



MTHS24 – Exercise sheet 2

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Lecture material

Discussed topics:

- Partial waves
- Analyticity
- Unitarity

References:

- Book 1, [inspire](#)
- Book 2, [amazon](#)
- A good review, [inspire](#)
- A good aper, [inspire](#)

Exercises

2.1 Two-body phase space

- (a) Particle A with mass M decays into two daughters with masses m_1 and m_2 . Derive the formula for the *break up momentum*, $p^* = |\mathbf{p}_1| = |\mathbf{p}_2|$, in the rest frame of A . Use your result to evaluate p^* for the decay $\Delta(1232) \rightarrow p\pi$.

Solution: First calculate E_1 :

$$\begin{aligned} P^2 = M^2 &= (p_1 + p_2)^2 = p_1^2 + p_2^2 + 2p_1 \cdot p_2 \\ &= m_1^2 + m_2^2 + 2(E_1 E_2 - \mathbf{p}_1 \cdot \mathbf{p}_2) \\ &= m_1^2 + m_2^2 + 2(E_1 E_2 + p_1^2) \quad \rightarrow \quad \text{with } \mathbf{p}_1 = -\mathbf{p}_2 \\ &= (2m_1^2 + 2p_1^2) - m_1^2 + m_2^2 + 2E_1(M - E_1) \quad \rightarrow \quad \text{with } M = E_1 + E_2 \\ &\rightarrow E_1 = \frac{M^2 + m_1^2 - m_2^2}{2M} \end{aligned}$$

Now calculate \mathbf{p}_1 :

$$\begin{aligned} \mathbf{p}_1^2 &= E_1^2 - m_1^2 \\ &= (E_1 + m_1)(E_1 - m_1) \\ &= \left(\frac{M^2 + m_1^2 - m_2^2 + 2Mm_1}{2M} \right) \left(\frac{M^2 + m_1^2 - m_2^2 - 2Mm_1}{2M} \right) \\ &\rightarrow \mathbf{p}_1 = \frac{1}{2M} \left[((M + m_1)^2 - m_2^2) ((M - m_1)^2 - m_2^2) \right]^{\frac{1}{2}} \end{aligned}$$

I believe this is equivalent to the PDG formula:

$$\mathbf{p}_1 = \frac{1}{2M} \left[(M^2 - (m_1 + m_2)^2) (M^2 - (m_1 - m_2)^2) \right]^{\frac{1}{2}}$$

For the decay $\Delta(1232) \rightarrow p\pi$, $p^* \approx 230$ MeV/c

(b) Starting from the formula for the 2-body decay rate,

$$\Gamma_{fi} = \frac{1}{2M} \int |\mathcal{M}_{fi}|^2 \frac{d^3 p_1}{(2\pi)^3 2E_1} \frac{d^3 p_2}{(2\pi)^3 2E_2} (2\pi)^4 \delta^4(P - p_1 - p_2),$$

perform integrations using δ functions to obtain $d\Gamma/d\Omega$ in the centre of mass frame, $P = \begin{pmatrix} M \\ \vec{0} \end{pmatrix}$, to obtain the 2-body phase-space factor.

Hint: You may want to use the following property of the delta function:

$$\int_{-\infty}^{+\infty} g(x) \delta(f(x)) dx = \sum_{x_0} \frac{g(x_0)}{|df/dx|_{x_0}}, \text{ where } x_0 \in \{x : f(x) = 0\}.$$

Solution: First integrate out \mathbf{p}_2 , imposing $\mathbf{p}_2 = -\mathbf{p}_1$ via the delta function:

$$\Gamma_{fi} = \frac{1}{8\pi^2 M} \int |\mathcal{M}_{fi}|^2 \delta(M - E_1 - E_2) \frac{d^3 p_1}{4E_1 E_2}$$

Substitute $d^3 p_1 = p_1^2 d\mathbf{p}_1 d\Omega$:

$$\Gamma_{fi} = \frac{1}{8\pi^2 M} \int |\mathcal{M}_{fi}|^2 \delta(M - E_1 - E_2) \frac{p_1^2 d\mathbf{p}_1 d\Omega}{4E_1 E_2}$$

Set:

$$\begin{aligned} f(\mathbf{p}_1) &= M - E_1 - E_2 = M - (m_1^2 + \mathbf{p}_1^2)^{\frac{1}{2}} - (m_2^2 + \mathbf{p}_1^2)^{\frac{1}{2}} \\ \frac{\partial f}{\partial \mathbf{p}_1} &= -\mathbf{p}_1 \left[(m_1^2 + \mathbf{p}_1^2)^{-\frac{1}{2}} + (m_2^2 + \mathbf{p}_1^2)^{-\frac{1}{2}} \right] \\ g(\mathbf{p}_1) &= \frac{\mathbf{p}_1^2}{4E_1 E_2} = \frac{\mathbf{p}_1^2}{4(m_1^2 + \mathbf{p}_1^2)^{\frac{1}{2}} (m_2^2 + \mathbf{p}_1^2)^{-\frac{1}{2}}} \\ \Gamma_{fi} &= \frac{1}{8\pi^2 M} \int |\mathcal{M}_{fi}|^2 \delta(f(\mathbf{p}_1)) g(\mathbf{p}_1) d\mathbf{p}_1 d\Omega \\ &= \frac{1}{8\pi^2 M} \left| \frac{df}{d\mathbf{p}_1} \right|_{\mathbf{p}^*}^{-1} \int |\mathcal{M}_{fi}|^2 g(\mathbf{p}_1) \delta(\mathbf{p}_1 - \mathbf{p}^*) d\mathbf{p}_1 d\Omega \\ &= \frac{1}{8\pi^2 M} \frac{E_1 E_2}{\mathbf{p}^*(E_1 + E_2)} \frac{(\mathbf{p}^*)^2}{4E_1 E_2} \int |\mathcal{M}_{fi}|^2 d\Omega \\ &= \frac{\mathbf{p}^*}{32\pi^2 M^2} \int |\mathcal{M}_{fi}|^2 d\Omega \end{aligned}$$