Integral exemplarisch für eine Diagonal und eine Nichtdiagonalkomponente

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\begin{aligned}
(I_i)_{11} =&\rho_i \int\limits_{x_i-r/2}^{x_i+r/2} \int\limits_{y_i-r/2}^{y_i+r/2} \int\limits_{z_i-r/2}^{z_i+r/2} (y-y_0)^2 + (z-z_0)^2 \, \mathrm{d}z \; \mathrm{d}y \; \mathrm{d}x \\
=&\rho_i \int\limits_{y_i-r/2}^{y_i+r/2} \int\limits_{z_i-r/2}^{z_i+r/2} \Big[((y-y_0)^2 + (z-z_0)^2)z\Big]_{z_i-r/2}^{z_i+r/2} \; \mathrm{d}z \; \mathrm{d}y \\
=&\rho_i r \int\limits_{y_i-r/2}^{y_i+r/2} \int\limits_{z_i-r/2}^{z_i+r/2} (y-y_0)^2 + (z-z_0)^2 \, \mathrm{d}z \; \mathrm{d}y \\
=&\rho_i r^2 (\int\limits_{y_i-r/2}^{y_i+r/2} (y-y_0)^2 \, \mathrm{d}y \; + \int\limits_{z_i-r/2}^{z_i+r/2} (z-z_0)^2 \, \mathrm{d}z \;) \\
=&\rho_i r^2 \left[\frac{(y-y_0)^3}{3} \right]_{y_i-r/2}^{y_i+r/2} + r^2 \left[\frac{(z-z_0)^3}{3} \right]_{z_i-r/2}^{z_i+r/2} \\
=&\rho_i r^3 (y_0^2 - y_0(2y_i) + z_0^2 - z_0(2z_i) + (r^2/6 + y_i^2 + z_i^2)) \\
=&y_0^2(m_i) - y_0(2y_i m_i) + z_0^2(m_i) - z_0(2z_i m_i) + (r^2/6 + y_i^2 + z_i^2) m_i \\
(I_i)_{12} =&\rho_i \int\limits_{x_i-r/2}^{x_i+r/2} \int\limits_{y_i-r/2}^{y_i+r/2} \int\limits_{z_i-r/2}^{z_i+r/2} -(x-x_0)(y-y_0) \, \mathrm{d}z \; \mathrm{d}y \; \mathrm{d}x \\
=&\rho_i r \int\limits_{x_i-r/2}^{x_i+r/2} \int\limits_{y_i-r/2}^{y_i+r/2} -(x-x_0)(y-y_0) \, \mathrm{d}y \; \mathrm{d}x \\
=&\rho_i r \int\limits_{x_i-r/2}^{x_i+r/2} -(x-x_0) \left[\frac{(y-y_0)^2}{2} \right]_{y_i-r/2}^{y_i+r/2} \; \mathrm{d}x \\
=& - \rho_i r \left[\frac{(x-x_0)^2}{2} \right]_{x_i-r/2}^{x_i+r/2} \left[\frac{(y-y_0)^2}{2} \right]_{y_i-r/2}^{y_i+r/2} \\
=& - \rho_i r^3 (x_i - x_0)(y_i - y_0) \\
=& - x_0 y_0 (m_i) + x_0 (y_i m_i) + y_0 (x_i m_i) - x_i y_i m_i
\end{aligned}
$$

Ergebniss:

$$
\begin{aligned}
I =& x_0^2 \sum_i m_i \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + y_0^2 \sum_i m_i \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + z_0^2 \sum_i m_i \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&+ x_0 y_0 \sum_i m_i \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + x_0 z_0 \sum_i m_i \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} + y_0 z_0 \sum_i m_i \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\
&+ x_0 \sum_i m_i \begin{pmatrix} 0 & y_i & z_i \\ y_i & -2x_i & 0 \\ z_i & 0 & -2x_i \end{pmatrix} + y_0 \sum_i m_i \begin{pmatrix} -2y_i & x_i & 0 \\ x_i & 0 & z_i \\ 0 & z_i & -2y_i \end{pmatrix} + z_0 \sum_i m_i \begin{pmatrix} -2z_i & 0 & x_i \\ 0 & -2z_i & y_i \\ x_i & y_i & 0 \end{pmatrix} \\
&+ \sum_i m_i \begin{pmatrix} y_i^2 + z_i^2 + r^2/6 & -x_i y_i & -x_i z_i \\ -x_i y_i & x_i^2 + z_i^2 + r^2/6 & -y_i z_i \\ -x_i z_i & -y_i z_i & x_i^2 + y_i^2 + r^2/6 \end{pmatrix}
\end{aligned}
$$

Inkrementell aktuallisiert werden also:

$$
\begin{gathered}
\sum_i m_i;\ \sum_i m_i x_i;\ \sum_i m_i y_i;\ \sum_i m_i z_i;\ \sum_i m_i x_i y_i;\ \sum_i m_i x_i z_i;\ \sum_i m_i y_i z_i; \\
\sum_i m_i (y_i^2 + z_i^2 + r^2/6);\ \sum_i m_i (x_i^2 + z_i^2 + r^2/6);\ \sum_i m_i (x_i^2 + y_i^2 + r^2/6)
\end{gathered}
$$