Solution to Homework Assignment No. 1

1. (a) Since

$$\operatorname{rect}(t) \rightleftharpoons \operatorname{sinc}(f)$$

~~and~~

$$\cancel{\frac{1}{2}\left[\delta(t-3)+\delta(t+3)\right] \rightleftharpoons \cos\left(2\pi \cdot 3 \cdot f\right) = \cos\left(6\pi f\right)}$$

we have

$$e^{-j6\pi f}\operatorname{sinc}(f) + e^{j6\pi f}\operatorname{sinc}(f)$$

請用 time shifting 來算

$$\begin{aligned} F\left[\operatorname{rect}(t-3)+\operatorname{rect}(t+3)\right] &= F\left[\cancel{2\cdot\operatorname{rect}(t)\star\frac{1}{2}\left[\delta(t-3)+\delta(t+3)\right]}\right] \\ &= 2\times\operatorname{sinc}(f)\times\cos\left(6\pi f\right). \end{aligned}$$

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(b) Since

$$\operatorname{sinc}(t) \rightleftharpoons \operatorname{rect}(f)$$

the Fourier transform of $\operatorname{sinc}^2(t)$ is given by

$$\operatorname{rect}(f)\star\operatorname{rect}(f) = \begin{cases} 1-|f|, & |f|<1 \\ 0, & \text{elsewhere.} \end{cases}$$

(c) First, the Fourier transform of $\operatorname{sinc}(t)\star\operatorname{sinc}(t)$ is given by

$$F\left[\operatorname{sinc}(t)\star\operatorname{sinc}(t)\right] = \operatorname{rect}^2(f) = \operatorname{rect}(f).$$

Then, taking the inverse Fourier transform on both sides, we get

$$\operatorname{sinc}(t)\star\operatorname{sinc}(t) = F^{-1}\left[\operatorname{rect}(f)\right] = \operatorname{sinc}(t).$$

2. (a) The ~~average~~ energy of $g_1(t)$ is given by

$$\begin{aligned} E_{g_1} &= \int_{-\infty}^{\infty}|g_1(t)|^2 dt = \int_0^{\infty} e^{-2t}\cos^2(t)dt \\ &= \int_0^{\infty} e^{-2t}\left[\frac{1+\cos(2t)}{2}\right] dt \\ &= \frac{1}{2}\int_0^{\infty} e^{-2t}+e^{-2t}\cos(2t)dt \\ &= \frac{1}{2}\left\{-\frac{1}{2}e^{-2t}+\frac{1}{4}e^{-2t}\left[\sin(2t)-\cos(2t)\right]\Big|_0^{\infty}\right\} \\ &= \frac{1}{2}\left(\frac{1}{2}+\frac{1}{4}\right) = \frac{3}{8}. \end{aligned}$$

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Since $g_1(t)$ has finite energy, we ~~have to~~ find its energy spectral density. First, the Fourier transform of $g_1(t)$ is given by

$$e^{-t}\cos(t)u(t) \rightleftharpoons \frac{1}{2}\left[\frac{1}{1+j(2\pi f-1)}+\frac{1}{1+j(2\pi f+1)}\right].$$

Then, the energy spectral density of $g_1(t)$ is ~~given by~~

$$\begin{aligned}|G_1(f)|^2 &= G_1(f)\cdot G_1^*(f)\\ &= \frac{1}{4}\left[\frac{1}{1+j(2\pi f-1)}+\frac{1}{1+j(2\pi f+1)}\right]\left[\frac{1}{1-j(2\pi f-1)}+\frac{1}{1-j(2\pi f+1)}\right]\\ &= \frac{1}{4}\left[\frac{1}{(2\pi f)^2-4\pi f+2}+\frac{1}{(2\pi f)^2+4\pi f+2}+\frac{2(2\pi f)^2+4}{(2\pi f)^4+4(2\pi f)^2+8}\right].\end{aligned}$$

這一項不太對，似乎是 $\frac{2(2\pi f)^2}{(2\pi f)^4+4}$ 請再驗算一次

(b) The ~~average~~ energy of $g_2(t)$ is given by

$$E_{g_2} = \int_{-\infty}^{\infty}|g_2(t)|^2dt = \int_{-\infty}^{\infty}1dt = \infty.$$

Since $g_2(t)$ has infinite energy, we ~~have~~ turn to find its average power, which is given by

$$\begin{aligned}P_{g_2} &= \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}|g_2(t)|^2dt\\ &= \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}1dt = 1.\end{aligned}$$

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(c) Since $g_3(t)$ is a periodic signal, which implies it has infinite energy, we ~~have~~ turn to find its average power. First, the Fourier transform of $g_3(t)$ is given by

$$2\cos(40\pi t)+6\cos(100\pi t) \rightleftharpoons \delta(f-20)+\delta(f+20)+3\delta(f-50)+3\delta(f+50).$$

Then, the average power is $1^2+1^2+3^2+3^2=20$.

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3. (a) ~~In time domain, we~~ We have

$$g_1(t) = \frac{A_c}{2}\left(e^{j(2\pi f_c t+\theta)}+e^{-j(2\pi f_c t-\theta)}\right).$$

Applying Fourier transform on $g_1(t)$, we get

$$G_1(f) = \frac{A_c}{2}\left(e^{j\theta}\delta(f-f_c)+e^{-j\theta}\delta(f+f_c)\right).$$

~~Applying~~ For Hilbert transform, we get

$$\begin{aligned}\hat{G}_1(f) &= -j\cdot\mathrm{sgn}(f)\cdot G_1(f)\\ &= \frac{A_c}{2j}\left(e^{j\theta}\delta(f-f_c)-e^{-j\theta}\delta(f+f_c)\right).\end{aligned}$$

2 (c) As $g_3(t)$ ~~is a~~ has period 1/10, the average power is given by

$$P_{g_3} = 10 \int_{-1/20}^{1/20} \left| 2\cos(40\pi t) + 6\cos(100\pi t) \right|^2 dt$$

$$= 10 \int_{-1/20}^{1/20} \left[4\cos^2(40\pi t) + 36\cos^2(100\pi t) + 24\cos(40\pi t)\cos(100\pi t) \right] dt$$

$$= \frac{4}{2} + \frac{36}{2} = 20.$$

Finally, taking inverse Fourier transform on both side, we ~~get~~ have

$$\begin{aligned}
\hat{g}_1(t) &= F^{-1}\{\hat{G}_1(f)\} \quad \text{中括號} \\
&= \frac{A_c}{2j}\left(e^{j\theta}e^{j2\pi f_c t} - e^{-j\theta}e^{-j2\pi f_c t}\right) \\
&= A_c \sin(2\pi f_c t + \theta).
\end{aligned}$$

(b) ~~Applying Fourier transform on $g_2(t)$, we get~~ The Fourier transform of $g_2(t)$ is

$$\begin{aligned}
G_2(f) &\cancel{= M(f) \star \left(\frac{1}{2}\delta(f - f_c) + \frac{1}{2}\delta(f + f_c)\right)} \\
&= \frac{1}{2}M(f - f_c) + \frac{1}{2}M(f + f_c).
\end{aligned}$$

As $f_c > W$, for ~~Applying~~ For Hilbert transform, we get

$$\begin{aligned}
\hat{G}_2(f) &= -j \cdot \mathrm{sgn}(f) \cdot G_2(f) \\
&= \frac{1}{2j}M(f - f_c) - \frac{1}{2j}M(f + f_c). \quad \cancel{(\because f_c > W)}.
\end{aligned}$$

Finally, taking inverse Fourier transform on both side, we ~~get~~ obtain

$$\begin{aligned}
\hat{g}_2(t) &= F^{-1}\{\hat{G}_2(f)\} \quad \text{中括號} \\
&\cancel{= F^{-1}\left\{M(f) \star \left(\frac{1}{2j}\delta(f - f_c) - \frac{1}{2j}\delta(f + f_c)\right)\right\}} \quad = \frac{1}{2j}\left[e^{j2\pi f_c t} m(t) - e^{-j2\pi f_c t} m(t)\right] \\
&= m(t)\sin(2\pi f_c t).
\end{aligned}$$

4. (a) For ~~$f \in [f_c - B, f_c + B]$~~ $f_c - B \le f \le f_c + B$, we have

$$\beta(f) \cancel{= 2\pi t_0 B - 2\pi t_0(f - f_c + B)} = -2\pi(f - f_c)t_0.$$

The ~~spectrum~~ Fourier transform of the pre-envelop ~~of $h(t)$~~ $h_+(t)$ is then given ~~as~~ by

$$H_+(f) = 2\,\mathrm{rect}\left(\frac{f - f_c}{2B}\right) e^{-j2\pi(f - f_c)t_0}.$$

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~~The spectrum~~ Hence the Fourier transform of ~~low-pass equivalence of $h(t)$~~ the complex envelope $\tilde{h}(t)$ is

$$\tilde{H}(f) = H_+(f + f_c) = 2\mathrm{rect}\left(\frac{f}{2B}\right) e^{-j2\pi f t_0}.$$

(b) ~~From the result of~~ from (a), by taking the ~~take~~ inverse Fourier transform ~~on both side~~, we obtain

$$\begin{aligned}
\tilde{h}(t) &= F^{-1}\{\tilde{H}(f)\} \\
&\cancel{= 2F^{-1}\left\{\mathrm{rect}\left(\frac{f}{2B}\right)\right\} \star F^{-1}\{e^{-j2\pi f t_0}\}} \\
&\cancel{= 2(2B\mathrm{sinc}(2Bt) \star \delta(t - t_0))} \\
&= 4B\mathrm{sinc}(2B(t - t_0)).
\end{aligned}$$

Since

$$A\,\mathrm{sinc}(2Bt) \rightleftharpoons \frac{A}{2B}\mathrm{rect}\left(\frac{f}{2B}\right)$$

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then

(c) The impulse response $h(t)$ is given ~~as~~ as

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$$h(t) = \text{Re}\left[\tilde{h}(t)e^{j2\pi f_c t}\right] = 4B\text{sinc}\,(2B(t-t_0))\cos(2\pi f_c t).$$

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5. (a) Starting from $X(f)$, we can deduce

$$\begin{aligned} X(f) &= F\{x(t)\} \\ &= F\{4A_cW\text{sinc}(2Wt)\cos(2\pi f_c t)\} \\ &= 4A_cW(F\{\text{sinc}(2Wt)\} \star F\{\cos(2\pi f_c t)\}) \\ &= 4A_cW[\frac{1}{2W}\text{rect}\left(\frac{f}{2W}\right) \star \frac{1}{2}(\delta(f-f_c)+\delta(f+f_c))] \\ &= A_c(\text{rect}\left(\frac{f-f_c}{2W}\right) + \text{rect}\left(\frac{f+f_c}{2W}\right)). \end{aligned}$$

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Then since $\tilde{X}(f) = 2X(f+f_c)u(f+f_c)$, we have

$$\tilde{X}(f) = 2A_c\text{rect}\left(\frac{f}{2W}\right).$$

(b) Since $\tilde{Y}(f) = \frac{1}{2}\tilde{X}(f)\tilde{H}(f)$ and from result of 4.(a), we have

$$\begin{aligned} \tilde{Y}(f) &= \frac{1}{2}\cdot 2A_c\text{rect}\left(\frac{f}{2W}\right)\cdot 2e^{-j2\pi f t_0}, |f|<B, W>B \\ &= 2A_c\text{rect}\left(\frac{f}{2B}\right)e^{-j2\pi f t_0}. \end{aligned}$$

V (c) From (b), ~~$\tilde{y}(t)$ is given by~~ we have

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$$\begin{aligned} \tilde{y}(t) &= F^{-1}\left[\tilde{Y}(f)\right] \\ &= 4BA_c\text{sinc}(2B(t-t_0)). \end{aligned}$$

then

~~Then~~ The the output signal $y(t)$ is given as

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$$y(t) = \text{Re}\left[\tilde{y}(t)e^{j2\pi f_c t}\right] = 4BA_c\text{sinc}(2B(t-t_0))\cos(2\pi f_c t).$$

6. (a) The Fourier transform of the input signal $x_1(t)$ is given by

$$X_1(f) = 4\delta(f) + 6[\frac{1}{2}\delta(f-800k) + \frac{1}{2}\delta(f+800k)] + 2[\frac{1}{2j}\delta(f-950k) + \frac{1}{2j}\delta(f+950k)].$$

Since $f_c = 1000kHz$ and $B = 100kHz$, only the frequency of $900kHz$ to $1100kHz$ can be passed. The function of phase of the band-pass filter is $\beta(f) = -2\pi t_0(f - f_c)$. Therefore, the phase of $f_c = 950kHz$ is π, and the phase of $f_c = -950kHz$ is $-\pi$. The Fourier transform of the output signal $y_1(t)$ is given by

$$\begin{aligned} Y_1(f) &= \frac{1}{j}e^{j\pi}\delta(f-950k) - \frac{1}{j}e^{-j\pi}\delta(f+950k) \\ &= -2[\frac{1}{2j}\delta(f-950k) - \frac{1}{2j}\delta(f+950k)]. \end{aligned}$$

5. (a) Since

$$x(t) = \mathrm{Re}\left[\tilde{x}(t)\, e^{j2\pi f_c t}\right]$$

~~we have~~ and $x(t) = 4A_c W \operatorname{sinc}(2Wt)\cos(2\pi f_c t)$, we have

$$\tilde{x}(t) = 4A_c W \operatorname{sinc}(2Wt).$$

The Fourier transform is then given as

$$\tilde{X}(f) = 2A_c \operatorname{rect}\left(\frac{f}{2W}\right).$$

~~(b) From Problem 4, We have~~ ~~Since~~ ~~From Problem 4, $\tilde{H}(f)=$~~

$$\cancel{\tilde{Y}(f) = \frac{1}{2}\tilde{X}(f)\tilde{H}(f)} =$$

(b) From Problem 4, we have

$$\tilde{H}(f) = 2\operatorname{rect}\left(\frac{f}{2B}\right) e^{-j2\pi f t_0}.$$

Hence

$$\tilde{Y}(f) = \frac{1}{2}\tilde{X}(f)\tilde{H}(f)$$

$$= 2A_c \operatorname{rect}\left(\frac{f}{2W}\right)\operatorname{rect}\left(\frac{f}{2B}\right) e^{-j2\pi f_0}$$

$$= 2A_c \operatorname{rect}\left(\frac{f}{2B}\right) e^{-j2\pi f t_0}$$

since $W > B$.

(c) ~~From (b), we have~~

$$\cancel{\tilde{y}(t) = \mathcal{F}^{-1}\left[\tilde{Y}(f)\right]} =$$

Hence, the output signal $y_1(t)$ is given by

$$\begin{aligned} y_1(t) &= -2\text{sin}(2\pi \cdot 950kt) \\ &= -2\text{sin}(1900000\pi t). \end{aligned}$$

(b) Since $x_2(t) = \text{rect}(\frac{t}{T_0/2}) \star \sum_{m=-\infty}^{\infty} \delta(t - mT_0)$. From class, the Fourier transform of $\sum_{m=-\infty}^{\infty} \delta(t - mT_0)$ is given by

$$\frac{1}{T_0} \sum_{n=-\infty}^{\infty} \delta(f - nf_0)$$

and the Fourier transform of $\text{rect}(t)$ is $\text{sinc}(f)$. Therefore, the Fourier transform of $\text{rect}(\frac{t}{T_0/2})$ is given by

$$\frac{T_0}{2}\text{sinc}(\frac{f}{2/T_0}).$$

Only the frequency of $1000kHz$ to $-1000kHz$ can be passed. Also, the phase of $f_c = 1000kHz$ and $f_c = -1000kHz$ is 0. Hence, The Fourier transform of the output signal $y_2(t)$ is given by

$$\begin{aligned} Y_2(f) &= \frac{T_0}{2}\text{sinc}(\frac{f}{2/T_0}) \cdot \frac{1}{T_0} \sum_{n=-\infty}^{\infty} \delta(f - nf_0) \cdot H(f) \\ &= \frac{1}{2}\text{sinc}(\frac{f}{2/T_0}) \cdot [\delta(f - 10^6) + \delta(f + 10^6)] \\ &= \frac{1}{2}\text{sinc}(\frac{1}{2})[\delta(f - 10^6) + \delta(f + 10^6)]. \end{aligned}$$

Hence, the output signal $y_2(t)$ is given by

$$y_2(t) = \text{sinc}(\frac{1}{2})\text{cos}(2\pi \cdot 10^6 t).$$

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In this problem, the ideal band-pass filter has mid-band frequency 1000 kHz and bandwidth 200 kHz.

6 (a) We have

$$x_1(t) = 4 + 6\cos(1600000\pi t) + 2\sin(1900000\pi t)$$

$$= 4 + 3e^{j1600000\pi t} + 3e^{-j1600000\pi t} + \frac{1}{j}e^{j1900000\pi t} - \frac{1}{j}e^{-j1900000\pi t}.$$

~~The output is then given by~~

$\cancel{y_1(t) =}$

As $H(800000) = 0$ and $H(950000) = e^{j\pi} = -1$, we have the output given by

$$y_1(t) = 4 \cdot H(0) + 3e^{j1600000\pi t} \cdot H(800000)$$

$$+ 3e^{-j1600000\pi t} \cdot H(-800000) + \frac{1}{j}e^{j1900000\pi t} \cdot H(950000)$$

$$- \frac{1}{j}e^{-j1900000\pi t} H(-950000)$$

$$= -\frac{1}{j}\left(e^{j1900000\pi t} - e^{-j1900000\pi t}\right)$$

$$= -2\sin(1900000\pi t).$$

(b) ~~Since We have~~ Since

$$F\left[\mathrm{rect}\left(\frac{t}{T_0/2}\right)\right] = \frac{T_0}{2}\,\mathrm{sinc}\left(fT_0/2\right) \times$$

~~The Fourier~~

we have

$$x_2(t) = \sum_{n=-\infty}^{\infty} \frac{1}{2}\,\mathrm{sinc}(n/2)\, e^{j2n\pi t/T_0}.$$

~~As $1/T_0 =$~~

Then the output is

$$y_2(t) = \sum_{n=-\infty}^{\infty} \frac{1}{2} \, \text{sinc}(n/2) \, H(n/T_0) \, e^{j2n\pi t/T_0}$$

$$= \frac{1}{2}\,\text{sinc}(1/2)\, e^{j2000000\pi t} + \frac{1}{2}\,\text{sin}(-1/2)\, e^{-j2000000\pi t}$$

$$= \frac{2}{\pi} \sin(2000000\pi t)$$

since $1/T_0 = 1000$ kHz and $\text{sinc}(1/2) = 2/\pi$.