在这一节中，取定测度空间$(X,\mathcal{M},\mu)$，定义L^+为所有$X\to[0,+\infty]$的可测函数空间

定义 0.0.1. 设$\phi=\sum_1^n a_j\mathcal{X}_{E_j}\in L^+$为简单函数，我们定义$\phi$的积分
$\int\phi\,d\mu=\sum\limits_1^n a_j\mu(E_j)$
约定$0*\infty=0$
设$A\in\mathcal{M}$，$\phi*\mathcal{X}_A=\sum\limits_1^n a_j\mathcal{X}_{E_j}\mathcal{X}_A=\sum\limits_1^n a_j\mathcal{X}_{E_j\cap A}$仍为简单函数
$\int_A\phi\,d\mu:=\int\phi*\mathcal{X}_A\,d\mu$
在这个约定下$\int\phi\,d\mu=\int_X\phi\,d\mu$
有时$\int_A\phi\,d\mu$也写作$\int_A\phi(x)\,d\mu(x)$或$\int_A\phi$

命题 0.0.2. 设$\phi,\psi\in L^+$为简单函数
a. $c\geq 0\Rightarrow\int c\phi=c\int\phi$
b. $\int(\phi+\psi)=\int\phi+\int\psi$
c. $\phi\leq\psi\Rightarrow\int\phi\leq\int\psi$
d. $A\mapsto\int_A\phi\,d\mu$是$\mathcal{M}$上的一个测度

证明. 设$\phi=\sum_1^n a_j\mathcal{X}_{E_j},\psi=\sum_1^m b_j\mathcal{X}_{F_j}$
a. $\int c\phi=\sum_1^n ca_j\mu(E_j)=c\sum_1^n a_j\mu(E_j)=c\int\phi$
b. 由于$X=\bigcup_1^n E_j=\bigcup_1^m F_j$为不交并，$E_j=\bigcup_{k=1}^m(E_j\cap F_k),F_j=\bigcup_{k=1}^n(E_k\cap F_j)$也是不交并
$\int\phi+\int\psi=\sum_{j=1}^n a_j\mu(E_j)+\sum_{k=1}^m b_k\mu(F_k)$
$=\sum_{j=1}^n a_j\mu(\bigcup_{k=1}^m(E_j\cap F_k))+\sum_{k=1}^m b_k\mu(\bigcup_{j=1}^n(E_j\cap F_k))$
$=\sum_{j,k}(a_j+b_k)\mu(E_j\cap F_k)$
$=\int(\phi+\psi)$
c. $E_j\cap F_k\neq\varnothing\Rightarrow a_j\leq b_k$
$\int\phi=\sum_{j=1}^n a_j\mu(E_j)=\sum_{j,k}a_j\mu(E_j\cap F_k)\leq\sum_{j,k}b_j\mu(E_j\cap F_k)=\int\psi$
d. $\int_\varnothing\phi\,d\mu=\int\phi*\mathcal{X}_\varnothing\,d\mu=\sum_1^n a_j\mu(\varnothing\cap E_j)=0$
设$\{A_k\}\subset\mathcal{M}$为一列不交集，$A=\bigcup_1^\infty A_k$

$\int_A \phi\, d\mu = \sum_{j=1}^n a_j \mu(A \cap E_j) = \sum_{j=1}^n a_j \mu(\bigcup_{k=1}^\infty (A_k \cap E_j))$
$= \sum_{j=1}^n a_j \sum_{k=1}^\infty \mu(A_k \cap E_j)$
$= \sum_{k=1}^\infty \sum_{j=1}^n a_j \mu(A_k \cap E_j)$
$= \sum_{k=1}^\infty \int_{A_k} \phi\, d\mu$
于是该映射是$\mathcal{M}$上的一个测度 □

现在我们将积分的定义扩张到L^+上

定义 0.0.3. *若*$f \in L^+$*，*$\int f\, d\mu := \sup\left\{\int \phi\, d\mu : 0 \le \phi \le f, \phi\text{是简单函数}\right\}$
由命题3.2.2.c.知，f*为简单函数时，两个定义是等价的*
容易验证$f \le g \Rightarrow \int f \le \int g, \int cf = c\int f, c \in [0, +\infty]$

定理 0.0.4. *The Monotone Convergence Theorem*
若$\{f_n\} \subset L^+, f_j \le f_{j+1}, f = \lim\limits_{n\to\infty} f_n (= \sup_n f_n)$
那么$\int f = \lim\limits_{n\to\infty} \int f_n$

证明. 首先$\int f_n$是单调递增数列，故极限存在或$= +\infty$
$f_n \le f \Rightarrow \int f_n \le \int f \Rightarrow \lim\limits_{n\to\infty} \int f_n \le \int f$
取$\alpha \in (0,1)$，$0 \le \phi \le f$为简单函数，$E_n := \{x : f_n(x) \ge \alpha\phi(x)\}$
则$E_n \in \mathcal{M}$($f - \alpha\phi$可测$(f-\alpha\phi)^{-1}[0, +\infty] \in \mathcal{M}$)，且$E_n \subset E_{n+1}, \bigcup_n E_n = X$，否则$\exists x \in X : \forall n, f_n(x) < \alpha\phi(x) \Rightarrow f(x) \le \alpha\phi(x) < \phi(x) \le f(x)$矛盾于是$\int_X f_n \ge \int_{E_n} f_n \ge \alpha \int_{E_n} \phi$
$\lim\limits_{n\to\infty} \int f_n \ge \alpha \lim\limits_{n\to\infty} \int_{E_n} \phi = \alpha \int_{\bigcup_1^\infty E_n} \phi = \alpha \int \phi$
以上讨论对任意α, ϕ成立，令$\alpha \to 1$，得$\lim\limits_{n\to\infty} \int f_n \ge \int \phi$
再对所有简单函数ϕ取上确界，得$\lim\limits_{n\to\infty} \int f_n \ge \int f$
综上，$\lim\limits_{n\to\infty} \int f_n = \int \phi$ □

对于$f \in L^+$，可取$\{\phi\} \subset L^+$为一单调简单函数且逐点收敛于f，$\int f = \lim_{n\to\infty} \int \phi_n$

定理 0.0.5. *若*$\{f_n\} \subset L^+$*是一列有限或可数函数，*$f = \sum_n f_n$
则$\int f = \sum_n \int f_n$

证明. 首先考虑f_1, f_2，取$\{\phi_j\}, \{\psi_j\} \subset L^+$为单调递增简单函数且分别逐点收敛于$f_1, f_2$，那么$\{\phi_j + \psi_j\}$为单调递增简单函数且逐点收敛于$f_1 + f_2$

$\int(f_1 + f_2) = \lim_{j\to\infty}\int(\phi_j + \psi_j) = \lim_{j\to\infty}(\int\phi_j + \int\psi_j) = \lim_{j\to\infty}\int\phi_j + \lim_{j\to\infty}\int\psi_j = \int f_1 + \int f_2$

这表明若$\{f_n\}$是有限的，$\int f = \sum_n \int f_n$

若是无穷的，$\sum_1^N f_n$是单调递增的，$\int f = \int\sum_1^\infty f_n = \int \lim\limits_{N\to\infty}\sum_1^N f_n$

$= \lim_{N\to\infty}\int\sum_1^N f_n = \lim_{N\to\infty}\sum_1^N\int f_n = \sum_1^\infty\int f_n$ □

命题 0.0.6. *若$f \in L^+$,那么$\int f = 0 \Leftrightarrow f = 0$ a.e.*

证明. 首先若$f = \sum_j a_j\mathcal{X}_{E_j}$是简单函数，$\int f = 0 \Leftrightarrow a_j = 0$或$\mu(E_j) = 0$对每个$j$成立，$\Leftrightarrow f = 0$ a.e.

$\Leftarrow$

$0 \le \phi \le f$为简单函数，则$\phi = 0$ a.e.$\Rightarrow \int\phi = 0$

于是$\int f = \sup_{0\le\phi\le f}\int\phi = 0$

$\Rightarrow$

$f^{-1}((0, +\infty]) = f^{-1}((\bigcup_1^\infty(\frac{1}{n}), +\infty]) = \bigcup_1^\infty f^{-1}((\frac{1}{n}, +\infty])$

若$f = 0$ a.e.不成立，必$\exists N : \mu(f^{-1}((\frac{1}{N}, +\infty])) > 0$，记为$E_N$

否则$\mu(f^{-1}((0, +\infty])) \le \sum_1^\infty \mu(f^{-1}((\frac{1}{n}, +\infty])) = 0$

这与$f = 0$ a.e.不成立矛盾

于是$\int f \ge \int f * \mathcal{X}_{E_N} > \int\frac{1}{N} * \mathcal{X}_{E_N} = \frac{1}{N} * \mu(E_N) > 0$

这又与$\int f = 0$矛盾 □

推论 0.0.7. *若$\{f_n\} \subset L^+, f \in L^+$，$f_n(x)$单调递增趋于$f(x)$ a.e.那么$\int f = \lim_{n\to\infty}\int f_n$*

证明. $\exists E \in \mathcal{M} : \mu(E^c) = 0, \forall x \in E, f_n(x)$单调递增趋于$f(x)$成立，那么$f - f * \mathcal{X}_E \in L^+, f_n - f_n * \mathcal{X}_E \in L^*$且$f - f * \mathcal{X}_E = 0, f_n - f_n * \mathcal{X}_E = 0$ a.e.，由命题3.2.6.知$\int f - f * \mathcal{X}_E = \int f_n - f_n * \mathcal{X}_E = 0$

又$f_n * \mathcal{X}_E$单调递增趋于$f * \mathcal{X}_E$

$\int f = \int f * \mathcal{X}_E = \lim_{n\to\infty} \int f_n * \mathcal{X}_E = \lim_{n\to\infty} \int f_n$ □

引理 0.0.8. *Fatou's Lemma*

若$\{f_n\} \subset L^+$是任意序列，则$\int (\liminf f_n) \leq \liminf \int f_n$

证明. $\forall j \geq k, \inf\limits_{n\geq k} f_n \leq f_j$,于是$\int \inf\limits_{n\geq k} f_n \leq \int f_j, \forall j \geq k$

$\Rightarrow \int \inf\limits_{n\geq k} f_n \leq \inf\limits_{j\geq k} \int f_j$

令$k \to \infty$，有$\int (\liminf f_n) = \lim_{k\to\infty} \int \inf\limits_{n\geq k} f_n \leq \liminf \int f_n$ □

推论 0.0.9. 若$\{f_n\} \subset L^+, f \in L^+, f_n \to f$ *a.e.*，那么$\int f \leq \liminf \int f_n$

证明. 若$f_n \to f$处处成立，那么根据Fatou's Lemma，$\int f = \int \liminf f_n \leq \liminf \int f_n$。一般的，$\exists E \in \mathcal{M} : \mu(E^c) = 0, \forall x \in E, f_n(x) \to f(x)$，即$f_n * \mathcal{X}_E \to f * \mathcal{X}_E$处处成立,$f - f * \mathcal{X}_E, f_n - f_n * \mathcal{X}_E \in L^+, f - f * \mathcal{X}_E = 0, f_n - f_n * \mathcal{X}_E = 0$ a.e.，于是$\int f = \int f * \mathcal{X}_E, \int f_n = \int f * \mathcal{X}_E$

$\int f = \int f * \mathcal{X}_E \leq \liminf \int f_n * \mathcal{X}_E = \liminf \int f_n$ □

命题 0.0.10. 若$f \in L^+, \int f < +\infty$，那么$\{x : f(x) = +\infty\}$是零测集，且$\{x : f(x) > 0\}$是$\sigma$-有限的

证明. 记$E_n = f^{-1}((n, n+1]), n = 1, \ldots, F_k = f^{-1}((\frac{1}{k+1}, \frac{1}{k}]), k = 1, 2, \ldots$，则$\{x : f(x) > 0\} = \bigcup_{n=1}^{\infty} E_n \cup \bigcup_{k=1}^{\infty} F_k$是不交并，若$\exists E_n$或$F_k$，使得$\mu(E_n) = +\infty$或$\mu(F_k) = +\infty$，则取$\phi = n\mathcal{X}_{E_n}$或$\frac{1}{k+1}\mathcal{X}_{F_k}$，均有$\phi \leq f$，于是$\int f \geq \int \phi = +\infty$，矛盾，从而$\{x : f(x) > 0\}$是$\sigma$-有限的

记$E = f^{-1}(\{+\infty\})$，若$\mu(E) = c > 0$，那么取$\phi_n = n\mathcal{X}_E$，有$\phi_n \leq f$

$\int f \geq \lim_{n\to\infty} \int \phi_n = \lim_{n\to\infty} nc = +\infty$，矛盾，故$\mu(E) = 0$ □