HPC_Project

HPC课程的Project， 关于一维传热方程的数值解(显式方法和隐式方法)

一维传热方程如下：

$$\begin{aligned} \rho c\frac{\partial u}{\partial t} - \kappa\frac{\partial^2 u}{\partial x^2} &= \sin(l\pi x) \\ \kappa\frac{\partial u}{\partial x}n_x &= h \\ \kappa &= 1.0 \end{aligned}$$

边界条件为：

$$\begin{aligned} u(0,t) = u(1,t) &= 0 \\ u|_{t=0} &= e^x \end{aligned}$$

传热方程改写成：

$$\frac{\partial u}{\partial t} - \frac{\kappa}{\rho c}\frac{\partial^2 u}{\partial x^2} = \frac{1}{\rho c}\sin(l\pi x)$$

显式格式(Adams-Bashforth)：

差分格式为：

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{u^{n+1}_j - u^n_j}{\Delta t} \\ \frac{\partial^2 u}{\partial x^2} = \frac{u^n_{j+1} - 2u^n_j + u^n_{j-1}}{\Delta x^2} \end{cases}$$

\begin{aligned} \therefore \frac{u^{n+1}_j-u^n_j}{\Delta t} &= \frac{\kappa}{\rho c} \frac{u^n_{j+1}-2u^n_j+u^n_{j-1}}{\Delta x^2} + \frac{1}{\rho c}\sin(l \pi j \Delta x) \\ \implies u^{n+1}_j &= u^n_j + \frac{\kappa \Delta t}{\rho c \Delta x^2} (u^n_{j+1}-2u^n_j+u^n_{j-1})+ \frac{\Delta t}{\rho c}\sin(l \pi j \Delta x)\\ \implies u^{n+1}_j &= \frac{\kappa \Delta t}{\rho c \Delta x^2} u^n_{j+1} + (1-\frac{2 \kappa \Delta t}{\rho c \Delta x^2}) u^n_j + \frac{\kappa \Delta t}{\rho c \Delta x^2}u^n_{j-1} + \frac{\Delta t}{\rho c}\sin(l \pi j \Delta x)\\ make \frac{\kappa}{\rho c} &= \alpha, \frac{\alpha \Delta t}{\Delta x^2} = \beta = \frac{\kappa \Delta t}{\rho c \Delta x^2}\\ \implies u^{n+1}_j &= \beta u^n_{j+1} + (1-2\beta) u^n_j + \beta u^n_{j-1} + \frac{\Delta t}{\rho c}\sin(l \pi j \Delta x)\\ \end{aligned}

隐式格式(Crank-Nicolson)：

差分格式为：

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{u^{n+1}_j - u^n_j}{\Delta t} \\ \dfrac{\partial^2 u}{\partial x^2} = \dfrac{u^{n+1}_{j+1} - 2u^{n+1}_j + u^{n+1}_{j-1}}{\Delta x^2} \end{cases}$$

$$\begin{aligned} \therefore \frac{u^{n+1}_j - u^n_j}{\Delta t} &= \frac{\kappa}{\rho c} \frac{u^{n+1}_{j+1} - 2u^{n+1}_j + u^{n+1}_{j-1}}{\Delta x^2} + \frac{1}{\rho c} \sin(l\pi j \Delta x) \\ make \frac{\kappa}{\rho c} &= \alpha, \frac{\alpha \Delta t}{\Delta x^2} = \beta = \frac{\kappa \Delta t}{\rho c \Delta x^2} \end{aligned}$$

\begin{aligned} \implies -\beta u^{n+1}_{j+1} + (1+2\beta)u^{n+1}_j - \beta u^{n+1}_{j+1} = u^n_j + \alpha \Delta t \sin(l \pi j \Delta x) \end{aligned}

解的格式为：

$$\begin{pmatrix} 1+2\beta & -\beta & 0 & 0 & \cdots & 0 & 0 & 0 \\ -\beta & 1+2\beta & -\beta & 0 & \cdots & 0 & 0 & 0 \\ 0 & -\beta & 1+2\beta & -\beta & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & -\beta & 1+2\beta \end{pmatrix} * \begin{pmatrix} u^{n+1}_2 \\ u^{n+1}_3 \\ \vdots \\ u^{n+1}_{n-1} \\ u^{n+1}_n \end{pmatrix} = \begin{pmatrix} u^n_2 + \alpha \Delta t \sin(l\pi \Delta x) \\ u^n_3 + \alpha \Delta t \sin(l\pi 2 \Delta x) \\ \vdots \\ u^{\ n}_{n-1} + \alpha \Delta t \sin(l\pi (n-2) \Delta x) \\ u^{n+1}_{\ n} + \alpha \Delta t \sin(l\pi (n-1) \Delta x) \end{pmatrix}$$