MTHS24 – Exercise sheet 6

Morning: Mikhail Mikhasenko / Sergi Gonzalez-Solis
Afternoon: Gloria Montana, Dhruvanshu Parmar

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Lecture material

Discussed topics:

- Lippmann-Schwinger equation, Bethe-Salpeter equation, and K-matrix
- Lineshape analysis and Briet-Wigner formula
- Complex algebra, dispersion relations
- Analytic continuation and pole search
- Khury-Treiman equations

References:

- A.D. Martin, T.D. Spearman, Elementary Particle Theory, inSpire
- Review on Novel approaches in hadron spectroscopy by JPAC, inspire

Exercices

6.1 Escape room in the complex plane

(a) Characterize the complex structure of functions $\sqrt{x}$ and $\log(-x)$ by finding the branch points, branch cuts and number of complex (Riemann) sheets in the complex plane.

(b) Repeat (a) for a function $f(x) = \sqrt{x} - \sqrt{x-1}$.

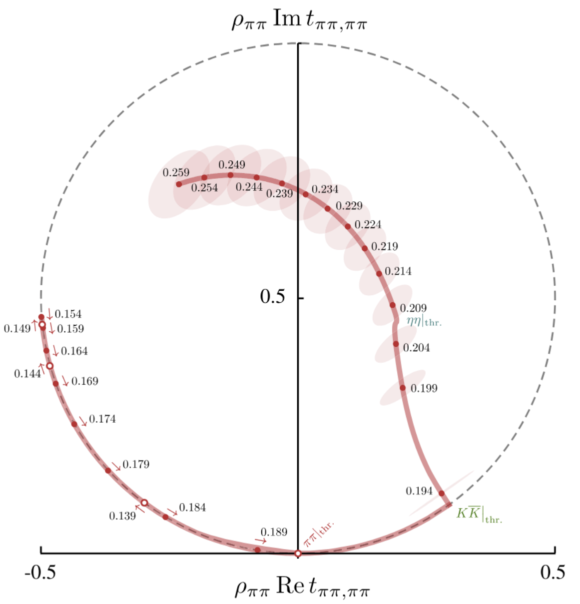
(c) Construct a complex function with two branch points at +i and -i connected by a branch cut.

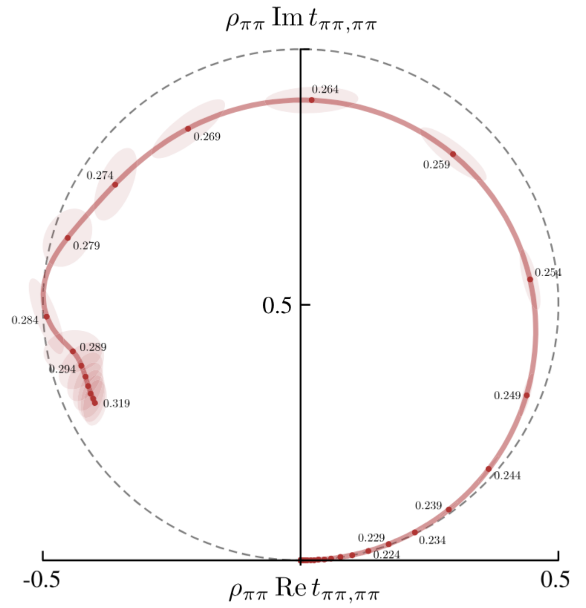
(d) Locate zeros of the function $g(z) = \sqrt{z} + i + 1$.

(e) Find residue of the function $1/g(z)$ by computing a circular integral about the complex pole.

6.2 Argand diagrams from lattice

The $\pi\pi$ scattering with unphysical pion mass ($m_\pi = 391\,\text{MeV}$) for S (left) and D (right) partial waves is studied using lattice calculations. Scattering amplitudes are presented on the Argand diagrams (parametric plot of energy in Real/Imaginary coordinates) as a function of energy of the system. The value are given units of $E_{\text{cm}} \cdot t$ where $t \cdot m_\pi = 0.06906$.





Using information on the diagrams, answer the following questions:

(a) Estimate masses of K and η particles.

(b) Find the elastic energy region for the S and D waves.

(c) Locate the energy value for which the S-wave peak.

(d) Estimate the mass and decay width for the D wave resonance.

(e) Sketch the amplitude phase versus energy of the system for both partial waves.

6.3 Cartesian tensors

Cartesian tensors are tensors in three-dimensional Euclidean space. The available tensors are:

- Rank 0: 1
- Rank 1: k^i
- Rank 2: $k^i k^j$, $delta^{ij}$, $\epsilon^{ijl} k_l$
- Rank 3: ϵ^{ijl}
- Rank 4: combinations of all the above

(a) Show that $\epsilon^{ijl} k_j k_l$ is not a rank 1 tensor.

(b) Show that $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$, using the following property: $(\mathbf{A} \times \mathbf{B})^i = \epsilon^{ijl} A_j B_i$
Hint:

$$\epsilon_{ijl}\epsilon_{ij'l'} = \delta_{jj'}\delta_{ll'} - \delta_{jl'}\delta_{j'l} \tag{1}$$

(c) Given $\int d^3k\, f(k)\, k^i = 0$ for a scalar function $f(k)$. Show that:

$$\int d^3k\, f(k)\, k^i k^j = \frac{1}{3}\delta^{ij} \int d^3k\, f(k)\, k^2 \,. \tag{2}$$

(d) Show that

$$\int d^3k\, f(k,\hat{p})\, k^i k^j = \frac{1}{2}\int d^3k\, f\left[k^2 - (k\cdot\hat{p})^2\right]\delta^{ij} + \frac{1}{2}\int d^3k\, f\left[3(k\cdot\hat{p})^2 - k^2\right]\hat{p}^i\hat{p}^j \,. \tag{3}$$

6.4 Photons in Cartesian Basis

Consider photons (or gluons) in Cartesian basis so $a^+_{k_i} = \sum_\lambda a^+_{k_\lambda}\epsilon^i(k_\lambda)\,$.

(a) Show that the "scalar photon ball" is just a scalar state $\gamma\gamma$ that can be written as:

$$|\gamma\gamma;0^+\rangle \propto \int d^3k\, \phi(k)\, a^+_{k_i}a^+_{-k_i}|0\rangle \,, \tag{4}$$

where $\phi(k)$ is the momentum wave function.

(b) Show that:

$$|\gamma\gamma;0^-\rangle \propto \int d^3k\, \phi(k)\, \epsilon_{ijl}k^l a^+_{k_i}a^+_{-k_j}|0\rangle \,. \tag{5}$$

(c) Prove the Lee-Yang theorem which states that one cannot construct a $J=1$, $\gamma\gamma$ state.

(d) Show that:

$$|\gamma\gamma\gamma;0^-\rangle = \int d^3k_1 d^3k_2 d^3k_3\, \phi(k_1k_2k_3)\, \epsilon_{i_1i_2i_3}\delta(k_1k_2k_3)a^+_{k_1i_1}a^+_{k_2i_2}a^+_{k_3i_3}|0\rangle \,, \tag{6}$$

is a viable state.

(e) Can we construct a $|\gamma\gamma\gamma;1^-\rangle$ state? (**Hint:** $Pa^+_{k_i}P^+ = -a^+_{-k_i}$)