Homework II

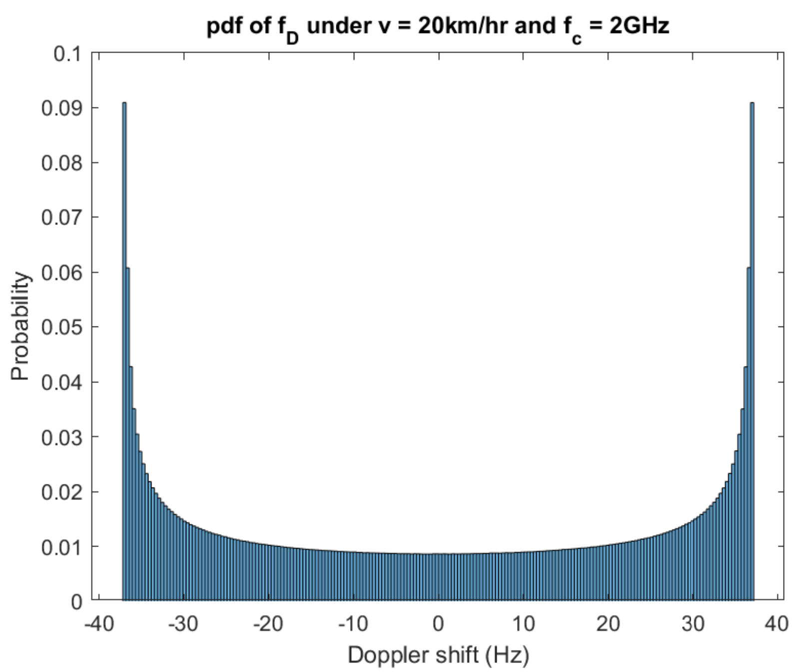
1. Doppler Shift

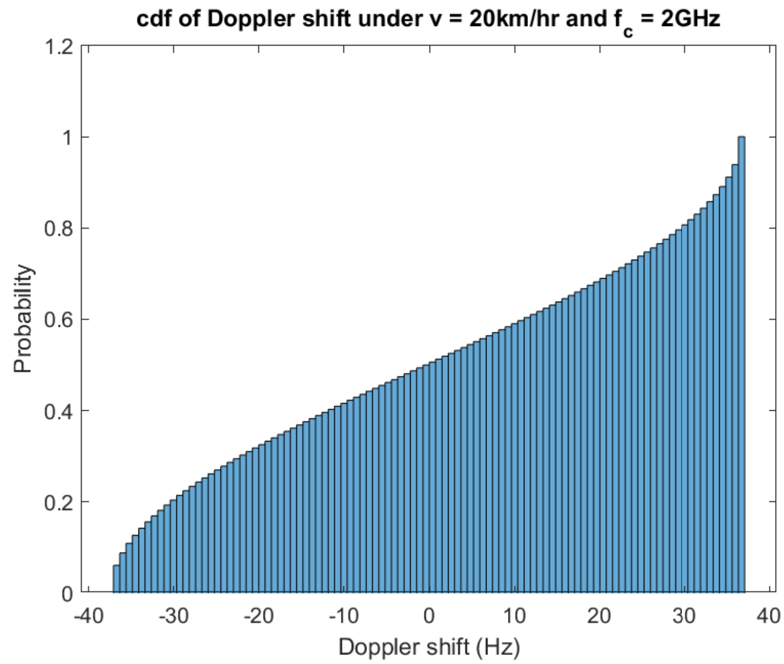
First we generate a large number of phase realization $\{\theta_i\}_{i=1}^{10^8}$ from uniform distribution $\mathcal{U}(-\pi, \pi)$

(a) Given $v = 20$ km/hr and $f_c = 2$GHz, the doppler shift for each θ_i is obtained by

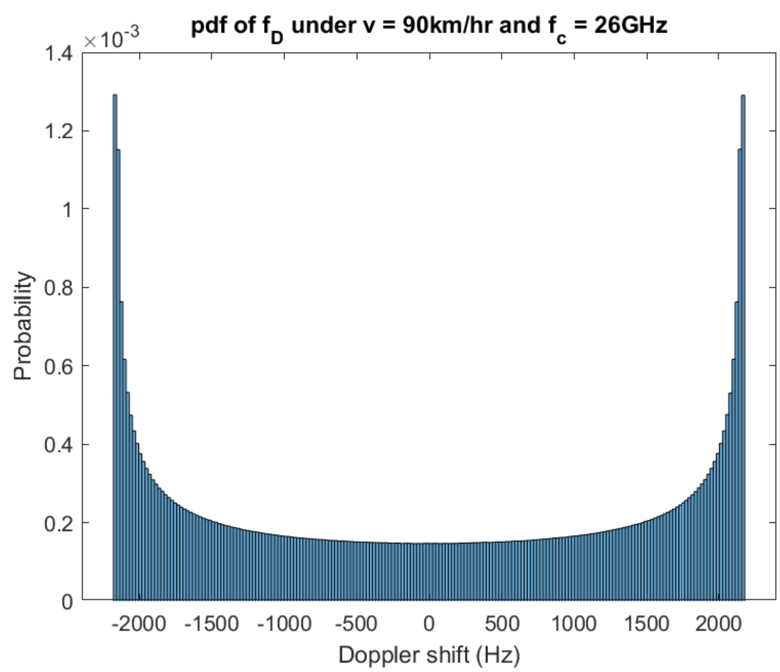
$$\begin{aligned} f_{D,i} &= f_m \cos(\theta_i) \\ &= \frac{v}{\lambda_c} \cos(\theta_i) \\ &= \frac{v}{v_c} f_c \cos(\theta_i) \quad \text{where} \quad v_c = 3 \cdot 10^8\, m/s \end{aligned}$$

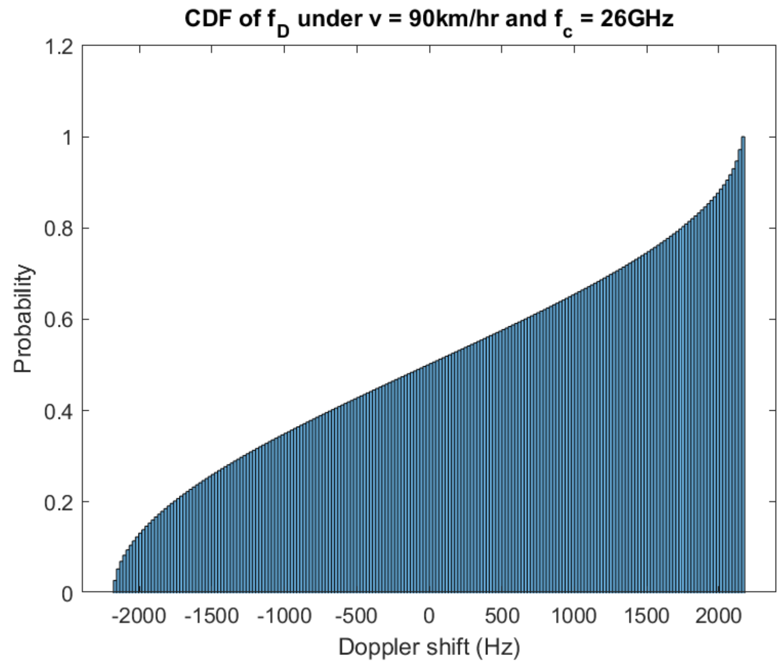
After calculating doppler shift for each θ_i, we can plot the histogram of $\{f_{D,i}\}$ to see its pdf and cdf.





(b) Totally the same as (a), just change v and f_c.

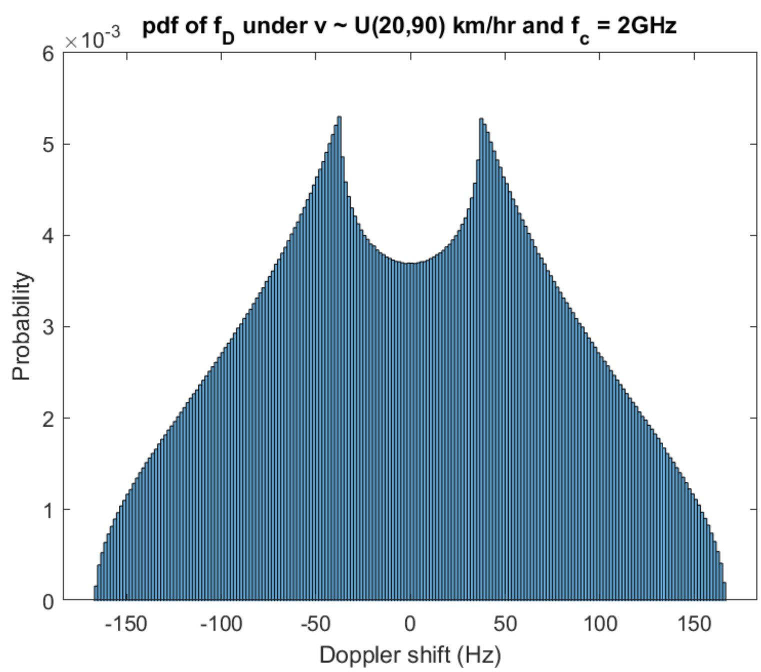


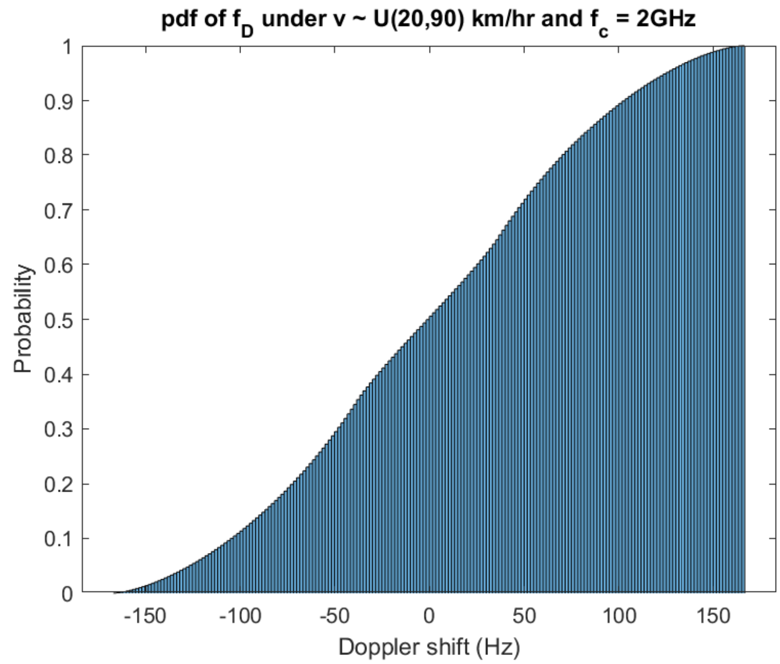


(c) In this part, v also becomes uniform distributed. Hence we generate same number of velocity realization $\{v_i\}_{i=1}^{10^8}$ and use the same formula to calculate doppler shift for each pair (v_i, θ_i).

$$f_{D,i} = \frac{v_i}{v_c} f_c \cos(\theta_i) \quad \text{where} \quad v_c = 3 \cdot 10^8 \, m/s$$

The results are shown below





(d) Now, we try to derive the pdf and cdf of f_D. Given

$$\theta \sim \mathcal{U}(-\pi, \pi) \tag{1}$$
$$f_D = f_m \cos(\theta) \tag{2}$$

For a certain f_D, the corresponding solution for (2) is

$$\theta_1 = \arccos\left(\frac{f_D}{f_m}\right) \quad \text{and} \quad \theta_2 = -\arccos\left(\frac{f_D}{f_m}\right).$$

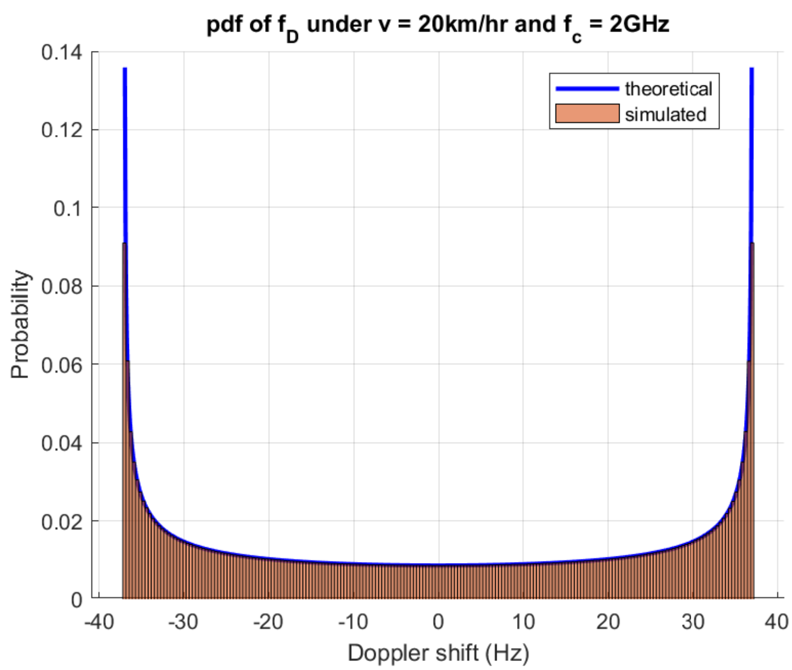
Therefore, the pdf of f_D is then given as

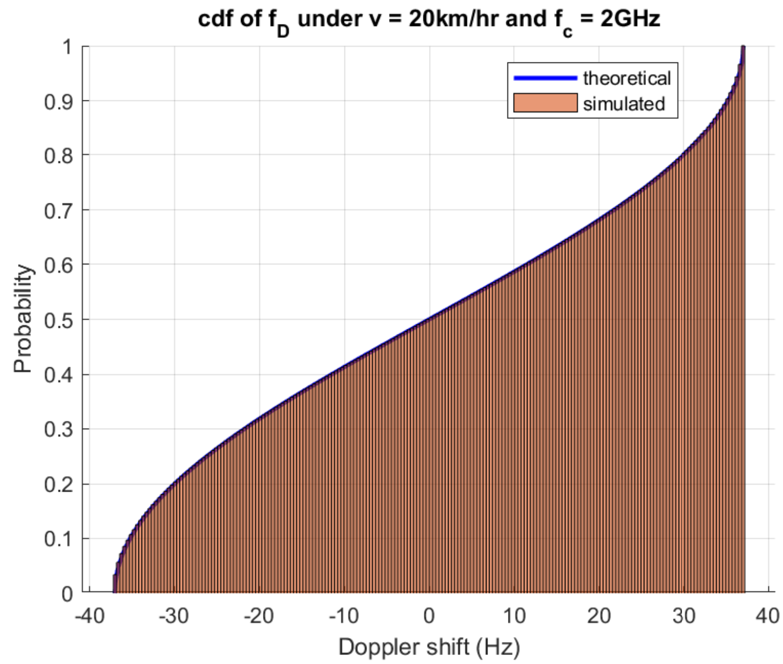
$$\begin{aligned} P(f_D) &= \frac{1/2\pi}{|-f_m \sin(\theta_1)|} + \frac{1/2\pi}{|-f_m \sin(\theta_2)|} \\ &= 2 \cdot \frac{1}{2\pi} \cdot \frac{1}{\sqrt{f_m^2 - f_D^2}} \quad \left(\because \sin\left(\arccos\left(\frac{f_D}{f_m}\right)\right) = \frac{\sqrt{f_m^2 - f_D^2}}{f_m}\right) \\ &= \frac{1}{\pi} \cdot \frac{1}{\sqrt{f_m^2 - f_D^2}} \end{aligned}$$

The cdf of f_D can further be derived

$$
\begin{aligned}
F(f_D) &= \int_{-f_m}^{f_D} P(x)\,dx \\
&= \frac{1}{\pi \cdot f_m} \int_{-f_m}^{f_D} \frac{1}{\sqrt{1-\left(\frac{x}{f_m}\right)^2}}\,dx \\
&= \frac{1}{\pi} \int_{-1}^{\frac{f_D}{f_m}} \frac{1}{\sqrt{1-u^2}}\,du \quad \left(\text{let } u = \frac{x}{f_m}\right) \\
&= \frac{1}{\pi} \arcsin(u) \Bigg|_{-1}^{\frac{f_D}{f_m}} \\
&= \frac{1}{\pi} \arcsin\left(\frac{f_D}{f_m}\right) + \frac{1}{2}
\end{aligned}
$$

Plot the results with MATLAB, we get





The solid line is the theretical result while the histogram is the simulated result. It can be seen that they look similiar.