



MTHS24 – Exercise sheet 8

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Lecture material

References:

- P.D.B. Collins, "An Introduction to Regge Theory and High Energy Physics", [Inspire](#)
- V.N. Gribov, ("Blue book") "The theory of complex angular momentum", [Inspire](#)
- V.N. Gribov, ("Gold book") "Strong interactions of hadrons at high energies", [Inspire](#)
- D. Sivers & J. Yellin, "Review of recent work on narrow resonance models" [Inspire](#)

Discussed topics:

- Regge theory
- High energy scattering
- Complex angular momentum
- Unitarity

Exercises

8.1 Unitarity and Reggeons

Van Hove proposed a physically intuitive picture of a Reggeon by relating it to Feynman diagrams in the cross-channels. We will explore this picture of Reggeization with a simple model.

(a) Elementary t -channel exchanges

Consider the amplitude corresponding to a particle with spin- J and mass m_J exchanged in the t -channel as:

$$A^J(s, t) = i g_J (q_1^{\mu_1} \dots q_1^{\mu_J}) \frac{P_{\mu_1 \dots \mu_J, \nu_1 \dots \nu_J}^J(k)}{m_J^2 - t} (q_2^{\nu_1} \dots q_2^{\nu_J}) \quad (1)$$

where g_J is a coupling constant with dimension $2-2J$ (i.e., $A^J(s, t)$ is dimensionless) and the projector of spin- J is defined from the polarization tensor of rank- $J \geq 1$ as

$$P_{\mu_1 \dots \mu_J, \nu_1 \dots \nu_J}^J(k) = \frac{(J+1)}{2} \sum_{\lambda} \epsilon^{\mu_1 \dots \mu_J}(k, \lambda) \epsilon^{*\nu_1 \dots \nu_J}(k, \lambda) . \quad (2)$$

Using the exchange momentum $k = q_1 + q_3 = q_1 - q_3$, calculate the amplitudes corresponding to $J = 0, 1, 2$ exchanges in terms of $t = k^2$, the modulus of 3-momentum and cosine of scattering angle in the t -channel frame, q_t and $\cos \theta_t$ respectively. Use the explicit forms of the projectors:

$$P^0(k^2) = 1 \quad (3)$$

$$P_{\mu\nu}^1(k^2) \equiv \tilde{g}_{\mu\nu} = \frac{k_\mu k_\nu}{k^2} - g_{\mu\nu} \quad (4)$$

$$P_{\mu\nu\alpha\beta}^2(k^2) = \frac{3}{4} (\tilde{g}_{\mu\alpha} \tilde{g}_{\nu\beta} + \tilde{g}_{\mu\beta} \tilde{g}_{\nu\alpha}) - \frac{1}{2} \tilde{g}_{\mu\nu} \tilde{g}_{\alpha\beta} , \quad (5)$$

and conjecture a generalization of the amplitude for arbitrary integer J .

Hint: Show that in the t-channel frame, the exchange particle is at rest and therefore $\tilde{g}_{\mu\nu}$ reduces to a δ_{ij} with respect to only spacial momenta.

Solution:

We start by considering the elastic scattering of two identical, spinless particles with 4-momentum q_i and with mass $q_i^2 = m^2$. We define the usual Mandelstam variables

$$\begin{aligned}s &= (q_1 + q_2)^2 = (q_3 + q_4)^2 \\t &= (q_1 - q_3)^2 = (q_4 - q_2)^2 \\u &= (q_1 - q_4)^2 = (q_2 - q_3)^2\end{aligned}$$

We refer to the s -channel as the physical region describing the process

$$1(q_1) + 2(q_2) \rightarrow 3(q_3) + 4(q_4),$$

while in the t -channel we consider

$$1(q_1) + \bar{3}(q_{\bar{3}}) \rightarrow \bar{2}(q_{\bar{2}}) + 4(q_4).$$

The $J = 0$ is trivial

$$A^0(s, t) = i g_0 \frac{1}{m_0^2 - t} = i g_0 \frac{P_0(\cos \theta_t)}{m_0^2 - t}. \quad (6)$$

For $J = 1$ use $q_1 = (\sqrt{t}/2, q_t \hat{z})$ and $q_{\bar{3}} = (\sqrt{t}/2, -q_t \hat{z})$. In the t -channel CM frame we have $k = (q_1 - q_3) = (q_1 + q_{\bar{3}}) = (\sqrt{t}, \vec{0})$ and

$$-\tilde{g}_{\mu\nu} = -g_{\mu\nu} + \frac{k_\mu k_\nu}{t} = -[\delta_{\mu 0} \delta_{\nu 0} - \delta_{ij}] + \frac{\sqrt{t}^2}{t} \delta_{\mu 0} \delta_{\nu 0} = +\delta_{ij}. \quad (7)$$

thus $q_1^\mu \tilde{g}_{\mu\nu} q_2^\nu = \vec{q}_1 \cdot \vec{q}_2 = q_t^2 \cos \theta_t$. Similarly $q_1^\mu \tilde{g}_{\mu\nu} q_1^\nu = q_2^\mu \tilde{g}_{\mu\nu} q_2^\nu = q_t^2$ and we have

$$A^1(s, t) = ig_1 q_t^2 \frac{\cos \theta_t}{m_1^2 - t} = ig_1 q_t^2 \frac{P_1(\cos \theta_t)}{m_1^2 - t}, \quad (8)$$

and finally also

$$A^2(s, t) = ig_2 q_t^4 \frac{\frac{1}{2}(3 \cos \theta_t - 1)}{m_2^2 - t} = ig_2 q_t^4 \frac{P_2(\cos \theta_t)}{m_2^2 - t}. \quad (9)$$

The generalization to arbitrary J is

$$A^J(s, t) = ig_J q_t^{2J} \frac{P_J(\cos \theta_t)}{m_J^2 - t}. \quad (10)$$

(b) Unitarity vs Elementary exchanges

Express the amplitude entirely in terms of invariants s and t . Use the optical theorem to relate the elastic amplitude to a total hadronic cross section:

$$\sigma_{\text{tot}} = \frac{1}{2q\sqrt{s}} \Im A^J(s, t = 0). \quad (11)$$

Unitarity (via the Froissart-Martin bound) prohibits σ_{tot} from growing faster than $\log^2 s$ as $s \rightarrow \infty$. What is then the maximal spin a single elementary exchange can have while satisfying this bound? Why is this a problem?

Solution: We have

$$\sigma_{\text{tot}} = \frac{1}{2q\sqrt{s}} \frac{g_J}{m_J^2} q_t^{2J} P_J(\cos \theta_t) \Big|_{t=0} . \quad (12)$$

We have $q_t^2 \cos \theta_t = (s - u)/4$ so that as $s \rightarrow \infty$, we have:

$$\sigma_{\text{tot}} \sim s^{J-1} . \quad (13)$$

To satisfy the Froissart bound, the maximally allowed spin then is $J = 1$.

(c) **Van Hove Reggeon**

Consider an amplitude of the form

$$A(s, t) = \sum_{J=0}^{\infty} g r^{2J} \frac{(q_t^2 \cos \theta_t)^J}{J - \alpha(t)} . \quad (14)$$

Here $\alpha(t) = \alpha(0) + \alpha' t$ is a real, linear Regge trajectory, g is a dimensionless coupling constant and $r \sim 1$ fm is a range parameter. Compare Eq. 14 with Eq. 1, write the mass of the J th pole, m_J^2 , as a function of the Regge parameters $\alpha(0)$ and α' . Interpret the pole structure in terms of the spectrum of particles in the model.

If the sum is truncated to a finite J_{\max} , and we take the $s \rightarrow \infty$ limit, what is the high energy behavior of the amplitude?

Solution: We can write

$$J - \alpha(t) = J - \alpha(0) - \alpha' t = \alpha' ((J - \alpha(0))/\alpha' - t) \quad (15)$$

and thus we have $m_J^2 = (J - \alpha(0))/\alpha'$.

We can use

$$(\cos \theta_t)^J = \sum_{J+J' \text{ even}}^J \frac{(J+1)!}{(J-J')!! (J+J'+1)!!} P_{J'}(\cos \theta_t) \quad (16)$$

$$= \sum_{J+J' \text{ even}}^J \mu_{JJ'} P_{J'}(\cos \theta_t) \quad (17)$$

to write

$$A(s, t) = \sum_J \sum_{J'=0}^J \left(\frac{g r^{2J} \mu_{JJ'}}{\alpha'} \right) q_t^{2J} \frac{P_{J'}(\cos \theta_t)}{m_J^2 - t} , \quad (18)$$

$$= \sum_J \sum_{J'=0}^J g_{JJ'} q_t^{2J} \frac{P_{J'}(\cos \theta_t)}{m_J^2 - t} , \quad (19)$$

Comparing with the form of our elementary exchanges, this amplitude is an infinite sum of particles with spin- J and mass m_J^2 but also all same parity daughters at the same mass.

If the sum is truncated at J_{\max} the $s \rightarrow \infty$ limit is dominated by the largest spin exchange and we have $A_{\text{trunc}}(s, t) \propto s^{J_{\max}}$.

(d) **Analytic continuation in J**

Show that if the summation is kept infinite, the amplitude can be re-summed to something that is entirely analytic in s , t , u , and J .

Hint: Use the Mellin transform

$$\frac{1}{J - \alpha(t)} = \int_0^1 dx x^{J-\alpha(t)-1}, \quad (20)$$

to express the amplitude in terms of the Gaussian hypergeometric function and the Euler Beta function

$$B(b, c-b) {}_2F_1(1, b, c; z) = \int_0^1 dx \frac{x^{b-1} (1-x)^{c-b-1}}{1-xz}. \quad (21)$$

Solution: Go back to the original form in terms of monomials, we can write

$$A(s, t) = \sum_{J=0} \int_0^1 dx g r^{2J} (q_t^2 \cos \theta_t)^J x^{J-\alpha(t)-1}. \quad (22)$$

Collecting all things with powers of J , we notice a geometric series which can be summed analytically

$$A(s, t) = g \int_0^1 dx \frac{x^{-\alpha(t)-1}}{1 - r^2 q_t^2 \cos \theta_t x}. \quad (23)$$

Comparing with the definition of the hypergeometric function, we can identify $z = r^2 q_t^2 \cos \theta_t$ and $b = -\alpha(t)$. Since there is no $(1-x)$ term we require $c = b+1 = 1-\alpha(t)$. Thus we have

$$A(s, t) = \frac{\Gamma(-\alpha(t))}{\Gamma(1-\alpha(t))} {}_2F_1(1, -\alpha(t), 1-\alpha(t), (q_t r)^2 \cos \theta_t) \quad (24)$$

$$= \Gamma(-\alpha(t)) {}_2\tilde{F}_1(1, -\alpha(t), 1-\alpha(t), (q_t r)^2 \cos \theta_t). \quad (25)$$

(e) Unitarity vs Reggeized exchanges

Revisit b) with the resummed amplitude. Take the $s \rightarrow \infty$ limit and set a limit on the maximal intercept $\alpha(0)$ which is allowed by unitarity.

Hint: Assume that $\alpha(0) > -1$ and use the asymptotic behavior of the hypergeometric function given by

$${}_2F_1(1, b, c; z) \rightarrow \frac{\Gamma(c)\Gamma(1-b)}{\Gamma(1)\Gamma(c-b)} (-z)^{-b}. \quad (26)$$

Solution: From the hypergeometric form we can take $s \rightarrow \infty$ which takes $q_t^2 \cos \theta_t = (s-u)/4 \rightarrow \infty$ and we can write

$$A(s, t) = g_0 \Gamma(-\alpha(t)) \Gamma(1+\alpha(t)) \left(\frac{u-s}{4r^2} \right)^{\alpha(t)}. \quad (27)$$

So, we have

$$\Im A(s, 0) \propto \Im(-s)^{\alpha(0)} \propto \sin \pi \alpha(0) s^{\alpha(0)} \quad (28)$$

so $\sigma_{\text{tot}} \sim s^{\alpha(0)-1}$ and unitarity requires $\alpha(0) \leq 1$.

(f) The Reggeon “propagator”

Modify Eq. 1 to have a definite signature by defining

$$A^\pm(s, t) = \frac{1}{2} [A(s, t) \pm A(u, t)]. \quad (29)$$

Repeat d) and e) with this signatured amplitude. Compare with the canonical form of the Reggeon exchange:

$$A_R^\pm(s, t) = \beta(t) \frac{1}{2} \left[\pm 1 + e^{-i\pi\alpha(t)} \right] \Gamma(-\alpha(t)) \left(\frac{s}{s_0} \right)^{\alpha(t)}. \quad (30)$$

Identify the Regge residue $\beta(t)$ and characteristic scale s_0 in terms of the parameters g_0 and r .

Solution: As we see above, switching $s \leftrightarrow u$ introduces a minus sign and we have

$$A^\pm(s, t) = g_0 \Gamma(1 + \alpha(t)) \frac{1}{2} [\pm 1 + e^{-i\pi\alpha(t)}] \Gamma(-\alpha(t)) \left(\frac{s - u}{4r^2} \right)^{\alpha(t)}, \quad (31)$$

and we can read off $\beta(t) = g_0 \Gamma(1 + \alpha(t))$. Using $s \sim -u$ we also see $s_0 = 2r^{-2}$

8.2 Veneziano Amplitude

The quintessential dual amplitude was first proposed by Veneziano for $\omega \rightarrow 3\pi$ and later applied to elastic $\pi\pi$ scattering by Shapiro and Lovelace. Consider the $\pi^+\pi^-$ scattering amplitude of the form

$$\mathcal{A}(s, t, u) = V(s, t) + V(s, u) - V(t, u). \quad (32)$$

with each

$$V(s, t) = \frac{\Gamma(1 - \alpha(s)) \Gamma(1 - \alpha(t))}{\Gamma(1 - \alpha(s) - \alpha(t))}, \quad (33)$$

where $\alpha(s) = \alpha(0) + \alpha's$ is a real, linear Regge trajectory with $\alpha' > 0$.

(a) Duality

Show that the function $V(s, t)$ is symmetric in $s \leftrightarrow t$ and dual, i.e., it can be written entirely as a sum of either s -channel poles OR t -channel poles but never both simultaneously. Compare with the Reggeized amplitude in the previous problem, was that amplitude dual?

Hint: Relate $V(s, t)$ to the Euler Beta function

$$B(x, y) = \frac{\Gamma(x) \Gamma(y)}{\Gamma(x + y)} \quad (34)$$

and use the identities $B(x, y) = B(y, x)$ and

$$B(p - x, q - y) = \sum_{J=1}^{\infty} \frac{\Gamma(J - p + 1 + x)}{\Gamma(J) \Gamma(-p + 1 + x)} \frac{1}{J - 1 + q - y}. \quad (35)$$

Solution: We can write:

$$V(s, t) = (1 - \alpha(s) - \alpha(t)) B(1 - \alpha(s), 1 - \alpha(t)). \quad (36)$$

Then using the expansion of the Beta function on its first argument, we have

$$V(s, t) = (1 - \alpha(s) - \alpha(t)) \sum_{J=1}^{\infty} \frac{\Gamma(J - 1 + \alpha(t))}{\Gamma(J) \Gamma(\alpha(t))} \frac{1}{J - \alpha(s)} \quad (37)$$

which only has poles in $\alpha(s)$. Because of the $s \leftrightarrow t$ symmetry, we can write the exact same expression with only poles in $\alpha(t)$.

(b) Isospin basis

Define the s -channel isospin basis through

$$\begin{pmatrix} \mathcal{A}^{(0)}(s, t, u) \\ \mathcal{A}^{(1)}(s, t, u) \\ \mathcal{A}^{(2)}(s, t, u) \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 3 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} \mathcal{A}(s, t, u) \\ \mathcal{A}(t, s, u) \\ \mathcal{A}(u, t, s) \end{pmatrix}. \quad (38)$$

Write down the definite-isospin amplitudes in terms of V 's. Comment on the symmetry properties of each isospin amplitude with respect to $t \leftrightarrow u$.

Solution: We have:

$$\mathcal{A}^{(0)}(s, t, u) = \frac{1}{2} [3V(s, t) + 3V(s, u) - V(t, u)] \quad (39)$$

$$\mathcal{A}^{(1)}(s, t, u) = V(s, t) - V(s, u) \quad (40)$$

$$\mathcal{A}^{(2)}(s, t, u) = V(t, u) . \quad (41)$$

$I = 0, 2$ are symmetric in $t \leftrightarrow u$ while $I = 1$ is anti-symmetric as required by Bose symmetry.

(c) **Chew-Frautschi plot**

Locate where each $\mathcal{A}^{(I)}(s, t, u)$ will have poles in the s -channel physical region. What is their residue? Draw a schematic Chew-Frautschi plot of the resonance spectrum in each isospin channel.

Solution: A single $V(s, t)$ will have poles at all $\alpha(s) = J \geq 1$ and all possible daughters. The residues are

$$-(J-1+\alpha(t)) \frac{\Gamma(J-1+\alpha(t))}{\Gamma(J)\Gamma(\alpha(t))} = \frac{-1}{\Gamma(J)} \frac{\Gamma(J+\alpha(t))}{\Gamma(\alpha(t))} = -\frac{(\alpha(t))_J}{(J-1)!}. \quad (42)$$

For a linear trajectory this is a order J polynomial in t and therefore in z_s .

The symmetry factors in $I = 0, 1$ will remove all odd (even) J daughters. The $I = 2$ amplitude has no s dependence and therefore no isospin-2 poles at all.

(d) **Regge limit**

Now consider the limit $t \rightarrow \infty$ and $u \rightarrow -\infty$ with $s \leq 0$ is fixed. What is the asymptotic behavior of $V(s, t)$ and $V(s, u)$? Assume that $V(t, u)$ vanishes faster than any power of s in this limit. What is the resulting behavior of the isospin amplitudes $\mathcal{A}^{(I)}(s, t, u)$ in this limit?

Hint: Use the Sterling approximation of the Γ function., i.e. as $|x| \rightarrow \infty$

$$\Gamma(x) \rightarrow \sqrt{\frac{2\pi}{x}} \left(\frac{x}{e}\right)^x . \quad (43)$$

Solution: Starting with

$$V(s, t) \rightarrow \Gamma(1-\alpha(s)) (-\alpha(t))^{\alpha(s)} \sim \Gamma(1-\alpha(s)) (-\alpha' t)^{\alpha(s)} . \quad (44)$$

Similarly

$$V(s, u) \rightarrow \Gamma(1-\alpha(s)) (-\alpha(u))^{\alpha(s)} \sim \Gamma(1-\alpha(s)) (-\alpha' t)^{\alpha(s)} . \quad (45)$$

Thus the combination

$$V(s, t) \pm V(s, u) \rightarrow \Gamma(1-\alpha(s)) \times \left[(-\alpha' t)^{\alpha(s)} \pm (-\alpha' u)^{\alpha(s)} \right] \quad (46)$$

$$= \Gamma(1-\alpha(s)) \times \left[1 \pm e^{-i\pi\alpha(s)} \right] (\alpha' t)^{\alpha(s)} \quad (47)$$

(e) **Ancestors and Strings**

Consider the model now with a complex trajectory $\alpha(s) = a_0 + \alpha' s + i\Gamma$ with $\Gamma > 0$ to move the poles off the real axis. Reexamine the the Chew-Frautschi plot for the $I = 1$ amplitude using this trajectory, why is the resulting spectrum problematic? Try a real but non-linear trajectory, say $\alpha(s) = a_0 + \alpha' s + \alpha'' s^2$, what is the spectrum like now?

Compare the requirements of the trajectory for $V(s, t)$ to make sense with the energy levels of a rotating relativistic string with a string tension T :

$$E_J^2 = \frac{1}{2\pi T} J . \quad (48)$$

What is a possible microscopic picture of hadrons if the Veneziano amplitude is believed?

Solution: If we allow $\alpha(t)$ to be complex, then at a pole $\alpha(s) \rightarrow J + i\Gamma$ and the residue we calculated

$$\frac{\Gamma(J + i\Gamma + \alpha(t))}{\Gamma(\alpha(t))}, \quad (49)$$

is no longer a fixed order polynomial in t . It will thus give contributions to ALL spins at each pole, i.e. introduce an infinite number of ancestors. Similarly if $\alpha(s)$ is non-linear, we will have finitely many ancestors but still unphysical poles nonetheless.

This means the Veneziano amplitude *only* gives a physical picture for real and linear trajectories. This means we require $J \propto s \sim m^2$ which mimics the spectrum of states in a relativistic rotating string. This gives rise to the stringy picture of a $q\bar{q}$ pair connected by a gluon flux tube and later the entire field of string theories.

8.3 Sommerfeld-Watson Transform

(a) Geometric series

Prove the well known resummation of the geometric series:

$$1 + x + x^2 + x^3 + \dots = \frac{1}{1-x} \quad \text{for } |x| < 1 \quad (50)$$

can be analytically continued to $|x| \geq 1$ with the Sommerfeld-Watson Transform.

Assume that $|x| > 1$ and show that the summation can be written as an integral over the complex plane

$$\int \frac{d\ell}{2i} \frac{(-x)^\ell}{\sin \pi \ell} = 1 + x + x^2 + \dots . \quad (51)$$

Draw the contour around which the above integration should be taken (careful with orientations and signs). Deform the contour such that you can relate Eq. 51 to the series

$$\frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} + \dots = \frac{1}{1 - \frac{1}{x}} \quad (52)$$

and arrive at Eq. 50.

Solution: We want to show that we can analytically continue the geometric series to $|x| > 1$ using the Sommerfeld-Watson Transform.

First, we will prove that the sum can be written as an integral over the complex plane:

$$\int_C \frac{d\ell}{2i} \frac{(-x)^\ell}{\sin \pi\ell}, \quad (53)$$

where the function $1/\sin \pi\ell$ has poles at integer values of $\ell = \dots, -2, -1, 0, 1, 2, \dots$. We use the Cauchy Residue Theorem:

$$\oint f(z) dz = \pm 2\pi i \sum_k \text{Res}_{z=z_k} f(z), \quad (54)$$

with sign + for a counterclockwise contour and - for a clockwise contour around the pole at z_k . The residue of $f(\ell) = (-x)^\ell / \sin \pi\ell$ at $\ell = k$ is given by

$$\text{Res}_{\ell=k} \frac{(-x)^\ell}{\sin \pi\ell} = \lim_{\ell \rightarrow k} (\ell - k) \frac{(-x)^\ell}{\sin \pi\ell} = \lim_{\ell \rightarrow k} \frac{(-x)^\ell}{\pi \cos \pi\ell} = \frac{x^k}{\pi} \quad (55)$$

Therefore, if we encircle all the poles at $\ell = k$ for $k \geq 0$ with counterclockwise contours C_k , which we can combine to a single counterclockwise contour C encircling all the poles at 0 and positive integers, we obtain the geometric series:

$$\sum_{k=0}^{\infty} \int_{C_k} \frac{d\ell}{2i} \frac{(-x)^\ell}{\sin \pi\ell} = \int_C \frac{d\ell}{2i} \frac{(-x)^\ell}{\sin \pi\ell} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \dots \quad (56)$$

Next, we deform the contour to a vertical line from $\sigma + i\infty$ to $\sigma - i\infty$, with $-1 < \sigma < 0$, and further deform it to enclose all negative integers in a clockwise contour C' , that we can split in individual clockwise contours $C_{k'}$:

$$\begin{aligned} \int_{C'} \frac{d\ell}{2i} \frac{(-x)^\ell}{\sin \pi\ell} &= \sum_{C_{k'}} \int_{C_{k'}} \frac{d\ell}{2i} \frac{(-x)^\ell}{\sin \pi\ell} = -\frac{1}{x} - \frac{1}{x^2} - \frac{1}{x^3} - \dots \\ &= -\frac{1}{x} \left(1 + \frac{1}{x} + \frac{1}{x^2} + \dots \right) \end{aligned} \quad (57)$$

and given the assumption $|x| > 0$, we have $|1/x| < 1$, and we can sum the geometric series:

$$-\frac{1}{x} \left(1 + \frac{1}{x} + \frac{1}{x^2} + \dots \right) = -\frac{1}{x} \frac{1}{1 - \frac{1}{x}} = \frac{1}{1 - x}. \quad (58)$$

(b) Van Hove Reggeon

Revisit the Regge behavior of Eq. 14 using the S-W transform. How does the inclusion of poles at $\alpha(s) = \ell$ change the contour of integration and the leading contribution to the asymptotic behavior?

Solution: If we include a Regge pole $[\ell - \alpha(t)]^{-1}$, in the process of deforming the contour from C to C' we have to pick the residue of the pole at $\ell = \alpha(t)$, with $\text{Re } \alpha(t) < 0$. This means we have an extra clockwise contour C_α , from which we obtain an extra contribution $\propto x^{\alpha(t)}$. Since $x \sim q_t^2 z_t \sim s$ this yields the $s^{\alpha(t)}$ behavior.

8.4 Finite Energy Sum Rules

Consider z a complex variable and α a real fixed parameter. What is the analytic structure of the function z^α ? What is the discontinuity across the cut?

Write a Cauchy contour C surrounding the cut and closing it with a circle of radius Λ in the complex z

plane, and check that

$$\oint_C z^\alpha dz = 0 \quad (59)$$

You can start with the simple case $\alpha = 1/2$, i.e. \sqrt{z} , then generalize to any real α .

Solution: The function z^α has a branch cut for $z \in [-\infty, 0]$. The Cauchy contour enclose the cut and the discontinuity across that cut is

$$(z + i\epsilon)^\alpha - (z - i\epsilon)^\alpha = (|z|e^{i\pi})^\alpha - (|z|e^{-i\pi})^\alpha \quad \text{for real negative } z \quad (60)$$

$$= |z|^\alpha (e^{i\pi\alpha} - e^{-i\pi\alpha}) \quad (61)$$

$$= 2i|z|^\alpha \sin \pi\alpha \quad (62)$$

The Cauchy contour is then, with C_Λ being the circle of radius Λ in the positive sense,

$$\oint z^\alpha dz = \int_{-\Lambda}^0 (z + i\epsilon)^\alpha dz + \int_0^{-\Lambda} (z - i\epsilon)^\alpha dz + \oint_{C_\Lambda} z^\alpha dz \quad (63)$$

$$= 2i \sin \pi\alpha \int_{-\Lambda}^0 |z|^\alpha dz + \oint_{C_\Lambda} z^\alpha dz \quad (64)$$

The first integral is easily done

$$2i \sin \pi\alpha \int_{-\Lambda}^0 |z|^\alpha dz = \frac{2i\Lambda^{\alpha+1}}{\alpha+1} \sin \pi\alpha. \quad (65)$$

For the second integral, we need the change of variable $z = \Lambda \exp(i\theta)$, with $\theta \in [-\pi, \pi]$. We obtain

$$\oint_{C_\Lambda} z^\alpha dz = i\Lambda^{\alpha+1} \int_{-\pi}^{\pi} e^{i\theta(\alpha+1)} d\theta = \frac{i\Lambda^{\alpha+1}}{\alpha+1} (e^{i\pi(\alpha+1)} - e^{-i\pi(\alpha+1)}) \quad (66)$$

$$= -\frac{2i\Lambda^{\alpha+1}}{\alpha+1} \sin \pi\alpha \quad (67)$$

We used $\exp(i\pi) = -1$.