBC = 32°$.
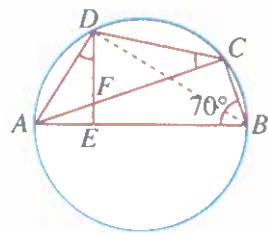
Find
(i) $\angle CQA$,
(ii) $\angle QCA$.

8. Find the value of x and of y in the following figure.
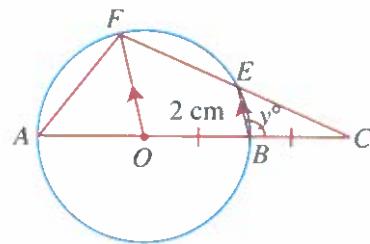
9. The figure shows a circle with AB as a diameter.

Given that $\angle ADE = \angle DCA$ and $\angle CBA = 70°$, find
(i) $\angle FEB$,
(ii) $\angle EFC$.
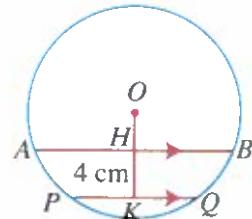
10. In the figure, O is the centre of the circle. FEC and $AOBC$ are two straight lines. BE is parallel to OF, $OB = BC = 2$ cm and $\angle CBE = y°$.

Find
(i) the length of BE,
(ii) $\angle FAO$ in terms of y.

11. In the figure, AB and PQ are parallel chords in a circle, centre O. H and K are the midpoints of AB and PQ respectively.

Given that $AB = 26$ cm, $PQ = 22$ cm and $HK = 4$ cm, find
(i) the length of OH,
(ii) the radius of the circle.
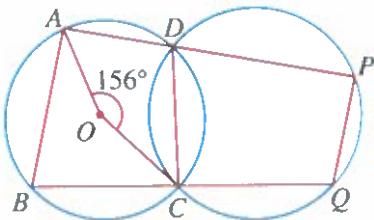
12. PA is a diameter of a circle and PT is a tangent. S is a point on the circle such that $\angle SPT = 46°$.

   (i) Find $\angle PAS$.

   (ii) Hence, find $\angle PRS$, where R is any other point on the minor arc PS of the circle.

13. L, M and N are three points on a circle. The tangents at L and M intersect at P. Given that $\angle LPM = 58°$, find $\angle LNM$.
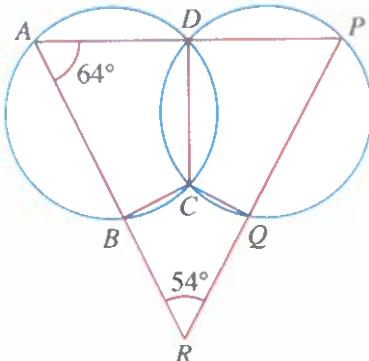
14. In the figure, O is the centre of the circle passing through the points A, B, C and D. The points C, D, P and Q lie on the circumference of another circle. ADP and BCQ are straight lines and $\angle AOC = 156°$.

Find

(i) $\angle PDC$, **(ii)** $\angle PQC$.
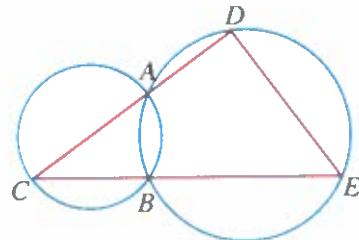
15. The figure shows two circles $ABCD$ and $CDPQ$ intersecting at C and D.

Given that ADP, ABR and PQR are straight lines, $\angle PAR = 64°$ and $\angle ARP = 54°$, find

(i) $\angle APR$, **(ii)** $\angle BCQ$.
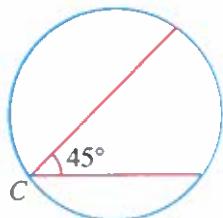
16. In the figure, CAD and CBE are straight lines.

Given that CA is a diameter of the circle ABC, determine if $\angle ADE$ is a right angle.

Challenge Yourself

The figure shows the plan of a circular hall of a jewellery exhibition. C is a hidden video camera which scans an angle of $45°$. How many more such video cameras must be installed on the walls of the hall so that they will cover the entire hall? Indicate the position where each video camera must be mounted.
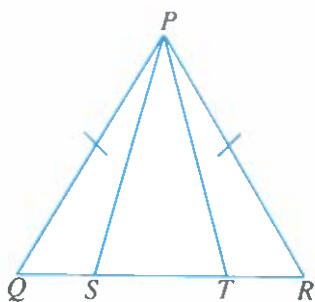
How many video cameras are required if each one can scan an angle of

(a) $35°$? **(b)** $60°$? **(c)** $90°$? **(d)** $100°$?
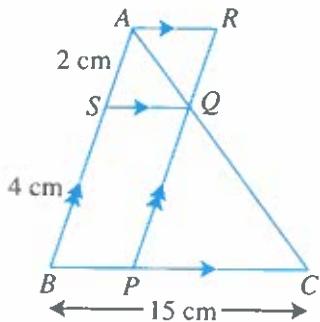
D1 Revision Exercise

1. In the figure, $PQ = PR$ and $\angle PST = \angle PTS$.

Determine if $\triangle PQS$ is congruent to $\triangle PRT$.

2. In the figure, AR, SQ and the straight line BPC are parallel. BSA is a straight line and is parallel to PQR.
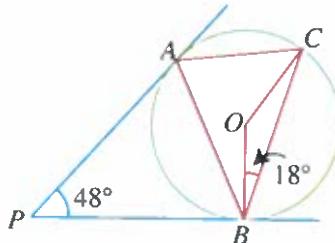
(a) Show that $\triangle ASQ$ is congruent to $\triangle QRA$.

(b) Name a triangle similar to $\triangle ABC$.

(c) Given that $AS = 2$ cm, $SB = 4$ cm and $BC = 15$ cm, find the length of SQ.

(d) Name two triangles similar to $\triangle PCQ$.

(e) Given that the area of $\triangle ABC$ is 36 cm², find the area of

   (i) $\triangle PCQ$,

   (ii) $\triangle BPQ$,

   (iii) quadrilateral $ASQR$.

3. A scale model of a warehouse is 45 cm high whereas the actual warehouse has a height of 30 m.

   (i) Find the scale of the model.

   (ii) Given that the floor area of the model is 810 cm², find the actual floor area of the house in m².

   (iii) If the volume of one of the rooms in the model is 162 cm³, find the volume of the corresponding room in the actual house in m³.
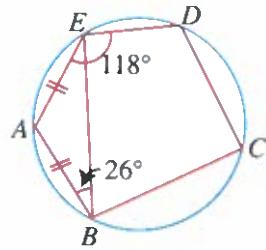
4. PA and PB are tangents to the circle with centre O.

Given that $\angle APB = 48°$ and $\angle OBC = 18°$, find

   (i) $\angle BAC$,   **(ii)** $\angle ABC$.

5. A, B, C, D and E are points on a circle.
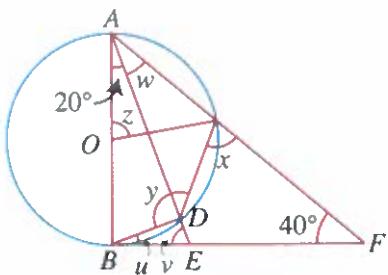
Given that $AB = AE$, $\angle ABE = 26°$ and $\angle AED = 118°$, find

   (i) $\angle BAE$,   **(ii)** $\angle BCD$.
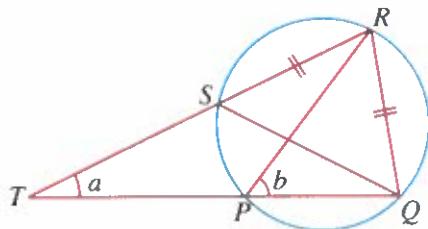
6. In the figure, *AB* is a diameter of the circle with centre *O*. *BEF* is a tangent to the circle at *B*, ∠*BAE* = 20° and ∠*AFB* = 40°.

Find the angles *u*, *v*, *w*, *x*, *y* and *z*.
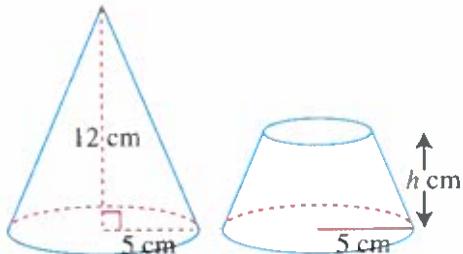
7. In the figure, *TPQ* and *TSR* are straight lines and *SR* = *QR*.

Given that ∠*QTR* = *a* and ∠*QPR* = *b*, express each of the following in terms of *a* and *b*.
- (i) ∠*RQS*
- (ii) ∠*PRS*
- (iii) ∠*PST*
- (iv) ∠*QRS*
- (v) ∠*PSQ*

8. The figure shows a wax candle in the shape of a right circular cone with base radius 5 cm and height 12 cm. It takes 1 hour 40 minutes to burn completely. After $12\frac{1}{2}$ minutes of burning, the candle is reduced to a frustrum with a height of *h* cm.
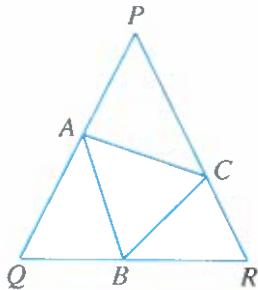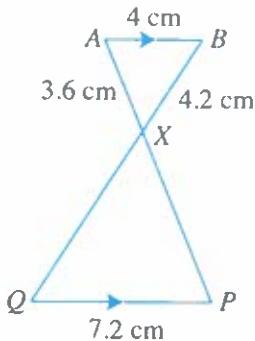
Find
- (i) the total surface area of the cone before burning starts,
- (ii) the value of *h*,
- (iii) the total surface area of the frustrum.

1. In the figure, $\triangle PQR$ is an equilateral triangle with sides of length 16 cm. A, B and C are points on PQ, QR and PR respectively such that $PA = QB = RC = 4$ cm.

   (i) Show that $\triangle APC$ is congruent to $\triangle BQA$.
   (ii) Name the third triangle which is congruent to $\triangle APC$ and $\triangle BQA$ and show that $\triangle ABC$ is an equilateral triangle.

2. The figure shows two triangles ABX and PQX. AB is parallel to QP, $AB = 4$ cm, $AX = 3.6$ cm, $BX = 4.2$ cm and $QP = 7.2$ cm.
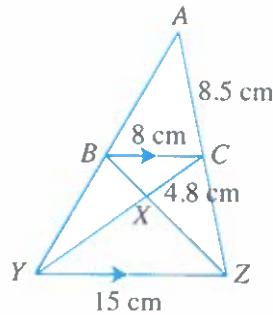
   Find
   (i) the length of PX and of QX,
   (ii) the ratio of the area of $\triangle ABX$ to that of $\triangle PQX$.

3. The surface area of two cups are in the ratio 9 : 64. If the smaller cup has a height of 25 cm and a volume of 2400 cm³, find
   (i) the height of the larger cup,
   (ii) the exact volume of the larger cup.
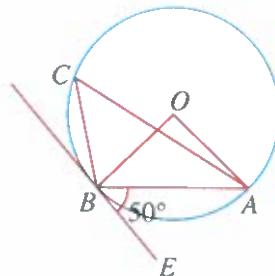
4. In the figure, $BC = 8$ cm, $CX = 4.8$ cm, $AC = 8.5$ cm and $YZ = 15$ cm. BC is parallel to YZ.

   Find
   (i) the length of XY and of CZ,
   (ii) the ratio of the area of $\triangle AYZ$ to the area of trapezium $BCZY$.
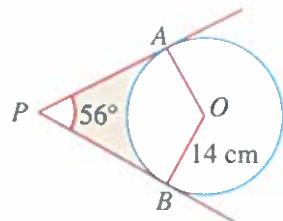
5. In the figure, EB is the tangent to the circle with centre O at B.

   Given that $\angle ABE = 50°$, find
   (i) $\angle AOB$,
   (ii) $\angle ACB$.
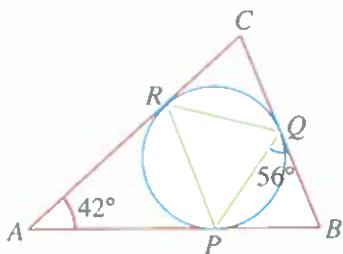
6. In the figure, PA and PB are tangents to the circle, centre O.

   Given that the radius of the circle is 14 cm and $\angle APB = 56°$, find the area of the shaded region.
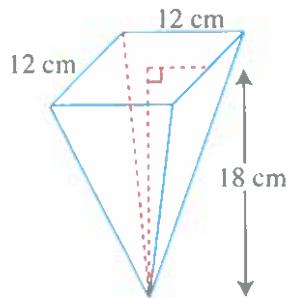
7. The tangents to a circle at P, Q and R intersect at A, B and C as shown.

Given that $\angle BAC = 42°$ and $\angle PQB = 56°$, find

(i) $\angle ACB$,

(ii) $\angle PQR$,

(iii) $\angle RPQ$.

8. The figure shows an empty inverted pyramid with a square base of length 12 cm and height 18 cm. A pipe can fill the pyramid with water in 4 minutes.

(i) Find the height of water in the pyramid after 30 seconds.

(ii) Calculate the ratio of the surface area in contact with the water to that of the surface area which is not in contact with the water after 30 seconds.

Problems in Real-World Contexts

PROBLEM 1: The Singapore Sports Hub

The Singapore Sports Hub is a newly-built sports complex in Singapore. Standing at 82.5 m tall, the hub is built in the shape of a dome with a diameter of 310 m.

(a)

(b)

Consider the 2-dimensional shape of the dome. It is similar to that of a parabola, which can be modelled by a quadratic equation $y = ax^2 + bx + c$, where x m and y m represent the horizontal and vertical distances respectively away from the point O (see Fig. **(b)**) and a, b and c are real numbers.

(a) Based on the information given, state the coordinates of two points (other than the point O) on the dome.

(b) Sketch the graph representing the dome and label the coordinates from your answer in **(a)**.

(c) Using the general form of a quadratic function, $y = ax^2 + bx + c$ and the coordinates stated in **(a)**,

   (i) write a system of three equations to solve for the three unknowns a, b and c,

   (ii) find the values of a, b and c,

   (iii) state the equation of the graph representing the dome.

(d) Suggest another way to orientate the coordinate axes to find the quadratic equation modelling the dome.

PROBLEM 2: Road Tax for Singapore-registered Vehicles

The road tax of a Singapore-registered car is calculated based on the car's engine capacity. The following table shows the formula of calculating the road tax per annum for petrol cars.

Engine Capacity (EC) in cc	6-Monthly Road Tax Formula (From 1 July 2008) in $
EC ⩽ 600	200×0.782
$600 < EC \leq 1000$	$[200 + 0.125 \times (EC - 600)] \times 0.782$
$1000 < EC \leq 1600$	$[250 + 0.375 \times (EC - 1000)] \times 0.782$
$1600 < EC \leq 3000$	$[475 + 0.75 \times (EC - 1600)] \times 0.782$
$EC > 3000$	$[1525 + 1 \times (EC - 3000)] \times 0.782$

(a) Calculate the 6-monthly road tax of a petrol car with an engine capacity of 1400 cc.

(b) How much road tax is payable per annum for a car with an engine capacity of 3000 cc?

(c) Using a suitable scale, draw the graph of road tax against engine capacity.

PROBLEM 3: High-Speed Chase

During a routine operation along an expressway one night, a car drove through a police road block without stopping. The police signalled for the car to stop but it accelerated and the police gave chase.

The speed and the time of the speeding car and the police car during the 3-minute high-speed chase along the expressway are recorded in the table.

Time	Speed of Speeding Car (km/h)	Speed of Police Car (km/h)
1st minute	110	95
2nd minute	145	140
3rd minute	160	185

(i) Based on the information given, using a distance-time graph, determine whether the police car will be able to overtake the speeding car and arrest the driver during the high-speed chase. Show how you arrive at your conclusion.

(ii) Are there any assumptions that you may have to make?

PROBLEM 4: Credit Card Debts

Many people have a credit card. Having a credit card allows people to track their expenses, enjoy promotions jointly offered by the merchant and the credit card companies and save them the hassle of having to carry a large amount of cash around. However, if the credit card is not being used wisely, an unpaid credit card bill may chalk up a debt that could snowball and cripple their finances.

Credit card companies typically impose a credit charge of 2% per month or 24% per annum for any unpaid bill plus a penalty of $50 per month if the bill is not paid or not paid in full.

Suppose a man owed a credit card company $100 at the beginning of January 2014 and did not pay a single cent to clear his debt.

The following table illustrates the working to obtain the amount of debt he owes at the end of 3 months. Complete the table using the formula given for each column below.

Month	Amount owed at the beginning of the month, A	Additional fee, $(A + 50)$	Interest, $I = \dfrac{2}{100}(A + 50)$	Amount owed at the end of the month, $D = A + 50 + I$
January	$100.00	$150.00	$3.00	$153.00
February	$153.00	$203.00	$4.06	$207.06
March	$207.06	$257.06	$5.14	$262.20
April				
May				
June				

(a) Using the above set of data and a spreadsheet (as described below), obtain different functions for estimating the amount of debt that the man will owe at the end of
(i) 1 year,   (ii) 3 years.

Open a spreadsheet and type in 2 columns of data. For the first column, use 1, 2, 3, 4, 5 and 6 to replace the months of January to June respectively. For the second column, enter the amount, D, owed at the end of each of the months. Select the entire table and insert a scatterplot with only the markers (or points). Right-click on the points and add a trendline to model or best fit the data: choose one of the functions given, i.e. exponential, linear, etc. For the forecast, select forward 36 periods, and choose to display the equation of the trendline on the scatterplot.

Use the equation of the trendline to obtain the estimated values of D at the end of 1 year and 3 years.

(b) The formula for D is given by

$$D = A(1.02)^n + 2550(1.02)^n - 2550,$$

where A is the initial amount that the man owes the credit card company and n is the number of months that he did not pay a single cent to the company.

Use the formula to obtain the value of D at the end of
(i) 1 year, **(ii)** 3 years.
Give your answers correct to the nearest cent.

(c) Compare the estimated values of D obtained using the different models in the spreadsheet with the actual values obtained using the formula. Which is a better model for the estimation of values for the different periods of time?

PROBLEM 5: Prices of Watermelons

A fresh fruit stall holder in a school canteen sells slices of watermelon or juice. The following table shows the sizes and prices of the watermelons that are on sale at a wholesale market.

Size	small	medium	large
Diameter (cm)	24	28	32
Price ($)	4.10	5.80	7.60

Which size of watermelon should the stall holder buy in order to maximise his profit? Show your working to support your answer and state any assumptions made.

PROBLEM 6: The Broken Plate
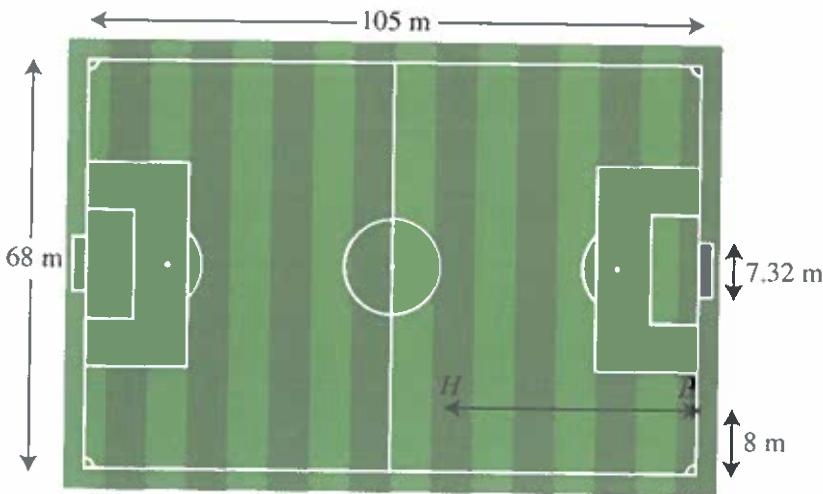
An archaeologist found a part of a broken circular plate, drawn to scale of 1 cm to represent 4 cm, as shown.

(i) Explain how you can help to find the diameter of the plate using geometrical properties of circles.

(ii) Using the method you have explained in **(i)**, find the diameter of the original circular plate.

(iii) Hence, find the area of the original circular plate.

PROBLEM 7: Penalty Shootout

The figure shows a soccer field measuring 105 m by 68 m and a goal post on each side with a width of 7.32 m.

(i) For a penalty shootout, the ball is placed at a distance of 11 m from the centre of the goal post. Find the widest angle for shooting that the player can make before the ball misses the goal post.

Hint: Let O be the point at which the ball is placed, and P and Q be the two ends of the goal post. Find angle POQ.

(ii) A winger dribbles a ball 8 m from the side of the field along the line HB. At which point along HB should he strike the ball so that he will get the widest angle for shooting?

(iii) Explain how the angle for shooting varies as the winger strikes the ball at different points along HB.

Problems in Real-World Contexts

PROBLEM 8: Income Tax

All income earned and received in Singapore are subjected to income tax. The income that will be taxed is referred to as 'chargeable income' and is given by:

Chargeable income = total gross income – total reliefs,

where total reliefs (i.e. amount that will not be taxed) = personal relief + wife relief + child relief + life insurance relief + Central Provident Fund (CPF) contributions + donations to charitable organisations + …

The table shows the income tax rates in Singapore in 2012.

	Chargeable Income ($)	Tax rate (%)	Gross income tax payable ($)
On the first	20 000	0	0
On the next	10 000	2	200
On the first	30 000	-	200
On the next	10 000	3.5	350
On the first	40 000	-	550
On the next	40 000	7	2800
On the first	80 000	-	3350
On the next	40 000	11.5	4600
On the first	120 000	-	7950
On the next	40 000	15	6000
On the first	160 000	-	13 950
On the next	40 000	17	6800
On the first	200 000	-	20 750
On the next	120 000	18	21 600
On the first	320 000	-	42 350
In excess of	320 000	20	

(i) In 2012, Mr Tan earned a gross annual income of $115 000. He is eligible for a personal relief of $3000, a wife relief of $2000 and a child relief of $4000. His CPF contributions amounted to $23 000 and he donated $1500 to charitable organisations. Based on the tax rates provided in the table, find his income tax payable.

(ii) Mdm Lee earned a gross annual income of $156 000 in 2012. She is eligible for a personal relief of $3000 and a parent relief of $5000. She has 2 children. She is entitled to a relief of 15% of her gross income for her first child and a relief of 20% of her gross income for her second child. Her CPF contributions amounted to $31 200. Based on the tax rates provided in the table, find her income tax payable.

(iii) Why are the income tax rates in Singapore staggered in this manner, i.e. 0% for the first $20 000, 2% for the next $10 000, 3.5% for the next $10 000, etc? Why do the income tax rates vary according to income earned?

(iv) Find out the income tax systems for other countries and compare them with that of Singapore. Which country's income tax system is fairer? Which system do you prefer? Explain your answers.

Problems in Real-World Contexts

PROBLEM 9: Purchasing a Flat (Stamp Duty)

(i) To purchase a flat, Mr Lee has to pay stamp duty to the Housing and Development Board (HDB). The flat costs $500 000 and his property agent tells him that the formula used to calculate the stamp duty is

$$\text{Stamp Duty} = 3\% \times \text{Selling Price of Flat} - \$5400.$$

How much stamp duty does Mr Lee have to pay?

(ii) Mr Lee checks the HDB website on the internet and finds that the formula used to calculate the stamp duty is different.

> First $180 000   : $1\% \times$ Selling Price of Flat
> Next $180 000   : $2\% \times$ Selling Price of Flat
> Thereafter   : $3\% \times$ Selling Price of Flat

Using this formula, calculate the stamp duty that Mr Lee has to pay. Is it the same as the answer you obtained in (i)?

(iii) Mr Lee is curious to know why the formula in (i) works. How do you prove to him that both formulae can be used to calculate the stamp duty?

(iv) What is the minimum selling price of the flat, below which the formula in (i) will not work? Explain your answer.

(v) Mr Lee notices that, according to the formula in (ii), the stamp duty for the first $360 000 is $5400, which also appears in the formula in (i). He wonders whether this is a coincidence or whether it will always work. Think of a counterexample to show that this is just a coincidence.

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Problem 10: Constructing A 3-Dimensional Star

To raise funds for charity, your class has been tasked to make and sell star-shaped paper lanterns. You will have to design 3-dimensional stars by creating a 2-dimensional template of each star and determining the amount of materials that are required.

You may have to consider the following:
- Creation of a 2-dimensional template
- Materials for lantern, e.g. coloured paper, glue, string, lightbulbs
- Size and colour of paper to be used and amount of paper needed
- Amount of money to be raised
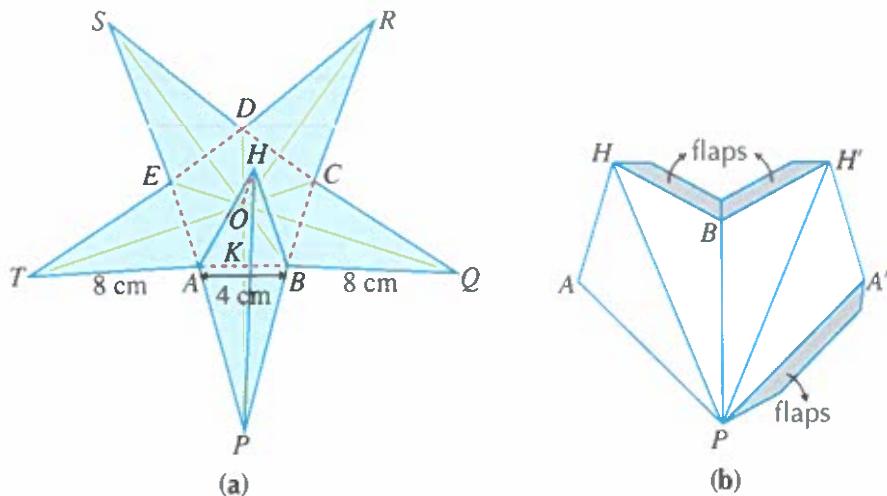
Guiding Questions:

1. How can we obtain the dimensions of the 2-dimensional template? Consider a star which is about 20 cm wide and 6 cm deep, with cross section and a 3-dimensional section $APBH$ as shown in Fig. (a). $TA = BQ = 8$ cm, $AB = 4$ cm and height of star above centre O, $OH = 3$ cm. Note that $TABQ$ is not a straight line.

   Fig. (b) shows the template used to make one part of the 3-dimensional star with cross section $OAPB$. You can fold the template along the lines and glue the flap on side PA' to PA to make this part of the 3-dimensional star. You need to make 5 such parts and glue each to the other along the remaining 2 flaps to make the 3-dimensional star.

Problem Solving Tip

The depth of the star is 6 cm because $OH = 3$ cm and $OH' = 3$ cm, where H' is the point directly below centre O on the other surface of the 3-dimensional star.

(a)

(b)

2. How do you calculate the amount of paper needed for each star? What size of paper is appropriate and available? How many sheets of paper would be needed?
3. What is the total cost of materials?
4. Propose a price for a lantern. Is the price proposed reasonable? How many lanterns have to be sold to raise the amount of money?

Practise Now Answers

CHAPTER 1

Practise Now 1

(a) −8 or 1

(b) $1\frac{1}{3}$ or $-2\frac{1}{2}$

Practise Now 2

(a) 3 or −17

(b) 8.32 or 1.68

Practise Now 3

(a) $(x+10)^2 - 100$

(b) $\left(x-\frac{7}{2}\right)^2 - \frac{49}{4}$

(c) $\left(x+\frac{1}{10}\right)^2 - \frac{1}{100}$

(d) $(x+3)^2 - 18$

Practise Now 4

1. (a) 0.61 or −6.61 (b) −0.81 or −6.19

   (c) 1.62 or −0.62

2. 4.72 or −5.72

Practise Now 5

(a) 1.27 or −2.77

(b) 1.72 or −0.117

(c) 4.73 or 1.27

(d) 5.24 or 0.764

Practise Now 6

1. (i) 15, 5, −1, −3, −1, 5, 15

   (iii) 2.2 or −0.2

2. −2.3 or 1

Practise Now 7

1. (i) 16, 9, 4, 1, 0, 1, 4, 9

   (iii) 3

2. 4

Practise Now 8

1. (a) −1 or −6 (b) 0.703 or −2.37

2. 2.35 or −0.851

Practise Now 9

1. (a) 3.46 or 2.14 (b) 3.58 or 1.92

2. 3 or 2.5

Practise Now 10

(i) $(9 - x)$ cm (iii) 7.928 or 1.072

(iv) 4.25 cm^2

Practise Now 11

(ii) 126.16 or −133.16 (iii) 4.51 h

Practise Now 13

1. (i) $(-1, 0), (5, 0); (0, 5)$

   (ii) $(2, 9)$

   (iv) $x = 2$

2. (i) $(-2, 0), (0, 0); (0, 0)$

   (ii) $(-1, -1)$

   (iv) $x = -1$

Practise Now 14

1. (i) $(x-3)^2 - 3$

   (ii) $(3, -3)$

   (iv) $x = 3$

2. (i) $\left(x+\frac{1}{2}\right)^2 + \frac{3}{4}$

   (ii) $\left(-\frac{1}{2}, \frac{3}{4}\right)$

   (iv) $x = -\frac{1}{2}$

CHAPTER 2

Practise Now 1

(i) $36x^2 - 15x + 2$ (ii) $16x^2 + 38x + 23$

(iii) $4x^4 - 29x^2 + 53$

Practise Now 2

$a = 1\frac{4}{5}, b = -\frac{2}{5}; \frac{7}{5}, 47$

Practise Now 3

$f^{-1}(x) = \frac{1}{8}(x-3)$

Practise Now 4

$2, 0, \frac{25}{49}$

Practise Now 5

$f^{-1}(x) = \frac{2}{x} + 5; x = 0; 5\frac{1}{3}, 4\frac{1}{3}, 13$

Practise Now 6

$p = -4, q = 27$

CHAPTER 3

Practise Now 1

1. (a) $x < 5$

   (b) $x \geqslant -4$

2. 5

Practise Now 2

(a) $x \geqslant 10$

(b) $y < \frac{1}{2}$

Practise Now 3

$x > 14$

(i) 17 (ii) 27

Practise Now 4

1. (a) $x < 1\frac{2}{5}$

   (b) $y \geqslant \frac{3}{29}$

   (c) $z \leqslant 26$

2. 4

Practise Now 5

60

Practise Now 6

6

Practise Now 7

$-1 < x \leqslant 5$

Practise Now 8

1. no solution

2. $-13 < y \leqslant 7$

Practise Now (Page 65)

(a) 375, 325 (b) 1800.5, 1799.5

(c) 85, 75

Practise Now 9

22 m

Practise Now 10

20 m, 18 m

Practise Now 11

$2350.25 \text{ m}^2 \leq \text{area} < 2450.25 \text{ m}^2$

Practise Now 12

3.10 l, 2.90 l

CHAPTER 4

Practise Now 1

(a) 4^{12}
(b) $(-3)^7$
(c) a^{20}
(d) $6x^6y^7$

Practise Now 2

(a) 9^4
(b) $(-4)^7$
(c) a^4
(d) $3x^3y$

Practise Now 3

1. (a) 6^{12}
   (b) k^{45}
   (c) 3^{10q}
2. 2

Practise Now 4

(a) 24^7
(b) $125b^{12}$
(c) $-32c^{10}d^{25}$
(d) $m^{28}n^{19}$
(e) $\dfrac{p^5q^4}{27}$

Practise Now 5

(a) 3^3
(b) 2^{15}
(c) $\dfrac{p^{11}}{q^{10}}$
(d) x^{11}

Practise Now 6

1. (a) 1
   (b) 1
   (c) 3
   (d) 1
2. (a) 3
   (b) 10

Practise Now 7

(a) $\dfrac{1}{36}$
(b) $-\dfrac{1}{8}$
(c) $1\dfrac{61}{64}$
(d) 9

Practise Now 8

(a) a^4
(b) $\dfrac{b^{15}}{c^6}$
(c) $\dfrac{2d}{e^2}$
(d) $\dfrac{5f^4}{3}$
(e) $\dfrac{6}{g^2}$
(f) $-2h^4$

Practise Now 9

(a) 4
(b) 4
(c) $\dfrac{2}{3}$

Practise Now 10

(a) 6
(b) $\dfrac{1}{2}$
(c) $-\dfrac{1}{5}$

Practise Now 11

1. (a) 16
   (b) $\dfrac{1}{8}$
   (c) 1000
2. (a) $a^{\frac{n}{3}}$
   (b) $x^{-\frac{2}{5}}$

Practise Now 12

(a) m^2
(b) $\dfrac{1}{m^{\frac{7}{15}}}$
(c) $\dfrac{m}{n^{\frac{5}{3}}}$
(d) $\dfrac{m^{\frac{11}{3}}}{n^{12}}$
(e) $\dfrac{5m^2}{n^{\frac{14}{5}}}$
(f) $\dfrac{m^3}{n^{\frac{8}{7}}}$

Practise Now 13

(a) 3
(b) -2
(c) $1\dfrac{1}{3}$

Practise Now 14

1. (a) 5.3×10^6
   (b) 6×10^8
   (c) 4.8×10^{-5}
   (d) 2.1×10^{-10}
2. (a) 1 325 000
   (b) 0.0044

Practise Now 15

(a) 2.54×10^{-5} m
(b) 2.34×10^2 cm
(c) 4.0×10^{12} bytes

Practise Now 16

(a) 5.20×10^9
(b) 1.09×10^{-1}
(c) 4×10^3
(d) 2.5×10^{-13}
(e) 1.60×10^5
(f) -2.56×10^6
(g) 6.57×10^{-9}
(h) 4.21×10^4

Practise Now 17

800

CHAPTER 5

Practise Now 1

1. (a) $33\dfrac{1}{3}\%$
   (b) 20%
2. (a) $635
   (b) $73 696

Practise Now 2

1. $950
2. $18.75

Practise Now 3

$0.14

Practise Now 4

1. 12%
2. $564

Practise Now 5

(i) $1400
(ii) No

Practise Now 6

1. $90.95
2. $600

Practise Now 7

1. $6.90
2. $25

Practise Now 8

$690

Practise Now 9

$28 130

Practise Now 10
1. $10 560
2. $725 000

Practise Now 11
(a) $24 750; $174 750
(b) 4 years

Practise Now 12
1. $646.52
2. (a) $60.60   (b) $61.16
3. 3%

Practise Now 13
(i) $21   (ii) $1188
(iii) $288

Practise Now 14
1. (a) (i) S$2230.44
      (ii) S$730.68
   (b) (i) €3229
      (ii) 177 046 baht
2. HK$13 436

CHAPTER 6
Practise Now 1
(a) $\frac{2}{3}$ (b) -1
(c) 0

Practise Now 2
12

Practise Now 3
(a) 5 units   (b) 11.4 units
(c) 6 units

Practise Now 4
(a) $\left(0, 2\frac{1}{4}\right)$ (b) $(-3, 0)$
$3\frac{3}{8}$ units^2

Practise Now 5
1. $\angle DEF$ 2. No

Practise Now 6
1. 7   2. 32

Practise Now 7
(a) $y = \frac{2}{7}x + \frac{11}{7}$ (b) $y = 4$
(c) $x = -3$

Practise Now 8
(a) 2 or 3   (b) -13

Practise Now 9
-3

Practise Now 10
$2y = x + 15$, $y = -2x + 40$

CHAPTER 7
Practise Now 1
(i) 17.5, -17.5 (ii) 2.35, -2.55

Practise Now 2
(i) 1.20   (ii) -2.5

Practise Now 3
(i) -0.9 (ii) 0.8 or -0.8

Practise Now 4
(i) 0.3   (ii) -0.3

Practise Now 5
(a) $a = 5, b = -3$ (c) 1.6
(d) (ii) $h = 2, k = -4$

Practise Now 6
(b) 3.5 minutes   (c) 27 km/h

Practise Now 7
(i) $2\frac{2}{3}$ m/s^2 (ii) 4.89 m/s
(iii) 2 m/s^2

Practise Now 8
(a) $a = 2, b = 17$
(c) (i) 0.7 or 3.8   (ii) $t = 2.25$
     (iii) 9 m/s^2 (iv) $0.25 < t < 4.2$

Practise Now 9
Carpark X

Practise Now 10
(a) 60 beats/minute
(b) 6 beats/minute^2
(c) 1 beat/minute^2

CHAPTER 8
Practise Now 1
1. (a) 0.995   (b) 0.629
2. 0.905

Practise Now 2
1. (a) $\frac{3}{5}$ (b) $-\frac{4}{5}$
   (c) $1\frac{1}{3}$
2. (a) 13 units
   (b) (i) $\frac{12}{13}$ (ii) $-\frac{12}{13}$
     (iii) $2\frac{2}{5}$

Practise Now 3
(a) 24.5° or 155.5°
(b) 103.5°
(c) 84.0°

Practise Now 4
298 m^2

Practise Now 5
1. 3.59   2. 53.1°

Practise Now 6
(i) 82.3°   (ii) 8.01 cm
(iii) 10.7 cm

Practise Now 7
1. (i) 34.7°   (ii) 103.3°
   (iii) 17.5 cm
2. (i) 52.1°   (ii) 31.1°
   (iii) 8.11 cm

Practise Now 8
83.2°, 50.8°, 7.65 cm or
96.8°, 37.2°, 5.96 cm

Practise Now 9
(i) 16.1 cm   (ii) 69.5°
(iii)39.5°

Practise Now 10
96.8°

CHAPTER 9

Practise Now 1
1. 63.8 m   2. 19.0 m

Practise Now 2
1. (i) 354 m   (ii) 14.3°
2. 45.2 m

Practise Now 3
1. (a) 050°   (b) 330°
   (c) 230°   (d) 150°
2. (a) 123°   (b) 231°
   (c) 303°   (d) 051°

Practise Now 4
1. 208 m
2. (i) 245°   (ii) 310 m, 317 m

Practise Now 5
(i) 49.4 km   (ii) 199.3°

Practise Now 6
(i) 200°   (ii) 2.92 km
(iii)4.92 km   (iv) 1.25 km

Practise Now 7
1. (i) 58.0°   (ii) 74.2°
   (iii) 28.1°
2. (i) 14.6 cm   (ii) 28.3°

Practise Now 8
1. (i) 32.0°   (ii) 35.3°
   (iii) 23.8°
2. (i) 33.7°   (ii) 53.1°
   (iii) 30.8°

Practise Now 9
(i) 170 m   (ii) 8.4°

Practise Now 10
68.8 m

CHAPTER 10

Practise Now 1
401 mm^2

Practise Now 2
1. (i) 99.5 cm   (ii) 108 cm
2. $\left(\dfrac{21}{2}\pi + 18\right)$ cm   3. 6.65

Practise Now 3
1. 13.2 cm   2. 34.0 cm

Practise Now 4
(i) 56.6 cm   (ii) 155 cm

Practise Now 5
(ii) 459 cm^2

Practise Now 6
(ii) 122 m^2

Practise Now 7
(a) 12°   (b) 270°
(c) 174.2°   (d) 458.4°

Practise Now 8
(a) 0.628 rad   (b) 5.03 rad
(c) 3.45 rad   (d) 6.98 rad

Practise Now 9
(a) 0.605   (b) 0.973
(c) 2.82

Practise Now 10
(a) 1.06 rad   (b) 1.22 rad
(c) 0.722 rad

Practise Now 11
(i) 7.00 cm   (ii) 11.9 cm

Practise Now 12
(i) 0.772 rad   (ii) 3.23 cm
(iii)5.30 cm

Practise Now 13
4.71 cm

Practise Now 14
1. (i) 0.75 rad   (ii) 54 m^2
2. 22 cm

Practise Now 15
(ii) 15.8 m   (iii) 93.4 m^2

Practise Now 16
(i) 12.96 cm   (ii) 43.4 cm^2

Practise Now 17
(ii) $(9.6\pi - 11.5)$ cm   (iii) 52.5 cm^2

CHAPTER 11

Practise Now 1
1. $E, F, D; EF, FD, 11, ED; EFD$, SSS
2. $\triangle WXY \equiv \triangle WZY$

Practise Now 2
2. No

Practise Now 3
1. (ii) 25°
2. (i) $\triangle PQS \equiv \triangle RSQ$
   (ii) 7 cm, 140°

Practise Now 6
1. (a) Yes   (b) No
   (c) Yes   (d) Yes
2. (ii) $DE = 10.5$ cm, $BD = 3$ cm
   (iii) $\dfrac{AB}{BD} = \dfrac{AC}{CE}$

Practise Now 7
(a) Yes   (b) No

Practise Now 8

(a) Yes   (b) No

Practise Now 9

1. (iii) 2 cm
2. $BQ = 20$ cm, $AC = 28$ cm

Practise Now 11

1. 56 m   2. 12 m

CHAPTER 12

Practise Now 1

(a) 98 cm^2 (b) 12.5 m^2

Practise Now 2

3.6 m

Practise Now 3

(i) 7 cm^2 (ii) 14.6 cm^2

Practise Now 4

1. 75 cm^3 2. 0.5 m

Practise Now 5

1. 328 g   2. 3000 kg

Practise Now 6

(i) 14.9 cm   (ii) 1 : 3.30

CHAPTER 13

Practise Now 1

1. 21.35 cm   2. 5.05 cm

Practise Now 2

32.6 cm or 6.14 cm

Practise Now 3

1. (i) 23.2°   (ii) 11.4 cm
   (iii) 23.625 cm^2
2. (i) 3.9   (ii) 64.0°
   (iii) 7.10 cm^2

Practise Now 4

1. (i) 28°   (ii) 59°
   (iii) 26.3 cm   (iv) 369 cm^2
2. 32°

Practise Now 5

1. (i) 56°   (ii) 118°
2. $x = 110°, y = 55°$
3. 146°

Practise Now 6

(i) 25°   (ii) 50°
(iii) 105°

Practise Now 7

1. (i) 44°   (ii) 25°
   (iii) 69°
2. $x = 50°, y = 25°$
3. 20°

Practise Now 8

1. (i) $(159 - x)°$ (ii) $(149 - x)°$
   (iii) 64   (iv) 85°
2. 34°
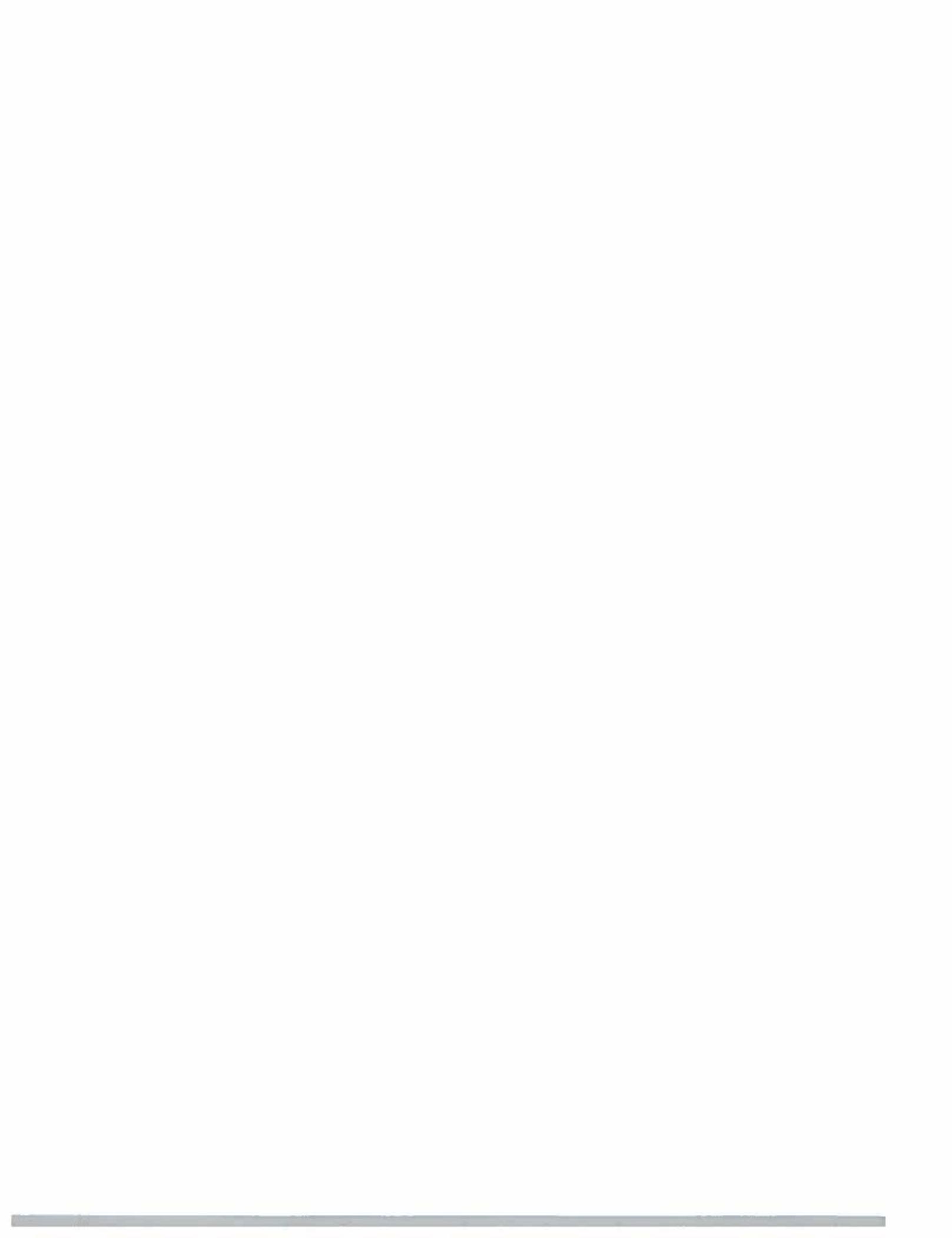
Practise Now 9

1. 24°
2. (i) 52°   (ii) 36°

Practise Now 10

48°

Practise Now 11

(ii) 7.14 cm

Answers

Exercise 1A

1. (a) 1 or $-3\frac{1}{2}$ (b) 2 or $-\frac{3}{4}$

   (c) 2 or -9 (d) 1 or -4

   (e) 1 or $-\frac{2}{3}$ (f) 2 or $-1\frac{2}{3}$

2. (a) 2 or -4 (b) $1\frac{1}{2}$ or $-2\frac{1}{2}$

   (c) $2\frac{3}{5}$ or -1 (d) $2\frac{7}{12}$ or $2\frac{1}{12}$

   (e) 0.32 or -6.32 (f) 3.90 or -0.90

   (g) 7.65 or 2.35 (h) 3.66 or -2.66

3. (a) $(x+6)^2 - 36$

   (b) $(x-3)^2 - 8$

   (c) $\left(x+\frac{3}{2}\right)^2 - \frac{17}{4}$

   (d) $\left(x+\frac{9}{2}\right)^2 - \frac{85}{4}$

   (e) $\left(x+\frac{1}{4}\right)^2 - \frac{1}{16}$

   (f) $\left(x-\frac{1}{9}\right)^2 - \frac{1}{81}$

   (g) $(x+0.1)^2 - 0.01$

   (h) $(x-0.7)^2 - 0.49$

4. (a) 1.45 or -3.45 (b) 1.61 or -18.61

   (c) 11.20 or 0.80 (d) 5.85 or -0.85

   (e) 1.61 or -1.86 (f) 0.81 or 0.05

   (g) 0.74 or -1.34 (h) 4.34 or 0.46

5. (a) 8.12 or -0.12 (b) 4.79 or 0.21

   (c) 8.22 or -1.22 (d) 7.80 or -1.80

6. $y = \dfrac{a \pm \sqrt{a^2 + 24}}{2}$

Exercise 1B

1. (a) -0.268 or -3.73 (b) 0.155 or -2.15

   (c) 3.19 or 0.314 (d) 3.36 or -1.69

   (e) 0.922 or -3.25 (f) 1.77 or 0.225

2. (a) 2.72 or -7.72 (b) 1.96 or -0.764

   (c) 1.07 or -0.699 (d) 1.60 or -1.10

   (e) 1.67 or -1.07 (f) 9.73 or 6.27

3. (a) 0.618 or -1.62 (b) 2.70 or -0.370

   (c) 3.73 or 0.268 (d) 4.54 or -1.54

   (e) 2.30 or -1.30 (f) 0.468 or -0.468

4. (a) 1 (b) -2.90 or 0.230

   (c) no real solution (d) no real solution

Exercise 1C

1. (i) $8, 1, -2, -1, 4, 13$

   (iii) 2.3 or 0.20

2. (i) $-5, 5, 9, 7, -1, -15$

   (iii) 0.90 or -2.55

3. (i) $4, 1, 0, 1, 4, 9$ (iii) -3

4. (ii) 0.80 or -2.10

5. 1.45 or -3.45

6. (ii) -1.5

7. 5

8. (b) 7.5

9. (b) (i) 6.4 (ii) 5.6 m

   (c) 6.4

Exercise 1D

1. (a) 1.77 or -2.27 (b) 0.818 or -4.48

   (c) 3.73 or 0.268 (d) 8.14 or 0.860

   (e) 5.54 or -0.541 (f) -2 or $-\frac{2}{3}$

2. (ii) -4 or 3 (iii) 3 and 4

3. (a) -2 or $\frac{3}{5}$ (b) 15.6 or 1.41

4. 10.6 or -0.141

5. (a) 2 or -4 (b) 2.70 or -3.70

   (c) 6.5 or 2 (d) 0.775 or -0.775

   (e) 6 or $-\frac{2}{3}$ (f) 6.43 or -2.43

   (g) 5.14 or 1.36 (h) 7.16 or 0.838

6. (i) $(56 - x)$ cm (iii) 41.67 or 14.33

   (iv) 44.1 cm

7. (ii) 0 or $\frac{1}{3}$ (iii) 4 cm

8. 150 cm

9. 5 cm, 20 cm

10. 10

11. 70

12. (i) $\dfrac{60}{x}$ (ii) $\dfrac{60}{x+2}$

   (iv) $\frac{1}{2}$ or $-1\frac{2}{3}$ (v) 6 minutes

13. (i) $\dfrac{350}{x}$ (ii) $\dfrac{350}{x+0.15}$

   (iv) $\$1.40$

14. (i) $\dfrac{2}{x} + \dfrac{8}{x+1}$ (iii) -4 or 3

   (iv) 2 hours 40 minutes

15. (ii) 88.08 or 11.92 (iii) 7.95 hours

16. (i) $\dfrac{1500}{x}$ (ii) $\dfrac{1500}{x+50}$

   (iv) 363.10 or -413.10

   (v) 3 minutes 38 seconds

17. (i) $\dfrac{2000}{x}$ (ii) $\dfrac{1000}{x+0.05}$

   (iv) 1.25 or -0.03 (v) US$1 $= S1.30$

18. (a) $-2\frac{6}{7}$ (b) $\frac{1}{2}$ or $-\frac{1}{3}$

   (c) 5.12 or -3.12 (d) 7.54 or 1.46

19. 15 km/h

Exercise 1E

2. (a) $(0, 2), x = 0$ (b) $(0, -6), x = 0$

   (c) $(3, 1), x = 3$ (d) $(-1, -3), x = -1$

   (e) $(-2, 3), x = -2$ (f) $(4, -1), x = 4$

3. (i) $x\left(x + \frac{3}{4}\right)$

5. (i) $(x+3)(x-2)$

7. (i) $(x-4)^2 - 11$

   (iii) $(4, -11)$

   (iv) $x = 4$

8. $\left(x + \frac{3}{2}\right)^2 - \frac{5}{4}, \left(-1\frac{1}{2}, -1\frac{1}{4}\right)$

9. (i) $h = -\frac{1}{2}, k = \frac{3}{4}$

10. $p = 5, q = 21; (5, 21)$

Review Exercise 1

1. (a) -0.683 or -7.32 (b) 0.405 or -7.41

   (c) 11.6 or -0.603 (d) 0.566 or -1.77

2. (a) -0.177 or -2.82 (b) 2.59 or -0.257

   (c) -1 or $1\frac{1}{4}$ (d) 1.85 or -0.180

3. (a) $3\frac{2}{5}$ or $2\frac{3}{5}$ (b) 7.46 or 0.536

   (c) 2.61 or -4.61 (d) 2.58 or -6.58

4. (i) 2.78 or 0.72 (ii) 3.78 or 1.72

5. (a) $7x^2 - 20x + 12 = 0$

   (b) $6x^2 + 7x + 2 = 0$

6. (a) 1.58 or -7.58 (b) 0.245 or -12.2

   (c) 0.171 or -1.17 (d) 5.68 or 1.32

   (e) 3.22 or -0.621 (f) 4.91 or -0.577

   (g) 19 or $-1\frac{1}{2}$ (h) 3.44 or -0.436

7. (i) $(x-3)(x-4)$

8. (i) $-\left(x-\dfrac{5}{2}\right)^2 + \dfrac{9}{4}$

9. 3 and 4

10. (i) $\dfrac{65000}{x-5} - \dfrac{65000}{x} = 20$

    (ii) \$1625

11. (i) $\dfrac{40}{x}$ h   (ii) $\dfrac{40}{x-30}$ h

    (iv) 101.17 or −71.17

    (v) 33.7 minutes

12. (i) $\dfrac{6000}{x}$ (ii) $\dfrac{6000}{x+10}$

    (iv) 200 or −210   (v) 16.2

13. (i) $(35 - 2x)$ m, $(22 - 2x)$ m

    (iii) 24.76 or 3.74

    (iv) 3.74 m

14. (a) (i) −2.2 or 27.2

    (c) (i) 216 m

        (ii) 6.5 m or 18.5 m

15. 20 minutes, 25 minutes

16. 17.4 km/h

Challenge Yourself

1. 15 or 24   **2.** $-\dfrac{b}{a}, \dfrac{c}{a}$

CHAPTER 2
Exercise 2A

1. (i) $a^2 + 5$ (ii) $a^2 + 2a + 6$

    (iii) $4a$ (iv) $a^4 + 5$

    (v) $a^6 + 5$ (vi) $a^6 + 2a^3 + 6$

2. (i) 1   (ii) 3

    (iii) 6   (iv) $\dfrac{1}{2}x^2(x-1)$

    (v) 5   (vi) 7

    (vii) $(x+1)(x+2)$ (viii) x

    (ix) $\dfrac{1}{2}x^2(x^2+1)$

3. (i) $3a^2 - 5a$ (ii) 1 or 4

    (iii) $a^4 - 4a^2 - 5a + 8$

4. $m = -2\dfrac{1}{4}, c = 7\dfrac{1}{4}; \dfrac{1}{2}, 16\dfrac{1}{4}$

5. $p = 5, q = 6; 106, 10$

Exercise 2B

1. $f^{-1}(x) = 4(x+3)$

2. $f^{-1} : x \mapsto x + 7; 10, 14, 2, 7\dfrac{1}{3}$

3. $g^{-1} : x \mapsto \dfrac{x-4}{3}; -\dfrac{1}{3}, -2\dfrac{2}{3}, -1\dfrac{1}{6}, -1\dfrac{7}{12}$

4. $h^{-1} : x \mapsto \dfrac{x+6}{5}; 2\dfrac{2}{5}, 3\dfrac{1}{5}, 1\dfrac{3}{25}, 3\dfrac{7}{10}$

5. $f^{-1}(x) = \dfrac{8-x}{3}; -\dfrac{1}{3}, 6\dfrac{2}{3}, 1\dfrac{5}{9}, 2\dfrac{35}{48}$

6. $3, 8, \dfrac{2}{3}, \dfrac{1}{3}$

7. (i) 632   (ii) 40

    (iii) 44   (iv) 0

    (v) 3613   (vi) −36

8. $a = 3, b = -4$

9. $p = 5, q = -3, 8, 14$

10. $f^{-1}(x) = \dfrac{2x}{5+4x}, x \neq -\dfrac{5}{4}; \dfrac{8}{21}, \dfrac{12}{19}$

11. $f^{-1}(x) = \dfrac{2x-1}{x-3}, x \neq 3; 4\dfrac{1}{2}, 3\dfrac{1}{4}$

12. $a = 4, b = -3$

    $f^{-1}(x) = \dfrac{x+3}{4}, 2\dfrac{1}{2}, -\dfrac{5}{8}$

13. $a = 1, b = 2$

    $f^{-1}(x) = x - 2$

14. $p = 1\dfrac{2}{3}, q = -6\dfrac{2}{3}$

    $f^{-1}(x) = \dfrac{3}{5}x + 4$

15. $7, 3\dfrac{1}{2}$

Review Exercise 2

1. (i) $9, -33$

    (ii) $f^{-1} : x \mapsto \dfrac{x+9}{6}$

    (iii) $2, \dfrac{1}{6}$

2. $p = 2, q = 3$

    (i) $25, -11$

    (ii) $f^{-1}(x) = \dfrac{1}{4}(x-9); \dfrac{1}{4}, -6\dfrac{1}{4}$

3. (i) $a^2 - 3a + 5$ (ii) $4a^2 - 6a + 5$

    (iii) $a^2 + 3a + 5$ (iv) $a^4 - 3a^2 + 5$

    (v) $a^4 + a^2 + 3$

4. (i) (a) 15   (b) −18

        (c) $64\dfrac{1}{2}$

    (ii) $1\dfrac{5}{21}$ (iii) $\dfrac{10}{21}$

    (iv) $f^{-1}(x) = \dfrac{x+7}{10}$

        $g^{-1}(x) = 12 - 4x$

    (v) 5

5. $p = -\dfrac{1}{12}, q = 4\dfrac{5}{12}$

    $g^{-1} : x \mapsto 53 - 12x$

6. (i) $a^4 - 8a^2 + 7a$ (ii) 1 or 6

    (iii) $3a^2 - 9a + 6$

7. (i) $20x^2 + 4x + 1$ (ii) $5x^2 + 12x + 8$

    (iii) $20x + 4$

Challenge Yourself

1. $(-3, 1, 3, 5\}, (2, 6, 7)$; No

CHAPTER 3
Exercise 3A

1. (a) <   (b) >

    (c) ≤   (d) ≥

    (e) <   (f) <

2. (a) $a < 1$ (b) $b \geq 7$

    (c) $c < -2$ (d) $d \geq 0$

    (e) $e \geq -1\dfrac{1}{2}$ (f) $f < -\dfrac{2}{5}$

    (g) $g \geq 4$ (h) $h > 4$

    (i) $j < 1\dfrac{1}{10}$ (j) $k \geq -\dfrac{5}{8}$

    (k) $m \leq 4$ (l) $n < -1$

3. $x \leq 4\dfrac{1}{2}$

    (i) 4   (ii) 4

4. $x < 3$

    (i) 2   (ii) Yes

5. (a) $x > -5$ (b) $x \leq -11\dfrac{1}{2}$

    (c) $y \geq 7$ (d) $y < 3\dfrac{8}{9}$

    (e) $p < 1\dfrac{1}{7}$ (f) $q \leq \dfrac{11}{13}$

6. (a) $a \geq 1\dfrac{1}{2}$ (b) $b > 5$

    (c) $c < 8$ (d) $d > -1$

    (e) $e < -10$ (f) $f \leq 2$

    (g) $g \geq -1\dfrac{8}{9}$ (h) $h > 108$

7. −10

8. (i) $x \geq -4$ (ii) 0

Exercise 3B

1. 19

2. $x = 24, y = 1; x = 12, y = 2; x = 8, y = 3;$
    $x = 6, y = 4$

3. (a) $-2 \leq x \leq 7$

    (b) $-1\dfrac{1}{3} < x < 5$

4. (a) $-2, -1, 0$

    (b) $3, 4, 5, 6, 7, 8$

5. (a) $x \geq 2$ (b) $3 < x \leq 5$
(c) $-9 < x < -3$ (d) $\frac{1}{2} < x \leq 6$

6. 10

7. $15\,625$

8. 18

9. 11

10. $29, 31, 37, 41, 43, 47$

11. 18

12. 5

13. (a) $3\frac{1}{2} \leq a \leq 4\frac{1}{3}$
(b) $1 < b < 4$
(c) $1\frac{1}{3} < c < 6$
(d) $0 \leq d < 3$

14. (a) $-4 \leq a \leq 4$
(b) $b \leq -6$
(c) $\frac{5}{6} \leq c < 1$
(d) $-2 \leq d < 8\frac{1}{3}$

15. (a) $8, 9, 10$
(b) 5
(c) $2, 3$
(d) $4, 5, 6, 7, 8$

16. (a) 12 (b) -5
(c) 35 (d) 0
(e) $0, 49$

17. (a) -10 (b) 5
(c) 2 (d) 2
(e) $1, 16$ (f) 40

18. (a) False   (b) True
(c) True

Exercise 3C

1. (i) (a) $4.55, 4.65$
(b) $4.55 \leq x < 4.65$
(ii) (a) $0.725, 0.735$
(b) $0.725 \leq x < 0.735$
(iii) (a) $675, 685$
(b) $675 \leq x < 685$
(iv) (a) $950, 1050$
(b) $950 \leq x < 1050$
(v) (a) $0.0345, 0.0355$
(b) $0.0345 \leq x < 0.0355$

(vi) (a) $71500, 72500$
(b) $71500 \leq x < 72500$

2. $620000.5, 619999.5$

3. $0.0035 \leq x < 0.0045$

4. (a) $775, 725$ (b) $35, 25$
(c) $758.5, 757.5$ (d) $1500.5, 1499.5$
(e) $497.5, 492.5$

5. 25.5 m, 22.5 m

6. 353

7. $24.9 \leq l < 25.1$

8. 125 cm

9. $137.5 \leq d < 227.5$

10. 14.0 cm³, 10.0 cm³

11. 280.21

12. 110

Review Exercise 3

1. (a) $a \leq 5$ (b) $b < \frac{2}{3}$
(c) $c \geq -2$ (d) $d > 6$
(e) $e \geq 3\frac{1}{2}$ (f) $f \leq 6\frac{2}{3}$
(g) $g < 1\frac{1}{3}$ (h) $h > 2\frac{3}{4}$

2. (a) $a < -24$ (b) $b < 7\frac{1}{5}$
(c) $c \leq -\frac{9}{20}$ (d) $d > -28\frac{3}{8}$
(e) $e \geq -3$ (f) $f \geq 2\frac{4}{9}$

3. (a) no solution
(b) $1\frac{2}{3} < b < 8$
(c) $-\frac{1}{2} \leq c < \frac{3}{8}$
(d) $-1 \leq d < 10$

4. (a) 14 (b) 13
(c) $14\frac{1}{2}$

5. (a) $9\frac{1}{2}$ (b) 10

6. (a) $10, 11$ (b) $-6, -5$

7. (a) $7, -3$ (b) $2\frac{1}{2}, -\frac{1}{2}$
(c) $12\frac{1}{2}, 0$

8. (a) -13 (b) $1\frac{3}{4}$
(c) 33 (d) 37

9. 410.1 cm²

10. 5

11. 18 years

12. 30

13. 1

14. (a) (i) $654, 652$
(ii) $315, 313$
(iii) $236.25, 206.25$
(iv) $569.25, 519.25$
(v) $3.24, 2.79$
(vi) $7.14, 6.87$
(b) (i) $652 \leq x < 654$
(ii) $313 \leq x < 315$
(iii) $206.25 \leq x < 236.25$
(iv) $519.25 \leq x < 569.25$
(v) $2.79 \leq x < 3.24$
(vi) $6.87 \leq x < 7.14$

15. 4.74 kg, 4.46 kg

16. (i) 290 kg   (ii) 350 kg

17. 9691.375 cm³

18. 19.5 cm

19. (a) 60.01 cm, 59.38 cm
(b) 286.56 cm², 280.59 cm²

Challenge Yourself

1. $12 \leq z \leq 40$

2. $x > 1\frac{2}{3}$

CHAPTER 4
Exercise 4A

1. (a) 2^{10} (b) $(-4)^{11}$
(c) x^{11} (d) $24y^9$

2. (a) 5^5 (b) $(-7)^7$
(c) $6x^4$ (d) $-3y^5$

3. (a) 9^8 (b) h^{10}
(c) 15^{14} (d) 18^3
(e) $8k^{18}$ (f) $81x^{24}y^8$

4. (a) 2^{13} (b) 3^{20}
(c) $\frac{m^5}{32}$ (d) $\frac{27}{n^6}$
(e) $\frac{p^{24}}{q^6}$ (f) $\frac{x^4}{y^8}$

5. (a) $h^{13}k^{10}$ (b) $-4m^{18}n^{12}$
(c) $22p^9q^{17}$ (d) h^4k^2
(e) $5m^6n^6$ (f) $5x^4y^3$

6. (a) a^{11} (b) b^{41}
(c) $-c^{28}$ (d) $\frac{9d^3}{8}$
(e) e^7 (f) $-8f^9$

7. (a) $8a^9b^9$ (b) $25c^8d^8$
(c) $64ef^3$ (d) $-\dfrac{2h}{g}$

8. (a) $\dfrac{2a^4}{b^5}$ (b) $\dfrac{c^9}{4d^8}$
(c) $3e^3f^3$ (d) $-\dfrac{8}{27g^3h^{12}}$

9. (a) $\dfrac{5x^6y^8}{2}$ (b) $\dfrac{32x^8y^2}{9}$
(c) $2y^5$ (d) $2x^2y^2$

10. $a=2, b=6$

Exercise 4B

1. (a) 1   (b) 1
(c) 4   (d) -8
(e) 1   (f) 7

2. (a) 16   (b) 7
(c) -63 (d) 211

3. (a) $\dfrac{1}{343}$ (b) $-\dfrac{1}{5}$
(c) $1\dfrac{7}{9}$ (d) $\dfrac{3}{5}$

4. (a) 1   (b) $\dfrac{24}{25}$
(c) $2\dfrac{2}{3}$ (d) 16

5. (a) 14   (b) 5
(c) $\dfrac{1}{2}$ (d) $\dfrac{2}{3}$

6. (a) $\sqrt{81}, 9$ (b) $\sqrt[3]{-27}, -3$
(c) $\dfrac{1}{\sqrt[4]{16}}, \dfrac{1}{2}$ (d) $(\sqrt{4})^3, 8$
(e) $\dfrac{1}{(\sqrt[3]{8})^5}, \dfrac{1}{32}$ (f) $(\sqrt[3]{-1000})^2, 100$

7. (a) $a^{\frac{1}{4}}$ (b) $b^{\frac{2}{3}}$
(c) $c^{\frac{4}{5}}$ (d) $d^{-\frac{1}{6}}$
(e) $e^{-\frac{1}{2}}$ (f) $f^{-\frac{5}{3}}$

8. (a) 3   (b) -7
(c) $2\dfrac{1}{2}$ (d) -2

9. (a) $15a^9$ (b) $-2\dfrac{2}{3}$
(c) $\dfrac{d^{10}}{c^6}$ (d) $\dfrac{2}{e^{24}}$
(e) $\dfrac{144}{h^6}$ (f) $\dfrac{k^{12}}{j^{12}}$
(g) m^3n (h) $61p^3$

10. (a) $a^{\frac{5}{6}}$ (b) $b^{\frac{1}{2}}$
(c) $c^{\frac{17}{10}}$ (d) $\dfrac{1}{d^{\frac{1}{6}}}$
(e) $\dfrac{e^{\frac{3}{2}}}{f^2}$ (f) $\dfrac{g}{h^{\frac{6}{5}}}$

11. (a) $\dfrac{a^{\frac{4}{3}}}{b^2}$ (b) $\dfrac{c^{10}}{d^{\frac{16}{5}}}$
(c) $\dfrac{e^{\frac{3}{11}}}{f^{12}}$ (d) $\dfrac{5g}{h}$
(e) $\dfrac{j^2k}{h^3}$ (f) $\dfrac{m^{\frac{56}{5}}n^{\frac{3}{5}}}{2}$

12. (a) $\dfrac{z^{13}}{x^{53}y^4}$ (b) $\dfrac{x^2z^{11}}{y^{10}}$
(c) $\dfrac{ac^{2n+1}}{b^3}$ (d) $\dfrac{a}{c(a+b)^3}$

Exercise 4C

1. (a) 8.53×10^4 (b) 5.27×10^7
(c) 2.3×10^{-4} (d) 9.4×10^{-8}

2. (a) 9600   (b) 400 000
(c) 0.000 28   (d) 0.000 001

3. (i) 3×10^2 MHz   (ii) 3×10^5 MHz

4. (i) 7×10^{-11} m   (ii) 7.4×10^{-11} m
(iii) 35 : 37

5. 3.94×10^5 %

6. (a) 1.67×10^2 (b) 1.41×10^{-9}
(c) 3.35×10^{-1} (d) 3.33×10^3
(e) 3.36×10^4 (f) -3.04×10^7
(g) 1.53×10^{-1} (h) 3.35×10^{-5}

7. (a) 2.46×10^{-12} (b) 6.94×10^7
(c) 1.1×10^4 (d) 2.1×10^2
(e) 5.41×10^{-2} (f) 1.99×10^5

8. (a) 3.15×10^9 (b) 4.5×10^4

9. 7.6×10^{-3}

10. (a) 1.6×10^{14} (b) 6.4×10^{-2}

11. 3.33×10^{-7}

12. (i) 3×10^8 m/s
(ii) 43 minutes 15 seconds

13. (i) 1.44×10^6 km   (ii) 400 days

14. (i) 1.25×10^8 (ii) 2.15
(iii) 1.69

Review Exercise 4

1. (a) a^7b^4 (b) $3a^2b^2$
(c) $-27a^9b^{15}$ (d) $\dfrac{a^2b}{2}$

2. (a) 5^{16} (b) 5^{-3}
(c) $5^{\frac{1}{5}}$

3. (a) 125   (b) $\dfrac{5}{36}$
(c) $2\dfrac{1}{9}$ (d) $1\dfrac{37}{125}$

4. (a) 3   (b) $\dfrac{3}{5}$
(c) 64   (d) $\dfrac{1}{64}$

5. (a) $\dfrac{x^4}{81}$ (b) $3x^3$

6. (a) y^3 (b) $\dfrac{4y^{10}}{x^6}$
(c) $\dfrac{y^{13}}{x^7}$ (d) $\dfrac{4}{x^2}$

7. (a) $2p^{14}$ (b) $\dfrac{p^{\frac{22}{5}}}{q^{\frac{5}{12}}}$
(c) $\dfrac{p^{\frac{20}{3}}}{q}$ (d) $\dfrac{3q^{\frac{32}{3}}}{p^{\frac{8}{3}}}$

8. (a) 6   (b) $\dfrac{1}{7}$
(c) 8

9. (a) $\dfrac{3}{4}$ (b) 0
(c) -1 (d) 16

10. (a) 3.26×10^4 (b) 3.1×10^{-10}
(c) 2.58×10^5 (d) 3.64×10^{-3}
(e) -2.47×10^6 (f) 7.42×10^{-4}

11. (a) 9.79×10^7 (b) 1.1×10^5
(c) 2.94×10^{-10} (d) 6.36×10^{-5}

12. (i) 2.20×10^{-8} m   (ii) 3.85×10^{-17} m²

13. (i) 1.496×10^{11} m   (ii) 499 s

14. (i) 2.4×10^8 m   (ii) 1920 s

15. (i) 2.99×10^{-23} (ii) 9.36×10^{24}

Challenge Yourself

1. 2^{34} 2. 7
3. 2

REVISION EXERCISE A1

1. (a) 8   (b) 125
(c) $\dfrac{4}{5}$ (d) 3
(e) $1\dfrac{1}{3}$

2. (a) a^7 (b) $\dfrac{1}{b^{3.5}}$
(c) $\dfrac{c^{25}}{d^{15}}$

3. (a) 0   (b) 11

4. (i) $p < -2.8$ (ii) -3

5. 5, 6, 7, 8

6. $f^{-1}(x) = \dfrac{1}{10}(x+3); 1\dfrac{1}{5}, 0, \dfrac{1}{4}$

7. (i) 2.76×10^{-8} m   (ii) 6.08×10^{-17} m²

8. 55 cm, 30 cm

9. 5.5 h, 6 h

10. (i) $y^2 + 16, y^2 - 16y + 73$

    (iii) 2 or 6

    (iv) 15 cm² or 13 cm²

11. (ii) 15 or 2.6   (iii) 2.6 cm

REVISION EXERCISE A2

1. (a) $8a^3b^6$ (b) c

    (c) $\dfrac{d^2}{4}$ (d) $\dfrac{1}{2}e^3$

2. 64

3. (a) -3 (b) ± 9

    (c) 4.5

4. (a) 6   (b) $\dfrac{1}{6}$

5. $-5 \leqslant q < 8$

6. (a) 11   (b) 10

    (c) $9\dfrac{3}{4}$

7. $f^{-1}(x) = \dfrac{2}{3}\left(\dfrac{1}{x}+1\right)$; $x = 0, \dfrac{7}{9}$

8. (i) $\dfrac{1}{2}(a+1)(5a+3)$ (iii) 1.11 or -2.71

    (iv) 5.32

9. 2 h, 4 h

10. (ii) 6.84 cm, 5.84 cm

11. 20 km/h

CHAPTER 5

Exercise 5A

1. (a) $5; 12.5\%$ (b) $120; 20\%$

    (c) $91\,520; \$3520$ (d) $4402; \$1278$

    (e) $24; \$4.14$ (f) $545; \$38.15$

2. (i) $800   (ii) $1080

3. $33\dfrac{1}{3}\%$

4. 25%

5. $384

6. $0.99

7. 12%

Exercise 5B

1. 20%

2. $39.60

3. (i) $700   (ii) $651

4. $288.90

5. $1300

---

6. (a) $108   (b) $494

    (c) $2790   (d) $6225

7. (i) $16\,250   (ii) $480\,000

8. (i) $800   (ii) No

9. $8.39

10. $12\,360

11. $1440

12. $18\,000

13. There is no difference.

14. $10.50

15. $100\,000

Exercise 5C

1. (a) $6720, \$18\,720 (b) 4 years, $720

    (c) $300, \$408   (d) $4\%, \$4200$

    (e) $3600, 5\%$ (f) $7\%, \$1989$

    (g) $6\%, 540$

2. $1100

3. $3600

4. $132

5. 10 years

6. $4\dfrac{1}{4}\%$

7. $53\,760

8. $2\dfrac{1}{2}$ years

9. $1298.56

10. (a) $94.50   (b) $257.34

    (c) $1244.03   (d) $149.84

    (e) $1556.25

11. $5829.57

12. $1019.50

13. $16

14. $6100

15. $20\,000

16. 2.5

17. (a) $21\,074.13   (b) $20\,991.14

18. 3.01%

19. $2264.09

20. $105.93

21. $9001.48

22. (i) Company B   (ii) $103.41

Exercise 5D

1. (a) (i) $90   (ii) 25%

    (b) (i) $150   (ii) $16\dfrac{2}{3}\%$

    (c) (i) $3000   (ii) 12%

2. (a) (i) $236   (ii) 18%

    (b) (i) $412.50   (ii) 17.9%

    (c) (i) $1960   (ii) 22.5%

3. (a) (i) $63   (ii) 7%

    (b) (i) $200   (ii) 15%

    (c) (i) $75   (ii) $16\dfrac{2}{3}\%$

4. (a) 18.2%   (b) $108

    (c) 5.13%

5. (i) $134.87   (ii) $3716.80

    (iii) $516.80

6. 1600

Exercise 5E

1. (a) 5215.50B   (b) US$46.51

2. (a) €380.80   (b) £803

3. (a) US$5000   (b) S$80

4. (a) S$1078.88   (b) S$281.60

5. (a) A$337   (b) P200 708

6. NZ$486.43

Review Exercise 5

1. $19\dfrac{11}{31}\%$

2. $2\dfrac{18}{41}\%$ loss

3. 5%

4. $3740

5. (a) $3

    (b) (i) $33   (ii) $322.50

6. $7.50

7. $1267.50

8. $6775

9. $1078

10. (a) $16\,969.85   (b) $16\,952.14

11. $45\,972

12. (a) $25.08   (b) $518.85

13. £3227

Challenge Yourself

1. (i) $35   (ii) $32.71

    (iii) $35

    (v) Yes, it makes a difference.

CHAPTER 6

Exercise 6A

1. (a) $-\dfrac{1}{2}$ (b) -10

   (c) $-\dfrac{4}{3}$ (d) -3

   (e) $\dfrac{11}{4}$ (f) 0

2. $0, \dfrac{1}{2}, -\dfrac{5}{6}, -\dfrac{1}{6}$ 3. $-2\dfrac{4}{5}$

4. $1\dfrac{5}{6}$ 5. -6 or 3

6. 9 7. 3

8. $1\dfrac{1}{2}$ or -1

9. (i) $-\dfrac{5}{2}, \dfrac{2}{5}, -\dfrac{5}{2}, \dfrac{2}{5}$

   (ii) They are equal.

Exercise 6B

1. (a) 8.06 units   (b) 8.54 units

   (c) 11.4 units   (d) 10.8 units

2. ± 7.07

3. (a) $\left(0, -8\dfrac{1}{3}\right)$ (b) $\left(2\dfrac{3}{11}, 0\right)$

4. $\left(0, -3\dfrac{21}{26}\right)$

5. (i) 32 units, 48 units^2

   (ii) 9.6 units

6. (i) 4.5 units^2 (ii) 5.83 units

   (iii) $(0, 4)$ (iv) 11 or -5

7. (i) 3 units, 4.12 units, 4.47 units

   (ii) 6 units^2

   (iii) 6 or -8

8. -2 or 1

9. (ii) 9 units^2

10. 2.57 units   11. 3.48 units

Exercise 6C

1. 3 2. 15

3. (a) $y = -x$ (b) $y = 2x + 1$

   (c) $y = \dfrac{1}{4}x + \dfrac{7}{2}$ (d) $y = \dfrac{9}{10}x + \dfrac{2}{5}$

   (e) $y = -x - 6$ (f) $y = \dfrac{2}{3}x - \dfrac{1}{3}$

   (g) $y = 0$ (h) $x = 0$

4. (a) $y = \dfrac{1}{3}x$ (b) $y = 3x - 2$

   (c) $y = -3x + 1$ (d) $y = -\dfrac{1}{2}x + \dfrac{19}{2}$

   (e) $y = 4$ (f) $y = ax + a$

5. $y = 2x$

6. (a) $0, 1; y = 1$

   (b) undefined; no y-intercept, $x = 1.5$

   (c) $1, -1; y = x - 1$

   (d) $-\dfrac{1}{2}, 1; y = -\dfrac{1}{2}x + 1$

7. (i) 4 units^2 (ii) $\dfrac{1}{2}$

   (iii) $y = x$

8. -4 or 3

9. (i) $y = -\dfrac{2}{3}x + 2$ (ii) $-\dfrac{2}{3}$

   (iii) $(3, 0)$

10. (i) $y = -\dfrac{2}{3}x + 3$ (ii) 0

11. $y = \dfrac{5}{2}x - \dfrac{19}{2}$

12. (i) $y = 3x - 8$ (ii) $(4, 4)$

13. (i) $(-6, 0)$ (ii) $\left(2, -6\dfrac{2}{3}\right)$

   (iii) $y = -\dfrac{5}{6}x + \dfrac{3}{2}$ (iv) $y = -1$

14. (a) (i) 3 (ii) $y = 3x + 3$

   (b) $(6, 3)$

15. $m = 0; n = 0$

16. (i) $A(8, 0), B(0, 6)$ (ii) 10 units

   (iii) $\left(3\dfrac{3}{7}, 3\dfrac{3}{7}\right)$ (iv) $y = x$

17. (i) $\left(-8\dfrac{1}{2}, 0\right)$ (ii) $(2, 7)$

   (iii) 7.28 units   (iv) 3.85 units

Exercise 6D

1. (a) 8 (b) $-\dfrac{1}{2}$

2. (a) $-\dfrac{1}{3}$ (b) $\dfrac{4}{3}$

3. (a) ± 1 (b) $\pm \dfrac{1}{3}$

4. $\dfrac{26}{3}$

5. (a) $3y = -7x + 1$ (b) $6y = -x - 37$

   (c) $y + 3x = 20$ (d) $2y = x - 8$

6. 1

7. Yes

8. (i) $(4, 0)$ or $(8, 0)$ (ii) $(0, 24)$

9. (i) $(4, 2)$ (ii) No

10. (a) -9 (ii) $\dfrac{4}{7}$

11. (i) 1 (ii) 17

12. Yes

13. Yes

14. (a) $1\dfrac{2}{3}$ (b) -55

   (c) $1 \pm 2\sqrt{19}$

15. (a) $y + x = 8$

   (b) $5y = x + 16, (4, 4)$

16. (i) $-\dfrac{4}{3}$ (iii) 50 units^2

Review Exercise 6

1. (i) $y = 2x - 3$ (ii) 5

2. (i) -3 (ii) 14

3. $y = -8x$

4. (i) $\dfrac{3}{4}$ (ii) 10 units

   (iii) $y = -\dfrac{3}{4}x - 6$

5. (i) $y = -\dfrac{3}{4}x + 6$ (ii) $\left(2\dfrac{6}{7}, 3\dfrac{6}{7}\right)$

   (iii) $y = 3\dfrac{6}{7}$ (iv) $x = 2\dfrac{6}{7}$

6. (i) $\dfrac{2}{3}$ (ii) $y = \dfrac{2}{3}x + \dfrac{5}{3}$

   (iii) 10 units^2 (iv) 4.12 units

7. (i) $(-4, 0)$ (ii) $(2, 4)$

   (iii) 16 units^2

8. (i) $y = 3x + 3$ (ii) 10

   (iii) -12 (iv) $(2, 1)$

9. (i) $P(5, 0), Q(0, 12)$

   (ii) 13 units

   (iii) $y = -\dfrac{12}{5}x - 2$

   (iv) $(-5, 0)$

10. $t = 15 - h$

11. (i) $y = -4$ (ii) 7.2 units^2

   (iii) 41.76 (iv) 2.23 units

12. (i) $y = -0.1x + 10$

13. $33y = 55x - 41$

14. $y = 2x - 1$

CHAPTER 7

Exercise 7A

1. (a) $-27, -1, 0, 8$

   (c) (i) 3.5 (ii) 2.3

2. (a) 2.75

   (c) (i) -0.5 (ii) 1.75

3. (a) $8, 4, 1.3, 0.8$

   (c) (i) 1.1 (ii) 2.65

4. (a) $a = 1.1, b = 0.4$

   (c) (i) 1.3 (ii) 1.5

5. (i) 1.5 (ii) 2.4

6. (a) (i) 9.5   (ii) 15
    (iii) 27
  (b) (i) 0.8   (ii) 3.5
    (iii) 4.15
7. (i) −1.8   (ii) 3.35
8. (a) $h = 2, k = 5.5$
  (c) (i) −0.3   (ii) 0.9
9. (i) 0.65   (ii) 2.45
10. (i) 5.5   (ii) 1, 2.75
11. (a) −1, −0.6, 1.6
  (b) (i) −1.5, −0.35, 1.9
    (ii) −1.5, −0.35, 1.9
12. (a) 1.7   (c) no solution
13. (a) −4.2
  (b) (i) 1.5 or 4
    (ii) 1.6 or 5.8
    (iii) 0.6 or 4.3

Exercise 7B

1. (a) 0.5, 8, 16, 32
  (c) (i) 12   (ii) −0.65
2. (a) 3, 4.2, 8.5
  (c) (i) 4.9, 14.8   (ii) −0.25, 1.30
3. (i) −7.4   (ii) 1.1
4. (a) $a = 2.5, b = 7.7$
  (c) (i) 2.6, 8.5   (ii) 1.725, 2.45
5. (a) 1.6
  (b) (i) (−1.15, 0.3)
    (ii) −1.15
6. 3
7. (a) 5, 8, 5   (c) 4
  (d) (ii) $h = 1, k = 9$
8. (b) −14   (c) 10
9. (a) (ii) −0.85
10. (c) (1, 2)

Exercise 7C

1. (ii) 1153
2. (i) 15 km   (iii) 65 minutes
3. (i) 1 h   (ii) 30 km/h
  (iii) 34.3 km/h
4. (i) 5 m/s²   (ii) 7.5 m/s
5. (ii) 7 m/s

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6. (ii) 3.54 m/s   (iii) $t = 25.5$
7. (b) (i) 2.3 minutes
    (ii) 0.44 km/minute
    (iii) 2.75 minutes
8. (b) (i) 1 h 11 minutes
    (ii) 1 h and 1 h 22 minutes
9. (b) (i) 1011   (ii) 8.3 km
11. (i) 9   (ii) 30 m/s
12. (i) 1.5 m/s²   (ii) 100 s
13. (a) $a = 4, b = 10$
  (c) (i) 1.7, 5.3   (ii) 3.5 s
    (iii) −3   (iv) $2.4 < t < 4.6$
14. (ii) 9.5 m/s, 34 m/s
  (iii) 2.5 m/s², 5 m/s²
15. (a) $h = 8, k = 10$
  (c) (i) 2.8 s   (ii) 10 km/h²
    (iii) $1.65 < t < 4$
  (d) 0.4
16. $6\frac{2}{3}$ m/s
17. (a) 55 cents   (b) Company B
18. (a) 30 minutes   (b) 20 km/h²
  (d) 1648
20. (b) (i) 3 cm/s², −1 cm/s²
    (ii) $3.3 < t < 6.7$
  (c) $a = −0.5, b = 5, c = 0$
22. (b) 12 m/s

Review Exercise 7

1. (a) −12, −8, −12, −8
  (c) (i) −9.5   (ii) 3.1
2. (a) 3, 0, 15
  (c) (i) −2.75   (ii) 2.4
    (iii) −2, 0, 2
3. (a) (i) 3.85   (ii) −0.375, −1.375
  (b) (−0.7, 3.8)
4. (i) 0.625   (ii) (0.55, −0.2)
5. (i) $y = 1.45, x = 1.75$
  (ii) $0.9 < x < 3.275$
  (iii) 0.67   (iv) 3.575
6. (ii) 27.7 km/h
7. (a) (i) $1 \leq t \leq 2$ (ii) $13\frac{1}{3}$ km/h
    (iii) 40 km/h   (iv) 17.1 km/h

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8. (i) 2 m/s²
  (ii) 3 m/s²
  (iii) 26 m
  (iv) $4\frac{1}{3}$ m/s
9. (i) 32.5 km
  (ii) 78 km/h
10. (i) 320 000
  (ii) 16, 33, 40; 3200, 512
  (iii) 350 units
11. (a) 5°C/minute   (c) 2°C/minute

REVISION EXERCISE B1

1. 15%
2. 25%
3. $3y = 5x − 4$
4. (i) 10 units²   (ii) (0, 6)
  (iii) (8, 6)
5. (i) $A(8, 0), B(0, 6)$ (ii) 39 units²
  (iii) 1.2   (iv) 7.8 units
6. (i) $13\frac{1}{3}$ km/h   (ii) 1030, 20 km
  (iii) 1 h   (iv) 10 km
7. (i) 16   (ii) 24
8. (a) $h = 3.5, k = 2.7$
  (c) (i) 1.5   (ii) 0.9
  (d) 0.5   (e) 1.4

REVISION EXERCISE B2

1. (i) $425   (ii) 20% gain
2. $1836
3. $5x + 7y + 25 = 0$
4. (i) $P(−4, 0), Q(0, 2)$
  (ii) $\left(−1\frac{1}{3}, 1\frac{1}{3}\right)$
  (iii) $2\frac{2}{3}$ units²
5. (i) (3, 4)   (ii) (0, −3)
  (iii) 21 units²   (iv) 3.61 units
  (v) $3y = 2x − 9$
6. (b) (i) 1233, 77 km
    (ii) 60 km
7. (i) 1 minute 35 seconds
8. (i) 6.5   (iii) 2.15 or 5.35
  (iv) −2.5
  (v) $y = 14.1, x = 3.75$
  (vi) 3.75 cm by 3.75 cm, square

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CHAPTER 8

Exercise 8A

1. (a) $\sin 70°$ (b) $\sin 4°$
   (c) $\sin 82°$ (d) $-\cos 81°$
   (e) $-\cos 73°$ (f) $-\cos 5°$
2. (a) 0.530   (b) 0.819
3. (a) 3.535   (b) 0.707
   (c) -2.121
4. (a) $\dfrac{4}{5}$ (b) $-\dfrac{3}{5}$
   (c) $1\dfrac{1}{3}$
5. (a) 9
   (b) (i) $\dfrac{9}{41}$ (ii) $-\dfrac{40}{41}$
   (iii) $\dfrac{9}{40}$
6. (a) $31.3°$ (b) $48.6°$
   (c) $61.0°$ (d) $20.2°$
7. (a) $148.7°$ (b) $131.4°$
   (c) $119.0°$ (d) $159.8°$
8. (a) $47.9°$ (b) $40.9°$
   (c) $60°$ (d) $9.9°$
9. (a) $48.9°$ or $131.1°$ (b) $72.2°$ or $107.8°$
   (c) $28.1°$ or $151.9°$ (d) $103.8°$
   (e) $141.5°$ (f) $58.4°$
10. (a) $\dfrac{8}{17}$ (b) $-\dfrac{15}{17}$
    (c) $\dfrac{8}{15}$
11. (a) $\dfrac{3}{5}$ (b) $-\dfrac{4}{5}$
    (c) $\dfrac{3}{8}$
12. (a) $\dfrac{5}{13}$ (b) $-\dfrac{12}{13}$
    (c) $\dfrac{5}{27}$
13. 27 or 153
14. (a) $18.0°$ or $142.0°$ (b) $134.1°$

Exercise 8B

1. (a) 34.2 cm^2 (b) 29.4 cm^2
   (c) 41.5 m^2 (d) 31.7 m^2
   (e) 27.4 cm^2 (f) 70.7 m^2
2. 117 cm^2 3. 9040 cm^2
4. (i) 633 cm^2 (ii) 29.5 cm
5. 23 000 m^2
6. (i) $27.5°$ (ii) 10.5 cm
   (iii) 6.22 cm^2

7. 3.23
8. (i) $30°$ (ii) 10 cm
   (iii) 346 cm^2
9. 116 cm^2 10. 22.7 cm
11. $10.2°$, $169.8°$
12. (i) 10.8 cm^2 (ii) $104.5°$

Exercise 8C

1. (a) $\angle C = 62°, b = 10.7$ cm, $c = 9.76$ cm
   (b) $\angle F = 79.3°, d = 4.43$ m, $f = 6.96$ m
   (c) $\angle H = 38°, g = 11.5$ mm, $i = 5.29$ mm
2. 11.8 cm   3. 15.6 cm
4. (a) $\angle B = 26.9°, \angle C = 61.1°,$
   $c = 13.4$ cm
   (b) $\angle A = 55.6°, \angle C = 26.4°, c = 7.81$ m
   (c) $\angle B = 31.7°, \angle A = 113.3°,$
   $a = 15.2$ cm
5. (i) $39.2°$ (ii) $39.8°$
   (iii) 13.6 cm
6. (i) $39.6°$ (ii) $49.4°$
   (iii) 8.80 cm
7. (i) 12.5 m   (ii) $41.7°$
   (iii) 13.4 m
8. (i) 6.92 m   (ii) 40.1 m^2
9. (i) 9.40 cm   (ii) 5 cm
   (iii) 4.92 cm
10. (i) No   (ii) 9.47 cm
    (iii) 5.62 cm
11. 14.7 km^2
12. (i) 2.64 cm   (ii) $55.8°$
    (iii) $49.4°$
13. (a) No   (b) Yes
    (c) No   (d) Yes
    (e) Yes   (f) No
14. (a) Not possible   (b) Possible
    (c) Not possible   (d) Possible
15. $\angle ABC = 68.9°, \angle ACB = 53.1°,$
    $AB = 13.2$ cm or $\angle ABC = 111.1°,$
    $\angle ACB = 10.9°, AB = 3.12$ cm
16. (i) 9.18 cm   (ii) 0.734 km
    (iii) 0.321 km^2
17. (i) $49.9°$ or $130.1°$ (ii) $6\dfrac{2}{3}$ cm

Exercise 8D

1. 6.24 cm
2. 4.57 cm
3. 9.45 cm
4. $\angle X = 48.2°, \angle Y = 58.4°, \angle Z = 73.4°$
5. $34.5°$
6. $88.5°$
7. (i) 9 m   (ii) 15.1 m
8. (i) 6.12 m   (ii) 7 m
9. (i) 3.46 cm   (ii) 5.29 cm
   (iii) $90°$
10. (i) $22.6°$ (ii) 4.84 m
    (iii) 6.86 m
11. (i) 7.94   (ii) $81.0°$
12. (i) $73.4°$ (ii) 1.92 cm
    (iii) 2.18 cm
13. (i) 20 km   (ii) $89.6°$
    (iii) 225 km^2
14. $93.8°$, 9.29 cm
15. (i) No   (ii) $\dfrac{131}{144}$
    (iii) 6.78
16. (i) $-\dfrac{1}{20}$ (ii) 6.57 cm
17. 7.09 cm

Review Exercise 8

1. (a) 25 cm
   (b) (i) $\dfrac{7}{24}$ (ii) $-\dfrac{24}{25}$
   (iii) $\dfrac{24}{25}$
2. (a) 37 cm
   (b) (i) $\dfrac{12}{37}$ (ii) $-\dfrac{35}{37}$
   (iii) $3\dfrac{109}{420}$
3. (i) $-\dfrac{1}{\sqrt{5}}$ or -0.4472
   (ii) $\dfrac{2}{\sqrt{5}}$ or 0.8944
   (iii) $\dfrac{3}{4}$
4. (i) 13 units, 14.4 units
   (ii) $\dfrac{12}{13}, -\dfrac{5}{13}$
   (iii) 18 units^2
   (iv) 2.50 units
5. (a) $24.8°$ or $155.2°$ (b) $21.3°$
   (c) $26.7°$ (d) $108.5°$

6. (b) (i) $-\dfrac{35}{37}, \dfrac{12}{37}$ **(ii)** 5.7°

7. (i) $\dfrac{21}{44}$ **(ii)** 20.1°

    (iii) 13.2 cm²   **(iv)** 5.04 cm

8. (i) 2.96 m   **(ii)** 1.19 m

    (iii) 6.00 m²   **(iv)** 2.52 m

    (v) 56.6°

9. (i) 377 m   **(ii)** 200 m

    (iii) 232 m   **(iv)** 85 900 m²

10. (i) 157 cm   **(ii)** 45.7°

    (iii) 2.33 m²

11. (i) 1810 m²   **(ii)** 66.8 m

12. (i) 43 100 m²   **(ii)** 277 m

    (iii) 193 m

Challenge Yourself

(a) 12.9 cm, 20.9 cm   **(b)** 64.5 cm²

CHAPTER 9
Exercise 9A

1. 119 m   **2.** 52.1 m

3. 27.6°   **4.** 63.1 m

5. 36.3 m   **6.** 68.7°

7. 9.74 m   **8.** 40.3 m

9. 35.0°   **10.** 210

11. 63.5 m   **12.** 10.3 m

13. 28.0

14. (i) 81.3 m   **(ii)** 93.5 m

Exercise 9B

1. (a) 033°   **(b)** 118°

    (c) 226°   **(d)** 321°

2. (a) 055°   **(b)** 165°

    (c) 317°   **(d)** 235°

    (e) 345°   **(f)** 137°

3. (a) 036°   **(b)** 216°

    (c) 073°   **(d)** 253°

    (e) 296°   **(f)** 116°

4. (a) 34.6 km   **(b)** 35.5 km

5. 40.2 km

6. (a) 310°   **(b)** 270°

    (c) 220°

---

7. (a) 315°   **(b)** 003° or 267°

    (c) 238° or 032°

8. 028° or 216°

9. (a) 218 m   **(b)** 180 m

    (c) 436 m

10. 7.97 km

11. (i) 696 m   **(ii)** 038.9°

12. (i) 126.6°   **(ii)** 16.9 m

    (iii) 305.8°

13. (i) 553 m   **(ii)** 40.5°

    (iii) 184.5°   **(iv)** 331 m

14. 31.2 km, 080.2°

15. (a) (i) 65.9 km   **(ii)** 139.0°

    (b) 1009

Exercise 9C

1. (i) 9.22 cm   **(ii)** 6 cm

    (iii) 33.1°   **(iv)** 45°

2. (i) 13 cm   **(ii)** 53.1°

    (iii) 67.4°

3. (i) 33.7°   **(ii)** 10 cm

    (iii) 21.8°

4. (i) 56.3°   **(ii)** 5.55 m

    (iii) 10.6 m   **(iv)** 52.0°

5. 38.6 m

6. (i) 14.1 cm   **(ii)** 28.7 cm

    (iii) 63.8°

7. (i) 113 cm   **(ii)** 15.7°

    (iii) 37.1°

8. (i) 22.4°   **(ii)** 44.8°

9. (i) 26.6°   **(ii)** 57.5°

10. (i) 053.1°   **(ii)** 61.2 m

    (iii) 4.7°

11. (a) (i) 227°   **(ii)** 120°

    (iii) 330°

    (b) 16.6 m

12. (a) (i) 053.1°   **(ii)** 326.3°

    (b) 12.1 m, 7.7°

13. (a) (i) 380 m   **(ii)** 33 400 m²

    (iii) 176 m

    (b) 7.8°

14. (i) 53.1°   **(ii)** 57.0°

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15. (i) 72.0°   **(ii)** 55.5 cm²

16. 39.3 m, 46.8 m

17. (i) 023.5°   **(ii)** 38.6°

Review Exercise 9

1. 40.2 m   **2.** 124 m

3. 207 m

4. (i) 14.8 cm   **(ii)** 46.6°

    (iii) 22.5 cm

5. (i) 17.1 m   **(ii)** 16.8 m

    (iii) 35.5°

6. (i) 24 m   **(ii)** 45.2°

    (iii) 24.7 m

7. 14.8 m, 18.1°

8. (a) (i) 269 m   **(ii)** 66.5°

    (iii) 011.2°   **(iv)** 56 700 m²

    (b) 8.8°

9. (a) (i) 100°   **(ii)** 126°

    (iii) 378 m   **(iv)** 370 m

    (b) 12.2°

10. (i) 9.02 km   **(ii)** 098°

    (iii) 5.76 km   **(iv)** 2.3°

    (v) 4.22 km

Challenge Yourself

1. $\sin\alpha\cos\beta$

2. $\dfrac{h\tan\alpha}{\cos\beta}$

CHAPTER 10
Exercise 10A

1. (a) 28.0 mm   **(b)** 31.5 m

    (c) 70.0 cm

2. (a) 44.0 cm²   **(b)** 0.495 m2²

3. (a) (i) 254 cm²   **(ii)** 113 cm

    (b) (i) 42.1 cm²   **(ii)** 44.0 cm

    (c) (i) 1230 cm²   **(ii)** 176 cm

    (d) (i) 61.5 cm²   **(ii)** 34.0 cm

4. (a) 672 cm²   **(b)** 19.3 cm²

    (c) 462 cm²   **(d)** 2328 cm²

5. (a) 28π m   **(b)** 52π m²

6. (a) 26 m²   **(b)** 30 m

7. (a) 11.4 cm   **(b)** 32.7 cm

    (c) 63.5 cm   **(d)** 53.7 cm

8. (a) (i) 11.9 cm (ii) 62.6 cm

(b) (i) 31.3 cm (ii) 101 cm

(c) (i) 44.5 cm (ii) 101 cm

9. (a) 16.0 cm (b) 28.0 cm

10. (a) 49° (b) 80°

(c) 263° (d) 346°

11. 1.18 m

12. 191.0°

13. (a) 17.0 cm (b) 37.0 cm

14. $\left(18 + \dfrac{25}{3}\pi\right)$ cm

15. (i) 105° (ii) 12.8 cm

16. (i) 61.8° (ii) 30.5 cm

17. (i) 67.7 cm (ii) 198 cm

18. (ii) 48.2 cm

19. (i) 27.3° (ii) 54.5°

(iii) 32.4 cm

20. 44.4 cm

21. 43.0 cm

Exercise 10B

1. (a) 8.80 cm, 30.8 cm², 22.8 cm

(b) 108.0°, 66 mm, 1150 mm²

(c) 28.0 mm, 132 mm, 188 mm

(d) 84.0 cm, 9240 cm², 388 cm

(e) 225.1°, 385 m², 83 m

(f) 15.3 cm, 20.1 cm, 50.8 cm

2. (a) (i) 17.7 cm (ii) 12.8 cm²

(b) (i) 8.22 cm (ii) 2.14 cm²

(c) (i) 26.7 cm (ii) 44.0 cm²

3. (a) 14.7 cm, 103 cm²

(b) 24.2 cm, 169 cm²

(c) 30.8 cm, 216 cm²

(d) 52.8 cm, 370 cm²

4. (a) 385 cm², 22.0 cm

(b) 898 cm², 51.3 cm

(c) 1150 cm², 66.0 cm

(d) 2120 cm², 121 cm

5. (a) 9.33 cm (b) 12.0 cm

6. (a) 60.3° (b) 165.8°

(c) 303.5° (d) 26.7°

7. (i) 43.6 cm, 118 cm²

(ii) 33.2 cm, 40.8 cm²

(iii) 263 cm, 1640 cm²

8. (i) 100° (ii) 42.0 cm

9. 84 cm²

10. (ii) 32.5 cm² **11.** 1.47p^2

12. (i) 132° (ii) 13.0 cm

(iii) 92.7 cm (iv) 200 cm²

13. (i) 8.49 cm (ii) 21.4 cm

(iii) 20.5 cm²

Exercise 10C

1. (a) 150° (b) 25.7°

(c) 183.3° (d) 146.7°

2. (a) 0.653 rad (b) 1.38 rad

(c) 2.48 rad (d) 5.38 rad

3. (a) $\dfrac{\pi}{12}$ rad (b) $\dfrac{\pi}{10}$ rad

(c) $\dfrac{5\pi}{12}$ rad (d) $\dfrac{5\pi}{4}$ rad

4. (a) 0.717 (b) 0.856

(c) 14.1 (d) 0.383

(e) 0.156 (f) 3.08

5. (a) 0.833 (b) 1.40

(c) 0.448 (d) 0.148

(e) 0.694 (f) 0.898

6. (i) 11.7 cm (ii) 14.6 cm

7. (i) 6.79 m (ii) 14.4 m

8. (i) 9.36 cm (ii) 14.0 cm

9. (i) 0.159 rad (ii) 11.2 m

(iii) 3.34 m

10. 13.8 m

Exercise 10D

1. (a) 9.6 cm (b) 3.5 cm

(c) 43.75 m (d) 9 mm

2. (a) 70.4 cm² (b) 66.47 cm²

(c) 108.9 m² (d) 2650.8 mm²

3. (i) $1\dfrac{1}{8}$ rad (ii) 144 cm²

4. (a) 5 cm, 10 cm² (b) 1.5 rad, 27 cm²

(c) 12 m, 57.6 m² (d) 10 m, 12 m

(e) 2 rad, 16 mm (f) 9 mm, $\dfrac{2}{3}$ rad

5. (i) 81.2 cm (ii) 243 cm²

6. 32 cm **7.** 117 m

8. (i) 22.5 cm² (ii) $(2\pi - 1.8)$ rad

(iii) $(10\pi - 9)$ cm

9. (i) $\dfrac{1}{2}r^2\theta = 8, 2r + r\theta = 18$

(ii) $r = 8, \theta = \dfrac{1}{4}$

10. (i) 15.6 cm (ii) 100 cm²

11. 292 cm²

12. (i) $\dfrac{\pi}{3}$ (ii) 12.6 cm

(iii) 43.5 cm²

13. (i) 18 cm (ii) 72.1 cm²

14. (i) 6.4 cm (ii) 19.7 cm²

15. (i) 6 (ii) 3.80 cm²

16. (i) 800 cm² (ii) 774 cm²

17. (i) 4 rad (ii) 10.9 m

(iii) 85.6 m²

18. (i) 4.80 (ii) 73.0 cm²

19. (i) 2.09 rad (ii) 22.1 cm²

20. 76.8 cm

21. (i) 1.32 (ii) 6.30 cm²

22. (i) 1.04 (ii) 44.7 cm²

23. (i) 0.284 rad (ii) 2.83 cm²

24. (i) 14.4 cm (ii) 19.3 cm²

(iii) 13.6 cm (iv) 35.8 cm²

25. (i) $\dfrac{3}{14}\pi$ (ii) 164 cm²

Review Exercise 10

1. (i) 50.3 cm (ii) 302 cm²

2. (i) 60.8 cm² (ii) $(2\pi - 1.9)$ rad

(iii) $(16\pi - 15.2)$ cm

3. (i) 0.453 (ii) 3.80 cm²

4. (i) 1.55 rad (ii) 27.6 cm²

5. 22%

6. (ii) 2.91×10^{-3}

7. (i) 23.9 cm (ii) 77.7 cm²

(iii) 98.6 cm² (iv) 17.1 cm

8. (i) 4.73 cm, 12.5 cm

(ii) 54.1 cm²

9. 0.433r^2

10. $50(\pi - 1)$ cm²

11. (a) (i) 17.6 cm (ii) 141 cm²

(iii) 107 cm²

(b) 80.0 cm²

Challenge Yourself

1. (a) $(7\pi + 14)$ cm   (b) $(7\pi + 14)$ cm
   (c) $(7\pi + 14)$ cm   (d) $(7\pi + 14)$ cm
2. (i) 3   (ii) 0.644 rad
   (iii) 6.99 cm^2
3. (ii) 22.1 cm^2
4. (i) 16   (ii) 175 cm^2

REVISION EXERCISE C1

1. 0.352, 2.79
2. (i) 1.9 rad   (ii) 60.8 cm^2
3. (i) 13.2 cm   (ii) 59.7 cm^2
   (iii) 8.57 cm^2
4. 35.3
5. (i) 5.14 cm   (ii) 69.1°
   (iii) 9.89 cm
6. (i) 97.2°   (ii) 17.9 cm^2
   (iii) 64.6 cm^2 (iv) 101.4°
   (v) 10.8 cm
7. (i) 005.4°   (ii) 078.6°
   (iii) 11.9 km
8. (i) 60°   (ii) 10.4 cm
   (iii) 46.1°
9. 56.3 m

REVISION EXERCISE C2

1. 2.09
2. (a) 0.643   (b) −0.966
3. 17.4 cm^2
4. $x = 7.37, y = 6.52$
5. (i) 9.77 cm   (ii) 13.1 cm
   (iii) 59.2°   (iv) 19.2°
6. (i) 7.81 cm   (ii) 33.7°
   (iii) 42.6°   (iv) 12.0 cm
   (v) 46.8 cm^2
7. (a) (i) 072°   (ii) 108°
   (iii) 036°
   (b) 1.101×10^6 m^2
8. (a) (i) 7.21 cm   (ii) 64.3°
   (b) 76.5°
9. (i) 3.56 m   (ii) 10.6 m
   (iii) 70.4°   (iv) 30.7°

CHAPTER 11

Exercise 11A

1. (a) (ii) and (vii)   (b) (iii) and (v)
   (c) (i) and (ix)   (d) (vi) and (viii)
2. (a) P, Q, R; PQ, QR, PR, 6; PQR, SSS
   (b) Z, Y, X; $ZY, Z\hat{Y}X, YX$, 5; ZYX, SAS
   (c) W, V, U; $W\hat{V}U, W\hat{U}V$, 70, VU, 7; NML, AAS
   (d) U, T, S; $U\hat{T}S$, 90, US, TS, 5; $\triangle IHG$, RHS
3. (a) No   (b) No
   (c) No   (d) No
4. (a) $\triangle ABD \equiv \triangle CBD$
   (b) $\triangle ABD \equiv \triangle CDB$
   (c) $\triangle ABC \equiv \triangle EDC$
   (d) $\triangle ABC \equiv \triangle CDA$
   (e) $\triangle ADE \equiv \triangle CDB$
   (f) $\triangle BCD \equiv \triangle EFD$
   (g) $\triangle ABD \equiv \triangle CBD$
   (h) $\triangle ABC \equiv \triangle CDA$
5. (ii) 4 cm   (iii) 80°
   (iv) RS is parallel to UV.
6. (i) $\triangle JIH$ (ii) 80°
7. (a) $\triangle ABC \equiv \triangle CDA$
   (b) $\triangle EFG \equiv \triangle GHE$
   (c) $\triangle IJK \equiv \triangle KLI$
   (d) $\triangle MNO \equiv \triangle OPM$
   (e) $\triangle QRS \equiv \triangle STQ$
   (f) $\triangle UVW \equiv \triangle WXU$

Exercise 11B

1. (a) (i) and (iii), (v) and (vii)
   (b) (ii) and (vi)
   (c) (iv) and (viii)
2. (a) 60; S, T, U; $T\hat{S}U, S\hat{T}U$, 60; STU, 3, $\angle$s
   (b) N, M, L; 8, 3; $NL, \frac{21}{7}$, 3; $ML, \frac{15}{5}$, 3; NML, 3, sides
   (c) G, I, H; $G\hat{I}H$, 90; $\frac{6}{9}, \frac{2}{3}$, $IH, \frac{4}{6}, \frac{2}{3}$; IH; GIH, 2, sides, included
3. (a) No   (b) No
   (c) No

4. (a) $\triangle ABC$ and $\triangle EDC$
   (b) $\triangle IJH$ and $\triangle IFG$
   (c) $\triangle PQR$ and $\triangle TSR$
   (d) $\triangle UVW$ and $\triangle UXY$
5. (a) $x = 9, y = 18$
   (b) $x = 4.8, y = 7.5$
   (c) $x = 12, y = 15$
   (d) $x = 3.2, y = 7.5$
6. (ii) 110.5°
7. $XY = 30$ cm, $WZ = 27$ cm
8. 16.2 cm
9. (i) $\triangle BAC, \triangle DBC$ and $\triangle DAB$
   (ii) $BC = 6\frac{2}{3}$ m, $CD = 5\frac{1}{3}$ m
10. (i) $RT = 6$ cm, $PR = 12$ cm
    (ii) 2 : 1
11. (ii) $\triangle ART$ and $\triangle MNT$
    (iii) 3.75 cm

Exercise 11C

3. 8 m
5. Length of AB'
6. 45

Review Exercise 11

1. (a) Yes   (b) Yes
   (c) No   (d) Yes
2. $\triangle DEF \equiv \triangle JLK$, AAS
3. (a) $\triangle ABC \equiv \triangle DEC$
   $A\hat{C}B = D\hat{C}E, BC = EC, AC = DC$
   (b) $\triangle FGH \equiv \triangle FIJ$
   $F\hat{G}H = F\hat{I}J, FG = FI, FH = FJ$
   (c) $\triangle KLN \equiv \triangle MNL$
   $K\hat{L}N = M\hat{N}L, K\hat{N}L = M\hat{L}N, KL = MN$
   (d) $\triangle SQP \equiv \triangle RPQ$
   $S\hat{Q}P = R\hat{P}Q, SQ = RP, SP = RQ$
   (e) $\triangle EBF \equiv \triangle ECD$
   $E\hat{B}F = E\hat{C}D, BF = CD, EF = ED$
   (f) $\triangle FHG \equiv \triangle FIJ$
   $F\hat{H}G = F\hat{I}J, FH = FI, GH = JI$
4. (a) Yes   (b) No
   (c) Yes   (d) No
   (e) No   (f) Yes
5. (i) $\triangle OAD \equiv \triangle OBC$
   (ii) $A\hat{O}D = B\hat{O}C, O\hat{A}D = O\hat{B}C$

6. (i) $\Delta PQR \cong \Delta SRQ$, SAS

   (ii) 5 cm, 50°

8. (a) $a = 5.92$

   (b) $b = 15\frac{5}{7}$, $c = 12\frac{4}{7}$

   (c) $d = 2\frac{2}{3}$, $e = 6\frac{3}{4}$

9. (i) ΔRQP (ii) 6.4 cm

10. (a) (i) ΔACE and ΔGFE

     (ii) 6 cm

   (b) $EG = 8$ cm, $FH = 18$ cm

11. (a) (i) ΔRLN (ii) 8 cm

   (b) (i) ΔNMS (ii) 9 cm

   (c) ΔPLM and ΔRLQ, ΔPQM and ΔSNM, ΔPQM and ΔRNQ

12. (i) $\Delta STR \cong \Delta STP$

   (ii) $UQ = 7$ cm, $PQ = 10$ cm

Challenge Yourself

2. 3 cm   3. 13.5 cm

CHAPTER 12

Exercise 12A

1. (a) 4 cm²   (b) 2.4 m²

   (c) 20 cm²   (d) 108 cm²

   (e) 27 m²   (f) 6 cm²

2. 16 : 49

3. (i) $66\frac{2}{3}$ m²   (ii) $42\frac{2}{3}$ m²

4. (a) 6   (b) 15

   (c) 20   (d) 4

5. 128 m²

6. 7 cm²

7. 812.5 m²

8. $\dfrac{p^2}{(p+q)^2}$

9. (i) 18 cm   (ii) 279 cm²

10. 3.62 cm

11. 3 : 5

12. 7.5 cm

13. (i) 40 cm²   (ii) 60 cm²

14. (i) $2\frac{2}{7}$ cm   (ii) 7 : 4

   (iii) 64 : 231

15. (i) 50 cm²   (ii) 12 cm²

   (iii) 30 cm²

Exercise 12B

1. (a) 576 cm³   (b) 162 cm³

   (c) 324 cm³   (d) 38.5 m³

   (e) 0.4 m³

2. (a) 125 : 64   (b) 27 : 64

   (c) 8 : 27

3. 160 cm³

4. (a) 4   (b) 9

   (c) 21   (d) 5

5. (i) 3 : 4   (ii) 189 cm³

6. 4 : 3

7. 20.8 cm

8. (i) 16 : 49   (ii) 53.6 g

9. 4.76 cm   10. 148 g

11. (i) 4.61 kg   (ii) 13 281 l

12. 8.15 m²

13. (i) 36 cm   (ii) 2 : 3

   (iii) 1134 cm³

14. (i) 10 cm   (ii) 20 cm²

   (iii) 225 cm³

15. 3.65

Review Exercise 12

1. (a) 9 : 25   (b) 1 : 4

   (c) 4 : 9

2. 49 : 64

3. (ii) 100 cm²   4. 2 : 3

5. (i) 3 : 4   (ii) 9 : 16

6. 6.144 tonnes   7. 16 : 25

8. (i) 9 : 1   (ii) 400 l

9. (a) ΔSLP

   (b) (i) $\dfrac{1}{4}$ (ii) $\dfrac{2}{3}$

    (iii) $\dfrac{2}{3}$

10. (a) $\dfrac{21}{25}$

   (b) (i) ΔNOM (ii) $\dfrac{2}{5}$

11. (ii) $\dfrac{25}{4}$ (iii) 31.5 cm²

12. 27 : 10

13. (i) 33.5 cm³   (ii) 14.1 cm³

   (iii) 232 g

14. (i) 1 : 7   (ii) 7 : 19

   (iii) 27 : 37

Challenge Yourself

1. 4 : 9

2. (ii) ΔEAS and ΔTHS, ΔTSA and ΔBSH

   (iii) 40 cm   (iv) 1 : 25

CHAPTER 13

Exercise 13A

1. (a) $a = 12$, $b = 67.4$

   (b) $c = 11.0$, $d = 61.9$

   (c) $e = 6$, $f = 50.2$

2. 15 cm

3. 13 m

4. 13.7 cm

5. (a) $a = 12$, $b = 90$ (b) $x = 11$, $y = 90$

6. 18.0 cm²

7. 17 cm

8. (i) 5.66 cm   (ii) 14.2 cm

9. 1 cm or 7 cm

10. 8.39 cm

11. 63.3 cm

Exercise 13B

1. 24°

2. (i) 26°   (ii) 122°

3. $45° + \dfrac{x}{2}$

4. (a) $a = 49$, $b = 14$

   (b) $c = 58$, $d = 15$

   (c) $e = 34$, $f = 14.8$

   (d) $g = 35$, $h = 55$

   (e) $i = 8$, $j = 67.4$

   (f) $k = 12.6$, $l = 50.0$

5. (i) 44°   (ii) 25°

6. (i) 7.5 cm   (ii) 67.4°

   (iii) 34.4 cm²

7. 138°

8. 9 cm

9. 7 m

10. 64°

11. 51°

12. 45 cm

Exercise 13C

1. (a) 80   (b) 30
   (c) 40   (d) 115
   (e) 125   (f) 50
   (g) 35   (h) 28
2. (a) 50   (b) 45
   (c) 30   (d) 60
3. (a) 50   (b) 12
4. $60°$ 5. $65°$
6. (a) 40   (b) 36
   (c) 47   (d) 130
7. (i) $70°$ (ii) $70°$
8. $270°$
9. (a) $40°$ (b) $50°$
   (c) $77°$ (d) $56°$
10. $37°$
11. (i) $62°$ (ii) $47°$
12. (i) $64°$ (ii) $64°$
13. 70
14. 78.5 cm^2
15. $31°$
16. $32°$
17. $45°$
18. $125°$
19. (i) $20°$ (ii) $50°$
20. (i) $90°$ (ii) $55°$
21. $40°$
22. (i) $35°$ (ii) $131°$
23. (b) (i) 8 cm   (ii) 9 cm
24. (ii) $\triangle PCB$
25. $18°$
26. $\angle P = 65°, \angle Q = 55°, \angle R = 60°$
27. (ii) $10\frac{1}{6}$ cm

Review Exercise 13

1. (a) $x = 25, y = 25$
   (b) $x = 34, y = 114$
   (c) $x = 28.5, y = 16.5$
   (d) $x = 64, y = 38$
   (e) $x = 26, y = 148$
   (f) $x = 90, y = 118$
2. (a) $x = 70, y = 35$
   (b) $x = 34, y = 56$

   (c) $x = 132, y = 114$
   (d) $x = 105, y = 30$
   (e) $x = 6.43, y = 25$
   (f) $x = 54, y = 72$
3. (a) $x = 62, y = 118$
   (b) $x = 116, y = 46$
   (c) $x = 115, y = 57.5$
   (d) $x = 50$
   (e) $x = 72, y = 28$
   (f) $x = 48, y = 22$
4. (a) $x = 41$
   (b) $x = 78, y = 30$
   (c) $x = 108, y = 144$
   (d) $x = 24, y = 42$
   (e) $x = 29, y = 59$
   (f) $x = 42, y = 90$
   (g) $x = 103, y = 45$
   (h) $x = 22.5, y = 135$
5. $x + y$
6. $20°$
7. (i) $24°$ (ii) $49°$
8. $x = 74, y = 103$
9. (i) $90°$ (ii) $110°$
10. (i) 1 cm   (ii) $\frac{1}{2}y$
11. (i) 4 cm   (ii) 13.6 cm
12. (i) $46°$ (ii) $134°$
13. $61°$ or $119°$
14. (i) $78°$ (ii) $102°$
15. (i) $62°$ (ii) $126°$
16. Yes

Challenge Yourself

(a) 6   (b) 3
(c) 2   (d) 2

REVISION EXERCISE D1

1. Yes
2. (b) $\triangle ASQ$ (c) 5 cm
   (d) $\triangle BCA, \triangle RAQ$
   (e) (i) 16 cm^2 (ii) 8 cm^2
      (iii) 8 cm^2
3. (i) $3 : 200$ (ii) 360 m^2
   (iii) 48 m^3

4. (i) $72°$ (ii) $42°$
5. (i) $128°$ (ii) $88°$
6. $u = 20°, v = 70°, w = 30°, x = 70°,$
   $y = 130°, z = 80°$
7. (i) b (ii) $b - a$
   (iii) $2b - a$ (iv) $180° - 2b$
   (v) $180° + a - 3b$
8. (i) 283 cm^2 (ii) 6
   (iii) 251 cm^2

REVISION EXERCISE D2

1. (ii) $\triangle CRB$
2. (i) $PX = 6.48 \text{ cm}, QX = 7.56 \text{ cm}$
   (ii) $25 : 81$
3. (i) $66\frac{2}{3}$ cm   (ii) $45\,511\frac{1}{9} \text{ cm}^3$
4. (i) 9 cm, $7\frac{7}{16}$ cm   (ii) $225 : 161$
5. (i) $100°$ (ii) $50°$
6. 157 cm^2
7. (i) $70°$ (ii) $69°$
   (iii) $55°$
8. (i) 9 cm   (ii) $1 : 3$

PROBLEMS IN REAL-WORLD CONTEXTS

1. (a) $(310, 0), (155, 82.5)$
   (c) (i) $c = 0,$
        $155^2a + 155b = 82.5,$
        $310^2a + 310b = 0$
      (ii) $a = -\frac{33}{9610}, b = 1\frac{2}{31}, c = 0$
      (iii) $y = -\frac{33}{9610}x^2 + 1\frac{2}{31}x$
2. (a) S\$312.80
   (b) $[S\$475 + 0.75 \times (EC - 1600)] \times 0.782;$
      S\$2385.10
3. (i) Yes
4. (b) (i) \$810.84   (ii) \$2855.70
5. The large watermelon
6. (ii) 16 cm   (iii) 201 cm^2
7. (i) $36.8°$
8. (i) \$3522.50   (ii) \$2104
9. (i) \$9600   (ii) \$9600; Yes
   (iv) \$360 000

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