An Algebraic Solution to the Multilateration Problem

Research · April 2015

DOI: 10.13140/RG.2.1.1681.3602

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***Abstract*—Across the spectrum of known algorithm for position estimation there is no favorite method. Some algorithms require intensive computation capabilities, while other algorithms could be implemented in devices with limited resources such as sensor nodes. In this paper an approach for solving nonlinear problems on the example of multilateration is presented in both cases with and without over determination. Thereby neither approximation, nor iterative solutions are used. In the proposed method, the nonlinear elements of the equations system are treated as additional unknowns, which represent simultaneously a constraint. Thus a new equations system is created, which is solved by mean of linear algebra methods with low computational complexity. The algorithm was implemented and tested in conjunction with a developed UWB indoor positioning system.**

***Index Terms*—Localization, Trilateration, Multilateration, non linear least square, Ultra Wide Band (UWB), sensor networks**

I. INTRODUCTION

The calculation of the spatial coordinates of unknown points from its distances to other known points is a common operation, known as multilateration. In recent years numerous studies on the solution of the multilateration range equation have been published. As an example, we can refer to [1] and [2] which listed a number of procedures and presented an algebraic approach. This paper shows an alternative method for solving the multilateration range equation with low computational complexity. The algorithm was first published by the author in German language [3]. In this contribution the algorithm is applied to real measurement data of an UWB indoor positioning system.

The suggested method is described in section II in the case of the trilateration problem. In section III the algorithm is extended to the multilateration problem based on range measurements to more than three reference points. Subsequently, as shown in section III, by using a recursive least square approach additional range measurements can be added gradually to lead to a balanced solution. In section IV two examples illustrate the application of the algorithm in conjunction with real measurement data.

II. SOLUTION BASED ON THREE REFERENCE POINTS

Given are the three reference points $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$, $P_3(x_3, y_3, z_3)$ and the range measurements s_1, s_2 and s_3 to the point N (cf. Fig.1). The determination of the coordinates

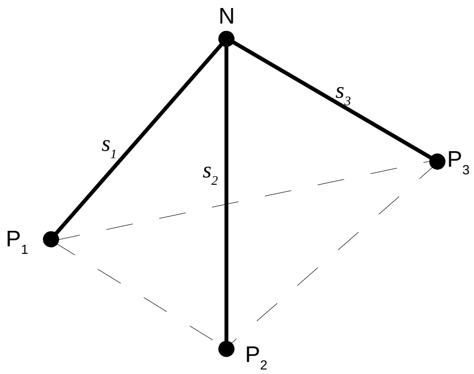

Figure 1: Trilateration problem

(x, y, z) of the point N is equivalent to finding the solutions to the following system of quadratic equations.

$$\begin{cases} (x-x_1)^2+(y-y_1)^2+(z-z_1)^2=s_1^2 \\ (x-x_2)^2+(y-y_2)^2+(z-z_2)^2=s_2^2 \\ (x-x_3)^2+(y-y_3)^2+(z-z_3)^2=s_3^2 \end{cases} \quad (1)$$

(1) can be arranged as:

$$\begin{cases} (x^2+y^2+z^2)-2x_1x-2y_1y-2z_1z=s_1^2-x_1^2-y_1^2-z_1^2 \\ (x^2+y^2+z^2)-2x_2x-2y_2y-2z_2z=s_2^2-x_2^2-y_2^2-z_2^2 \\ (x^2+y^2+z^2)-2x_3x-2y_3y-2z_3z=s_3^2-x_3^2-y_3^2-z_3^2 \end{cases} \quad (2)$$

Or in matrix representation:

$$\begin{bmatrix} 1 & -2x_1 & -2y_1 & -2z_1 \\ 1 & -2x_2 & -2y_2 & -2z_2 \\ 1 & -2x_3 & -2y_3 & -2z_3 \end{bmatrix} \begin{bmatrix} x^2+y^2+z^2 \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} s_1^2-x_1^2-y_1^2-z_1^2 \\ s_2^2-x_2^2-y_2^2-z_2^2 \\ s_3^2-x_3^2-y_3^2-z_3^2 \end{bmatrix} \quad (3)$$

Thus, (3) is represented in the known form

$$A_0 \cdot \boldsymbol{x} = \boldsymbol{b_0} \quad (4)$$

With the constraint: $\boldsymbol{x} \in E$

Where $E=\left\{(x_0,x_1,x_2,x_3)^T \in \mathbb{R}^4 \,/\, x_0={x_1}^2+{x_2}^2+{x_3}^2\right\}$

A. *Solution of the equation system (4)*

Case 1: P_1, P_2 and P_3 do not lie on a straight line.
Then Rang(A_0)=3 and dim(Kern(A_0))=1.

The general solution of (4) is then

$$\boldsymbol{x} = \boldsymbol{x_p} + t \cdot \boldsymbol{x_h} \tag{5}$$

With the real parameter t. Where $\boldsymbol{x_p}$ is a particular solution of (4) and $\boldsymbol{x_h}$ is a solution of the homogeneous system $A_0.\boldsymbol{x}=\boldsymbol{0}$ i.e. $\boldsymbol{x_h}$ is a Basis of Kern(A_0).

The vectors $\boldsymbol{x_p}$ and $\boldsymbol{x_h}$ can be computed using the Gaussian elimination method. The particular solution $\boldsymbol{x_p}$ can also be determined using the pseudo inverse of the matrix A_0. The pseudo-inverse gives the solution with the minimum norm [4].

Determination of the parameter t:

Let $\boldsymbol{x_p} = \left(x_{p0}, x_{p1}, x_{p2}, x_{p3}\right)^T$, $\boldsymbol{x_h} = \left(x_{h0}, x_{h1}, x_{h2}, x_{h3}\right)^T$ and $\boldsymbol{x} = \left(x_0, x_1, x_2, x_3\right)^T$

When inserted into (5) then it becomes:

$$\begin{cases} x_0 = x_{p0} + t \cdot x_{h0} \\ x_1 = x_{p1} + t \cdot x_{h1} \\ x_2 = x_{p2} + t \cdot x_{h2} \\ x_3 = x_{p3} + t \cdot x_{h3} \end{cases} \tag{6}$$

By using the constraint $\boldsymbol{x} \in E$ it follows:

$$x_{p0} + t \cdot x_{h0} = \left(x_{p1} + t \cdot x_{h1}\right)^2 + \left(x_{p2} + t \cdot x_{h2}\right)^2 + \left(x_{p3} + t \cdot x_{h3}\right)^2$$

and thus

$$\begin{aligned} & t^2\left(x_{h1}^2 + x_{h2}^2 + x_{h3}^2\right) + t\left(2 \cdot x_{p1}x_{h1} + 2 \cdot x_{p2}x_h + 2 \cdot x_{p3}x_{h3} - x_{h0}\right) \\ & + x_{p1}^2 + x_{p2}^2 + x_{p3}^2 - x_{p0} = 0 \end{aligned} \tag{7}$$

This is a quadratic equation in the form $at^2 + bt + c = 0$ with the solutions

$$t_{1/2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{8}$$

The solutions of the equation system (4) are:

$$\begin{aligned} \boldsymbol{x_1} &= \boldsymbol{x_p} + t_1 \cdot \boldsymbol{x_h} \\ \boldsymbol{x_2} &= \boldsymbol{x_p} + t_2 \cdot \boldsymbol{x_h} \end{aligned} \tag{9}$$

If the multilateration problem cannot be solved (too short distances), so there are no real solutions. In this case, the real part is used as an approximation for the solution. With this approximation, the constraint $\boldsymbol{x_{1/2}} \in E$ is not met. Thus the difference:

$$d = x_0 - \left(x_1^2 + x_2^2 + x_3^2\right) \tag{10}$$

is a measure of the solvability of the multilateration problem, where x_0 , x_1 , x_2 and x_3 the coordinates of the solution $\boldsymbol{x}$ of (4). Solutions of the multilateration problem are the points: $N_1 = \boldsymbol{x_1} \cdot I$ and $N_2 = \boldsymbol{x_2} \cdot I$, where

$$I = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Case 2: P_1, P_2 and P_3 lie on a straight line.

Then Rang(A_0)=2 and dim(Kern(A_0))=2. The general solution of (4) is then

$$\boldsymbol{x} = \boldsymbol{x_p} + t \cdot \boldsymbol{x_{h1}} + k \cdot \boldsymbol{x_{h2}} \tag{11}$$

With real parameters t and k

$\boldsymbol{x_p}$ is a particular solution of (4) and $\boldsymbol{x_{h1}}$ and $\boldsymbol{x_{h2}}$ are two solutions of the homogeneous system $A_0.\boldsymbol{x}=\boldsymbol{0}$. They are linearly independent solutions and form therefore a basis of Kern(A_0). Since there is only one constraint equation, the multilateration problem has infinitely many solutions.

III. SOLUTION BASED ON MORE THAN THREE REFERENCE POINTS

With additional distances s_4, s_5 …. s_n to the reference points P_4, P_5 … P_n , (3) can be extend as follows:

$$\begin{bmatrix} 1 & -2x_1 & -2y_1 & -2z_1 \\ 1 & -2x_2 & -2y_2 & -2z_2 \\ 1 & -2x_3 & -2y_3 & -2z_3 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & -2x_n & -2y_n & -2z_n \end{bmatrix} \begin{bmatrix} x^2 + y^2 + z^2 \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} s_1^2 - x_1^2 - y_1^2 - z_1^2 \\ s_2^2 - x_2^2 - y_2^2 - z_2^2 \\ s_3^2 - x_3^2 - y_3^2 - z_3^2 \\ \vdots \\ s_n^2 - x_n^2 - y_n^2 - z_n^2 \end{bmatrix} \tag{12}$$

That means in the known form

$$A \cdot \boldsymbol{x} = \boldsymbol{b} \tag{13}$$

With the constraint $\boldsymbol{x} \in E$. The solution $\hat{\boldsymbol{x}}$ of (13) in the sense of least squares method is:

$$\hat{\boldsymbol{x}} = \left(A^T A\right)^{-1} A^T \boldsymbol{b} \tag{14}$$

The projection of p on the column space of A is:

$$\boldsymbol{p} = A\left(A^T A\right)^{-1} A^T \boldsymbol{b} \tag{15}$$

The coordinates of $\boldsymbol{p}$ on the column space Col(A) represent the solution $\hat{\boldsymbol{x}}$

To note is that all elements in matrix $L = (A^TA)^{-1}A^T$ are derived from reference points coordinates only. Moreover, vector $\boldsymbol{b}$ consists of distances between the unknown point N and all the reference points. Especially in static sensor networks the computation of the entire localization can be accomplished in the nodes themselves, since the computation is restricted to a matrix vector multiplication of matrix L and vector $\boldsymbol{b}$.

If, however, the measurements are uncorrelated but have different uncertainties, the Weighted Least Square (WLS) is used [5]. The solution $\hat{\boldsymbol{x}}$ is given by the following equation:

$$\hat{\boldsymbol{x}} = \left(A^T V^{-1} A\right)^{-1} A^T V^{-1} \boldsymbol{b} \qquad (16)$$

V is the covariance matrix of random errors.

Since the solution should also belong to E, and because E does not form a vector space, a closed formula for the projection of $\boldsymbol{p}$ on $Col(A) \cap E$ is not given. Thus only candidates for approximate solutions are selected. The solution candidate N, which minimizes the error square sum (17) is then chosen

$$\min_N \left\{ \left(\|N - P_1\|^2 - s_1 \right)^2 + \ldots + \left(\|N - P_n\|^2 - s_n \right)^2 \right\} \qquad (17)$$

First candidate:

Equation (13) is solved using (14) or (16).

Further candidates by using the recursive least square:

First the non over determined system of equations (4) have to be solved. Since (4) has two solutions, the solution which is closer to the first candidate is selected. This solution is used as a starting point for the Recursive Least Square (RLS) [5]. Let x_0 be the initial solution. By every incoming distance, x_0 is updated in x_1 by using RLS.

The approach enables a simultaneous execution of distance measurement and positioning calculation. Hence a position assignment can be started, although not all distances are available. Thereby unnecessary waiting time is avoided and the positioning calculation is speed up.

IV. NUMERICAL EXAMPLES

To test the numerical method, distance measurements in a specially created test field at the University Institute were performed using a UWB Indoor Local Positioning System (UWB-ILPS). The main hardware components of UWB-ILPS are Time Domain PulsON 210 UWB Radios [6]. Fig. 2 shows the mobile station. Distance measurement between the stations takes place via UWB, communication and control is via WLAN. For more details on UWB-ILPS, please refer to [7] and [8].

Fig. 3 demonstrates the location of the reference stations as well as the mobile station located on points P_{36} and P_{38} , where the x-axis points to north direction, y-axis to east direction and z-axis points down to the center of earth.

A. *Solution based on three reference points:*

The coordinates of three reference points and the measured distances to the point P_{36} are listed in Table I. The true coordinate of the unknown point are

P_{36} = (24,335 ; -2,506 ; 1,130).

The particular solution of (4) and the solution of the homogeneous system are respectively:

$\boldsymbol{x_p}$ = (0,0000 ; 13,2890 ; -13,4807 ; -35,9246) and

$\boldsymbol{x_h}$ = (0,9977 ; 0.0183 ; 0,0188 ; 0,0623)

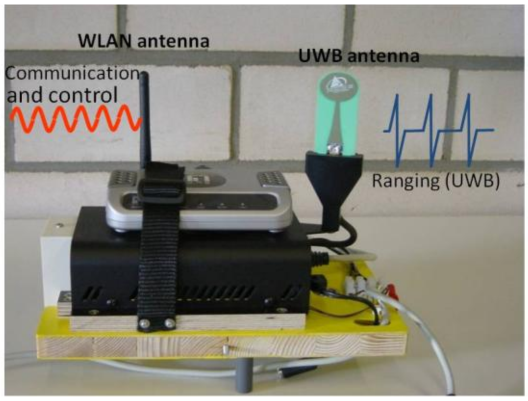

Figure 2: Mobile station with attached WLAN bridge

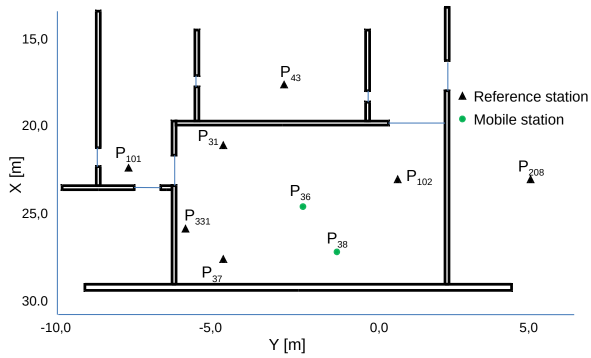

Figure 3: Test field at the Geodetic Institute (2nd floor)

TABLE I
DISTANCE MEASUREMENTS TO THREE REFERENCE POINTS

Reference points	X[m]	Y [m]	Z [m]	$d_{measured}$ [m]	d_{true} [m]	d_{true} - $d_{measured}$ [cm]
P_{37}	27,297	-4,953	1,470	3,851	3,857	0,6
P_{331}	25,475	-6,124	2,360	3,875	3,988	-11,3
P_{102}	22,590	0,524	1,200	3,514	3,497	-1,7

The general solution of (4) is $\boldsymbol{x} = \boldsymbol{x_p} + t\, \boldsymbol{x_h}$, where t is the solution of the quadratic equation (7):

$$(0{,}00457\ldots)\cdot t^2 + (-5{,}5087\ldots)\cdot t + 1658{,}74\ldots = 0$$

In this case t_1 = -603,2647 and t_2 = -601,1773. (4) has then the following solutions:

$\boldsymbol{x_1}$ = $(601{,}8863\ ;\ 24{,}3506\ ;\ -2{,}4811\ ;\ 1{,}6667)^T$ and $\boldsymbol{x_2}$ = $(599{,}8010\ ;\ 24{,}3123\ ;\ -2{,}5205\ ;\ 1{,}5365)^T$

The corresponding differences according to (10) are d_1=d_2=0. The small size of the differences confirmed the solvability of the problem.

The solutions of the multilateration problem are in meter:

N_1 = (24,3506 ; −2,4811 ; 1,6667) and

N_2 = (24,3123 ; −2,5205 ; 1,5365)

B. Solution based on more than three reference points:

The coordinates of six reference points and the measured distances to the point P_{38} are listed in Table II. The true coordinate of the mobile station are

P_{38} = (26,7590 ; -1,342 ; 1,130)

The first three distances provide the following solutions:

x_{01} = (720,6931 ; 26,7726 ; −1,3389 ; 1,4584) and

x_{02} = (716,7319 ; 26,7234 ; −1,4152 ; 0,7685)

The starting value for the RLS is taken from the x_{01} and x_{02} solution that is compatible to the fourth distance (distance to P_{43}), in this case $x_0 = x_{01}$

Furthermore it can be shown from the location of the solutions that the lines to the points P_{43} and P_{208} run through walls and therefore the distances to these points have to be corrected according to UWB wave excess delay estimation equation proposed in [9]. The calculated delays are in this case 12 cm and 14 cm respectively.

By using the RLS we get the following solutions:

x_0 = (720,6931 ; 26,7726 ; −1,3389 ; 1,4584)
x_1 = (720,6933 ; 26,7659 ; −1,3377 ; 1,4576)
x_2 = (720,6934 ; 26,7680 ; −1,3267 ; 1,4564)
x_3 = (720,6877 ; 26,7629 ; −1,3189 ; 1,5005)

TABLE II
DISTANCE MEASUREMENTS TO SIX REFERENCE POINTS

Reference points	X[m]	Y [m]	Z [m]	$d_{measured}$ [m]	d_{true} [m]	d_{true} - $d_{measured}$ [cm]
P_{37}	27,297	-4,953	1,470	3,652	3,666	1,47
P_{31}	20,693	-4.849	1,93	7,036	7,052	1,63
P_{102}	22,590	0,524	1,200	4,586	4,568	1,79
P_{43}	17,113	-3,003	2,17	9,960	9,843	11,69
P_{208}	22,554	4,727	1,77	7,542	7,411	13,09
P_{101}	22,45	-7,880	1,6	7,883	7,846	3,7

TABLE III
CALCULATED DIFFERENCES AND SUM OF SQUARES

Solutions	x_{01}	x_{02}	x_1	x_2	x_3
Difference d [m^2]	0	0	0,37	0,29	0,44
Error square sum [m^2]	0,0421	0,1246	0,0485	0,0432	0,0537

Table III shows the difference d and the sum of squares by using (10) and (17) respectively

The solution which minimizes the error square sum is x_{01}. Therefore The solution of the multilateration problem is the point N = (26,7726 ; -1,3389 ; 1,4584)

C. Computational efficiency

Generally for indoor positioning, the limited number of participating reference stations doesn't cause a big challenge to the computational resource for multilateration. However, for example, in the case of a sensor network node, every reduction in the computation time can lead to an increase in battery lifetime.

The computational efficiency of the algorithm is compared to the Gauss–Newton algorithm (GNA) applied on (1). The results indicate that the average execution of the proposed algorithm is significantly faster than GNA. However the convergence of GNA depends on the start solution. Assuming that the point N is within the reference point's volume, the test field center is selected as start solution for GNA. For comparison, the execution time of the algorithm based on the last numerical example average to 1.9 µs while GNA needs 18 µs. To note the evaluation is based on the execution time performed with MATLAB on a 2.53 GHz dual-core PC.

V. CONCLUSION

In this contribution, an alternative solution for the multilateration problem, with and without over determination, is proposed. Thereby neither approximation, nor iterative solutions are used. The algorithm is based on linear algebraic method, has low computational complexity and can be applicable to real-time applications and in wireless sensor nodes. The measurement results performed with our UWB positioning system show that the algorithm is highly effective and has low estimation error. However the raging update rate of the used UWB transceivers is relatively low, and doesn't permit to test the algorithm by moving objects. Therefore further works consist of the expansion of the positioning algorithm, regarding kinematic scenarios.

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