Robust regression using iteratively reweighted least-squares

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ROBUST REGRESSION USING ITERATIVELY REWEIGHTED LEAST-SQUARES

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ABSTRACT

The rapid development of the theory of robust estimation (Huber, 1973) has created a need for computational procedures to produce robust estimates. We will review a number of different computational approaches for robust linear regression but focus on one--iteratively reweighted least-squares (IRLS). The weight functions that we discuss are a part of a semi-portable subroutine library called ROSEPACK (RObust Statistical Estimation PACKage) that has been developed by the authors and Virginia Klema at the Computer Research Center of the National Bureau of Economic Research, Inc. in Cambridge, Mass. with the support of the National Science Foundation. This library (Klema, 1976) makes it relatively simple to implement an IRLS regression package.

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INTRODUCTION

We will consider the standard regression model

$$y = X \beta + \epsilon \ , \tag{1}$$

where y and ϵ are n by 1, X is n by p, and β is p by 1. A robust estimate for β, $\hat{\beta}$, minimizes

$$\sum_{i=1}^{n} \rho\left(\frac{y_i - x_i \beta}{\sigma}\right) \ , \tag{2}$$

where σ is a known or previously estimated scale parameter, ρ is a robust loss function, and x_i is the ith row of X. If we let $\psi = \rho'$, then a necessary condition for a minimum is that $\hat{\beta}$ satisfy

$$\sum_{i=1}^{n} x_{ij} \ \psi\left(\frac{y_i - x_i \beta}{\sigma}\right) = 0 \text{ for all } j. \tag{3}$$

In general (3) is a set of nonlinear equations and iterative methods are required.

We define the weight function $w(r)$ as $\psi(r)/r$, and let $< >$ denote an n by n diagonal matrix. If we have a starting value $\hat{\beta}_0$ then there are three fairly obvious iteration schemes for finding the solution to (3):

$$\hat{\beta}_1 = \hat{\beta}_0 + \sigma\left(X^T <\psi'\left(\frac{y - X\hat{\beta}_0}{\sigma}\right)> X\right)^{-1} X^T \ \psi\left(\frac{y - X\hat{\beta}_0}{\sigma}\right) \ , \tag{4}$$

$$\hat{\beta}_1 = \hat{\beta}_0 + \sigma(X^T X)^{-1} X^T \ \psi\left(\frac{y - X\hat{\beta}_0}{\sigma}\right) \ , \tag{5}$$

$$\hat{\beta}_1 = \hat{\beta}_0 + \left(X^T <w\left(\frac{y - X\hat{\beta}_0}{\sigma}\right)> X\right)^{-1} X^T <w\left(\frac{y - X\hat{\beta}_0}{\sigma}\right)> (y - X\hat{\beta}_0) \ . \tag{6}$$

The first is Newton's method, the second has been discussed extensively by Huber (1975) and Bickel (1975) and the third, reweighted least-squares, is generally attributed to Beaton and Tukey (1974).

Newton's method is theoretically the most desirable, but it is difficult to implement because it requires ψ'; also $X^T <\psi'> X$ may be negative definite. The Huber method has desirable

computational properties since $(X^T X)^{-1} X^T$, the generalized inverse, need only be computed once. It does, however, require more iterations and is not as easy to use with existing least-squares regression packages.

The third method only requires knowing how to compute the weight function, $w(r)$, and then it is possible to use an existing weighted least-squares algorithm or to compute the square root of $w(r)$, form $<w^{1/2}>X$ and $<w^{1/2}>y$ and use a standard least-squares program for each step. The IRLS method generally converges somewhat faster than the Huber method, but more slowly than Newton's method.

With nonconvex ρ functions we can only hope for convergence to a local minimum and therefore a good starting value is important. We propose using the least absolute residuals estimator, $\hat{\beta}_L$ which minimizes

$$\sum_{i=1}^{n} |y_i - x_i \beta| \quad . \tag{7}$$

This can be obtained from algorithms described by Barrodale and Roberts (1973) and Bartels (1975).

Clearly a question arises about estimating σ. We have done this by using

$$\sigma = 1.48 \; [\text{med}_i \; |(y_i - x_i \hat{\beta}_o) - \text{med}_i \; (y_i - x_i \hat{\beta}_o)|] \tag{8}$$

just once, before starting the iterations. (The factor 1.48 makes this an approximately unbiased estimate of scale when the error model is Gaussian.) We do not iterate scale because there is no convergence theory when the scale is iterated except for the Huber loss function (Huber, 1975). If scale is to be iterated and/or a least-squares start is used, the best procedure is to iterate to convergence with the Huber ρ function and the Huber scaling and then use a nonconvex ρ function without further scale iteration. Hill and Holland (1977) discuss other aspects of σ defined in (8).

We will not discuss convergence criteria in this paper.
Useful references are Dennis and Welsch (1976) and Huber (1975).

WEIGHT FUNCTIONS

After some years of experience with robust regression and a
survey of the literature, we chose to restrict our attention to
the eight weight functions given in Table I. They can be classi-
fied by the behavior of $\psi(t)$. The hard redescenders; A (Andrews
et al., 1972), B (Beaton and Tukey, 1974), and T (Hinich and
Talwar, 1975), all have $\psi(r) = 0$ for $|r|$ sufficiently large. The

TABLE I

Weight Functions

Name	$\rho(r)$	$\psi(r)$	$w(r)$	Range
A	$A^2[1-\cos(r/A)]$	$A\sin(r/A)$	$(r/A)^{-1}\sin(r/A)$	$|r|\leq\pi A$
	$2A^2$	0	0	$|r|>\pi A$
B	$(B^2/2)[1-[1-(r/B)^2]^3]$	$r[1-(r/B)^2]^2$	$(1-(r/B)^2)^2$	$|r|\leq B$
	$B^2/2$	0	0	$|r|>B$
T	$r^2/2$	r	1	$|r|\leq T$
	$T^2/2$	0	0	$|r|>T$
C	$(C^2/2)\log[1+(r/C)^2]$	$r[1+(r/C)^2]^{-1}$	$[1+(r/C)^2]^{-1}$	
W	$(W^2/2)[1-\exp[-(r/W)^2]]$	$r\exp[-(r/W)^2]$	$\exp[-(r/W)^2]$	
H	$r^2/2$	r	1	$|r|\leq H$
	$H|r| - H^2/2$	$\text{sgn}\cdot(r)\,H$	$H|r|^{-1}$	$|r|>H$
L	$L^2\log[\cosh(r/L)]$	$L\tanh(r/L)$	$(r/L)^{-1}\tanh(r/L)$	
F	$F^2[|r|/F-\log(1+|r|/F)]$	$r(1+|r|/F)^{-1}$	$(1+|r|/F)^{-1}$	

soft redescenders, C (Cauchy or t-likelihood) and W (Dennis and
Welsch, 1976) are asymptotic to zero for large $|r|$. The last three,
H (Huber, 1964), L (logistic), and F (Fair, 1974) have monotone ψ
functions.

   With some abuse of notation we will use the same letters
to refer to the ψ-function and to the "tuning constants" for
each ψ-function. Tuning constants may be used to adjust the
efficiency of the resulting estimators for specific distributions.

   Functions A and B are essentially the same except that ψ' is
continuous for B but not for A. The same comment applies to L and
H respectively. The function F was created as an approximation to
least absolute residuals (LAR) for nonlinear models but can be
useful for linear problems as well. Performing LAR regression using
iteratively reweighted least-squares (Armstrong and Frome, 1976)
is not recommended. The algorithms described for finding $\hat{\beta}_L$ in (7)
are better.

   The C class is easily seen to give the maximum likelihood
estimates for t distributions where the degrees of freedom can be
related to the tuning constant, C. We should note that H and L
can also be derived using maximum likelihood, but we will not
discuss this aspect of robust estimation here.

   The function W was developed mainly to simplify certain
theoretical computations relating to the asymptotic efficiencies
of robust estimators. Recent work by Schumaker and Paulson (1976)
on characteristic functions may give it a more important theoretical
foundation.

   For monotone ψ-functions (H, L, and F), Huber (1973) has shown
that under reasonable conditions fully iterated estimators defined
by (2) are asymptotically Gaussian with covariance

$$V(F,\psi) = \frac{E_F\ (\psi^2)}{[E_F\ (\psi')]^2} \times (X^TX)^{-1} \quad , \tag{9}$$

when the underlying population has a cumulative distribution func-
tion, F. For nonmonotone ψ-functions, much less is known, but
Collins (1976) gives some results. Monte Carlo studies of the

covariance matrix are discussed in Welsch (1975) and Gross (1977).
In order to provide some guidance about the use of the eight weight
functions, we have listed in Table II the values of the tuning
constants such that the asymptotic efficiency in the location model,
$nV(F,\psi)^{-1}$, is 95% when F is the unit Gaussian distribution.

TABLE II

Tuning Constants for 95% Asymptotic
Efficiency at the Gaussian Distribution

Weight Function	A	B	C	F	H	L	T	W
Tuning Constant	1.339	4.685	2.385	1.400	1.345	1.205	2.795	2.985

Bickel (1975) has shown that the same asymptotic formulas apply
in most cases to estimators which take one Newton step (4) or one
Huber step (5) from a consistent starting value such as (7). These
results do not appear to apply to one IRLS step (6) and we would
expect these one-step estimators to be less efficient than the Newton
one-step. The T estimator is a special case, since $\psi' = w$ and
Table IV (in the next section) shows a high one-step efficiency.

In order to obtain asymptotic efficiencies at the unit Gaussian
we needed to compute integrals of the form

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi^2(x) \, e^{-x^2/2} dx \tag{10}$$

and

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi'(x) \, e^{-x^2/2} \, dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(x) \, xe^{-x^2/2} \, dx. \tag{11}$$

Only the ψ' for T requires some care in evaluating the jump discon-
tinuities and this can be avoided by using the right-hand side of
(11).

The B, H, T, and W estimators have formulas for their asymp-
totic efficiencies which may be expressed in terms of incomplete
chi-square (or Gamma) functions and Gaussian integrals and may,
therefore, be evaluated fairly easily.

The other four estimators--A, C, F, L--required numerical integration in order to compute their asymptotic efficiencies. Step-function approximations were used with a sufficiently fine grid so that the accuracy indicated in Table II was obtained. The Monte Carlo discussed in the next section provides a crude check on these tuning constant values.

MONTE CARLO

A small Monte Carlo study was performed to check the values of the tuning constants given above. For three sample sizes (n = 10, 20 and 40) we obtained Gaussian efficiencies for location estimates based on these w-functions. (The eff_E = $Var(\overline{X})/Var(E)$ where E is the estimate being studied and $\overline{X}$ is the sample mean.) These efficiencies were computed for 1 through 5 iterations of reweighted least-squares using each of the eight w-functions. In these iterations σ was not changed from iteration to iteration to insure convergence.

Table III gives the results of this Monte Carlo study. We see that multiple iterations improve the efficiency of these estimators in the desired direction (i.e., increases them towards the n = ∞ value of 95%) but even after convergence (which is operationally defined here as five iterations) there is still wide disparity among the efficiencies of the eight estimators, and none of these exceed 94% for n = 10. The five iteration data from Table III is graphed in Figure 1. This graph shows the disparity in the efficiencies of the fully iterated estimates for n = 10, 20 and 40. While there is a clear trend for the efficiencies to increase towards the asymptotic value of 95% as n → ∞ , even by n = 40 only one of them (F) is over 94% efficient. The eight curves in Figure 1 group neatly if we examine them as we did earlier by the type of ψ-function that motivates each w-function.

The first impression one gets from Figure 1 is that the w-functions approach 95% efficiency at different rates as n → ∞ . But the other possibility is that they are differentially sensitive to the estimated scale. To test this, we ran a second Monte Carlo

study with $\sigma^2 = 1$ which was the correct variance for the Gaussian
variables used in the Monte Carlo. The results are contained in
Table IV. This allowed us to see the effect of estimating σ
compared to knowing the correct scale. Figure 2 summarizes the
results of that study for the fully iterated estimators (i.e., after
5 iterations). Except for estimator T at n = 10, all of the estima-
tors are very nearly 95% efficient at all values of n. Hence we
conclude that the variation exhibited across the eight estimators in
Figure 1 is primarily due to differential sensitivity to the effect
of estimating the scale. In general, the effect of estimating the
scale has been swept under the rug in previous studies of robust
estimation and perhaps these results will bring attention to the

TABLE III

Gaussian Efficiencies with Estimated Scale

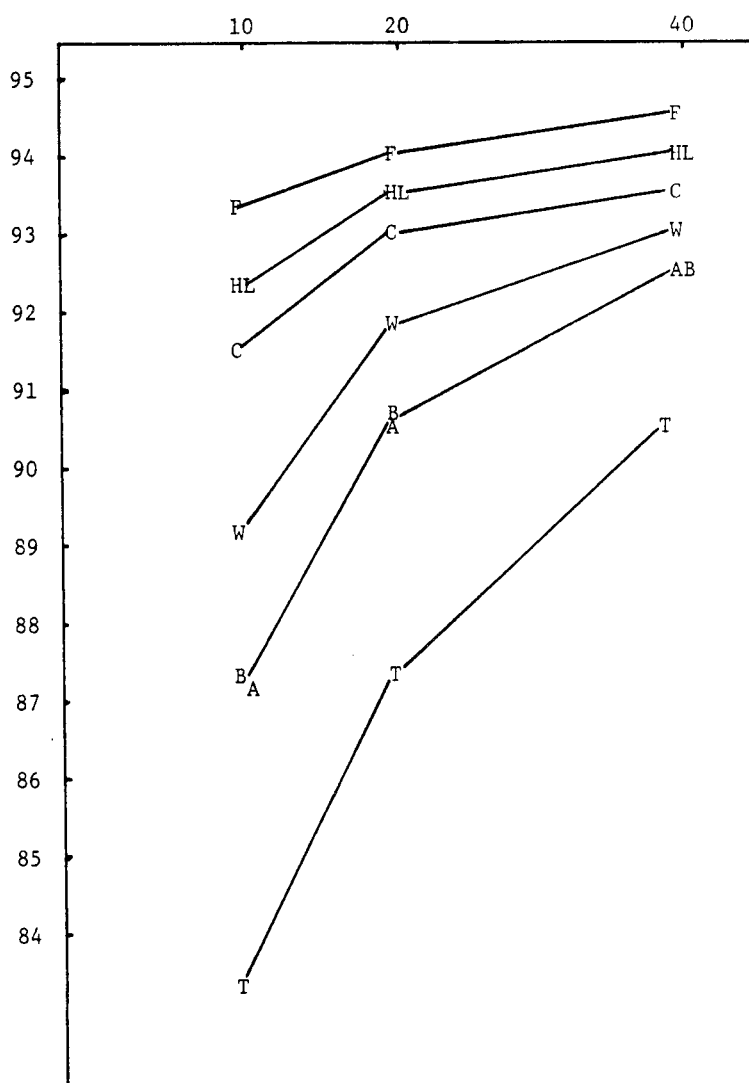| Iteration	Weight Function								
	A	B	C	F	H	L	T	W	M
				n=10 (1000 replications)					
1	85.5	85.7	88.2	87.8	90.3	88.8	83.9	86.8	70.2
2	86.8	87.0	90.8	92.0	91.9	91.7	83.7	88.6	
3	87.0	87.2	91.3	93.1	92.2	92.3	83.7	89.0	
4	87.0	87.3	91.5	93.4	92.3	92.5	83.7	89.1	
5	87.1	87.3	91.5	93.5	92.3	92.5	83.7	89.1	
				n=20 (1500 replications)					
1	88.3	88.4	89.5	88.1	91.4	89.5	87.1	89.0	67.6
2	90.2	90.3	92.4	92.6	93.1	92.7	87.1	91.3	
3	90.5	90.6	92.9	93.7	93.4	93.4	87.1	91.7	
4	90.5	90.7	93.0	94.0	93.4	93.5	87.1	91.8	
5	90.5	90.7	93.1	94.1	93.4	93.5	87.1	91.8	
				n=40 (750 replications)					
1	90.3	90.3	90.3	88.4	92.2	90.2	91.1	90.4	65.8
2	92.3	92.3	93.2	93.0	93.8	93.4	90.6	92.8	
3	92.6	92.6	93.7	94.0	94.0	93.9	90.6	93.1	
4	92.6	92.7	93.8	94.3	94.1	94.0	90.6	93.2	
5	92.6	92.7	93.8	94.4	94.1	94.0	90.6	93.2	

FIG. 1

Gaussian Efficiencies with Estimated Scale (5 Iterations)

TABLE IV

Gaussian Efficiencies with Known Scale

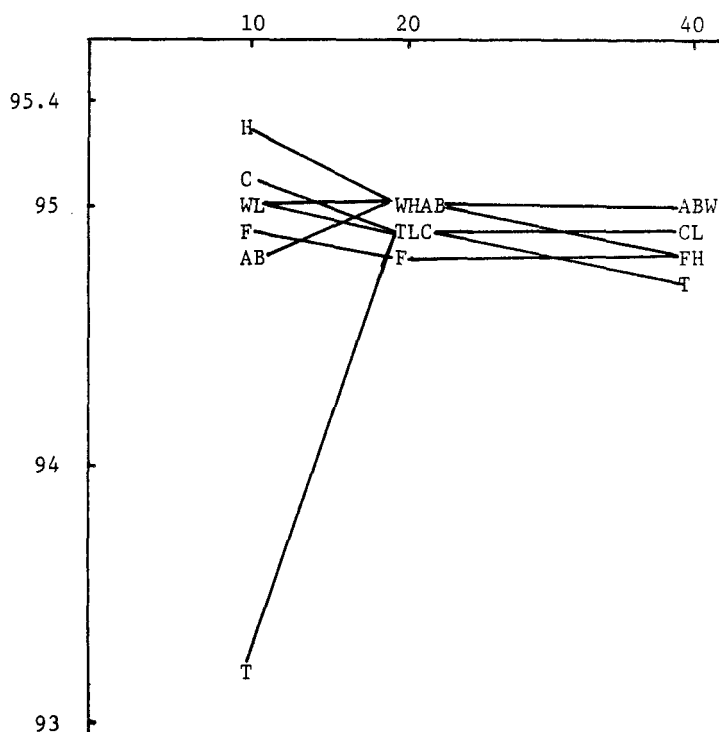| Iteration	Weight Function								
	A	B	C	F	H	L	T	W	M
n=10 (1000 replications)									
1	92.2	92.2	91.6	89.4	93.4	91.3	92.8	92.1	70.2
2	94.4	94.4	94.4	93.5	95.0	94.3	93.2	94.5	
3	94.7	94.7	94.9	94.5	95.2	94.9	93.2	94.9	
4	94.7	94.8	95.0	94.8	95.3	95.0	93.2	95.0	
5	94.8	94.8	95.1	94.9	95.3	95.0	93.2	95.0	
n=20 (1500 replications)									
1	92.3	92.3	91.3	88.9	93.1	90.9	94.7	92.0	67.6
2	94.6	94.6	94.3	93.4	94.8	94.1	94.9	94.5	
3	94.9	94.9	94.8	94.4	94.9	94.7	94.9	94.9	
4	95.0	95.0	94.9	94.7	95.0	94.8	94.9	94.9	
5	95.0	95.0	94.9	94.8	95.0	94.9	94.9	95.0	
n=40 (750 replications)									
1	92.8	92.8	91.6	89.0	93.3	91.2	94.6	92.4	65.8
2	94.8	94.8	94.4	93.6	94.7	94.3	94.7	94.7	
3	95.0	95.0	94.8	94.5	94.8	94.7	94.7	94.9	
4	95.0	95.0	94.9	94.8	94.8	94.8	94.7	95.0	
5	95.0	95.0	94.9	94.8	94.8	94.9	94.7	95.0	

FIG. 2

Gaussian Efficiencies with Known Scale (5 Iterations)

fact that it should be more carefully considered. One implication
might be that each w-function needs it own version of the median
absolute deviations scaling (8) with a multiplier that depends on
the w-function and n.

So far we have only given results at the Gaussian distribution.
We also did a small study of the SLASH distribution which has the
form

$$Y = \frac{G}{U} \qquad\qquad (12)$$

where G is Gaussian (0,1) and U is uniform on (0,1) and independent
of G. Table V gives the variance of each estimator for 1 to 5
iterations. For the SLASH, the median (M) does better than all of

TABLE V

Slash Variances with Estimated Scale

Iteration	Weight Function								
	A	B	C	F	H	L	T	W	M
	n=10 (1000 replications)								
1	7.70	7.70	8.27	9.98	9.51	9.23	9.25	7.76	7.39
2	8.12	8.12	8.78	11.38	10.07	10.01	9.33	8.17	
3	8.24	8.24	8.92	11.90	10.17	10.23	9.34	8.28	
4	8.27	8.26	9.96	12.09	10.19	10.29	9.34	8.31	
5	8.28	8.27	8.97	12.15	10.20	10.31	9.34	8.31	
	n=20 (1000 replications)								
1	6.16	6.16	6.62	8.41	7.82	7.65	7.28	6.19	6.26
2	6.48	6.48	6.96	9.55	8.28	8.25	7.56	6.49	
3	6.57	6.57	7.05	9.94	8.36	8.41	7.56	6.57	
4	6.60	6.59	7.07	10.06	8.37	8.44	7.56	6.59	
5	6.60	6.60	7.07	10.10	8.38	8.45	7.56	6.60	
	n=40 (500 replications)								
1	5.98	5.99	6.74	8.50	7.80	7.72	7.13	6.11	6.63
2	6.20	6.21	7.04	9.55	8.25	8.29	7.38	6.32	
3	6.27	6.28	7.12	9.91	8.33	8.43	7.40	6.38	
4	6.28	6.29	7.14	10.02	8.34	8.47	7.40	6.39	
5	6.29	6.30	7.14	10.05	8.34	8.48	7.40	6.40	

the fully iterated estimators except for A, B and W when n=40.
These tables also illustrate the phenomenon that multiple itera-
tions tend to increase the variance under highly non-Gaussian
error (like the SLASH). Figure 3, graphs the data from the 5
iteration columns of Table V for n=10, 20, 40. It shows that under
the SLASH, the ordering of non-Gaussian variances is in agreement
with the Gaussian efficiencies. Thus except for T, the ordering
of the variances under the SLASH is exactly the opposite of the
ordering of the Gaussian efficiencies in Figure 1. This suggests
that if the differential sensitivity to scale estimation were
removed for these estimators, their behavior under the SLASH would
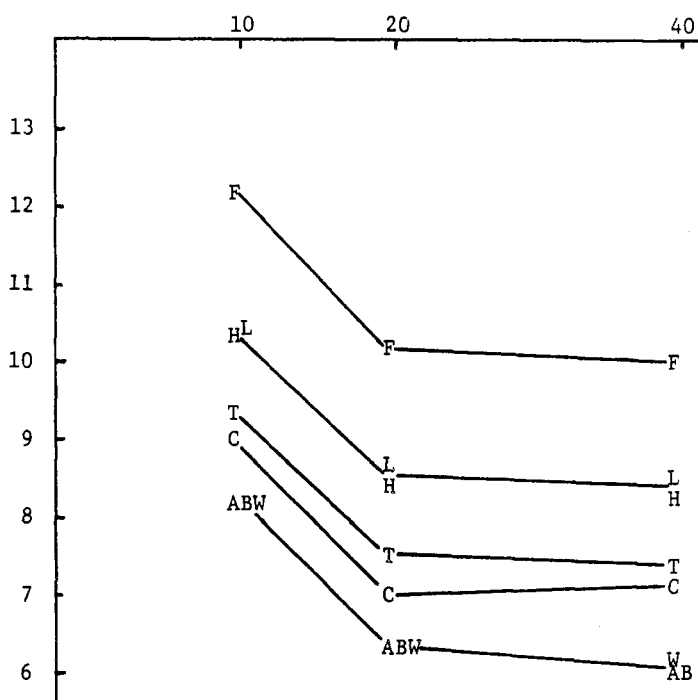be the same. The estimator T is a possible exception and its

FIG. 3

SLASH Variances with Estimated Scale (5 Iterations)

poorer performance under both the Gaussian and the SLASH distribution suggests that a more detailed examination is warranted.

The Gaussian part of this study used the swindles discussed in Holland (1973). The random number generators were obtained from Marsaglia et al. (1975).

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