



MTHS24 – Exercise sheet 2

Morning: Alessandro Pilloni

Afternoon: Daniel Winney, Vanamali Shastry



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Lecture material

Discussed topics:

- Amplitude generalities
- Canonical and helicity states
- Analyticity
- Unitarity
- Crossing symmetry

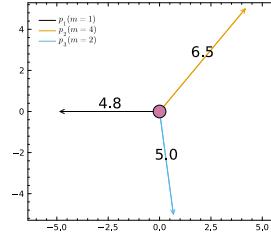
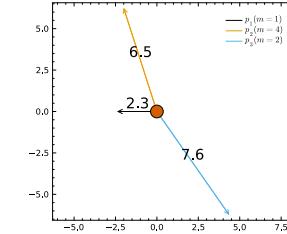
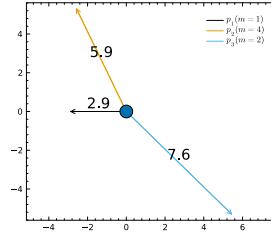
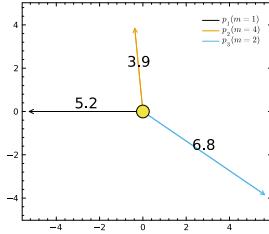
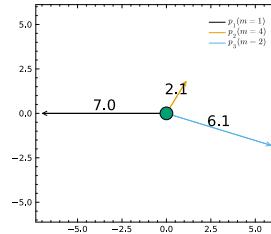
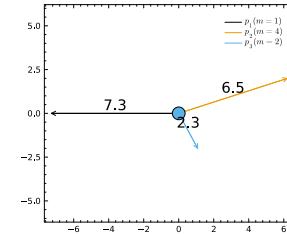
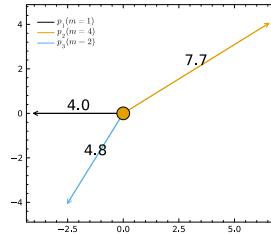
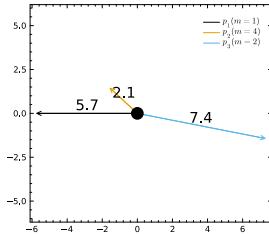
References:

- A.D. Martin, T.D. Spearman, Elementary Particle Theory, [inSpire](#)
- Eero Byckling, K. Kajantie, Particle Kinematics, [inSpire](#)

Exercises

2.1 Decay kinematics

Relate the values of momenta and their orientation to location on the Dalitz plot for the decay $p_0 \rightarrow p_1 + p_2 + p_3$ using values indicated on the figures below.

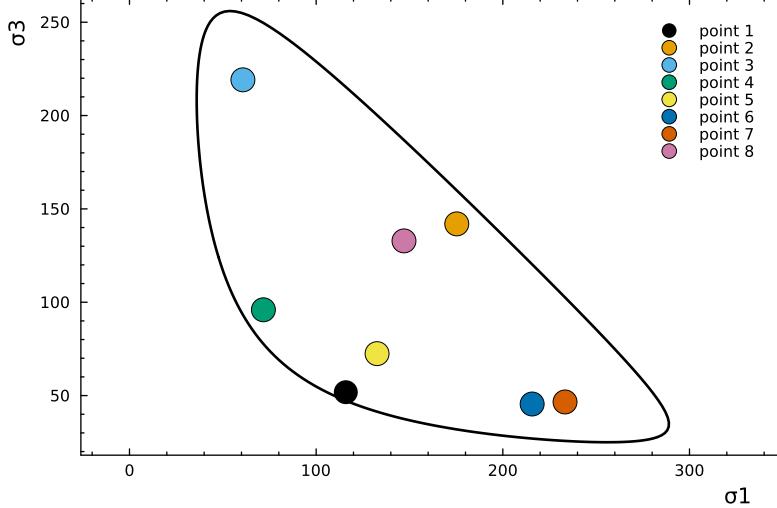


Solution: Mass is computed as

$$\begin{aligned} m_{ij} &= (p_i + p_j)^2 = m_i^2 + m_j^2 + 2E_i E_j - 2p_i p_j \cos \angle \vec{p}_i \vec{p}_j \\ &= m_i^2 + m_j^2 + 2E_i E_j + p_i^2 + p_j^2 - p_k^2 \\ &= (E_i + E_j)^2 - p_k^2 \end{aligned}$$

the angle between the two momenta is traded to magnitude of momenta using algebra of three-vectors, $-\vec{p}_k = \vec{p}_i + \vec{p}_j$

$$p_k^2 = p_i^2 + p_j^2 + 2p_i p_j \cos \angle \vec{p}_i \vec{p}_j$$



2.2 2-to-2 Kinematics

In hadron physics, the Mandelstam invariants s , t , and u are essential for describing the kinematics of scattering processes. Consider a $2 \rightarrow 2$ scattering process where two particles with four-momenta p_1 and p_2 scatter into two particles with four-momenta p_3 and p_4 . Consider the particles have different masses. The Mandelstam invariants are defined as $s = (p_1 + p_2)^2$, $t = (p_1 - p_3)^2$, and $u = (p_1 - p_4)^2$. The scattering angle θ is the angle between the momenta of the incoming (\vec{p}_1) and outgoing (\vec{p}_3) particle in the center of mass frame.

- (a) Verify the relation $s + t + u = \sum m_i^2$, where m_i are the masses of the particles.
- (b) Calculate the particle energies and 3-momenta in the center of mass frame, as a function of s
- (c) Calculate the scattering angle θ in terms of s , t , and u
- (d) How would those relation change if one considered the crossed reaction $p_1 p_3 \rightarrow p_2 p_4$?

2.3 Canonical and Helicity states

Consider a particle of spin $1/2$ at rest with spin up $u_+ = \sqrt{2m} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ or spin down, $u_- = \sqrt{2m} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$.

Remember that the generator of Lorentz group is $\sigma^{\mu\nu} = \frac{i}{2} [\gamma^\mu, \gamma^\nu]$. That implies that

$$R_y(\theta) = \exp\left(-i\frac{\theta}{2}\sigma_y\right) = \begin{pmatrix} \cos\frac{\theta}{2} - i\sigma_y \sin\frac{\theta}{2} & 0 \\ 0 & \cos\frac{\theta}{2} + i\sigma_y \sin\frac{\theta}{2} \end{pmatrix} \quad (1)$$

$$B(0 \rightarrow \vec{p}) = \exp\left(\frac{\vec{\eta}}{2} \cdot \vec{\alpha}\right) = \begin{pmatrix} \cosh\frac{\eta}{2} & \hat{\eta} \cdot \vec{\sigma} \sinh\frac{\eta}{2} \\ \hat{\eta} \cdot \vec{\sigma} \sinh\frac{\eta}{2} & \cosh\frac{\eta}{2} \end{pmatrix} \quad (2)$$

where $\tanh \eta = p/E$, η being the particle rapidity.

- (a) Derive the spinors in canonical and helicity basis for a particle with momentum lying in the xz plane.

2.4 Boosting spinning particles

A particle of spin 1 in the helicity basis lying in the xz plane is described by the following polarization vectors:

$$\epsilon_{\pm 1}^\mu(\theta) = \left(0, \mp \frac{\cos \theta}{\sqrt{2}}, -\frac{i}{\sqrt{2}}, \pm \frac{\sin \theta}{\sqrt{2}} \right) \quad (3)$$

$$\epsilon_0^\mu(\theta) = \left(\frac{p}{m}, \sin \theta \frac{E}{m}, 0, \cos \theta \frac{E}{m} \right) \quad (4)$$

$$p^\mu(\theta) = (E, \sin \theta p, 0, \cos \theta p) \quad (5)$$

Consider a particle with mass m and momentum $p = m\hat{z}$ and with helicity +1.

- (a) Boost the momentum and the polarization in the x direction of a boost $\beta = 1/\sqrt{2}$.
- (b) Compare with the polarization vectors according to the new momentum. Decompose it in the basis of the new polarization vectors.