Ruhr-University Bochum — Modern Techniques in Hadron Physics

MTHS24 – Exercise sheet 11

Morning: Christian Fischer

Afternoon:

Friday, 26 July 2024

Lecture material

Discussed topics:

- Functional methods
- Dynamical Chiral Symmetry Breaking
- Spectra of conventional and exotic hadrons
- (optional: g-2, form factors,...)

References:

- Eichmann et al., "Baryons as relativistic three-quark bound states," PPNP **91** (2016), 1-100 arXiv:1606.09602 [hep-ph]].
- Eichmann et al. "Four-Quark States from Functional Methods," FBS **61** (2020) no.4, 38 arXiv:2008.10240 [hep-ph].
- Michele Maggiore, "Modern Introduction to Quantum Field Theory", Oxford University Press

Exercices

11.1 Diquarks

Write down spin, color and flavour wave functions for a scalar and an axialvector diquark built from

(a) two light quarks (what is the resulting isospin ?)

(b) two strange, charm or bottom quarks

(c) a heavy-(not-so-heavy) combination such as bc, bs or cs.

Hint: carefully think about symmetries...

Solution: **I.) Scalar diquarks:**
Spin S=0 $\rightarrow \frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)$ and antisymmetric
Color: From Young-Tableaux we find $3 \bigotimes 3 = 6 \bigoplus \bar{3}$ and $\bar{3}$ is antisymmetric, while 6 is symmetric.
Thus we need an antisymmetric flavour wave function together with $\bar{3}$-color and a symmetric flavour wave function together with 6-color.
We obtain for $\bar{3}$-color:
(a) $\frac{1}{\sqrt{2}}(ud - du)$ and we have I=0.
(b) not possible
(c) $\frac{1}{\sqrt{2}}(bc - cb)$ and analogously for the others.
We obtain for 6-color:
(a) $\{\frac{1}{\sqrt{2}}(ud + du), uu, dd\}$ and we have I=1.
(b) ss, cc, bb
(c) $\frac{1}{\sqrt{2}}(bc + cb)$ and analogously for the others.

II.) Axialvector diquarks:
Spin S=1 $\rightarrow \{\frac{1}{\sqrt{2}}(\uparrow\downarrow + \downarrow\uparrow), \uparrow\uparrow, \downarrow\downarrow\}$ and symmetric
Color: Same as above. But now we need a symmetric flavour wave function together with $\bar{3}$-color and an antisymmetric flavour wave function together with 6-color.
Thus, flavour/color combinations are interchanged as compared to scalar diquark.

11.2 Four-quark states

Now think about a four-quark state with two heavy quarks and two light anti-quarks in the two flavour combinations $bb\bar{q}\bar{q}$ and $bc\bar{q}\bar{q}$. Suppose, the quarks and antiquarks are arranged in scalar (S) and axialvector (A) diquarks. Which diquark combinations are possible for the following quantum numbers?

(a) $I(J) = 0(1)$

Solution: $J = 1 \rightarrow$ we need at least one axialvector diquark, i.e. only combinations AA, SA, AS are possible.
color $3 \bigotimes \bar{3}$:
$I = 0 \rightarrow$ light diquark needs to be S $\rightarrow$ heavy diquark needs to be A and indeed, this is possible
color $6 \bigotimes \bar{6}$:
$I = 0 \rightarrow$ light diquark needs to be A $\rightarrow$ heavy diquark needs to be S and indeed, this is possible

(b) $I(J) = 1(1)$

Solution: $J = 1 \rightarrow$ we need at least one axialvector diquark, i.e. only combinations AA, SA, AS are possible.
color $3 \bigotimes \bar{3}$:
$I = 1 \rightarrow$ light diquark needs to be A. Heavy diquark also needs to be A (S is not possible).
color $6 \bigotimes \bar{6}$:
$I = 1 \rightarrow$ light diquark needs to be S $\rightarrow$ heavy diquark needs to be A, but this is not possible.

(c) $I(J) = 0(0)$

Solution: $J = 0 \rightarrow$ we need either SS or AA (from rules of adding angular momenta).
color $3 \bigotimes \bar{3}$:
$I = 0 \rightarrow$ light diquark needs to be S $\rightarrow$ heavy diquark also needs to be S, but this is not possible.
color $6 \bigotimes \bar{6}$:
$I = 0 \rightarrow$ light diquark needs to be A $\rightarrow$ heavy diquark also needs to be A, but this is not possible.

Hint: again carefully think about symmetries...

11.3 Pion Bethe-Salpeter Equation

The Bethe-Salpeter vertex function Γ_π of a pion can be expressed most generally by

$$\Gamma_\pi(P,p) = \sum_{i=1}^{4} f_i(P,p)\, T_i \tag{1}$$

with tensors $T_1 = \gamma_5\,\mathbb{1}$, $T_2 = \gamma_5\,\hat{\not{P}}$, $T_3 = \gamma_5\,\not{p}$ and $T_4 = \gamma_5\,[\hat{\not{P}}, \not{p}]$ in Dirac-space. Here we use the normalised total momentum $\hat{P}$ of the pion and the orthogonalised relative momentum p between quark and antiquark, i.e. $\hat{P}\cdot\hat{P} = 1$ and $p \cdot P = 0$.
The Pauli-Lubanski vector (see e.g. Maggiore, chapter 2.7) can be used to determine the spin and angular momentum quantum numbers of these tensors in the rest frame of the pion. Its square can be separated in a part referring to angular momentum and a part referring to spin:

$$L^2 = 2p^\alpha \frac{\partial}{\partial p^\alpha} + \left(p_T^\alpha\, p_T^\beta - p_T^2\, T_p^{\alpha\beta}\right) \frac{\partial}{\partial p^\alpha}\frac{\partial}{\partial p^\beta} \tag{2}$$

$$[S^2]_{i,j}^{k,l} = \frac{3}{2}\mathbb{1}_{i,j} \otimes \mathbb{1}_{k,l} - \frac{1}{2}\left(\gamma_T^\mu \gamma_5 \hat{\not{P}}\right)_{i,j} \otimes \left(\hat{\not{P}}\gamma_5\gamma_T^\mu\right)_{k,l} \tag{3}$$

Here, $\gamma_T^\mu = \gamma^\mu - \hat{P}^\mu \hat{\not{P}}$, $p_T^\alpha = p^\alpha - \hat{P}^\alpha\, p\cdot P$ and $T_p^{\alpha\beta} = \delta^{\alpha\beta} - p^\alpha p^\beta/p^2$.

(a) Show that T_1 and T_2 are s-waves (eigenvalue 0 of L^2), whereas T_3 and T_3 are p-waves (eigenvalue 1 of L^2).

> **Solution:**
>
> $$L^2 T_1 = L^2 \gamma_5\,\mathbf{1} = 0 \tag{4}$$
> $$L^2 T_2 = L^2 \gamma_5\,\hat{\not{P}} = 0 \tag{5}$$
> $$L^2 T_3 = L^2 \gamma_5\,\not{p} = \gamma_5\, 2p^\alpha \gamma^\mu \delta_{\alpha\mu} = 2\gamma_5\,\not{p} \tag{6}$$
> $$L^2 T_4 = L^2 \gamma_5\,[\hat{\not{P}}, \not{p}] = \gamma_5\, 2[\hat{\not{P}}, \gamma^\mu\delta_{\alpha\mu}] = 2\gamma_5\,[\hat{\not{P}}, \not{p}] \tag{7}$$

(b) Show that T_1 has eigenvalue 0 wrt. S^2. (The same is true for T_2.)
Hint: $\gamma_T^\mu \gamma_T^\mu = 3$

> **Solution:**
>
> $$[S^2]_{i,j}^{k,l}[\gamma_5]_{j,k} = \frac{3}{2}[\gamma_5]_{i,l} - \frac{1}{2}[\gamma_T^\mu\gamma_5\hat{\not{P}}\gamma_5\hat{\not{P}}\gamma_5\gamma_T^\mu]_{i,l} \tag{8}$$
> $$= \frac{3}{2}[\gamma_5]_{i,l} - \frac{1}{2}[\gamma_T^\mu\gamma_T^\mu\gamma_5]_{i,l} \tag{9}$$
> $$= \frac{3}{2}[\gamma_5]_{i,l} - \frac{3}{2}[\gamma_5]_{i,l} = 0 \tag{10}$$

(c) Show that T_3 has eigenvalue 1 wrt. S^2. (The same is true for T_4, but this calculation is rather lengthy...)
Hint: Use $p\cdot P = 0$ *and* $\not{p}\gamma_T^\mu = -\gamma_T^\mu\not{p} + 2p^\mu$

Solution:

$$[S^2]^{k,l}_{i,j}[\gamma_5 \not{p}]_{j,k} = \frac{3}{2}[\gamma_5 \not{p}]_{i,l} - \frac{1}{2}[\gamma^\mu_T \gamma_5 \hat{\not{P}} \gamma_5 \not{p} \hat{\not{P}} \gamma_5 \gamma^\mu_T]_{i,l} \tag{11}$$

$$= \frac{3}{2}[\gamma_5 \not{p}]_{i,l} - \frac{1}{2}[\gamma^\mu_T \hat{\not{P}} \not{p} \hat{\not{P}} \gamma^\mu_T \gamma_5]_{i,l} \tag{12}$$

$$= \frac{3}{2}[\gamma_5 \not{p}]_{i,l} - \frac{1}{2}[\gamma^\mu_T \gamma^\mu_T \gamma_5 \not{p}]_{i,l} + [\gamma^\mu_T p_\mu \gamma_5] \tag{13}$$

$$= (\frac{3}{2} + \frac{3}{2} - 1)[\gamma_5 \not{p}]_{i,l} = 2[\gamma_5 \not{p}]_{i,l} \tag{14}$$