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Q1) Consider a system with 1 transmit antenna and L receive antennas. The transmitter has a power constraint of P. The signal received by each receive antenna is corrupted by AWGN independently. Assume the noise follows complex Gaussian distribution with zero mean and variance of N_0.
a) [10%] Suppose the gain between the transmit antenna and each of the receive antennas is constant, equal to 1. What is the capacity of the channel? What is the power gain compared to a single receive antenna system?

(a) $y_i = h_i x + w_i$ with $|h_i| = 1$ for $i = 1, 2 \dots L$

$\Rightarrow y = h x + w$ with $w \sim CN(0, N_0 I_L)$

$\Rightarrow \frac{h^H}{|h|} y = |h| x + \frac{h^H}{|h|} w$

$= \sqrt{L} x + \tilde{w}$ with $\tilde{w} \sim CN(0, N_0)$

$\Rightarrow C = \log\left(1 + \frac{LP}{N_0}\right)$

For single received antenna case, $C_{single} = \log\left(1 + \frac{P}{N_0}\right)$. Therefore there is a power gain of L.

Q2) [20%] Reproduce Figure 5.15 in the textbook. In your figure, plot both the theoretical curves and the simulated ones. The theoretical curve can be obtained by using Eq. (5.57) in the textbook. Notice that the exact cumulative distribution function should be used instead of the high-SNR approximation. As to the simulated curve, it should be produced by generating a large number of channel realizations.

In the submitted figure, the theoretical curves should be plotted with solid lines, and use different symbols (like circles, squares, triangles, etc.) to plot the simulated results. Colors should be used with cautions as they are not distinguishable in the printed hard-copy.

2. ϵ-outage probability

The simulation consists of two parts:

(1) **Generate theoretical curve**

Generate the theoretical curve under different L and SNR by

$$C_{\epsilon,\text{theoretical}} = \log\left(1 + F^{-1}(1-\epsilon)\,\text{SNR}\right)$$

where $F(x) = P\left(\|h\|^2 > x\right)$ and $\|h\|^2$ is chi-square distribution with $2L$ degrees of freedom.

(2) **Generate Simulated curve**

To generate the simulated curve, recall the definition of ϵ-outage probability:

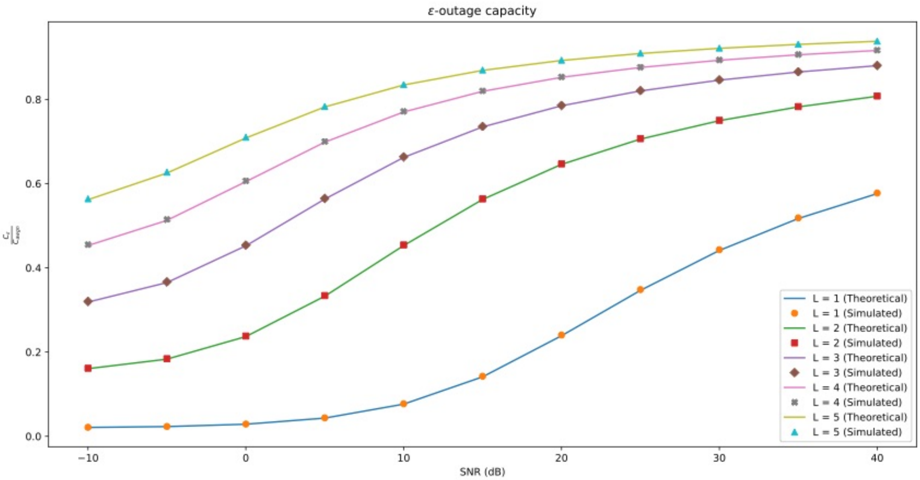
$$\epsilon\,(C) = P\left(\log(1 + \|h\|^2\,\text{SNR}) < C\right)$$

Given ϵ, L, and SNR, we first generate a large number ($N = 2L \cdot 10^6$) of channel realizations from $\mathcal{CN}(0,1)$. For every $2L$ channel realization, we obtain $\|h\|^2$ by

$$\|h\|^2 = \sum_{l=1}^{2L} x_l^2$$

Calculate $\log(1+\|h\|^2\,\text{SNR})$ for each $\|h\|^2$ and sort them into nondecreasing order. Let $k = 10^6 \cdot \epsilon$. From the definition above we conclude that the kth smallest value from the sorted data is the simulated result.

The comparison between theoretical computation and simulation is shown below. Solid lines represent the theoretical result and marked points show the simulated result.



Q4) [20%] In this exercise, we derive some properties a code construction must satisfy to mimic the Alamouti scheme behavior for more than two transmit antennas. Consider communication over n time slots on the L transmit antenna channel (cf. (3.80)):

$$\mathbf{y}^T = \mathbf{h}^*\mathbf{X} + \mathbf{w}^T, \tag{6}$$

where $\mathbf{X}$ is the $L \times n$ space-time code. Over n time slots, we want to communicate L independent constellation symbols, $d_1, \cdots, d_L$; the space-time code $\mathbf{X}$ is a deterministic function of these symbols.

a) [10%] Consider the following property for every channel realization $\mathbf{h}$ and space-time codeword $\mathbf{X}$

$$(\mathbf{h}^*\mathbf{X})^T = \mathbf{A}\mathbf{d}. \tag{7}$$

Here we have written $\mathbf{d} = [d_1, \cdots, d_L]^T$ and $\mathbf{A} = [\mathbf{a}_1, \cdots, \mathbf{a}_L]$, a matrix with orthogonal columns. The vector $\mathbf{d}$ depends solely on the codeword $\mathbf{X}$ and the matrix $\mathbf{A}$ depends solely on the channel $\mathbf{h}$. Show that, if the space-time codeword $\mathbf{X}$ satisfies the property in (7), the joint receiver to detect $\mathbf{d}$ separates into individual linear receivers, each separately detecting $d_1, \cdots, d_L$.

Hint: Think of $\mathbf{a}_i$, the i-th column in $\mathbf{A}$ as a receive beamforming vector and project $\mathbf{y}$ along $\mathbf{a}_i$.

b) [10%] We would like the effective channel (after the linear receiver) to provide each symbol d_i ($i = 1, \cdots, L$)) with full diversity. Show that, if we impose the condition that

$$\|\mathbf{a}_i\| = \|\mathbf{h}\|, \quad i = 1, \cdots, L, \tag{8}$$

then each data symbol d_i has full diversity.

Hint: By having full diversity for each symbol, it means each symbol can benefit from all the diversity branches.

(a) To detect d_i, we project y onto a_i that is

$$\frac{a_i^H}{|a_i|} y = \frac{a_i^H}{|a_i|} A d + \frac{a_i^H}{|a_i|} w$$

$$= |a_i| d_i + \frac{a_i^H}{|a_i|} w \sim |a_i| d_i + w$$

which is sufficient statistic for d_i. Therefore, the joint receiver can be separated into L individual linear receiver

(b) For each symbol the typical error is

$$P(|a_i|^2 < 1/SNR) = P(|h|^2 < 1/SNR)$$

$$\approx \frac{1}{L!} \frac{1}{SNR^L}$$

(same as that for repetition code)

The diversity gain for each symbol is L. Hence, each symbol has full diversity.

Q5) [10%] We have studied the performance of the Alamouti scheme in a channel with two transmit and one receive antenna. Suppose now we have an additional receive antenna. Derive the ML detector for the symbols based on the received signals at both receive antennas. Show that the scheme effectively provides two independent scalar channels.

For Alamouti scheme

$$\begin{cases} \begin{pmatrix} y_{11} \\ y_{12}^* \end{pmatrix} = \begin{pmatrix} h_{11} & h_{12} \\ h_{12}^* & -h_{11}^* \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} w_{11} \\ w_{12}^* \end{pmatrix} \Rightarrow y_1 = H_1 u + w_1 \\ \begin{pmatrix} y_{21} \\ y_{22}^* \end{pmatrix} = \begin{pmatrix} h_{21} & h_{22} \\ h_{22}^* & -h_{21}^* \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} w_{21} \\ w_{22}^* \end{pmatrix} \Rightarrow y_2 = H_2 u + w_2 \end{cases}$$

$$\Rightarrow \underbrace{\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}}_{y} = \underbrace{\begin{pmatrix} H_1 \\ H_2 \end{pmatrix}}_{H} u + \underbrace{\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}}_{w} \quad \text{with } w \sim CN(0, N_0 I_4)$$

The columns of H are mutually orthogonal. Let $H = (h_1, h_2)$ with $|h_1| = |h_2| = |h| = \sqrt{\sum_{i=1}^{2} \sum_{j=1}^{2} |h_{ij}|^2}$, we have

$$\begin{cases} \frac{h_1^*}{|h|} y_1 = |h| u_1 + \frac{h_1^*}{|h|} w_1 \\ \frac{h_2^*}{|h|} y_2 = |h| u_2 + \frac{h_2^*}{|h|} w_2 \end{cases} \Rightarrow \begin{matrix} \tilde{y}_1 = |h| u_1 + \tilde{w} \\ \tilde{y}_2 = |h| u_2 + \tilde{w} \end{matrix} \quad \text{with } \tilde{w} \sim CN(0, N_0)$$

This shows that the scheme provides two independent channel with channel gain $|h| = \sqrt{\sum_{i=1}^{2} \sum_{j=1}^{2} |h_{ij}|^2}$. To detect u_1, u_2 we can perform ML on both channel. That is

$$\hat{u}_j = \underset{u}{\arg\min} \, |\tilde{y}_j - |h| u_j| \text{ for } j = 1, 2$$

$$\left(\underline{\text{Ex}} \text{ For BPSK, we have } \hat{u}_j = \begin{matrix} 1 \\ -1 \end{matrix} \text{ if } \tilde{y}_j \begin{matrix} > 0 \\ < 0 \end{matrix} \right)$$