原 机器人运动学中李代数se(3)与李群SE(3)的基本概念与联系

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1.Three-dimensional Euclidean space

1.three-dimensional Euclidean space can be represented by a Cartesian coordinate frame:

$p \in \mathbb{E}^3$ can be identied with a point in $\mathbb{R}^3$ by three coordinates : $[X, Y, Z]^T$

2. $\mathbb{E}^3$ use vector to define the metric and define inner product to measure distances and angle between two point:

$$\|v\| = \sqrt{\langle v, v \rangle} \qquad \|v\| = \sqrt{v_1^2 + v_2^2 + v_3^2}$$

$$\langle u, v \rangle = u^T v \qquad \langle u, v \rangle = u^T v = u_1 v_1 + u_2 v_2 + u_3 v_3$$

3.Corss product:

$$u \times v = \begin{bmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{bmatrix} \in \mathbb{R}^3$$

3.Cross product: orthogonal to each of its facter

$$u \times v = \begin{bmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{bmatrix} \in \mathbb{R}^3 \quad \langle u \times v, u\rangle = \langle u \times v, v\rangle = 0, \quad u \times v = -v \times u$$

interpretion: fix u, **the cross product is as a map:** $v \mapsto u \times v$
to linear map between vector spaces, represented by matrix: $\widehat{u} \in \mathbb{R}^{3\times 3}$

$$\widehat{u} = \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix} \in \mathbb{R}^{3\times 3} \qquad u \times v = \widehat{u}v$$

skew-symmetric, confirm with right-hand rule

2.Rigid body motion

1.Rigid body displacement :

$$\begin{aligned} g : \mathbb{R}^3 &\to \mathbb{R}^3 \\ \mathrm{X} &\mapsto g(\mathrm{X}) \end{aligned}$$

g **also induces a transformation on vectors:**

$$g_*(v) \doteq g(\mathrm{Y}) - g(\mathrm{X})$$

preserve the norm of vectors , their cross product and inner product:

$$\textit{Norm: } \|g_*(v)\| = \|v\|, \ \forall v \in \mathbb{R}^3.$$

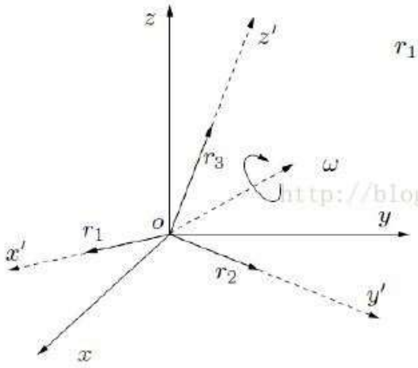
$$\textit{Cross product: } g_*(u) \times g_*(v) = g_*(u \times v), \ \forall u, v \in \mathbb{R}^3$$

$$u^Tv = g_*(u)^T g_*(v), \quad \forall u, v \in \mathbb{R}^3$$

the set of rigid body motions , or special Euclidean transformation is a (Lie) group, denoted by SE(3).

3. Rotation motion and its representations

1.Rotation of a rigid body about a fixed point o:



$$r_1 = g_*(e_1), r_2 = g_*(e_2), r_3 = g_*(e_3) \in \mathbb{R}^3$$

$$R_{wc} = [r_1, r_2, r_3] \in \mathbb{R}^{3\times 3}$$

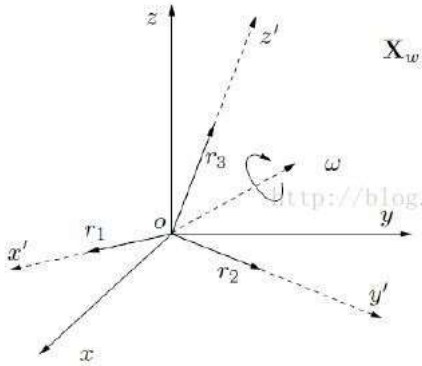
$$R_{wc}^T R_{wc} = R_{wc} R_{wc}^T = I.$$

R_{wc} **is an orthogonal matrix**

det $R_{wc} = 1$

such special orthogonal matrix can be denoted by:

$$\boxed{SO(3) = \{R \in \mathbb{R}^{3\times 3} \mid R^T R = I, \det(R) = +1\}}.$$



$$\mathbf{X}_w = X_{1c} r_1 + X_{2c} r_2 + X_{3c} r_3 = R_{wc} \mathbf{X}_c$$

$$\mathbf{X}_c = R_{wc}^{-1} \mathbf{X}_w = R_{wc}^T \mathbf{X}_w$$

$$\mathbf{X}_c(t) = R_{cw}(t) \mathbf{X}_w$$

$$\mathbf{X}_w(t) = R_{wc}(t) \mathbf{X}_c$$

Canonical exponential coordinates:

a rotational rigid body motion in $\mathbb{E}^3$ **can be represented by 3x3 rotation matrix** $R \in SO(3)$ **. 9 entries, 6 constraints, dimension is 3, 6 redundant. give a few more parameterizations for rotation matrix:**

$$R(t)R^T(t) = I$$

$$\dot{R}(t)R^T(t) + R(t)\dot{R}^T(t) = 0 \quad \Rightarrow \quad \dot{R}(t)R^T(t) = -(\dot{R}(t)R^T(t))^T$$

$$\widehat{\omega}(t) = \dot{R}(t)R^T(t) \qquad \dot{R}(t) = \widehat{\omega}(t)R(t)$$

$$R(t_0 + dt) \approx I + \widehat{\omega}(t_0)\, dt$$

$$\boxed{so(3) = \{\widehat{\omega} \in \mathbb{R}^{3\times 3} \mid \omega \in \mathbb{R}^3\}}$$

$$\dot{R}(t) = \widehat{\omega} R(t)$$

$$\dot{x}(t) = \widehat{\omega} x(t), \quad x(t) \in \mathbb{R}^3. \qquad x(t) = e^{\widehat{\omega} t} x(0)$$

$$e^{\widehat{\omega} t} = I + \widehat{\omega} t + \frac{(\widehat{\omega} t)^2}{2!} + \cdots + \frac{(\widehat{\omega} t)^n}{n!} + \cdots$$

$$\boxed{R(t) = e^{\widehat{\omega} t}}$$

$$\boxed{R(t) = e^{\widehat{\omega}t}}$$

$$(e^{\widehat{\omega}t})^T e^{\widehat{\omega}t} = I \qquad \det(e^{\widehat{\omega}t}) = +1$$

A physical interpretation of the equation is: if $\|\omega\| = 1$, then $R(t) = e^{\widehat{\omega}t}$ is simply a rotation around the axis $\omega \in \mathbb{R}^3$ by t radians

the exponential map:

$$\begin{aligned} \exp: so(3) &\rightarrow SO(3) \\ \widehat{\omega} \in so(3) &\mapsto e^{\widehat{\omega}} \in SO(3) \end{aligned}$$

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \quad \boxed{t = \cos^{-1}\left(\frac{\text{trace}(R) - 1}{2}\right), \quad \omega = \frac{1}{2\sin(t)} \begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix}}$$

How to compute the rotation matrix given (ω, t) ? $\boxed{R(t) = e^{\widehat{\omega}t}}$

Use Rodrigues' formula for simplicity, given $\|\omega\| = 1$ and $t \in \mathbb{R}$

$$\boxed{e^{\widehat{\omega}t} = I + \widehat{\omega}\sin(t) + \widehat{\omega}^2(1 - \cos(t))}$$

Quanternions

$$q = q_0 + q_1 i + (q_2 + i q_3) j = q_0 + q_1 i + q_2 j + q_3 ij, \quad q_0, q_1, q_2, q_3 \in \mathbb{R}.$$

Consider a unit quaternions:

$$\mathbb{S}^3 = \{q \in \mathbb{H} \mid \|q\|^2 = q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1\}$$

We associate it to rotation matrix $R = e^{\widehat{\omega}t}$ with $\omega = [\omega_1, \omega_2, \omega_3]^T \in \mathbb{R}^3$ and $t \in \mathbb{R}$

$$q(R) = \cos(t/2) + \sin(t/2)(\omega_1 i + \omega_2 j + \omega_3 ij) \in \mathbb{S}^3$$

$$t = 2\arccos(q_0), \quad \omega_m = \begin{cases} q_m/\sin(t/2), & t \neq 0 \\ 0, & t = 0 \end{cases}, \quad m = 1, 2, 3$$

4. Rigid body motion and its representations

$$\mathbf{X}_w = R_{wc}\mathbf{X}_c + T_{wc}$$

denote the full rigid motion as $g_{wc} = (R_{wc}, T_{wc})$

$$\mathbf{X}_w = g_{wc}(\mathbf{X}_c)$$

Set of all possible congurations of rigid body can then be described as

$$\boxed{SE(3) = \{g = (R,T) \mid R \in SO(3), T \in \mathbb{R}^3\} = SO(3) \times \mathbb{R}^3}$$

$g = (R,T)$ is not yet a matrix representation for the group SE(3) , so Introduce Homogeneous representation:

$$\bar{\mathbf{X}}_w = \begin{bmatrix} \mathbf{X}_w \\ 1 \end{bmatrix} = \begin{bmatrix} R_{wc} & T_{wc} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{X}_c \\ 1 \end{bmatrix} =: \bar{g}_{wc}\bar{\mathbf{X}}_c$$

$$\boxed{SE(3) = \left\{ \bar{g} = \begin{bmatrix} R & T \\ 0 & 1 \end{bmatrix} \middle| R \in SO(3), T \in \mathbb{R}^3 \right\} \subset \mathbb{R}^{4\times 4}}$$

Canonical exponential coordinates

$$g(t) = \begin{bmatrix} R(t) & T(t) \\ 0 & 1 \end{bmatrix} \in \mathbb{R}^{4\times 4}$$

$$\dot{g}(t)g^{-1}(t) = \begin{bmatrix} \dot{R}(t)R^T(t) & \dot{T}(t) - \dot{R}(t)R^T(t)T(t) \\ 0 & 0 \end{bmatrix}$$

$\dot{R}(t)R^T(t)$ skew symmetric, define $\widehat{\omega}(t) = \dot{R}(t)R^T(t)$ and $v(t) = \dot{T}(t) - \widehat{\omega}(t)T(t)$

$$\dot{g}(t)g^{-1}(t) = \begin{bmatrix} \widehat{\omega}(t) & v(t) \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{4\times 4} \qquad \widehat{\xi}(t) = \begin{bmatrix} \widehat{\omega}(t) & v(t) \\ 0 & 0 \end{bmatrix}$$

$$\dot{g}(t) = (\dot{g}(t)g^{-1}(t))g(t) = \widehat{\xi}(t)g(t)$$

$\widehat{\xi}$ is the tangent vector of $g(t)$, used for approximate $g(t)$ locally:

$$g(t+dt) \approx g(t) + \widehat{\xi}(t)g(t)dt = \left(I + \widehat{\xi}(t)dt\right) g(t)$$

In robotics literature a 4x4 matrix of the form as $\widehat{\xi}$ is called a twist. The set of all twist is then denoted as:

$$\boxed{se(3) = \left\{ \widehat{\xi} = \begin{bmatrix} \widehat{\omega} & v \\ 0 & 0 \end{bmatrix} \middle| \widehat{\omega} \in so(3), v \in \mathbb{R}^3 \right\} \subset \mathbb{R}^{4\times 4}}$$

In the twist coordinates , we will refer to v as the linear velocity and ω as the angular velocity

Solve the ODE $\dot{g}(t)=\widehat{\xi}g(t)$ $\quad g(t)=e^{\widehat{\xi}t}g(0)$

$$\boxed{g(t)=e^{\widehat{\xi}t}}$$

$$e^{\widehat{\xi}t}=I+\widehat{\xi}t+\frac{(\widehat{\xi}t)^2}{2!}+\cdots+\frac{(\widehat{\xi}t)^n}{n!}+\cdots$$

Using Rodrigues' formula: $\boxed{e^{\widehat{\omega}t}=I+\widehat{\omega}\sin(t)+\widehat{\omega}^2(1-\cos(t))}$

$$\boxed{e^{\widehat{\xi}t}=\begin{bmatrix} e^{\widehat{\omega}t} & (I-e^{\widehat{\omega}t})\widehat{\omega}v+\omega\omega^T vt \\ 0 & 1\end{bmatrix}}$$

The exponential map from se(3) to SE(3):

$$\begin{aligned} \exp:\ se(3) &\rightarrow SE(3) \\ \widehat{\xi}\in se(3) &\mapsto e^{\widehat{\xi}}\in SE(3)\end{aligned}$$

5. Summery

Two most commonly used representation of elements of $g\in SE(3)$ are:

1. Homogeneous representation:

$$\bar{g}=\begin{bmatrix} R & T \\ 0 & 1\end{bmatrix}\in\mathbb{R}^{4\times 4} \text{ with } R\in SO(3) \text{ and } T\in\mathbb{R}^3$$

2. Twist representation:

$$g(t)=e^{\widehat{\xi}t} \text{ with the twist coordinates } \xi=(v,\omega)\in\mathbb{R}^6 \text{ and } t\in\mathbb{R}$$