# Wilson-loop One-point Functions in ABJM Theory

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Joint HEP-TH Seminars

June 7, 2023

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- In may cases, this correspondence is a strong-weak duality.
- So we can use weakly coupled gravity/string theory to compute quantities in strongly coupled gauge theory in the large N limit.
- The quantities includes amplitudes, correlation functions of local operators, vacuum expectation values of loop operators, entanglement entropy...

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- The non-perturbative tools in the field theory side of gauge/gravity correspondence include integrability, supersymmetric localization, bootstrap...
- Integrability makes people be able to compute many quantities in the large N limit, even beyond the BPS sectors.

# Integrability in $AdS_5/CFT_4$

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- Integrability is an important non-pertubative tool in $AdS_5/CFT_4$ correspondence.

# Integrability in $AdS_4/CFT_3$

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- The integrable structure was also found in this $AdS_4/CFT_3$ correspondence. [Minahan, Zarembo, 08][Bak, Rey, 08][Gromov, Vieira, 08]
- Almost every aspect of integrability in this case is more complicated and difficult.

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- Such states also appear in the Wilson-loop/domain wall one-point functions [Kristjansen, Vu, Zarembo, 21] and three-point functions involving two BPS determinant operators and one non-BPS single-trace operator in ABJM theory [Yang, Jiang, Komatsu, **JW**, 21].

## Heisenberg XXX spin chain

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- The Hamiltonian

$$H = J \sum_{j=1}^{L} \left( S_j^x S_{j+1}^x + S_j^y S_{j+1}^y + S_j^z S_{j+1}^z \right) , \tag{2}$$

with periodic boundary condition,

$$S_{L+1}^{\alpha} = S_1^{\alpha}, \; \alpha = x, y, z. \tag{3}$$

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- Here $U = T(0) = Q_1$ is a shift operator.

# IBS for XXX chain

- The definition of IBS [Piroli, Pozsgay, Vernier, 17] for XXX chain is that the state $|B\rangle$ satisfying

$$Q_{2l-1}|B\rangle = 0,\ l = 1, 2, \cdots \tag{5}$$

- This is equivalent to

$$T(u)|B\rangle = T(-u)|B\rangle\,. \tag{6}$$

- For ABJM theory, since there are two sets of conserved charges, the definition of the integrble boundary states are more complicated. (More on this later.)

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- Eigenstates of integrable Hamiltonian can be labelled by Bethe roots, solutions to certain Bethe ansatz equations (BAEs).
- A selection rule for the overlap of an integrable boundary state and a Bethe state: the overlap is nonzero only when the Bethe roots satisfy certain pairing conditions.
- When this selection rule is satisfied, the overlap can often be expressed as a product of super-Gaudin-determinant and a prefactor. Great simplification!

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- This theory should be low energy effective theory of $N$ M2-branes putting at the tip of $\mathbf{C}^4/\mathbf{Z}_k$.

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- Two limits:
  't Hooft limit (planar limit): $N, k \to \infty$, $\lambda \equiv \frac{N}{k}$ fixed;
  M-theory limit: $N \to \infty$, $k$ fixed.

# Holographic dual

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- When $N \gg k^5$, this theory is dual to M-theory on $AdS_4 \times S^7/Z_k$.
- When $k \ll N \ll k^5$, a better description is in terms of IIA superstring theory on $AdS_4 \times \mathbf{CP}^3$.

# Bosonic $1/6$-BPS circular WLs

- We consider the loops along $x^\mu = (R\cos\tau, R\sin\tau, 0), \tau \in [0, 2\pi]$.
- The construction is the following,

$$W^B_{1/6} = \mathrm{Tr}\mathcal{P}\,\exp\left(-i\oint d\tau \mathcal{A}^B_{1/6}(\tau)\right)\,, \tag{7}$$

$$\hat{W}^B_{1/6} = \mathrm{Tr}\mathcal{P}\,\exp\left(-i\oint d\tau \hat{\mathcal{A}}^B_{1/6}(\tau)\right)\,, \tag{8}$$

$$\mathcal{A}^B_{1/6} = A_\mu \dot{x}^\mu + \frac{2\pi}{k} R_I{}^J Y^I Y^\dagger_J |\dot{x}|\,, \tag{9}$$

$$\hat{\mathcal{A}}^B_{1/6} = \hat{A}_\mu \dot{x}^\mu + \frac{2\pi}{k} R^J{}_I Y^\dagger_J Y^I |\dot{x}|\,, \tag{10}$$

with $R^I{}_J = \mathrm{diag}(i, i, -i, -i)$. [Drukker, Plefka, Young, 08][Chen, **JW**, 08][Rey, Suyama, Yamaguchi, 08]

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- We focus a class of fermionic $1/6$-BPS WL with $R$-matrix above replaced by ${U_I}^J = \mathrm{diag}(i, i - 2\bar{\alpha}^1\beta_1, -i, -i)$. (Here and below, we omit the details about the suitable coupling of WLs with fermions.)

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- When $\bar{\alpha}^1\beta_1 = i$, the WL becomes half-BPS and now ${U_I}^J = \mathrm{diag}(i, -i, -i, -i)$.

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- When $C$ is symmetric and traceless, $\mathcal{O}_C$ is a chiral primary operator.
- Here we take $\mathcal{O}_C$ to be a generic local operator which is eigen-operator of the planar two-loop anomalous dimension matrix.

# Wick contraction

- At tree-level, the correlator $\langle W(\mathcal{C})^B_{1/6}\mathcal{O}_C(0)\rangle$ only gets contributions from

$$\oint\cdots\oint d\tau_{1>2>\cdots>L}\left(\frac{2\pi}{k}\right)^L\langle \mathrm{tr}(R^{\tilde{J}_1}{}_{\tilde{I}_1}Y^{\tilde{I}_1}(x_1)Y^\dagger_{\tilde{J}_1}(x_1)\cdots R^{\tilde{J}_L}{}_{\tilde{I}_L}Y^{\tilde{I}_L}(x_L)Y^\dagger_{\tilde{J}_L}(x_L))C^{J_1\cdots J_L}_{I_1\cdots I_L}\mathrm{tr}(Y^{I_1}(0)Y^\dagger_{J_1}(0)\cdots Y^{I_L}(0)Y^\dagger_{J_L}(0))\rangle\,, \quad (11)$$

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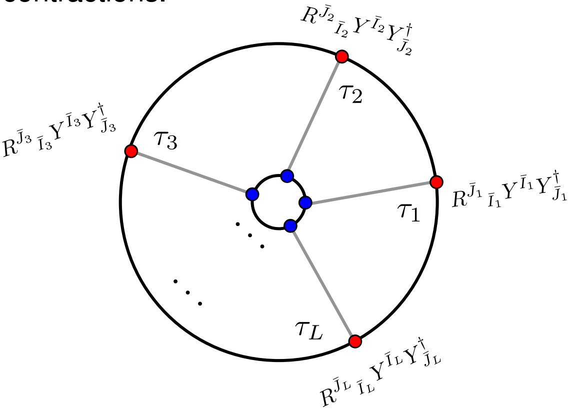
- where $x_i=(R\cos\tau_i,R\sin\tau_i,0)$, $i=1,\cdots,L$, and

$$\oint\cdots\oint d\tau_{1>2>\cdots>L}=\int_0^{2\pi}d\tau_1\int_0^{\tau_1}d\tau_2\cdots\int_0^{\tau_{L-1}}d\tau_L\,. \tag{12}$$

- In the large $N$ limit, we only take into account planar Wick contractions.




Planar Wick contractions between the local operator and the Wilson loop.

# Wick contraction

- One can easily obtain

$$\langle W(\mathcal{C})^{B}_{1/6}\mathcal{O}_C(0)\rangle = \frac{\lambda^{2L}k^{L}}{(L-1)!(2R)^{2L}}C^{J_1\cdots J_L}_{I_1\cdots I_L}R^{I_L}{}_{J_L}\cdots R^{I_1}{}_{J_1}\,, \qquad (13)$$

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- where $\lambda \equiv \frac{N}{k}$ is the 't Hooft coupling of ABJM theory and the tree-level propagators of the scalar fields

$$\langle Y^{I\alpha}{}_{\bar\beta}(x) Y^{\dagger\ \bar\gamma}_{J\ \ \rho}(y)\rangle = \frac{\delta^I_J\delta^\alpha_\rho\delta^{\bar\gamma}_{\bar\beta}}{4\pi|x-y|}\,, \tag{14}$$

have been used.

# Boundary state

- In the spin chain language, we can introduce the following boundary state

$$|\mathcal{B}^B_{1/6}\rangle = |\mathcal{B}_R\rangle\,, \tag{15}$$

where, for a four-dimensional matrix $R$, we define the boundary state $|\mathcal{B}_R\rangle$ as

$$|\mathcal{B}_R\rangle \equiv R^{I_1}{}_{J_1} R^{I_2}{}_{J_2} \cdots R^{I_L}{}_{J_L} |I_1, J_1, \cdots, I_L, J_L\rangle = \left(R^I{}_J |I, J\rangle\right)^{\otimes L}, \tag{16}$$

which is a two-site state.

# Overlap

- Then the above correlation function can be expressed as

$$\langle W(\mathcal{C})^{B}_{1/6}\mathcal{O}_C(0)\rangle = \frac{\lambda^{2L}k^{L}}{(L-1)!(2R)^{2L}}\langle\mathcal{B}^{B}_{1/6}|\mathcal{O}_C\rangle\,, \tag{17}$$

where $|\mathcal{O}_C\rangle$ is the spin chain state corresponding to the operator $\mathcal{O}_C$.

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  where $|\mathcal{O}_C\rangle$ is the spin chain state corresponding to the operator $\mathcal{O}_C$.

- Our convention for the Hermitian conjugation and the overlap of the spin chain states is

$$\left(\langle I_1\bar{J}_1\cdots I_L\bar{J}_L|\right)^\dagger = |I_1\bar{J}_1\cdots I_L\bar{J}_L\rangle\,, \tag{18}$$

$$\langle I_1\bar{J}_1\cdots I_L\bar{J}_L|M_1\bar{N}_1\cdots M_L\bar{N}_L\rangle = \delta_{I_1M_1}\delta^{J_1N_1}\cdots$$
$$\delta_{I_LM_L}\delta_{J_LN_L} \tag{19}$$

# Norm

- Let us define the normalization factor $\mathcal{N}_{\mathcal{O}}$ using the two-point function of $\mathcal{O}$ and $\mathcal{O}^\dagger$ as

$$\langle \mathcal{O}(x)\mathcal{O}^\dagger(y)\rangle = \frac{\mathcal{N}_{\mathcal{O}}}{|x-y|^{2\Delta_{\mathcal{O}}}}\,, \tag{20}$$

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  where $\Delta_{\mathcal{O}}$ is the conformal dimension of $\mathcal{O}$.
- At tree level and the planar limit, we have

$$\mathcal{N}_{\mathcal{O}} = \left(\frac{N}{4\pi}\right)^{2L} L\langle\mathcal{O}|\mathcal{O}\rangle\,. \tag{21}$$

# WL one-point function

- We define the Wilson-loop one-point function as

$$\langle\!\langle \mathcal{O} \rangle\!\rangle_{W(\mathcal{C})} \equiv \frac{\langle W(\mathcal{C})\mathcal{O}\rangle}{\sqrt{\mathcal{N}_{\mathcal{O}}}} \,. \tag{22}$$

- Then for $W^{B}_{1/6}$ we have

$$\langle\!\langle \mathcal{O} \rangle\!\rangle_{W(\mathcal{C})^{B}_{1/6}} = \frac{\pi^L \lambda^L}{R^{2L}(L-1)!\sqrt{L}} \frac{\langle \mathcal{B}^{B}_{1/6}|\mathcal{O}\rangle}{\sqrt{\langle \mathcal{O}|\mathcal{O}\rangle}} \,. \tag{23}$$

- The computation of the Wilson loop one-point function thus amounts to the calculation of

$$\frac{\langle \mathcal{B}^{B}_{1/6}|\mathcal{O}\rangle}{\sqrt{\langle \mathcal{O}|\mathcal{O}\rangle}} \,, \tag{24}$$

which will be done by integrability in some cases.

## ABJM spin chain

- The operator $\mathcal{O}_C = C^{J_1 \cdots J_L}_{I_1 \cdots I_L} \mathrm{Tr}(Y^{I_1} Y^\dagger_{J_1} \cdots Y^{I_L} Y^\dagger_{J_L})$ can be mapped to a state $|C\rangle := C^{J_1 \cdots J_L}_{I_1 \cdots I_L} |I_1 \bar{J}_1 \cdots I_L \bar{J}_L\rangle$ on an alternative closed $SU(4)$ spin chain with length $2L$.

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- The Hibert space of this chain is $\mathbf{C}^{8L} = \otimes_{i=1}^{2L} \mathbf{C}^4$.

# ABJM spin chain

- The operator $\mathcal{O}_C = C^{J_1 \cdots J_L}_{I_1 \cdots I_L} \mathrm{Tr}(Y^{I_1} Y^\dagger_{J_1} \cdots Y^{I_L} Y^\dagger_{J_L})$ can be mapped to a state $|C\rangle := C^{J_1 \cdots J_L}_{I_1 \cdots I_L} |I_1 \bar{J}_1 \cdots I_L \bar{J}_L\rangle$ on an alternative closed $SU(4)$ spin chain with length $2L$.
- The Hibert space of this chain is $\mathbf{C}^{8L} = \otimes_{i=1}^{2L} \mathbf{C}^4$.
- The odd site of the chain is in the $\mathbf{4}$ representation of $SU(4)$, while the even site is in the $\bar{\mathbf{4}}$ representation.

# Hamiltonian

- The planar two-loop anomalous dimensional matrix can be map to the following Hamiltonian on the above chain ([Minahan, Zarembo, 08][Bak, Rey, 08]),

$$\mathbb{H} = \frac{\lambda^2}{2} \sum_{l=1}^{2L} \left(2 - 2\mathrm{P}_{l,l+2} + \mathrm{P}_{l,l+2}\mathrm{K}_{l,l+1} + \mathrm{K}_{l,l+1}\mathrm{P}_{l,l+2}\right) , \qquad (25)$$

where $\mathrm{P}_{ab}$ and $\mathrm{K}_{ab}$ are permutation and trace operators acting on the $a$-th and $b$-th sites. We denote the set of orthonormal basis of the Hilbert space at each site by $|i\rangle$, $i = 1, \cdots, 4$. The two operators act as

$$\mathrm{P}|i\rangle \otimes |j\rangle = |j\rangle \otimes |i\rangle, \qquad \mathrm{K}|i\rangle \otimes |j\rangle = \delta_{ij} \sum_{k=1}^{4} |k\rangle \otimes |k\rangle \,. \qquad (26)$$

# Integrability

- In the algebraic Bethe ansatz (ABA) approach, we introduce the following R-matrices

$$\begin{aligned} R_{12}^{\bullet\bullet}(u) &= R_{12}^{\circ\circ}(u) = u + P_{12} \equiv R_{12}(u)\,, \\ R_{12}^{\bullet\circ}(u) &= R_{12}^{\circ\bullet}(u) = -u - 2 + K_{12} \equiv \bar{R}_{12}(u)\,, \end{aligned} \tag{27}$$

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- where $\bullet$ denotes the states in the $\mathbf{4}$ representation of $SU(4)_R$, while $\circ$ denotes the states in the $\bar{\mathbf{4}}$ representation.
- These $R$-matrices satisfy a set of Yang-Baxter equations and the following crossing symmetry relation,

$$R_{12}(u)^{t_1} = \bar{R}_{12}(-u-2), \qquad \bar{R}_{12}(u)^{t_1} = R_{12}(-u-2)\,. \tag{28}$$

# Integrability

- Using these $R$-matrices one can constructed two transfer matrices $\tau(u)$ and $\bar{\tau}(u)$, satisfying

$$[\tau(u), \tau(v)] = [\tau(u), \bar{\tau}(v)] = [\bar{\tau}(u), \bar{\tau}(v)] = 0\,. \tag{29}$$

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- They are generating functions of commuting conserved charges, among whom there is the Hamiltonian.
- This proves the integrability of two-loop ABJM spin chain. [Minahan, Zarembo, 08][Bak, Rey, 08]

# Bethe roots

- Eigenstates of $\mathbb{H}$ can be constructed using $R$-matrices and the states are parameterized by three set of Bethe roots,

$$u_1, \cdots, u_{K_{\mathbf{u}}}\,, \tag{30}$$

$$v_1, \cdots, v_{K_{\mathbf{v}}}\,, \tag{31}$$

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- One selection rule for $\langle \mathcal{B}_R | \mathbf{u}, \mathbf{v}, \mathbf{w} \rangle$ being nonzero is that $K_{\mathbf{u}} = K_{\mathbf{v}} = K_{\mathbf{w}} = L$.

# Bethe ansatz equations

- These Bethe roots should satisfy the following Bethe ansatz equations,

$$1 = \left(\frac{u_j + \frac{i}{2}}{u_j - \frac{i}{2}}\right)^L \prod_{\substack{k=1\\k\neq j}}^{K_\mathrm{u}} S\left(u_j, u_k\right) \prod_{k=1}^{K_\mathrm{w}} \tilde{S}\left(u_j, w_k\right), \tag{33}$$

$$1 = \prod_{\substack{k=1\\k\neq j}}^{K_\mathrm{w}} S\left(w_j, w_k\right) \prod_{k=1}^{K_\mathrm{u}} \tilde{S}\left(w_j, u_k\right) \prod_{k=1}^{K_\mathrm{v}} \tilde{S}\left(w_j, v_k\right), \tag{34}$$

$$1 = \left(\frac{v_j + \frac{i}{2}}{v_j - \frac{i}{2}}\right)^L \prod_{\substack{k=1\\k\neq j}}^{K_\mathrm{v}} S\left(v_j, v_k\right) \prod_{k=1}^{K_\mathrm{w}} \tilde{S}\left(v_j, w_k\right), \tag{35}$$

## Bethe ansatz equations

- In the previous page, the S-matrices $S(u,v)$ and $\tilde{S}(u,v)$ are given by

$$S(u,v) \equiv \frac{u-v-i}{u-v+i}, \quad \tilde{S}(u,v) \equiv \frac{u-v+\frac{i}{2}}{u-v-\frac{i}{2}}. \tag{36}$$

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- The eigenvalues of $\tau(u), \bar{\tau}(u), \mathbb{H}$ on the Bethe state $|\mathbf{u},\mathbf{v},\mathbf{w}\rangle$ can be expressed in terms of the Bethe roots, $\mathbf{u},\mathbf{v},\mathbf{w}$.

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### Theorem

*If a four-dimensional matrix $K(u)$ satisfies the following boundary Yang-Baxter equation,*

$$R_{12}(u-v)K_1(u)R_{12}(u+v)K_2(v) = K_2(v)R_{12}(u+v) \\ K_1(u)R_{12}(u-v)\,, \tag{38}$$

*the boundary state*

$$|\mathcal{B}_M\rangle \equiv M^{I_1}{}_{J_1} M^{I_2}{}_{J_2} \cdots M^{I_L}{}_{J_L} |I_1, J_1, \cdots, I_L, J_L\rangle = \left(M^I{}_J |I, J\rangle\right)^{\otimes L}\,, \tag{39}$$

*with $M = K(-1)$ is integrable in the sense explained in the next page.*

-

# A key selection rule

- When the condition of the theorem is satisfied, we have that $|\mathcal{B}_M\rangle$ satisfying the following untwisted integrable condition,

$$\tau(-u-2)|\mathcal{B}_M\rangle = \tau(u)|\mathcal{B}_M\rangle. \tag{40}$$

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- This leads to the pairing condition which states that $\langle\mathcal{B}_M|\mathbf{u},\mathbf{v},\mathbf{w}\rangle$ is non-zero only when the selection rule

$$\mathbf{u} = -\mathbf{v}\,, \qquad \mathbf{w} = -\mathbf{w} \tag{41}$$

is satisfied.

# IBS from WLs

- Using this theorem, we can prove that the boundary state from bosonic $1/6$-BPS Wilson loop, $|\mathcal{B}_R\rangle$ is integrable.
- We just take $K(u) = R$. (Notice this $R$ is the one appearing in the definition of $|\mathcal{B}_R\rangle$, it is not the $R$-matrices in the ABA approach. )
- Similarly we proved that the half-BPS and one-third BPS WLs give integrable boundary state.

# Non-integrable boundary states

- For the boundary state from a generic$(^{*})$ fermionic $1/6$-BPS WL, we found numerically a Bethe state whose Bethe roots do not satisfy the pairing condition, but has non-zero overlap.

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# Non-integrable boundary states

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- This shows that this boundary state is not integrable.
- $*$ By ‘generic’, we mean that this WL is neither half-BPS nor bosonic $1/6$-BPS.

# Overlaps

- We obtained the following formula for the normalized overlap between $|\mathcal{B}_R\rangle$ and a Bethe state,

$$\frac{|\langle \mathcal{B}_R|\mathbf{u},\mathbf{v},\mathbf{w}\rangle|^2}{\langle \mathbf{u},\mathbf{v},\mathbf{w}|\mathbf{u},\mathbf{v},\mathbf{w}\rangle} = \prod_{i=1}^{K_{\mathbf{w}}/2} \frac{w_i^2}{w_i^2+1/4} \times \frac{\det G^+}{\det G^-}. \tag{42}$$

- Here the Bethe roots satisfy the pairing condition, $G^\pm$ are Gaudin determinants depending on $\mathbf{u},\mathbf{v},\mathbf{w}$.
- This result was obtained using [Gombor, Bajnok, 20][Gombor, Kristjansen, 22] and passed non-trivial checks based on numerical computations.

# Summary

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# Summary

- By studying WL one-point function at tree level, we found that bosonic $1/6$-BPS, half-BPS and $1/3$-BPS WLs lead to integrable boundary states.
- For generic fermionic $1/6$-BPS WLs, the corresponding boundary states are not integrable.
- We computed the norm of the overlap of some integrable boundary states from WLs and the Bethe states.

# Outlook

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- Finite size effects from TBA?
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- Correlators of BPS WLs and CPOs from localization and/or holography?

# Thanks for Your Attention !