



MTHS24 – Exercise sheet 6

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Lecture material

Discussed topics:

- Lippmann-Schwinger equation, Bethe-Salpeter equation, and K-matrix
- Lineshape analysis and Breit-Wigner formula
- Complex algebra, dispersion relations
- Analytic continuation and pole search
- Khury-Treiman equations

References:

- A.D. Martin, T.D. Spearman, Elementary Particle Theory, [inSpire](#)
- Review on Novel approaches in hadron spectroscopy by JPAC, [inspire](#)

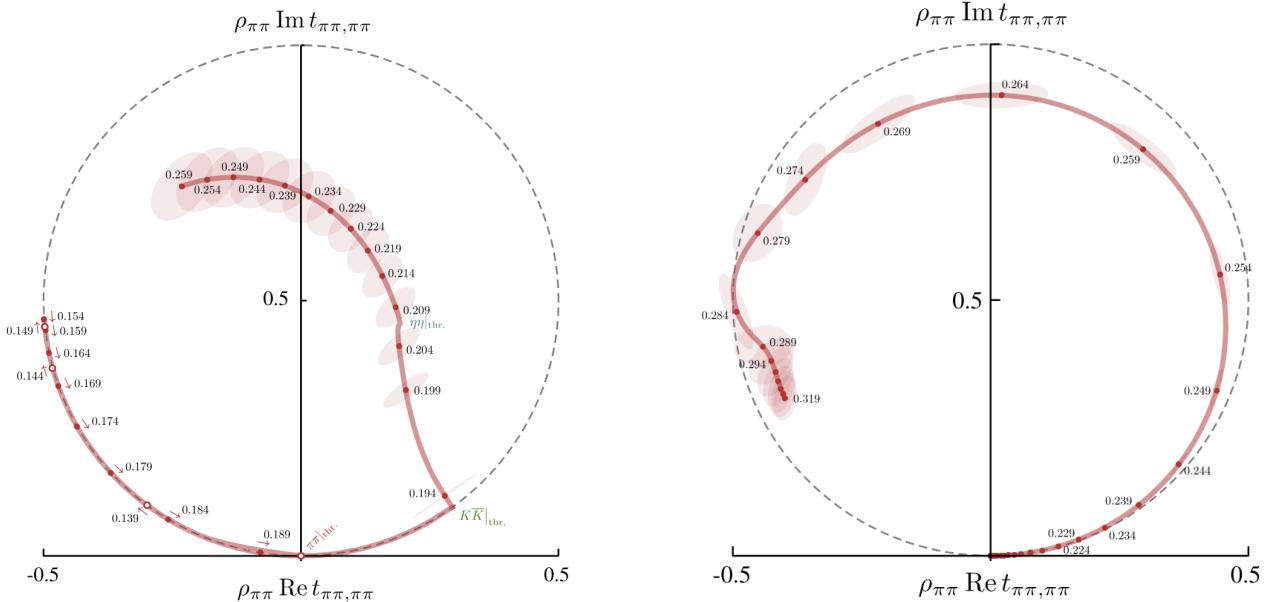
Exercises

6.1 Escape room in the complex plane

- (a) Characterize the complex structure of functions \sqrt{x} and $\log(-x)$ by finding the branch points, branch cuts and number of complex (Riemann) sheets in the complex plane.
- (b) Repeat (a) for a function $f(x) = \sqrt{x} - \sqrt{x-1}$.
- (c) Construct a complex function with two branch points at $+i$ and $-i$ connected by a branch cut.
- (d) Locate zeros of the function $g(z) = \sqrt{z} + i + 1$.
- (e) Find residue of the function $1/g(z)$ by computing a circular integral about the complex pole.

6.2 Argand diagrams from lattice

The $\pi\pi$ scattering with unphysical pion mass ($m_\pi = 391$ MeV) for S (left) and D (right) partial waves is studied using [lattice calculations](#). Scattering amplitudes are presented on the Argand diagrams (parametric plot of energy in Real/Imaginary coordinates) as a function of energy of the system. The values are given in units of $E_{\text{cm}} \cdot t$ where $t \cdot m_\pi = 0.06906$.



Using information on the diagrams, answer the following questions:

- Estimate masses of K and η particles.
- Find the elastic energy region for the S and D waves.

Solution: Elastic region is defined as the range of energy values for which $\pi\pi \rightarrow \pi\pi$ process dominates. For the S-wave, the elastic region lies for the $E_{\text{cm}t}$ range of [0.139, 0.189] (after this point, the amplitude hits the $K\bar{K}$ threshold). Also, after this point, the curve starts going inside the unitarity circle). For the D-wave, this region exists until value of 0.229.

- Locate the energy value for which the S-wave peak.

Solution: The S-wave peaks at the point $E_{\text{cm}t} = 0.154$, which is at energy $E_{\text{cm}} = 0.872$ GeV.

- Estimate the mass and decay width for the D wave resonance.

Solution: The D-wave resonance is observed at $E_{\text{cm}t} = 0.284$, this is the point where there is a kink in the argand diagram. Note : I can locate where the resonance is. I am confused about how to proceed from there, because I can get center of mass energy from the point, and maybe equate it to the pole value. But I would expect a complex output but I cannot read it out properly from the Argand diagram.

- Sketch the amplitude phase versus energy of the system for both partial waves.

6.3 Cartesian tensors

Cartesian tensors are tensors in three-dimensional Euclidean space. The available tensors are:

- Rank 0: 1
- Rank 1: k^i
- Rank 2: $k^i k^j$, δ^{ij} , $\epsilon^{ijl} k_l$

- Rank 3: ϵ^{ijl}
 - Rank 4: combinations of all the above
- (a) Show that $\epsilon^{ijl}k_j k_l$ is not a rank 1 tensor.
- (b) Show that $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$, using the following property: $(\mathbf{A} \times \mathbf{B})^i = \epsilon^{ijl} A_j B_l$
Hint:

$$\epsilon_{ijl}\epsilon_{ij'l'} = \delta_{jj'}\delta_{ll'} - \delta_{jl'}\delta_{j'l} \quad (1)$$

(c) Given $\int d^3k f(k) k^i = 0$ for a scalar function $f(k)$. Show that:

$$\int d^3k f(k) k^i k^j = \frac{1}{3} \delta^{ij} \int d^3k f(k) k^2. \quad (2)$$

(d) Show that

$$\int d^3k f(k, \hat{p}) k^i k^j = \frac{1}{2} \int d^3k f [k^2 - (k \cdot \hat{p})^2] \delta^{ij} + \frac{1}{2} \int d^3k f [3(k \cdot \hat{p})^2 - k^2] \hat{p}^i \hat{p}^j. \quad (3)$$

6.4 Photons in Cartesian Basis

Consider photons (or gluons) in Cartesian basis so $a_{k_i}^+ = \sum_\lambda a_{k_\lambda}^+ \epsilon^i(k_\lambda)$.

(a) Show that the “scalar photon ball” is just a scalar state $\gamma\gamma$ that can be written as:

$$|\gamma\gamma; 0^+\rangle \propto \int d^3k \phi(k) a_{k_i}^+ a_{-k_i}^+ |0\rangle, \quad (4)$$

where $\phi(k)$ is the momentum wave function.

(b) Show that:

$$|\gamma\gamma; 0^-\rangle \propto \int d^3k \phi(k) \epsilon_{ijl} k^l a_{k_i}^+ a_{-k_j}^+ |0\rangle. \quad (5)$$

(c) Prove the Lee-Yang theorem which states that one cannot construct a $J = 1$, $\gamma\gamma$ state.

(d) Show that:

$$|\gamma\gamma\gamma; 0^-\rangle = \int d^3k_1 d^3k_2 d^3k_3 \phi(k_1 k_2 k_3) \epsilon_{i_1 i_2 i_3} \delta(k_1 k_2 k_3) a_{k_1 i_1}^+ a_{k_2 i_2}^+ a_{k_3 i_3}^+ |0\rangle, \quad (6)$$

is a viable state.

(e) Can we construct a $|\gamma\gamma\gamma; 1^-\rangle$ state? (**Hint:** $P a_{k_i}^+ P^+ = -a_{-k_i}^+$)