



## MTHS24 – Exercise sheet 6

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## Lecture material

### Discussed topics:

- Lippmann-Schwinger equation, Bethe-Salpeter equation, and K-matrix
- Lineshape analysis and Breit-Wigner formula
- Complex algebra, dispersion relations
- Analytic continuation and pole search
- Khury-Treiman equations

### References:

- A.D. Martin, T.D. Spearman, Elementary Particle Theory, [inSpire](#)
- Review on Novel approaches in hadron spectroscopy by JPAC, [inspire](#)

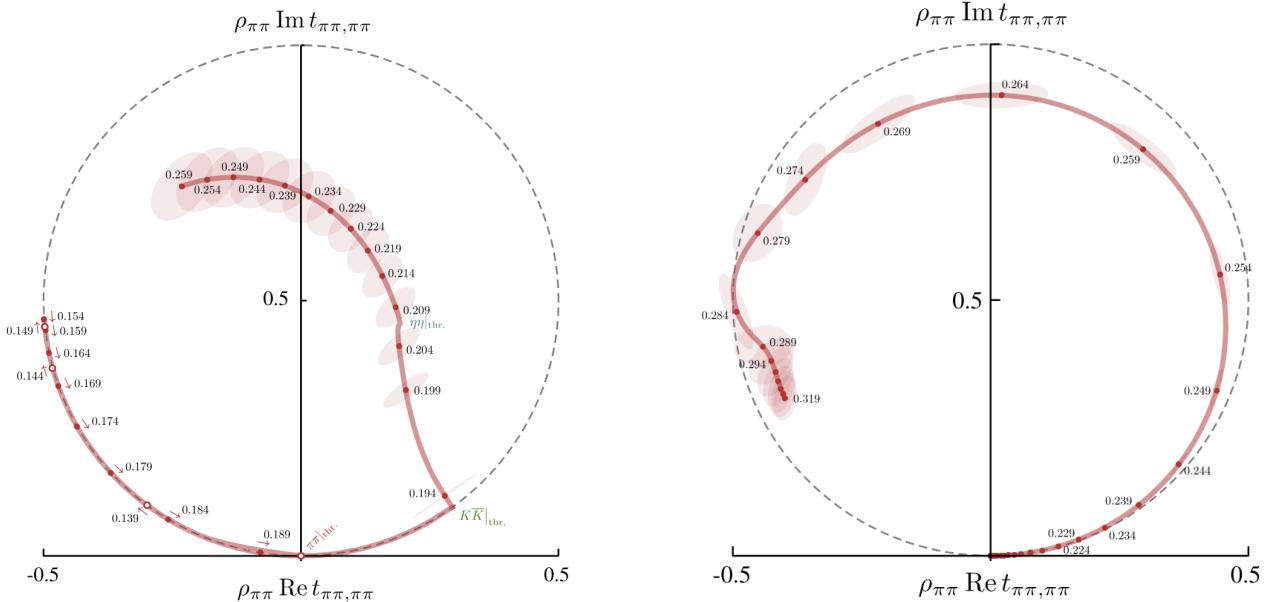
## Exercises

### 6.1 Escape room in the complex plane

- (a) Characterize the complex structure of functions  $\sqrt{x}$  and  $\log(-x)$  by finding the branch points, branch cuts and number of complex (Riemann) sheets in the complex plane.
- (b) Repeat (a) for a function  $f(x) = \sqrt{x} - \sqrt{x-1}$ .
- (c) Construct a complex function with two branch points at  $+i$  and  $-i$  connected by a branch cut.
- (d) Locate zeros of the function  $g(z) = \sqrt{z} + i + 1$ .
- (e) Find residue of the function  $1/g(z)$  by computing a circular integral about the complex pole.

## 6.2 Argand diagrams from lattice

The  $\pi\pi$  scattering with unphysical pion mass ( $m_\pi = 391$  MeV) for S (left) and D (right) partial waves is studied using [lattice calculations](#). Scattering amplitudes are presented on the Argand diagrams (parametric plot of energy in Real/Imaginary coordinates) as a function of energy of the system. The values are given in units of  $E_{\text{cm}} \cdot t$  where  $t \cdot m_\pi = 0.06906$ .



Using information on the diagrams, answer the following questions:

- Estimate masses of  $K$  and  $\eta$  particles.
- Find the elastic energy region for the S and D waves.

**Solution:** Elastic region is defined as the range of energy values for which  $\pi\pi \rightarrow \pi\pi$  process dominates. For the S-wave, the elastic region lies for the  $E_{\text{cm}t}$  range of [0.139, 0.189] (after this point, the amplitude hits the  $K\bar{K}$  threshold). Also, after this point, the curve starts going inside the unitarity circle). For the D-wave, this region exists until value of 0.229.

- Locate the energy value for which the S-wave peak.

**Solution:** The S-wave peaks at the point  $E_{\text{cm}t} = 0.154$ , which is at energy  $E_{\text{cm}} = 0.872$  GeV.

- Estimate the mass and decay width for the D wave resonance.

**Solution:** The D-wave resonance is observed at  $E_{\text{cm}t} = 0.284$ , this is the point where there is a kink in the argand diagram. Note : I can locate where the resonance is. I am confused about how to proceed from there, because I can get center of mass energy from the point, and maybe equate it to the pole value. But I would expect a complex output but I cannot read it out properly from the Argand diagram.

- Sketch the amplitude phase versus energy of the system for both partial waves.

## 6.3 Cartesian tensors

Cartesian tensors are tensors in three-dimensional Euclidean space. The available tensors are:

- Rank 0: 1
- Rank 1:  $k^i$
- Rank 2:  $k^i k^j$ ,  $\delta^{ij}$ ,  $\epsilon^{ijl} k_l$

- Rank 3:  $\epsilon^{ijl}$
  - Rank 4: combinations of all the above
- (a) Show that  $\epsilon^{ijl}k_j k_l$  is not a rank 1 tensor.

**Solution:** The Levi-Civita tensor  $\epsilon^{ijl}$  is antisymmetric under the exchange of any two indices, i.e.  $\epsilon^{ijl} = -\epsilon^{ilj}$ . Then

$$\epsilon^{ijl}k_j k_l = -\epsilon^{ilj}k_l k_j$$

and since the product of two components of the same vector is commutative thus  $k_j k_l = k_l k_j$  we have:

$$\epsilon^{ijl}k_j k_l = -\epsilon^{ilj}k_l k_j$$

This implies:

$$\epsilon^{ijl}k_j k_l = 0$$

Therefore,  $\epsilon^{ijl}k_j k_l$  is identically zero and does not form a valid rank 1 tensor (does not behave as rank 1 tensor).

- (b) Show that  $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$ , using the following property:  $(\mathbf{A} \times \mathbf{B})^i = \epsilon^{ijl} A_j B_l$
- Hint:**

$$\epsilon_{ijl}\epsilon_{ij'l'} = \delta_{jj'}\delta_{ll'} - \delta_{jl'}\delta_{j'l} \quad (1)$$

**Solution:** Starting with the vector identity:

$$(\mathbf{A} \times \mathbf{B})^i = \epsilon^{ijl} A_j B_l$$

Let  $\mathbf{C} = \nabla \times \mathbf{A}$ , then:

$$C^i = (\nabla \times \mathbf{A})^i = \epsilon^{ijl} \partial_j A_l$$

Now, consider the curl of  $\mathbf{C}$ , and use the property of the Levi-Civita symbol:

$$\begin{aligned} (\nabla \times \mathbf{C})^i &= \epsilon^{imn} \partial_m C_n = \epsilon^{imn} \partial_m (\epsilon^{njl} \partial_j A_l) \\ &= \epsilon^{imn} \epsilon^{njl} \partial_m \partial_j A_l \\ &= (\delta^{ij} \delta^{ml} - \delta^{il} \delta^{mj}) \partial_m \partial_j A_l \\ &= \partial_i \partial_l A_l - \partial_j \partial_j A_i \\ &= \nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} \end{aligned}$$

- (c) Given  $\int d^3 k f(k) k^i = 0$  for a scalar function  $f(k)$ . Show that:

$$\int d^3 k f(k) k^i k^j = \frac{1}{3} \delta^{ij} \int d^3 k f(k) k^2. \quad (2)$$

**Solution:** By symmetry, the integral  $\int d^3k f(k) k^i k^j$  must be proportional to  $\delta^{ij}$  because the left-hand side is a rank 2 tensor and the only isotropic rank 2 tensor is proportional to  $\delta^{ij}$ , then:

$$\int d^3k f(k) k^i k^j = A \delta^{ij}$$

To find  $A$ , take the trace:

$$\int d^3k f(k) k^i k^i = A \delta^{ii}$$

Since  $\delta^{ii} = 3$ , we have:

$$\int d^3k f(k) k^2 = 3A$$

So,

$$A = \frac{1}{3} \int d^3k f(k) k^2$$

Thus,

$$\int d^3k f(k) k^i k^j = \frac{1}{3} \delta^{ij} \int d^3k f(k) k^2$$

(d) Show that

$$\int d^3k f(k, \hat{p}) k^i k^j = \frac{1}{2} \int d^3k f [k^2 - (k \cdot \hat{p})^2] \delta^{ij} + \frac{1}{2} \int d^3k f [3(k \cdot \hat{p})^2 - k^2] \hat{p}^i \hat{p}^j. \quad (3)$$

**Solution:** The given integral can be split into parts proportional to  $\delta^{ij}$  and  $\hat{p}^i \hat{p}^j$ :

$$\int d^3k f(k, \hat{p}) k^i k^j = A\delta^{ij} + B\hat{p}^i \hat{p}^j$$

To find  $A$  and  $B$ , we take the trace and the contraction with  $\hat{p}^i \hat{p}^j$ :

(a) Trace:

$$\begin{aligned} \delta^{ij} \int d^3k f(k, \hat{p}) k^i k^j &= A\delta^{ii} + B(\hat{p}^i \hat{p}^i) \\ &\rightarrow \int d^3k f(k, \hat{p}) k^2 = 3A + B \end{aligned}$$

(b) Contraction with  $\hat{p}^i \hat{p}^j$ :

$$\begin{aligned} \hat{p}^i \hat{p}^j \int d^3k f(k, \hat{p}) k^i k^j &= A(\hat{p} \cdot \hat{p}) + B(\hat{p}^i \hat{p}^j \hat{p}^i \hat{p}^j) \\ &\rightarrow \int d^3k f(k, \hat{p}) (k \cdot \hat{p})^2 = A + B \end{aligned}$$

(c) Solving for  $A$  and  $B$ :

$$\begin{aligned} 3A + B &= \int d^3k f(k, \hat{p}) k^2 \\ A + B &= \int d^3k f(k, \hat{p}) (k \cdot \hat{p})^2 \end{aligned}$$

(d) Subtract the second equation from the first:

$$\begin{aligned} 2A &= \int d^3k f(k, \hat{p}) (k^2 - (k \cdot \hat{p})^2) \\ \text{Thus } A &= \frac{1}{2} \int d^3k f(k, \hat{p}) (k^2 - (k \cdot \hat{p})^2) \end{aligned}$$

(e) Using  $A$  in the second equation:

$$\begin{aligned} B &= \int d^3k f(k, \hat{p}) (k \cdot \hat{p})^2 - A \\ B &= \int d^3k f(k, \hat{p}) (k \cdot \hat{p})^2 - \frac{1}{2} \int d^3k f(k, \hat{p}) (k^2 - (k \cdot \hat{p})^2) \\ \text{So } B &= \frac{1}{2} \int d^3k f(k, \hat{p}) (3(k \cdot \hat{p})^2 - k^2) \end{aligned}$$

Therefore,

$$\int d^3k f(k, \hat{p}) k^i k^j = \frac{1}{2} \int d^3k f [k^2 - (k \cdot \hat{p})^2] \delta^{ij} + \frac{1}{2} \int d^3k f [3(k \cdot \hat{p})^2 - k^2] \hat{p}^i \hat{p}^j$$

## 6.4 Photons in Cartesian Basis

Consider photons (or gluons) in Cartesian basis so  $a_{k_i}^+ = \sum_\lambda a_{k_\lambda}^+ \epsilon^i(k_\lambda)$ .

(a) Show that the “scalar photon ball” is just a scalar state  $\gamma\gamma$  that can be written as:

$$|\gamma\gamma; 0^+\rangle \propto \int d^3k \phi(k) a_{k_i}^+ a_{-k_i}^+ |0\rangle, \quad (4)$$

where  $\phi(k)$  is the momentum wave function.

**Solution:** Apply the parity operator  $P$ :

$$\int d^3k \phi(k) P a_{k_i}^+ P^+ P a_{k_i}^+ P^+ |0\rangle = \int d^3k \phi(k) (-a_{-k_i}^+) (-a_{k_i}^+) |0\rangle$$

Now take  $k \rightarrow -k$  then:

$$\int d^3k \phi(-k) (-a_{k_i}^+) (-a_{-k_i}^+) |0\rangle = (+) \int d^3k \phi(k) (a_{k_i}^+) (a_{-k_i}^+) |0\rangle$$

since  $\phi(k)$  is a symmetric function (scalar),  $\phi(k) = \phi(-k)$ . Thus, we arrived to the desired state  $|\gamma\gamma; 0^+\rangle$  which is invariant under parity as expected for scalar states.

(b) Show that:

$$|\gamma\gamma; 0^-\rangle \propto \int d^3k \phi(k) \epsilon_{ijl} k^l a_{k_i}^+ a_{-k_j}^+ |0\rangle. \quad (5)$$

**Solution:** Apply the parity operator  $P$ :

$$\int d^3k \phi(k) \epsilon_{ijl} k^l P a_{k_i}^+ P^+ P a_{-k_j}^+ P^+ |0\rangle = \int d^3k \phi(k) \epsilon_{ijl} k^l (-a_{-k_i}^+) (-a_{k_j}^+) |0\rangle$$

(a) First way: we take  $k \rightarrow -k$  then

$$\begin{aligned} \int d^3k \phi(k) \epsilon_{ijl} k^l P a_{k_i}^+ P^+ P a_{-k_j}^+ P^+ |0\rangle &= \int d^3k \phi(k) \epsilon_{ijl} (-k^l) (a_{k_i}^+) (a_{-k_j}^+) |0\rangle \\ &= - \int d^3k \phi(k) \epsilon_{ijl} k^l (a_{k_i}^+) (a_{-k_j}^+) |0\rangle \end{aligned}$$

(b) Second way: Use the antisymmetric property of  $\epsilon_{ijl}$  i.e. (flipping  $i \leftrightarrow j$ ) where:

$$\epsilon_{ijl} k^l (a_{-k_i}^+) (a_{k_j}^+) = -\epsilon_{ijl} k^l (a_{k_i}^+) (a_{-k_j}^+)$$

Thus, we arrived to the desired state:  $|\gamma\gamma; 0^-\rangle$  which is also invariant under parity.

(c) Prove the Lee-Yang theorem which states that one cannot construct a  $J = 1, \gamma\gamma$  state.

**Solution:** We seek a rank 1 Cartesian tensor then:

$$|\gamma\gamma; J=1;l\rangle = \int d^3k \phi_{lij}(k) a_{k_i}^+ a_{-k_j}^+ |0\rangle,$$

We must have:

$$\phi_{lij}(k) = \phi(k)t_{lij} + \chi(k)\delta_{ij}k^l,$$

then

$$\begin{aligned} |\gamma\gamma; J=1;l\rangle &= \int d^3k [\phi(k)t_{lij} + \chi(k)\delta_{ij}k^l] a_{k_i}^+ a_{-k_j}^+ |0\rangle \\ k \rightarrow -k &= \int d^3k [\phi(-k)t_{lij} + \chi(-k)\delta_{ij}(-k^l)] a_{-k_i}^+ a_{k_j}^+ |0\rangle \\ &= \int d^3k [-\phi(k)t_{lij} - \chi(k)\delta_{ij}k^l] a_{-k_i}^+ a_{k_j}^+ |0\rangle \\ &= - \int d^3k [\phi(k)t_{lij} + \chi(k)\delta_{ij}k^l] a_{k_i}^+ a_{-k_j}^+ |0\rangle \\ &= -|\gamma\gamma; J=1;l\rangle \end{aligned}$$

Therefore  $|\gamma\gamma; J=1;l\rangle = 0$ , which validates Lee-Yang theorem that we cannot construct a  $J=1$   $\gamma\gamma$  state.

**Remark:** Note that here  $\phi(-k) = -\phi(k)$  since it is not scalar in this case,  $\chi(-k) = \chi(k)$  (scalar), and  $a_{-k_i}^+ a_{k_j}^+ = a_{k_i}^+ a_{-k_j}^+$  since the creation operators commutes in the case of Bosons (photons).

(d) Show that:

$$|\gamma\gamma\gamma; 0^-\rangle = \int d^3k_1 d^3k_2 d^3k_3 \phi(k_1 k_2 k_3) \epsilon_{i_1 i_2 i_3} \delta(k_1 k_2 k_3) a_{k_1 i_1}^+ a_{k_2 i_2}^+ a_{k_3 i_3}^+ |0\rangle, \quad (6)$$

is a viable state.

**Solution:** Apply the parity operator  $P$ :

$$\begin{aligned} &\int d^3k_1 d^3k_2 d^3k_3 \phi(k_1 k_2 k_3) \epsilon_{i_1 i_2 i_3} \delta(k_1 + k_2 + k_3) P a_{k_1 i_1}^+ P^+ P a_{k_2 i_2}^+ P^+ P a_{k_3 i_3}^+ P^+ |0\rangle \\ &= - \int d^3k_1 d^3k_2 d^3k_3 \phi(k_1 k_2 k_3) \epsilon_{i_1 i_2 i_3} \delta(k_1 + k_2 + k_3) a_{-k_1 i_1}^+ a_{-k_2 i_2}^+ a_{-k_3 i_3}^+ |0\rangle \\ k \rightarrow -k &= - \int d^3k_1 d^3k_2 d^3k_3 \phi((-k_1)(-k_2)(-k_3)) \epsilon_{i_1 i_2 i_3} \delta(-k_1 - k_2 - k_3) a_{k_1 i_1}^+ a_{k_2 i_2}^+ a_{k_3 i_3}^+ |0\rangle \\ &= - \int d^3k_1 d^3k_2 d^3k_3 \phi(k_1 k_2 k_3) \epsilon_{i_1 i_2 i_3} \delta(k_1 k_2 k_3) a_{k_1 i_1}^+ a_{k_2 i_2}^+ a_{k_3 i_3}^+ |0\rangle \end{aligned}$$

Thus, this state is a valid state.

(e) Can we construct a  $|\gamma\gamma\gamma; 1^-\rangle$  state? (**Hint:**  $P a_{k_i}^+ P^+ = -a_{-k_i}^+$ )

**Solution:** We seek to combine 3 vectors to form a vector state. Consider the angular momentum of three photons.

- Combining Angular Momenta:

- Start by combining two vectors to form an intermediate state.
- In the case of three vectors  $\mathbf{k}_1$ ,  $\mathbf{k}_2$ , and  $\mathbf{k}_3$ , we can form the combinations:

$$(\mathbf{k}_1 \times \mathbf{k}_2)_s = (\mathbf{k}_1 \mathbf{k}_2) + (\mathbf{k}_2 \mathbf{k}_1)$$

- This is symmetric in space.

- Cross Product for Antisymmetry:

- To obtain a  $1^-$  state, we need an antisymmetric combination.
- Take the cross product of the intermediate state with the third vector  $\mathbf{k}_3$ :  $(\mathbf{k}_1 \times (\mathbf{k}_2 \times \mathbf{k}_3))$

- Constructing the State:

- Combine all three vectors in such a way that respects the antisymmetry needed for a  $1^-$  state.

$$\int d^3k_1 d^3k_2 d^3k_3 \phi(k_1 k_2 k_3) \epsilon_{i_1 i_2 i_3} \delta(k_1 + k_2 + k_3) a_{k_1 i_1}^+ a_{k_2 i_2}^+ a_{k_3 i_3}^+ |0\rangle$$

- Symmetry Considerations:

- This state is symmetric under permutations of the three momenta  $\mathbf{k}_1$ ,  $\mathbf{k}_2$ , and  $\mathbf{k}_3$ .
- The use of  $\epsilon_{i_1 i_2 i_3}$  ensures the antisymmetry necessary for a pseudoscalar state.

- Parity Check:

- Under parity, the creation operator transforms as

$$P a_{k_i}^+ P^+ = -a_{-k_i}^+$$

- . - The state under parity transforms as:

$$P |\gamma\gamma\gamma; 1^-\rangle = (-1)^3 |\gamma\gamma\gamma; 1^-\rangle = -|\gamma\gamma\gamma; 1^-\rangle$$

(same as part(d))

This confirms that possibility of constructing a  $|\gamma\gamma\gamma; 1^-\rangle$  state using the above considerations.