Ruhr-University Bochum — Modern Techniques in Hadron Physics

MTHS24 – Exercise sheet 8

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Lecture material

Discussed topics:

- Regge theory
- High energy scattering
- Complex angular momentum
- Unitarity

References:

- P.D.B. Collins, "An Introduction to Regge Theory and High Energy Physics", Inspire
- V.N. Gribov, ("Blue book") "The theory of complex angular momentum", Inspire
- V.N. Gribov, ("Gold book") "Strong interactions of hadrons at high energies", Inspire
- D. Sivers & J. Yellin, "Review of recent work on narrow resonance models" Inspire

Exercices

8.1 Unitarity and Reggeons

Van Hove proposed a physically intuitive picture of a Reggeon by relating it to Feynman diagrams in the cross-channels. We will explore this picture of Reggeization with a simple model.

(a) **Elementary t-channel exchanges**

Consider the amplitude corresponding to a particle with spin-J and mass m_J exchanged in the t-channel as:

$$A^J(s,t) = i\, g_J\, (q_1^{\mu_1} \dots q_1^{\mu_J})\, \frac{P^J_{\mu_1\dots\mu_J,\nu_1\dots\nu_J}(k)}{m_J^2 - t}\, (q_{\bar{2}}^{\nu_1} \dots q_{\bar{2}}^{\nu_J}) \tag{1}$$

where g_J is a coupling constant with dimension $2-2J$ (i.e., $A^J(s,t)$ is dimensionless) and the projector of spin-J is defined from the polarization tensor of rank-$J \geq 1$ as

$$P^J_{\mu_1\dots\mu_J,\nu_1\dots\nu_J}(k) = \frac{(J+1)}{2} \sum_\lambda \epsilon^{\mu_1\dots\mu_J}(k,\lambda)\, \epsilon^{*\nu_1\dots\nu_J}(k,\lambda)\ . \tag{2}$$

Using the exchange momentum $k = q_1 + q_{\bar{3}} = q_1 - q_3$, calculate the amplitudes corresponding to $J = 0, 1, 2$ exchanges in terms of $t = k^2$, the modulus of 3-momentum and cosine of scattering angle in the t-channel frame, q_t and $\cos\theta_t$ respectively. Use the explicit forms of the projectors:

$$P^0(k^2) = 1 \tag{3}$$

$$P^1_{\mu\nu}(k^2) \equiv \tilde{g}_{\mu\nu} = \frac{k_\mu\, k_\nu}{k^2} - g_{\mu\nu} \tag{4}$$

$$P^2_{\mu\nu\alpha\beta}(k^2) = \frac{3}{4}\,(\tilde{g}_{\mu\alpha}\, \tilde{g}_{\nu\beta} + \tilde{g}_{\mu\beta}\, \tilde{g}_{\nu\alpha}) - \frac{1}{2}\, \tilde{g}_{\mu\nu}\, \tilde{g}_{\alpha\beta}\ , \tag{5}$$

and conjecture a generalization of the amplitude for arbitrary integer J.

Hint: Show that in the t-channel frame, the exchange particle is at rest and therefore $\tilde{g}_{\mu\nu}$ reduces to a δ_{ij} with respect to only spacial momenta.

Solution:

We start by considering the elastic scattering of two identical, spinless particles with 4-momentum q_i and with mass $q_i^2 = m^2$. We define the usual Mandelstam variables

$$s = (q_1 + q_2)^2 = (q_3 + q_4)^2$$
$$t = (q_1 - q_3)^2 = (q_4 - q_2)^2$$
$$u = (q_1 - q_4)^2 = (q_2 - q_3)^2$$

We refer to the s-channel as the physical region describing the process

$$1\,(q_1) + 2\,(q_2) \to 3\,(q_3) + 4\,(q_4)\ ,$$

while in the t-channel we consider

$$1\,(q_1) + \bar{3}\,(q_{\bar{3}}) \to \bar{2}\,(q_{\bar{2}}) + 4\,(q_4)\ .$$

The $J = 0$ is trivial

$$A^0(s,t) = i\,g_0\,\frac{1}{m_0^2 - t} = i\,g_0\,\frac{P_0(\cos\theta_t)}{m_0^2 - t}\ . \tag{6}$$

For $J = 1$ use $q_1 = (\sqrt{t}/2, q_t\,\hat{z})$ and $q_{\bar{3}} = (\sqrt{t}/2, -q_t\,\hat{z})$. In the t-channel CM frame we have $k = (q_1 - q_3) = (q_1 + q_{\bar{3}}) = (\sqrt{t}, \vec{0})$ and

$$-\tilde{g}_{\mu\nu} = -g_{\mu\nu} + \frac{k_\mu\,k_\nu}{t} = -\left[\delta_{\mu 0}\,\delta_{\nu 0} - \delta_{ij}\right] + \frac{\sqrt{t}^2}{t}\,\delta_{\mu 0}\,\delta_{\nu 0} = +\delta_{ij}\ . \tag{7}$$

thus $q_1^\mu\,\tilde{g}_{\mu\nu} q_{\bar{2}}^\nu = \vec{q}_1 \cdot \vec{q}_{\bar{2}} = q_t^2\,\cos\theta_t$. Similarly $q_1^\mu\,\tilde{g}_{\mu\nu} q_1^\nu = q_{\bar{2}}^\mu\,\tilde{g}_{\mu\nu} q_{\bar{2}}^\nu = q_t^2$ and we have

$$A^1(s,t) = i g_1\,q_t^2\,\frac{\cos\theta_t}{m_1^2 - t} = i g_1\,q_t^2\,\frac{P_1(\cos\theta_t)}{m_1^2 - t}\ , \tag{8}$$

and finally also

$$A^2(s,t) = i g_2\,q_t^4\,\frac{\frac{1}{2}(3\cos\theta_t - 1)}{m_2^2 - t} = i g_2\,q_t^4\,\frac{P_2(\cos\theta_t)}{m_2^2 - t}\ . \tag{9}$$

The generalization to arbitrary J is

$$A^J(s,t) = i g_J\,q_t^{2J}\,\frac{P_J(\cos\theta_t)}{m_J^2 - t}\ . \tag{10}$$

(b) **Unitarity vs Elementary exchanges**

Express the amplitude entirely in terms of invariants s and t. Use the optical theorem to relate the elastic amplitude to a total hadronic cross section:

$$\sigma_{\text{tot}} = \frac{1}{2q\sqrt{s}}\,\Im A^J(s, t=0)\ . \tag{11}$$

Unitarity (via the Froissart-Martin bound) prohibits σ_{tot} from growing faster than $\log^2 s$ as $s \to \infty$. What is then the maximal spin a single elementary exchange can have while satisfying this bound? Why is this a problem?

Solution: We have

$$\sigma_{\text{tot}} = \frac{1}{2q\sqrt{s}} \frac{g_J}{m_J^2} q_t^{2J} P_J(\cos\theta_t)\Big|_{t=0} . \tag{12}$$

We have $q_t^2 \cos\theta_t = (s-u)/4$ so that as $s \to \infty$, we have:

$$\sigma_{\text{tot}} \sim s^{J-1} . \tag{13}$$

To satisfy the Froissart bound, the maximally allowed spin then is $J = 1$.

(c) **Van Hove Reggeon**

Consider an amplitude of the form

$$A(s,t) = \sum_{J=0}^{\infty} g\, r^{2J} \frac{(q_t^2 \cos\theta_t)^J}{J - \alpha(t)} . \tag{14}$$

Here $\alpha(t) = \alpha(0) + \alpha'\, t$ is a real, linear Regge trajectory, g is a dimensionless coupling constant and $r \sim 1$ fm is a range parameter. Compare Eq. 14 with Eq. 1, write the mass of the Jth pole, m_J^2, as a function of the Regge parameters $\alpha(0)$ and α'. Interpret the pole structure in terms of the spectrum of particles in the model.

If the sum is truncated to a finite J_{max}, and we take the $s \to \infty$ limit, what is the high energy behavior of the amplitude?

Solution: We can write

$$J - \alpha(t) = J - \alpha(0) - \alpha'\, t = \alpha' \left((J - \alpha(0))/\alpha' - t \right) \tag{15}$$

and thus we have $m_J^2 = (J - \alpha(0))/\alpha'$.
We can use

$$(\cos\theta_t)^J = \sum_{J+J' \text{ even}}^{J} \frac{(J+1)!}{(J-J')!!\,(J+J'+1)!!} P_{J'}(\cos\theta_t) \tag{16}$$

$$= \sum_{J+J' \text{ even}}^{J} \mu_{JJ'}\, P_{J'}(\cos\theta_t) \tag{17}$$

to write

$$A(s,t) = \sum_J \sum_{J'=0}^{J} \left(\frac{g\, r^{2J} \mu_{JJ'}}{\alpha'} \right) q_t^{2J} \frac{P_{J'}(\cos\theta_t)}{m_J^2 - t} , \tag{18}$$

$$= \sum_J \sum_{J'=0}^{J} g_{JJ'}\, q_t^{2J} \frac{P_{J'}(\cos\theta_t)}{m_J^2 - t} , \tag{19}$$

Comparing with the form of our elementary exchanges, this amplitude is an infinite sum of particles with spin-J and mass m_J^2 but also all same parity daughters at the same mass.
If the sum is truncated at J_{max} the $s \to \infty$ limit is dominated by the largest spin exchange and we have $A_{\text{trunc}}(s,t) \propto s^{J_{\text{max}}}$.

(d) **Analytic continuation in J**

Show that if the summation is kept infinite, the amplitude can be re-summed to something that is entirely analytic in s, t, u, and J.

Hint: Use the Mellin transform

$$\frac{1}{J-\alpha(t)} = \int_0^1 dx\, x^{J-\alpha(t)-1} \,, \tag{20}$$

to express the amplitude in terms of the Gaussian hypergeometric function and the Euler Beta function

$$B(b, c-b)\,{}_2F_1(1,b,c;z) = \int_0^1 dx\, \frac{x^{b-1}\,(1-x)^{c-b-1}}{1-x\,z} \,. \tag{21}$$

Solution: Go back to the original form in terms of monomials, we can write

$$A(s,t) = \sum_{J=0} \int_0^1 dx\, g\, r^{2J} (q_t^2\, \cos\theta_t)^J\, x^{J-\alpha(t)-1} \,. \tag{22}$$

Collecting all things with powers of J, we notice a geometric series which can be summed analytically

$$A(s,t) = g \int_0^1 dx\, \frac{x^{-\alpha(t)-1}}{1-r^2\, q_t^2\, \cos\theta_t\, x} \,. \tag{23}$$

Comparing with the definition of the hypergeometric function, we can identify $z = r^2\, q_t^2\, \cos\theta_t$ and $b = -\alpha(t)$. Since there is no $(1-x)$ term we require $c = b+1 = 1-\alpha(t)$. Thus we have

$$\begin{aligned} A(s,t) &= \frac{\Gamma(-\alpha(t))}{\Gamma(1-\alpha(t))}\,{}_2F_1\left(1, -\alpha(t), 1-\alpha(t), (q_t\, r)^2\, \cos\theta_t\right) \qquad (24)\\ &= \Gamma(-\alpha(t))\,{}_2\tilde{F}_1\left(1, -\alpha(t), 1-\alpha(t), (q_t\, r)^2\, \cos\theta_t\right) \,. \qquad (25) \end{aligned}$$

(e) **Unitarity vs Reggeized exchanges**
Revisit **b**) with the resummed amplitude. Take the $s \to \infty$ limit and set a limit on the maximal intercept $\alpha(0)$ which is allowed by unitarity.

Hint: Assume that $\alpha(0) > -1$ and use the asymptotic behavior of the hypergeometric function given by

$${}_2F_1(1,b,c;z) \to \frac{\Gamma(c)\,\Gamma(1-b)}{\Gamma(1)\,\Gamma(c-b)}\,(-z)^{-b} \,. \tag{26}$$

Solution: From the hypergeometric form we can take $s \to \infty$ which takes $q_t^2\, \cos\theta_t = (s-u)/4 \to \infty$ and we can write

$$A(s,t) = g_0\, \Gamma(-\alpha(t))\, \Gamma(1+\alpha(t)) \left(\frac{u-s}{4r^{-2}}\right)^{\alpha(t)} \,. \tag{27}$$

So, we have

$$\Im A(s,0) \propto \Im(-s)^{\alpha(0)} \propto \sin\pi\alpha(0)\, s^{\alpha(0)} \tag{28}$$

so $\sigma_{\text{tot}} \sim s^{\alpha(0)-1}$ and unitarity requires $\alpha(0) \leq 1$.

(f) **The Reggeon "propagator"**
Modify Eq. 1 to have a definite signature by defining

$$A^{\pm}(s,t) = \frac{1}{2}\left[A(s,t) \pm A(u,t)\right] \,. \tag{29}$$

Repeat **d)** and **e)** with this signatured amplitude. Compare with the canonical form of the Reggeon exchange:

$$A^{\pm}_{\mathbb{R}}(s,t) = \beta(t)\, \frac{1}{2}\left[\pm 1 + e^{-i\pi\alpha(t)}\right]\, \Gamma(-\alpha(t)) \left(\frac{s}{s_0}\right)^{\alpha(t)} \,. \tag{30}$$

Identify the Regge residue $\beta(t)$ and characteristic scale s_0 in terms of the parameters g_0 and r.

Solution: As we see above, switching $s \leftrightarrow u$ introduces a minus sign and we have

$$A^{\pm}(s,t) = g_0\,\Gamma(1+\alpha(t))\,\frac{1}{2}[\pm 1 + e^{-i\pi\alpha(t)}]\,\Gamma(-\alpha(t))\left(\frac{s-u}{4r^{-2}}\right)^{\alpha(t)} , \tag{31}$$

and we can read off $\beta(t) = g_0\,\Gamma(1+\alpha(t))$. Using $s \sim -u$ we also see $s_0 = 2\,r^{-2}$

8.2 Veneziano Amplitude

The quintessential dual amplitude was first proposed by Veneziano for $\omega \to 3\pi$ and later applied to elastic $\pi\pi$ scattering by Shapiro and Lovelace. Consider the $\pi^+\pi^-$ scattering amplitude of the form

$$\mathcal{A}(s,t,u) = V(s,t) + V(s,u) - V(t,u) \ . \tag{32}$$

with each

$$V(s,t) = \frac{\Gamma(1-\alpha(s))\,\Gamma(1-\alpha(t))}{\Gamma(1-\alpha(s)-\alpha(t))} , \tag{33}$$

where $\alpha(s) = \alpha(0) + \alpha' s$ is a real, linear Regge trajectory with $\alpha' > 0$.

(a) **Duality**

Show that the function $V(s,t)$ is symmetric in $s \leftrightarrow t$ and dual, i.e., it can be written entirely as a sum of either s-channel poles OR t-channel poles but never both simultaneously. Compare with the Reggeized amplitude in the previous problem, was that amplitude dual?

Hint: Relate $V(s,t)$ to the Euler Beta function

$$B(x,y) = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)} \tag{34}$$

and use the identities $B(x,y) = B(y,x)$ and

$$B(p-x, q-y) = \sum_{J=1}^{\infty} \frac{\Gamma(J-p+1+x)}{\Gamma(J)\,\Gamma(-p+1+x)}\,\frac{1}{J-1+q-y} \ . \tag{35}$$

Solution: We can write:

$$V(s,t) = (1-\alpha(s)-\alpha(t))\,B(1-\alpha(s), 1-\alpha(t)) \ . \tag{36}$$

Then using the expansion of the Beta function on its first argument, we have

$$V(s,t) = (1-\alpha(s)-\alpha(t))\sum_{J=1}^{\infty} \frac{\Gamma(J-1+\alpha(t))}{\Gamma(J)\,\Gamma(\alpha(t))}\,\frac{1}{J-\alpha(s)} \tag{37}$$

which only has poles in $\alpha(s)$. Because of the $s \leftrightarrow t$ symmetry, we can write the exact same expression with only poles in $\alpha(t)$.

(b) **Isospin basis**

Define the s-channel isospin basis through

$$\begin{pmatrix} \mathcal{A}^{(0)}(s,t,u) \\ \mathcal{A}^{(1)}(s,t,u) \\ \mathcal{A}^{(2)}(s,t,u) \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 3 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} \mathcal{A}(s,t,u) \\ \mathcal{A}(t,s,u) \\ \mathcal{A}(u,t,s) \end{pmatrix} \ . \tag{38}$$

Write down the definite-isospin amplitudes in terms of V's. Comment on the symmetry properties of each isospin amplitude with respect to $t \leftrightarrow u$.

Solution: We have:

$$\mathcal{A}^{(0)}(s,t,u) = \frac{1}{2}\left[3\,V(s,t) + 3\,V(s,u) - V(t,u)\right] \tag{39}$$

$$\mathcal{A}^{(1)}(s,t,u) = V(s,t) - V(s,u) \tag{40}$$

$$\mathcal{A}^{(2)}(s,t,u) = V(t,u)\ . \tag{41}$$

$I = 0, 2$ are symmetric in $t \leftrightarrow u$ while $I = 1$ is anti-symmetric as required by Bose symmetry.

(c) **Chew-Frautshi plot**
Locate where each $\mathcal{A}^{(I)}(s,t,u)$ will have poles in the s-channel physical region. What is their residue? Draw a schematic Chew-Frautschi plot of the resonance spectrum in each isospin channel.

Solution: A single $V(s,t)$ will have poles at all $\alpha(s) = J \geq 1$ and all possible daughters. The residues are

$$-(J-1+\alpha(t))\,\frac{\Gamma(J-1+\alpha(t))}{\Gamma(J)\,\Gamma(\alpha(t))} = \frac{-1}{\Gamma(J)}\,\frac{\Gamma(J+\alpha(t))}{\Gamma(\alpha(t))} = -\frac{(\alpha(t))_J}{(J-1)!}. \tag{42}$$

For a linear trajectory this is a order J polynomial in t and therefore in z_s.
The symmetry factors in $I = 0, 1$ will remove all odd (even) J daughters. The $I = 2$ amplitude has no s dependence and therefore no isospin-2 poles at all.

(d) **Regge limit**
Now consider the limit $t \to \infty$ and $u \to -\infty$ with $s \leq 0$ is fixed. What is the asymptotic behavior of $V(s,t)$ and $V(s,u)$? Assume that $V(t,u)$ vanishes faster than any power of s in this limit. What is the resulting behavior of the isospin amplitudes $\mathcal{A}^{(I)}(s,t,u)$ in this limit?

Hint: Use the Sterling approximation of the Γ function., i.e. as $|x| \to \infty$

$$\Gamma(x) \to \sqrt{\frac{2\pi}{x}}\left(\frac{x}{e}\right)^x\ . \tag{43}$$

Solution: Starting with

$$V(s,t) \to \Gamma(1-\alpha(s))\,(-\alpha(t))^{\alpha(s)} \sim \Gamma(1-\alpha(s))\,\left(-\alpha'\,t\right)^{\alpha(s)}\ . \tag{44}$$

Similarly

$$V(s,u) \to \Gamma(1-\alpha(s))\,(-\alpha(u))^{\alpha(s)} \sim \Gamma(1-\alpha(s))\,\left(-\alpha'\,t\right)^{\alpha(s)}\ . \tag{45}$$

Thus the combination

$$V(s,t) \pm V(s,u) \to \Gamma(1-\alpha(s)) \times \left[\left(-\alpha'\,t\right)^{\alpha(s)} \pm \left(-\alpha'\,u\right)^{\alpha(s)}\right] \tag{46}$$

$$= \Gamma(1-\alpha(s)) \times \left[1 \pm e^{-i\pi\alpha(s)}\right]\,\left(\alpha'\,t\right)^{\alpha(s)} \tag{47}$$

(e) **Ancestors and Strings**
Consider the model now with a complex trajectory $\alpha(s) = a_0 + \alpha'\,s + i\,\Gamma$ with $\Gamma > 0$ to move the poles off the real axis. Reexamine the the Chew-Frautshi plot for the $I = 1$ amplitude using this trajectory, why is the resulting spectrum problematic? Try a real but non-linear trajectory, say $\alpha(s) = \alpha_0 + \alpha'\,s + \alpha''\,s^2$, what is the spectrum like now?

Compare the requirements of the trajectory for $V(s,t)$ to make sense with the energy levels of a rotating relativistic string with a string tension T:

$$E_J^2 = \frac{1}{2\pi\,T}\,J\ . \tag{48}$$

What is a possible microscopic picture of hadrons if the Veneziano amplitude is believed?

Solution: If we allow $\alpha(t)$ to be complex, then at a pole $\alpha(s) \to J + i\Gamma$ and the residue we calculated

$$\frac{\Gamma(J + i\,\Gamma + \alpha(t))}{\Gamma(\alpha(t))}\,, \tag{49}$$

is no longer a fixed order polynomial in t. It will thus give contributions to ALL spins at each pole, i.e. introduce an infinite number of ancestors. Similarly if $\alpha(s)$ is non-linear, we will have finitely many ancestors but still unphysical poles nonetheless.
This means the Veneziano amplitude *only* gives a physical picture for real and linear trajectories. This means we require $J \propto s \sim m^2$ which mimics the spectrum of states in a relativistic rotating string. This gives rise to the stringy picture of a $q\bar{q}$ pair connected by a gluon flux tube and later the entire field of string theories.

8.3 Sommerfeld-Watson Transform

(a) **Geometric series**
Prove the well known resummation of the geometric series:

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x} \quad \text{for } |x| < 1 \tag{50}$$

can be analytically continued to $|x| \geq 1$ with the Sommerfeld-Watson Transform.

Assume that $|x| > 1$ and show that the summation can be written as an integral over the complex plane

$$\int \frac{d\ell}{2i} \frac{(-x)^\ell}{\sin \pi\ell} = 1 + x + x^2 + \dots\ . \tag{51}$$

Draw the contour around which the above integration should be taken (careful with orientations and signs). Deform the contour such that you can relate Eq. 51 to the series

$$\frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} + \cdots = \frac{1}{1 - \frac{1}{x}} \tag{52}$$

and arrive at Eq. 50.

Solution: We want to show that we can analytically continue the geometric series to $|x| > 1$ using the Sommerfeld-Watson Transform.
First, we will prove that the sum can be written as an integral over the complex plane:

$$\int_C \frac{d\ell}{2i} \frac{(-x)^\ell}{\sin \pi\ell} , \tag{53}$$

where the function $1/\sin \pi\ell$ has poles at integer values of $\ell = \ldots, -2, -1, 0, 1, 2, \ldots$ We use the Cauchy Residue Theorem:

$$\oint f(z)dz = \pm 2\pi i \sum_k \mathrm{Res}_{z=z_k} f(z) , \tag{54}$$

with sign $+$ for a counterclockwise contour and $-$ for a clockwise contour around the pole at z_k. The residue of $f(\ell) = (-x)^\ell / \sin \pi\ell$ at $\ell = k$ is given by

$$\mathrm{Res}_{\ell=k} \frac{(-x)^\ell}{\sin \pi\ell} = \lim_{\ell \to k} (\ell - k) \frac{(-x)^\ell}{\sin \pi\ell} = \lim_{\ell \to k} \frac{(-x)^\ell}{\pi \cos \pi\ell} = \frac{x^k}{\pi} \tag{55}$$

Therefore, if we encircle all the poles at $\ell = k$ for $k \geq 0$ with counterclockwise contours C_k, which we can combine to a single counterclockwise contour C encircling all the poles at 0 and positive integers, we obtain the geometric series:

$$\sum_{k=0}^{\infty} \int_{C_k} \frac{d\ell}{2i} \frac{(-x)^\ell}{\sin \pi\ell} = \int_C \frac{d\ell}{2i} \frac{(-x)^\ell}{\sin \pi\ell} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \cdots \tag{56}$$

Next, we deform the contour to a vertical line from $\sigma + i\infty$ to $\sigma - i\infty$, with $-1 < \sigma < 0$, and further deform it to enclose all negative integers in a clockwise contour C', that we can split in individual clockwise contours $C_{k'}$:

$$\begin{aligned} \int_{C'} \frac{d\ell}{2i} \frac{(-x)^\ell}{\sin \pi\ell} &= \sum_{C_{k'}} \int_{C_{k'}} \frac{d\ell}{2i} \frac{(-x)^\ell}{\sin \pi\ell} = -\frac{1}{x} - \frac{1}{x^2} - \frac{1}{x^3} - \cdots \\ &= -\frac{1}{x} \left(1 + \frac{1}{x} + \frac{1}{x^2} + \cdots \right) \end{aligned} \tag{57}$$

and given the assumption $|x| > 0$, we have $|1/x| < 1$, and we can sum the geometric series:

$$-\frac{1}{x} \left(1 + \frac{1}{x} + \frac{1}{x^2} + \cdots \right) = -\frac{1}{x} \frac{1}{1 - \frac{1}{x}} = \frac{1}{1 - x} . \tag{58}$$

(b) **Van Hove Reggeon**
Revisit the Regge behavior of Eq. 14 using the S-W transform. How does the inclusion of poles at $\alpha(s) = \ell$ change the contour of integration and the leading contribution to the asymptotic behavior?

Solution: If we include a Regge pole $[\ell - \alpha(t)]^{-1}$, in the process of deforming the contour from C to C' we have to pick the residue of the pole at $\ell = \alpha(t)$, with $\mathrm{Re}\,\alpha(t) < 0$. This means we have an extra clockwise contour C_α, from which we obtain an extra contribution $\propto x^{\alpha(t)}$. Since $x \sim q_t^2\, z_t \sim s$ this yields the $s^{\alpha(t)}$ behavior.

8.4 Finite Energy Sum Rules

Consider z a complex variable and α a real fixed parameter. What is the analytic structure of the function z^α? What is the discontinuity across the cut?

Write a Cauchy contour C surrounding the cut and closing it with a circle of radius Λ in the complex z

plane, and check that

$$\oint_C z^\alpha \mathrm{d}z = 0 \tag{59}$$

You can start with the simple case $\alpha = 1/2$, *i.e.* $\sqrt{z}$, then generalize to any real α.

Solution: The function z^α has a branch cut for $z \in [-\infty, 0]$. The Cauchy contour enclose the cut and the discontinuity across that cut is

$$\begin{aligned}
(z+i\epsilon)^\alpha - (z-i\epsilon)^\alpha &= (|z|e^{i\pi})^\alpha - (|z|e^{-i\pi})^\alpha \qquad \text{for real negative} z && (60)\\
&= |z|^\alpha \left(e^{i\pi\alpha} - e^{-i\pi\alpha}\right) && (61)\\
&= 2i|z|^\alpha \sin\pi\alpha && (62)
\end{aligned}$$

The Cauchy contour is then, with C_Λ being the circle of radius Λ in the positive sense,

$$\begin{aligned}
\oint z^\alpha \mathrm{d}z &= \int_{-\Lambda}^{0} (z+i\epsilon)^\alpha \mathrm{d}z + \int_{0}^{-\Lambda} (z-i\epsilon)^\alpha \mathrm{d}z + \oint_{C_\Lambda} z^\alpha \mathrm{d}z && (63)\\
&= 2i\sin\pi\alpha \int_{-\Lambda}^{0} |z|^\alpha \mathrm{d}z + \oint_{C_\Lambda} z^\alpha \mathrm{d}z && (64)
\end{aligned}$$

The first integral is easily done

$$2i\sin\pi\alpha \int_{-\Lambda}^{0} |z|^\alpha \mathrm{d}z = \frac{2i\Lambda^{\alpha+1}}{\alpha+1}\sin\pi\alpha. \tag{65}$$

For the second integral, we need the change of variable $z = \Lambda\exp(i\theta)$, with $\theta \in [-\pi, \pi]$. We obtain

$$\begin{aligned}
\oint_{C_\Lambda} z^\alpha \mathrm{d}z = i\Lambda^{\alpha+1} \int_{-\pi}^{\pi} e^{i\theta(\alpha+1)} \mathrm{d}\theta &= \frac{i\Lambda^{\alpha+1}}{\alpha+1}\left(e^{i\pi(\alpha+1)} - e^{-i\pi(\alpha+1)}\right) && (66)\\
&= -\frac{2i\Lambda^{\alpha+1}}{\alpha+1}\sin\pi\alpha && (67)
\end{aligned}$$

We used $\exp(i\pi) = -1$.