MATHEMATICAL METHODS II
MATH 266

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June 21, 2022

Course Outline

1. Improper Integration

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2. Integral depending on a Parameter

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2. Integral depending on a Parameter
3. Differentiation and Integration under the Integral sign

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4. Gamma and Beta Functions , Stirling's Formula

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6. Fourier Series
7. Fourier Transforms

Recommended Textbooks/Readings

- G. Arfken and H.J Weber Mathematical Methods for Physicist
- M.L. Boas Mathematical Methods in the Physical Sciences
- Erwin Kreyszig Advanced Engineering Mathematics

AWESOME COEXISTENCE

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- At all times Kindly make sure the lecture hall is free from noise. Feel free to ask as many questions as possible.

Improper Integral

An integral is said to be Improper if

- Either the interval of integration is infinite or
- The function to be integrated is not continuous on the given interval

Examples

1. $$\int_0^{\infty} e^{-x} dx$$
2. $$\int_1^{\infty} \frac{1}{x^2} dx$$
3. $$\int_1^{\infty} \frac{1}{x} dx$$

Type 1: Infinite Intervals

With this kind of improper integral, we will take a look at integrals in which one or both of the limits of integration are infinity. In this case, the interval of integration is said to be over an infinite interval.

Eg. 1: Evaluate the integral

$$\int_{1}^{\infty} \frac{1}{x^2} dx$$

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Eg. 1: Evaluate the integral

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Solution

NB: Because infinity is not a real number we can't just integrate as normal and then "plug in" the infinity to get an answer. We instead think of this as an 'area under the curve' problem.

Infinite Interval

So instead of asking what the integral is, let's instead ask what the area under $f(x) = \frac{1}{x^2}$ on the interval $[1, \infty)$ is. We still aren't able to do this, however, let's step back a little and instead ask what the area under $f(x)$ is on the interval $[1, t]$, where $t > 1$ and t is finite. This is given by

$$\int_1^t \frac{1}{x^2} dx = \frac{-1}{x}\Big|_1^t = 1 - \frac{1}{t}$$

Now, we can get the area under $f(x)$ on $[1, \infty]$ simply by taking the limit of the answer as t approaches infinity.

$$\lim_{t \to \infty} (1 - \frac{1}{t}) = 1$$

Infinite Interval

This is then how we will do the integral itself;

$$\int_1^\infty \frac{1}{x^2}dx = \lim_{t\to\infty}\int_1^t \frac{1}{x^2}dx$$

$$\lim_{t\to\infty} \frac{-1}{x}\Big|_1^t = \lim_{t\to\infty}(1-\frac{1}{t}) = \mathbf{1}$$

So in general, to handle these kinds of integrals, we will replace the infinity with a variable (say t), perform the integration, and then take the limit of the results as the variable (say t) goes to infinity.

NOTE

We call the integrals ***convergent*** if the associated limit exists and is a finite number (i.e it's not plus or minus infinity), and ***divergent*** if the associated limit either doesn't exist or is (plus or minus) infinity.

Example 1:

Determine if the following integral is convergent or divergent, and if convergent find its value.

$$\int_1^\infty \frac{1}{x}dx$$

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Example 1:

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$$\int_1^{\infty} \frac{1}{x}\,dx$$

Solution:

So, the first thing we do is convert the integral to a limit;

$$\lim_{t\to\infty}\int_1^{t} \frac{1}{x}\,dx = \lim_{t\to\infty}(\ln x)\big|_1^t = \lim_{t\to\infty}\ln(t) - \ln(1) = \lim_{t\to\infty}\ln t = \infty$$

So, the limit is infinite and so the integral is ***divergent***.

Example 2:

For what value of p is the following integral convergent?

$$\int_{1}^{\infty} \frac{1}{x^p}\,dx$$

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Solution:

We know from 'Example 1' that if $p = 1$, then the integral is divergent, so let's assume that $p \neq 1$, Then

$$\int_1^\infty \frac{1}{x^p} dx = \lim_{t\to\infty} \int_1^t x^{-p} dx = \lim_{t\to\infty} \left[\frac{x^{-p+1}}{-p+1}\right]_1^t \tag{1}$$

$$= \lim_{t\to\infty} \frac{1}{1-p}\left[\frac{1}{t^{p-1}} - 1\right] \tag{2}$$

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$$= \lim_{t\to\infty} \frac{1}{1-p} \left[\frac{1}{t^{p-1}} - 1 \right] \tag{2}$$

If $p > 1$, then $p - 1 > 0$, so as $t \to \infty$, $t^{p-1} \to \infty$ and $\frac{1}{t^{p-1}} \to 0$. Therefore

$$\int_1^{\infty} \frac{1}{x^p} dx = \frac{1}{p-1} \text{ if } p > 1. \tag{3}$$

Thus the integral converges.

But if $p < 1$, the $p - 1 < 0$ and so $\dfrac{1}{t^{p-1}} = t^{1-p} \to \infty$ as $t \to \infty$ and the integral diverges.

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Summary of Results

If $a > 0$ then

$$\int_a^\infty \frac{1}{x^p} dx$$

is ***convergent*** if $p > 1$ and ***divergent*** if $p \leq 1$.

Example 3:

Determine whether the following integral is convergent or divergent

$$\int_{-\infty}^{0} \frac{1}{\sqrt{3-x}}dx$$

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Solution:

We'll convert the integral to a limit/integral pair, evaluate the integral and then the limit.

$\int_{-\infty}^{0} \frac{1}{\sqrt{3-x}}dx = \lim_{t\to-\infty} \int_{t}^{0} \frac{1}{\sqrt{3-x}}dx = \lim_{t\to-\infty} -2\sqrt{3-x}\Big|_{t}^{0}$
$\lim_{t\to-\infty} (-2\sqrt{3} + 2\sqrt{3-t}) = -2\sqrt{3} + \infty$
$= \infty$

So, the limit is infinite and so this integral is ***divergent.***

Example 4

$$\int_{-1}^{1} \frac{1}{x^{\frac{2}{3}}}\,dx$$

Example 4

$\int_{-1}^{1} \frac{1}{x^{\frac{2}{3}}} dx$

Solution : Here, although none of the limits of integration are infinity but its still an improper integral because the integrand $(\frac{1}{x^{\frac{2}{3}}})$ becomes ∞ over the interval [a,b]

$$= \lim_{a\to 0} \int_{-1}^{a} \frac{1}{x^{\frac{2}{3}}} dx + \lim_{a\to 0} \int_{a}^{1} \frac{1}{x^{\frac{2}{3}}} dx$$
$$= \lim_{a\to 0} \left(3x^{\frac{1}{3}}\right) \Big|_{-1}^{a} + \lim_{a\to 0} \left(3x^{\frac{1}{3}}\right) \Big|_{a}^{1}$$
$$= \lim_{a\to 0} (3a^{\frac{1}{3}} - 3(-1)^{\frac{1}{3}}) + \lim_{a\to 0} (3(1)^{\frac{1}{3}} - 3(a)^{\frac{1}{3}})$$
$$= \lim_{a\to 0} \left(3(a)^{\frac{1}{3}} + 3\right) + \lim_{a\to 0} \left(3 - 3(a)^{\frac{1}{3}}\right)$$

As $a \to 0$

$= (0 + 3) + (3 - 0) = 3 + 3 = 6$

Hence the integral converges..(***Convergent***)

Example 5

Evaluate $\int_{-\infty}^{0} xe^{x}\,dx$

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Solution

$\int_{-\infty}^{0} xe^x dx = \lim_{t \to -\infty} \int_{t}^{0} xe^x dx$

We integrate by parts with $u = x$,$dv = e^x dx$ so that $du = dx,\ v = e^x$;

$\int_{t}^{0} xe^x dx = xe^x|_t^0 - \int_{t}^{0} e^x dx$
$= -te^t - 1 + e^t$

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$= -te^t - 1 + e^t$

We know that $e^t \to 0$ as $t \to -\infty$, and by L'hospital's Rule we have

$\lim_{t \to -\infty} te^t = \lim_{t \to -\infty} \dfrac{t}{e^{-t}} = \lim_{t \to -\infty} \dfrac{1}{-e^{-t}}$
$= \lim_{t \to -\infty} (-e^t) = 0$

...continued

Therefore

$$\int_{-\infty}^{0} xe^{x} dx = \lim_{t \to -\infty} (-te^{t} - 1 + e^{t})$$

$$-0 - 1 + 0 = -1$$

Example 6

$$\int_{-\infty}^{\infty} xe^{-x^2}dx$$

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$\int_{-\infty}^{\infty} xe^{-x^2} dx$

Solution

In this case we've got infinities in both limits and so we'll need to split the integral up into two separate integrals.

The integral is then,

$\int_{-\infty}^{\infty} xe^{-x^2} dx = \int_{-\infty}^{0} xe^{-x^2} dx + \int_{0}^{\infty} xe^{-x^2} dx$

We then look at each of the individual limits.

$= \lim_{t \to -\infty} \int_{t}^{0} xe^{-x^2} dx + \lim_{t \to \infty} \int_{0}^{t} xe^{-x^2} dx$

$= \lim_{t \to -\infty} \left(\frac{-1}{2} e^{-x^2} \right) \Big|_{t}^{0} + \lim_{t \to \infty} \left(\frac{-1}{2} e^{-x^2} \right) \Big|_{0}^{t}$

$= \lim_{t \to -\infty} (\frac{-1}{2} - \frac{-1}{2} e^{-(t)^2}) + \lim_{t \to \infty} (\frac{-1}{2} e^{-(t)^2} - \frac{-1}{2})$

As $t \to -\infty$ and ∞

$(\frac{-1}{2} + 0) + (0 + \frac{1}{2})$

$= \frac{-1}{2} + \frac{1}{2} = \mathbf{0}$; Hence the integral is convergent..

Example 7

$\int_{-2}^{\infty} \sin x dx$

Example 7

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Solution

First convert to a limit

$\int_{-2}^{\infty} \sin x dx = \lim_{t \to \infty} \int_{-2}^{t} \sin x dx$

$= \lim_{t \to \infty} (-\cos x) \Big|_{-2}^{t} = \lim_{t \to \infty} (-\cos t - - \cos 2)$

$= \lim_{t \to \infty} (\cos 2 - \cos t)$

As $t \to \infty$

The limit does not exist and so the integral is divergent.

Example 8

Evaluate $\int_{-\infty}^{\infty} \dfrac{1}{1+x^2} dx$

Example 8

Evaluate $\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx$

Solution

It's convenient to choose $a = 0$

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx = \int_{-\infty}^{0} \frac{1}{1+x^2} dx + \int_{0}^{\infty} \frac{1}{1+x^2} dx$$

We must now evaluate the integrals on the right side separately:

$$\int_{0}^{\infty} \frac{1}{1+x^2} dx = \lim_{t\to\infty} \int_{0}^{t} \frac{dx}{1+x^2} = \lim_{t\to\infty} \tan^{-1} x|_{0}^{t}$$

$$= \lim_{t\to\infty} (\tan^{-1} t - \tan^{-1} 0) = \lim_{t\to\infty} \tan^{-1} t = \frac{\pi}{2}$$

$$\int_{-\infty}^{0} \frac{1}{1+x^2} dx = \lim_{t\to-\infty} \int_{t}^{0} \frac{dx}{1+x^2} = \lim_{t\to-\infty} \tan^{-1} x|_{t}^{0}$$

$$= \lim_{t\to-\infty} (\tan^{-1} 0 - \tan^{-1} t) = 0 - (-\frac{\pi}{2}) = \frac{\pi}{2}$$

...continued

Since both of these integrals are convergent, the given integral is convergent and

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}dx = \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

TESTS FOR CONVERGENCE AND DIVERGENCE

When we cannot evaluate an improper integral directly, we try to determine whether or not it converges or diverges. If the integral diverges, then that ends the story. We can employ numerical methods to approximate the value of the integral if it converges. The principal tests for convergence or divergence are the Direct Comparison Test and the Limit Comparison Test.

Eg.1. Investigating Convergence

Does the integral $\int_1^\infty e^{-x^2}\,dx$

Solution

By Definition,

$\int_1^\infty e^{-x^2}\,dx = \lim_{b\to\infty} \int_1^b e^{-x^2}\,dx$

We cannot evaluate the latter integral directly because it is non-elementary.

...continued

But we can show that its limit as $b \to \infty$ is finite. We know that $\int_1^b e^{-x^2} dx$ is an increasing function of b. Therefore either it becomes infinite as $b \to \infty$ or it has a finite limit as $b \to \infty$. It does not become infinite: $\forall\ x \geq 1\ \exists\ e^{-x^2} \leq e^{-x}$ such that
$\int_1^b e^{-x^2} dx \leq \int_1^b e^{-x} dx = e^{-b} + e^{-1} < e^{-1} \approx 0.36788$ Hence

$\int_1^\infty e^{-x^2} dx = \lim\limits_{b \to \infty} \int_1^b e^{-x^2} dx$ converges to some definite finite value. We do nit know exactly what the value is except that it is something positive and less than 0.37. Here we are relying on the completeness property of the real numbers.

Convergence Tests for Improper Integrals of the First Kind

The following tests are given for cases where an integration limit is ∞. Similar tests exist where an integration limit is $-\infty$ (a change of variable $x = -y$ then makes the integration limit ∞).
Unless otherwise specified we shall assume that $f(x)$ is continuous and thus integrable in every finite interval $a \leq x \leq b$

Theorem 1: Direct Comparison Test

Let f and g be continuous on $[a, \infty)$ with $0 \leq f(x) \leq g(x) \ \forall x \geq a$.
Then:
1. $\int_a^\infty f(x)dx$ converges if $\int_a^\infty g(x)dx$ converges.
2. $\int_a^\infty g(x)dx$ diverges if $\int_a^\infty f(x)dx$ diverges.

The reasoning behind the argument establishing **Theorem 1** is similar to that in **Example 1** where it can be argued that $\displaystyle\int_a^{\infty} f(x)dx$ converges if $\displaystyle\int_a^{\infty} g(x)dx$ converges.

Turning this around says that $\displaystyle\int_a^{\infty} g(x)dx$ diverges if $\displaystyle\int_a^{\infty} f(x)dx$ diverges.

Using the Direct Comparison Test

$a)$ $\displaystyle\int_1^{\infty} \frac{\sin^2 x}{x^2} dx$ converges because $0 \leq \dfrac{\sin^2 x}{x^2} \leq \dfrac{1}{x^2}$ on $[1, \infty)$ and $\displaystyle\int_1^{\infty} \frac{1}{x^2} dx$ converges.

$b)$ $\displaystyle\int_1^{\infty} \frac{1}{\sqrt{x^2-0.1}}dx$ diverges because $\displaystyle\frac{1}{\sqrt{x^2-0.1}} \geq \frac{1}{x}$ on $[1,\infty)$ and $\displaystyle\int_1^{\infty} \frac{1}{x}dx$ diverges.

$c)$ Since $\displaystyle\frac{1}{e^x+1} \leq \frac{1}{e^x} = e^{-x}$ and $\displaystyle\int_1^{\infty} e^{-x}dx$ converges, $\displaystyle\int_1^{\infty} \frac{1}{e^x+1}dx$ converges.

$d)$ Since $\displaystyle\int_1^{\infty} \frac{1}{\ln x} > \frac{1}{x}$ for $x \geq 2$ and $\displaystyle\int_2^{\infty} \frac{dx}{x}$ diverges (p integral with $P=1$) $\displaystyle\int_2^{\infty} \frac{dx}{\ln x}$ also diverges.

Theorem 2: Limit Comparison Test (Quotient Test)

If the positive functions f and g are continuous on $[a, \infty)$ or non-negative integrands and

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L,\ 0 \leq L \leq \infty \tag{4}$$

a) If L is finite and positive then $\int_a^\infty f(x)dx$ $\int_a^\infty g(x)dx$ either both converge or both diverge.

b) If $L = 0$ in (4), and $\int_a^\infty g(x)dx$ converges, then $\int_a^\infty f(x)dx$ converges.

c) If $L = \infty$ in (4), and $\int_a^\infty g(x)dx$ diverges, then $\int_a^\infty f(x)dx$ diverges.

Although the improper integrals of two functions from a to ∞ may both converge , this does not mean that their integrals necessarily have the same value, as the next example shows

Example Using the Limit Comparison Test

1. Show that $\displaystyle\int_1^{\infty} \frac{dx}{1+x^2}$ converges by comparison with $\displaystyle\int_1^{\infty} \frac{1}{x^2}\,dx$

Although the improper integrals of two functions from a to ∞ may both converge , this does not mean that their integrals necessarily have the same value, as the next example shows

Example Using the Limit Comparison Test

1. Show that $\displaystyle\int_1^{\infty} \frac{dx}{1+x^2}$ converges by comparison with $\displaystyle\int_1^{\infty} \frac{1}{x^2}dx$

Solution The functions $f(x) = \dfrac{1}{x^2}$ and $g(x) = \dfrac{1}{1+x^2}$ are positive and continuous on $[1, \infty)$. Also,

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{\frac{1}{x^2}}{\frac{1}{1+x^2}} = \lim_{x\to\infty} \frac{1+x^2}{x^2} = \lim_{x\to\infty} \frac{1}{x^2}+1 = 0+1 = 1 \tag{5}$$

A positive finite limit. Therefore $\displaystyle\int_1^{\infty} \frac{dx}{1+x^2}$ converge because $\displaystyle\int_1^{\infty} \frac{dx}{x^2}$ converges.

The integrals converge to different values, however.

$$\int_1^{\infty} \frac{dx}{x^2} = \frac{1}{2-1} = 1 \text{ and}$$

$$\int_1^{\infty} \frac{dx}{1+x^2} = \lim_{x\to\infty} \int_1^{b} \frac{dx}{1+x^2} = \lim_{x\to\infty} [\tan^{-1}(b) - \tan^{-1}(1)]$$

$$= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

2. Show that $\displaystyle\int_1^{\infty} \frac{3}{e^x - 5} dx$ converges.

Solution

From the previous example, it is very easy to see that $\int_1^{\infty} e^{-x} dx$ converges. Because

$$\lim_{x\to\infty} \frac{\frac{1}{e^x}}{\frac{3}{(e^x-5)}} = \lim_{x\to\infty} \frac{e^x - 5}{3e^x} = \lim_{x\to\infty} \left(\frac{1}{3} - \frac{5}{3e^x} \right) = \frac{1}{3},$$

the integral $\int_1^{\infty} \frac{3}{e^x - 5} dx$ also converges.

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integral $\int_1^\infty \frac{3}{e^x-5} dx$ also converges.

The test is related to the comparison test and is often a very useful alternative to it. In particular, taking $g(x) = \frac{1}{x^p}$, we have from known facts about the p integral , the

Theorem 3 Let $\lim_{x\to\infty} x^p f(x) = L$. Then

1. $\displaystyle\int_a^\infty f(x)dx$ converges if $p > 1$ and L is finite.
2. $\displaystyle\int_a^\infty f(x)dx$ diverges if $p \leq 1$ and $L \neq 0$ (L may be finite)

Example

1. $\int_a^{\infty} f(x)dx$ converges if $p > 1$ and L is finite.
2. $\int_a^{\infty} f(x)dx$ diverges if $p \leq 1$ and $L \neq 0$ (L may be finite)

Example

1. $\int_0^{\infty} \frac{x^2}{4x^4 + 25} dx$ converges since $\lim_{x\to\infty} x^2 . \frac{x^2}{4x^4+25} = \frac{1}{4}$

   Explanation: In doing the limit, the denominator polynomial has a degree of 4 so $f(x)$ has to be multiplied by x^2 to make the denominator degree also 4. Hence $p = 2 > 1$ and L is finite.

2. $\int_0^{\infty} \frac{xdx}{\sqrt{x^4+x^2+1}}$ diverges since $\lim_{x\to\infty} x . \frac{x}{\sqrt{x^4+x^2+1}} = 1$

Explanation: Since $\dfrac{1}{\sqrt{x^4+x^2+1}} \approx \dfrac{1}{x^2}$ for large x, we can say that the denominator polynomial has a degree 2. So $f(x)$ has to be multiplied by x to make the numerator degree also 2. hence, $p = 1 \leq 1$ and $L \neq 0$ is finite.

Series test for integrals with non-negative integrands

$\int_a^\infty f(x)dx$ converges or diverges according as $\sum u_n$ where $u_n = f(n)$, converges or diverges.

Absolute and conditional convergence.

$\int_a^\infty f(x)dx$ is called absolutely convergent if $\int_a^\infty |f(x)|dx$ converges but if $\int_a^\infty |f(x)|dx$ diverges, then $\int_a^\infty f(x)dx$ is called conditionally convergent.

Theorem 4

If $\displaystyle\int_a^\infty |f(x)|dx$ converges, then $\displaystyle\int_a^\infty f(x)dx$ converges. In words, an absolutely convergent integral converges.

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Example

1. $\displaystyle\int_0^\infty \frac{\cos x}{x^2+1}dx$ is absolutely convergent and thus convergent since $\displaystyle\int_0^\infty \left|\frac{\cos x}{x^2+1}\right|dx \leq \int_0^\infty \frac{dx}{x^2+1}$ and $\displaystyle\int_0^\infty \frac{dx}{x^2+1}$ converges

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Example

1. $\int_0^\infty \frac{\cos x}{x^2+1}dx$ is absolutely convergent and thus convergent since $\int_0^\infty |\frac{\cos x}{x^2+1}|dx \leq \int_0^\infty \frac{dx}{x^2+1}$ and $\int_0^\infty \frac{dx}{x^2+1}$ converges
2. $\int_0^\infty \frac{\sin x}{x}dx$ converges , but $\int_0^\infty |\frac{\sin x}{x}|dx$ does not converge.

Thus $\int_0^\infty \frac{\sin x}{x}dx$ is conditionally convergent.

Any of the tests used for integrals with non-negative integrands can be used to test for absolute convergence.

Example

1. Test for convergence :a) $\displaystyle\int_1^{\infty} \frac{xdx}{3x^4 + 5x^2 + 1}$ b) $\displaystyle\int_2^{\infty} \frac{x^2 - 1}{\sqrt{x^6 + 16}} dx$

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Example

1. Test for convergence :a) $\displaystyle\int_1^\infty \frac{xdx}{3x^4+5x^2+1}$ b) $\displaystyle\int_2^\infty \frac{x^2-1}{\sqrt{x^6+16}}dx$

Solution (a)

Method 1. For large x, the integrand is approximately $\dfrac{x}{3x^4} = \dfrac{1}{3x^3}$.

Since $\dfrac{x}{3x^4+5x^2+1} \le \dfrac{1}{3x^3}$, and $\dfrac{1}{3}\displaystyle\int_1^\infty \frac{dx}{x^3}$ converges (p integral with $p=3$), it follows by the comparison test that $\displaystyle\int_1^\infty \frac{xdx}{3x^4+5x^2+1}$ also converges.

Note

Note that the purpose of examining the integrand for large x is to obtain a suitable comparison integral.

Method 2. Let $f(x) = \dfrac{x}{3x^4 + 5x^2 + 1}$ and let $g(x) = \dfrac{1}{x^3}$.

Since $\lim\limits_{x \to \infty} \dfrac{f(x)}{g(x)} = \dfrac{1}{3}$, and $\displaystyle\int_1^\infty g(x)dx$ converges, $\displaystyle\int_1^\infty f(x)dx$ also converges by the quotient test.

Note

Note that in the comparison function $g(x)$, we have discarded the factor $\frac{1}{3}$. It could, however, just as well have been included.

Method 3. $\lim_{x\to\infty} x^3\left(\dfrac{x}{3x^4+5x^2+1}\right)=\dfrac{1}{3}$. Hence by Theorem 3,
the required integral converges.

Solution (b)

Method 1. For large x, the integrand is approximately $\dfrac{x^2}{\sqrt{x^6}}=\dfrac{1}{x}$.

For $x\geq 2$, $\dfrac{x^2-1}{\sqrt{x^6+16}}\dfrac{x^2-1}{\sqrt{x^6+16}}\geq\dfrac{1}{2}.\dfrac{1}{x}$.

Since $\dfrac{1}{2}\displaystyle\int_2^\infty \dfrac{dx}{x}$ diverges , $\displaystyle\int_2^\infty \dfrac{x^2-1}{\sqrt{x^6+16}}dx$ also diverges.

Method 2. Let $f(x) = \dfrac{x^2 - 1}{\sqrt{x^6 + 16}}$ and let $g(x) = \dfrac{1}{x}$. Since $\lim_{x\to\infty} \dfrac{f(x)}{g(x)} = 1$,
and $\int_2^\infty g(x)dx$ diverges, $\int_2^\infty f(x)dx$ also diverges.

Method 3. Since $\lim_{x\to\infty} x\left(\dfrac{x^2 - 1}{\sqrt{x^6 + 16}}\right) = 1$, the required integral diverges by **Theorem**.

Note

That method 1 may (and often does) require one to obtain a suitable inequality factor (in this case $\frac{1}{2}$) before the comparison test can be applied. Method 2 and 3 however, do not require this.

Example

2. Test for convergence: a) $\int_{-\infty}^{-1} \frac{e^x}{x} dx$ b) $\int_{-\infty}^{\infty} \frac{x^3+x^2}{x^6+1} dx$

Solution (a)

Let $x = -y$. then the integral becomes $-\int_1^{\infty} \frac{e^{-y}}{y} dy$

Method 1. $\frac{e^{-y}}{y} \leq e^{-y}$ for $y \geq 1$. Then since $\int_1^{\infty} e^{-y} dy$ converges, $\int_1^{\infty} \frac{e^{-y}}{y} dy$ converges; hence the given integral converges.

Method 2. $\lim_{y\to\infty} y^2\left(\frac{e^{-y}}{y}\right) = \lim_{y\to\infty} ye^{-y} = 0.$ Then the given integral converges by **Theorem 3** $L = 0$ and $p = 2.$

Solution (b)

Write the integral as $\int_{-\infty}^{0} \frac{x^3+x^2}{x^6+1}dx + \int_{0}^{\infty} \frac{x^3+x^2}{x^6+1}dx$

Letting $x = -y$ in the first integral, it becomes

$-\int_{0}^{\infty} \frac{y^3-y^2}{y^6+1}dy.$ Since $\lim_{y\to\infty} y^3\left(\frac{y^3-y^2}{y^6+1}\right) = 1,$ This integral converges.

Since $\lim_{x\to\infty} x^3\left(\frac{x^3-x^2}{x^6+1}\right) = 1,$ the second integral converges.

Thus the given integral converges.

3. Prove that $\int_{-\infty}^{0} e^{-x^2} dx$ converges.

Solution

$\lim_{x \to \infty} x^2 e^{-x^2} = 0$ (by L' hôspital's Rule or otherwise). Then by **theorem 3** with $A = 0$, $p = 2$, the given integral converges.

Convergence Tests for Improper Integrals of the Second Kind

p integrals of the second kind

1. $\displaystyle\int_a^b \frac{dx}{(x-a)^p}$ converges if $p < 1$ and diverges if $p \geq 1$
2. $\displaystyle\int_a^b \frac{dx}{(b-x)^p}$ converges if $p < 1$ and diverges if $p \geq 1$

Convergence Tests for Improper Integrals of the Second Kind

The following tests are given for the case where $f(x)$ is unbounded only at $x = a$ in the interval $a \leq x \leq b$. Similar tests are available if $f(x)$ is unbounded at $x = b$ or at $x = x_0$ where $a < x_0 < b$.

Theorem 5: Direct Comparison Test

1. Let $g(x) \geq 0$ for $a < x \leq b$, and suppose that $\int_a^b g(x)dx$ converges. Then if

   $0 \leq f(x) \leq g(x) \ \ \forall \ \ a < x \leq b, \quad \int_a^b f(x)dx$ converges.

2. Let $g(x) \geq 0$ for $a < x \leq b$, and suppose that $\int_a^b g(x)dx$ diverges. Then if

   $f(x) \geq g(x)$ *for* $a < x \leq b, \quad \int_a^b f(x)dx$ also diverges.

Example ;Using the Direct Comparison Test

a) $\dfrac{1}{\sqrt{x^4-1}} < \dfrac{1}{\sqrt{x-1}}$ for $x > 1$. Then since

$\displaystyle\int_1^5 \frac{dx}{\sqrt{x-1}}$ converges (p integral with $a = 1,\ p = \dfrac{1}{2}$),

$\displaystyle\int_1^5 \frac{1}{\sqrt{x^4-1}}$ also converges.

b) $\dfrac{\ln x}{(x-3)^4} < \dfrac{1}{(x-3)^4}$ for $x > 3$. Then since

$\displaystyle\int_3^6 \frac{dx}{(x-3)^4}$ diverges (p integral with $a = 3,\ p = 4,$)

$\displaystyle\int_3^6 \frac{\ln x}{(x-3)^4}\,dx$ also diverges.

Theorem : Limit Comparison Test (Quotient Test)

a) If $f(x) \geq 0$ and $g(x) \geq 0$ for $a < x \leq b$, that is non-negative integrands and if

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = L \neq 0 \;\; or \;\; \infty, \tag{6}$$

then $\int_a^b f(x)dx$ and $\int_a^b g(x)dx$ either both converge or both diverge.

b) If $L = 0$ in (1) and $\int_a^b g(x)dx$ converges, then $\int_a^b f(x)dx$ converges.

c) If $L = \infty$ in (1) and $\int_a^b g(x)dx$ diverges, then $\int_a^b f(x)dx$ diverges.

This test is related to the comparison test and is often a very useful alternative to it. In particular, taking $g(x) = \dfrac{1}{(x-a)^p}$, we have from known facts about the p integral, the

Theorem 6

Let $\lim\limits_{x \to a^+} (x-a)^p f(x) = L$. Then

i) $\displaystyle\int_a^b f(x)dx$ converges if $p < 1$ and L is finite.

ii) $\displaystyle\int_a^b f(x)dx$ diverges if $p \geq 1$ and $L \neq 0$ (L may be finite).

If $f(x)$ becomes unbounded only at the upper limit these conditions are replaced by those in

Theorem 7

Let $\lim_{x \to b^+} (b-x)^p f(x) = M$. Then

i) $\int_a^b f(x)dx$ converges if $p < 1$ and M is finite.

ii) $\int_a^b f(x)dx$ diverges if $p \geq 1$ and $M \neq 0$ (M may be finite).

Example

1. $\displaystyle\int_1^5 \frac{dx}{\sqrt{x^4-1}}$ converges since

$$\lim_{x\to 1^+} (x-1)^{\frac{1}{2}}.\frac{1}{\sqrt{x^4-1}} = \lim_{x\to 1^+} \sqrt{\frac{x-1}{x^4-1}} = \frac{1}{2}$$

2. $\displaystyle\int_0^3 \frac{xdx}{(3-x)\sqrt{x^2+1}}$ diverges since

$$\lim_{x\to 3^-} (3-x).\frac{1}{(3-x)\sqrt{x^2+1}} = \frac{1}{\sqrt{10}}$$

Absolute and conditional convergence

$\displaystyle\int_a^b f(x)dx$ is called absolutely convergent if $\displaystyle\int_a^b |f(x)|dx$ converges. If $\displaystyle\int_a^b f(x)dx$ converges but $\displaystyle\int_a^b |f(x)|dx$ diverges, then $\displaystyle\int f(x)dx$ is called conditionally convergent.

Theorem 8

If $\int_a^b |f(x)|dx$ converges, then $\int_a^b f(x)dx$ converges. In words, an absolutely convergent integral converges.

Example

Since $\left|\frac{\sin x}{\sqrt[3]{x-\pi}}\right| \leq \frac{1}{\sqrt[3]{x-\pi}}$ and $\int_\pi^{4\pi} \frac{dx}{\sqrt[3]{x-\pi}}$ converges (p integral with $a = \pi$ and $p = \frac{1}{3}$), it follows that $\int_\pi^{4\pi} \left|\frac{\sin x}{\sqrt[3]{x-\pi}}\right| dx$ converges and thus $\int_\pi^{4\pi} \frac{\sin x}{\sqrt[3]{x-\pi}} dx$ converges (absolutely).

Any of the tests used for integrals with non-negative integrands can be used to test for absolute convergence.

Test for convergence of the following.

Example

Investigate the convergence of

a) $\displaystyle\int_2^3 \frac{dx}{x^2(x^3-8)^{2/3}}$

b) $\displaystyle\int_0^{\pi} \frac{\sin x}{x^3}\,dx$

Solution

a) $\displaystyle\lim_{x\to 2^+}(x-2)^{2/3}.\frac{1}{x^2(x^3-8)^{2/3}} =$ $\displaystyle\lim_{x\to 2^+}\frac{1}{x^2}.\left(\frac{1}{x^2+2x+4}\right)^{2/3} = \frac{1}{8\sqrt[3]{18}}.$ Hence The integral converges by **Theorem 6 (ii)**

b) $\displaystyle\lim_{x\to 0^+} x^2.\frac{\sin x}{x^3} = 1.$ Hence the integral converges by **Theorem 6 (ii)**

Example

Prove that $\int_0^\infty e^{-x^2} dx = \sqrt{\pi}/2$

Solution

Let $I_M = \int_0^M e^{-x^2} dx = \int_0^M e^{-y^2} dy$ and let $\lim_{M\to\infty} I_M = I$, the required value of the integral. Then

$$I_M^2 = \left(\int_0^M e^{-x^2} dx \right) \left(\int_0^M e^{-y^2} dy \right)$$

$$= \int_0^M \int_0^M e^{-(x^2+y^2)} dx dy$$

Since the integrand is positive, the regions are in the first quadrant and using the polar coordinates, we have

$$I_M^2 = \int_0^{\pi/2} \int_0^M e^{-r^2} r dr d\theta$$

$$I_M^2 = -\frac{1}{2} \int_0^{\pi/2} \left[e^{-M} - 1\right] d\theta \implies I_M^2 = -\frac{1}{2}\left[e^{-M} - 1\right] \left(\theta\Big|_0^{\pi/2}\right)$$

$$\implies I_M^2 = \frac{\pi}{4}\left[1 - e^{-M}\right] = \frac{\pi}{4} - \frac{\pi}{4}e^{-M}$$

$$\lim_{M\to\infty} I_M^2 = I^2 = \lim_{M\to\infty} \left[\frac{\pi}{4} - \frac{\pi}{4}e^{-M}\right] = \frac{\pi}{4} \implies I^2 = \frac{\pi}{4}$$

$$\implies I = \sqrt{\pi}/2$$

Use integration, the Direct Comparison Test or the Limit comparison Test to test the integrals for convergence.

1. $$\int_0^{\pi} \frac{\sin\theta\, d\theta}{\sqrt{\pi-\theta}}$$
2. $$\int_0^{\pi/2} \tan\theta\, d\theta$$
3. $$\int_0^{\pi/2} \cot\theta\, d\theta$$
4. $$\int_0^{\pi} \frac{dt}{\sqrt{t}+\sin t}$$
5. $$\int_{-1}^{1} \ln|x|\,dx$$
6. $$\int_0^{2} \frac{dx}{1-x}$$

Test for convergence of the following

1. $$\int_{-\infty}^{\infty} \frac{2+\sin x}{x^2+1}dx$$
2. $$\int_{2}^{\infty} \frac{xdx}{(\ln x)^3}$$
3. $$\int_{0}^{\infty} \frac{\sin^2 x}{x^2}dx$$
4. $$\int_{0}^{\infty} \frac{dx}{(x+1)\sqrt{1-x^2}}$$
5. $$\int_{0}^{2} \frac{\ln x}{\sqrt[3]{8-x^3}}dx$$
6. $$\int_{0}^{1} \frac{dx}{x^x}$$

1. $$\int_0^\infty e^{-x} \ln x dx$$
2. $$\int_0^\infty \frac{e^x dx}{\sqrt{x \ln(x+1)}}$$
3. $$\int_0^\infty \frac{dx}{\sqrt[3]{x^4 + x^2}}$$

Introduction

In this unit we look at integral depending on a parameter that has a bearing on Laplace transform and Gamma and Beta function. We also look at differentiation under the integral sign to help us solve some difficult integrals.

Integral depending on a parameter

Let $f : [a, b] \times [c, d] \to \mathbb{R}$, if for each fixed $t \epsilon [c, d]$ the function $f(x, t)$ is integrable over $[a, b]$ on the x variable, we define the following function $F : [a, b] \to \mathbb{R}$ *as*

$$F(t) = \int_a^b f(x, t)dx$$

We call $F(t)$ **an integral depending on a parameter**.(where t is the parameter)

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Example

Let $F(t) = \int_0^{\pi} \sin xt dx$,then

$$\int_0^{\pi} \sin xt dx = \left[-\frac{1}{t} \cos xt \Big|_0^{\pi} \right] = -\tfrac{1}{t}[\cos \pi t - \cos 0]$$

$= \frac{1}{t}(1 - \cos \pi t)$

Therefore ,$F(t) = \frac{1}{t}(1 - \cos \pi t)$

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$$= \tfrac{1}{t}(1 - \cos \pi t)$$

Therefore ,$F(t) = \frac{1}{t}(1 - \cos \pi t)$

Theorem 1

f continuous on $[a, b] \times [c, d] \Rightarrow F$ continuous on $[c, d]$

Theorem 2

f and $f_t = \frac{\partial f}{\partial t}$ continuous on $[a, b] \times [c, d] \Rightarrow F$ differentiable on $[c, d]$

$$F'(t) = \int_a^b f_t(x, t)dx$$

Theorem 2

f and $f_t = \frac{\partial f}{\partial t}$ continuous on $[a, b] \times [c, d] \Rightarrow F$ differentiable on $[c, d]$

$$F'(t) = \int_a^b f_t(x, t)dx$$

Example

Given that $f(x, t) = \sin xt$ and $f_t(x, t) = x \cos xt$
also $F(t) = \frac{1}{t}(1 - \cos \pi t)$,
then we have : $F'(t) = \frac{1}{t^2}(\cos \pi t - 1) + \frac{\pi}{t} \sin \pi t$

For $\displaystyle\int_a^b f_t(x, t)dx = \int_0^\pi f_t(x, t)dx = \int_0^\pi x \cos xtdx$

We have $\displaystyle\int_0^\pi x \cos xtdx = \frac{x}{t} \sin xt\Big|_0^\pi - \int_0^\pi \frac{1}{t} \sin xtdx$

$$= \frac{\pi}{t}\sin \pi t - \left[-\frac{1}{t^2}\cos xt \Big|_0^{\pi} \right]$$
$$= \frac{\pi}{t}\sin \pi t - \frac{1}{t^2}[-\cos \pi t + 1]$$
$$\Rightarrow \int_0^{\infty} x\cos xt dx = \frac{1}{t^2}(\cos \pi t - 1) + \frac{\pi}{t}\sin \pi t$$ Which verifies the theorem as :
$$F'(t) = \int_a^b f_t(x,t)dx$$

Differentiation under the Integral sign

By the fundamental theorem of calculus, if $f(x)$ is a continuous function on the
interval $[a, b]$ and letting $F(x)$ be an antiderivative of $f(x)$;
then $\int_a^b f(x)dx = F(b) - F(a)$. The expression $F(b) - F(a)$
is written more compactly as $F(x)\Big|_a^b$, so this
equation becomes $\int_a^b f(x)dx = F(x)\Big|_a^b$
We know that differentiation and integration are inverse operations.
So, if $f(x)$ is continuous function on the interval $[a, b]$, then
for any x in this interval : $\int_a^x f(t)dt = F(x) - F(a)$.

Where $F(x)$ (and antiderivative of $f(x)$) is a variable and $F(a)$ a constant.

Differentiating both sides with respect to x, we have:

$$\frac{d}{dx}\int_a^x f(t)dt = \frac{d}{dx}\Big[F(x) - F(a)\Big] = \frac{dF(x)}{dx} - \frac{dF(a)}{dx}$$

$$\Rightarrow \frac{d}{dx}\int_a^x f(t)dt = \frac{dF(x)}{dx} - 0$$

$$\therefore \frac{d}{dx}\int_a^x f(t)dt = f(x)$$

Example

1. If
$$\int_0^x \sin t dt = -\cos\Big|_0^x = -\cos x - (-\cos 0) = -\cos x + 1$$
Then
$$\frac{d}{dx}\int_0^x \sin t dt = \frac{d}{dx}(-\cos x + 1) = \sin x + 0 = \sin x$$
Relating it to $\displaystyle \frac{d}{dx}\int_a^x f(t)dt = f(x)$

We conclude that $\displaystyle \frac{d}{dx}\int_0^x \sin t dt = \sin x.$

2. If $\displaystyle \int_5^x e^{2t}dt = \frac{1}{2}e^{2t}\Big|_5^x = \frac{1}{2}\left[e^{2x} - e^{10}\right]$

Then
$$\frac{d}{dx}\int_5^x e^{2t}dt = \frac{d}{dx}\left[\frac{1}{2}\left(e^{2x} - e^{10}\right)\right] = \frac{2}{2}e^{2x} - 0 = e^{2x}$$

$$\text{Hence } \frac{d}{dx}\int_5^x e^{2t}\,dt = e^{2x}$$

3. Simplify the following $\dfrac{d}{dx}\displaystyle\int_2^x \cos t\,dt$

$$\text{Hence } \frac{d}{dx}\int_5^x e^{2t}dt = e^{2x}$$

3. Simplify the following $\frac{d}{dx}\int_2^x \cos t dt$

<u>Solution</u>

Let $f(t) = \cos t$, and let $F(t)$ be an antiderivative of $f(t)$.
Then :

$$\int_2^x \cos t dt = F(t)\Big]_2^x = F(x) - F(2)$$

$$\frac{d}{dx}\int_2^x \cos t dt = F'(x) - F'(2) = f(x) - 0 = f(x)$$

Hence , if we dummy t to x, we have $f(x) = \cos x$

Theorem 3 : Leibniz's Theorem

Let f and $f_t = \dfrac{\delta f}{\delta t}$ be continuous on $[a, b]$ x $[c, d]$ and α, β differentiable functions on [c,d] with image on [a,b], that is, $\alpha(t), \beta(t); [c, d] \to [a, b]$, $x \in [\alpha(t), \beta(t)] \subset [a, b]$

We define $G(t) = \int_{\alpha(t)}^{\beta(t)} f(x, t)dx$

Then G is differentiable on $[c, d]$ and

$$G^{'}(t) = f(\beta(t), t).\beta^{'}(t) - f(\alpha(t), t).\alpha^{'}(t) + \int_{\alpha(t)}^{\beta(t)} f_t(x, t)dx$$

The previous theorems cannot be used when the integrals are improper. So we have the following results that are valid if the integrals are improper or not.

Theorem 4

Let $f(x, t)$ be a continuous function of t on $[c, d]$ for almost every $x \in [a, b]$, if there exists $g(x)$ integral on $[a, b]$ s.t. $|F(x, t)| \leq g(x), \forall t \in [c, d]$ and for every $x \in [a, b]$, then F is continuous on $[c, d]$.

Theorem 5

Let $f_t(x, t)$ be a continuous function of t on $[c, d]$ for almost every $x \in [a, b]$, if there exists $g(x)$ integrable on $[a, b]$ s.t. $|f_t(x, t)| \leq g(x), \forall t \in [c, d]$ and for almost every $x \in [a, b]$, then F is differentiable on $[c, d]$ and $F'(t) = \int_a^b f_t(x, t)dx$

Differentiation under the integral sign is a very useful operation in calculus.

Suppose it is required to differentiate with respect to t the function $F(t) = \int_{a(t)}^{b(t)} f(x, t)dx$, where the function $f(x, t)$ and

$\frac{\partial}{\partial t} f(x,t)$ are both continuous in both t and x in some region of the (x,t) plane, including $a(t) \leq x \leq b(t), t_0 \leq t \leq t_1$, and the function $a(t)$ and $b(t)$ are both continuous and both have continuous derivative for $t_0 \leq t \leq t_1$. Then for $t_0 \leq t \leq t_1$

$$\frac{d}{dt}F(t) = f(t, b(t))b^{'}(t) - f(t, a(t))a^{'}(t) + \int_{a(t)}^{b(t)} \frac{\partial}{\partial t} f(x,t)dx$$

This formula is the general form of the **Leibniz integral rule** and can be derived using fundamental theorem of calculus, which we have already done. The fundamental theorem of calculus is just a particular case of the above formula, $a(t) = a$, a constant, $b(t) = x$ and $f(x,t) = f(x)$. If both

upper and lower limits are taken as constants, then the formula takes the shape of an operator equation $I_x D_t = D_t I_x$, where D_t is the partial derivative with respect to t and I_x is the integral operator with respect to x over a fixed interval.

That is, it is related to the symmetry of second derivatives, but involving integrals as well as derivatives. This is also known as the **Leibniz integral rule**.

Example

1. If $F(t) = \int_t^{t^2} \frac{\sin(tx)}{x} dx$, Find $F^{'}(t)$ where $t \neq 0$.

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Example

1. If $F(t) = \int_t^{t^2} \frac{\sin(tx)}{x} dx$, Find $F^{'}(t)$ where $t \neq 0$.
   Solution:
   By Leibniz's rule,
   $F^{'}(t) =$
   $\int_t^{t^2} \frac{\partial}{\partial t}(\frac{\sin(tx)}{x})dx + \frac{\sin(t.t^2)}{t^2}\frac{d}{dt}(t^2) - \frac{\sin(t.t)}{t}\frac{d}{dt}(t)$

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1. If $F(t) = \int_t^{t^2} \frac{\sin(tx)}{x} dx$, Find $F^{'}(t)$ where $t \neq 0$.
   Solution:
   By Leibniz's rule,
   $$F^{'}(t) =$$
   $$\int_t^{t^2} \frac{\partial}{\partial t}(\frac{\sin(tx)}{x})dx + \frac{\sin(t.t^2)}{t^2}\frac{d}{dt}(t^2) - \frac{\sin(t.t)}{t}\frac{d}{dt}(t)$$
   $$= \int_t^{t^2} \cos(tx)dx + \frac{2\sin t^3}{t} - \frac{sint^2}{t}$$
   $$= \frac{\sin tx}{t}\bigg|_t^{t^2} + \frac{2\sin t^3}{t} - \frac{\sin t^3}{t}$$
   $$= \frac{3\sin t^3 - 2\sin t^2}{t}$$

2. If $\displaystyle\int_0^{\pi} \frac{dx}{t - \cos x} = \frac{\pi}{\sqrt{t^2 - 1}}$, $t > 1$ find $\displaystyle\int_0^{\pi} \frac{dx}{(2 - \cos x)^2}$

2. If $\displaystyle\int_0^{\pi} \frac{dx}{t-\cos x} = \frac{\pi}{\sqrt{t^2-1}}$, $t > 1$ find $\displaystyle\int_0^{\pi} \frac{dx}{(2-\cos x)^2}$

Solution

By Leibnitz's rule, if $\displaystyle F(t) = \int_0^{\pi} \frac{dx}{t-\cos x} = \pi(t^2-1)^{-1/2}$,

Then $\displaystyle F'(t) = -\int_0^{\pi} \frac{dx}{(t-\cos x)^2} = -\frac{1}{2}\pi(t^2-1)^{-3/2}2t = \frac{-\pi t}{(t^2-1)^{3/2}}$

Thus $\displaystyle\int_0^{\pi} \frac{dx}{(t-\cos x)^2} = \frac{\pi t}{(t^2-1)^{3/2}}$ from which

$$\int_0^{\pi} \frac{dx}{(2-\cos x)^2} = \frac{2\pi}{3\sqrt{3}}$$

Integration under the integral sign is the use of the identity

$$\int_a^b F(t)dt = \int_a^b \left(\int_{t_0}^{t_1} f(x,t)dx \right) dt$$

$$= \int_{t_0}^{t_1} \left(\int_a^b f(x,t)dt \right) dx$$

This result is known as *interchange of the order of integration or integration under the integral sign*

Theorem 6

$f(x,t)$ be integrable on $[a,b]\times[c,d] \Rightarrow F(t)$ is integrable on $[c,d]$ and

$$\int_a^b F(t)dt = \int_c^d \int_a^b f(x,t)dxdt = \int_a^b \int_c^d f(x,t)dtdx$$

Theorem 6

$f(x,t)$ be integrable on $[a,b]\times[c,d] \Rightarrow F(t)$ is integrable on $[c,d]$ and

$$\int_a^b F(t)dt = \int_c^d \int_a^b f(x,t)dxdt = \int_a^b \int_c^d f(x,t)dtdx$$

Example

$\displaystyle\int_0^1 x^t dx = \frac{1}{t+1}$ for $t > -1$. Multiply by dt and integrating between a and b gives:

$$\int_a^b dt \int_0^1 x^t dx = \int_a^b \frac{dt}{t+1} = \ln\left|\frac{b+1}{a+1}\right|$$

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$f(x,t)$ be integrable on $[a,b] \times [c,d] \Rightarrow F(t)$ is integrable on $[c,d]$ and

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Example

$\displaystyle\int_0^1 x^t dx = \frac{1}{t+1}$ for $t > -1$. Multiply by dt and integrating between a and b gives:

$$\int_a^b dt \int_0^1 x^t dx = \int_a^b \frac{dt}{t+1} = \ln\left|\frac{b+1}{a+1}\right|$$

But the left-hand side is equal to

$\displaystyle\int_0^1 dx \int_a^b x^t dt = \int_0^1 \frac{x^b - x^a}{\ln x} dx$ so it follows that :

$$\int_0^1 \frac{x^b - x^a}{\ln x}dx = \ln\left|\frac{b+1}{a+1}\right|$$

$$\int_0^1 \frac{x^b - x^a}{\ln x} dx = \ln\left|\frac{b+1}{a+1}\right|$$

Problem

Prove that $\displaystyle\int_0^{\pi} \ln\left(\frac{b-\cos x}{a-\cos x}\right) dx = \pi\left(\frac{b+\sqrt{b^2-1}}{a+\sqrt{a^2-1}}\right)$ if a,b>1

$$\int_0^1 \frac{x^b - x^a}{\ln x} dx = \ln\left|\frac{b+1}{a+1}\right|$$

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Proof

From the previous example $\displaystyle\int_0^\pi \frac{dx}{(t-\cos x)} = \frac{\pi}{\sqrt{t^2-1}}, t > 1$

$$\int_0^1 \frac{x^b - x^a}{\ln x} dx = \ln \left| \frac{b+1}{a+1} \right|$$

Problem

Prove that $\displaystyle\int_0^{\pi} \ln\left(\frac{b-\cos x}{a-\cos x}\right) dx = \pi \left(\frac{b+\sqrt{b^2-1}}{a+\sqrt{a^2-1}}\right)$ if a,b>1

Proof

From the previous example $\displaystyle\int_0^{\pi} \frac{dx}{(t-\cos x)} = \frac{\pi}{\sqrt{t^2-1}}, t > 1$

Integrating the left side with respect to t from a to b yields:

$$\int_0^{\pi} \left(\int_a^b \frac{dt}{t-\cos x}\right) dx = \int_0^{\pi} \ln(t-\cos x)\Big|_a^b dx =$$

$$\int_0^{\pi} \ln\left(\frac{b-\cos x}{a-\cos x}\right) dx$$

Integrating the right side with respect to t from a to b yields:

$$\int_a^b \frac{\pi}{\sqrt{t^2-1}}dt = \pi \ln(t+\sqrt{t^2-1})|_a^b = \pi \left(\frac{b+\sqrt{b^2-1}}{a+\sqrt{a^2-1}} \right)$$

and the required result follows.

Integrating the right side with respect to t from a to b yields:

$$\int_a^b \frac{\pi}{\sqrt{t^2-1}}dt = \pi \ln(t+\sqrt{t^2-1})|_a^b = \pi \left(\frac{b+\sqrt{b^2-1}}{a+\sqrt{a^2-1}} \right)$$

and the required result follows.

Exercise

1. If $F(t) = \displaystyle\int_{\sqrt{t}}^{1/t} \cos tx^2 dx, \frac{dF}{dt}.$

Integrating the right side with respect to t from a to b yields:

$$\int_a^b \frac{\pi}{\sqrt{t^2-1}}dt = \pi \ln(t+\sqrt{t^2-1})|_a^b = \pi\left(\frac{b+\sqrt{b^2-1}}{a+\sqrt{a^2-1}}\right)$$

and the required result follows.

Exercise

1. If $F(t) = \displaystyle\int_{\sqrt{t}}^{1/t} \cos tx^2 dx, \frac{dF}{dt}$.

Ans: $-\displaystyle\int_{\sqrt{t}}^{1/t} x^2 \sin tx^2 dx - \frac{1}{t^2}\cos\frac{1}{t} - \frac{1}{2\sqrt{t}}\cos t^2$

2. (a) $F(t) = \displaystyle\int_0^{t^2} \tan^{-1}\frac{x}{t}dx$, find $\dfrac{dF}{dt}$ by Leibnitz's rule.

(b) Check the result in (a) by direct integration.

Ans: $2t\tan^{-1} t - \dfrac{1}{2}\ln(t^2+1)$

Introduction

The theory of "Laplace Transforms" to be discussed in the following notes will be for the purpose of solving certain kinds of "differential equation"; that is, an equation which involves a derivative or derivatives.

Roughly speaking, the method of Laplace Transforms is used to convert a calculus problem (the differential equation) into an algebra problem (frequently an exercise on partial fractions and/or completing the square). The solution of the algebra problem is then fed backwards through what is called an "Inverse Laplace Transform" and the solution of the differential equation is obtained.

Laplace Transforms

The Laplace Transform of a function $f(t)$, defined for $t > 0$, is denoted by C and is defined as;

$L\{f(t)\} = \int_0^\infty f(t)\, e^{-st} dt$
where $'s'$ is an arbitrary positive number i.e. $s > 0$

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The Laplace Transform of a function $f(t)$, defined for $t > 0$, is denoted by C and is defined as;

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where $'s'$ is an arbitrary positive number i.e. $s > 0$

NB

The Laplace Transform is usually denoted by $L\{f(t)\}$ or $L[f(t)]$ or $F(s)$, since the result of the definite integral in the definition will be an expression involving $'s'$.

Now, the integral in the definition of the transform is called an improper integral and it would probably be best to recall how these kinds of integrals work before we actually jump into computing some transforms

Examples

Example 1

Find the Laplace Transform of $f(t) = a$ (constant)

Examples

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Find the Laplace Transform of $f(t)= a$ (constant)

solution

L{a}= $\int_0^\infty ae^{-st}dt$

[We just plug in 'a' in place of f(t)]

$=a\int_0^\infty e^{-st}dt$

$=a\left[\frac{e^{-st}}{-s}\right]_0^\infty = \frac{-a}{s}[e^{-st}]_0^\infty$

$= \frac{-a}{s}[0-1] = \frac{a}{s}$

$\therefore L\{a\} = \frac{a}{s}$

Examples

Example 1

Find the Laplace Transform of $f(t) = a$ (constant)

solution

L{a}= $\int_0^{\infty} ae^{-st} dt$
[We just plug in 'a' in place of f(t)]
$= a\int_0^{\infty} e^{-st} dt$
$= a\left[\frac{e^{-st}}{-s}\right]_0^{\infty} = \frac{-a}{s}[e^{-st}]_0^{\infty}$
$= \frac{-a}{s}[0 - 1] = \frac{a}{s}$
$\therefore L\{a\} = \frac{a}{s}$

This implies the Laplace Transform of any constant is the "constant over s"

Eg. **(a)** L{5}= $\frac{5}{s}$ **(b)** L{7}= $\frac{7}{s}$
(c) L{59}= $\frac{59}{s}$ **(d)** L{85}= $\frac{85}{s}$

Example 2

To find the Laplace Transform of e^{at} ;*where 'a' is a constant*

Example 2

To find the Laplace Transform of e^{at} ;*where 'a' is a constant*

Solution

$L\{e^{at}\}= \int_0^\infty e^{at}e^{-st}dt$
$= \int_0^\infty e^{at-st}dt = \int_0^\infty e^{-(s-a)t}dt$
$= \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty = -\frac{1}{s-a}[0-1]$
$\frac{1}{s-a}$
$\therefore L\{e^{at}\} = \frac{1}{s-a}$

Eg. **(a)** $L\{e^{5t}\} = \frac{1}{s-5}$
(b) $L\{e^{8t}\} = \frac{1}{s-8}$
(c) $L\{e^{-2t}\} = \frac{1}{s--2} = \frac{1}{s+2}$

Example 3

To find the Laplace Transform of $f(t) = \sin at$

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To find the Laplace Transform of $f(t) = \sin at$

Solution

$L\{\sin at\} = \int_0^\infty \sin ate^{-st}dt$
We do this using integration by parts, however, it is much easier to use the fact that; $e^{i\theta} = \cos\theta t + i\sin\theta$;
So that $\sin\theta$ is the imaginary part of $e^{i\theta}$, written as $Im(e^{i\theta})$

$L\{\sin at\} = L\{Im(e^{iat})\}$
$= Im\int_0^\infty e^{iat}e^{-st}dt = Im\int_0^\infty e^{-(s-ia)t}dt$
$= Im(\left[\frac{e^{-(s-ia)t}}{-(s-ia)}\right]_0^\infty) = Im(0 - -\frac{1}{s-ia}) = \frac{1}{s-ia}$
(Multiply both the numerator and denominator by the conjugate of $s - ia$)

$$= Im\left[\frac{s+ia}{s^2+a^2}\right] \quad \frac{a}{s^2+a^2} \quad ; \textit{ i.e. we take the Imaginary part}$$

Hence , $L\{\sin at\} = \dfrac{a}{s^2+a^2}$

E.g

(a) $L\{\sin 2t\} = \dfrac{2}{s^2+4}$ (b) $L\{\sin 5t\} = \dfrac{2}{s^2+25}$

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Example 4

To find the Laplace Transform of $f(t) = \cos at$

E.g

(a) $L\{\sin 2t\} = \dfrac{2}{s^2+4}$ (b) $L\{\sin 5t\} = \dfrac{2}{s^2+25}$

Example 4

To find the Laplace Transform of $f(t) = \cos at$

Solution

$L\{\cos at\} = L\{Re(e^{iat})\}$
It is almost the same as that of the sin but here we take the Real part; written $Re(e^{iat})$;

$\therefore$ from $Re\left[\dfrac{s+ia}{s^2+a^2}\right]$

$$= L\{\cos at\} = Re\left[\frac{s+ia}{s^2+a^2}\right] = \frac{s}{s^2+a^2}$$

E.g

(a) $L\{\cos 2t\} = \dfrac{s}{s^2+4}$ (b) $L\{\cos 4t\} = \dfrac{s}{s^2+16}$

Example 5

To find the Laplace Transform of $f(t) = t^n$, where n is a positive integer.

Solution

By Definition;

$L\{t^n\} = \int_0^\infty t^n e^{-st} dt$

Integrating by parts:

$L\{t^n\} = \left[t^n \left(\frac{e^{-st}}{-s}\right)\right]_0^\infty + \frac{n}{s}\int_0^\infty t^{n-1} e^{-st} dt$

$= \frac{-1}{s}[t^n e^{-st}]_0^\infty + \frac{n}{s}\int_0^\infty t^{n-1} e^{-st} dt$

In a product such as $t^n e^{-st}$, the numerical value of s is large enough to make the product converge to zero as $t \to \infty$

$\therefore [t^n e^{-st}]_0^\infty = 0 - 0$

$L\{t^n\} = \frac{n}{s}\int_0^\infty t^{n-1} e^{-st} dt$

Notice that $\int_0^\infty t^{n-1} e^{-st} dt$ is identical to $\int_0^\infty t^n e^{-st} dt$ except that n is replaced by $(n-1)$.

If $I_n = \int_0^\infty t^n e^{-st} dt$, then $I_{n-1} = \int_0^\infty t^{n-1} e^{-st} dt$ and the result becomes $I_n = \frac{n}{s} I_{n-1}$. Replace n by $n-1$ repeatedly.

$$I_n = \int_0^\infty t^n e^{-st} dt = \tfrac{n}{s} I_{n-1} = \frac{n}{s} \cdot \frac{n-1}{s} I_{n-2}$$

$$= \frac{n}{s} \cdot \frac{n-1}{s} \cdot \frac{n-2}{s} I_{n-3} \ ...etc$$

$$I_n = \frac{n}{s} \cdot \frac{n-1}{s} \cdot \frac{n-2}{s} \ .... \ \frac{n-(n-1)}{s} I_0$$

where $I_o = L\{t^o\} = L\{1\} = \dfrac{1}{s}$

$$\therefore I_n = \frac{n(n-1)(n-2)(n-3)...(3)(2)(1)}{s^{n+1}} = \frac{n!}{s^{n+1}}$$

Hence, $L\{t^n\} = \dfrac{n!}{s^{n+1}}$

E.g

(a) $L\{t\} = \dfrac{1!}{s^{1+1}} = \dfrac{1}{s^2}$ **(b)** $L\{t^6\} = \dfrac{6!}{s^{6+1}} = \dfrac{720}{s^7}$

Example 6

To find the Laplace Transform of $f(t) = \sinh at$

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Solution

Note that

$\sinh at = \frac{1}{2}(e^{at} - e^{-at})$

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To find the Laplace Transform of $f(t) = \sinh at$

<u>Solution</u>

Note that

$\sinh at = \frac{1}{2}(e^{at} - e^{-at})$

$\therefore L\{\sinh at\} = \frac{1}{2}\int_0^\infty (e^{at} - e^{-at})e^{-st}dt$

$=\frac{1}{2}\int_0^\infty (e^{-(s-a)t} - e^{-(s+a)t})dt$

$$=\frac{1}{2}\left(\left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty - \left[\frac{e^{-(s+a)t}}{-(s+a)}\right]_0^\infty\right)$$

$$= \frac{1}{2}\left[0 - -\frac{1}{s-a}\right] - \left[0 - -\frac{1}{s+a}\right]$$

$$= \frac{1}{2}\left[\left(\frac{1}{s-a}\right) - \left(\frac{1}{s+a}\right)\right] = \frac{1}{2}\left[\frac{2a}{s^2-a^2}\right] = \frac{a}{s^2-a^2}$$

Hence, $L\{\sinh at\} = \dfrac{a}{s^2-a^2}$

E.g

(a) $L\{\sinh 2t\} = \dfrac{2}{s^2 - 4}$ **(b)** $L\{\sinh 7t\} = \dfrac{7}{s^2 - 49}$

Example 7

To find the Laplace Transform of $f(t) = \cosh at$

Solution

Note that

$\cosh at = \frac{1}{2}(e^{at} + e^{-at})$

$\therefore L\{\cosh at\} = \frac{1}{2}\int_0^{\infty}(e^{at} + e^{-at})e^{-st}dt$

$$= \frac{1}{2}\left(\left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^{\infty} + \left[\frac{e^{-(s+a)t}}{-(s+a)}\right]_0^{\infty}\right)$$

$$= \frac{1}{2}\left[\left(\frac{1}{s-a}\right) + \left(\frac{1}{s+a}\right)\right] = \frac{1}{2}\left[\frac{2s}{s^2 - a^2}\right] = \frac{s}{s^2 - a^2}$$

Hence, $L\{\cosh at\} = \dfrac{s}{s^2 - a^2}$

E.g

(a) $L\{\cosh 2t\} = \dfrac{s}{s^2-4}$ **(b)** $L\{\cosh 3t\} = \dfrac{s}{s^2-9}$

Computing Laplace Transforms

NOTE

Given $f(t)$ and $g(t)$ then,

$L\{af(t)+bg(t)\} = aF(s)+bG(s)$ for any constants a and b.

In other words, we don't worry about constants and we don't worry about sums or differences of functions in taking Laplace transforms. All that we need to do is take the transform of the individual functions, then put any constants back in and add or subtract the results back up.

E.g

(a) $L\{\cosh 2t\} = \dfrac{s}{s^2 - 4}$ **(b)** $L\{\cosh 3t\} = \dfrac{s}{s^2 - 9}$

Computing Laplace Transforms

NOTE

Given $f(t)$ and $g(t)$ then,

$L\{af(t) + bg(t)\} = aF(s) + bG(s)$ for any constants a and b.

In other words, we don't worry about constants and we don't worry about sums or differences of functions in taking Laplace transforms. All that we need to do is take the transform of the individual functions, then put any constants back in and add or subtract the results back up.

Examples

(1) $L\{2\sin 3t + \cos 3t\} =$

$$2L\{\sin 3t\} + L\{\cos 3t\} = 2 \cdot \left(\frac{3}{s^2 + 9}\right) + \left(\frac{s}{s^2 + 9}\right)$$

$$\frac{6}{s^2+9} + \frac{s}{s^2+9} = \frac{6+s}{s^2+9}$$

$$\frac{6}{s^2+9} + \frac{s}{s^2+9} \;=\; \frac{6+s}{s^2+9}$$

(2) $L\{4e^{2t} + 3\cosh 4t\}$

$$\frac{6}{s^2+9} + \frac{s}{s^2+9} = \frac{6+s}{s^2+9}$$

(2) $L\{4e^{2t} + 3\cosh 4t\}$

$$= 4L\{e^{2t}\} + 3L\{\cosh 4t\} = 4 \cdot \left(\frac{1}{s-2}\right) + 3 \cdot \left(\frac{s}{s^2-16}\right)$$

$$= \frac{4}{s-2} + \frac{3s}{s^2-16}$$

$$= \frac{7s^2 - 6s - 64}{(s-2)(s^2-16)}$$

Exercises

(1)$L\{2\sin 3t + 4\sinh 3t\}$ (2) $L\{5e^{4t} + \cosh 2t\}$
(3)$L\{t^3 + t^2 - 4t + 1\}$ (4) $L\{6e^{-5t} + e^{3t} + 5t^3 - 9\}$

THEOREM 1 : THE FIRST SHIFT THEOREM

If the $L\{f(t)\} = F(s)$ then
$L\{e^{at}f(t)\} = F(s-a)$

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(1) Find the Laplace of $f(t) = e^{-3t}\sin 2t$

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Examples

(1) Find the Laplace of $f(t) = e^{-3t}\sin 2t$

Solution

$$L\{\sin 2t\} = \frac{2}{s^2+4} \text{ and } L\{e^{-3t}\} = \frac{1}{s+3} \text{ then ;}$$

$$L\{e^{-3t}\sin 2t\} = \frac{2}{(s+3)^2+4} = \frac{2}{s^2+6s+13}$$

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$$L\{e^{-3t}\sin 2t\} = \frac{2}{(s+3)^2+4} = \frac{2}{s^2+6s+13}$$

(2) $L\{2e^{3t}\sin 3t\}$

Solution

$$L\{e^{3t}\} = \frac{1}{s-3} \text{ and } L\{\sin 3t\} = \frac{3}{s^2+9}$$

$$\therefore L\{2e^{3t}\sin 3t\} = 2\cdot\frac{3}{(s-3)^2+9} = \frac{6}{(s-3)^2+9}$$

Exercises

(1) $L\{e^{-2t}\cosh 3t\}$ (2) $L\{e^{3t}\sinh 2t\}$
(3) $L\{4e^{-t}\}$ (4) $L\{e^{3t}\sinh 2t\}$
(5) $L\{t^3e^{-4t}\}$

THEOREM 2

Multiplying by 't'and 't^n'

If the $L\{f(t)\} = F(s)$ then

$$L\{tf(t)\} = -\frac{d}{ds}F(s)$$

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(1) Find the Laplace of $f(t) = t \sin 2t$

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Multiplying by 't'and 't^n'

If the $L\{f(t)\} = F(s)$ then

$$L\{tf(t)\} = -\frac{d}{ds}F(s)$$

Examples

(1) Find the Laplace of $f(t) = t\sin 2t$

Solution

$$L\{\sin 2t\} = \frac{2}{s^2+4}$$

Now $L\{t\sin 2t\} = -\dfrac{d}{ds}\left(\dfrac{2}{s^2+4}\right)$

$$= -\left[\frac{(s^2+4)(0) - 2(2s)}{(s^2+4)^2}\right] = -\frac{-4s}{(s^2+4)^2} = \frac{4s}{(s^2+4)^2}$$

Eg.2 Find $L\{t\cosh 3t\}$

Eg.2 Find $L\{t\cosh 3t\}$

<u>Solution</u>

$$L\{\cosh 3t\} = \frac{s}{s^2-9}$$

$$L\{t\cosh 3t\} = -\frac{d}{ds}\left(\frac{s}{s^2-9}\right) = -\left[\frac{(s^2-9)(1)-(s)(2s)}{(s^2-9)^2}\right]$$

$$= -\left(\frac{s^2-9-2s^2}{(s^2-9)^2}\right) = -\left(\frac{-s^2-9}{(s^2-9)^2}\right)$$

$$= \frac{s^2+9}{(s^2-9)^2}$$

What we do is that, anytime we are multiplying a function $f(t)$ by t , we differentiate the Laplace of the function f(t) and then negate the results (theorem 2)

Exercises

(1) $L\{t^2 \cosh 3t\}$

Hint: Since 'n' in t^n is 2, we differentiate and negate twice

(2) $L\{t^2 \sinh 2t\}$

(3) $L\{t \sinh 4t\}$

(4) $L\{t^2 \sin 2t\}$

(5) $L\{t^3 e^{-4t}\}$

Generally

If $L\{f(t)\} = F(s)$ then $L\{t^n f(t)\} = (-1)^n \dfrac{d^n}{ds^n} F(s)$

THEOREM 3

Dividing by 't'
If $L\{f(t)\} = F(s)$ then
$L\{\frac{f(t)}{t}\} = \int_s^{\infty} F(s)ds$

This rule is somewhat restricted in use, since it is applicable if and only if the limit of $\left[\frac{f(t)}{t}\right]$ as $t \to 0$ exists.
In indeterminate case, we use L'Hopital's rule to find out.

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In indeterminate case, we use L'Hopital's rule to find out.

Examples

Determine $L\{\frac{\sin at}{t}\}$

<u>Solution</u>

First, we have to test if the limits exists. If it does, then we continue but if it doesn't we can't continue.

So $\lim\limits_{t\to 0}\left[\frac{\sin at}{t}\right] = \frac{0}{0}$, which is indeterminate..

Using L'hospitals rule..

$$\lim_{t \to 0} \left[\frac{\sin at}{t} \right] = \frac{a \cos at}{1}$$

$$\lim_{t \to 0} \left[\frac{a \cos at}{1} \right] = a \quad \textit{i.e the limit exists ..}$$

Then we continue using the theorem 3

Using L'hospitals rule..

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$$\lim_{t\to 0}\left[\frac{a\cos at}{1}\right] = a \quad \textit{i.e the limit exists ..}$$

Then we continue using the theorem 3

Eg.(2) $L\{\frac{1-\cos 2t}{t}\}$

Using L'hospitals rule..

$$\lim_{t\to 0}\left[\frac{\sin at}{t}\right] = \frac{a\cos at}{1}$$

$$\lim_{t\to 0}\left[\frac{a\cos at}{1}\right] = a \quad \textit{i.e the limit exists ..}$$

Then we continue using the theorem 3

Eg.(2) $L\{\frac{1-\cos 2t}{t}\}$

<u>Solution</u>

First, we test $\lim\limits_{t\to 0}\left[\dfrac{1-\cos 2t}{t}\right]$

$$\lim_{t\to 0}\left[\frac{1-1}{0}\right] = \frac{0}{0}$$

By L'hospitals rule..

$$\lim_{t\to 0}\left[0+\frac{2\sin 2t}{1}\right] = \frac{0}{1} = 0$$

hence it exists..

Then we continue...

$$L\{1-\cos 2t\} = \frac{1}{s} - \frac{s}{s^2+4}$$

By Theorem 3

$$L\{\tfrac{1-\cos 2t}{t}\} = \int_s^\infty \left(\tfrac{1}{s} - \tfrac{s}{s^2+4}\right) ds$$
$$= \left[\ln s - \tfrac{1}{2}\ln|s^2+4|\right]_s^\infty$$
$$= \frac{1}{2}\left[\ln\left(\frac{s^2}{s^2+4}\right)\right]_s^\infty$$

As $s \to \infty$

$$\ln\left(\frac{s^2}{s^2+4}\right) \to \ln(1) = 0$$

$$\therefore L\{\tfrac{1-\cos 2t}{t}\} = \frac{-1}{2}\ln\left(\frac{s^2}{s^2+4}\right)$$
$$= \ln\left(\frac{s^2}{s^2+4}\right)^{\frac{-1}{2}} = \ln\sqrt{\frac{s^2+4}{s^2}}$$

Exercises

Determine the Laplace Transforms of the ff. functons.

(1) $\sin 3t$

(2) $\cos 2t$

(3)e^{4t}

(4)$6t^2$

(5) $\sinh 3t$

(6)$t \cosh 4t$

(7)$\dfrac{e^{3t}-1}{t}$

(8)$e^{3t}\cos 4t$

(9)t^2-3t+4

(10)$t^2 \sin t$

INVERSE TRANSFORMS

Suppose the Laplace transform $L\{f(t)\} = F(s)$, then the inverse transform is given as $L^{-1}\{F(s)\} = f(t)$

INVERSE TRANSFORMS

Suppose the Laplace transform $L\{f(t)\} = F(s)$, then the inverse transform is given as $L^{-1}\{F(s)\} = f(t)$

Note

Given the two functions $F(s)$ and $G(s)$ then
$L^{-1}\{aF(s) + bG(s)\} = aL^{-1}\{F(s)\} + bL^{-1}\{G(s)\}$
for any constants a and b.

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Examples

Find the Inverse transforms of the following

$$(a) F(s) = \frac{6}{s} - \frac{1}{s-8} + \frac{4}{s-3}$$

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for any constants a and b.

Examples

Find the Inverse transforms of the following

(a) $F(s) = \dfrac{6}{s} - \dfrac{1}{s-8} + \dfrac{4}{s-3}$

<u>Solution</u>

$$f(t) = 6\left(\frac{1}{s}\right) - \frac{1}{s-8} + 4\left(\frac{1}{s-3}\right)$$
$$= 6(1) - e^{8t} + 4(e^{3t}) = 6 - e^{8t} + 4e^{3t}$$

$(b) L^{-1}\{\frac{1}{s-2}\} = e^{2t}$
$(c) L^{-1}\{\frac{4}{s}\} = 4$
$(d) L^{-1}\{\frac{s}{s^2+25}\} = \cos 5t$
$(e) L^{-1}\{\frac{12}{s^2-9}\} = 4\left(\frac{3}{s^2-3^2}\right) = 4\sinh 3t$

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Rules in Partial Fractions

1. The numerator must be of lower degree than the denominator. If it is not, then we first divide out.
2. Factorize the denominator into prime factors. These determine the shapes of the partial fractions.
3. A linear factor $(s+a)$ gives a partial fraction $\dfrac{A}{s+a}$ where A is a constant to be determined.
4. A repeated factor $(s+a)^2$ gives $\dfrac{A}{s+a} + \dfrac{B}{(s+a)^2}$

5. Similarly, $(s+a)^3$ gives $\dfrac{A}{s+a}+\dfrac{B}{(s+a)^2}+\dfrac{C}{(s+a)^3}$
6. A quadratic factor (s^2+ps+q) gives $\dfrac{Ps+Q}{s^2+ps+q}$
7. Repeated quadratic factors $(s^2+ps+q)^2$ gives
$\dfrac{Ps+Q}{s^2+ps+q}+\dfrac{Rs+T}{(s^2+ps+q)^2}$

Examples

Find
$L^{-1}\left\{\frac{5s+1}{s^2-s-12}\right\}$

5. Similarly, $(s+a)^3$ gives $\dfrac{A}{s+a} + \dfrac{B}{(s+a)^2} + \dfrac{C}{(s+a)^3}$
6. A quadratic factor (s^2+ps+q) gives $\dfrac{Ps+Q}{s^2+ps+q}$
7. Repeated quadratic factors $(s^2+ps+q)^2$ gives $\dfrac{Ps+Q}{s^2+ps+q} + \dfrac{Rs+T}{(s^2+ps+q)^2}$

Examples

Find

$L^{-1}\left\{\frac{5s+1}{s^2-s-12}\right\}$

Solution

Here, we have to split the denominator into two to get ones of the form in which we can get our transform.

...continued

$$\frac{5s+1}{s^2-s-12} \equiv \frac{5s+1}{(s-4)(s+3)} \equiv \frac{A}{(s-4)} + \frac{B}{(s+3)}$$

$$\frac{5s+1}{s^2-s-12} \equiv \frac{A}{(s-4)} + \frac{B}{(s+3)}$$

$$5s+1 \equiv A(s+3) + B(s-4)$$

when s=4

$\frac{21}{7} = \frac{7A}{7} \therefore A = 3$

when s=-3

$\frac{-14}{-7} = \frac{-7B}{-7} \therefore B = 2$

Hence, $\dfrac{5s+1}{s^2-s-12} \equiv \dfrac{3}{(s-4)} + \dfrac{2}{(s+3)}$

$\therefore L^{-1}\{\frac{5s+1}{s^2-s-12}\} = 3e^{4t} + 2e^{-3t}$

Determine $L^{-1}\left\{\frac{9s-8}{s^2-2s}\right\}$

Determine $L^{-1}\{\frac{9s-8}{s^2-2s}\}$

Solution

$$\frac{9s-8}{s^2-2s} \equiv \frac{A}{(s-2)} + \frac{B}{(s)}$$

$9s-8 \equiv A(s) + B(s-2)$

when s=2

$10 = 2A \therefore A = 5$

when s=0

$-8 = -2B \therefore B = 4$

Hence, $\dfrac{9s-8}{s^2-2s} \equiv \dfrac{5}{(s-2)} + \dfrac{4}{s}$

$\therefore L^{-1}\{\frac{9s-8}{s^2-2s}\} = 5e^{2t} + 4$

Express $F(s) = \dfrac{s^2 - 15s + 41}{(s+2)(s-3)^2}$ in partial fractions and hence determine its inverse transform.

Express $F(s) = \dfrac{s^2 - 15s + 41}{(s+2)(s-3)^2}$ in partial fractions and hence determine its inverse transform.

<u>Solution</u>

$$\frac{s^2 - 15s + 41}{(s+2)(s-3)^2} \equiv \frac{A}{(s+2)} + \frac{B}{(s-3)} + \frac{C}{(s-3)^2}$$

$$s^2 - 15s + 41 = A(s-3)^2 + B(s+2)(s-3) + C(s+2)$$

$A = 3, B = -2, C = 1$

Hence $\dfrac{s^2 - 15s + 41}{(s+2)(s-3)^2} = \dfrac{3}{(s+2)} - \dfrac{2}{(s-3)} + \dfrac{1}{(s-3)^2}$

$L^{-1}\{\frac{s^2-15s+41}{(s+2)(s-3)^2}\} = 3e^{-2t} - 2e^{3t} + te^{3t}$

NB; For $L^{-1}\{\frac{1}{(s-3)^2}\}$; Remember $L^{-1}\{\frac{1}{s^2}\} = t$

and by the **first shift theorem** $\dfrac{1}{(s-3)^2}$ is just like $\dfrac{1}{s^2}$ with s replaced by $(s-3)$ thus $a = 3$ ..Hence we have te^{3t}

Exercises

1. Determine $L^{-1}\left\{\frac{4s^2-5s+6}{(s+1)(s^2+4)}\right\}$
2. Find the inverse transforms of ;

$(a)\dfrac{1}{2s-3}$

$(b)\dfrac{5}{(s-4)^3}$

$(c)\dfrac{3s+4}{s^2+9}$

3. Express in partial fractions and their inverse transforms

$(a)\dfrac{22s+16}{(s+1)(s-2)(s+3)}$

$(b)\dfrac{s^2-11s+16}{(s+1)(s-2)^2}$

4. Determine

$(a) L^{-1}\left\{\frac{4s^2-17s-24}{s(s+3)(s-4)}\right\}$

$(b) L^{-1}\left\{\frac{5s^2-4s-7}{(s-3)(s^2+4)}\right\}$

SOLUTIONS TO DIFFERENTIAL EQUATION BY LAPLACE TRANSFORMS

To solve a differential equation by Laplace transforms, we go through **four distinct** stages

a. Rewrite the equation in terms of Laplace transforms.
b. Insert the given initial conditions.
c. Rearrange the equation algebraically to give the transform of the solution.
d. Determine the inverse transform to obtain the particular solution.

Laplace Transform of derivatives

$$\bullet L\{y''\} = s^2 L\{y\}(s) - sy(0) - y'(0)$$
$$= s^2 Y(s) - sy(0) - y'(0)$$

$$\bullet L\{y'\} = sL\{y\}(s) - y(0)$$
$$= sY(s) - y(0)$$

NB; $L\{y\} = Y(s)$

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$$= sY(s) - y(0)$$

NB; $L\{y\} = Y(s)$

Generally

$$L\{y^{(k)}\} = s^{(k)} Y(s) - s^{(k-1)} y(0) - \ldots - sy^{(k-2)}(0) - y^{(k-1)}(0)$$

Examples

Use the Laplace transform to solve the initial value problem $y'' - 2y' - 3y = 0$ with $y(0) = 1$ and $y'(0) = 0$

Examples

Use the Laplace transform to solve the initial value problem $y'' - 2y' - 3y = 0$ with $y(0) = 1$ and $y'(0) = 0$

Solution

$L\{y'' - 2y' - 3y = 0\}$
$= [s^2Y(s) - sy(0) - y'(0)] - 2[sY(s) - y(0)] - 3[Y(s)] = 0$
$= s^2Y(s) - sy(0) - y'(0) - 2sY(s) + 2y(0) - 3Y(s) = 0$
$= Y(s)(s^2 - 2s - 3) + y(0)(-s + 2) - y'(0) = 0$
substitution for $y(0) = 1$ and $y'(0) = 0$ gives
$= Y(s)(s^2 - 2s - 3) = s - 2$

$$Y(s) = \frac{s-2}{s^2 - 2s - 3}$$

Finding the Laplace inverse gives ;

$$\frac{s-2}{s^2 - 2s - 3} = \frac{\frac{1}{4}}{s-3} + \frac{\frac{3}{4}}{s+1}$$

..continued

Hence ;

$$L^{-1}\{\frac{s-2}{s^2-2s-3}\} = \frac{1}{4}e^{3t} + \frac{3}{4}e^{-t}$$

Try the following

1. $y' + 3y = e^{2t}, \quad y(0) = -1$
2. $y'' - 4y = e^{-t}, \quad y(0) = -1, y'(0) = 0$
3. $y'' - 10y' + 9y = 5t, \quad y(0) = -1, y'(0) = 2$
4. $y'' - 2y' + y = 3e^{t}, \quad y(0) = 1, y'(0) = 1$

THE GAMMA AND BETA FUNCTIONS

THE GAMMA FUNCTION

The gamma function may be regarded as a generalization of $n!$ (n-factorial), where n is any positive integer to $x!$, where x is any real number. (With limited exceptions, the discussion that follows will be restricted to positive real numbers.)

The Gamma function is defined by the improper integral

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt \text{ ——-(1) ; it is convergent for } x > 0$$

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The gamma function may be regarded as a generalization of $n!$ (n-factorial), where n is any positive integer to $x!$, where x is any real number. (With limited exceptions, the discussion that follows will be restricted to positive real numbers.)

The Gamma function is defined by the improper integral

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} \mathrm{d}t \text{ ———-(1) ; it is convergent for } x > 0$$

From (1) $\Gamma(x+1) = \displaystyle\int_0^\infty t^x e^{-t} \mathrm{d}t$

Integrating by parts

$$\Gamma(x+1) = \left[t^x \left(\frac{e^{-t}}{-1} \right) \right]_0^\infty + x \int_0^\infty e^{-t} t^{x-1} dt$$

$$= (0-0) + x\Gamma(x)$$

$\therefore \Gamma(x+1) = x\Gamma(x)$

This is the fundamental recurrence relation for the gamma functions. It can also be written as
$\Gamma(x) = (x-1)\Gamma(x-1)$

With it we can derive a number of other results;
For instance, when $x = n$, a positive integer ≥ 1 ,then
$\Gamma(n+1) = n\Gamma(n)$ But $\Gamma(n) = (n-1)\Gamma(n-1)$
$\quad = n(n-1)\Gamma(n-1) \quad \Gamma(n-1) = (n-2)\Gamma(n-2)$
$\quad = n(n-1)(n-2)\Gamma(n-2)$

- - - - - - - - - -

$\quad = n(n-1)(n-2)(n-3)...1\Gamma(1) = n!\Gamma(1)$
but from the definition $\Gamma(1) = 1$ because

$$\Gamma(1) = \int_0^\infty t^0 e^{-t} dt = \left[-e^{-t}\right]_0^\infty = 0 + 1 = 1$$

therefore, we have $\Gamma(1) = 1$ and $\Gamma(n+1) = n!$ provided n is a positive integer.

$\therefore \Gamma(7) = \Gamma(6+1) = 6! = 720$
$\Gamma(8) = \Gamma(7+1) = 7! = 5040$
or $\Gamma(7+1) = 7\Gamma(7) = 7(720) = 5040$
$\Gamma(9) = \Gamma(8+1) = 8! = 40320$

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$\Gamma(8) = \Gamma(7+1) = 7! = 5040$
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$\Gamma(9) = \Gamma(8+1) = 8! = 40320$

We can also use the recurrence relation in reverse
$\Gamma(x+1) = x\Gamma(x)$

$$\therefore \Gamma(x) = \frac{\Gamma(x+1)}{x}$$

For example ,given that $\Gamma(7) = 720$, we can determine $\Gamma(6)$

$$\Gamma(6) = \frac{\Gamma(6+1)}{6} = \frac{\Gamma(7)}{6} = \frac{720}{6} = 120$$

Now we have used the original definition

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} \mathrm{dt}$$

What happens when $x = \frac{1}{2}$?
Lets Note that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

Examples

evaluate

- $\Gamma(0)$
- $\Gamma\left(\frac{3}{2}\right)$
- $\Gamma\left(\frac{5}{2}\right)$
- $\Gamma\left(\frac{7}{2}\right)$
- $\Gamma\left(\frac{9}{2}\right)$

Solution

- $\Gamma(0) = \frac{1}{0}\Gamma(0+1) = \frac{1}{0} = \infty$

Solution

- $\Gamma(0) = \frac{1}{0}\Gamma(0+1) = \frac{1}{0} = \infty$
- $\Gamma\left(\frac{3}{2}\right) = \Gamma\left(\frac{1}{2}+1\right)$ since $\Gamma(x+1) = x\Gamma(x)$

$$= \left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right) \text{ but } \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

$$= \frac{1}{2}\sqrt{\pi}$$

Solution

- $\Gamma(0) = \dfrac{1}{0}\Gamma(0+1) = \dfrac{1}{0} = \infty$
- $$\begin{aligned} \Gamma\left(\frac{3}{2}\right) &= \Gamma\left(\frac{1}{2}+1\right) \text{ since } \Gamma(x+1) = x\Gamma(x) \\ &= \left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right) \text{ but } \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \\ &= \frac{1}{2}\sqrt{\pi} \end{aligned}$$
- $$\begin{aligned} \Gamma\left(\frac{5}{2}\right) &= \Gamma\left(\frac{3}{2}+1\right) = \frac{3}{2}\Gamma\left(\frac{3}{2}+1\right) = \frac{3}{2}\Gamma\left(\frac{1}{2}+1\right) \\ &= \frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{3}{4}\Gamma\left(\frac{1}{2}\right) \\ &= \frac{3}{4}\sqrt{\pi} \end{aligned}$$

Solution Cont'd

- $\Gamma\left(\frac{7}{2}\right) = \Gamma\left(\frac{5}{2}+1\right) = \frac{5}{2}\Gamma\left(\frac{5}{2}\right)$ but $\Gamma\left(\frac{5}{2}\right) = \frac{3}{4}\sqrt{\pi}$
  hence $\frac{5}{2} \cdot \frac{3}{4}\sqrt{\pi} = \frac{15}{8}\sqrt{\pi}$
- $$\begin{aligned}\Gamma\left(\frac{9}{2}\right) &= \Gamma\left(\frac{7}{2}+1\right) \\ &= \frac{7}{2}\Gamma\left(\frac{7}{2}\right) = \frac{7}{2} \cdot \frac{15}{8}\sqrt{\pi} \\ &= \frac{105}{6}\sqrt{\pi}\end{aligned}$$

In summary

- $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} \mathrm{dt}$
- $\Gamma(x+1) = x\Gamma(x)$
- $\Gamma(n+1) = n!$, where $n \in z^+$
- $\Gamma(1) = 1, \Gamma(0) = \infty, \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}, \Gamma(-n) = \pm\infty$

Now let us evaluate some integrals

1.Evaluate $\int_0^{\infty} x^7 e^{-x} \mathrm{dx}$

Now let us evaluate some integrals

1.Evaluate $\int_0^\infty x^7 e^{-x} \mathrm{d}x$

solution

We recognize this as a standard form of the gamma function $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$ just that the variables have been changed. So it is often convenient to change it to the form $\Gamma(v) = \int_0^\infty x^{v-1} e^{-x} dx$

solution cont'd

Our example then becomes

$$= \int_0^{\infty} x^7 e^{-x} dx = \int_0^{\infty} x^{v-1} e^{-x} dx$$

by comparing $v = 8$,

$$\int_0^{\infty} x^7 e^{-x} dx = \Gamma(8) = 7! = 5040$$

2. Evaluate $\int_0^{\infty} x^3 e^{-4x} dx$

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Solution

If we compare the above to $\Gamma(v) = \int_0^\infty x^{v-1} e^{-x} dx$ we must reduce the power of e to a single variable. Let $y = 4x$

Then $x = \dfrac{y}{4}$

$$\int_0^\infty \left(\frac{y}{4}\right)^3 e^{-y} \frac{dy}{4}$$

$$= \frac{1}{4} \cdot \frac{1}{4^3} \int_0^\infty y^3 e^{-y} dy$$

Now comparing; $v = 4$

$$\frac{1}{256}\Gamma(4) = \frac{1}{256} \times 6 = \frac{3}{128}$$

Example

Evaluate $\displaystyle\int_0^\infty x^{\frac{1}{2}} e^{-x^2}\, dx$

Example

Evaluate $\displaystyle\int_0^{\infty} x^{\frac{1}{2}} e^{-x^2}\,dx$

Solution

To make the power of e a single variable,

Let $y = x^2$

$\dfrac{dy}{dx} = 2x = dx = \dfrac{dy}{2x}$

and since $y = x^2; x = y^{\frac{1}{2}}$

The limits remain unchanged because, when
$x = 0, y = 0; x = \infty, y = \infty$

$$\int_0^\infty \left(y^{\frac{1}{2}}\right)^{\frac{1}{2}} e^{-y} \frac{dy}{2x}$$

$$= \int_0^\infty y^{\frac{1}{4}} e^{-y} \frac{dy}{2x}$$

$$= \int_0^\infty y^{\frac{1}{4}} e^{-y} \frac{dy}{2y^{\frac{1}{2}}}$$

$$= \frac{1}{2} \int_0^\infty y^{-\frac{1}{4}} e^{-y} dy$$

Now by comparing to $\int_0^\infty x^{v-1} e^{-x} dx$

$$v = \frac{3}{4}$$

$$\therefore \int_0^\infty x^{\frac{1}{2}} e^{-x^2} dx = \frac{1}{2} \Gamma\left(\frac{3}{4}\right)$$

Example

Evaluate $\int_0^{\infty} x^6 e^{-4x^2}\, dx$

Solution

Let $y = 4x^2 dx, \dfrac{dy}{dx} = 8x$

$= dx = \dfrac{dy}{8x}$ and also $x = \dfrac{y^{\frac{1}{2}}}{2}$

Limits remain unchanged

$$\int_0^{\infty} x^6 e^{-4x^2}\, dx = \int_0^{\infty} \left(\frac{y^{\frac{1}{2}}}{2}\right)^6 e^{-y} \frac{dy}{8x}$$

Solution cont'd

$$\frac{1}{2^6}\int_0^{\infty} y^3 e^{-y}\frac{dy}{8\left(\frac{y^{\frac{1}{2}}}{2}\right)}$$

$$\frac{1}{2^6}\cdot\frac{1}{4}\int_0^{\infty} y^3 e^{-y}\frac{dy}{\frac{y^{\frac{1}{2}}}{2}}$$

$$\frac{1}{256}\int_0^{\infty} y^{\frac{5}{2}} e^{-y}\,dy$$

By comparing to $\int_0^{\infty} x^{v-1}e^{-x}dx;\ v=\frac{7}{2}$

$$\int_0^{\infty} x^6 e^{-4x^2}dx = \frac{1}{256}\cdot\Gamma\left(\frac{7}{2}\right)$$

$$= \frac{1}{256}\cdot\frac{15}{8}\sqrt{\pi} = \frac{15}{2048}\sqrt{\pi}$$

Exercise

Solve the following:

- $\displaystyle\int_0^{\infty} x^5 e^{-x}\,dx$
- $\displaystyle\int_0^{\infty} x^4 e^{-x}\,dx$
- $\displaystyle\int_0^{\infty} x^8 e^{-2x}\,dx$
- $\displaystyle\int_0^{\infty} x^2 e^{-2x^2}\,dx$

BETA FUNCTIONS

The beta function $B(m, n)$is defined by:

$B(m, n) = \int_0^1 x^{(m-1)}(1-x)^{n-1}dx$ which converges for $m > 0$ and $n > 0$

NOTE

$B(m, n) = B(n, m)$

Proof

$$B(m,n) = \int_0^1 x^{(m-1)}(1-x)^{n-1}dx$$

Let $u = 1-x; x = 1-u; \dfrac{du}{dx} = -1; dx = -du$

When $x = 0, u = 1; x = 1, u = 0$

$$= B(m,n) = -\int_1^0 (1-u)^{(m-1)}u^{n-1}du$$

$$= \int_0^1 (1-u)^{(m-1)}u^{n-1}du$$

$$= \int_0^1 u^{(n-1)}(1-u)^{m-1}du = B(n,m)$$

$\therefore B(m,n) = B(n,m)$

Alternative form of the Beta Function

$$B(m,n) = \int_0^1 x^{(m-1)}(1-x)^{n-1}dx$$

If we put $x = sin^2\theta$

Then $\dfrac{dx}{d\theta} = 2sinxcosx$

when $x = 0, \theta = 0; x = 1, \theta = \dfrac{\pi}{2}$

Again $1 - x = 1 - sin^2\theta = cos^2\theta$

$$B(m,n) = \int_0^{\frac{\pi}{2}} sin^{2m-2}\theta cos^{2n-2}\theta \cdot 2sin\theta cos\theta d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} sin^{2m-2}\theta cos^{2n-2}\theta sin\theta cos\theta d\theta$$

Proof cont'd

From laws of indices

$$sin^{2m-2}\theta = \frac{sin^{2m}\theta}{sin^2\theta} \text{ and}$$

$$cos^{2n-2}\theta = \frac{cos^{2n}\theta}{cos^2\theta}$$

$$= 2\int_0^{\frac{\pi}{2}} \frac{sin^{2m}\theta}{sin^2\theta} \cdot \frac{cos^{2n}\theta}{cos^2\theta} d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} sin^{2m-1}\theta cos^{2n-1}\theta d\theta$$

$$\therefore B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx \text{ can be written as:}$$

$$= 2\int_0^{\frac{\pi}{2}} sin^{2m-1}\theta cos^{2n-1}\theta d\theta$$

Reduction Formulas

- $$\int_0^{\frac{\pi}{2}} sin^n x dx = \frac{n-1}{n} \int_0^{\frac{\pi}{2}} sin^{n-2} x dx$$

- $$\int_0^{\frac{\pi}{2}} cos^n x dx = \frac{n-1}{n} \int_0^{\frac{\pi}{2}} cos^{n-2} x dx$$
- $$\int_0^{\frac{\pi}{2}} sin^m x cos^n x dx = \frac{m-1}{m+n} \int_0^{\frac{\pi}{2}} sin^{m-2} x cos^n x dx$$

If we denote $\int_0^{\frac{\pi}{2}} sin^m x cos^n x dx$ by $I_{m,n}$ then the above can be written as;

$$\frac{m-1}{m+n} \cdot I_{m-2,n}$$

Alternatively

$$\int_0^{\frac{\pi}{2}} sin^m x cos^n x dx = \frac{n-1}{m+n}\int_0^{\frac{\pi}{2}} sin^m x dx cos^{n-2} x dx$$

i.e $I_{m,n} = \dfrac{n-1}{m+n} \cdot I_{m,n-2}$ Now, we know that

$$B(m,n) = \int_0^{\frac{\pi}{2}} sin^{2m-1}\theta cos^{2n-1}\theta dx$$

If we apply

$\displaystyle\int_0^{\frac{\pi}{2}} sin^{2m-1} x cos^{2n-1} x dx = \frac{m-1}{m+n}\int_0^{\frac{\pi}{2}} sin^{m-2} x cos^n x dx$ to $B(m,n)$, we have

$$\int_0^{\frac{\pi}{2}} sin^{2m-1}\theta cos^{2n-1}\theta d\theta =$$

$$\frac{(2m-1)-1}{(2m-1)+(2n-1)}\int_0^{\frac{\pi}{2}} sin^{2m-3}\theta cos^{2n-1}\theta d\theta$$

$$\frac{m-1}{m+n-1}\int_0^{\frac{\pi}{2}} sin^{2m-3}\theta cos^{2n-1}\theta d\theta$$

Using the right hand integral, i.e $I_{m,n} = \dfrac{n-1}{m+n} \cdot I_{m,n-2}$

$$= \int_0^{\frac{\pi}{2}} sin^{2m-1}\theta cos^{2n-1}\theta d\theta =$$

$$\frac{m-1}{m+n-1} \cdot \frac{(2n-1)-1}{(2m-3)+(2n+1)} \times \int_0^{\frac{\pi}{2}} sin^{2m-3}\theta cos^{2n-3}\theta d\theta$$

$$= \frac{(m-1)}{m+n-1} \cdot \frac{(n-1)}{m+n-2} \times \int_0^{\frac{\pi}{2}} sin^{2m-3}\theta cos^{2n-3}\theta d\theta$$

$B(m,n) =$

$$\frac{(m-1)(n-1)}{(m+n-1)(m+n-2)} \times 2\int_0^{\frac{\pi}{2}} sin^{2m-3}\theta cos^{2n-3}\theta d\theta$$

$B(m,n) = \frac{(m-1)(n-1)}{(m+n-1)(m+n-2)} B(m-1,n-1)$The above is a reduction formula for $B(m,n)$ and the process can be repeated as required

Example

Evaluate $B(4,3)$, where $m = 4$ and $n = 3$

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Example

Evaluate $B(4,3)$, where $m = 4$ and $n = 3$

Solution

$$B(4,3) = \frac{3(2)}{6(5)} B(3,2)$$

$= \dfrac{3(2)}{6(5)} \dfrac{2(1)}{4(3)} B(2,1)$ We can go no further in the reduction process from $B(2,1)$ since from the definition of $B(m,n)$,'m' and 'n' must be greater than zero

Solution cont'd

$$B(2,1) = 2\int_0^{\frac{\pi}{2}} sin^{2m-1}\theta cos^{2n-1}\theta d\theta$$

$$= \int_0^{\frac{\pi}{2}} sin^3\theta cos\theta d\theta$$

$$= 2\left[\frac{sin^4\theta}{4}\right]_0^{\frac{\pi}{2}}$$

$$= \frac{sin^4\theta}{2}\bigg|_0^{\frac{\pi}{2}} = \left[\frac{1}{2} - 0\right] = \frac{1}{2}$$

$$B(4,3) = \frac{3(2)}{6(5)}\frac{2(1)}{4(3)} \cdot \frac{1}{2}$$

Note that, by rearranging

$$B(4,3) = \frac{3(2)(1) \times 2(1)}{6(5)(4)(3)(2)(1)} = \frac{(3!)(2!)}{6!} \equiv \frac{\Gamma(4)\Gamma(3)}{\Gamma(7)}$$

That is, to find the $B(m,n)$ we find the gamma of m, n and we divide by the gamma of $(m+n)$

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That is, to find the $B(m,n)$ we find the gamma of m, n and we divide by the gamma of $(m+n)$

Similarly $B(5,3) = \dfrac{\Gamma(5)\Gamma(3)}{\Gamma(8)} = \dfrac{(4!)(2!)}{7!}$

So in generic terms:
$$B(m,n) = \frac{(m-1)!(n-1)!}{(m+n-1)!} = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

Note

$$B(k,1) = 2\int_0^{\frac{\pi}{2}} sin^{2k-1}\theta cos\theta d\theta$$
$$= 2\left[\frac{sin^{2k}\theta}{2k}\right]_0^{\frac{\pi}{2}} = \frac{1}{k}$$

Therefore $B(k,1) = B(1,k) = \dfrac{1}{k}$

Question

Evaluate $B\left(\dfrac{1}{2}, \dfrac{1}{2}\right)$

Solution

$$B\left(\frac{1}{2},\frac{1}{2}\right)=\frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(1)}=\frac{\sqrt{\pi}\cdot\sqrt{\pi}}{1}=\pi$$

Alternatively

$$B\left(\frac{1}{2},\frac{1}{2}\right)=2\int_0^{\frac{\pi}{2}} sin^{2m-1}\theta cos^{2n-1}\theta d\theta$$
$$=2\int_0^{\frac{\pi}{2}} sin^0\theta cos^0\theta d\theta$$
$$=2\int_0^{\frac{\pi}{2}} 1 d\theta$$
$$=2\left[\theta\right]_0^{\frac{\pi}{2}}$$
$$=[\pi-0]=\pi$$

Summary

- $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$
- $B(m,n) = B(n,m)$
- $B(m,n) = 2\int_0^{\frac{\pi}{2}} sin^{2m-1}\theta cos^{2n-1}\theta d\theta$
- $B(m,n) = \frac{(m-1)(n-1)}{(m+n-1)(m+n-2)} \cdot B(m-1,n-1)$
- $B(m,n) = \frac{(m-1)!(n-1)!}{(m+n-1)!}; m,n \in z^+$
- $B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$
- $B(k,1) = B(1,k) = \frac{1}{k}$
- $B(1,1) = 1$

- $B(\frac{1}{2}, \frac{1}{2}) = \pi$

RELATIONSHIP BETWEEN THE GAMMA AND THE BETA FUNCTION

If m and n are positive integers, $B(m,n) = \dfrac{(m-1)!(n-1)!}{(m+n-1)!}$

Also, we have previously established in the gamma functions that, for n, a positive integer;

$n! = \Gamma(n+1)$

$\therefore (m-1)! = \Gamma(m), (n-1)! = \Gamma(n)$

And also $(m+n-1)! = \Gamma(m+n)$

$\therefore \dfrac{(m-1)!(n-1)!}{(m+n-1)} = \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

The relation $B(m,n) = \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$ holds even when m and n are not necessarily integers

Example

$$B(\tfrac{3}{2}, \tfrac{1}{2}) = \frac{\Gamma(\frac{3}{2})\Gamma(\frac{1}{2})}{\Gamma(2)} = \frac{\frac{\sqrt{\pi}}{2} \times \sqrt{\pi}}{1} = \frac{\pi}{2}$$

APPLICATION OF GAMMA AND BETA FUNCTIONS

In evaluation of definite integrals using gamma and beta functions, we try to express the integral in the basic form of the beta function

$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$ or its trigonometric form

$2\int_0^{\frac{\pi}{2}} sin^{2m-1}\theta cos^{2n-1}\theta d\theta$

Examples

1. Evaluate $I = \int_0^1 x^5(1-x)^4 dx$

Solution

Comparing with $\int_0^1 x^{m-1}(1-x)^{n-1} dx$

$m-1=5, m=6; n-1=4, n=5$

$\therefore B(6,5) = \dfrac{\Gamma(6)\Gamma(5)}{\Gamma(11)} = \dfrac{5!4!}{10!} = \dfrac{1}{1260}$

2. Evaluate $\int_0^1 x^4\sqrt{(1-x^2)}dx$

Solution

$$= \int_0^1 x^4(1-x^2)^{\frac{1}{2}}\,dx$$

Let $y = x^2 = x = y^{\frac{1}{2}}$

Then $\dfrac{dy}{dx} = 2x, dx = \dfrac{dy}{2x}$

Now when $x = 0, y = 0; x = 1, y = 1$

$$= \int_0^1 y^2(1-y)^{\frac{1}{2}}\frac{dy}{2y^{\frac{1}{2}}}$$

$$= \frac{1}{2}\int_0^1 y^2(1-y)^{\frac{1}{2}}\frac{dy}{y^{\frac{1}{2}}}$$

$$= \frac{1}{2}\int_0^1 y^{\frac{3}{2}}(1-y)^{\frac{1}{2}}\,dy$$

Solution cont'd

By comparison $m-1=\frac{3}{2}=m=\frac{5}{2}$

$n-1=\frac{1}{2}=n=\frac{3}{2}$

$$B(\tfrac{5}{2},\tfrac{3}{2})=\frac{1}{2}\frac{\Gamma(\frac{5}{2})\Gamma(\frac{3}{2})}{\Gamma(4)}=\frac{\frac{3}{4}\sqrt{\pi}}{3!}\cdot\sqrt{\frac{\pi}{2}}$$

$$=\frac{1}{2}\times\frac{\pi}{16}=\frac{\pi}{32}$$

3. Evaluate $\int_0^3 \frac{x^3dx}{\sqrt{3-x}}$

Solution

$$\int_0^3 \frac{x^3 dx}{(3-x)^{\frac{1}{2}}} dx$$

$$3^{-\frac{1}{2}} \int_0^3 x^3 (1 - \frac{x}{3})^{-\frac{1}{2}} dx$$

Let $y = \dfrac{x}{3} = x = 3y; \dfrac{dy}{dx} = \dfrac{1}{3}; dx = 3dy$

When $x = 0, y = 0; x = 3, y = 1$

$$= 3^{-\frac{1}{2}} \int_0^1 (3y)^3 (1-y)^{-\frac{1}{2}} \cdot 3dy$$

$$= 3^{-\frac{1}{2}} \cdot 3^3 \cdot 3 \int_0^1 (y)^3 (1-y)^{-\frac{1}{2}} \cdot dy$$

Solution cont'd

By Comparison $m-1=3, m=4; n-1=-\frac{1}{2}, n=\frac{1}{2}$

$$\frac{81}{\sqrt{3}}B(4,\frac{1}{2})=\frac{81}{\sqrt{3}}\frac{\Gamma(d)\Gamma(\frac{1}{2})}{\Gamma(\frac{9}{2})}=27\sqrt{3}\left(\frac{6\sqrt{\pi}}{\frac{105}{16}\sqrt{\pi}}\right)$$

$$=\frac{864\sqrt{3}}{35}=42.76$$

3. Evaluate $\int_0^{\frac{\pi}{2}} sin^5\theta cos^4\theta d\theta$

Solution cont'd

$$\int_0^{\frac{\pi}{2}} sin^5\theta cos^4\theta d\theta = \frac{1}{2}B\left(3, \frac{5}{2}\right)$$

But $\frac{1}{2}B\left(3, \frac{5}{2}\right) = \dfrac{\Gamma(3)\Gamma\left(\frac{5}{2}\right)}{\Gamma\frac{11}{2}}$

But $\Gamma(3) = 3; \Gamma\left(\frac{5}{2}\right) = \frac{3}{2}\Gamma\left(\frac{3}{2}\right)$

$$\Gamma\left(\frac{11}{2}\right) = \frac{9}{2}\cdot\frac{7}{2}\cdot\frac{5}{2}\Gamma\left(\frac{5}{2}\right) = \frac{945}{32}\sqrt{\pi}$$

Finally

$$\frac{1}{2}B\left(3, \frac{5}{2}\right) = \frac{2\cdot\frac{3}{4}\sqrt{\pi}}{\frac{945}{32}\sqrt{\pi}} = \frac{8}{315}$$

4. Evaluate $\int_0^{\frac{\pi}{2}} \sqrt{tan\theta} d\theta$

4. Evaluate $\int_0^{\frac{\pi}{2}} \sqrt{tan\theta} d\theta$

Solution

$$\sqrt{tan\theta} = \sqrt{\frac{sin\theta}{cos\theta}} = sin^{\frac{1}{2}}\theta cos^{-\frac{1}{2}}\theta$$

So we have

$$\int_0^{\frac{\pi}{2}} \sqrt{tan\theta} d\theta = \int_0^{\frac{\pi}{2}} sin^{\frac{1}{2}}\theta cos^{-\frac{1}{2}}\theta d\theta$$

By Comparison

$$2m - 1 = \frac{1}{2}, m = \frac{3}{4}; 2n - 1 = -\frac{1}{2}, n = \frac{1}{4}$$

$$\int_0^{\frac{\pi}{2}} sin^{\frac{1}{2}}\theta cos^{-\frac{1}{2}}\theta d\theta = \frac{1}{2} B\left(\frac{3}{4}, \frac{1}{4}\right)$$

solution cont'd

$$= \frac{\frac{1}{2}\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma(1)} = \frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)}{2}$$

We cannot continue further since we are not given the table

Try these:

- $$\int_0^1 x^5(2-x)^4dx$$
- $$\int_0^{\frac{\pi}{2}} sin^7\theta cos^3\theta d\theta$$
- $$\int_0^{\frac{\pi}{2}} sin^2 4\theta cos^5 4\theta d\theta$$

Stirling Formula

Stirling's Formula is a formula which is used in probability as well as in applied mathematics. Stirling's formula is denoted as follows;

$$n! \sim \sqrt{2\pi} n^{(n+1/2)} e^{-n}$$

$$\lim_{n\to\infty} \frac{n!}{\sqrt{2\pi} n^{(n+1/2)} e^{-n}} = 1$$

$$\lim_{n\to\infty} \frac{n!}{n^{(n+1/2)} e^{-n}} = \sqrt{2\pi}$$

Stirling Formula Proof

For the proof of Stirling formula, lets take log of $n!$ both side. Then,

$$\log(n!) = \log(1) + \log(2) + ... + \log(n)$$

We know that in the interval $(0, \infty)$ log is an increasing function. Hence, we can show that the above expression in such a way that

$$\int_{n-1}^{n} \log(x)dx < \log(n) < \int_{n}^{n+1} \log(x)dx \text{ where } n \geq 1$$

If we take $n \in N$, then

$$\int_{0}^{N} \log(x)dx < \log(N!) < \int_{1}^{N+1} \log(x)dx$$

Here, the first integral is convergent. Now, we can use the integration of $\log(x)$, which is $x\log(x) - x$.So, the above expression becomes

$$n\log(n) - n < \log(n!) < (n+1)\log(n+1) - n$$

So, here we can define

$$d_n = \log(n!) - (n + \tfrac{1}{2})\log(n) + n$$

Then we get

$$d_n - d_{n+1} = (n + \tfrac{1}{2})\log(\frac{n+1}{n}) - 1$$

It gives an algebraic expression

$$\frac{n+1}{n} = \frac{1+\frac{1}{2n+1}}{1-\frac{1}{2n+1}}$$

Now, at this point,we can use Taylor rule.Then,

$$\frac{1}{2}\log\left(\frac{1+t}{1-t}\right) = t + \frac{t^3}{3} + \frac{t^5}{5} + ...$$

For $-1 < t < 1$ we define

$$d_n - d_{n+1} = \frac{1}{3(2n+1)^2} + \frac{1}{5(2n+1)^4} + ...$$

This shows

$$0 < d_n - d_{n+1} < \frac{1}{3}\left[\frac{1}{(2n+1)^2} + \frac{1}{(2n+1)^4} + ...\right]$$

It gives

$$0 < d_n - d_{n+1} < \frac{1}{3} \cdot \frac{1}{(2n+1)^2 - 1} = \frac{1}{2}\left(\frac{1}{n} - \frac{1}{n+1}\right)$$

This will imply that d_n converges to a number C with $\lim_{n\to\infty} d_n = \lim_{n\to\infty}(d_n - \dfrac{1}{12n}) = C$

$C > d_1 - 1/12 = 1 - 1/12 = 11/12$. Now, taking exponent of both sides of d_n, then

$$\lim_{n\to\infty} \frac{n!}{n^{n+\frac{1}{2}}} e^{-n} = e^{C}$$

The final step of proof that $e^{C} = 2\pi$. For this, we can use Wallis formula, which gives

$$\lim_{n\to\infty} \frac{2.2.4.4.....(2n)(2n)}{1.1.3.3....(2n-1)(2n-1)(2n+1)} = \frac{\pi}{2}$$

We can rewrite this as ;

$$\frac{2.4.6..(2n)(2n)}{1.3.5...(2n-1)\sqrt{2n}} \sim \sqrt{\frac{\pi}{2}}$$

Now, rearrange the numbers,

$$\frac{(2^n n!)^2}{(2n)!} \cdot \frac{1}{\sqrt{2n}} \sim \sqrt{\frac{\pi}{2}}$$

$$n! \sim n^{n+\frac{1}{2}} e^{-n} e^{C}$$

$$\frac{2^{2n}(n^{2n+1})e^{-2n}e^{2C}}{(2n)(2n+1)e^{-2n}e^{C}} \cdot \frac{1}{\sqrt{2n}} \sim \sqrt{\frac{\pi}{2}}$$

Hence, we get the Stirling's formular

$$\lim_{n\to\infty} \frac{n!}{n^{n+\frac{1}{2}}e^{-n}} = \sqrt{2\pi}$$

Examples

Given below are some examples that use Stirling's formula to solve problems.

1. Find the factorial of 4 using Stirling's formula

<u>Solution</u>

$=n! \sim \sqrt{2\pi} n^{(n+1/2)} e^{-n}$

Here, $n = 4$

Now, we apply the Stirling formula.

Substituting $n = 4$ we get ;

$= \sqrt{2\pi} \times 4^{(4+\frac{1}{2})} e^{-4}$

$= \sqrt{2 \times 3.14} \times 4^{\frac{9}{2}} \times 0.0183$

$= \sqrt{6.28} \times 512 \times 0.0183$

$= 2.505 \times 512 \times 0.0183$

$= 1283.06 \times 0.0183$

$= 23.48$

2. Find the factorial of 2 using Stirling's formula

Solution

$=n! \sim \sqrt{2\pi} n^{(n+1/2)} e^{-n}$

$n = 2$

Now, we apply the Stirling formula.

Substituting $n = 2$ we get ;

$= \sqrt{2\pi} \times 2^{(2+\frac{1}{2})} e^{-2}$

$= \sqrt{2 \times 3.14} \times 5.656 \times 0.1353$

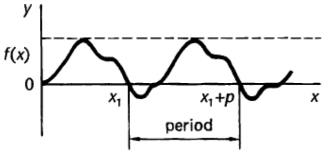
$= \sqrt{6.28} \times 5.656 \times 0.1353$

$= 2.505 \times 5.656 \times 0.1353$

$= 14.1682 \times 0.1353$

$= 1.9169$

A function $f(x)$ is said to be periodic if its functions values repeat at regular intervals of the independent variable. The regular interval between the repetitions is the **period** of the oscillations...



Graphs of $y = A \sin nx$

(a) '$y = \sin x$' is an obvious example of a periodic function which goes through its complete range of values while n increases from 0° to 360°.

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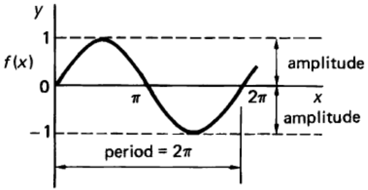
(a) '$y = \sin x$' is an obvious example of a periodic function which goes through its complete range of values while n increases from 0° to 360°.

- The period is therefore 360° or 2π radians whiles the
- Amplitude(A), which is the maximum displacement from the position of rest, is 1.

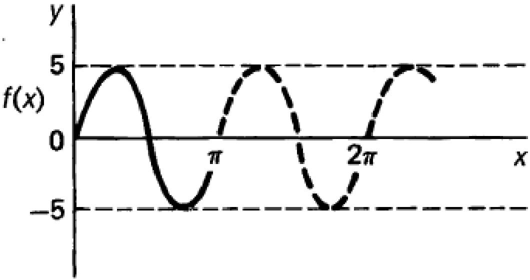


NB; *The period is the interval the graph repeats itself. For the Amplitude,we take from 0 to 1*

(b) $y = 5\sin 2x$

(b) $y = 5\sin 2x$

The *Amplitude* is 5. The *Period* is 180° and there are thus 2 complete cycles in 360°



General formular

(c) $y = A \sin nx$

Thinking along the same lines, the function '$y = A \sin nx$' has;

- Amplitude = A
- Period $\dfrac{360^\circ}{n} = \dfrac{2\pi}{n}$; n cycles in 360°

Graphs of $y = A \cos nx$ have the same characteristics.

Exercises

In each of the following, state (a) the amplitude and (b) the period

(1) $y = 3\sin 5x$
(2) $y = 2\cos 3x$
(3) $y = \sin \frac{x}{2}$
(4) $y = 4\sin 2x$
(5) $y = 5\cos 4x$
(6) $y = 2\sin x$
(7) $y = 3\cos 6x$
(8) $y = 6\sin \frac{2}{3}x$

Answers

No.	Amplitude	Period	No.	Amplitude	Period
1	3	$2\pi/5$	**5**	5	$\pi/2$
2	2	$2\pi/3$	**6**	2	π
3	1	4π	**7**	3	$\pi/3$
4	4	π	**8**	6	3π

Harmonics

A function $f(x)$ is sometimes expressed as a series of a number of different ***sine*** components. The component with the largest period is the **first harmonic** or fundamental of $f(x)$.
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- $y = A_2 \sin 2x$ is the second harmonic because the period is the second largest
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In general;
$y = A_n \sin nx$ is the nth harmonic with Amplitude A_n and period $= \frac{2\pi}{n}$ or $\frac{360°}{n}$

Non-sinusoidal periodic functions

Although, we introduced the concept of a periodic function via a *sine* curve, a function can be periodic without being obviously sinusoidal in appearance.

Non-sinusoidal periodic functions

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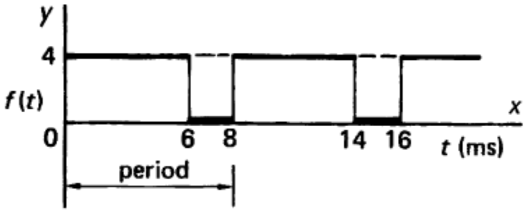
Examples

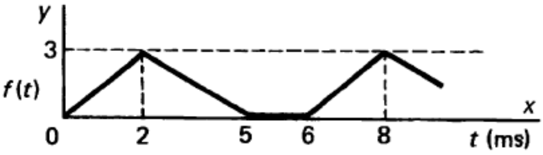
In the following cases, the x-axis carries a scale of t in milliseconds.

(a)



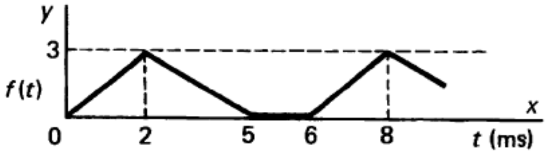
Period=**8ms** (*the interval inwhich the graph repeats itself*)

(b)



Period= **6ms**

(b)



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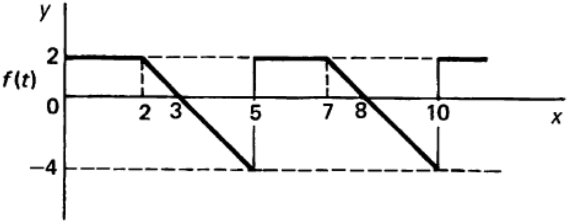
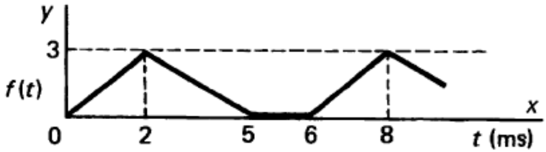


(b)



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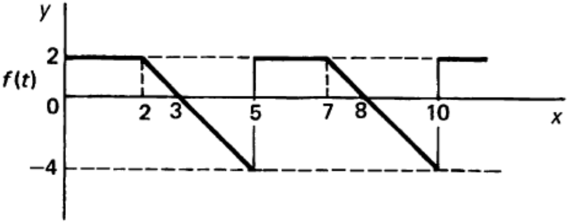
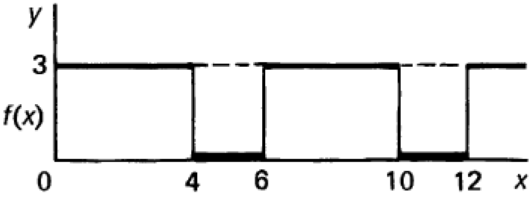


Period= **5ms**

Analytic description of a periodic function

A periodic function can be defined analytically in many cases;

<u>Example</u>



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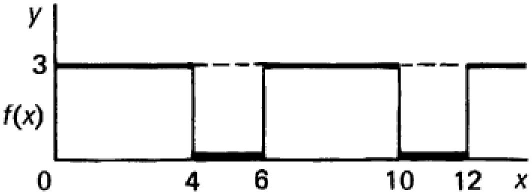

(a)Between $x = 0$ and $x = 4, y = 3$ i.e. $f(x) = 3, 0 < x < 4$
(b)Between $x = 4$ and $x = 6, y = 0$ i.e. $f(x) = 0, 4 < x < 6$

So we can define the function by;

$$f(x) = \begin{cases} 3 & 0 < x < 4 \\ 0 & 4 < x < 6 \end{cases}$$

$f(x+6) = f(x)$ means the function is periodic with period 6 units.

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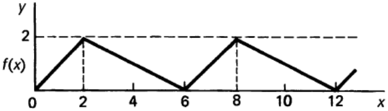
Example 2



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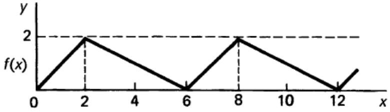


In this case,

(a) Between $x = 0$ and $x = 2, y = x$ i.e. $f(x) = x, 0 < x < 2$

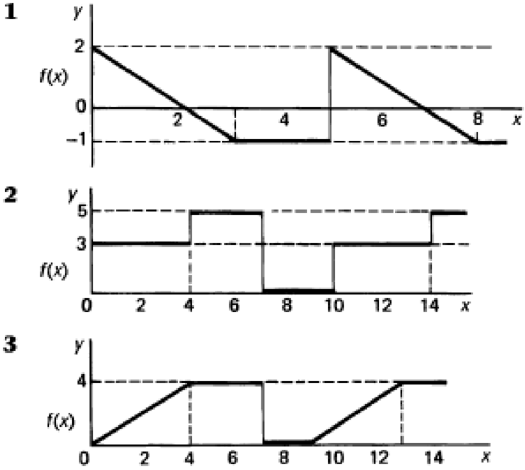
(b)Between $x = 2$ and $x = 6, y = -\frac{x}{2} + 3$ i.e. $f(x) = 3 - \frac{x}{2}, 2 < x < 6$

(c) The period is 6 units i.e $f(x+6) = f(x)$
So we have;

$$f(x) = \begin{cases} x & 0 < x < 2 \\ 3 - \dfrac{x}{2} & 2 < x < 6 \end{cases}$$

Short Exercise

Define analytically the periodic functions shown.



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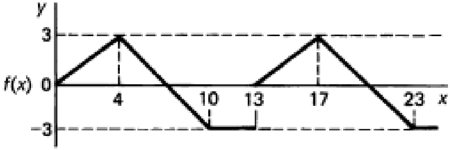
4



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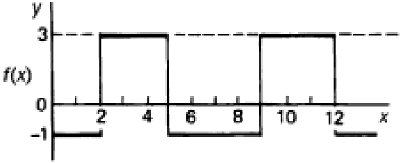

Sketch the graphs of the following, inserting relevant values

1
$$f(x) = \begin{cases} 4 & 0 < x < 5 \\ 0 & 5 < x < 8 \end{cases}$$
$f(x+8) = f(x)$.

2
$$f(x) = 3x - x^2 \qquad 0 < x < 3$$
$f(x+3) = f(x)$.

3
$$f(x) = \begin{cases} 2\sin x & 0 < x < \pi \\ 0 & \pi < x < 2\pi \end{cases}$$
$f(x+2\pi) = f(x)$.

4
$$f(x) = \begin{cases} \dfrac{x}{2} & 0 < x < \pi \\ \pi - \dfrac{x}{2} & \pi < x < 2\pi \end{cases}$$
$f(x+2\pi) = f(x)$.

5
$$f(x) = \begin{cases} \dfrac{x^2}{4} & 0 < x < 4 \\ 4 & 4 < x < 6 \\ 0 & 6 < x < 8 \end{cases}$$
$f(x+8) = f(x)$.

INTEGRALS OF PERIODIC FUNCTIONS

The integrals that we are concerned with are those of sines, cosines and their combinations where the integration is over a single period from $-\pi$ to π. First, though, we list the integral of the unit constant over the period.

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(1) $\displaystyle\int_{-\pi}^{\pi} dx = [x]_{-\pi}^{\pi} = 2\pi$

(2) $\displaystyle\int_{-\pi}^{\pi} \cos nx dx = \left[\frac{\sin nx}{n}\right]_{-\pi}^{\pi} = \frac{\sin n\pi}{n} - \frac{\sin(-n\pi)}{n} = 0$

because $\sin\pi = 0$, hence $\sin n\pi$ is also 0.

(3)

$$\int_{-\pi}^{\pi} \sin nx dx = \left[-\frac{\cos nx}{n}\right]_{-\pi}^{\pi} = \frac{-\cos n\pi}{n} - \frac{-\cos(-n\pi)}{n} = 0$$

because $\cos(-x) = \cos x$

(4) $\int_{-\pi}^{\pi} \sin^2 nx dx =$

(4) $\displaystyle\int_{-\pi}^{\pi} \sin^2 nx dx = \int_{-\pi}^{\pi} \frac{1-\cos 2nx}{2} dx$

Because $\cos 2A = 1 - 2\sin^2 A$

$= \left[\frac{x}{2}\right]_{-\pi}^{\pi} - \left[\frac{\sin 2nx}{4n}\right]_{-\pi}^{\pi} = \pi \quad$ i.e $\sin 2n\pi = 0$

(5) $\displaystyle\int_{-\pi}^{\pi} \cos^2 nx dx =$

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(5) $\int_{-\pi}^{\pi} \cos^2 nx dx = \int_{-\pi}^{\pi} \frac{\cos 2nx + 1}{2} dx$

Because $\cos 2A = 2\cos^2 A - 1$

$= \left[\frac{\sin 2nx}{4n} + \frac{x}{2}\right]_{-\pi}^{\pi} = \frac{\sin 2n\pi}{4n} + \frac{\pi}{2} - \frac{\sin(-2n\pi)}{4n} + \frac{\pi}{2} = \pi$

(6) $\displaystyle\int_{-\pi}^{\pi} \cos mx \cos nx dx$

Recall that

$$2\cos A\cos B = \cos(A+B) + \cos(A-B)$$

$$\therefore \cos A\cos B = \frac{\cos(A+B) + \cos(A-B)}{2}$$

$$= \int_{-\pi}^{\pi}\left[\frac{\cos(m+n)x + \cos(m-n)x}{2}\right] dx$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} \cos(m+n)x + \cos(m-n)x dx$$

$$= \left[\frac{\sin(m+n)x}{m+n} + \frac{\sin(m-n)x}{m-n}\right]_{-\pi}^{\pi} \quad (m \neq n)$$

$$= \frac{\sin(m+n)\pi}{m+n} + \frac{\sin(m-n)\pi}{m-n} - \frac{\sin(m+n)(-\pi)}{m+n} - \frac{\sin(m-n)(-\pi)}{m-n} = 0$$

(7) $\displaystyle\int_{-\pi}^{\pi} \sin mx \sin nx dx$

Recall that

$2 \sin A \sin B = \cos(A - B) - \cos(A + B)$

$\therefore \sin A \sin B = \dfrac{\cos(A - B) - \cos(A + B)}{2}$

$$= \frac{1}{2}\int_{-\pi}^{\pi} \cos(m-n)x - \cos(m+n)x dx$$

$$= \left[\frac{\sin(m-n)x}{m-n} - \frac{\sin(m+n)x}{m+n}\right]_{-\pi}^{\pi} \quad (m \neq n)$$

$$= \frac{\sin(m-n)\pi}{m-n} - \frac{\sin(m+n)\pi}{m+n} - \frac{\sin(m-n)(-\pi)}{m-n} +$$

$$\frac{\sin(m+n)(-\pi)}{m+n} = 0$$

(8) $\displaystyle\int_{-\pi}^{\pi} \cos mx \sin nx dx$

Since $2\cos A \sin B = \sin(A+B) - \sin(A-B)$

$$= \frac{1}{2}\int_{-\pi}^{\pi} \sin(m+n)x - \sin(m-n)x dx$$

$$= \frac{1}{2}\left[\frac{-\cos(m+n)x}{m+n} + \frac{\cos(m-n)x}{m-n}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2}\left[\frac{-\cos(m+n)\pi}{m+n} + \frac{\cos(m-n)\pi}{m-n} + \frac{\cos(m+n)(-\pi)}{m-n} - \frac{\cos(m-n)(-\pi)}{m-n}\right]$$

$= 0$

because $\cos(-x) = \cos(x)$

And finally when $m = n$

(9) $\displaystyle\int_{-\pi}^{\pi} \cos mx \sin mx dx$

$$= \frac{1}{2}\int_{-\pi}^{\pi} \sin 2mx dx$$

Since $\sin 2A = 2\sin A\cos A$

$$= \frac{1}{2}\left[\frac{-\cos 2mx}{2m}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2}\left[\frac{-\cos 2m\pi}{2m} + \frac{\cos 2m(-\pi)}{2m}\right]$$

$$= 0$$

because $\cos(-x) = \cos(x)$

NOTE

Note that the same results are obtained no matter the end points of the integrals; provided that the interval between them is one period.

Eg. $\displaystyle\int_{k}^{k+2\pi} \cos nx dx$

$= \left[\dfrac{\sin nx}{n}\right]_{k}^{k+2\pi}$

$= \dfrac{\sin(nk + 2n\pi)}{n} - \dfrac{\sin nk}{n}$

$= 0$

because $\sin(x + 2n\pi) = \sin(x)$

Orthogonal functions

If two functions $f(x)$ and $g(x)$ are defined on the interval $a \leq x \leq b$ and $\int_b^a f(x)g(x)dx = 0$ then we say that the two functions are **orthogonal** to each other on the interval $a \leq x \leq b$.

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i.e.

$$\int_{-\pi}^{\pi} \cos mx \cos nx dx = 0$$

$$\int_{-\pi}^{\pi} \sin mx \sin nx dx = 0$$

$$\int_{-\pi}^{\pi} \cos mx \sin nx dx = 0$$

FOURIER SERIES

Given that certain conditions are satisfied then it is possible to write a periodic function of period 2π as a series expansion of the orthogonal periodic function just discussed.

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Given that certain conditions are satisfied then it is possible to write a periodic function of period 2π as a series expansion of the orthogonal periodic function just discussed.
That is, if $f(x)$ is defined on the interval $-\pi \leq x \leq \pi$ where $f(x+2n\pi) = f(x)$ then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

where 'a_n' and 'b_n' are constants called the Fourier constants.
The above series is called the **Fourier Series of expansion of** $f(x)$.
NB: We make use of the mutual orthogonality of the trigonometric functions in the expansion to find the Fourier coefficients.

To find 'a_n'and 'b_n';

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx$$

n= 0,1,2,...

For instance,

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) dx$$

$a_{10} = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\cos 10x dx$ and so on...

where $f(x)$ is given as $f(x) = \dfrac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$

Dirichlet Conditions

If a function $f(x)$ is such that ;
(a) $f(x)$ is defined , single-valued and periodic with period 2π
(b) $f(x)$ and $f'(x)$ have at most a finite number of finite discontinuities over a single period- that is they are piecewise continuous then the series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

where

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx dx \text{ and}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx dx$$

converges to $f(x)$ when $(x, f(x))$ is a point continuity.

Exercises

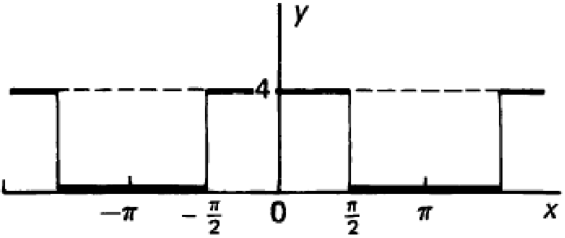
If the following functions are defined over the interval $-\pi < x < \pi$ and $f(x + 2\pi) = f(x)$, state whether or not each function can be represented by a Fourier series.

1. $f(x) = x^3$
2. $f(x) = 4x - 5$
3. $f(x) = \frac{2}{x}$
4. $f(x) = \frac{1}{x-5}$
5. $f(x) = \tan x$
6. $f(x) = y$ where $x^2 + y^2 = 9$

Answers

1. Yes
2. Yes
3. No: infinite discontinuity at $x = 0$
4. No: infinite discontinuity at $x = 5$
5. No: infinite discontinuity at $x = \pi/2$
6. No : Two valued

Example 1



Find the fourier series for the function shown. Consider one cycle between $x = -\pi$ and $x = \pi$
The function can be defined by;

$$f(x) = \begin{cases} 0 & -\pi < x < -\frac{\pi}{2} \\ 4 & -\frac{\pi}{2} < x < \frac{\pi}{2} \\ 0 & \frac{\pi}{2} < x < \pi \end{cases}$$

$$f(x + 2\pi) = f(x)$$

(a) $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$

where $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$

(a) $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$

where $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$

Now for a_0 we have,

$$a_0 = \frac{1}{\pi}\left(\int_{-\pi}^{-\pi/2} 0dx + \int_{-\pi/2}^{\pi/2} 4dx + \int_{\pi}^{\pi/2} 0dx\right)$$

$$= \frac{1}{\pi}\left[4x\right]_{-\pi/2}^{\pi/2}$$

$$= \frac{1}{\pi}\left[4(\tfrac{\pi}{2}) - 4(\tfrac{-\pi}{2})\right] = \tfrac{1}{\pi}\left[2\pi + 2\pi\right]$$

$$= \frac{1}{\pi}(4\pi) = 4$$

$\therefore a_0 = 4$

(b) To find a_n ;

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx$$

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$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nxdx$$

$$= \frac{1}{\pi}\left(\int_{-\pi}^{-\pi/2}(0)\cos nxdx + \int_{-\pi/2}^{\pi/2} 4\cos nxdx + \int_{\pi}^{\pi/2}(0)\cos nxdx\right)$$

$$a_n = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} 4\cos nxdx$$

$$\therefore\ a_n = \frac{8}{\pi n}\sin\frac{n\pi}{2}$$

Then considering different integer values of n, we have;

if n is even ; $a_n = 0$

if $n = 1, 5, 9, \dots$; $a_n = \dfrac{8}{n\pi}$

if $n = 3, 7, 11, \dots$; $a_n = \dfrac{-8}{n\pi}$

We keep these in mind while we find b_n.

(c) To find b_n

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx$$

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(c) To find b_n

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nxdx$$

$$= \frac{1}{\pi}\left(\int_{-\pi}^{-\pi/2}(0)\sin nxdx + \int_{-\pi/2}^{\pi/2} 4\sin nxdx + \int_{\pi}^{\pi/2}(0)\sin nxdx\right)$$

$$b_n = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} 4\sin nxdx = \frac{4}{\pi}\int_{-\pi/2}^{\pi/2}\sin nxdx$$

$$\frac{4}{\pi}\left[\frac{-\cos nx}{n}\right]_{-\pi/2}^{\pi/2}$$

$$\frac{-4}{\pi}\left[\cos\tfrac{n\pi}{2} - \cos\tfrac{-n\pi}{2}\right] = 0$$

$$\therefore b_n = 0$$

Therefore with ;

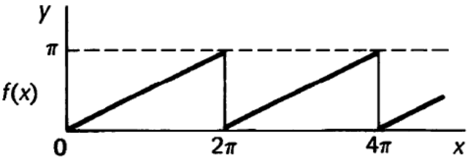
$$a_0 = 4 \ , \ a_n = \frac{8}{\pi n} \sin \frac{n\pi}{2} \text{ and } b_n = 0 \ ;$$

The Fourier series is ;

$$f(x) = 2 + \frac{8}{\pi}\left(\cos x - \frac{1}{3}\cos 3x + \frac{1}{5}\cos 5x - \frac{1}{7}\cos 7x + ...\right)$$

With this particular example,there are no *sine* terms since b_n is 0.

Try this...



Determine the Fourier Series to represent the periodic function shown above;
Take the limits as 0 to 2π

Hint:
The function can be defined as ;
$f(x) = \frac{x}{2} \quad 0 < x < 2\pi$

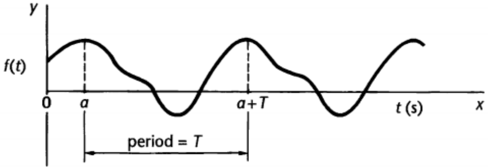
$f(x + 2\pi) = f(x)$ i.e. period $= 2\pi$

Functions with periods other than 2π

So far, we have considered functions $f(x)$ with periods 2π. In practice, we often encounter functions defined over periodic intervals other than 2π, e.g from 0 to T, $-\frac{T}{2}$ to $\frac{T}{2}$, etc.

Functions with period of T

If $y = f(x)$ is defined in the range $-\frac{T}{2}$ to $\frac{T}{2}$, i.e. has period T, we can convert this to an interval of 2π by changing the units of the independent variable.
In many practical cases involving physical oscillation, the independent variable is time(t) and the periodic interval is normally denoted by T, i.e.
$f(t + T) = f(t)$



Each cycle is therefore completed in T seconds and the frequency f **hertz** (oscillations per second) of the periodic function is therefore given by $f = \frac{1}{T}$. If the angular velocity, ω radians per second, is defined by $\omega = 2\pi f$, then

$$\omega = \frac{2\pi}{T} \quad \text{and} \quad T = \frac{2\pi}{\omega}$$

The angle, x radians, at any time t is therefore $x = \omega t$ and the Fourier series to represent the function can be expressed as

$$\begin{aligned} f(t) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left\{ a_n \cos n\omega t + b_n \sin n\omega t \right\} \\ &= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left\{ a_n \cos \frac{2n\pi t}{T} + b_n \sin \frac{2n\pi t}{T} \right\} \end{aligned}$$

Fourier Coefficient

With the new variable, the Fourier coefficients become

$$a_0 = \frac{2}{T}\int_0^T f(t)dt \qquad\qquad = \frac{\omega}{\pi}\int_0^{\frac{2\pi}{\omega}} f(t)dt$$

$$a_n = \frac{2}{T}\int_0^T f(t)\cos n\omega t dt = \frac{\omega}{\pi}\int_0^{\frac{2\pi}{\omega}} f(t)\cos n\omega t dt$$

$$b_n = \frac{2}{T}\int_0^T f(t)\sin n\omega t dt = \frac{\omega}{\pi}\int_0^{\frac{2\pi}{\omega}} f(t)\sin n\omega t dt$$

We can see that there is very little difference between these expressions and those that have gone before. The limits can, of course, be 0 to T,$-\frac{T}{2}$ to $\frac{T}{2}$, $-\frac{\pi}{\omega}$ to $\frac{\pi}{\omega}$, 0 to $\frac{2\pi}{\omega}$ etc. as is convenient, so long as they cover a complete period.

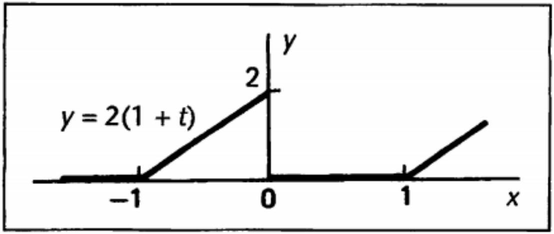
Example

Determine the Fourier series for a periodic function defined by

$$f(t) = \begin{cases} 2(1+t) & -1 < t < 0 \\ 0 & 0 < t < 1 \end{cases}$$

$f(t+2) = f(t)$

Solution



We have

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left\{a_n \cos\frac{2n\pi t}{T} + b_n \sin\frac{2n\pi t}{T}\right\}$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left\{a_n \cos n\pi t + b_n \sin n\pi t\right\} \quad \text{because}\, T = 2$$

Therefore

$$a_0 = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)dt = \int_{-1}^{1} f(t)dt = \int_{-1}^{0} 2(1+t)dt + \int_{0}^{1}(0)dt$$

$$= \left[2t + t^2\right]_{-1}^{0} \quad = 1$$

and

$$a_n = \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\cos n\pi t dt = \int_{-1}^{1} f(t)\cos n\pi t dt$$

...continued

$$a_n = \int_{-1}^{0} 2(1+t)\cos n\pi t dt$$

$$= 2\left\{\left[(1+t)\frac{\sin n\pi t}{n\pi}\right]_{-1}^{0} - \frac{1}{n\pi}\int_{-1}^{0} \sin n\pi t dt\right\}$$

$$= 2\left\{(0-0) - \frac{1}{n\pi}\left[-\frac{\cos n\pi t}{n\pi}\right]_{-1}^{0}\right\} = \frac{2}{n^2\pi^2}(1-\cos n\pi)$$

$$= \frac{2}{n^2\pi^2}(1-(-1)^n)$$

...continued

so that

$a_n = 0 \quad$ (n even), $a_n = \frac{4}{n^2\pi^2} \quad$ (n odd)

Now, for b_n

$$b_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) \sin \frac{2n\pi}{T} dt = ...$$

$$b_n = \int_{-1}^{0} 2(1+t) \sin n\pi t dt$$

$$= 2\left\{ \left[(1+t)\frac{-\cos n\pi t}{n\pi} \right]_{-1}^{0} + \frac{1}{n\pi}\int_{-1}^{0} \cos n\pi t dt \right\}$$

$$= 2\left\{-\frac{1}{n\pi} + \left[\frac{\sin n\pi t}{n\pi}\right]_{-1}^{0}\right\} = -\frac{2}{n\pi} + \frac{2}{n^2\pi^2}(\sin n\pi)$$
$$= -\frac{2}{n\pi}$$

So the few terms of the series give;

$$f(t) = \frac{1}{2} + \frac{4}{\pi^2} + \left\{\cos \pi t + \frac{1}{9}\cos 3\pi t + \frac{1}{25}\cos 5\pi t + \ldots\right\}$$
$$- \frac{2}{\pi}\left\{\sin \pi t + \frac{1}{2}\sin 2\pi t + \frac{1}{3}\sin 3\pi t + \frac{1}{4}\sin 4\pi t \ldots\right\}$$

...continued

The Fourier series

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left\{ a_n \cos n\omega t + b_n \sin n\omega t \right\}$$

can also be written in the form

$$f(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} B_n \sin \left\{ n\omega t + \phi_n \right\}$$

comparing these two expressions we see that

$$A_0 = a_0, \;\; B_n \sin \phi_n = a_n \; \textit{and} \; B_n \cos \phi_n = b_n$$

From this, it follows that
$B_n = \sqrt{{a_n}^2 + {b_n}^2}$; $\phi_n = \arctan\left(\frac{a_n}{b_n}\right)$

So
$B_1 \sin(\omega t + \phi_1)$ is the first harmonic or fundamental (lowest frequency)
$B_2 \sin(2\omega t + \phi_2)$ is the second harmonic (frequency twice that of the fundamental)
$B_n \sin(n\omega t + \phi_n)$ is the *nth* harmonic (frequency n times that of the fundamental)

And for the series to converge, the values of B_n must eventually decrease with higher-order harmonics, i.e.
$B_n \rightarrow 0 \; as \; n \rightarrow \infty$

Here, we are going to convert the infinite Fourier series in terms of sines and cosines into a doubly infinite series involving complex exponentials.

Complex exponentials

Recall the exponential form of a complex number and its relationship to the polar form, namely

$z = r(\cos\theta + j\sin\theta) = re^{j\theta}$

From this equation, we can see that

$z = \cos\theta + j\sin\theta = e^{j\theta}$

and so

$\cos(-\theta) + j\sin(-\theta) = e^{-j\theta} = \cos\theta - j\sin\theta$

Using these two equations we can find the complex exponential form of the trigonometric functions as ;

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2} \text{ and } \sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

Thus; we know

$\cos\theta + j\sin\theta = e^{j\theta}$ and $\cos\theta - j\sin\theta = e^{-j\theta}$

so adding these two equations gives

$$2\cos\theta = e^{j\theta} + e^{-j\theta} \text{ that is } \cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2} \quad — (1)$$

and subtracting the two equations gives

$$2j\sin\theta = e^{j\theta} - e^{-j\theta} \text{ that is } \sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j} \quad — (2)$$

These two equations permit us to develop an alternate representation of a Fourier series.

Previously,we found out that the Fourier series of the piecewise continuous function $f(t)$ with piecewise continuous derivative and where $f(t+T)=f(t)$ is given as ;

$$f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos n\omega_0 t+b_n\sin n\omega_0 t) \quad — (3)$$

$\omega_0=\dfrac{2\pi}{T}$ and where $a_n=\dfrac{2}{T}\displaystyle\int_{-T/2}^{T/2}f(t)\cos n\omega_0 t dt$ and

$$b_n=\frac{2}{T}\int_{-T/2}^{T/2}f(t)\sin n\omega_0 t dt$$

If we substitute the right-hand sides of equation (1) and (2) into equation (3) we obtain

$$f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\frac{e^{jn\omega_0 t}+e^{-jn\omega_0 t}}{2}+b_n\frac{e^{jn\omega_0 t}-e^{-jn\omega_0 t}}{2j}\right)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[\left(\frac{a_n + b_n/j}{2} \right) e^{jn\omega_0 t} + \left(\frac{a_n - b_n/j}{2} \right) e^{-jn\omega_0 t} \right]$$
$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[\left(\frac{a_n - jb_n}{2} \right) e^{jn\omega_0 t} + \left(\frac{a_n + jb_n}{2} \right) e^{-jn\omega_0 t} \right]$$

Next, we shall make some notational changes to simplify the expression above;

If we now define $c_n = \dfrac{a_n - jb_n}{2}$ so that the complex conjugate of c_n is $c_n^* = \dfrac{a_n + jb_n}{2}$ we can write this sum as

$$f(t) = c_0 + \textstyle\sum_{n=1}^{\infty} (c_n e^{jn\omega_0 t} + c_n^* e^{-jn\omega_0 t})$$
$$= c_0 + \textstyle\sum_{n=1}^{\infty} c_n e^{jn\omega_0 t} + \sum_{n=1}^{\infty} c_n^* e^{-jn\omega_0 t}$$

Note that we have taken $b_0 = 0$. There is no problem about this.There is no term $\sin 0\omega_0 t$ in the Fourier series and so $b_0 = 0$

$= c_0 + \sum_{n=1}^{\infty} c_n e^{jn\omega_0 t} + \sum_{n=1}^{\infty} c_{-n} e^{-jn\omega_0 t}$
For notational convenience we denote c_n^* by c_{-n}. This means that $a_{-n} = a_n$ and $b_{-n} = -b_n$
$= c_0 + \sum_{n=1}^{\infty} c_n e^{jn\omega_0 t} + \sum_{n=-1}^{-\infty} c_n e^{jn\omega_0 t}$

As n ranges from 1 to ∞ so $-n$ ranges from -1 to $-\infty$
$= \sum_{n=-\infty}^{-1} c_n e^{jn\omega_0 t} + c_0 + \sum_{n=1}^{\infty} c_n e^{jn\omega_0 t}$

Notice the reversed order of summation in the first sum
$= \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$

Combining all three terms into the *doubly infinite sum*

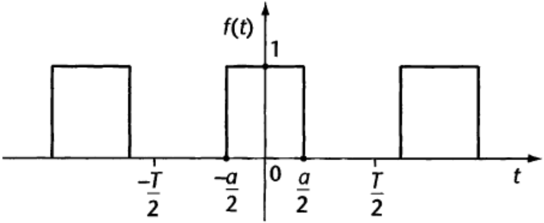
where $c_n = \dfrac{a_n - jb_n}{2} = \dfrac{1}{T} \displaystyle\int_{-T/2}^{T/2} f(t)(\cos n\omega_0 t - j \sin n\omega_0 t) dt.$
That is $\dfrac{1}{T} \displaystyle\int_{-T/2}^{T/2} f(t) e^{-jn\omega_0 t} dt$

Examples

(1). To find the complex Fourier series for the function

$$f(t) = \begin{cases} 0 & -T/2 < t < -a/2 \\ 1 & -a/2 < t < a/2 \\ 0 & a/2 < t < T/2 \end{cases}$$

where $f(t + T) = f(t)$



$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t} \text{ where } \omega_0 = \frac{2\pi}{T} \text{ and}$$

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-jn\omega_0 t}$$

$$= \frac{1}{T}\int_{-a/2}^{a/2} e^{-jn\omega_0 t} dt \quad \text{Because } f(t) = 1 \text{ for } -a/2 < t < a/2$$

$$= \frac{1}{T}\left[\frac{e^{-jn\omega_0 t}}{-jn\omega_0}\right]_{-a/2}^{a/2} \quad \text{Provided } n \neq 0$$

$$= \left(\frac{e^{-jn\omega_0 a/2} - e^{jn\omega_0 a/2}}{-j2n\pi}\right) \quad \text{Since } \omega_0 = \frac{2\pi}{T}$$

$$= \frac{\sin n\omega_0 a/2}{n\pi} \quad \text{Recall that } \sin\theta = \frac{e^{j\theta} - e^{j\theta}}{2j}$$

$$= \frac{\sin na/T}{n\pi} \quad \text{Since } \omega_0 = \frac{2\pi}{T}$$

$$= \frac{a}{T}\left(\frac{\sin n\pi a/T}{n\pi a/T}\right) \quad \text{Provided } n \neq 0$$

When $n = 0$

$$c_0 = \frac{1}{T}\int_{-T/2}^{T/2} f(t)dt = \frac{1}{T}\int_{-a/2}^{a/2} dt = \frac{a}{T}$$

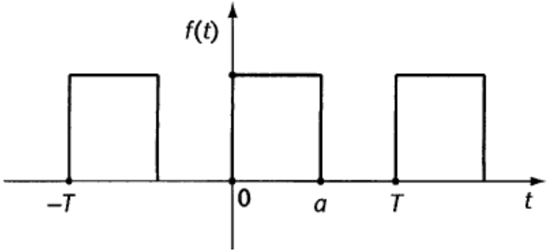
Therefore

$$f(t) = \frac{a}{T} + \sum_{n=-\infty, n\neq 0}^{\infty} \frac{a}{T}\left(\frac{\sin n\pi a/T}{n\pi a/T}\right) e^{jn\omega_0 t}$$

(2) To find the complex Fourier series for the function

$$f(t) = \begin{cases} 1 & 0 < t < a \\ 0 & a < t < T \end{cases}$$

where $f(t + T) = f(t)$



$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t} \quad \text{where } \omega_0 = \frac{2\pi}{T} \text{ and}$$

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-jn\omega_0 t}dt$$

$$= \frac{1}{T}\int_0^a e^{-jn\omega_0 t}dt$$

$$= \frac{1}{T}\left[\frac{e^{-jn\omega_0 t}}{-jn\omega_0}\right]_0^a \quad \text{Provided } n \neq 0$$

$$= \left(\frac{e^{-jn\omega_0 a} - 1}{-j2n\pi}\right)$$

$$= e^{-jn\omega_0 a/2}\left(\frac{e^{-jn\omega_0 a/2} - e^{jn\omega_0 a/2}}{-j2n\pi}\right)$$

$$= e^{-jn\omega_0 a/T}\frac{a}{T}\left(\frac{\sin n\pi a/T}{n\pi a/T}\right) \quad \text{Provided } n \neq 0$$

and c_0 will then be;

$$c_0 = \frac{1}{T}\int_{-T/2}^{T/2} f(t)dt$$

$$= \frac{1}{T}\int_0^a dt = \frac{a}{T}$$

Therefore

$$f(t) = \frac{a}{T} + \sum_{n=-\infty, n\neq 0}^{\infty} e^{-jn\pi a/T}\frac{a}{T}\left(\frac{\sin n\pi a/T}{n\pi a/T}\right)e^{jn\omega_0 t}$$

Fourier's Integral Theorem

Given function $f(t)$ with derivative $f'(t)$ where

1. $f(t)$ and $f'(t)$ are piecewise continuous in every finite interval
2. $f(t)$ is absolutely integrable in $(\infty, -\infty)$, that is $\displaystyle\int_{-\infty}^{\infty} |f(t)|dt$ is finite

then

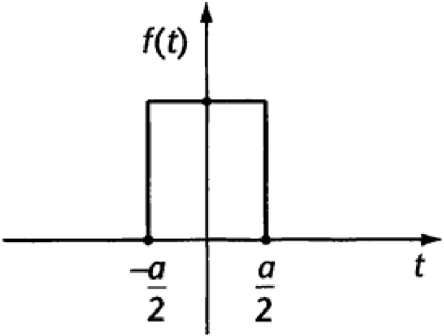
$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega$$

where $\displaystyle F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$

Example

Find the Fourier transform of

$$f(t) = \begin{cases} 0 & t < -a/2 \\ 1 & -a/2 < t < a/2 \\ 0 & a/2 < t \end{cases}$$



$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{\frac{-a}{2}}^{\frac{a}{2}} e^{-j\omega t} dt$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{-j\omega t}}{-j\omega} \right]_{\frac{-a}{2}}^{\frac{a}{2}}$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{e^{-j\omega a/2} - e^{j\omega a/2}}{-j\omega} \right)$$

$$= \sqrt{\frac{2}{\pi}} \left(\frac{e^{-j\omega a/2} - e^{j\omega a/2}}{-2j\omega} \right)$$

$$= \sqrt{\frac{2}{\pi}} \frac{\sin \omega a/2}{\omega}$$

$$= \frac{a}{\sqrt{2\pi}} \frac{\sin \omega a/2}{\omega a/2}$$

$$= \frac{a}{\sqrt{2\pi}} \sin c(\omega a/2)$$