Formula Derivation

HPC课程的Project， 关于一维传热方程的数值解(显式方法和隐式方法)

一维传热方程如下：

$$
\begin{aligned}
\rho c \frac{\partial u}{\partial t} - \kappa \frac{\partial^2 u}{\partial x^2} &= \sin(l\pi x) \\
\kappa \frac{\partial u}{\partial x} n_x &= h \\
\kappa &= 1.0
\end{aligned}
$$

边界条件为：

$$
\begin{aligned}
u(0,t) = u(1,t) = 0 \\
u|_{t=0} = e^x
\end{aligned}
$$

传热方程改写成：

$$
\frac{\partial u}{\partial t} - \frac{\kappa}{\rho c} \frac{\partial^2 u}{\partial x^2} = \frac{1}{\rho c} \sin(l\pi x)
$$

显式格式(Adams-Bashforth)：

差分格式为：

$$
\begin{cases}
\frac{\partial u}{\partial t} = \frac{u_j^{n+1} - u_j^n}{\Delta t} \\
\frac{\partial^2 u}{\partial x^2} = \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2}
\end{cases}
$$

$$
\begin{aligned}
\therefore \frac{u_j^{n+1} - u_j^n}{\Delta t} &= \frac{\kappa}{\rho c} \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2} + \frac{1}{\rho c} \sin(l\pi j \Delta x) \\
\implies u_j^{n+1} &= u_j^n + \frac{\kappa \Delta t}{\rho c \Delta x^2}(u_{j+1}^n - 2u_j^n + u_{j-1}^n) + \frac{\Delta t}{\rho c} \sin(l\pi j \Delta x) \\
\implies u_j^{n+1} &= \frac{\kappa \Delta t}{\rho c \Delta x^2} u_{j+1}^n + (1 - \frac{2\kappa \Delta t}{\rho c \Delta x^2}) u_j^n + \frac{\kappa \Delta t}{\rho c \Delta x^2} u_{j-1}^n + \frac{\Delta t}{\rho c} \sin(l\pi j \Delta x) \\
make \frac{\kappa}{\rho c} &= \alpha, \frac{\alpha \Delta t}{\Delta x^2} = \beta = \frac{\kappa \Delta t}{\rho c \Delta x^2} \\
\implies u_j^{n+1} &= \beta u_{j+1}^n + (1 - 2\beta) u_j^n + \beta u_{j-1}^n + \frac{\Delta t}{\rho c} \sin(l\pi j \Delta x)
\end{aligned}
$$

隐式格式(Crank-Nicolson):

差分格式为:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{u_j^{n+1} - u_j^n}{\Delta t} \\ \dfrac{\partial^2 u}{\partial x^2} = \dfrac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{\Delta x^2} \end{cases}$$

$$\therefore \frac{u_j^{n+1} - u_j^n}{\Delta t} = \frac{\kappa}{\rho c}\frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{\Delta x^2} + \frac{1}{\rho c}\sin(l\pi j\Delta x)$$

$$make\frac{\kappa}{\rho c} = \alpha, \frac{\alpha\Delta t}{\Delta x^2} = \beta = \frac{\kappa\Delta t}{\rho c\Delta x^2}$$

$$\Longrightarrow -\beta u_{j+1}^{n+1} + (1+2\beta)u_j^{n+1} - \beta u_{j+1}^{n+1} = u_j^n + \alpha\Delta t\sin(l\pi j\Delta x)$$

解的格式为:

$$\begin{pmatrix} 1+2\beta & -\beta & 0 & 0 & \cdots & 0 & 0 & 0 \\ -\beta & 1+2\beta & -\beta & 0 & \cdots & 0 & 0 & 0 \\ 0 & -\beta & 1+2\beta & -\beta & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & -\beta & 1+2\beta \end{pmatrix} * \begin{pmatrix} u_2^{n+1} \\ u_3^{n+1} \\ \vdots \\ u_{n-1}^{n+1} \\ u_n^{n+1} \end{pmatrix} = \begin{pmatrix} u_2^n + \alpha\Delta t\sin(l\pi\Delta x) \\ u_3^n + \alpha\Delta t\sin(l\pi 2\Delta x) \\ \vdots \\ u_{n-1}^n + \alpha\Delta t\sin(l\pi(n-2)\Delta x) \\ u_n^{n+1} + \alpha\Delta t\sin(l\pi(n-1)\Delta x) \end{pmatrix}$$