6. Biased vs. Unbiased estimation

(a) We first generate stationary gaussian samples $\{x(n)\}_{n=0}^{M-1}$ with $M = 100$. Then we use the following formula to estimate the autocorrelation for $-M+1 \le m \le M-1$

$$\begin{aligned} r_{unbiased}(m) &= \frac{1}{M-|m|} \sum_{n=0}^{M-|m|-1} x(n)x(n+|m|) \\ r_{biased}(m) &= \frac{1}{M} \sum_{n=0}^{M-|m|-1} x(n)x(n+|m|). \end{aligned}$$

The estimations are shown below

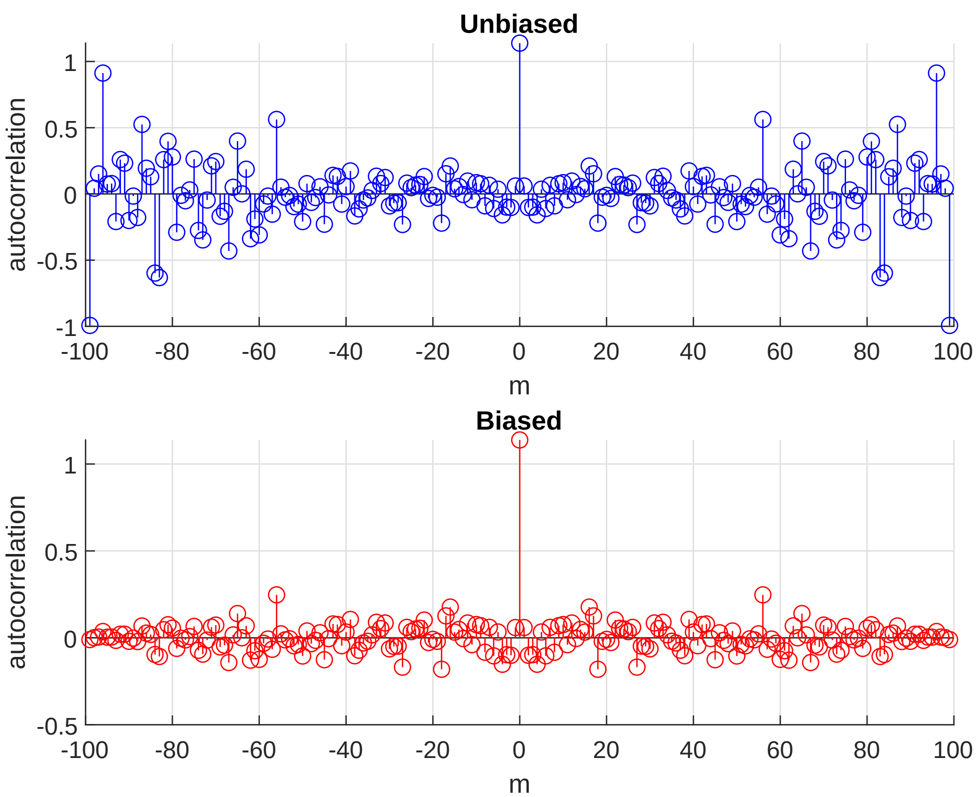
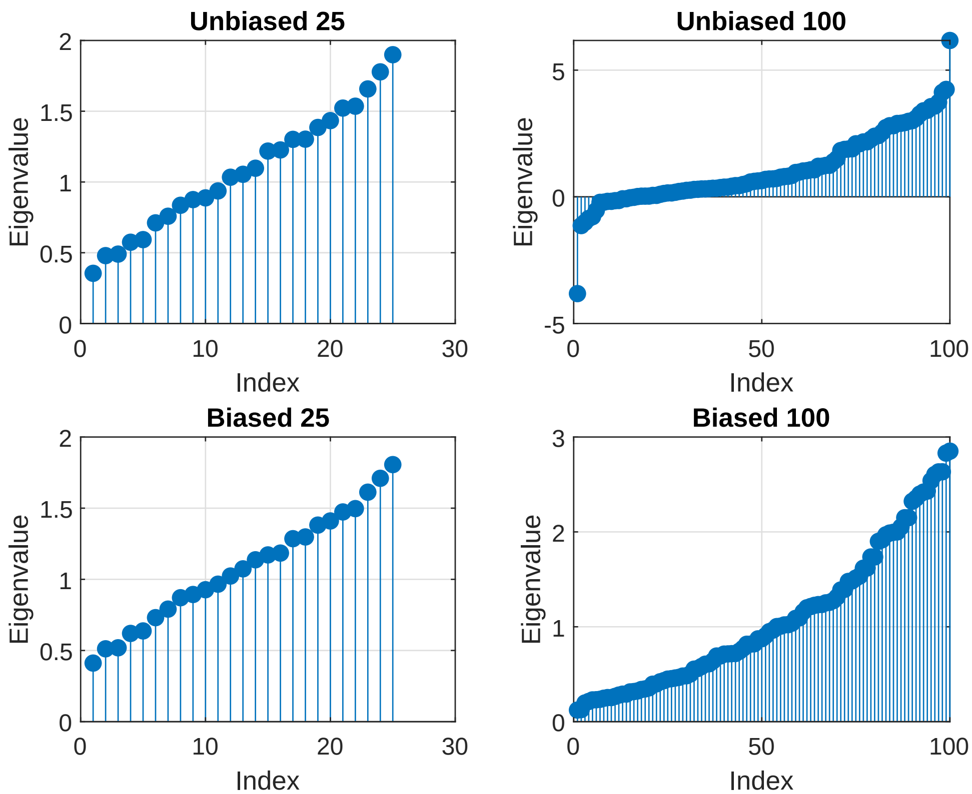


Figure 1. Biased and Unbiased estimation

It can be seen that the unbiased estimator has higher variance for $m > \frac{M}{4} = 25$.

(b) We first find the correlation matrix R_{25} and R_{100} formed by results of biased and unbiased estimator we obtain in (a). Then we check the eigenvalues of each correlation matrix.



It can be seen that for $M = 25$, the correlation matrixs formed by the results of both estimator are PSD (Positive Semi-Definite). For $M = 100$, the correlation matrix formed by the results of unbiased estimator is not PSD since there are negative eigenvalues. In the following, we will prove that the correlation matrix formed by the results of the biased estimator is always PSD.

Consider a random signal $\{x(n)\}_{n=0}^{M-1}$. Suppose we want to estimate a $K \times K$ correlation matrix. Let

$$X = \begin{pmatrix} x[0] & 0 & \dots & 0 \\ x[1] & x[0] & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ x[K-1] & x[K-2] & \dots & x[0] \\ \vdots & \vdots & \ddots & \vdots \\ x[M-1] & x[M-2] & \dots & x[M-P] \\ 0 & x[M-1] & \dots & x[M-P-1] \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & [M-1] \end{pmatrix}.$$

Then the correlation matrix of biased estimator is given by

$$R_P = \frac{1}{M} X^H X$$

It can be seen that

$$R(k,l) = \frac{1}{M} \sum_{n=0}^{M-|k-l|-1} x(n)x(n+|k-l|) = r_{biased}(|k-l|)$$

Therefore, R is a Toeplitz matrix formed by $\{r_{biased}(m)\}_{m=0}^{M-1}$. For any vector a, we have

$$a^H R a = \frac{1}{M} a^H X^H X a = \frac{1}{M} \|Xa\|^2 \geq 0$$

Therefore, the correlation matrix formed by the results of the biased estimator is always PSD.