



## MTHS24 – Exercise sheet 8

Morning: Vincent Mathieu / Andrew Jackura / Arkaitz Rodas

Afternoon: Daniel Winney, Gloria Montana



Tuesday, 23 July 2024

## Lecture material

### References:

- P.D.B. Collins, "An Introduction to Regge Theory and High Energy Physics", [Inspire](#)
- V.N. Gribov, ("Blue book") "The theory of complex angular momentum", [Inspire](#)
- V.N. Gribov, ("Gold book") "Strong interactions of hadrons at high energies", [Inspire](#)
- D. Sivers & J. Yellin, "Review of recent work on narrow resonance models" [Inspire](#)

### Discussed topics:

- Regge theory
- High energy scattering
- Complex angular momentum
- Unitarity

## Exercises

### 8.1 Unitarity and Reggeons

Van Hove proposed a physically intuitive picture of a Reggeon by relating it to Feynman diagrams in the cross-channels. We will explore this picture of Reggeization with a simple model.

#### (a) Elementary $t$ -channel exchanges

Consider the amplitude corresponding to a particle with spin- $J$  and mass  $m_J$  exchanged in the  $t$ -channel as:

$$A^J(s, t) = i g_J (q_1^{\mu_1} \dots q_1^{\mu_J}) \frac{P_{\mu_1 \dots \mu_J, \nu_1 \dots \nu_J}^J(k)}{m_J^2 - t} (q_2^{\nu_1} \dots q_2^{\nu_J}) \quad (1)$$

where  $g_J$  is a coupling constant with dimension  $2-2J$  (i.e.,  $A^J(s, t)$  is dimensionless) and the projector of spin- $J$  is defined from the polarization tensor of rank- $J \geq 1$  as

$$P_{\mu_1 \dots \mu_J, \nu_1 \dots \nu_J}^J(k) = \frac{(J+1)}{2} \sum_{\lambda} \epsilon^{\mu_1 \dots \mu_J}(k, \lambda) \epsilon^{*\nu_1 \dots \nu_J}(k, \lambda) . \quad (2)$$

Using the exchange momentum  $k = q_1 + q_3 = q_1 - q_3$ , calculate the amplitudes corresponding to  $J = 0, 1, 2$  exchanges in terms of  $t = k^2$ , the modulus of 3-momentum and cosine of scattering angle in the  $t$ -channel frame,  $q_t$  and  $\cos \theta_t$  respectively. Use the explicit forms of the projectors:

$$P^0(k^2) = 1 \quad (3)$$

$$P_{\mu\nu}^1(k^2) \equiv \tilde{g}_{\mu\nu} = \frac{k_\mu k_\nu}{k^2} - g_{\mu\nu} \quad (4)$$

$$P_{\mu\nu\alpha\beta}^2(k^2) = \frac{3}{4} (\tilde{g}_{\mu\alpha} \tilde{g}_{\nu\beta} + \tilde{g}_{\mu\beta} \tilde{g}_{\nu\alpha}) - \frac{1}{2} \tilde{g}_{\mu\nu} \tilde{g}_{\alpha\beta} , \quad (5)$$

and conjecture a generalization of the amplitude for arbitrary integer  $J$ .

*Hint: Show that in the t-channel frame, the exchange particle is at rest and therefore  $\tilde{g}_{\mu\nu}$  reduces to a  $\delta_{ij}$  with respect to only spacial momenta.*

**Solution:**

We start by considering the elastic scattering of two identical, spinless particles with 4-momentum  $q_i$  and with mass  $q_i^2 = m^2$ . We define the usual Mandelstam variables

$$\begin{aligned}s &= (q_1 + q_2)^2 = (q_3 + q_4)^2 \\t &= (q_1 - q_3)^2 = (q_4 - q_2)^2 \\u &= (q_1 - q_4)^2 = (q_2 - q_3)^2\end{aligned}$$

We refer to the  $s$ -channel as the physical region describing the process

$$1(q_1) + 2(q_2) \rightarrow 3(q_3) + 4(q_4),$$

while in the  $t$ -channel we consider

$$1(q_1) + \bar{3}(q_{\bar{3}}) \rightarrow \bar{2}(q_{\bar{2}}) + 4(q_4).$$

The  $J = 0$  is trivial

$$A^0(s, t) = i g_0 \frac{1}{m_0^2 - t} = i g_0 \frac{P_0(\cos \theta_t)}{m_0^2 - t}. \quad (6)$$

For  $J = 1$  use  $q_1 = (\sqrt{t}/2, q_t \hat{z})$  and  $q_{\bar{3}} = (\sqrt{t}/2, -q_t \hat{z})$ . In the  $t$ -channel CM frame we have  $k = (q_1 - q_3) = (q_1 + q_{\bar{3}}) = (\sqrt{t}, \vec{0})$  and

$$-\tilde{g}_{\mu\nu} = -g_{\mu\nu} + \frac{k_\mu k_\nu}{t} = -[\delta_{\mu 0} \delta_{\nu 0} - \delta_{ij}] + \frac{\sqrt{t}^2}{t} \delta_{\mu 0} \delta_{\nu 0} = +\delta_{ij}. \quad (7)$$

thus  $q_1^\mu \tilde{g}_{\mu\nu} q_2^\nu = \vec{q}_1 \cdot \vec{q}_2 = q_t^2 \cos \theta_t$ . Similarly  $q_1^\mu \tilde{g}_{\mu\nu} q_1^\nu = q_2^\mu \tilde{g}_{\mu\nu} q_2^\nu = q_t^2$  and we have

$$A^1(s, t) = ig_1 q_t^2 \frac{\cos \theta_t}{m_1^2 - t} = ig_1 q_t^2 \frac{P_1(\cos \theta_t)}{m_1^2 - t}, \quad (8)$$

and finally also

$$A^2(s, t) = ig_2 q_t^4 \frac{\frac{1}{2}(3 \cos \theta_t - 1)}{m_2^2 - t} = ig_2 q_t^4 \frac{P_2(\cos \theta_t)}{m_2^2 - t}. \quad (9)$$

The generalization to arbitrary  $J$  is

$$A^J(s, t) = ig_J q_t^{2J} \frac{P_J(\cos \theta_t)}{m_J^2 - t}. \quad (10)$$

**(b) Unitarity vs Elementary exchanges**

Express the amplitude entirely in terms of invariants  $s$  and  $t$ . Use the optical theorem to relate the elastic amplitude to a total hadronic cross section:

$$\sigma_{\text{tot}} = \frac{1}{2q\sqrt{s}} \Im A^J(s, t = 0). \quad (11)$$

Unitarity (via the Froissart-Martin bound) prohibits  $\sigma_{\text{tot}}$  from growing faster than  $\log^2 s$  as  $s \rightarrow \infty$ . What is then the maximal spin a single elementary exchange can have while satisfying this bound? Why is this a problem?

**Solution:** We have

$$\sigma_{\text{tot}} = \frac{1}{2q\sqrt{s}} \frac{g_J}{m_J^2} q_t^{2J} P_J(\cos \theta_t) \Big|_{t=0} . \quad (12)$$

We have  $q_t^2 \cos \theta_t = (s - u)/4$  so that as  $s \rightarrow \infty$ , we have:

$$\sigma_{\text{tot}} \sim s^{J-1} . \quad (13)$$

To satisfy the Froissart bound, the maximally allowed spin then is  $J = 1$ .

(c) **Van Hove Reggeon**

Consider an amplitude of the form

$$A(s, t) = \sum_{J=0}^{\infty} g r^{2J} \frac{(q_t^2 \cos \theta_t)^J}{J - \alpha(t)} . \quad (14)$$

Here  $\alpha(t) = \alpha(0) + \alpha' t$  is a real, linear Regge trajectory,  $g$  is a dimensionless coupling constant and  $r \sim 1$  fm is a range parameter. Compare Eq. 14 with Eq. 1, write the mass of the  $J$ th pole,  $m_J^2$ , as a function of the Regge parameters  $\alpha(0)$  and  $\alpha'$ . Interpret the pole structure in terms of the spectrum of particles in the model.

If the sum is truncated to a finite  $J_{\max}$ , and we take the  $s \rightarrow \infty$  limit, what is the high energy behavior of the amplitude?

**Solution:** We can write

$$J - \alpha(t) = J - \alpha(0) - \alpha' t = \alpha' ((J - \alpha(0))/\alpha' - t) \quad (15)$$

and thus we have  $m_J^2 = (J - \alpha(0))/\alpha'$ .

We can use

$$(\cos \theta_t)^J = \sum_{J+J' \text{ even}}^J \frac{(J+1)!}{(J-J')!! (J+J'+1)!!} P_{J'}(\cos \theta_t) \quad (16)$$

$$= \sum_{J+J' \text{ even}}^J \mu_{JJ'} P_{J'}(\cos \theta_t) \quad (17)$$

to write

$$A(s, t) = \sum_J \sum_{J'=0}^J \left( \frac{g r^{2J} \mu_{JJ'}}{\alpha'} \right) q_t^{2J} \frac{P_{J'}(\cos \theta_t)}{m_J^2 - t} , \quad (18)$$

$$= \sum_J \sum_{J'=0}^J g_{JJ'} q_t^{2J} \frac{P_{J'}(\cos \theta_t)}{m_J^2 - t} , \quad (19)$$

Comparing with the form of our elementary exchanges, this amplitude is an infinite sum of particles with spin- $J$  and mass  $m_J^2$  but also all same parity daughters at the same mass.

If the sum is truncated at  $J_{\max}$  the  $s \rightarrow \infty$  limit is dominated by the largest spin exchange and we have  $A_{\text{trunc}}(s, t) \propto s^{J_{\max}}$ .

(d) **Analytic continuation in  $J$**

Show that if the summation is kept infinite, the amplitude can be re-summed to something that is entirely analytic in  $s$ ,  $t$ ,  $u$ , and  $J$ .

*Hint: Use the Mellin transform*

$$\frac{1}{J - \alpha(t)} = \int_0^1 dx x^{J-\alpha(t)-1}, \quad (20)$$

*to express the amplitude in terms of the Gaussian hypergeometric function and the Euler Beta function*

$$B(b, c-b) {}_2F_1(1, b, c; z) = \int_0^1 dx \frac{x^{b-1} (1-x)^{c-b-1}}{1-xz}. \quad (21)$$

**Solution:** Go back to the original form in terms of monomials, we can write

$$A(s, t) = \sum_{J=0} \int_0^1 dx g r^{2J} (q_t^2 \cos \theta_t)^J x^{J-\alpha(t)-1}. \quad (22)$$

Collecting all things with powers of  $J$ , we notice a geometric series which can be summed analytically

$$A(s, t) = g \int_0^1 dx \frac{x^{-\alpha(t)-1}}{1 - r^2 q_t^2 \cos \theta_t x}. \quad (23)$$

Comparing with the definition of the hypergeometric function, we can identify  $z = r^2 q_t^2 \cos \theta_t$  and  $b = -\alpha(t)$ . Since there is no  $(1-x)$  term we require  $c = b+1 = 1-\alpha(t)$ . Thus we have

$$A(s, t) = \frac{\Gamma(-\alpha(t))}{\Gamma(1-\alpha(t))} {}_2F_1(1, -\alpha(t), 1-\alpha(t), (q_t r)^2 \cos \theta_t) \quad (24)$$

$$= \Gamma(-\alpha(t)) {}_2\tilde{F}_1(1, -\alpha(t), 1-\alpha(t), (q_t r)^2 \cos \theta_t). \quad (25)$$

#### (e) Unitarity vs Reggeized exchanges

Revisit b) with the resummed amplitude. Take the  $s \rightarrow \infty$  limit and set a limit on the maximal intercept  $\alpha(0)$  which is allowed by unitarity.

*Hint: Assume that  $\alpha(0) > -1$  and use the asymptotic behavior of the hypergeometric function given by*

$${}_2F_1(1, b, c; z) \rightarrow \frac{\Gamma(c)\Gamma(1-b)}{\Gamma(1)\Gamma(c-b)} (-z)^{-b}. \quad (26)$$

**Solution:** From the hypergeometric form we can take  $s \rightarrow \infty$  which takes  $q_t^2 \cos \theta_t = (s-u)/4 \rightarrow \infty$  and we can write

$$A(s, t) = g_0 \Gamma(-\alpha(t)) \Gamma(1+\alpha(t)) \left( \frac{u-s}{4r^2} \right)^{\alpha(t)}. \quad (27)$$

So, we have

$$\Im A(s, 0) \propto \Im(-s)^{\alpha(0)} \propto \sin \pi \alpha(0) s^{\alpha(0)} \quad (28)$$

so  $\sigma_{\text{tot}} \sim s^{\alpha(0)-1}$  and unitarity requires  $\alpha(0) \leq 1$ .

#### (f) The Reggeon “propagator”

Modify Eq. 1 to have a definite signature by defining

$$A^\pm(s, t) = \frac{1}{2} [A(s, t) \pm A(u, t)]. \quad (29)$$

Repeat d) and e) with this signatured amplitude. Compare with the canonical form of the Reggeon exchange:

$$A_R^\pm(s, t) = \beta(t) \frac{1}{2} \left[ \pm 1 + e^{-i\pi\alpha(t)} \right] \Gamma(-\alpha(t)) \left( \frac{s}{s_0} \right)^{\alpha(t)}. \quad (30)$$

Identify the Regge residue  $\beta(t)$  and characteristic scale  $s_0$  in terms of the parameters  $g_0$  and  $r$ .

**Solution:** As we see above, switching  $s \leftrightarrow u$  introduces a minus sign and we have

$$A^\pm(s, t) = g_0 \Gamma(1 + \alpha(t)) \frac{1}{2} [\pm 1 + e^{-i\pi\alpha(t)}] \Gamma(-\alpha(t)) \left( \frac{s - u}{4r^2} \right)^{\alpha(t)}, \quad (31)$$

and we can read off  $\beta(t) = g_0 \Gamma(1 + \alpha(t))$ . Using  $s \sim -u$  we also see  $s_0 = 2r^{-2}$

## 8.2 Veneziano Amplitude

The quintessential dual amplitude was first proposed by Veneziano for  $\omega \rightarrow 3\pi$  and later applied to elastic  $\pi\pi$  scattering by Shapiro and Lovelace. Consider the  $\pi^+\pi^-$  scattering amplitude of the form

$$\mathcal{A}(s, t, u) = V(s, t) + V(s, u) - V(t, u). \quad (32)$$

with each

$$V(s, t) = \frac{\Gamma(1 - \alpha(s)) \Gamma(1 - \alpha(t))}{\Gamma(1 - \alpha(s) - \alpha(t))}, \quad (33)$$

where  $\alpha(s) = \alpha(0) + \alpha's$  is a real, linear Regge trajectory with  $\alpha' > 0$ .

### (a) Duality

Show that the function  $V(s, t)$  is symmetric in  $s \leftrightarrow t$  and dual, i.e., it can be written entirely as a sum of either  $s$ -channel poles OR  $t$ -channel poles but never both simultaneously. Compare with the Reggeized amplitude in the previous problem, was that amplitude dual?

*Hint: Relate  $V(s, t)$  to the Euler Beta function*

$$B(x, y) = \frac{\Gamma(x) \Gamma(y)}{\Gamma(x + y)} \quad (34)$$

and use the identities  $B(x, y) = B(y, x)$  and

$$B(p - x, q - y) = \sum_{J=1}^{\infty} \frac{\Gamma(J - p + 1 + x)}{\Gamma(J) \Gamma(-p + 1 + x)} \frac{1}{J - 1 + q - y}. \quad (35)$$

**Solution:** We can write:

$$V(s, t) = (1 - \alpha(s) - \alpha(t)) B(1 - \alpha(s), 1 - \alpha(t)). \quad (36)$$

Then using the expansion of the Beta function on its first argument, we have

$$V(s, t) = (1 - \alpha(s) - \alpha(t)) \sum_{J=1}^{\infty} \frac{\Gamma(J - 1 + \alpha(t))}{\Gamma(J) \Gamma(\alpha(t))} \frac{1}{J - \alpha(s)} \quad (37)$$

which only has poles in  $\alpha(s)$ . Because of the  $s \leftrightarrow t$  symmetry, we can write the exact same expression with only poles in  $\alpha(t)$ .

### (b) Isospin basis

Define the  $s$ -channel isospin basis through

$$\begin{pmatrix} \mathcal{A}^{(0)}(s, t, u) \\ \mathcal{A}^{(1)}(s, t, u) \\ \mathcal{A}^{(2)}(s, t, u) \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 3 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} \mathcal{A}(s, t, u) \\ \mathcal{A}(t, s, u) \\ \mathcal{A}(u, t, s) \end{pmatrix}. \quad (38)$$

Write down the definite-isospin amplitudes in terms of  $V$ 's. Comment on the symmetry properties of each isospin amplitude with respect to  $t \leftrightarrow u$ .

**Solution:** We have:

$$\mathcal{A}^{(0)}(s, t, u) = \frac{1}{2} [3V(s, t) + 3V(s, u) - V(t, u)] \quad (39)$$

$$\mathcal{A}^{(1)}(s, t, u) = V(s, t) - V(s, u) \quad (40)$$

$$\mathcal{A}^{(2)}(s, t, u) = V(t, u) . \quad (41)$$

$I = 0, 2$  are symmetric in  $t \leftrightarrow u$  while  $I = 1$  is anti-symmetric as required by Bose symmetry.

(c) **Chew-Frautschi plot**

Locate where each  $\mathcal{A}^{(I)}(s, t, u)$  will have poles in the  $s$ -channel physical region. What is their residue? Draw a schematic Chew-Frautschi plot of the resonance spectrum in each isospin channel.

**Solution:** A single  $V(s, t)$  will have poles at all  $\alpha(s) = J \geq 1$  and all possible daughters. The residues are

$$-(J-1+\alpha(t)) \frac{\Gamma(J-1+\alpha(t))}{\Gamma(J)\Gamma(\alpha(t))} = \frac{-1}{\Gamma(J)} \frac{\Gamma(J+\alpha(t))}{\Gamma(\alpha(t))} = -\frac{(\alpha(t))_J}{(J-1)!}. \quad (42)$$

For a linear trajectory this is a order  $J$  polynomial in  $t$  and therefore in  $z_s$ .

The symmetry factors in  $I = 0, 1$  will remove all odd (even)  $J$  daughters. The  $I = 2$  amplitude has no  $s$  dependence and therefore no isospin-2 poles at all.

(d) **Regge limit**

Now consider the limit  $t \rightarrow \infty$  and  $u \rightarrow -\infty$  with  $s \leq 0$  is fixed. What is the asymptotic behavior of  $V(s, t)$  and  $V(s, u)$ ? Assume that  $V(t, u)$  vanishes faster than any power of  $s$  in this limit. What is the resulting behavior of the isospin amplitudes  $\mathcal{A}^{(I)}(s, t, u)$  in this limit?

*Hint: Use the Sterling approximation of the  $\Gamma$  function., i.e. as  $|x| \rightarrow \infty$*

$$\Gamma(x) \rightarrow \sqrt{\frac{2\pi}{x}} \left(\frac{x}{e}\right)^x . \quad (43)$$

**Solution:** Starting with

$$V(s, t) \rightarrow \Gamma(1-\alpha(s)) (-\alpha(t))^{\alpha(s)} \sim \Gamma(1-\alpha(s)) (-\alpha' t)^{\alpha(s)} . \quad (44)$$

Similarly

$$V(s, u) \rightarrow \Gamma(1-\alpha(s)) (-\alpha(u))^{\alpha(s)} \sim \Gamma(1-\alpha(s)) (-\alpha' t)^{\alpha(s)} . \quad (45)$$

Thus the combination

$$V(s, t) \pm V(s, u) \rightarrow \Gamma(1-\alpha(s)) \times \left[ (-\alpha' t)^{\alpha(s)} \pm (-\alpha' u)^{\alpha(s)} \right] \quad (46)$$

$$= \Gamma(1-\alpha(s)) \times \left[ 1 \pm e^{-i\pi\alpha(s)} \right] (\alpha' t)^{\alpha(s)} \quad (47)$$

(e) **Ancestors and Strings**

Consider the model now with a complex trajectory  $\alpha(s) = a_0 + \alpha' s + i\Gamma$  with  $\Gamma > 0$  to move the poles off the real axis. Reexamine the the Chew-Frautschi plot for the  $I = 1$  amplitude using this trajectory, why is the resulting spectrum problematic? Try a real but non-linear trajectory, say  $\alpha(s) = a_0 + \alpha' s + \alpha'' s^2$ , what is the spectrum like now?

Compare the requirements of the trajectory for  $V(s, t)$  to make sense with the energy levels of a rotating relativistic string with a string tension  $T$ :

$$E_J^2 = \frac{1}{2\pi T} J . \quad (48)$$

What is a possible microscopic picture of hadrons if the Veneziano amplitude is believed?

**Solution:** If we allow  $\alpha(t)$  to be complex, then at a pole  $\alpha(s) \rightarrow J + i\Gamma$  and the residue we calculated

$$\frac{\Gamma(J + i\Gamma + \alpha(t))}{\Gamma(\alpha(t))}, \quad (49)$$

is no longer a fixed order polynomial in  $t$ . It will thus give contributions to ALL spins at each pole, i.e. introduce an infinite number of ancestors. Similarly if  $\alpha(s)$  is non-linear, we will have finitely many ancestors but still unphysical poles nonetheless.

This means the Veneziano amplitude *only* gives a physical picture for real and linear trajectories. This means we require  $J \propto s \sim m^2$  which mimics the spectrum of states in a relativistic rotating string. This gives rise to the stringy picture of a  $q\bar{q}$  pair connected by a gluon flux tube and later the entire field of string theories.

### 8.3 Sommerfeld-Watson Transform

#### (a) Geometric series

Prove the well known resummation of the geometric series:

$$1 + x + x^2 + x^3 + \dots = \frac{1}{1-x} \quad \text{for } |x| < 1 \quad (50)$$

can be analytically continued to  $|x| \geq 1$  with the Sommerfeld-Watson Transform.

Assume that  $|x| > 1$  and show that the summation can be written as an integral over the complex plane

$$\int \frac{d\ell}{2i} \frac{(-x)^\ell}{\sin \pi \ell} = 1 + x + x^2 + \dots . \quad (51)$$

Draw the contour around which the above integration should be taken (careful with orientations and signs). Deform the contour such that you can relate Eq. 51 to the series

$$\frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} + \dots = \frac{1}{1 - \frac{1}{x}} \quad (52)$$

and arrive at Eq. 50.

**Solution:** We want to show that we can analytically continue the geometric series to  $|x| > 1$  using the Sommerfeld-Watson Transform.

First, we will prove that the sum can be written as an integral over the complex plane:

$$\int_C \frac{d\ell}{2i} \frac{(-x)^\ell}{\sin \pi\ell}, \quad (53)$$

where the function  $1/\sin \pi\ell$  has poles at integer values of  $\ell = \dots, -2, -1, 0, 1, 2, \dots$ . We use the Cauchy Residue Theorem:

$$\oint f(z) dz = \pm 2\pi i \sum_k \text{Res}_{z=z_k} f(z), \quad (54)$$

with sign + for a counterclockwise contour and - for a clockwise contour around the pole at  $z_k$ . The residue of  $f(\ell) = (-x)^\ell / \sin \pi\ell$  at  $\ell = k$  is given by

$$\text{Res}_{\ell=k} \frac{(-x)^\ell}{\sin \pi\ell} = \lim_{\ell \rightarrow k} (\ell - k) \frac{(-x)^\ell}{\sin \pi\ell} = \lim_{\ell \rightarrow k} \frac{(-x)^\ell}{\pi \cos \pi\ell} = \frac{x^k}{\pi} \quad (55)$$

Therefore, if we encircle all the poles at  $\ell = k$  for  $k \geq 0$  with counterclockwise contours  $C_k$ , which we can combine to a single counterclockwise contour  $C$  encircling all the poles at 0 and positive integers, we obtain the geometric series:

$$\sum_{k=0}^{\infty} \int_{C_k} \frac{d\ell}{2i} \frac{(-x)^\ell}{\sin \pi\ell} = \int_C \frac{d\ell}{2i} \frac{(-x)^\ell}{\sin \pi\ell} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \dots \quad (56)$$

Next, we deform the contour to a vertical line from  $\sigma + i\infty$  to  $\sigma - i\infty$ , with  $-1 < \sigma < 0$ , and further deform it to enclose all negative integers in a clockwise contour  $C'$ , that we can split in individual clockwise contours  $C_{k'}$ :

$$\begin{aligned} \int_{C'} \frac{d\ell}{2i} \frac{(-x)^\ell}{\sin \pi\ell} &= \sum_{C_{k'}} \int_{C_{k'}} \frac{d\ell}{2i} \frac{(-x)^\ell}{\sin \pi\ell} = -\frac{1}{x} - \frac{1}{x^2} - \frac{1}{x^3} - \dots \\ &= -\frac{1}{x} \left( 1 + \frac{1}{x} + \frac{1}{x^2} + \dots \right) \end{aligned} \quad (57)$$

and given the assumption  $|x| > 0$ , we have  $|1/x| < 1$ , and we can sum the geometric series:

$$-\frac{1}{x} \left( 1 + \frac{1}{x} + \frac{1}{x^2} + \dots \right) = -\frac{1}{x} \frac{1}{1 - \frac{1}{x}} = \frac{1}{1 - x}. \quad (58)$$

### (b) Van Hove Reggeon

Revisit the Regge behavior of Eq. 14 using the S-W transform. How does the inclusion of poles at  $\alpha(s) = \ell$  change the contour of integration and the leading contribution to the asymptotic behavior?

**Solution:** If we include a Regge pole  $[\ell - \alpha(t)]^{-1}$ , in the process of deforming the contour from  $C$  to  $C'$  we have to pick the residue of the pole at  $\ell = \alpha(t)$ , with  $\text{Re } \alpha(t) < 0$ . This means we have an extra clockwise contour  $C_\alpha$ , from which we obtain an extra contribution  $\propto x^{\alpha(t)}$ . Since  $x \sim q_t^2 z_t \sim s$  this yields the  $s^{\alpha(t)}$  behavior.

## 8.4 Finite Energy Sum Rules

Consider  $z$  a complex variable and  $\alpha$  a real fixed parameter. What is the analytic structure of the function  $z^\alpha$ ? What is the discontinuity across the cut?

Write a Cauchy contour  $C$  surrounding the cut and closing it with a circle of radius  $\Lambda$  in the complex  $z$

plane, and check that

$$\oint_C z^\alpha dz = 0 \quad (59)$$

You can start with the simple case  $\alpha = 1/2$ , i.e.  $\sqrt{z}$ , then generalize to any real  $\alpha$ .

**Solution:** The function  $z^\alpha$  has a branch cut for  $z \in [-\infty, 0]$ . The Cauchy contour enclose the cut and the discontinuity across that cut is

$$(z + i\epsilon)^\alpha - (z - i\epsilon)^\alpha = (|z|e^{i\pi})^\alpha - (|z|e^{-i\pi})^\alpha \quad \text{for real negative } z \quad (60)$$

$$= |z|^\alpha (e^{i\pi\alpha} - e^{-i\pi\alpha}) \quad (61)$$

$$= 2i|z|^\alpha \sin \pi\alpha \quad (62)$$

The Cauchy contour is then, with  $C_\Lambda$  being the circle of radius  $\Lambda$  in the positive sense,

$$\oint z^\alpha dz = \int_{-\Lambda}^0 (z + i\epsilon)^\alpha dz + \int_0^{-\Lambda} (z - i\epsilon)^\alpha dz + \oint_{C_\Lambda} z^\alpha dz \quad (63)$$

$$= 2i \sin \pi\alpha \int_{-\Lambda}^0 |z|^\alpha dz + \oint_{C_\Lambda} z^\alpha dz \quad (64)$$

The first integral is easily done

$$2i \sin \pi\alpha \int_{-\Lambda}^0 |z|^\alpha dz = \frac{2i\Lambda^{\alpha+1}}{\alpha+1} \sin \pi\alpha. \quad (65)$$

For the second integral, we need the change of variable  $z = \Lambda \exp(i\theta)$ , with  $\theta \in [-\pi, \pi]$ . We obtain

$$\oint_{C_\Lambda} z^\alpha dz = i\Lambda^{\alpha+1} \int_{-\pi}^{\pi} e^{i\theta(\alpha+1)} d\theta = \frac{i\Lambda^{\alpha+1}}{\alpha+1} (e^{i\pi(\alpha+1)} - e^{-i\pi(\alpha+1)}) \quad (66)$$

$$= -\frac{2i\Lambda^{\alpha+1}}{\alpha+1} \sin \pi\alpha \quad (67)$$

We used  $\exp(i\pi) = -1$ .