HW1

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1. Consider a discrete-time random signal $x(n) = A\sin(\omega nT + \theta)$, where A is a constant, ω is a fixed radial frequency, T is a sampling interval, and θ is a phase angle uniformly distributed over $[-\pi, \pi]$. Determine the mean, correlation, and covariance of $x(n)$.

① $$E[X(n)] = \int_{-\pi}^{\pi} \frac{1}{2\pi} A \sin \omega n T + \theta \, d\theta$$

$$= 0$$

② $$R_{xx}(m,n) = E[X(m) X^*(n)]$$

$$= E[A \sin(\omega m T + \theta) A \sin(\omega n T + \theta)]$$

$$= A^2 \cdot \frac{1}{2} E[\cos(\omega(m-n)T) - \cos(\omega(m+n)T + 2\theta)]$$

$$= \frac{A^2}{2} \cos(\omega(m-n)T)$$

② $$C_{xx}(m,n) = R_{xx}(m,n) - E[X(m)] E[X(n)]$$

$$= \frac{A^2}{2} \cos(\omega(m-n)T)$$

2. Let $y(n) = \alpha v^2(n)$, where α is a constant and $v(n)$ is an independent, identically distributed random signal with unit variance. Determine the condition under which $y(n)$ is wide-sense stationary.

① $E[Y(n)] = \alpha E[v^2(n)]$

$$= \alpha \{1 + (E[v(n)])^2\}$$

To make $y(n)$ WSS, $E[v(n)]$ should be constant for all n

② Let $E[v(n)] = C$, which is a constant

$$R_{YY}(m,n) = E[Y(m) Y(n)]$$

$$= \alpha^2 E[v^2(m) v^2(n)]$$

$$= \alpha^2 E[v^2(m)] E[v^2(n)]$$

$$= \alpha^2 [1 + (E[v(m)])^2][1 + (E[v(n)])^2]$$

$$= \alpha^2 (1+C^2)(1+C^2)$$

Therefore, $Y(n)$ is WSS iff $E[v(n)]$ is constant

3. Show that if $r(i)$ is a real and even function of i, then $R(e^{j\omega})$ is a real, even, and non-negative function of ω, as stated on Page 2-6 in the Lecture Notes of Chapter 2.

① $R(e^{j\omega}) = \sum_{n=-\infty}^{\infty} r(n) e^{-j\omega n}$

$= \sum_{n=-\infty}^{\infty} r(-n) e^{j\omega n}$

$= \sum_{n=-\infty}^{\infty} r(n) e^{j\omega n} \quad (\because \; r(n) = r(-n))$

$= R(e^{-j\omega}) \Rightarrow R(e^{j\omega})$ is even

② $R^*(e^{j\omega}) = \left(\sum_{n=-\infty}^{\infty} r(n) e^{-j\omega n} \right)^*$

$= \sum_{n=-\infty}^{\infty} r^*(n) e^{j\omega n}$

$= \sum_{n=-\infty}^{\infty} r(n) e^{j\omega n} = R(e^{-j\omega}) \overset{\text{by ①}}{=} R(e^{j\omega}) \Rightarrow R(e^{j\omega})$ is real

③ Consider the rectangular window

$$w(n) = \begin{cases} 1 & |n| \le N \\ 0 & \text{otherwise} \end{cases}$$

Let $X_N(e^{j\omega}) = F\{ x(n) w(n) \}$, then we have

$$0 \le \frac{1}{2N+1} E\left[|X_N(e^{j\omega})|^2 \right] = \frac{1}{2N+1} E\left[\sum_{k=-N}^{N} \sum_{l=-N}^{N} x(k) x^*(l) e^{-j\omega k} e^{j\omega l} \right]$$

$$= \frac{1}{2N+1} \sum_{k=-N}^{N} \sum_{l=-N}^{N} r(k-l) e^{-j\omega(k-l)}$$

$$\left(\begin{array}{l} m = k-l \Rightarrow k' = l+m \\ \Rightarrow \max(-N, m-N) \le k' \le \min(N, m+N) \end{array} \right) = \frac{1}{2N+1} \sum_{m=-2N}^{2N} (2N+1-|m|)\, r(m) e^{-j\omega m}$$

$$= \sum_{m=-2N}^{2N} \left(1 - \frac{|m|}{2N+1} \right) r(m) e^{-j\omega m}$$

$$\Rightarrow R(e^{j\omega}) = \sum_{m=-\infty}^{\infty} r(m) e^{-j\omega m}$$

$$= \lim_{N\to\infty} \sum_{m=-2N}^{N} r(m) \left(1 - \frac{|m|}{2N+1}\right) e^{-j\omega m}$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} E\left[|X_N(e^{j\omega})|^2 \right] \geq 0$$

4. Show that $|r_{xy}(i)| \leq [r_{xx}(0)\, r_{yy}(0)]^{1/2}$, as stated on Page 2-7 in the Lecture Notes of Chapter 2.

Lemma For any two r.v X and Y we have

$$\left(E[XY^*]\right)^2 \leq E[|X|^2]\, E[|Y|^2]$$

Apply the lemma directly, we get

$$R_{XY}(i) = E[X(n+i)\, Y^*(n)]$$

$$\leq \left[E[X(n+i)\, X^*(n+i)]\right]^{1/2} \left[E[Y(n)\, Y^*(n)]\right]^{1/2}$$

$$= \left[R_{XX}(0)\, R_{YY}(0)\right]^{1/2}$$

pf of Lemma

For any real constant t, we have

$$E[|tX+Y|^2] = t^2 E[|X|^2] + 2t\, E[XY^*] + E[|Y|^2]$$

The left hand side is non-negative. The quadratic equation at^2+bt+c is non-negative iff $a \geq 0$ and $b^2-4ac \leq 0$

$$\left(2E[XY^*]\right)^2 - 4E[|X|^2]\, E[|Y|^2] \leq 0$$

$$\Rightarrow \left(E[XY^*]\right)^2 \leq E[|X|^2]\, E[|Y|^2]$$

5. A random signal $x(n)$ is passed through a linear system with impulse response

$$h(n) = \delta(n) - 2\delta(n-1)$$

(a) Find the cross-correlation function between input and output, $r_{xy}(i)$, if the input is white noise with variance σ_0^2.

(b) Find the correlation function of the output $r_{yy}(i)$.

(c) Find the output power spectral density $R_{yy}(e^{j\omega})$.

(a) $R_{XX}(i) = E[X(n+i)X(n)]$

$$= \begin{cases} \sigma_0^2 & \text{if } i = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$= \sigma_0^2\,\delta(i)$$

$$Y(n) = X(n) \otimes h(n) = X(n) - 2X(n-1)$$

$$R_{XY}(i) = E[X(n+i)\,Y(n)]$$

$$= E[X(n+i)\,X(n)] - 2E[X(n+i)\,X(n-1)]$$

$$= R_{XX}(i) - 2R_{XX}(i+1)$$

$$= [\delta(i) - 2\delta(i+1)]\,\sigma_0^2$$

(b) $R_{YY}(i) = E[Y(n+i)\,Y(n)]$

$$= E[(X(n+i) - 2X(n+i-1))(X(n) - 2X(n-1))]$$

$$= R_{XX}(i) - 2R_{XX}(i-1) - 2R_{XX}(i+1) + 4R_{XX}(i)$$

$$= \sigma_0^2[5\delta(i) - 2\delta(i-1) - 2\delta(i+1)]$$

(c) $S_{XX}(e^{j\omega}) = \sigma_0^2[5 - 2e^{j\omega} - 2e^{-j\omega}]$

$$= \sigma_0^2[5 - 4\cos\omega]$$

6. A random process $x(n)$ consists of independent random variables each with uniform density

$$f(x) = \begin{cases} 1/2, & -1 \le x \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

This process is applied to a linear shift-invariant system with impulse response $h(n) = (1/2)^n u(n)$, where $u(n)$ is the unit step function. Let the output process be denoted by $y(n)$.

(a) Compute the cross-correlation function $r_{yx}(i)$.

(b) Find the correlation function of output $r_{yy}(i)$.

(c) Find the output power spectrum $R_{yy}(e^{j\omega})$.

(a) $$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h(n) e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} (1/2)^n e^{-j\omega n} = \frac{1}{1 - \frac{1}{2} e^{-j\omega}}$$

$$E[X(n)] = \int_{-1}^{1} \frac{1}{2} x \, dx = 0 \qquad \frac{1}{6} x^3 \Big|_{-1}^{1}$$

$$E[X^2(n)] = \int_{-1}^{1} \frac{1}{2} x^2 \, dx = \frac{1}{3}$$

$$\Rightarrow R_{XX}(i) = E[X(n+i) X(n)] = \begin{cases} 0 & \text{if } i \neq 0 \\ 1/3 & \text{if } i = 0 \end{cases} = \frac{1}{3} \delta(i)$$

$$\Rightarrow R_{YX}(i) = R_{XX}(i) \otimes h(-i)$$

$$= \frac{1}{3} (1/2)^{-i} u(-i) = \frac{1}{3} 2^{i} u(-i)$$

(c) First, we find the PSD of Y

$$S_{YY}(e^{j\omega}) = |H(e^{j\omega})|^2 S_{XX}(e^{j\omega})$$

$$= \frac{1/3}{(1 - \frac{1}{2} e^{-j\omega})(1 - \frac{1}{2} e^{j\omega})} = \frac{1/3}{5/4 - \cos\omega}$$

(b) The autocorrelation of Y is given as

$$R_{YY}(i) = F^{-1}\{S_{YY}(e^{j\omega})\}$$

$$= \frac{4}{3} F^{-1}\left\{\frac{1-(\frac{1}{2})^2}{1-2\cdot\frac{1}{2}\cos\omega+(\frac{1}{2})^2}\right\} \cdot \frac{1}{3}$$

$$= \frac{4}{9} \cdot 2^{-|i|}$$