# MTHS24 – Exercise sheet 6

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Saturday, 20 July 2024

## Lecture material

**Discussed topics**:

- Lippmann-Schwinger equation, Bethe-Salpeter equation, and K-matrix
- Lineshape analysis and Briet-Wigner formula
- Complex algebra, dispersion relations
- Analytic continuation and pole search
- Khury-Treiman equations

**References**:

- A.D. Martin, T.D. Spearman, Elementary Particle Theory, inSpire
- Review on Novel approaches in hadron spectroscopy by JPAC, inspire

## Exercices

### 6.1 Escape room in the complex plane

(a) Characterize the complex structure of functions $\sqrt{x}$ and $\log(-x)$ by finding the branch points, branch cuts and number of complex (Riemann) sheets in the complex plane.

(b) Repeat (a) for a function $f(x) = \sqrt{x} - \sqrt{x-1}$.

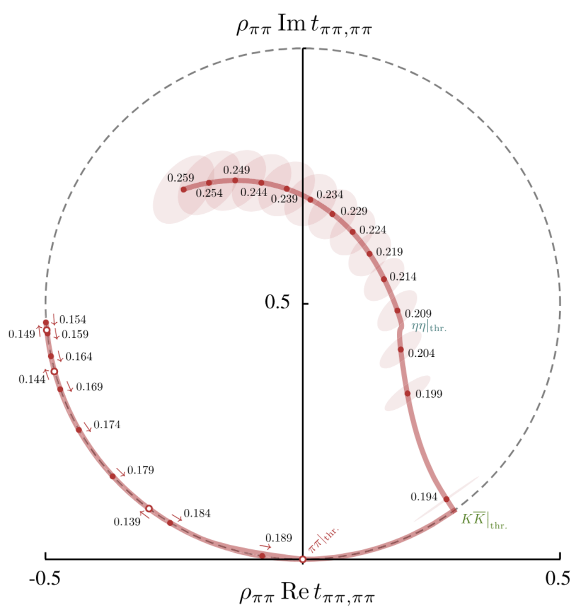
(c) Construct a complex function with two branch points at +i and -i connected by a branch cut.

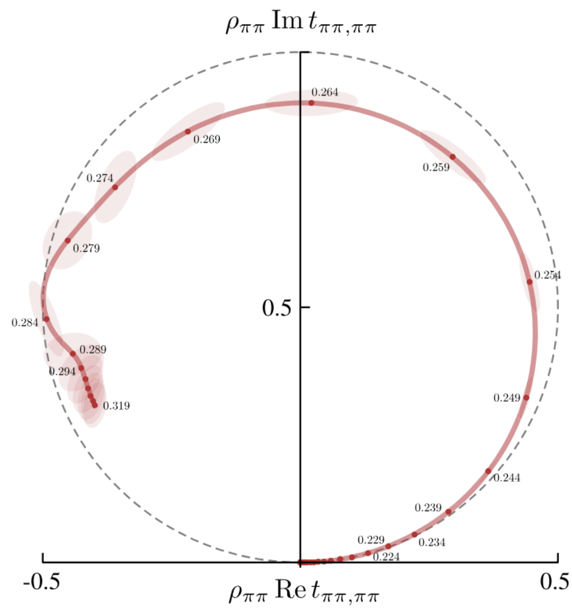
(d) Locate zeros of the function $g(z) = \sqrt{z} + i + 1$.

(e) Find residue of the function $1/g(z)$ by computing a circular integral about the complex pole.

## 6.2 Argand diagrams from lattice

The $\pi\pi$ scattering with unphysical pion mass ($m_\pi = 391\,\text{MeV}$) for S (left) and D (right) partial waves is studied using lattice calculations. Scattering amplitudes are presented on the Argand diagrams (parametric plot of energy in Real/Imaginary coordinates) as a function of energy of the system. The value are given units of $E_{\text{cm}} \cdot t$ where $t \cdot m_\pi = 0.06906$.




Using information on the diagrams, answer the following questions:

(a) Estimate masses of $K$ and $\eta$ particles.

(b) Find the elastic energy region for the S and D waves.

> **Solution:** Elastic region is defined as the range of energy values for which $\pi\pi \to \pi\pi$ process dominates. For the S-wave, the elastic region lies for the $E_{\text{cm}}t$ range of [0.139,0.189] (after this point, the amplitude hits the $K\bar{K}$ threshold. Also, after this point, the curve starts going inside the unitarity circle). For the D-wave, this region exists until value of 0.229.

(c) Locate the energy value for which the S-wave peak.

> **Solution:** The S-wave peaks at the point $E_{\text{cm}}t$ = 0.154, which is at energy $E_{\text{cm}} = 0.872$ GeV.

(d) Estimate the mass and decay width for the D wave resonance.

> **Solution:** The D-wave resonance is observed at $E_{\text{cm}}t = 0.284$, this is the point where there is a kink in the argand diagram. *Note : I can locate where the resonance is. I am confused about how to proceed from there, because I can get center of mass energy from the point, and maybe equate it to the pole value. But I would expect a complex output but I cannot read it out properly from the Argand diagram.*

(e) Sketch the amplitude phase versus energy of the system for both partial waves.

## 6.3 Cartesian tensors

Cartesian tensors are tensors in three-dimensional Euclidean space. The available tensors are:

- Rank 0: 1
- Rank 1: $k^i$
- Rank 2: $k^i k^j$, $delta^{ij}$, $\epsilon^{ijl}k_l$

- Rank 3: $\epsilon^{ijl}$
- Rank 4: combinations of all the above

(a) Show that $\epsilon^{ijl}k_j k_l$ is not a rank 1 tensor.

(b) Show that $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$, using the following property: $(\mathbf{A} \times \mathbf{B})^i = \epsilon^{ijl} A_j B_i$
**Hint**:

$$\epsilon_{ijl}\epsilon_{ij'l'} = \delta_{jj'}\delta_{ll'} - \delta_{jl'}\delta_{j'l} \tag{1}$$

(c) Given $\int d^3k\, f(k)\, k^i = 0$ for a scalar function $f(k)$. Show that:

$$\int d^3k\, f(k)\, k^i k^j = \frac{1}{3}\delta^{ij} \int d^3k\, f(k)\, k^2\,. \tag{2}$$

(d) Show that

$$\int d^3k\, f(k,\hat{p})\, k^i k^j = \frac{1}{2}\int d^3k\, f\left[k^2 - (k\cdot\hat{p})^2\right]\delta^{ij} + \frac{1}{2}\int d^3k\, f\left[3(k\cdot\hat{p})^2 - k^2\right]\hat{p}^i\hat{p}^j\,. \tag{3}$$

## 6.4 Photons in Cartesian Basis

Consider photons (or gluons) in Cartesian basis so $a^+_{k_i} = \sum_\lambda a^+_{k_\lambda}\epsilon^i(k_\lambda)\,$.

(a) Show that the "scalar photon ball" is just a scalar state $\gamma\gamma$ that can be written as:

$$|\gamma\gamma; 0^+\rangle \propto \int d^3k\, \phi(k)\, a^+_{k_i} a^+_{-k_i}|0\rangle\,, \tag{4}$$

where $\phi(k)$ is the momentum wave function.

(b) Show that:

$$|\gamma\gamma; 0^-\rangle \propto \int d^3k\, \phi(k)\, \epsilon_{ijl}k^l a^+_{k_i} a^+_{-k_j}|0\rangle\,. \tag{5}$$

(c) Prove the Lee-Yang theorem which states that one cannot construct a $J=1$, $\gamma\gamma$ state.

(d) Show that:

$$|\gamma\gamma\gamma; 0^-\rangle = \int d^3k_1 d^3k_2 d^3k_3\, \phi(k_1k_2k_3)\, \epsilon_{i_1i_2i_3}\delta(k_1k_2k_3) a^+_{k_1i_1} a^+_{k_2i_2} a^+_{k_3i_3}|0\rangle\,, \tag{6}$$

is a viable state.

(e) Can we construct a $|\gamma\gamma\gamma; 1^-\rangle$ state? (**Hint:** $Pa^+_{k_i}P^+ = -a^+_{-k_i}$)