



## MTHS24 – Exercise sheet 2

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## Lecture material

### Discussed topics:

- Amplitude generalities
- Canonical and helicity states
- Analyticity
- Unitarity
- Crossing symmetry

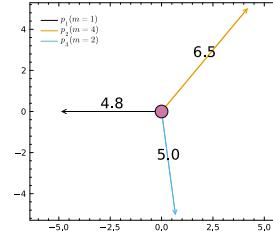
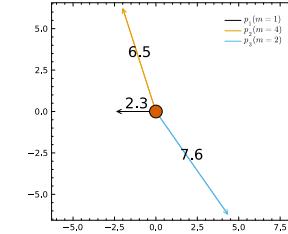
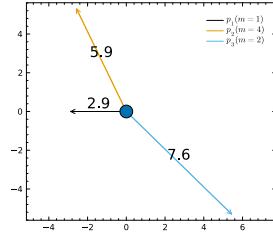
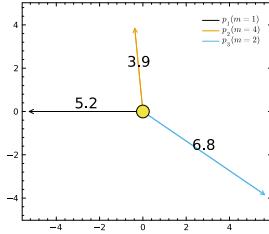
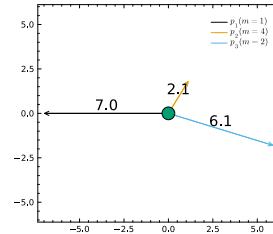
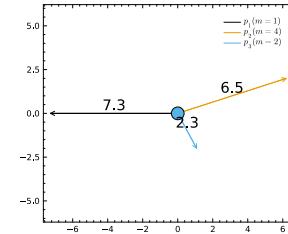
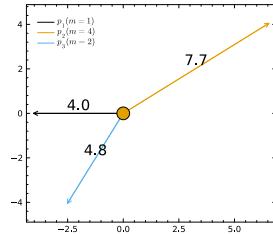
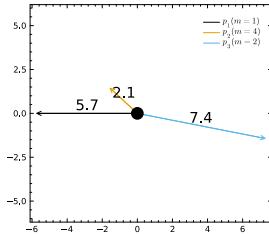
### References:

- A. Martin and D. Spearman, Elementary Particle Theory, [Google Books](#)

## Exercises

### 2.1 Decay kinematics

Relate the values of momenta and their orientation to location on the Dalitz plot for the decay  $p_0 \rightarrow p_1 + p_2 + p_3$  using values indicated on the figures below.

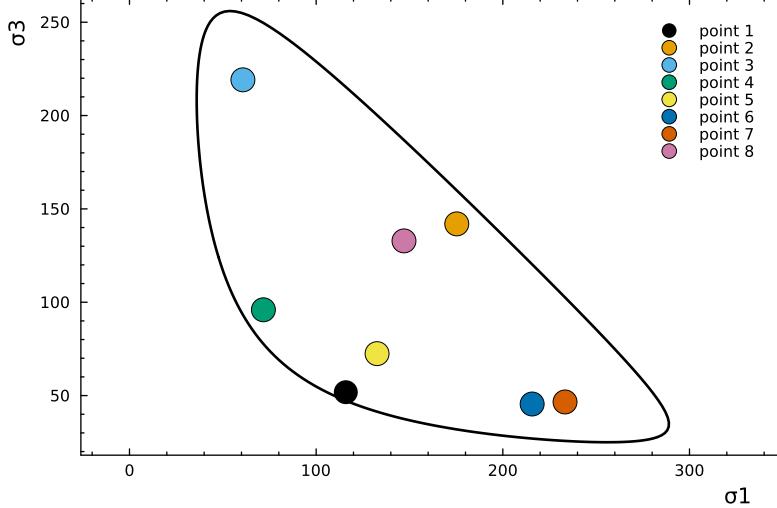


**Solution:** Mass is computed as

$$\begin{aligned} m_{ij} &= (p_i + p_j)^2 = m_i^2 + m_j^2 + 2E_i E_j - 2p_i p_j \cos \angle \vec{p}_i \vec{p}_j \\ &= m_i^2 + m_j^2 + 2E_i E_j + p_i^2 + p_j^2 - p_k^2 \\ &= (E_i + E_j)^2 - p_k^2 \end{aligned}$$

the angle between the two momenta is traded to magnitude of momenta using algebra of three-vectors,  $-\vec{p}_k = \vec{p}_i + \vec{p}_j$

$$p_k^2 = p_i^2 + p_j^2 + 2p_i p_j \cos \angle \vec{p}_i \vec{p}_j$$



## 2.2 2-to-2 Kinematics

In hadron physics, the Mandelstam invariants  $s$ ,  $t$ , and  $u$  are essential for describing the kinematics of scattering processes. Consider a  $2 \rightarrow 2$  scattering process where two particles with four-momenta  $p_1$  and  $p_2$  scatter into two particles with four-momenta  $p_3$  and  $p_4$ . Consider the particles have different masses. The Mandelstam invariants are defined as  $s = (p_1 + p_2)^2$ ,  $t = (p_1 - p_3)^2$ , and  $u = (p_1 - p_4)^2$ . The scattering angle  $\theta$  is the angle between the momenta of the incoming and outgoing particles in the center of mass frame.

- (a) Verify the relation  $s + t + u = \sum m_i^2$ , where  $m_i$  are the masses of the particles.
- (b) Calculate the particle energies and 3-momenta in the center of mass frame, as a function of  $s$
- (c) Calculate the scattering angle  $\theta$  in terms of  $s$ ,  $t$ , and  $u$
- (d) How would those relation change if one considered the crossed reaction  $p_1 p_3 \rightarrow p_2 p_4$ ?

## 2.3 Canonical and Helicity states

Consider a particle of spin  $1/2$  at rest with spin up  $u_+ = \sqrt{2m} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$  or spin down,  $u_- = \sqrt{2m} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ .

Remember that the generator of Lorentz group is  $\sigma^{\mu\nu} = \frac{i}{2} [\gamma^\mu, \gamma^\nu]$ . That implies that

$$R_y(\theta) = \exp\left(-i\frac{\theta}{2}\sigma_y\right) = \begin{pmatrix} \cos\frac{\theta}{2} - i\sigma_y \sin\frac{\theta}{2} & 0 \\ 0 & \cos\frac{\theta}{2} + i\sigma_y \sin\frac{\theta}{2} \end{pmatrix} \quad (1)$$

$$B(0 \rightarrow \vec{p}) = \exp\left(\frac{\vec{\eta}}{2} \cdot \vec{\alpha}\right) = \begin{pmatrix} \cosh\frac{\eta}{2} & \hat{\eta} \cdot \vec{\sigma} \sinh\frac{\eta}{2} \\ \hat{\eta} \cdot \vec{\sigma} \sinh\frac{\eta}{2} & \cosh\frac{\eta}{2} \end{pmatrix} \quad (2)$$

where  $\tanh \eta = p/E$ ,  $\eta$  being the particle rapidity.

- (a) Derive the spinors in canonical and helicity basis for a particle with momentum lying in the  $xz$  plane.

## 2.4 Boosting spinning particles

A particle of spin 1 in the helicity basis lying in the  $xz$  plane is described by the following polarization vectors:

$$\epsilon_{\pm 1}^\mu(\theta) = \left( 0, \mp \frac{\cos \theta}{\sqrt{2}}, -\frac{i}{\sqrt{2}}, \pm \frac{\sin \theta}{\sqrt{2}} \right) \quad (3)$$

$$\epsilon_0^\mu(\theta) = \left( \frac{p}{m}, \sin \theta \frac{E}{m}, 0, \cos \theta \frac{E}{m} \right) \quad (4)$$

$$p^\mu(\theta) = (E, \sin \theta p, 0, \cos \theta p) \quad (5)$$

Consider a particle with mass  $m$  and momentum  $p = m\hat{z}$  and with helicity +1.

- (a) Boost the momentum and the polarization in the  $x$  direction of a boost  $\beta = 1/\sqrt{2}$ .
- (b) Compare with the polarization vectors according to the new momentum. Decompose it in the basis of the new polarization vectors.