$$S = X^Y, \text{ where } X, Y \in \mathbb{R}$$

BECAUSE computers cannot actually resolve all real numbers, we are justified in approximating any real (and computer-compatible) number as:

$$X = (-1)^m \left(\frac{a}{b}\right)$$
$$Y = (-1)^n \left(\frac{c}{d}\right), \text{ where } m, n = 0 \text{ or } 1; \quad a, b, c, d \text{ integers} > 0$$

Except in the (trivial) case where x or y = 0, any floating-point number can be resolved as a ratio of positive integers and a (-1), when needed. For example:

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          314159
3.14159 = -------
          100000
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Using exponent rules, the real number exponent operation can be decomposed as follows:

$$S = \left[(-1)^m \frac{a}{b}\right]^{\left[(-1)^n \frac{c}{d}\right]}$$

$$S = \left[(-1)^m\right]^{\left[(-1)^n \frac{c}{d}\right]} \left[a\right]^{\left[(-1)^n \frac{c}{d}\right]} \left[\frac{1}{b}\right]^{\left[(-1)^n \frac{c}{d}\right]}$$

$$S = \left\{\left[\left[(-1)^{m(-1)^n}\right]^c\right]^{\frac{1}{d}}\right\}\left\{\left[\left[a^{(-1)^n}\right]^c\right]^{\frac{1}{d}}\right\}\left\{\left[\left[\left(\frac{1}{b}\right)^{(-1)^n}\right]^c\right]^{\frac{1}{d}}\right\} = \{T_1\}\{T_2\}\{T_3\}$$

Beginning with the first of three major terms, the four permutations of m and n are:

$m = 0$, $n = 0$: $(-1)^0 = 1$ $\qquad$ $m = 1$, $n = 0$: $(-1)^1 = -1$

$m = 1$, $n = 1$: $(-1)^{-1} = -1$ $\qquad$ $m = 0$, $n = 1$: $(-1)^0 = 1$

In the trivial case that m=0 (i.e. X is positive), the entire first term works out to 1, and the result S is guaranteed to be real. If X is negative, however, there are some conditions that result in nonreal answers.

If c is even, then the -1 is flipped positive and the first term works out to 1. However, if c is odd, then the first term becomes:

$$T_1 = \left[(-1)^{\frac{1}{d}}\right]$$

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In order to resolve the dth-root of -1, we first represent -1 as a complex number using de Moivre's theorem:

In polar form, a complex number is described by:

$$z = r(\cos x + i\sin x)$$
$$-1 + i0 = 1(\cos\pi + i\sin\pi) \qquad (r = 1,\ x = \pi)$$
$$-1 = 1(-1 + i0)$$
$$-1 = -1$$

An extension of de Moivre's theorem states that:

$$r^{1/n}\left[\cos\left(\frac{x+2\pi k}{n}\right) + i\sin\left(\frac{x+2\pi k}{n}\right)\right]$$

Complexnumbers have n nth roots, and $0 \le k \le n-1$ are used to compute each one. For our purposes, we are justified in taking the first convenient one, k=0. Using this and returning to the definitions of our particular problem:

$$T_1 = (-1)^{1/d} = (1)^{1/d}\left[\cos\left(\frac{\pi}{d}\right) + i\sin\left(\frac{\pi}{d}\right)\right]$$
$$T_1 = (-1)^{1/d} = \cos\left(\frac{\pi}{d}\right) + i\sin\left(\frac{\pi}{d}\right)$$

Cosine and sine both have very well-defined Taylor series expansions, which means T_1 can be computed easily.

To compute the second and third terms, it is useful to recognize the fact that they can be treated similarly, both being positive integers.

If n = 0: $(a^1)^{c/d}$, $(1/b)^{c/d}$

If n = 1: $(1/a)^{c/d}$, $(b)^{c/d}$

Therefore, there is always going to be a 1/ term to deal with. However, using the power of a quotient rule:

$$\left(\frac{1}{a}\right)^{\frac{c}{d}} = \frac{1}{a^{\frac{c}{d}}}$$

Thus, both T_2 and T_3 can both be calculated the same way, and if one of them is a 1/ term, then it can be reckoned with as a final step.

The full decomposition is (ignoring a -1 exponent, as it has been shown to be a trivial step):

$$T_2 = a^{c/d}$$
$$\ln(T_2) = \ln(a^{c/d})$$
$$\ln(T_2) = \frac{c}{d}\ln a$$
$$e^{\ln(T_2)} = T_2 = e^{\frac{c}{d}\ln a}$$

It is important to note that, because of the constraints put on a, b, c, and d (that is, that they are integers > 0), the use of the natural logarithm is legal under its particular domain.

To finally resolve T_2 (and T_3, by extension), first the natural log must be calculated. To do so, we use Halley's method, which is an augmented form of Newton's method:

$$y = \ln x$$

$$y_{n+1} = y_n + 2\left(\frac{x - e^{y_n}}{x + e^{y_n}}\right)$$

This will converge rather quickly, with high precision. Obviously, a method for the natural exponent is also needed, and there is a convenient power series expansion that will also converge very quickly:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

What the preceding groundwork ultimately accomplishes is decomposing the real number exponent, in any possible computer-understandable form, into a set of smaller computations that are easily programmed into a computer. Every computation needed relies only on:

- Addition, subtraction, multiplication, division
- Whole number, positive exponents (repeated multiplication)
- Direct case checking (i.e. $m=0$, $T_1=1$)
- "infinite" series (that break on convergence)

Of course, it remains to be seen whether the efficiency of these methods are within acceptable bounds.

$T_1 = (-1)^{1/d}$

$z = r(\cos x + i \sin x)$

$\cancel{-1 + 0j = r(\cos x + i \sin x)}$

$\cancel{\Rightarrow x = 0, \pi, (\pi), \ldots}$

$\cancel{\Rightarrow r = -1}$

$\cancel{-1 + 0j = (-1)(\cos 0 + i \sin 0)}$

$\cancel{z^{1/n} = r^{1/n}\left(\cos \frac{x+2\pi k}{n} + i \sin \frac{x+2\pi k}{n}\right)}$

$\Rightarrow r = 1, \; x = \pi$

DE MOIVRE'S THM:

$-1 + 0j = (1)(\underbrace{\cos \pi}_{-1} + i \underbrace{\sin \pi}_{0}) = -1$ ✓

$z^{1/n} = r^{1/n}\left(\cos \frac{\pi+2\pi k}{n} + i \sin \frac{\pi+2\pi k}{n}\right)$

TAKE $k = 0$

$(-1)^{1/d} = (1)^{1/d}\left(\cos\left(\frac{\pi}{n}\right) + i \sin\left(\frac{\pi}{n}\right)\right)$

$(-1)^{1/d} = \cos(\pi/n) + i \sin(\pi/n)$

$v \quad a^{c/d} = (a^c)^{1/d} =$ $\qquad x^{2.5} = x^{2+0.5} = x^2 x^{0.5}$

$v = (a^c)^{1/d}$

$\ln v = \ln((a^c)^{1/d})$ $\qquad \ln(a^c) =$

$e^{\ln v} = e^{d \ln a^c}$