



## MTHS24 – Exercise sheet 8

Morning: Vincent Mathieu / Andrew Jackura / Arkaitz Rodas

Afternoon: Daniel Winney, Gloria Montana



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## Lecture material

### References:

- P.D.B. Collins, "An Introduction to Regge Theory and High Energy Physics", [Inspire](#)
- V.N. Gribov, ("Blue book") "The theory of complex angular momentum", [Inspire](#)
- V.N. Gribov, ("Gold book") "Strong interactions of hadrons at high energies", [Inspire](#)
- D. Sivers & J. Yellin, "Review of recent work on narrow resonance models" [Inspire](#)

### Discussed topics:

- Regge theory
- High energy scattering
- Complex angular momentum
- Unitarity

## Exercises

### 8.1 Unitarity and Reggeons

Van Hove proposed a physically intuitive picture of a Reggeon by relating it to Feynman diagrams in the cross-channels. We will explore this picture of Reggeization with a simple model.

#### (a) Elementary $t$ -channel exchanges

Consider the amplitude corresponding to a particle with spin- $J$  and mass  $m_J$  exchanged in the  $t$ -channel as:

$$A^J(s, t) = i g_J (q_1^{\mu_1} \dots q_1^{\mu_J}) \frac{P_{\mu_1 \dots \mu_J, \nu_1 \dots \nu_J}^J(k)}{m_J^2 - t} (q_2^{\nu_1} \dots q_2^{\nu_J}) \quad (1)$$

where  $g_J$  is a coupling constant with dimension  $2-2J$  (i.e.,  $A^J(s, t)$  is dimensionless) and the projector of spin- $J$  is defined from the polarization tensor of rank- $J \geq 1$  as

$$P_{\mu_1 \dots \mu_J, \nu_1 \dots \nu_J}^J(k) = \frac{(J+1)}{2} \sum_{\lambda} \epsilon^{\mu_1 \dots \mu_J}(k, \lambda) \epsilon^{*\nu_1 \dots \nu_J}(k, \lambda) . \quad (2)$$

Using the exchange momentum  $k = q_1 + q_3 = q_1 - q_3$ , calculate the amplitudes corresponding to  $J = 0, 1, 2$  exchanges in terms of  $t = k^2$ , the modulus of 3-momentum and cosine of scattering angle in the  $t$ -channel frame,  $q_t$  and  $\cos \theta_t$  respectively. Use the explicit forms of the projectors:

$$P^0(k^2) = 1 \quad (3)$$

$$P_{\mu\nu}^1(k^2) \equiv \tilde{g}_{\mu\nu} = \frac{k_\mu k_\nu}{k^2} - g_{\mu\nu} \quad (4)$$

$$P_{\mu\nu\alpha\beta}^2(k^2) = \frac{3}{4} (\tilde{g}_{\mu\alpha} \tilde{g}_{\nu\beta} + \tilde{g}_{\mu\beta} \tilde{g}_{\nu\alpha}) - \frac{1}{2} \tilde{g}_{\mu\nu} \tilde{g}_{\alpha\beta} , \quad (5)$$

and conjecture a generalization of the amplitude for arbitrary integer  $J$ .

*Hint: Show that in the t-channel frame, the exchange particle is at rest and therefore  $\tilde{g}_{\mu\nu}$  reduces to a  $\delta_{ij}$  with respect to only spacial momenta.*

(b) **Unitarity vs Elementary exchanges**

Express the amplitude entirely in terms of invariants  $s$  and  $t$ . Use the optical theorem to relate the elastic amplitude to a total hadronic cross section:

$$\sigma_{\text{tot}} = \frac{1}{2q\sqrt{s}} \Im A^J(s, t=0) . \quad (6)$$

Unitarity (via the Froissart-Martin bound) prohibits  $\sigma_{\text{tot}}$  from growing faster than  $\log^2 s$  as  $s \rightarrow \infty$ . What is then the maximal spin a single elementary exchange can have while satisfying this bound? Why is this a problem?

(c) **Van Hove Reggeon**

Consider an amplitude of the form

$$A(s, t) = \sum_{J=0}^{\infty} g r^{2J} \frac{(q_t^2 \cos \theta_t)^J}{J - \alpha(t)} . \quad (7)$$

Here  $\alpha(t) = \alpha(0) + \alpha' t$  is a real, linear Regge trajectory,  $g$  is a dimensionless coupling constant and  $r \sim 1$  fm is a range parameter. Compare Eq. 7 with Eq. 1, write the mass of the  $J$ th pole,  $m_J^2$ , as a function of the Regge parameters  $\alpha(0)$  and  $\alpha'$ . Interpret the pole structure in terms of the spectrum of particles in the model.

If the sum is truncated to a finite  $J_{\max}$ , and we take the  $s \rightarrow \infty$  limit, what is the high energy behavior of the amplitude?

(d) **Analytic continuation in  $J$**

Show that if the summation is kept infinite, the amplitude can be re-summed to something that is entirely analytic in  $s$ ,  $t$ ,  $u$ , and  $J$ .

*Hint: Use the Mellin transform*

$$\frac{1}{J - \alpha(t)} = \int_0^1 dx x^{J - \alpha(t) - 1} , \quad (8)$$

*to express the amplitude in terms of the Gaussian hypergeometric function and the Euler Beta function*

$$B(b, c - b) {}_2F_1(1, b, c; z) = \int_0^1 dx \frac{x^{b-1} (1-x)^{c-b-1}}{1 - x z} . \quad (9)$$

(e) **Unitarity vs Reggeized exchanges**

Revisit b) with the resummed amplitude. Take the  $s \rightarrow \infty$  limit and set a limit on the maximal intercept  $\alpha(0)$  which is allowed by unitarity.

*Hint: Assume that  $\alpha(0) > -1$  and use the asymptotic behavior of the hypergeometric function given by*

$${}_2F_1(1, b, c; z) \rightarrow \frac{\Gamma(c) \Gamma(1-b)}{\Gamma(1) \Gamma(c-b)} (-z)^{-b} . \quad (10)$$

(f) **The Reggeon “propagator”**

Modify Eq. 1 to have a definite signature by defining

$$A^\pm(s, t) = \frac{1}{2} [A(s, t) \pm A(u, t)] . \quad (11)$$

Repeat **d)** and **e)** with this signatured amplitude. Compare with the canonical form of the Reggeon exchange:

$$A_{\mathbb{R}}^{\pm}(s, t) = \beta(t) \frac{1}{2} \left[ \pm 1 + e^{-i\pi\alpha(t)} \right] \Gamma(-\alpha(t)) \left( \frac{s}{s_0} \right)^{\alpha(t)}. \quad (12)$$

Identify the Regge residue  $\beta(t)$  and characteristic scale  $s_0$  in terms of the parameters  $g_0$  and  $r$ .

## 8.2 Veneziano Amplitude

The quintessential dual amplitude was first proposed by Veneziano for  $\omega \rightarrow 3\pi$  and later applied to elastic  $\pi\pi$  scattering by Shapiro and Lovelace. Consider the  $\pi^+\pi^-$  scattering amplitude of the form

$$\mathcal{A}(s, t, u) = V(s, t) + V(s, u) - V(t, u). \quad (13)$$

with each

$$V(s, t) = \frac{\Gamma(1 - \alpha(s)) \Gamma(1 - \alpha(t))}{\Gamma(1 - \alpha(s) - \alpha(t))}, \quad (14)$$

where  $\alpha(s) = \alpha(0) + \alpha' s$  is a real, linear Regge trajectory with  $\alpha' > 0$ .

**(a) Duality**

Show that the function  $V(s, t)$  is symmetric in  $s \leftrightarrow t$  and dual, i.e., it can be written entirely as a sum of either  $s$ -channel poles OR  $t$ -channel poles but never both simultaneously. Compare with the Reggeized amplitude in the previous problem, was that amplitude dual?

*Hint: Relate  $V(s, t)$  to the Euler Beta function*

$$B(x, y) = \frac{\Gamma(x) \Gamma(y)}{\Gamma(x + y)} \quad (15)$$

and use the identities  $B(x, y) = B(y, x)$  and

$$B(p - x, q - y) = \sum_{J=1}^{\infty} \frac{\Gamma(J - p + 1 + x)}{\Gamma(J) \Gamma(-p + 1 + x)} \frac{1}{J - 1 + q - y}. \quad (16)$$

**(b) Isospin basis**

Define the  $s$ -channel isospin basis through

$$\begin{pmatrix} \mathcal{A}^{(0)}(s, t, u) \\ \mathcal{A}^{(1)}(s, t, u) \\ \mathcal{A}^{(2)}(s, t, u) \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 3 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} \mathcal{A}(s, t, u) \\ \mathcal{A}(t, s, u) \\ \mathcal{A}(u, t, s) \end{pmatrix}. \quad (17)$$

Write down the definite-isospin amplitudes in terms of  $V$ 's. Comment on the symmetry properties of each isospin amplitude with respect to  $t \leftrightarrow u$ .

**(c) Chew-Frautschi plot**

Locate where each  $\mathcal{A}^{(I)}(s, t, u)$  will have poles in the  $s$ -channel physical region. What is their residue? Draw a schematic Chew-Frautschi plot of the resonance spectrum in each isospin channel.

**(d) Regge limit**

Now consider the limit  $t \rightarrow \infty$  and  $u \rightarrow -\infty$  with  $s \leq 0$  is fixed. What is the asymptotic behavior of  $V(s, t)$  and  $V(s, u)$ ? Assume that  $V(t, u)$  vanishes faster than any power of  $s$  in this limit. What is the resulting behavior of the isospin amplitudes  $\mathcal{A}^{(I)}(s, t, u)$  in this limit?

*Hint: Use the Sterling approximation of the  $\Gamma$  function., i.e. as  $|x| \rightarrow \infty$*

$$\Gamma(x) \rightarrow \sqrt{\frac{2\pi}{x}} \left( \frac{x}{e} \right)^x. \quad (18)$$

(e) **Ancestors and Strings**

Consider the model now with a complex trajectory  $\alpha(s) = a_0 + \alpha' s + i\Gamma$  with  $\Gamma > 0$  to move the poles off the real axis. Reexamine the the Chew-Frautschi plot for the  $I = 1$  amplitude using this trajectory, why is the resulting spectrum problematic? Try a real but non-linear trajectory, say  $\alpha(s) = a_0 + \alpha' s + \alpha'' s^2$ , what is the spectrum like now?

Compare the requirements of the trajectory for  $V(s, t)$  to make sense with the energy levels of a rotating relativistic string with a string tension  $T$ :

$$E_J^2 = \frac{1}{2\pi T} J . \quad (19)$$

What is a possible microscopic picture of hadrons if the Veneziano amplitude is believed?

### 8.3 Sommerfeld-Watson Transform

(a) **Geometric series**

Prove the well known resummation of the geometric series:

$$1 + x + x^2 + x^3 + \dots = \frac{1}{1-x} \quad \text{for } |x| < 1 \quad (20)$$

can be analytically continued to  $|x| \geq 1$  with the Sommerfeld-Watson Transform.

Assume that  $|x| > 1$  and show that the summation can be written as an integral over the complex plane

$$\int \frac{d\ell (-x)^\ell}{2i \sin \pi \ell} = 1 + x + x^2 + \dots . \quad (21)$$

Draw the contour around which the above integration should be taken (careful with orientations and signs). Deform the contour such that you can relate Eq. 21 to the series

$$\frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} + \dots = \frac{1}{1 - \frac{1}{x}} \quad (22)$$

and arrive at Eq. 20.

(b) **Van Hove Reggeon**

Revisit the Regge behavior of Eq. 7 using the S-W transform. How does the inclusion of poles at  $\alpha(s) = \ell$  change the contour of integration and the leading contribution to the asymptotic behavior?

### 8.4 Finite Energy Sum Rules

Consider  $z$  a complex variable and  $\alpha$  a real fixed parameter. What is the analytic structure of the function  $z^\alpha$ ? What is the discontinuity across the cut?

Write a Cauchy contour  $C$  surrounding the cut and closing it with a circle of radius  $\Lambda$  in the complex  $z$  plane, and check that

$$\oint_C z^\alpha dz = 0 \quad (23)$$

You can start with the simple case  $\alpha = 1/2$ , i.e.  $\sqrt{z}$ , then generalize to any real  $\alpha$ .