Quadboost: a Scalable Concurrent Quadtree

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Abstract—Building concurrent spatial trees is more complicated than binary search trees since a space hierarchy should be preserved during modifications. We present a non-blocking quadtree–*quadboost*–that supports concurrent insert, remove, move, and contain operations. To increase its concurrency, we propose a decoupling approach that separates the physical adjustment from the logical removal within the remove operation. Besides, we design a continuous find mechanism to reduce its search cost. The move operation combines the searches for different keys together and modifies different positions with atomicity. The experimental results show that quadboost scales well on a multi-core system with 32 hardware threads. More than that, it outperforms existing concurrent trees in retrieving two-dimensional keys with up to 109% improvement when the number of threads is large. The move operation performs better than the best-known algorithm in most cases, with up to 47%.

Index Terms—Computer Society, IEEE, IEEEtran, journal, LATEX, paper, template.

1 Introduction

MULTI-CORE processors have been the universal computing engine in computer systems. Therefore, it is urgent to develop data structures that provide an efficient and scalable multi-thread execution. At present, concurrent data structures [1] such as stacks, linked-lists, queues have been extensively investigated. As a fundamental building block of many parallel programs, these concurrent data structures provide significant performance benefits [2], [3].

Recently, research on concurrent trees has been focusing on binary search trees (BSTs) [4], [5], [6], [7], [8], [9], [10], which are at the heart of many tree-based algorithms. The concurrent paradigms of BSTs were extended to design concurrent spatial trees like R-Tree [11], [12]. However, there remains another unaddressed spatial tree–quadtree, which is widely used in applications for multi-dimensional data. For instance, spatial databases, like PostGIS [13], adopt octree, a three-dimensional variant of quadtree, to build spatial indexes. Video games apply quadtree to handle collision detection [14]. In image processing [15], quadtree is used to decompose pictures into separate regions.

There are different categories of quadtree according to the type of data a node represents, where two major types are region quadtree and point quadtree [16]. Point quadtree stores actual keys in each node. It is hard to design concurrent algorithms for point quadtree since an insert operation might involve re-balance issues, and a remove operation needs to re-insert the whole subtree under a removed node. The region quadtree divides a given region into several sub-regions, where internal nodes represent regions and leaf nodes store actual keys. Our work focuses on region quadtree because: (i) The shape of region quadtree is independent of insert/remove operations' order. Hence, we could either avoid complex re-balance rules and devise specific concurrent techniques for it. (ii) Further, region quadtree could be regarded as a typical external tree that we can adjust existing concurrent techniques from BSTs. Therefore, we'll refer to region quadtree as *quadtree* in the following context.

In this paper, we design a non-blocking quadtree, referred to as *quadboost*, that supports concurrent insert, remove, contain, and move operations. Our key contributions are as follows:

- We propose the first non-blocking quadtree. It records traversal paths, compresses *Empty* nodes if necessary, adopts a decoupling technique to increase the concurrency, and devises a continuous find mechanism to reduce the cost of retries induced by `CAS` failures.
- We design a lowest common ancestor (LCA) based move operation, which traverses a common path for two different keys and modifies two distinct nodes with atomicity.
- We prove the correctness of quadboost algorithms and evaluate them on a multi-core system. The experiments demonstrate that quadboost is highly efficient for concurrent updating at different contention levels.

The rest of this paper is organized as follows. In Section 2, we overview some basic operations. Section 3 describes a simple `CAS` quadtree to motivate this work. Section 4 provides detailed algorithms for quadboost. We provide a sketch of correctness proof in Section 5. Experimental results are discussed in Section 6. Section 7 summarizes related works. Section 8 concludes the paper.

2 Preliminary

Quadtree can be considered as a dictionary for retrieving two-dimensional keys, where $<$*keyX, keyY*$>$ is never duplicated. Figure 1 illustrates a sample quadtree and its corresponding region, where we use numbers to indicate keys. Labels on edges are routing directions–Southwest (sw), Northwest (nw), Southeast (se), and Northeast (ne). The right picture is a mapping of the quadtree on a two-dimensional region, where keys are located according to its coordinate, and regions are divided by their corresponding width (w) and height (h). There are three types of nodes in quadtree, which represent different regions in the right figure. *Internal* nodes are circles on the left figure, and each of them has four children which indicate four equal sub-regions on different directions. The root node is an *Internal* node, and it is the largest region. *Leaf* nodes and *Empty* nodes are located at the terminal of quadtree; they indicate the smallest regions on the right figure. *Leaf* nodes are solid rectangles that store keys; they represent regions with the same numbers on the right figure. *Empty* nodes are dashed rectangles without any key.

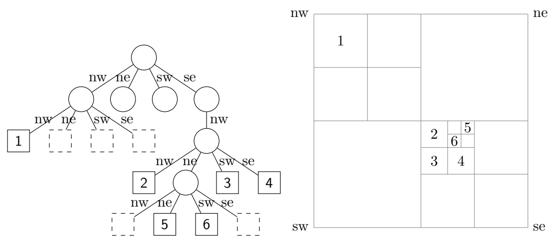


Fig. 1: A sample quadtree and its corresponding region

We describe the detailed structures of quadtree in figure 2. An *Internal* node maintains its four children and a two-dimensional routing structure–<*x, y, w, h*>, where <*x, y*> stands for the upper left coordinate and <*w, h*> are the width and height of the region. A *Leaf* node contains a key <*keyX, keyY*> and its corresponding *value*. An *Empty* node does not have any field. To avoid some corner cases, we initially split the root node and its children to form two layers of dummy *Internal* nodes with a layer of *Empty* nodes at the terminal. Also, we present routing functions *find* and *getQuadrant* in the figure. For instance, if we have to locate key 1 in Figure 1, we start with the root node and compare key 1 with its routing structures by *getQuadrant*. Then we reach its *nw* and perform a comparison again. In the end, we find the terminal node that contains key 1.

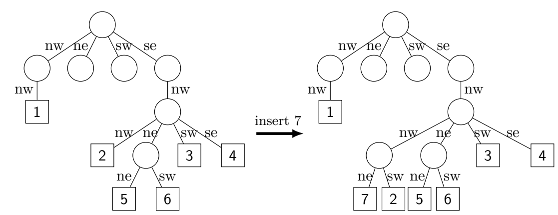
```
class Node<V> {}

class Internal<V> extends Node<V> {
    final double x, y, w, h;
    Node nw, ne, sw, se;
}

class Leaf<V> extends Node<V> {
    final double keyX, keyY;
    final V value;
}

class Empty<V> extends Node<V> {}

void find(Node& l, double keyX, double keyY) {
    while (l.class() == Internal) {
        getQuadrant(l, keyX, keyY);
    }
}

void getQuadrant(Node& l, double keyX, double keyY) {
    if (keyX < l.x + l.w / 2) {
        if (keyY < l.y + l.h / 2) l = l.nw;
        else l = l.sw;
    } else {
        if (keyY < l.y + l.h / 2) l = l.ne;
        else l = l.se;
    }
}
```

Fig. 2: Quadtree nodes and routing functions

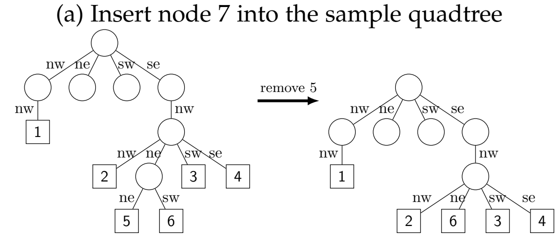
There are four basic operations that rely on *find* for quadtree:

- *insert(key, value)* adds a node that contains *key* and *value* into quadtree.
- *remove(key)* deletes an existing node with *key* from quadtree.
- *contain(key)* checks whether *key* is in quadtree.
- *move(oldKey, newKey)* replaces an existing node with *oldKey* and *value* by a node with *newKey* and *value* that is not in quadtree.

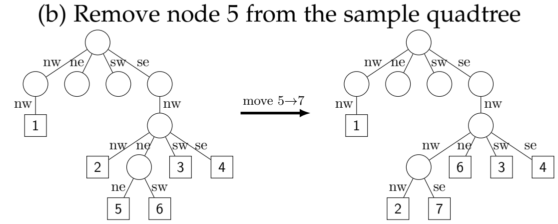
To insert a node, we first locate it's position by calling *find*. If we find an *Empty* node at the terminal, we directly replace it by the node. Otherwise, we recursively divide a *Leaf* node into four corresponding sub-regions until the candidate region contains no *Leaf* node. Figure 3a illustrates a scenario of inserting node 7 (*insert(7)*) as a neighborhood of node 2. The parent node is split, and node 7 is added on the *ne* direction. Likewise, to remove a node, we also begin by locating it and then erase it from quadtree. In the next, we check whether its parent contains a single *Leaf* node. If so, we record the node and traverse up until reaching a node that contains at least a *Leaf* node or the child of the root node. Finally, we use that node as a new parent and re-insert the recorded node as its child. Take Figure 3b as an example, if we remove node 5 (*remove(5)*) from the quadtree, node 6 will be linked to the upper level. The move operation is a combination of the insert and move operation. It first removes the node with *oldKey* and then adds a new node with *newKey* into quadtree. Consider the scenario in Figure 3c, after removing node 5 and inserting node 7 with the same value (*move(1, 7)*), the new tree appears on the right part. Because *contain* function just checks whether the node returned by *find* has the same *key*, it does not need extra explanation.



(a) Insert node 7 into the sample quadtree



(b) Remove node 5 from the sample quadtree



(c) Move the value of node 5 to node 7 from the sample quadtree

Fig. 3: Sample quadtree operations[1]

3 CAS QUADTREE

There are a plenty of concurrent tree algorithms, yet a formal concurrent quadtree algorithm has not been studied. Intuitively, we can devise concurrent quadtree by modifying the

1. *Empty* nodes are not drawn

```
25 bool contain(double keyX, double keyY) {
26     Node p, l = root;
27     // l: terminal node for retrieving <keyX, keyY>
28     // p: parent of l
29     find(p, l, keyX, keyY);
30     if (inTree(l, keyX, keyY)) return true;
31     return false;
32 }

33 bool insert(double keyX, double keyY, V value) {
34     while (true) {
35         Node p, l = root;
36         find(p, l, keyX, keyY);
37         if (inTree(l, keyX, keyY)) return false;
38         Node newNode = createNode(l, p, keyX, keyY, value);  // creates a
               sub-tree by split l until the candidate region of <keyX, keyY> is
               not a Leaf node and returns the sub-tree's root node.
39         if (helpReplace(p, l, newNode)) return true;
40     }
41 }

42 bool remove(double keyX, double keyY, V value) {
43     Node newNode = new Empty();
44     while (true) {
45         Node p, l = root;
46         find(p, l, keyX, keyY);
47         if (!inTree(l, keyX, keyY)) return false;
48         if (helpReplace(p, l, newNode)) return true;
49     }
50 }

51 void find(Node& p, Node& l, double keyX, double keyY) {
52     while (l.class() == Internal) {
53         p = l;  // record the parent node
54         getQuadrant(l, keyX, keyY);
55     }
56 }

57 bool helpReplace(Internal p, Node oldChild, Node newChild) {
58     if (p.nw == oldChild) return CAS(p.nw, oldChild, newChild);
59     else if (p.ne == oldChild) return CAS(p.ne, oldChild, newChild);
60     else if (p.sw == oldChild) return CAS(p.sw, oldChild, newChild);
61     else if (p.se == oldChild) return CAS(p.se, oldChild, newChild);
62     return false;
63 }

64 bool inTree(Node node, double keyX, double keyY) {
65     return node.class == Leaf && node.keyX == keyX && node.keyY == keyY;
66 }
```

Fig. 4: CAS quadtree algorithm

sequential algorithm described in Section 2 and adopting CAS instruction, which we name CAS quadtree. Figure 4 depicts the CAS quadtree algorithm. Both the insert operation and the remove operation follow such a paradigm: it starts by locating a terminal node. Then, it checks whether the node satisfies some conditions. If conditions are not satisfied, it returns false. Otherwise, it tries to apply a single CAS to replace the terminal node by a new node. After a successful CAS, it returns true. Or else it restarts locating a terminal node from the root node. For the insert operation, if <keyX, keyY> is not in the tree, it creates a corresponding sub-tree that contains <keyX, keyY> and $value$ (line 38). Then it uses the root of the sub-tree to replace the terminal node by a single CAS (line 39). To remove a node, it creates an *Empty* node at the beginning (line 43). If <keyX, keyY> is in the tree, it uses the *Empty* node to replace the terminal node (line 48). Different with the sequential algorithm, we do not adjust the structure as it involves several steps that cannot be implemented with atomicity. The algorithm is non-blocking. In other words, the algorithm provides a whole progress guarantee even if some threads starve.

However, the CAS quadtree algorithm has several limitations. First, consider if there are a considerable proportion of remove operations. By applying the simple mechanism, we still have a large number of nodes in the quadtree because we substitute existing nodes with *Empty* nodes without structural adjustment. Figure 5 illustrates a detailed example to show that there remains a chain of *Empty* nodes. Hence, not only we have to traverse a long path to locate the terminal node for basic operations, but also plenty of nodes

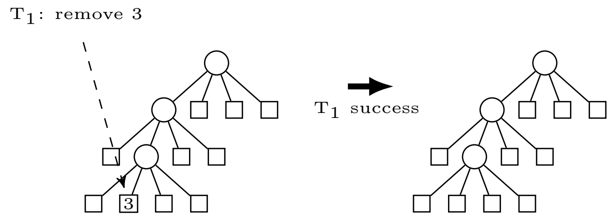


Fig. 5: Thread T_1 intends to remove node 3 from quadtree. After its removal, there remains a chain of *Empty* nodes

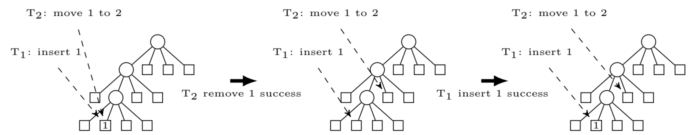


Fig. 6: An example of the incorrect move operation. Thread T_1 intends to insert node 1 into a quadtree, and thread T_2 plans to move the value from node 1 to node 2. T_2 first successfully removes node 1 and then attempts to insert node 2 into the quadtree. In the interval node 1 is added back by T_1, but T_2 is not aware of the action and reports success.

are maintained in the memory.

Second, we cannot implement the move operation in the simple algorithm. There might be two different nodes in a quadtree were under modifying. If we apply the remove operation to erase the node with *oldKey*, following the insert operation to add the node with *newKey*, the move operation cannot be correctly linearized. Figure 6 shows an incorrect scenario of the simple move operation by combining the insert operation and the remove operation.

Therefore, these drawbacks motivate us to develop a new concurrent algorithm to make the move operation correct and employ an efficient mechanism to compress nodes.

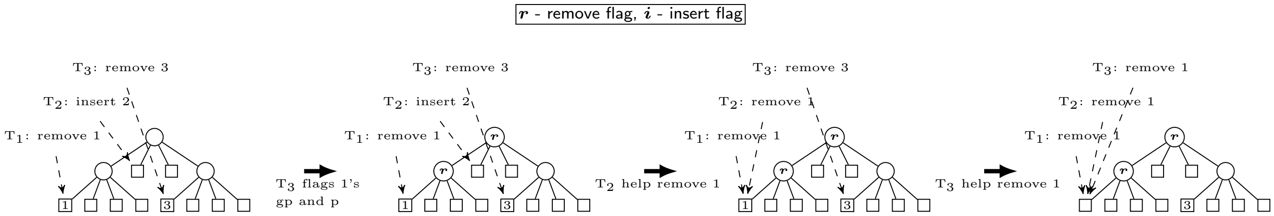
4 QUADBOOST

4.1 Rationale

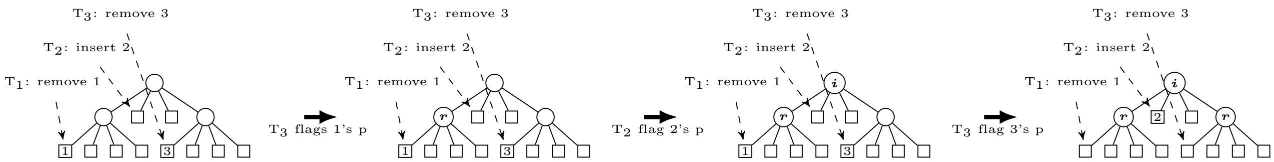
In this section, we describe how to design the *quadboost* algorithm to solve the two problems addressed in Section 3.

To make the move operation correct, we should ensure that other threads know whether a terminal node is under moving. Hence, we attach an internal node with a separate object–*Operation* (*op*) to represent a node's state and record sufficient information to complete the operation. We instantiate the attachment behavior as a CAS and call it the *flag* operation. We design different *op*s for insert, remove and move operations. The detailed description of structures and a state transition mechanism is presented in Section 4.2.

To erase *Empty* nodes from quadtree, we can apply a similar paradigm in concurrent BST's removal [4] as shown in Figure 7a, which flags both the parent and the grandparent of a terminal node. This mechanism mixes logical removal and physical removal together. Different with BST's removal that every time the parent node has to be adjusted, we only have to compress the parent when there is only a single *Leaf* node. Hence, we could separate the removal of a node and



(a) Apply traditional removal



(b) Our decoupling approach

Fig. 7: At the beginning, three threads are performing different operations concurrently: (a) T_1 removes key 1 in the lower level. (b) T_2 inserts key 2 in the upper level. (c) T_3 removes key 3 in the lower level. Consider the scenario that T_1 precedes others threads, and then both 1's parent and grandparent will be flagged. Hence, T_2 and T_3 will help T_1 remove key 1 before restarting their operations. By applying our decoupling method, only the parent node should be flagged. Hence three threads could run without intervention.

the adjustment of the structure into two phases. Meanwhile, we design an *op–Compress* to indicate a node is underlying the structural adjustment. Figure 7b shows how this method increases concurrency; three threads that handle different *op*s could run in parallel. Further, for simplicity, we relax the adjustment condition to that if all children are *Empty*, the parent could be compressed.

There's still a problem left after applying the above two methods. Recall the example in Figure 5. We shall flag the bottom *Internal* node to indicate that it should be compressed. But after replacing the bottom *Internal* node with an *Empty* node, it results in four *Empty* nodes in the last level. How to remove a series of nodes from quadtree in a bottom-up way? We record the entire traversal path from the root to a terminal node in a stack. Besides, since the traversal path will be altered when a node is compressed, we only have to restart locating the terminal node from the parent node if any flag operation other than the flag of *Compress* fails. It's called the continuous find mechanism.

4.2 Structures and State Transitions

As mentioned before, we add an *Operation* object to handle concurrency issues. Figure 8 shows the data structure of *quadboost*. Four sub-classes of *Operation*, including *Substitute*, *Compress*, *Move*, and *Clean*, describe all states in our algorithm. *Substitute* provides information on the insert operation and the remove operation that are designed to replace an existing node by a new node. Hence, we shall let other threads be aware of its parent, child, and a new node for substituting. *Compress* provides information on quadtree's physical adjustment. We erase the parent node, previously connected by the grandparent, by swinging the link to an *Empty* node. *Move* stores both *oldKey*'s and *newKey*'s terminal nodes, their parents, their parents' prior *op*s, and a new node. Moreover, we use a bool variable–*allFlag* to indicate whether two parents have been attached on a *Move op* or not. Another bool variable–*iFirst* is used to indicate the attaching order. For instance, if *iFirst* is true, *iParent* will be attached with a *Move op* before *rParent*. *Clean* means there's no thread modifying the node. In contrast to Figure 4, the *Internal* class adds an *op* field to hold its state and related information. Note that the flag operation is only applied on *Internal* nodes. We atomically set *Move op*s in *Leaf* nodes to linearize the move operation correctly.

```
class Node<V> {}

class Internal<V> extends Node<V> {
    final double x, y, w, h;
    volatile Node nw, ne, sw, se;
    volatile Operation op = new Clean();
}

class Leaf<V> extends Node<V> {
    final double keyX, keyY;
    final V value;
    volatile Move op;
}

class Empty<V> extends Node<V> {}

class Operation {}

class Substitute extends Operation {
    Internal parent;
    Node oldChild, newNode;
}

class Compress extends Operation {
    Internal grandparent, parent;
}

class Move extends Operation {
    Internal iParent, rParent;
    Node oldIChild, oldRChild, newIChild;
    Operation oldIOp, oldROp;
    volatile bool allFlag = false, iFirst = false;
}

class Clean extends Operation {}
```

Fig. 8: Quadtree structures

Each basic operation, except for the contain operation, starts by changing an *Internal* node's *op* from *Clean* to other states. The three basic operations, therefore, generate a corresponding state transition diagram which provides a high-level description of our algorithm. After locating a terminal node, insert, remove, compress, and move transitions execute as shown in Figure 9. In the figure, we specify the flag operation that restores a *op* to *Clean* as *unflag*, the flag operation that changes a *op* from *Clean* to *Substitute* in *insert* as *iflag*, the flag operation that changes a *op* from

Clean to *Substitute* in *remove* as *rflag*, and the flag operation that changes a *op* from *Clean* to *Compress* in *remove* as *cflag*. We describe how these transitions execute when a thread detects a state as follows:

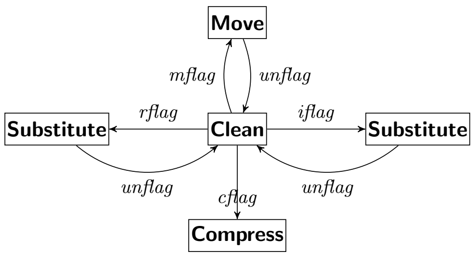


Fig. 9: State transition diagram

- **Clean**. For the insert transition, a thread constructs a new node and changes the parent's *op* to *Substitute* by *iflag*. For the remove transition, the thread creates an *Empty* node and changes the parent's *op* from *Clean* to *Substitute* by *rflag*. For the move transition, the thread flags both *newKey*'s and *oldKey*'s parents by *mflag*. For the compress circle, the thread uses a *cflag* operation to flag the parent if necessary.
- **Substitute**. The thread uses a CAS to change the existing node by a given node stored in the *op*. It then restores the parent's state to *Clean* by *unflag*.
- **Move**. The thread first determines the flag order by *oldKey*'s parent and *newKey*'s parent. Suppose it flags *newKey*'s parent first, it will flag *oldKey*'s parent later. Then, the thread replaces *oldKey*'s terminal with an *Empty* node and replaces *newKey*'s terminal with a new node. Finally, it *unflag*s their parents' *op* to *Clean* in the reverse order.
- **Compress**. The thread erases the node from the tree so that it cannot be detected. Different with other states, the node with a *Compress op* cannot be set to *Clean* by *unflag*.

4.3 Concurrent Algorithms

Figure 10 reflects quadboost's insert and contain operations. Both of them start by a find process that locates the terminal node. The find operation (line 98) pushes *Internal* nodes into a stack (line 95) and keeps recording the parent node's *op* (line 96).

The contain operation executes in a similar way as the CAS quadtree. It calls the find function to locate a terminal node (line 98). The *op* at line 96 and the stack at line 95 are created for a modular presentation. We can omit them in a real implementation.

In the insert operation, we create a stack at the beginning to record the traversal path (line 98) and a *pOp* to record the parent node's *op* (line 96). After locating a terminal node, we check whether the key of the node is in the tree and whether the node is moved at line 108. We will show the reason why we have to check whether a node is moved in Section 4.4. Then, we flag the parent of the terminal node before replacing it. In the next, we call the helpSubstitute function at line 114, which first invokes the helpReplace function at line 145 to replace the terminal node and then *unflag* the parent node at line 146). If the flag operation fails, we update *pOp* at line **??** and help it finish at line 118. More than that, we have to execute the continueFind function to restart from the nearest parent. The continueFind function (line 122) pops nodes from the path until reaching a node whose *op* is not *Compress*. If the node's *op* is *Compress*, it helps the *op* finish. Or else it breaks the loop and performs the find operation from the last node popped at line 132.

```
94  bool contain(double keyX, double keyY) {
95      Stack<Node> path;
96      Operation pOp;
97      Node l = root;
98      find(l, pOp, path, keyX, keyY);
99      if (inTree(l, keyX, keyY) && !moved(l)) return true;
100     return false;
101 }

102 bool insert(double keyX, double keyY, V value) {
103     Stack<Node> path;
104     Operation pOp;
105     Node l = root;
106     find(l, pOp, path, keyX, keyY);
107     while (true) {
108         if (inTree(l, keyX, keyY) && !moved(l)) return false;
109         p = path.pop();
110         if (pOp.class == Clean) {
111             Node newNode = createNode(l, p, keyX, keyY, value);
112             Operation op = new Substitute(p, l, newNode);
113             if (helpFlag(p, pOp, op)) {
114                 helpSubstitute(op);
115                 return true;
116             } else pOp = p.op;
117         }
118         help(pOp);    // help complete the operation of pOp
119         continueFind(pOp, path, l, p);  // find a new terminal in a bottom up
                way
120     }
121 }

122 void continueFind(Operation& pOp, Stack& path, Node& l, Internal p) {
123     if (pOp.class != Compress) l = p;  // start from the parent node
124     else {
125         while (!path.isEmpty()) {  // find a node in the path that is not in
                the Compress state
126             l = path.pop();
127             pOp = l.op;
128             if (pOp.class == Compress) helpCompress(pOp);
129             else break;  // find the node
130         }
131     }
132     find(l, pOp, path, keyX, keyY);
133 }

134 void find(Node& l, Operation& pOp, Stack& path, double keyX, double keyY)
135     while (l.class() == Internal) {
136         path.push(l);
137         pOp = l.op;
138         getQuadrant(l, keyX, keyY);
139     }
140 }

141 bool helpFlag(Internal node, Operation oldOp, Operation newOp) {
142     return CAS(node.op, oldOp, newOp);
143 }

144 void helpSubstitute(Substitute op) {
145     helpReplace(op.parent, op.oldChild, op.newNode);
146     helpFlag(op.parent, op, new Clean());    // unflag the parent node to Clean
147 }

148 void help(Operation op) {
149     if (op.class == Compress) helpCompress(op);
150     else if (op.class == Substitute) helpSubstitute(op);
151     else if (op.class == Move) helpMove(op);
152 }

153 bool moved(Node l) {
154     return l.class == Leaf && l.op != null && !hasChild(l.op.iParent,
            l.op.oldIChild);
155 }

156 bool hasChild(Internal parent, Node oldChild) {
157     return parent.nw == oldChild || parent.ne == oldChild || parent.sw ==
            oldChild || parent.se == oldChild;
158 }
```

Fig. 10: quadboost *insert* and *contain*

The remove operation has a similar paradigm as the insert operation. It first locates a terminal node and checks whether the node is in the tree and not moved. After that, it flags the parent node and replaces the terminal node with an *Empty* node. An extra step in the remove operation is the compress function at line 171. In the algorithm, we perform the compress function before the remove operation returns true. In this way, the linearization point of the remove operation belongs to the execution of itself, and extra adjustments induced by the compress operation do affect the effectiveness. The compress function should examine three conditions before compressing the parent node. First, because the remove operation must return true, we do not compress and even not help if the state of the parent is not *Clean* (line 182). Second, we check if the grandparent node (*gp*) is the root (line 184) as we maintain two layers of dummy *Internal* nodes. At last, we check whether four children of a parent node are all *Empty* (line 186).

```
159 bool remove(double keyX, double keyY, V value) {
160     Stack<Node> path;
161     Node l = root, newNode = new Empty();
162     Operation pOp;
163     find(l, keyX, keyY, pOp, path);
164     while (true) {
165         if (!inTree(l, keyX, keyY) || moved(l)) return false;
166         p = path.pop();
167         if (pOp.class == Clean) {
168             Operation op = new Substitute(p, l, newNode);
169             if (helpFlag(p, pOp, op) {
170                 helpSubstitute(op);
171                 compress(path, p);  // compress the path if necessary
172                 return true;
173             } else pOp = p.op;
174         }
175         help(pOp);
176         continueFind(pOp, path, l, p);
177     }
178 }

179 void compress(Stack& path, Internal p) {
180     while (true) {
181         Operation pOp = p.op;
182         if (pOp.class == Clean) {
183             gp = path.pop();
184             if (gp == root) return;  // two layers of Internal nodes are
                    maintained
185             Operation op = new Compress(gp, p);
186             if (!check(p) || !helpFlag(p, pOp, op)) return;
187             else helpCompress(op);
188             p = gp;
189         } else return;
190     }
191 }

192 bool helpCompress(Compress op) {
193     return helpReplace(op.grandparent, op.parent, new Empty<V>());
194 }

195 bool check(Internal node) {
196     return node.nw.class == Empty && node.ne.class == Empty && node.sw.class ==
            Empty && node.se.class == Empty;
197 }
```

Fig. 11: quadboost *remove*

4.4 LCA-based Move Operation

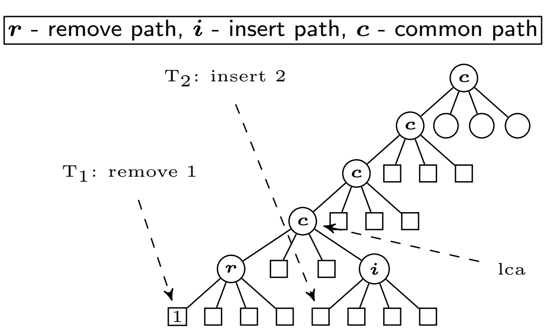


Fig. 12: Paths of two operations–insert node 2 and remove node 1 that share the LCA node

Tree-based structures share a property that two nodes in the tree have a common path starting from the root, and the lowest node in the path is called the lowest common ancestor (LCA). Figure 12 demonstrates the LCA node of a quadtree by a concrete example. Based on the observation, our LCA-based move operation is defined to find two different terminal nodes sharing a common path, remove the node with *oldKey*, and insert the node with *newKey*.

```
198 bool move(double oldKeyX, double oldKeyY, double newKeyX, double newKeyY) {
199     Stack rPath, iPath;
200     // rl: terminal node of the remove path
201     // il: terminal node of the insert path
202     // rp: parent node of the remove path
203     // ip: parent node of the insert path
204     Node rl = root, il, rp, ip;
205     // rOp: op object of rp
206     // iOp: op object of ip
207     Operation rOp, iOp;
208     // rFail: whether the remove path fail
209     // iFail: whether the insert path fail
210     // cFail: whether the common path fail
211     bool rFail = false, iFail = false, cFail = false;
212     if (!findCommon(il, rl, lca, rOp, iOp, iPath, rPath, oldKeyX, oldKeyY,
            newKeyX, newKeyY)) return false;
213     ip = iPath.pop();
214     rp = rPath.pop();
215     while (true) {
216         if (rOp.class != Clean) rFail = true;
217         if (iOp.class != Clean) iFail = true;
218         if (iOp != rOp && ip == rp) cFail = true;  // the same parent node has
                two different states
219         if (!cFail && !iFail && !rFail) {
220             if (il.class == Empty || il == rl) newNode = new Leaf(newKeyX,
                    newKeyY, rl.value);
221             else newNode = createNode(il, ip, newKeyX, newKeyY, rl.value);
222             //iFirst: flag which node first
223             bool iFirst = getSpatialOrder(ip, rp);
224             Operation op = new Move(ip, rp, il, rl, newNode, iOldOp, rOldOp,
                    iFirst);
225             if (rp != ip) {  // two parents are different
226                 bool hasFlag = iFirst ? helpFlag(ip, iOp, op) : helpFlag(rp,
                        rOp, op);
227                 if (hasFlag) {
228                     if (helpMove(op)) {
229                         compress(rp, rPath);
230                         return true;
231                     } else {  // all flag operation fail
232                         rFail = iFail = true;
233                     }
234                 } else {  // one of two paths fail
235                     if (iFirst) {iOp = ip.op; iFail = true;}
236                     else {rOp = rp.op; rFail = true;}
237                 }
238             } else {  // two parents are the same
239                 if (helpMove(op)) return true;
240                 else {rOp = iOp = rp.op; cFail = true;}
241             }
242         }
243         if (!continueFindCommon(iFail, rFail, cFail, il, rl, ip, rp, lca, rOp,
                iOp, oldKeyX, oldKeyY, newKeyX, newKeyY, iPath, rPath)) return
                false;  // continuous find insert and remove paths
244         cFail = iFail = rFail = false;
245     }
246 }

247 bool helpMove(Move op) {
248     if (op.iFirst) helpFlag(op.rParent, op.oldROp, op);  // flag the parent of
            the oldKey's terminal first
249     else helpFlag(op.iParent, op.oldIOp, op);  // flag the parent of the
            newKey's terminal first
250     bool doCAS = op.iFirst ? op.rParent.op == op: op.iParent.op == op;  //
            whether the flag operation succeed
251     if (doCAS) {  // all flags have been done
252         op.allFlag = true;
253         op.oldRChild.op = op;
254         if (op.oldRChild == op.oldRChild) {  // combine two CASes in two one
255             helpReplace(op.rParent, op.oldRChild, op.newIChild);
256         } else {
257             helpReplace(op.iParent, op.oldIChild, op.newIChild);
258             helpReplace(op.rParent, op.oldRChild, new Empty());
259         }
260     }
261     if (op.iFirst) {  // unflag in a reverse order
262         if (op.allFlag) helpFlag(op.rParent, op, new Clean());
263         if (op.rParent != op.rParent) helpFlag(op.iParent, op, new Clean());
264     } else {
265         if (op.allFlag) helpFlag(op.iParent, op, new Clean());
266         if (op.rParent != op.rParent) helpFlag(op.rParent, op, new Clean());
267     }
268     return op.allFlag;
269 }
```

Fig. 13: quadboost *move*

Figure 13 and Figure 14 present the algorithm of the move operation. In contrast with the insert operation and the remove operation, the move operation begins by calling the findCommon function (line 212) that combines searches for two nodes together. The two searches share a common path such that we could record them only once. We use two stacks to record nodes at line 199. The shared path and the remove path are pushed into the *rPath*; the insert path is pushed into the *iPath*. By doing this, we could avoid considering complicated corner cases in terms of node compressions in the *rPath*. The findCommon function begins by reading a child node from the parent for *oldKey* and checks whether *newKey* is in the same direction (line 271). If not, it terminates the traversal, and then searches for individual keys separately (line 282 and line 279). If *oldKey* is not in the tree or it is not moved, or *newKey* is in the tree but moved, it returns false.

The move operation then checks two parents' *op* (*iOp* and *rOp*) before flag operations. If neither of them is *Clean* (line 216-217), or *rp* and *ip* are the same but their *op*s are different (line 218), it starts the continueFindCommon function at line 243 which we will discuss later in the section. Otherwise, it creates a new node for inserting and a *op* to hold essential information for a CAS at line 220-224. There are two specific cases. If two terminal nodes share a common parent, we directly call the helpMove function at line 228. Or else, to avoid live locks, we shall flag two nodes in a specific order. In our algorithm, we use the getSpatialOrder function at line 223 to compare *ip* and *rp* in the following order: $x \rightarrow y \rightarrow w$, where x, y, w are the fields of an *Internal* node. We prove that this method produces a unique order among all *Internal* nodes in quadtree in the appendix A.

To start with, the helpMove function flags a parent and checks whether both parents are successfully flagged at line 248-250. If both flag operations succeed, it sets *allFlag* to true at line 252. Note we assign the *op* to *oldRChild* at line 253, letting other threads know whether the CAS on *iParent* has done, which is the linearization point of the move operation. Because simultaneously $oldRChild$ is aware that it is removed, it also explains why we should examine whether a node is moved. Then, we check whether two terminal nodes are the same. If so, we could combine two CASs into a single one. If not, we should remove *oldKey*'s terminal and later replace *newKey*'s terminal. In the end, we reset the parents' *op* to *Clean* in the reverse order (line 261-267).

The continueFindCommon function is also invoked when one of the flag operations fails. If *rp* cannot be flagged, it pops nodes *rPath* until it reaches the LCA node (line 291) or a node's *op* is *Compress* (line 297). It then starts from the last popped node to search for a new terminal of *oldKey* at line 302. If *ip* cannot be flagged, it pops nodes from *iPath* until it's empty (line 315) or a node's *op* is not *Compress* (line 317). It again continues to search for *newKey* at line 323. If either *rPath* has popped the LCA node or *iPath* is empty, it clears *iPath* at line 335 and pops all nodes above the LCA node from *rPath* at line 334. In the end, it calls the findCommon function to locate terminal nodes again at line 342. We prove that if *oldKey*'s terminal and *newKey*'s terminal share an LCA node, the common path will never be changed unless the LCA is altered.

```
270 bool findCommon(Node& il, Node& rl, Internal& lca, Operation& rOp, Operation&
        iOp, Stack& iPath, Stack& rPath, double oldKeyX, double oldKeyY, double
        newKeyX, double newKeyY) {
271     while (rl.class == Internal) {
272         rPath.push(rl);
273         rOp = rl.op;
274         getQuadrant(rl, oldKeyX, oldKeyY);
275         if (!sameDirection(oldKeyX, oldKeyY, newKeyX, newKeyY, il, rl))
276             break // check whether two nodes are in the same direction
277     }
278     lca = rPath.top();
279     find(rl, oldKeyX, oldKeyY, rOp, rPath);  // find oldKey's terminal
280     if (!inTree(rl, oldKeyX, oldKeyY) || moved(rl)) return false;
281     il = lca;
282     find(il, newKeyX, newKeyY, iOp, iPath);  // find newKey's terminal
283     if (inTree(il, newKeyX, newKeyY) && !moved(il)) return false;
284 }

285 bool continueFindCommon(Node& il, Node& rl, Internal& lca, Operation& rOp,
        Operation& iOp, Stack& iPath, Stack& rPath, Internal ip, Internal rp,
        double oldKeyX, double oldKeyY, double newKeyX, double newKeyY, bool
        iFail, bool rFail, bool cFail) {
286     if (rFail && !cFail) {  // restart to find oldKey's terminal in cast that
            the lca's op remains the same
287         help(rOp);
288         if (rOp.class != Compress) rl = rp;
289         else {
290             while (!rPath.empty()) {
291                 if (rPath.size() <= indexOf(lca)) {  // reach the lca node,
292                     cFail = true;  // break the loop the find the common path
                            again
293                     break;
294                 }
295                 rl = rPath.pop();
296                 rOp = rl.op;
297                 if (rOp.class == Compress) helpCompress(rOp);
298                 else break;
299             }
300         }
301         if (!cFail) {  // the lca node has not been changed
302             find(rl, oldKeyX, oldKeyY, rOp, rPath);
303             if (!inTree(rl, oldKeyX, oldKeyY) || moved(rl)) return false;
304             else {
305                 rp = rPath.pop();
306                 return true;
307             }
308         }
309     }
310     if (iFail && !cFail) {
311         help(iOp);
312         if (iOp.class != Compress) il = ip;
313         else {
314             while (!iPath.empty()) {
315                 il = iPath.pop();
316                 iOp = il.op;
317                 if (iOp.class == Compress) helpCompress(iOp);
318                 else break;
319             }
320         }
321         if (iOp.class == Compress) cFail = true;  // check the last op which
                must be the op of the lca node
322         if (!cFail) {
323             find(il, newKeyX, newKeyY, iOp, iPath);
324             if (inTree(il, newKeyX, newKeyY) && !moved(il)) return false;
325             else {
326                 ip = iPath.pop();
327                 return true;
328             }
329         }
330     }
331     if (cFail) {
332         help(iOp);
333         help(rOp);
334         rPath.setIndex(indexOf(lca));  // pop out all nodes above the lca node
335         iPath.clear();  // clean the input path
336         while (!rPath.empty()) {//first time must be not empty
337             rl = rPath.pop();
338             rOp = rl.op;
339             if (rOp.getClass() == Compress) helpCompress(rOp);
340             else break;
341         }  // pop up nodes in the common path
342         if (!findCommon(il, rl, lca, rOp, iOp, iPath, rPath, oldKeyX, oldKeyY,
                newKeyX, newKeyY) return false;
343         else {  // find two nodes
344             rp = rPath.pop();
345             ip = iPath.pop();
346         }
347     }
348 }
```

Fig. 14: quadboost findCommon and continueFindCommon

4.5 One Parent Optimization

In practice, we notice that pushing a whole stack during a traversal is highly expensive. On detecting a failed flag operation, many times we only have to restart from the parent of a terminal node because we do not change *Internal* nodes unless the compress function erases them from quadtree. Thus, to reduce the pushing cost, we could only record the parent of the terminal node during a traversal.

Meanwhile, we have to change the continuous find mechanism. For the insert and the remove operation, encountering a *Compress op*, we straightforwardly restart from the root. For the move operation, if either *oldKey*'s or *newKey*'s parent is under compression, we restart from the LCA node. If the LCA node also has a *Compress op*, we restart from the root.

5 Proof Sketch

In this section, we prove that quadboost is both linearizable and non-blocking, and we propose a lengthy proof in the appendix A.

There are four kinds of *basic operations* in quadboost, i.e. the insert operation, the remove operation, the move operation, and the contain operation. Other functions are called *subroutines*, which are invoked by basic functions. The insert operation and the remove operation only modifies one terminal node, whereas the move operation might operate two different terminals–one for inserting a node with $newKey$, the other is for removing a node with $oldKey$. We call them $newKey$'s terminal and $oldKey$'s terminal, and we call their parents $newKey$'s parent and $oldKey$'s parent accordingly. We define $snapshot_{T_i}$ as the state of our quadtree at some time T_i.

In our proofs, a CAS that changes a node's *op* is a *flag* operation and a CAS that changes a node's child is a *replace* operation. Specifically, we use $iflag$, $rflag$, $mflag$, and $cflag$ to denote flag operations for the insert operation, the remove operation, the move operation, and the compress operation separately. Likewise, we use $ireplace$, $rreplace$, $mreplace$, and $creplace$ for replace operations. Moreover, we specify a flag operation which attaches a *Clean op* on a node as an unflag operation.

First we present some observations from quadboost. Then, we propose lemmas to show that our subroutines satisfy their pre-conditions and post-conditions because later proofs on basic operations depend on these conditions. Next, we demonstrate there are three categories of successful CAS transitions according to Figure 9. We also derive some invariants of these transitions. Using above post-conditions of subroutines and invariants, we could demonstrate that quadtree's structure maintains during concurrent modifications. In following proofs, we show quadboost is linearizable because it can be ordered equivalently as a sequential one by its linearization points. In the last part, we prove the non-blocking progress condition of quadboost.

Observation 1. *The key field of a Leaf node is never changed. The op field of a Leaf node is initially null. The space information of an Internal node is never changed.*

Observation 2. *The root node is never changed.*

Observation 3. *A flag operation attach an op on a Internal node.*

Observation 4. *The $allFlag$ and $iFirst$ field in Move are initially false, and they will never be set back after assigning to true.*

Observation 5. *If a Leaf node is moved, its op is set before the replace operation on $op.iParent$, which is before the replace operation on $op.rParent$.*

Observation 6. *The help function, The helpCompress function, The helpMove function, and the helpSubstitute function are not called in a mutual way. (If method A calls method B, and method B also calls method A, we say A and B are called in a mutual way)*

5.1 Basic Invariants

We use $find(keys)$ to denote a set of find operations for keys: $find$, $continueFind$, $findCommon$, and $continueFindCommon$. The find function and the continueFind function return a tuple $\langle l, pOp, path\rangle$. The findCommon function and the continueFindCommon function return two such tuples. We specify functions outside the while loop in the insert operation, the remove operation, and the move operation are at $iteration_0$, and functions inside the while loop are at $iteration_i, 0 < i$ ordered by their invocation sequence.

We suppose that $find(keys)$ executes from a valid $snapshot_{T_i}$ to derive the following conditions. Proofs for conditions of other subroutines are included in the appendix.

Lemma 1. *The post-conditions of $find(keys)$ returned at T_i, with tuples $\langle l^k, pOp^k, path^k\rangle$, $0 \le k < |keys|$.*

1) *l^k is a Leaf node or an Empty node.*
2) *At some $T_{i1} < T_i$, the top node in $path^k$ has contained pOp^k.*
3) *At some $T_{i2} < T_i$, the top node in $path^k$ has contained l^k.*
4) *If pOp^k is read at T_{i1}, and l^k is read at T_{i2}, then $T_{i1} < T_{i2} < T_i$.*
5) *For each node n in the $path^k$, $size(path^k) \ge 2$, n_t is on the top of n_{t-1}, and n_t is on the direction $d \in \{nw, ne, sw, se\}$ of n_{t-1} at $T_{i1} < T_i$.*

Based on these post-conditions, we show that each *op* created at T_i store their corresponding information.

Lemma 2. *For op created at T_i:*

1) *If op is Substitute, $op.parent$ has contained $op.oldChild$ that is a Leaf node at $T_{i1} < T_i$ from the results of $find(keys)$ at the prior iteration.*
2) *If op is Compress, $op.grandparent$ has contained $op.parent$ that is an Internal node at $T_{i1} < T_i$ from the results of $find(keys)$ at the prior iteration.*
3) *If op is Move, $op.iParent$ has contained $op.oldIChild$ that is a Leaf node and $op.oldIOp$ before T_i, and $op.rParent$ has contained $op.oldRChild$ that is a Leaf and $op.oldROp$ before T_i from the results of $find(keys)$ at the prior iteration.*

Though we have not presented details of the createNode function, our implementation could guarantee that it has following conditions.

Lemma 3. *For $createNode(l, p, newKeyX, newKeyY, value)$ that returns a $newNode$ invoked at T_i, it has the post-condition:*

1) *The $newNode$ returned is either a Leaf node with $newKey$ and $value$, or a sub-tree that contains both $l.key$ node and $newKey$ node with the same parent.*

Using prior conditions, we derive some invariants during concurrent executions and prove that there are three kinds of successful CAS transitions. We put successful flag operations that attach *op*s on nodes at the beginning of each CAS transition. We say every successive replace operations that read *op* belongs to it and follows the flag operations.

Let $flag_0$, $flag_1$, ..., $flag_n$ be a sequence of successful flag operations. $flag_i$ reads pOp_i and attaches op_i. $replace_i$ and $unflag_i$ read op_i and come after it. Therefore, we say $flag_i$, $replace_i$, and $unflag_i$ belong to the same *op*. In addition, if there are more than one replace operation belongs to the same op_i, we denote them as $replace_i^0$, $replace_i^1$, ..., $replace_i^n$ ordered by their successful sequence. Similarly, if there are more than one flag operation that belongs to op_i on different nodes, we denote them as $flag_i^0$, $flag_i^1$, ..., $flag_i^n$. A similar notation is used for $unflag_i$.

The following lemmas prove the correct ordering of three different transitions.

Lemma 4. *For a new node n:*

1) *It is created with a Clean op.*
2) *$rflag$, $iflag$, $mflag$ or $cflag$ succeeds only if n's op is Clean.*
3) *$unflag$ succeeds only if n's op is Substitute or Move.*
4) *Once n's op is Compress, its op will never be changed.*

Lemma 5. *For an Internal node n, it never reuse an op that has been set previously.*

Lemma 6. *$replace_i^k$ will not occur before $flag_i^k$ that belongs to the same op has been done.*

Lemma 7. *The $flag \rightarrow replace \rightarrow unflag$ transition occurs when $rflag_i$, $iflag_i$ or $mflag_i$ succeeds, and it has following properties:*

1) *$replace_i$ never occurs before $flag_i$.*
2) *$flag_i^k, 0 \leq k < |flag_i|$ is the first successful flag operation on $op_i.parent^k$ after T_{i1} when pOp_i^k is read.*
3) *$replace_i^k, 0 \leq k < |replace_i|$ is the first successful replace operation on $op_i.parent^k$ after T_{i2} when $op_i.oldChild^k$ is read.*
4) *$replace_i^k, 0 \leq k < |replace_i|$ is the first successful replace operation on $op_i.parent^k$ that belongs to op_i.*
5) *$unflag_i^k, 0 \leq k < |unflag_i|$ is the first successful unflag operation on $op_i.parent^k$ after $flag_i^k$.*
6) *There is no successful unflag operation occurs before $replace_i$.*
7) *The first replace operation on $op_i.parent^k$ that belongs to op_i must succeed.*

Lemma 8. *The $flag \rightarrow replace$ transition occurs only when $cflag_i$ succeeds, and it has following properties:*

1) *$creplace_i$ never occurs before $cflag_i$.*
2) *$cflag_i$ is the first successful flag operation on $op_i.parent$ after T_{i1} when pOp_i is read.*
3) *$creplace_i$ is the first successful replace operation on $op_i.grandparent$ after T_{i2} when $op_i.parent$ is read.*
4) *$creplace_i$ is the first successful replace operation on $op_i.grandparent$ that belongs to op_i.*
5) *There is no unflag operation after $creplace_i$.*
6) *The first replace operation on $op_i.grandparent$ that belongs to op_i must succeed.*

Lemma 9. *For the $flag \rightarrow unflag$ transition, it only results from $mflag$ such that (Suppose $iFirst$ is false):*

1) *$unflag_i$ is the first successful unflag operation on $op_i.rParent$.*
2) *The first flag operation on $op_i.iParent$ must fail, and no later flag operation succeeds.*
3) *$op_i.iParent$ and $op_i.rParent$ are different.*

Claim 1. *There are three kinds of successful transitions belong to an op: (1) $flag \rightarrow replace \rightarrow unflag$, (2) $flag \rightarrow unflag$, (3) $flag \rightarrow replace$.*

Then, we prove that quadtree maintains its properties during concurrent modifications.

Definition 1. *Our quadtree has these properties:*

1) *Two layers of dummy Internal nodes are never changed.*
2) *An Internal node n has four children, which locate in the direction $d \in \{nw, ne, sw, se\}$ respectively according to their $\langle x, y, w, h \rangle$, or $\langle keyX, keyY \rangle$.*
   For Internal nodes reside on four directions:
   - $n.nw.x = n.x, n.nw.y = n.y;$
   - $n.ne.x = n.x + w/2, n.ne.y = n.y;$
   - $n.sw.x = n.x, n.sw.y = n.y + n.h/2;$
   - $n.se.x = n.x + w/2, n.se.y = n.y + h/2,$

   All children have their $w' = n.w/2$, $h' = n.h/2$.
   For Leaf nodes reside on four directions:
   - $n.x \leq n.nw.keyX < n.x + n.w/2,$ $n.y \leq n.nw.keyY < n.y + n.h/2;$
   - $n.x + n.w/2 \leq n.ne.keyX < n.x + n.w,$ $n.y \leq n.ne.keyY < n.y + n.h/2;$
   - $n.x \leq n.sw.keyX < n.x + n.w/2,$ $n.y + n.h/2 \leq n.sw.keyY < n.y + n.h;$
   - $n.x + n.w/2 \leq n.se.keyX < n.x + n.w,$ $n.y + n.h/2 \leq n.se.keyY < n.y + n.h.$

To help clarify quadtree's properties during concurrent executions, we define *active* set and *inactive* set for different kinds of nodes. For an *Internal* node or an *Empty* node, if it is reachable from the root in $snapshot_{T_i}$, it is active; otherwise, it is inactive. For a *Leaf* node, if it is reachable from the root in $snapshot_{T_i}$ and not moved, it is active; otherwise, it is *inactive*. We say a node n is moved in $snapshot_{T_i}$ if the function moved(n) returns true at T_i. We denote $path(keys^k), 0 \leq k < |replace_i|$ as a stack of nodes pushed by $find(keys)$ in a snapshot. We define $physical\_path(keys^k)$ to be the path for $keys^k$ in $snapshot_{T_i}$, consisting of a sequence of *Internal* nodes with a *Leaf* node or an *Empty* node at the end. We say a subpath of $path(keys^k)$ is an $active\_path$ if all nodes from the root to node $n \in path(keys^k)$ are active. Hence a $physical\_path(keys^k)$ is active only if the end node is not moved.

Lemma 10. *Two layers of dummy nodes are never changed.*

Lemma 11. *Children of a node with a Compress op will not be changed.*

Lemma 12. *Only an Internal node with all children Empty could be attached with a Compress op.*

Lemma 13. *An Internal node whose op is not Compress is active.*

Lemma 14. *After the invocation of $find(keys)$ which reads l^k, there is a snapshot that the path from the root to it is $physical\_path(keys^k)$.*

Lemma 15. *After $ireplace$, $rreplace$, $mreplace$, and $creplace$, quadtree's properties remain.*

Proof. We shall prove that in any $snapshot_{T_i}$, quadtree's properties remain.

First, Lemma 10 shows that two layers of dummy nodes remain in the tree. We have to consider other layers of nodes which are changed by replace operations.

Consider $creplace$ that replaces an *Internal* node by an *Empty* node. Because the *Internal* node has been flagged on a *Compress op* before $creplace$ (Lemma 8), all of its children are *Empty* and not changed (Lemma 11 and Lemma 12). Thus, $creplace$ does not affect the second claim of Definition 1.

Consider $ireplace$, $rreplace$, or $mreplace$ that replaces a terminal node by an *Empty* node, a *Leaf* node, or a sub-tree. By Lemma 7, before $replace^k$, $op.parent^k$ is flagged with op such that no successful replace operation could happen on $op.parent^k$. Therefore, if the new node is *Empty*, it does not affect the tree property. If the new node is a *Leaf* node or a sub-tree, based on the post-conditions of the createNode function (Lemma 3), after replace operations the second claim of Definition 1 still holds. □

Claim 2. *Quadtree maintains its properties in every snapshot.*

5.2 Linearizability

In this Section, we define linearization points for basic operations. As the compress function is included in the move operation and the remove operation that returns true, it does not affect the linearization points of them. If an algorithm is linearizable, its result be ordered equivalently as a sequential history by the linearization points. Since all modifications depend on $find(keys)$, we first point out its linearization point. For $find(keys)$, we define its linearization point at T_i such that l^k returned is on the $physical\_path(keys^k)$ in $snapshot_{T_i}$.

For the contain operation that returns true, we show there is a corresponding snapshot that l^k in $physical\_path(keys^k)$ is active. For the contain operation, the insert operation, the remove operation, and the move operation that returns false, we show there is a corresponding snapshot that l^k in $physical\_path(keys^k)$ is inactive. For the insert operation, the remove operation, and the move operation that returns true, we define linearization points to be their first successful replace operation—$replace_i^0$. To make a reasonable demonstration, we first show that $replace_i^k, 0 \leq k < |replace_i|$ belongs to each operation that creates op_i, and illustrate that op is unique for each operation.

Lemma 16. *For $find(keys)$ that returns tuples $\langle l^k$, pOp^k, $path^k\rangle$, there is a $snapshot_{T_i}$ such that $path^k$ returned with l^k at the end is $physical\_path(key^k)$ in $snapshot_{T_i}$.*

Lemma 17. *If the insert operation, the remove operation, and the move operation that returns true, the first successful replace operation occurs before returning, and it belongs to the op created by the operation itself at the last iteration in the while loop.*

Lemma 18. *If the insert operation, the remove operation, and the move operation that return false, there is no successful $replace$ happens during the execution.*

The next lemma points out the **linearization points** of the contain operation.

Lemma 19. *For the contain operation that returns true, there is a corresponding snapshot that l^k in $physical\_path(keys^k)$ is active in $snapshot_{T_i}$. For the contain operation that returns false, there is a corresponding snapshot that l^k in $physical\_path(keys^k)$ is inactive in $snapshot_{T_i}$.*

We list out the **linearization points** of other operations as follows:

- *insert(key)*. The linearization point of the insert operation that returns false is at T_i after calling $find(key)$ that l at the end of $physical\_path(key)$ does not contain the key, or it contains the key but is moved. For the insert operation that returns true, the linearization point is at the first successful replace operation (line 145).
- *remove(key)*. The linearization point of the remove operation that returns false is at T_i after calling $find(key)$ that l at the end of $hysical\_path(key)$ contains the key and is not moved. For the successful remove operation, we define the linearization point at where the node with key is replaced by an *Empty* node (line 145).
- *move(oldKey, newKey)*. For the unsuccessful move operation, the linearization point depends on both $newKey$ an $oldKey$. If rl does not contain $oldKey$, or rl is moved, the linearization point is at T_{i1} after calling $find(keys)$. Or else, if il contains $oldKey$, or il is not moved, the linearization point is at T_{i2} after calling $find(keys)$.
  For the successful move operation, the linearization point is the first successful replace operation. (line 255 or line 257)

Claim 3. *quadboost is linearizable.*

5.3 Non-blocking

Finally, we prove that quadboost is non-blocking, which means that the system as a whole is making progress even if some threads are starving.

Lemma 20. *A node with a Compress op will not be pushed into path more than once.*

Lemma 21. *For $path(keys^k)$, if n_t is active in $snapshot_{T_i}$, then $n_0, ..., n_{t-1}$ pushed before n_t are active.*

Lemma 22. *If in $snapshot_{T_i}$, n is the LCA node on $physical\_path$ for $oldKey$ and $newKey$. Then at $T_{i1}, T_{i1} > T_i$, n is still the LCA on $active\_path$ for both $oldKey$ and $newKey$ if it is active.*

Lemma 23. *$path^k$ returned by $find(keys)$ consists of finite number of keys.*

Lemma 24. *There is a unique spatial order among nodes in quadtree in every snapshot.*

Lemma 25. *There are a finite number of successful $flag \rightarrow replace \rightarrow$, $flag \rightarrow replace$, $flag \rightarrow unflag$ transitions.*

Lemma 26. *If the help function returned at T_i, and $find(keys)$ at the prior iteration reads $p^0.op$ at $T_{i1} < T_i$, Leaf nodes in $snapshot_{T_i}$ and $snapshot_{T_{i1}}$ are different.*

Claim 4. *quadboost is non-blocking.*

Proof. We have to prove that no process will execute loops infinitely without changing keys in quadtree. First, we shall prove that $path$ is terminable. Next, we shall prove that $find(keys^k)$ starts from an active node in $physical\_path(keys^k)$ in $snapshot_{T_i}$ between i_{th} iteration and $i+1_{th}$ iteration. Finally, *Leaf* nodes in $snapshot_{T_{i1}}$ at the returning of $find(keys)$ at i_{th} iteration is different from $snapshot_{T_{i2}}$ that at the returning of $find(keys)$ at the $i+1_{th}$ iteration.

For the first part, initially we start from the root node. Therefore, $path$ is empty. Moreover, as Lemma 22 shows that $path^k$ consists of finite number of keys, we establish this part.

For the second part, the continueFind function and the continueFindCommon function pops all nodes with *Compress op* from $path$. For the insert and remove operation, since Lemma 13 shows that an *Internal* nodes whose *op* is not *Compress* is active and Lemma 21 shows that nodes above the active node are also active, there is a snapshot such that the top node of $path$ is still in $physical\_path(keys^k)$. For the move operation, if either $rFail$ or $iFail$ is true, it is equivalent with the prior case. Or if $cFail$ is true, Lemma 22 illustrates that if the LCA node is active, it is in $physical\_path$ for both $oldKey$ and $newKey$. Thus, there is also a snapshot that the start node is in $physical\_path$.

For the third part, we prove it by contradiction. Assuming that quadtree is stabilized at T_i, and all invocations after T_i are looping infinitely without changing *Leaf* nodes.

For the insert operation and the remove operation, before the invocation of $find(keys)$ at the next iteration, they must execute the help function at line 118 and line 175 accordingly. In both cases, the help function changes the snapshot (Lemma 26).

For the move operation, consider different situations of the continueFindCommon function. If $rFail$ or $iFail$ is true, there are two situations: (1) iOp or rOp is *Clean* but $mflag$ fails. (2) iOp or rOp is not *Clean*. Consider the first case that $mflag$ fails, iOp and rOp are updated respectively at line 236 and line 235. If ip and rp are different, before its invocation of $find(keys^k)$, the help function is performed at line 311 and line 287. Thus by Lemma 26, the snapshot is changed between two iterations. Now consider if $cFail$ is true. If ip and rp are the same, it could result from the difference between rOp and iOp. In this case, the snapshot might be changed between reading rOp and iOp. It could also result from the failure of $mflag$. For above cases, the help function at line 332 would change quadtree. If ip and rp are different, it results from that either $iPath$ or $rPath$ has popped the LCA node. We have proved the case in the former paragraph. Thus, we derives a contradiction.

From above discussions, we prove that quadboost is non-blocking. □

6 EVALUATION

We run experiments on a machine with 64GB main memory. It has two 2.6GHZ Intel(R) Xeon(R) 8-core E5-2670 processors with hyper-threading enabled, rendering 32 hardware threads in total. Besides, we use RedHat Enterprise Server 6.3 with Linux core 2.6.32. All experiments were run under Sun Java SE Runtime Environment (build 1.8.0_65). To avoid significant run-time garbage collection cost, we set up the initial heap size to 6GB.

For each experiment, we run eight 1-second cases, where the first 3 cases are served to warm up JVM, and the median of the last 5 cases is used as the real performance. Before the start of each case, we insert half keys from the key set into quadtree to guarantee that initially the insert and the remove operation have equal success opportunity.

We apply uniformly distributed key sets that contain two-dimensional points within a square. We use the $range$ to denote the border of a square. Thus, points are located inside a $range * range$ square. In our experiments, we use two different key sets: 10^2 keys to measure the performance under high contention, 10^6 keys to measure the performance under low contention. For simplicity, we let the $range$ of the first category experiment be 10, rendering $1-10^2$ consecutive keys for one-dimensional structure. For the second category experiment, we let the $range$ To be 1000, generating $1-10^6$ consecutive keys.

TABLE 1: The concurrent quadtree algorithms with different optimization strategies.

type	insert	remove	move
qc	single CAS	single CAS	**not support**
qb-s	flag CAS, continuous find, stack	flag CAS, continuous find, decoupling stack, recursive compression	flag CAS, continuous find, decoupling stack, recursive compression
qb-o	flag CAS, continuous find	flag CAS, continuous find, decoupling compression	flag CAS, continuous find, decoupling compression

We evaluate quadboost algorithms by comparing with the state-of-the-art concurrent trees (kary, ctrie, and patricia) for throughput (Section 6.1) and presenting the incremental effects of the optimization strategies proposed in this work (Section 6.2). Table 1 lists the concurrent quadtree algorithms. **qb-o** (quadboost-one parent) is the one parent optimization based on **qb-s** (quadboost-stack) mentioned in Section 4.4. **qc** is the CAS quadtree introduced in Section 3.

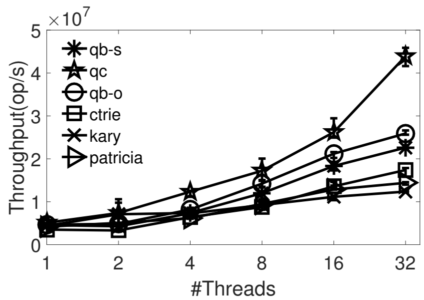
6.1 Throughput

To the best of our knowledge, a formal concurrent quadtree has not been published yet. Hence, we compare our quadtrees with three one-dimensional non-blocking trees:

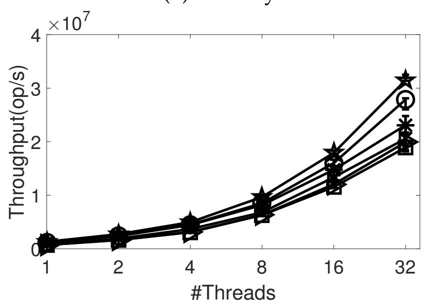
- **kary** is a non-blocking k-way search tree, where k represents the number of branches maintained by an internal node. Like the non-blocking BST [4], keys are kept in leaf nodes. When $k=2$, the structure is similar as the non-blocking BST; when $k=4$, each internal node has four children, it has a similar structure as quadtree. However, kary's structure depends on the modification order, and it doesn't have a series of internal nodes representing the two-dimensional space hierarchy.
- **ctrie** is a concurrent hash trie, where each node can store up to 2^k children. We use $k=2$ to make a 4-way hash trie that resembles quadtree. The hash trie also incorporates a compression mechanism to reduce unnecessary nodes. Whereas different with quadtree, it uses a control node (`INODE` as the paper indicates) to coordinate concurrent updates. Hence, the search depth could be longer than quadtree.
- **patricia** is a binary search tree, which adopts ellen's BST techniques [4]. As the author points out, it can be used as quadtree by interleaving the bits of x and y. It also supports the move (replace) operation like quadboost. But unlike our LCA-based operation, it

searches two positions separately without a continuous find mechanism.

Since the above structures only store one-dimensional keys, we have to transform a two-dimensional key to a one-dimensional key for comparison. Though patricia could store two-dimensional keys using the mentioned method, ctrie and kary cannot use it. Thus, we devise a general formula: $key^1 = key_x^2 * range + key_y^2$. Given a two-dimensional key-key^2, and $range$, we can transform it into a one-dimensional key-key^1. To refrain from trivial transformations by floating numbers, we only consider integer numbers in this section.



(a) 10^2 keys

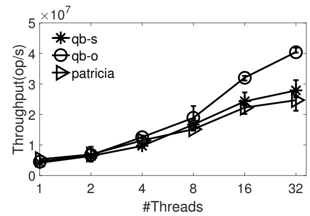


(b) 10^6 keys

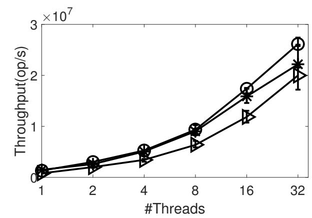
Fig. 15: Throughput of different concurrent trees under both high and low contention (50% *insert*, 50% *remove*).

Due to the lack of the move operation in the simple CAS quadtree algorithm (**qc**), we compare throughput with-/without the move operation respectively. Figure 15 plots throughput without any move operation for the concurrent algorithms. It's unsurprising to observe that **qc** achieves the highest throughput. To some extent, **qc** represents an upper bound of throughput because it maintains the hierarchy without physical removal, i.e., its remove operation only applies a CAS on edges to change links, which leads to less contention than other practical concurrent algorithms. However, both **qb-s** and **qb-o** can achieve comparable throughput when the key set becomes larger. This phenomenon is because: (i) Given the large key set, fewer thread interventions results in less number of CAS failures on nodes. (ii) Both algorithms compress nodes and use the continuous find mechanism to reduce the length of traverse path.

As a comparison, **ctrie**, **kary**, and **patricia** show lower performances with the increasing number of threads. For instance, in Figure 15a at 32 threads, **qb-o** outperforms **ctrie** by 49%, **patricia** by 79%, and **kary** by 109%. Note that **qb-o** and **qb-s** incorporate the continuous find mechanism to reduce the length of traverse path. Further, both **kary** and **patricia** flag the grandparent node in the remove operation, which allows less concurrency than **ctrie**, **qb-o** and **qb-s** with the decoupling approach shown in Figure 7. **qb-s** is worse than **qb-o** due to its extra cost of recording elements and compressing nodes recursively. Figure 15b exhibits results when the key set is large. There's less collision among threads but deeper depths of trees than the small key set. In the scenario, **qb-o** and **qb-s** show a similar performance as **qc** because of less number of CAS failures caused by thread interventions. **qb-o** is only 12% worse than **qc** at 32 threads, but 47% better than **ctrie**, 35% better than **kary**, and 39% better than **patricia** mainly for its shorter traversal paths caused by its static representation and the continuous find mechanism. As we point in the next section, **qb-s** and **qb-o** save a significant number of nodes as shown in Figure 18. It implies that **qb-s** and **qb-o** occupy less memory and result in a shorter path for traversal.



(a) 10^2 keys



(b) 10^6 keys

Fig. 16: Comparison of the move operation's throughput between quadboost and patricia in both small and large range (10% *insert*, 10% *remove*, 80% *move*).

Figure 16b demonstrates that quadboost has an efficient move operation. Using the small key set, where the depth is not a significant impact, Figure 16a shows that **qb-o** is more efficient than **patricia** especially when contention is high. For example, it performs better than **patricia** by 47% at 32 threads. Because it adopts the continuous find mechanism to traverse less path and decouples physical adjustment for higher concurrency. But **qb-s** is similar as **patricia** since it has to maintain a stack and recursively compress nodes in quadtree. Figure 16b illustrates that **qb-s** and **qb-o** have a similar throughput for the large key set. **qb-o** outperforms **patricia** by 31% at 32 threads. When the key set is large, the depth becomes a more significant factor due to less contention. Since each *Internal* node in quadtree maintains four children while **patricia** maintains two, the depth of **patricia** is deeper than quadboost. Further, the combination of the LCA node and the continuous find mechanism ensures that **qb-o** and **qb-s** do not need to restart from the root even if flags on two different nodes fail.

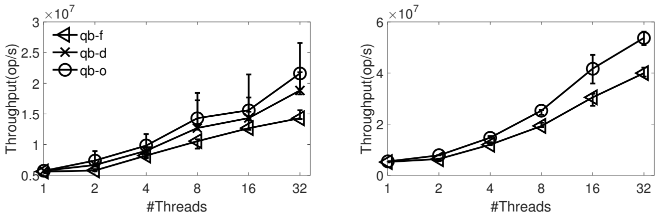
6.2 Analysis

To figure out how quadboost algorithms improve the performance, we devise two algorithms that incrementally use parts of techniques in **qb-o**:

- **qb-f** flags the parent of a terminal node in move, insert, and remove operations. It restarts from the root without a continuous find mechanism. Besides, it adopts the traditional *remove* mechanism mentioned in Figure 7a.

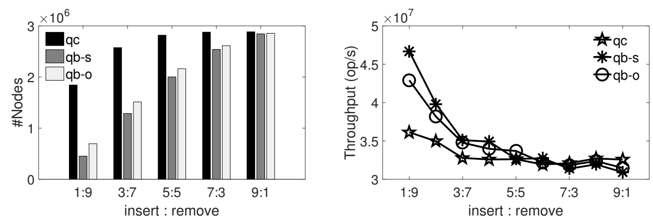
- **qb-d** decouples the physical adjustment in the remove operation based on **qb-f**.

$range$ here is set to $2^{32}-1$, and both key_x and key_y could be floating numbers. We use an insert dominated and a remove dominated experiment to demonstrate the effect of different techniques. In the insert dominated experiment, the *insert:remove* ratio is 9:1, hence there are far more insert operations. Since fewer compress operations are induced, the experiment will show the effect of the continuous find mechanism. We use a remove dominated experiment to show the effect of decoupling, where *insert:remove* ratio is 1:9. Figure 17a illustrates that quadtrees with decoupling exhibit a higher throughput than **qb-f**, the basic flag concurrent quadtree. Besides, **qb-o** which incorporates the continuous find is more efficient than **qb-d**. Specifically, at 32 threads, **qb-o** performs 15% better than **qb-d** and 51% better than **qb-f**. From Figure 17b, we figure out that **qb-o** outperforms **qb-f** by up to 35%. Thereforeit demonstrates that the continuous find mechanism and the decoupling approach play a significant role in our algorithm.



(a) 10% *insert*, 90% *remove* (b) 90% *insert*, 10% *remove*

Fig. 17: Throughput comparison in insert dominated and remove dominated cases (10^2 keys).



(a) Nodes (b) Throughput

Fig. 18: Number of nodes left and throughput under different ratio of insert:remove from 9:1 to 1:9, with fixed 90% contain, under 32 threads and 10^6 keys.

Another advantage of quadboost results from the compression technique, which reduces the search path for each operation and the memory consumption. Figure 18[2] plots the number of nodes left and the throughput of each quadtree at different *insert:remove* ratio. As **qc** only replaces the terminal node with an *Empty* node without compression, it results in the greatest number of nodes in the memory (Figure 18a). In contrast, **qb-s** and **qb-o** compress quadtree if necessary. With the increment of the remove ratio, **qc** contains more nodes than other quadtrees. In the case the remove operation dominates (the first group of bars to left), it has three times more nodes than **qb-s**. The result also indicates that **qb-o** contains few nodes more than **qb-s** despite it only compresses one layer of nodes. Figure 18b illustrates the effectiveness of compression in the face of tremendous contain operations. **qb-s** outperforms **qc** by 30% at 9:1 *insert:remove* ratio because **qb-s** adjusts the quadtree's structure by compression to reduce the length of the search path. With the increment of the insert ratio, **qb-s** performs similarly as **qc** due to the extra cost of maintaining a stack and the recursive compression. However, **qb-o** achieves good balance between **qb-s** and **qc**, which compresses one layer of nodes without recording the whole traverse path.

2. Unlike previous experiments, we run eight 3-second cases in the experiment to ensure stable amount of modifications

7 Related Works

Because there are few formal works related to concurrent quadtrees, we present a roadmap to show the development of state-of-the-art concurrent trees in Figure 19.

Ellen [4] provided the first non-blocking BST and proved it correct. Their work is dependent on the cooperative method described in Turek [17] and Barnes [18]. Brown [19] used a similar approach for the concurrent k-ary tree. Shafiei [20] also applied the method for the concurrent patricia trie. It also showed how to design a concurrent operation where two pointers need to be changed. Above concurrent trees have an external structure, where only leaf nodes contain actual keys. Howley [5] designed the first internal BST built by the technique. Recently, Brown2014general [21] presented a generalized template for all concurrent downtrees. Ellen [22] exhibited how to incorporate a stack to reduce the original complexity from $O(ch)$ to $O(h+c)$. Our quadboost is a hybrid of above techniques. It uses a cooperative method for concurrent coordination, changes two different positions with atomicity, devises a continuous find mechanism to reduce restart cost.

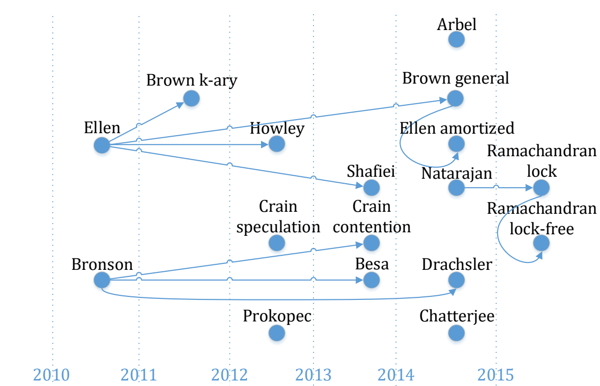


Fig. 19: Concurrent trees roadmap

Different with the mentioned method which applies flags on nodes, Natarajan2014fast [6] illustrated how to apply flags on edges for a non-blocking external BST. Ramachandran [7] adopted CAS locks on edges to design a concurrent internal BST, and they later extended the work to non-blocking in [8]. Their experiments showed that internal trees are more scalable than external trees with large key range. On the one hand, internal trees take up less memory. On the other hand, the remove operation is more complicated than that in external trees. Chatterjee [9] provided a threaded-BST with edge flags, and they claimed it has a lower theoretical complexity. Unlike the above trees that have to flag their edges before removal, our CAS quadtree uses a single CAS in both the insert operation and the remove operation.

The first balanced concurrent BST is proposed by Bronson [23]. They used an optimistic and relaxed balance method to build an AVL tree. Besa [24] employed a similar method for a red-black tree. Crain [25] proposed a method that decouples physical adjustment from logical removal by a background thread. Drachsler [10] mentioned an alternative technique called logical ordering, which uses the key order of the BST to optimize the contain operation. All of these works are built on fine-grained locks, and they are deadlock free. Based on special properties of quadtree, we also decouple the physical adjustment from logical removal and achieve a higher throughput.

There are other studies on concurrent trees. Crain [26] designed a concurrent AVL tree based on STM. Prokopec [27] used a control node, which is similar as our *Operation* object to develop a concurrent trie. Arbel [28] provided a balanced BST with RCU and fine-grained locks.

8 Conclusions

In this paper, we present a set of concurrent quadtree algorithms–*quadboost*, which supports concurrent insert, remove, contain, and move operations. In the remove operation, we decouple the physical update from the logical removal to improve concurrency. The continuous find mechanism checks out flags on quadtree to decide whether to move down or up. Further, our LCA-based move operation modifies two pointers with atomicity. The experimental results demonstrate that quadboost outperforms existing one-dimensional tree structures while maintaining a two-dimensional hierarchy. The quadboost algorithms are scalable with a variety of workloads and thread counts.

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APPENDIX

We provide a detailed proof of quadboost in this section. We follow the same naming convention in the paper. There are four kinds of *basic operations* in quadboost, i.e. the insert operation, the remove operation, the move operation, and the contain operation. Other functions are called *subroutines*, which are invoked by *basic operations*. The insert operation and the remove operation only operate on one terminal node, whereas the the move operation operates two different terminals–one for inserting a node with $newKey$, the other is for removing a node with $oldKey$. We call them $newKey$'s terminal and $oldKey$'s terminal, and we call their parents $newKey$'s parent and $oldKey$'s parent accordingly. Moreover, we name a CAS that changes a node's op a $flag$ operation and a CAS that changes a node's child a $replace$ operation. We define $snapshot_{T_i}$ as the state of our quadtree at some time T_i.

.1 Subroutines

To begin with, we have following observations from quadboost.

Observation 1. *The key field of a Leaf node is never changed. The op field of a Leaf node is initially null.*

Observation 2. *The space information–$\langle x, y, w, h\rangle$ of an Internal node is never changed.*

Observation 3. *The root node is never changed.*

Based on Observation 3, we derive a Corollary as follows:

Corollary 1. *Two nodes in quadtree must share a common search path starting from the root.*

Using these observations, we prove that each subroutine satisfies their specific pre-conditions and post-conditions. Because all basic operations invoke the find function at line 134 that returns $\langle l, pOp, path\rangle$ and the findCommon function at line 270 that returns $\langle il, iOp, iPath, rl, rOp, rPath\rangle$, we prove that the two functions satisfy their pre-conditions and post-conditions beforehand. We suppose that they execute from a $snapshot_{T_i}$ to derive these conditions.

Definition 1. *The pre-condition of find(l, pOp, path, keyX, keyY) invoked at T_i:*

1) *If path is not empty, l was on the direction $d \in \{nw, ne, sw, se\}$ of the top node in path at $T_{i1} \leq T_i$.*

The post-conditions of find(l, pOp, path, keyX, keyY) that returns a tuple $\langle l, pOp, path\rangle$ at T_i:

1) *l is a $Leaf$ node or an $Empty$ node.*
2) *At some $T_{i1} < T_i$, the top node in path has contained pOp.*
3) *At some $T_{i1} < T_i$, the top node in path has contained l.*
4) *If pOp was read at T_{i1}, and l was read at T_{i2}, then $T_{i1} < T_{i2} < T_i$.*
5) *For each node n in path, $size(path) \geq 2$, n_t is on the top of n_{t-1}, and n_t was on the direction $d \in \{nw, ne, sw, se\}$ of n_{t-1} at $T_{i1} leq T_i$.*

Definition 2. *The pre-conditions of findCommon(il, rl, lca, rOp, iOp, iPath, rPath, oldKeyX, oldKeyY, newKeyX, newKeyY) invoked at T_i:*

1) *If $iPath$ is not empty, il was on the direction $d \in \{nw, ne, sw, se\}$ of the top node in $iPath$ at $T_{i1} \leq T_i$.*
2) *If $rPath$ is not empty, rl was on the direction $d \in \{nw, ne, sw, se\}$ of the top node in $rPath$ at $T_{i1} \leq T_i$.*

The post-conditions of findCommon(il, rl, lca, rOp, iOp, iPath, rPath, oldKeyX, oldKeyY, newKeyX, newKeyY) that returns two tuples $\langle il, iOp, iPath\rangle$ and $\langle rl, rOp, rPath\rangle$ at T_i:

1) *rl was a Leaf or an Empty node.*
2) *il was a Leaf or an Empty node.*
3) *At some $T_{i1} < T_i$, the top node in $rPath$ has contained rOp.*
4) *At some $T_{i1} < T_i$, the top node in $iPath$ has contained iOp.*
5) *At some $T_{i1} < T_i$, the top node in $rPath$ has contained rl.*
6) *At some $T_{i1} < T_i$, the top node in $iPath$ has contained il.*
7) *If iOp was read at T_{i1}, and il was read at T_{i2}, then $T_{i1} < T_{i2} < T_i$.*
8) *If rOp was read at T_{i1}, and rl was read at T_{i2}, then $T_{i1} < T_{i2} < T_i$.*
9) *For each node n in $rPath$, $size(rPath) \geq 2$, n_t is on the top of n_{t-1}, and n_t was on the direction $d \in \{nw, ne, sw, se\}$ of n_{t-1} at $T_{i1} \leq T_i$.*
10) *For each node n in $iPath$, $size(iPath) \geq 2$, n_t is on the top of n_{t-1}, and n_t was on the direction $d \in \{nw, ne, sw, se\}$ of n_{t-1} at $T_{i1} \leq T_i$.*

By observing the find function and the findCommon function, we have following lemmas.

Lemma 1. *At the first time calling the find function, the contain operation, the insert operation, and the remove function start with l as an Internal node and an empty path.*

Proof. The contain operation at line 97, the insert operation at line 105 and the remove operation at line 161 start with the root node by $l = root$, an *Internal* node that is never changed (Observation 3).

Besides, the contain operation at line 95, the insert operation at line 103, and the remove operation at line 160 start with an empty stack to record nodes. □

Lemma 2. *At the first time the move operation calls the findCommon function, it starts with rl as an Internal node and $rPath$ and $iPath$ are empty.*

Proof. The move operation begins with the root node at line by $rl = root$ that is never changed (Observation 3) at line 204. Both $rPath$ and $iPath$ are initialized as empty stacks at line 199. □

Lemma 3. *All nodes pushed by the find function (line 134) and the continueFind function (line 122) are Internal nodes.*

Proof. Before pushing into the stack, the find function at line 135 first check the class of a node. Also, the continueFind function calls the find function at line 132 to push nodes into $path$. Thus, nodes other than *Internal* cannot be pushed. □

Lemma 4. *All nodes pushed by the findCommon function (line 270) the continueFindCommon function (line 285) are Internal.*

Proof. Before pushing into the stack, the findCommon function first checks the class of a node at line 271. Also, the continueFindCommon function calls the findCommon function at line 342. Or it calls the find function at line 323 and line 302 to push *Internal* nodes according to Lemma 3. Thus, nodes other than *Internal* cannot be pushed. □

Lemma 5. *For the loop in the find function at line 134 and the findCommon function at line 270, we suppose that path($rPath$) is empty and refer l(rl) and pOp(rOp) to each field updated by the loop. If the loop executes at least once and breaks at T_i, l(rl), pOp(rOp), and path($rPath$) satisfy following conditions:*

1) *l(rl) is a Leaf node or an Empty node.*
2) *At some $T_{i1} < T_i$, the top node in path($rPath$) has contained pOp(rOp).*
3) *At some $T_{i1} < T_i$, the top node in path($rPath$) has contained l(rl).*
4) *If pOp(rOp) was read at T_{i1}, and l(rl) was read at T_{i2}, then $T_{i1} < T_{i2} < T_i$.*
5) *For each node n in path($rPath$), $size(path) \geq 2$, n_t is on the top of n_{t-1}, and n_t was on the direction $d \in \{nw, ne, sw, se\}$ of n_{t-1} at $T_{i1} \leq T_i$.*

Proof. Apart from the terminate condition that the findCommon function judges whether two directions are the same or not, it performs in the same pattern as the find function. Both functions first push the previous *Internal* node l' into $path$, then get its *op*, and read a child pointer l finally.

The first condition always holds because by Lemma 4 and Lemma 3 *Leaf* nodes and *Empty* nodes will never be pushed into the $path(rPath)$.

We then prove part 2-4. At the last iteration, $pOp(rOp)$ and $l(rl)$ are read from l', which has already been pushed into the $path(rPath)$. Hence, part 2 and part 3 are correct. Besides, $pOp(rOp)$ is read before $l(rl)$ so that part 4 holds.

For the last part, we assume that at T_{i1} the lemma holds, so at $T_{i2} > T_{i1}$ l is read from l' which has already been pushed into the $path(rPath)$. As $path(rPath)$ is initially empty, and T_i is at the end of the last iteration, l which on the top of l' has been a child of it. □

Lemma 5 proves loop within the find function and the findCommon function satisfy post-conditions in Definition 1 and Definition 2. Next parts are going to show that other statements do not change these properties. We consider the first invocation as the base case, and prove lemmas by induction.

Lemma 6. *Every call to the find function satisfies its pre-condition and post-conditions.*

Proof. Now we prove the base case. We prove that initially the lemma holds.

The pre-condition:

By Lemma 1, the contain operation, the insert operation, and the remove operation start with an empty stack to record nodes.

For the move operation, by Lemma 2 it starts with rl as an *Internal* node and empty paths. Therefore it executes the loop more than once to push *Internal* nodes into $path$. In the last iteration rl is pushed at line 272, and the parent of it has already been push at the prior iteration. Then, rl is read from at line 278. Hence, rl at line 279 is a child of the top node in $rPath$ if it is not empty. $iPath$ is empty since it starts with an empty stack.

Therefore, initially the pre-conditions are satisfied.

The post-conditions:

The first condition always satisfies as the class of a node is checked at line 135.

By Lemma 1, the contain operation, the insert operation, and the remove operation start with an empty stack. Thu Lemma 5 indicates that post-conditions are satisfied.

For the find function within the findCommon function, by Lemma 5 it satisfies post-conditions when it breaks out. Hence, at line 282, post-conditions are satisfied because $iPath$ is empty. At line 279, if rl starts as a *Leaf* node, the post-conditions are also satisfied. Otherwise, there are two parts of $rPath$ at the time the loop breaks out, where we have proved each of them satisfy post-conditions by Lemma 5. Moreover, by the pre-condition, rl has been a child of the top node in $rPath$ of the find function. Thus, all nodes pushed later also satisfy the post-conditions.

We have proved the base case. Then we assume that for invocations $find_0, find_1, find_2 ... find_k$, the first k invocations satisfy the pre-conditions and post-conditions. We should prove that $find_{k+1}$ also satisfy the conditions. We have to consider all places where the find function is invoked.

The pre-condition:

First we consider the contain operation, the insert operation, and the move operation.

At line 98, line 106, line 163, the invocations follow the base case which we have proved satisfy the pre-condition.

At line 132, there are two scenarios l is read. l could either be read from the $path$ at line 126 or be assigned to its parent node at line 123. In the latter case, p was at the top of $path$ at $T_{i1} < T_i$ (line 109 and line 166) by the hypothesis. Otherwise l is assigned to a node in $path$. As there's no other push operations, n_t on the top of n_{t-1} must be its child at some $T_{i1} < T_i$ by the hypothesis.

Then, we discuss the move operation.

At line 279, the find function is wrapped by the findCommon function. Since Lemma 5 shows that the loop set rl as a child of the top node in $rPath$, we have to prove that either it is an *Internal* node or an *Leaf* node that was a child of the top node before entering the loop. At line 212, the findCommon function follows the base case. At line 342, Corollary 1 shows that the root node will always be a LCA node. Hence, after setting $rPath$ to its lca index at line 334, it contains at least one node. By Lemma 3 and Lemma 4, rl is an *Internal* node popped from $rPath$ at line 337.

At line 282, the find function is also wrapped by the findCommon function. We could prove that $iPath$ is empty. At line 212, the findCommon function follows the base case. At line 342, $iPath$ is set to empty before the invocation (line 335).

At line 302, rl is either assigned to rp (line 288) or a node popped from $rPath$ (line 295). If it is popped from $rPath$, our hypothesis ensures that it was a child of the top node from $rPath$. If it is read from rp, which is an *Internal* node popped from $rPath$ at line 214, line 344, or line 305, it also satisfies the claim.

At line 323, likewise, il is either assigned to ip (line 312) or a node popped from $iPath$ (line 315). If it is popped from $iPath$, our hypothesis ensures that il was a child node of the top node from $iPath$ if it is not empty. If it is read from ip, which is an *Internal* node popped from $iPath$ at line 213, line 345, or line 326, it also satisfies the claim.

The post-conditions:

The first condition always holds. Because by Lemma 3, if l is an *Internal* node, it will be pushed into $path$.

We then prove other conditions.

First we consider the contain operation, the insert operation, and the move operation.

At line 98, line 106, line 163, the invocations follow the base case which we have proved satisfy the post-conditions.

At line 132, there are two scenarios l is read. l could either be read from the $path$ at line 126 or be assigned to its parent node at line 123. In both cases, l was an *Internal* node in $path$. Therefore Lemma 5 indicates that all nodes followed by l satisfy the post-conditions.

Then, we discuss the move operation.

At line 279, the find function is wrapped by the findCommon function. The pre-condition indicates rl was a child of $rPath$ if it is not empty. Hence, it suffices to show that the post-conditions are satisfied by Lemma 5.

At line 282, the find function is also wrapped by the findCommon function. The pre-condition indicates $iPath$ is an empty. In this way, by Lemma 5 we prove the post-conditions.

At line 302, likewise, the pre-condition suggests that rl was an *Internal* node in $rPath$. By Lemma 5 we also prove the post-conditions.

At line 323, the pre-condition suggests that il was an *Internal* node in $iPath$. Therefore, by Lemma 5 we prove the post-conditions. □

After showing the pre-condition and post-conditions of the find function, we the prove that the findCommon function that returns $\langle il, iOp, iPath, rl, rOp, rPath\rangle$ satisfies its pre-conditions and post-conditions.

Lemma 7. *Every call to the findCommon function satisfies its pre-conditions and post-conditions.*

Proof. The pre-condition:

We prove the lemma by induction. First we prove the base case.

At line 212, the findCommon function is initially invoked, and both $iPath$ and $rPath$ are empty (line 199).

Then we assume that for invocations $findCommon_0$, $findCommon_1$, $findCommon_2$ ... $findCommon_k$, the first k invocations satisfy the pre-conditions. We prove that $findCommon_{k+1}$ also satisfies the conditions.

There are two places the findCommon function is invoked. At line 212, the base case proves the post-conditions. At line 342, Corollary 1 shows that the root node will always be a LCA node. Hence, after setting $rPath$ to its lca index at line 334, it contains at least one node. By Lemma 3 and Lemma 4, rl is an *Internal* node popped from $rPath$ at line 337. Hence, we prove the pre-condition.

The post-conditions:

The findCommon function wraps two find functions at line 282 and line 279. Because $\langle rl, rOp, rPath\rangle$ that passed from the find function for rl and $\langle il, iOp, iPath\rangle$ that passed from the find function for il hold the post-conditions, the lemma is true. □

After proving the pre-conditions and post-conditions of the find function and the findCommon function, we shall prove that the continuous find mechanism (i.e. the continueFind function and the continueFindCommon function) satisfies its pre-conditions and post-conditions.

Definition 3. *The pre-conditions of continueFind(pOp, path, l, p):*

1) *l was child of p.*
2) *p was child of the top node in $path$ if it is not empty.*

The post-conditions are the same as the find function.

Definition 4. *The pre-conditions of continueFindCommon(il, rl, lca, rOp, iOp, iPath, rPath, ip, rp, oldKeyX, oldKeyY, newKeyX, newKeyY, iFail, rFail, cFail):*

1) *At least one of $iFail$, $rFail$ or $cFail$ is true.*
2) *il was a child of ip.*
3) *ip was a child of the top node in $iPath$ if it is not empty.*
4) *rl was a child of rp.*
5) *rp was a child of the top node in $rPath$ if it is not empty.*

The post-conditions are the same as the findCommon function.

We now prove that the continueFind function satisfies its pre-conditions and post-conditions.

Lemma 8. *Every call to the continueFind function satisfies its pre-condition and post-conditions.*

Proof. The post-conditions:

At line 132 of the continueFind function, it executes the find function. Because the find operation satisfies its post-conditions by Lemma 6, the continueFind function also satisfies the same conditions.

The pre-condition:

The continueFind function is invoked at line 119 and line 176.

Initially, the continueFind function follows the find function at line 106 and line 163. For the first part, the post-condition of the find function shows that l was a child of p which popped at line 109 or line 166(Lemma 5). For the second part, after popping p, it becomes a child of the top node in $path$ if it is not empty.

Otherwise, it reads results from the continueFind invocation at the prior iteration. As the post-conditions of the continueFind function is the same as the find function, we prove the lemma. □

We now prove that the continueFindCommon function satisfies its pre-conditions and post-conditions.

Lemma 9. *Every call to the continueFindCommon function satisfies its pre-conditions and post-conditions*

Proof. The post-conditions:

There are three cases, the first part of the pre-condition shows that it must enter one of the loop. We consider each case by the program execution order.

1) If $rFail$ is true and $cFail$ is false, it executes the case starts from line 236. If $cFail$ is not set to true, it executes the find function at line 302. Because the find function satisfies its post-conditions (Lemma 6), the lemma holds. If $iFail$ is true, it comes to the second part of the proof. If $cFail$ is set to true, it comes to the third part of the proof.
2) If $iFail$ is true and $cFail$ is false, it executes the case starts from line 235. If $cFail$ is not set to true, it executes the find function at line 323. Because the find function satisfies its post-conditions, part 1 and Lemma 6 indicate the Lemma holds. If $cFail$ is true, it comes to the third part of the proof.
3) If $cFail$ is true, it executes the case starts from line 240. Because the findCommon function satisfies its post-conditions (Lemma 7), it establishes this part of the lemma.

The pre-condition:

The continueFindCommon function is invoked at line 243.

For the first condition, line 216, line 217, line 218, line 232, line 235, line 236, line 240 indicate that at least on of three flags is set to true.

For the next conditions, initially the continueFindCommon follows the findCommon function at line 212. As ip and rp are popped at line 213 and line 214, il was a child of ip and rl was a child of rp. Further, ip and rp was a child of the top node in $iPath$ and $rPath$. All these claims are based on Lemma 8. Otherwise, the continueFindCommon function reads results from its invocation at the prior iteration. In such a case, our post-conditions prove the pre-conditions. □

We observe that the find function, the continueFind function, the findCommon function, and the continueFindCommon function return a similar pattern of results. The find function and the continueFind function return a tuple $\langle l, pOp, path\rangle$. The findCommon function and the continueFindCommon function return two such tuples. Hence We use $find(keys)$ to denote such a set of find operations for searching keys. Further, we number $find(keys)$ by their invocation orders. Invocations at line 106, line 163, and line 212 are at $iteration_0$. Invocations at line 106, line 163, and line 212 are at $iteration_i, 0 < i$.

Definition 5. *For $compress(path, p)$ invoked at T_i, it has following pre-conditions:*

1) *p is an Internal node.*
2) *$path$ is read from the result of $find(keys)$ at the prior iteration.*
3) *If $path$ is not empty, p was a child node of the top node in $path$ at $T_{i1} < T_i$.*

Lemma 10. *Every call to the compress function satisfies its pre-conditions.*

Proof. The compress function is called at line 171 or line 229. At line 171, p is a node popped from $path$ (line 166. At line 229, rp is an *Internal* node popped from $rPath$ (line 214, line 305 or line 344. This proves the first part.

For last two parts, since the compress function reads result following $find(keys)$, it suffices to prove the post-conditions of $find(keys)$ satisfy the preconditions. By Lemma 8, Lemma 9, Lemma 6, and Lemma 7, the post-conditions of the find function, the findCommon function, the continueFind function, and the continueFindCommon function establish the lemma. □

Corollary 2. *Every call to the check function satisfies its pre-conditions that p is an Internal node.*

Definition 6. *For $moved(node)$ invoked at T_i, it has following the pre-conditions that $node$ is a Leaf node or an Empty node.*

Lemma 11. *Every call to the moved function satisfies its pre-condition.*

Proof. The moved function is called at line 108, line 165, line 283, line 280, line 324, line 303.

At line 108, it reads l from line 106 or line 119. Hence, according to Lemma 6, l is a *Leaf* node or an *Empty* node.

At line 165, it reads l from line 163 or line 176. Hence, according to Lemma 6, l is a *Leaf* node or an *Empty* node.

At line 283 and line 280, it reads il from line 282 and rl from line 279. Therefore, according to Lemma 6, il and rl are *Leaf* node or *Empty* nodes.

At line 324, it reads il from line 323. Therefore, according to Lemma 6, il is a *Leaf* node or an *Empty* node.

At line 303, it reads rl from line 302. Therefore, according to Lemma 6, rl is a *Leaf* node or an *Empty* node. □

Based on these post-conditions, we show that each op created at T_i store their corresponding information.

Lemma 12. *For op created at T_i:*

1) *If op is a Substitute object, $op.parent$ has contained $op.oldChild$, a Leaf node, at $T_{i1} < T_i$ from the result of $find(keys)$ at the previous iteration.*
2) *If op is a Compress object, $op.grandparent$ has contained $op.parent$, an Internal node, at $T_{i1} < T_i$ from the result of $find(keys)$ at the previous iteration.*
3) *If op is a Move object, $op.iParent$ has contained $op.oldIChild$, a Leaf node, $op.rParent$ has contained $op.oldRChild$, a Leaf node, $op.iParent$ has contained $op.oldIOp$, and $op.rParent$ has contained $op.oldROp$ before T_i from the result of $find(keys)$ at the previous iteration.*

Proof.
1) A *Substitute* object is created at line 112 or line 168 by assigning $\langle p, l, newNode\rangle$. Since p read from the top node in $path$ at line 109 or line 166, and l is returned by the find function or findCommon function, the post-conditions of them establish the claim according to Lemma 6, Lemma 7, Lemma 9, and Lemma 8.
2) A *Compress* object is created at line 185 by assigning $\langle path, p\rangle$. p is passed from the top of $path$ at line 166 or line 214, the post-conditions of $find(keys)$ and the property of $path$ establishes the claim (Lemma 6, Lemma 7, Lemma 9, Lemma 8, Lemma 3, and Lemma 4).
3) A *Move* object is created at line 224 by assigning $\langle ip, rp, il, rl, newNode, iOldOp, rOldOp\rangle$. ip is assigned to the top node of $iPath$ at line 213, rp is

assigned to the top node of $rPath$ at line 214, il was a child of the top node of $iPath$, rl was a child of the top node of $rPath$, and iOp and rOp are read from ip and rp by the post-conditions of the findCommon function and the continueFindCommon function (Lemma 9 and Lemma 7.

□

Then we prove the pre-conditions of other functions which use the result of above functions to modify quadtree.

The following lemmas satisfy a basic pre-condition that their arguments are the same type as indicated in algorithms.

Lemma 13. 1) *Every call to the help function satisfies its pre-conditions.*
2) *Every call to the helpSubstitute function satisfies its pre-conditions.*
3) *Every call to the helpCompress function satisfies its pre-conditions.*
4) *Every call to the helpMove function satisfies its pre-conditions.*

Proof. 1) The help function is called at line 118, line 175, line 311, line 287, line 333, and line 332.
At line 118 and line 175, pOp might be obtained from the result of the find function and the continueFind function. Hence, the post-conditions of them ensures that pOp is an *Operation* object read from p. Otherwise, pOp might be obtained from line 116, or line 173 by reading a node's *op*.
At line 311 and line 287, rOp and iOp are passed the move operation. The post-conditions of the continueFindCommon function and the findCommon function (Lemma 7 and Lemma 9), with line 235, line 236, and line 240 guarantee that arguments' type are the same.
At line 333 and line 332, rOp and iOp could be passed from the move function or read from ip and rp in the continueFindCommon function at lines that have been mentioned.
2) The helpSubstitute function is called at line 114, line 170, and line 150. At line 114 and line 170, *op* is created at line 112 or line 168 respectively. At line 150, it checks whether *op* is *Substitute* before calling the helpSubstitute function.
3) The helpCompress function is called at line 185, line 317, line 297, and line 339. For each invocation, it checks whether *op* is *Compress* beforehand.
4) The helpMove function is invoked at line 228, line 239, and line 151. At line 228 and line 239, *op* is created at line 224. At line 151, it checks whether *op* is *Move* before the invocation.

□

Corollary 3. *For $help(op)$, op is read from an Internal node.*

Definition 7. *For $helpFlag(p, oldOp, newOp)$ invoked at T_i, it has the pre-conditions:*

1) *p is an Internal node.*
2) *For $cflag$, p has contained pOp at $T_{i1} < T_i$ after reading p; otherwise, p has contained pOp at $T_{i1} < T_i$ from the result of $find(keys)$ at the prior iteration.*

Lemma 14. *Every call to the helpFlag function satisfies its pre-conditions.*

Proof. The helpFlag function is invoked at line 113, line 169, line 186, line 226, line 248-line 250, and line 261-line 267.

At line 113 and line 169, the post-conditions of the find function and the continueFind function (Lemma 6 and Lemma 8 imply that p is an *Internal* node popped at line 109 and has contained pOp.

At line 186, the pre-condition of the compress function (Lemma 10 shows that p and nodes from $path$ are read from $find(keys)$ at the prior iteration, and pOp is read at line 181. Thus, p has pOp at $T_{i1} < T_i$ after reading p.

At line 226, ip and rp are popped from $iPath$ at line 213 and line 214 or read from the continueFindCommon function. By Lemma 7 and Lemma 9, either $op.iParent$ has contained $op.iOp$ or $op.rParent$ has contained $op.rOp$.

At line 248-line 250 and line 261-line 267, by the pre-condition of the helpMove function 13, *op* is a *Move* object. Therefore, according to Lemma 12, either $op.iParent$ has contained $op.iOp$ or $op.rParent$ has contained $op.rOp$. □

Lemma 15. *Every call to the hasChild function satisfies its pre-conditions that $parent$ is an Internal node.*

Proof. According to Lemma 11, $node$ is a *Leaf* node or an *Empty* node. According to Lemma 14, flag operations only perform on *Internal* nodes. Therefore line is the only place that set a *Move op* on a *Leaf* node. By Observation 1, *op* is initially null. After checking the condition, *op* is assigned to a *Leaf* node where $op.iParent$ is an *Internal* node by Lemma 12 □

Definition 8. *For $helpReplace(p, oldChild, newChild)$ invoked at T_i, it has the pre-conditions:*

1) *p, $oldChild$, and $newChild$ are read from the same op.*
2) *$newChild$ is a node that has not been in quadtree.*
3) *p is an $Internal$ node that has contained $oldChild$ at $T_{i1} < T_i$.*

Lemma 16. *Every call to the helpReplace function satisfies its pre-conditions.*

Proof. For the first condition:

The helpReplace function is invoked at line 145, line 193, line 254-259.

By Lemma 13, at line 193, it reads a *Compress* object; at line 145, it reads a *Substitute* object; at line 254-259. it reads a *Move* object.

For the second condition:

Since part 1 illustrates that p, $oldChild$, and $newChild$ are read from the same *op*, if *op* is a *Substitute* object, $op.newNode$ is created at line 111 or line 161; if *op* is a *Compress* object, each time the helpCompress function creates a new Empty node at line 193; if *op* is a *Move* object, $op.newIChild$ is created at line 220, and an empty node is created at line 145.

For the third condition:

Since part 1 illustrates that p, $oldChild$, and $newChild$ are read from the same *op*, Lemma 12 shows that p has contained $oldChild$. □

Definition 9. *For $createNode(l, p, newKeyX, newKeyY, value)$ that returns a $newNode$ invoked at T_i, it has the pre-conditions:*

1) *p is an Internal node.*
2) *p has contained l at $T_{i1} < T_i$ from the result of $find(keys)$ at the prior iteration.*

post-conditions:

The $newNode$ returned is either a Leaf node with $newKey$ and $value$, or a sub-tree that contains both $l.key$ node and $newKey$ node with the same parent.

Lemma 17. *Every call to the createNode function satisfies its pre-conditions and post-condition.*

Proof. The createNode function is invoked at line 111 and line 221.

At line 111, the post-conditions of the find function and the continueFind function ensure that l has been a child of p that popped from $path$ at line 109 (Lemma 6 and Lemma 8).

At line 221, the post-conditions of the findCommon function and the continueFindCommon function ensure that l has been a child of p popped at line 213. (Lemma 7, Lemma 9).

Though we have not presented the details of the createNode function, our implementation guarantee that the post-condition must be satisfied. □

.2 Flag and replace operations

We argue that each successful CAS operation occurs in a correct order. At outset, we argue that CAS behaviours for each helpFlag function. A flag operation involves three arguments: *node*, *oldOp*, *newOp*. A replace operation involves three arguments: *p*, *oldChild*, *newChild*. From Figure 9, we denote each flag operation accordingly. $iflag$ occurs in the insert operation, $rflag$ occurs in the remove operation, and $mflag$ occurs in the move operation. Moreover, we specify a flag operation which attaches a *Clean op* on a node as an unflag operation. We call *ireplace* as the replace operation occurs in the insert operation, *rreplace* as the replace operation occurs in the remove operation, *mreplace* as the replace operation occurs in the move operation, and *creplace* as the replace operation occurs in the compress operation. The next lemmas describe the behaviours of the helpFlag function according to the state transition diagram.

Observation 4. *op is only attached on an Internal node.*

Lemma 18. *For a new node n:*

1) *It is created with a Clean op.*
2) *$rflag$, $iflag$, $mflag$ or $cflag$ succeeds only if n's op is Clean.*
3) *$unflag$ succeeds only if n's op is Substitute or Move.*
4) *Once n's op is Compress, its op will never be changed.*

Proof.

1) As shown at line 71, *Internal* nodes are assigned a *Clean op* when they are created.
2) Before $iflag$, *op* is checked at line 110; before $rflag$, *op* is checked at line 167; before $cflag$, *op* is checked at line 182. For the move operation, before $mflag$ on *oldKey*'s parent and $newKey$'s parent, they are checked at line 217 and line 216.
3) $unflag$ is called at line 146 and line 261-line 267, where the pre-conditions specify that *op* is *Move* or *Substitute* according to Lemma 13.
4) By part 3, there's no unflag operation happens on an *Internal* node with a *Compress op*.

□

Next we illustrate that there's no ABA problem on any *op*. That is to say, in terms of an *Internal* node n, its op_i at T_i has not appeared at $T_{i1} < T_i$, $op_{i1} \neq op_i$. We prove the following lemma:

Lemma 19. *For an Internal node n, it never reuse a op that has been set previously.*

Proof. If a node's *op* is set to op_i at T_i, it has never been appeared before. We prove the lemma by discussing different types of $flag$ operations.

For $iflag$, a *Substitute op* is created at line 112 before its invocation. For $rflag$, a *Substitute op* is created at line 168 before its invocation. For a $cflag$, a Compress *op* is created at line 185 before its invocation. For $mflag$, a *Move op* is created at line 224 before its invocation. Because each *newOp* is newly created, it has not been set before. For $unflag$, each time it creates a new *Clean op* to replace the prior *Move op* or *Substitute op* by Lemma 18. Therefore, the new *Clean op* has never appeared before. □

We have shown that for each successful flag operation, it never set an *op* that has been used before. Every *Internal* node is initialized with *Clean op* by Lemma 18 and we use a flag operation to change *op* at the beginning of each CAS transition. We say successive replace operations and unflag operations that read the *op* belongs to it. According to Figure 9, there are threes categories of successful CAS transitions. We denote them as $flag \rightarrow unflag$, $flag \rightarrow replace$, and $flag \rightarrow replace \rightarrow unflag$.

Let $flag_0$, $flag_1$, ..., $flag_n$ be a sequence of successful flag operations; let $unflag_0$, $unflag_1$, ..., $unflag_n$ be a sequence of successful unflag operations. $flag_i$ attaches op_i, and $replace_i$ and $unflag_i$ read op_i and come after it. Therefore we say $flag_i$, $replace_i$, and $unflag_i$ belong to the same *op*. In addition, if there are more than one replace operations belong to the same op_i, we denote them as $replace_i^0$, $replace_i^1$, ..., $replace_i^k$ ordered by their successful sequence. Similarly, if there are more than one flag operations that belong to op_i on different nodes, we denote them as $flag_i^0$, $flag_i^1$, ..., $flag_i^k$. A similar notation is used for $unflag_i$. The following lemmas prove the correct ordering of three different transitions.

Lemma 20.

1) *ireplace will not be done before the success of $iflag$ which belongs to the same op.*
2) *rreplace will not be done before the success of $rflag$ which belongs to the same op.*

Proof. For the insert operation and the remove operation, at line 145 the helpReplace function is wrapped by the helpSubstitute function. The helpSubstitute function is called at line 114, line 170, and line 150. At line 114, it is called after the success of $iflag$ at line 113. At line 170, it is called after the success of $rflag$ at line 169. At line 150, by Corollary 3 and Lemma 18, there must have been $iflag$ or $rflag$ that change the node's *op* from *Clean* to *Substitute*. □

Lemma 21. *creplace will not be done before the success of $cflag$ which belongs to the same op.*

Proof. The helpCompress function which wraps $creplace$ is called at line 185, line 317, line 297, and line 339, and line 149. At line 185, the helpCompress function follows the successful helpFlag function at line 186. At line 317, line 297, and line 339, a pOp is read from *Internal* nodes. At line 149, Lemma 3 shows that *op* is read from an *Internal* node. Hence, by Lemma 18, successful $cflag$ must have changed the node's *op* from *Clean* to *Compress*. □

Lemma 22. *$mreplace$ will not be done before the success of $mflag$ which belongs to the same op.*

Proof. The helpMove function which wraps two $mreplace$s is called at line 228, line 239, and line 151. At line 228 and line 239, the *Move op* is created at line 267. Otherwise, at line 151, by Corollary 3, *op* is read from an *Internal* node. Hence, Lemma 18 demonstrates that a successful $mflag$ must have changed the node's *op* from *Clean* to *Move*. □

From the above lemmas, we have the following corollary:

Corollary 4. *A successful replace operation which belongs to an op will not succeed before it's flag operations have been done.*

We discuss each CAS transition accordingly. We begin by discussing the $flag \rightarrow replace \rightarrow unflag$ transition. We refer $\langle oldIOp, oldIChild, newIChild\rangle$ and $\langle oldROp, oldRChild, newRChild\rangle$ to different tuples of $\langle oldOp, oldChild, newChild\rangle$.

Lemma 23. *$flag \rightarrow replace \rightarrow unflag$ occurs when $rflag_i$, $iflag_i$ or $mflag_i$ succeeds, and it has following properties:*

1) *$replace_i$ never occurs before $flag_i$.*
2) *$flag_i^k, 0 \leq k < |flag_i|$ is the first successful flag operation on $op_i.parent^k$ after T_{i1} when pOp_i is read.*
3) *$replace_i^k, 0 \leq k < |replace_i|$ is the first successful replace operation on $op_i.parent^k$ after T_{i2} when $op_i.oldChild^k$ is read.*
4) *$replace_i^k, 0 \leq k < |replace_i|$ is the first successful replace operation on $op_i.parent^k$ that belongs to op_i.*
5) *$unflag_i^k, 0 \leq k < |unflag_i|$ is the first successful unflag operation on $op_i.parent^k$ after $flag_i^k$.*
6) *There is no successful unflag operation occurs before $replace_i$.*
7) *The first replace operation on $op_i.parent^k$ that belongs to op_i must succeed.*

Proof.
1) Corollary 4 proves the claims.
2) By Lemma 14, for each flag operation, p has contained pOp at some time T_{i1}. Suppose that there's another $flag^k$ that occurs after T_{i1} but before $flag_i^k$, then $flag_i^k$ fails because Lemma 19 shows that *op* is not reused. Therefore, it contradicts to the definition that $flag_i^k$ is a successful flag operation.
3) By Lemma 16, for each replace operation, p has contained $oldChild$ at some time T_{i1}. Suppose that there's another $replace$ that occurs after T_{i1} but before $replace_i^k$, then $replace_i^k$ fails because Lemma 16 shows that $newChild$ is never a node in the tree. Therefore, it contradicts to the definition that $replace_i^k$ is a successful flag operation.
4) Suppose there's $replace^k$ that belongs to op_i and occurs before $creplace_i^k$. It must follow $flag^k$ according to Corollary 4. If $flag^k$ happens before $flag_i^k$, it fails because $fail_i^k$ will fail by consequence (Lemma 19). If $flag^k$ happens after $flag_i^k$, it contradicts Lemma 18 that a flag operation starts Because $flag^k$ is done after reading $op_i.oldChild^k$, $replace^k$ is also after reading it. Therefore, if $flag^k$ succeeds, it contradicts part 3 that $replace_i^k$ is the first successful replace operation after reading $op_i.oldChild^k$.
5) Based on Lemma 14, if there is another $unflag$ changes $p.op$, $unflag_i^k$ that use op_i as the *oldOp* will fail. This contradicts to the definition that $unflag_i^k$ is a successful unflag operation that reads op_i.
6) We consider each operation separately.
For the insert operation and the remove operation, $unflag$ follows $replace$ immediately at line 146. Assuming that $replace_i^k$ occurs after $unflag_i^k$, there must be another $replace$ reads op_i and changes the link. It contradicts part 3 of the proof that $replace_i^k$ is the first successful replace operation that belongs to op_i.
For the move operation, because we assume that replace operations exist, $doCAS$ is set to true before performing replace operations. An unflag operation follows replace operations from line 261-267. Consider if a $replace_i^k, 0 \leq k < |replace_i|$ happens after any unflag operation, then another $replace$ which use the same *op* must have succeed as the program order specifies that $replace$ execute before $unflag$ for a process. Hence, $replace_i^k$ fails and contradicts the definition.
Therefore all successful unflag operations occur after $replace$.
7) According to Corollary 4, $replace_i^k$ follows $flag_i^k$ belongs to op_i. Suppose there exists $replace$ after reading $op_i.parent^k$ and before $replace_i^k$. Then, there must be $flag$ that precedes $replace$ by Corollary 4. By part 1, $flag_i$ is the first successful flag operation after reading op_i. By Lemma 14, op_i is read before $op_i.l$. Hence, there does not exist any replace operation between T_{i1} reading $op_i.oldChild^k$ and $flag_i^k$. If $flag$ happens between $flag_i^k$ and $replace_i^k$, it will also fail because $op_i.parent^k.op$ is not *Clean*. Then, the first replace operation that belongs to op_i must succeed.

□

We then discuss the ordering of the $flag \rightarrow replace$ transition.

Lemma 24. *$flag \rightarrow replace$ occurs only when $cflag_i$ succeeds, it has following properties:*

1) *$creplace_i$ never occurs before $cflag_i$.*
2) *$cflag_i$ is the first successful flag operation on $op_i.parent$ after T_{i1} when rOp_i is read.*
3) *$creplace_i$ is the first successful replace operation on $op_i.grandparent$ after T_{i1} when $op_i.parent$ is read.*
4) *$creplace_i$ is the first successful replace operation on $op_i.grandparent$ that belongs to op_i.*
5) *There is no unflag operation after $creplace_i$.*

6) *The first replace operation that belongs to op_i must succeed.*

Proof. 1) Corollary 4 proves the claims.

2) By Lemma 14, for each flag operation, p has contained pOp at some time T_{i1}. Suppose that there's another $cflag$ that occurs after T_{i1} but before $cflag_i$, then $cflag_i$ fails because Lemma 19 shows that *op* is not reused. Therefore, it contradicts to the definition that $cflag_i$ is a successful flag operation.
3) By Lemma 16, for each replace operation, p has contained $oldChild$ at some time T_{i1}. Suppose that there's another $creplace$ that occurs after T_{i1} but before $replace_i$, then $creplace_i$ fails because $oldChild$ remains the same but the link has been changed to another node. Therefore, it contradicts to the definition that $creplace_i$ is a successful flag operation.
4) Suppose there's a replace operation that belongs to op_i and occurs before $creplace_i$. It must follow $cflag$ according to Corollary 4. If $cflag$ happens before $cflag_i$, it fails because $cfail_i$ will fail by consequence (Lemma 19. If $cflag$ happens after $cflag_i$, it contradicts Lemma 18 that a flag operation starts with a *Clean op*. Therefore, $cflag$ is the same as $cflag_i$. Because $cflag$ is done after reading $op_i.parent$, $creplace$ is also after reading it. Therefore, if $cflag$ succeeds, it contradicts part 3 that $creplace_i$ is the first successful replace operation after reading $op_i.parent$.
5) By Lemma 18, once a node's *op* is *Compress*, it will never be changed.
6) Suppose the first replace operation that belongs to op_i fails, there must exist some successful $replace$ after reading $op_i.parent$ and before $creplace_i$. Further, there is $flag$ that precedes $replace$ by Corollary 4. If $flag$ happens between $creplace_i$ and $cflag_i$, $flag$ will fail by Lemma 18. If $flag$ happens between reading op_i and $cflag_i$, it contradicts part 1 that $cflag_i$ is the first successful flag operation after reading op_i. Because op_i is read after $op_i.parent$, we also have to consider if $flag$ happens before reading op_i. Consider if $cflag$ happens, $flag_i$ will fail because $op_i.parent.op$ will not be set back to *Clean* according to the fourth part. Consider if $mflag$, $iflag$ or $rflag$ happens, it will not change the link from $op_i.grandparent$ to $op_i.p$ according to Lemma 23.

□

Corollary 5. *The first replace operation belongs to op_i must succeed.*

Next we discuss the $flag \rightarrow unflag$ transition, where replace operation occurs between a successful flag operation and an unflag operation that belong to the same *op*. We suppose that $op.iFirst$ is true. The claim for $op.iFirst = false$ is symmetric.

Observation 5. *$allFlag$ and $iFirst$ in a Move op are initially false, and they will never be set false after assigning to true.*

Lemma 25. *For $flag \rightarrow unflag$ circle, it only results from a Move object op_i such that:*

1) *$unflag_i$ is the first successful unflag operation on $op_i.rParent$.*
2) *The first flag operation on $op_i.iParent$ must fail, and no later flag operation will succeed.*
3) *$op_i.iParent$ and $op_i.rParent$ are different.*

Proof. 1) We prove it by contradiction. Suppose there's another unflag operation that reads op_i and succeeds before $unflag_i$ on $op_i.rParent$. It must have changed $op_i.rParent.op$ to a new *op*, by Lemma 19 $unflag_i$ will fail. This contradicts the definition of $unflag_i$.

2) For the former part of the claim, we prove it by contradiction. Assuming that $op_i.iParent$ is flagged on op_i, and will be unflagged without performing replace operations. However, unflag operations occur after the judgement at line 254. The first process must set $doCAS$ to true. In the next step, it executes line 254-259. According to Corollary 5, the first $replace_i^t$ must succeed. Therefore it contradicts the definition that there's no replace operation before an unflag operation. For the latter part, by Lemma 12 $op_i.iParent$ has contained $op_i.oldIOp$. Because the first flag operation on $op_i.iParent$ fails, latter flag operations read the $op_i.oldIOp$ also fail.
3) Assuming $op_i.iParent$ and $op_i.rParent$ are the same. By part 2, flag operations on $op_i.iParent$ must fail, hence $doCAS$ will never be true. $op_i.allFlag$ is always false by by Observation 5. In this way, line 261-267 prevents any unflag operation. This derives a contradiction.

□

.3 Quadtree properties

In the Section, we use above lemmas to show that quadtree's properties are maintained during concurrent modifications.

Definition 10. *Our quadtree has these properties:*

1) *Two layers of dummy Internal nodes are never changed.*
2) *An Internal node n has four children, which locate in the direction $d \in \{nw, ne, sw, se\}$ respectively according to their $\langle x, y, w, h \rangle$, or $\langle keyX, keyY \rangle$.*
For Internal nodes reside on four directions:
   - $n.nw.x = n.x$, $n.nw.y = n.y$;
   - $n.ne.x = n.x + w/2$, $n.ne.y = n.y$;
   - $n.sw.x = n.x$, $n.sw.y = n.y + n.h/2$;
   - $n.se.x = n.x + w/2$, $n.se.y = n.y + h/2$,

   and all children have their $w' = n.w/2$, $h' = n.h/2$.
   For Leaf nodes reside on four directions:
   - $n.x \leq n.nw.keyX < n.x + n.w/2$, $n.y \leq n.nw.keyY < n.y + n.h/2$;
   - $n.x+n.w/2 \leq n.ne.keyX < n.x+n.w$, $n.y \leq n.ne.keyY < n.y + n.h/2$;
   - $n.x \leq n.sw.keyX < n.x + n.w/2$, $n.y + n.h/2 \leq n.sw.keyY < n.y + n.h$;
   - $n.x+n.w/2 \leq n.se.keyX < n.x+n.w$, $n.y+n.h/2 \leq n.se.keyY < n.y + n.h$.

Lemma 26. *Two layers of dummy nodes are never changed.*

Proof. We prove the lemma by induction. Only replace operations will affect the structure of quadtree as we define in Definition 10.

Initially the property holds as we initialize quadtree with two layers of dummy nodes and a layer of *Empty* nodes.

Suppose that after $replace_i$ the lemma holds, we shall prove that after $replace_{i+1}$ the lemma is still true. For $ireplace$, $rrepalce$, and $mreplace$, the pre-condition 16 ensures that three nodes are from the same *op*. Thus, as Lemma 12 further ensures that they only swing *Leaf* nodes and *Empty* node, dummy node are not changed. For $creplace$, since at $replace_i$ the property is true, the root is connected to a layer of dummy nodes. Lemma 12 shows that $op.oldChild$ is an *Internal* node, and line 186 shows that the root node's child pointer will never be changed. Therefore, $replace_{i+1}$ will not occur on the root node. □

Lemma 27. *Children of a node with Compress op will not be changed.*

Proof. Based on Lemma 18, the node's *op* will never be changed after being set to *Compress*. Lemma 24 and Lemma 23 indicate that for a node flagged with *Compress op*, only itself will be unlinked. This establishes the Lemma. □

Lemma 28. *Only an Internal node with all children Empty could be attached with a Compress object.*

Proof. Corollary 2 ensures that the pre-conditions of the check function is satisfied. It checks whether all children are *Empty* before calling the helpFlag function at line 186. This establishes the Lemma. □

Lemma 29. *An Internal node whose op is not Compress is reachable from the root.*

Proof. We have to consider all kinds of replace operations.

For $ireplace$, $rrepalce$, and $mreplace$, by Lemma 16 and Lemma 12 they use a new node to replace a *Leaf* node or an *Empty* node. Thus, *Internal* nodes are still reachable from the root.

For $creplace$, by Lemma 16 and Lemma 12, it replaces *Internal* nodes. Lemma 24 shows that it is preceded by $cflag$. Lemma 28 implies that all children are *Empty* nodes. Therefore, an *Internal* node with a *Compress op* and its children are not reachable from the root only after the success of $creplace$. □

Next we define *active* set and *inactive* set for different kinds of nodes. We say a node is *moved* if the moved function turns true. For an *Internal* node or an *Empty* node, if it is reachable from the root in $snapshot_{T_i}$, it is active; otherwise, it is *inactive*. For a *Leaf* node, if it is reachable from the root in $snapshot_{T_i}$ and not moved, it is active; otherwise, it is *inactive*. We denote $path(keys^k), 0 \leq k < |replace_i|$ as a stack of nodes pushed by $find(keys)$ in a snapshot. We define $physical\_path(keys^k)$ to be the path for $keys^k$ in $snapshot_{T_i}$, consisting of a sequence of *Internal* nodes with a *Leaf* node or an *Empty* node at the end, which is the actual path in the snapshot. We say a subpath of $path(keys^k)$ is an $active\_path$ if all nodes from the root to node $n \in path(keys^k)$ are active due to their pushing order. Hence a physical path with a *Leaf* node is active only if $path(keys^k)$ is an active path and the *Leaf* node is active.

Lemma 30. *There's at most a node with Compress op in the subpath of $path(keys^k)$ that is active, and it resides in the end of the path if exists.*

Proof. Assume that there are two nodes n and n' reside in the $active\_path(keys^k)$ with *Compress op*. We prove the lemma by contradiction. Because n and n' are active with *Compress op*, all children of n and n' are *Empty* (Lemma 28. Since n and n' are *Internal* nodes in the same $path$, it derives a contradiction.

For the second part of the proof, if n resides in other places in $path$, its children should be *Empty* by Lemma 28. Therefore, only if n is on the end of path, conditions are satisfied. □

Lemma 31. *For $path(keys^k)$, if n_t is active in $snapshot_{T_i}$, then $n_0, ..., n_{t-1}$ above n_t are active.*

Proof. By Lemma 29, nodes with *op* other than *Compress* are reachable from the root. Lemma 30 indicates that a node with a *Compress op* is on the end of the path. Therefore, other nodes do not have a *Compress op*, which establishes the lemma. □

Lemma 32. *If in the $snapshot_{T_i}$, n is the LCA on active_path for $oldKey$ and $newKey$. Then at $T_{i1}, T_{i1} > T_i$, n is still the LCA on $active\_path$ for both $oldKey$ and $newKey$ if it is active.*

Proof. Nodes from root to the LCA node shares a common path (Observation 1. Because the LCA node is active, nodes above the LCA node are active. Hence, the subpath from the root to the LCA node is active by Lemma 31. □

Lemma 33. *Nodes a with a Compress op will not be pushed into $path$ more than once.*

Proof. Supposing that a node n is active in $snapshot_{T_i}$, and it becomes inactive at $T_{i1} > T_i$. Besides, suppose that *op* of n is *Compress* and pushed at $T_{i2}, T_i < T_{i2} < T_{i1}$. We prove that after T_{i2}, n will never be pushed again.

Lemma 24 shows that a node with a *Compress op* will never be unflagged. Hence, based on Lemma 6 and Lemma 6), we have to prove that nodes in $path$ with a *Compress op* will not be pushed into again. For the insert operation and the remove operation, pOp is checked at line 123 and line 127 before the next find operation. For the move operation, rOp is checked at line 312, line 316, line 339, and iOp is checked at line 296 and line 288 before calling the findCommon function and the find function. Hence, at $T_{i2} < T_{i1}$, *op* must be checked before $find(keys)$ so that nodes with a *Compress op* will not be pushed more than once. □

Lemma 30 shows that for $active\_path$, active nodes with a *Compress op* will only reside at the end. If other nodes pushed before n will become inactive, we can pop them beforehand until reaching the last node a *op* other than *Compress*.

Lemma 34. *After the invocation of $find(keys)$ which reads l^t, there is a snapshot, where the path from the root to l^t is $physical\_path(keys^t)$.*

Proof. We prove the lemma by induction.

In the base case, where $path$ starts from the root node, the claim is true.

We consider the pushing sequence as $n_0, n_1, ..., n_k$. Suppose that for first k nodes, the path from the root to n_k is the $physical\_path$. We shall prove that for n_{k+1}, the lemma is true. If there is no $replace$ operation before reading n_{k+1}, it is obvious that we can linearized it as the same snapshot before pushing n_k.

Next, we assume that $replace$ occurs on n_k before reading n_{k+1}, and results in $snapshot$. There are two cases:

1) Consider if $replace$ occurs after reading n_{k+1}. We could have $snapshot_{k+1} = snapshot_k$ because n_{k+1} is connected to an active node in a snapshot so that n_{k+1} is also reachable from the root.
2) Consider if $replace$ occurs before reading n_{k+1}. If the replace operation changes n_k, then n_k is flagged with a *Compress op* by Lemma 24. Then we have $snapshot_{k+1} = snapshot_k$ because Lemma 27 demonstrates that a node flagged with a *Compress* has no successive replace operations on its child pointers. Otherwise, if the replace operation changes n_{k+1}, n_k is not changed by Lemma 23. Hence, we have $snapshot_{k+1} = snapshot$, where n_{k+1} is reachable from the root in the $snapshot$ just after a replace operation.

□

Lemma 35. *After $ireplace$, $rreplace$, $mreplace$, and $creplace$, quadtree's properties remain.*

Proof. We shall prove that in any $snapshot_{T_i}$, quadtree's properties remain.

First, Lemma 10 shows that two layers of dummy nodes remain in the tree. We have to consider other layers of nodes which are changed by replace operations.

Consider $creplace$ that replaces an *Internal* node by an *Empty* node. Because the *Internal* node has been flagged on a *Compress op* before $creplace$ (Lemma 24), all of its children are *Empty* and not changed (Lemma 28 and Lemma 27). Thus, $creplace$ does not affect the second claim of Definition 10.

Consider $ireplace$, $rreplace$, or $mreplace$ that replaces a terminal node by an *Empty* node, a *Leaf* node, or a sub-tree. By Lemma 23, before $replace^k$, $op.parent^k$ is flagged with *op* such that no successful replace operation could happen on $op.parent^k$. Therefore, if the new node is *Empty*, it does not affect the tree property. If the new node is a *Leaf* node or a sub-tree, based on the post-conditions of the createNode function (Lemma 17), after replace operations the second claim of Definition 10 still holds. □

In this Section, we define linearization points for *basic operations*. As the compress operation is included in the move operation and the remove operation that returns true, it does not affect the linearization points of them. If an algorithm is linearizable, it could be ordered equivalently as a sequential one by their linearization points. Since all the modifications depend on $find(keys)$, we first point out its linearization point.

To prove that our linearization points are correct, we shall demonstrate that for some time T_i, the key set in $snapshot_{T_i}$ is the same as the results of modifications linearized before T_i. For $find(keys)$, we should define the linearization point at some $snapshot_{T_i}$ so that l^t returned is on the $pysical\_path$ for key^k in $snapshot_{T_i}$.

Lemma 36. *For $find(keys)$ that returns tuples $\langle l^k, pOp^k, path^k \rangle$, there's a snapshot after its invocation and before reading l^k, such that $path(keys^k)$ returned with l^k at the end is a $physical\_path$ in $snapshot_{T_i}$.*

Proof. Since l^k is a *Leaf* node or an *Empty* node, by Lemma 34 there is a $snapshot_{T_i}$ that l^k as the last node from l_{i-1}, and l_0 to n_{i-1} in the $path(keys^k)$ are active. Therefore $path(keys^k)$ with l^k is $physical\_path(keys^k)$ in $snapshot_{T_i}$. We define $snapshot_{T_i}$ as $find(keys^k)$'s linearization point. □

In the next, we define linearization points for *basic operations*. For the insert operation, the remove operation, and the move operation that returns true, we define it to be the first $replace_i^k, 0 \leq k < |replace_i|$ belongs to op_i. To make a reasonable demonstration, we first show that the first $replace_i^k$ belongs the op_i created by each operation itself, and then illustrate that *op* is unique for each operation.

Lemma 37. *If the insert operation, the remove operation, or the move operation that returns true, the first $replace_i^k$ occurs before the returning, and it belongs to op_i created by the operation itself.*

Proof. First we prove that $replace_i^k$ belongs to op_i created by the operation itself.

For the insert operation, it returns true after successful $iflag$ and $ireplace$ using *op* created at line 112.

For the remove operation, it returns true after successful $rflag$ and $rreplace$ using *op* created at line 168.

For the move operation, it returns true after successful $mflag$ and $mreplace$ using *op* created at line 224.

Corollary 5 indicates that the first replace operation must succeed, therefore all successful replace operations happen before returning. □

Then, we define the linearization points of the insert operation, the remove operation, and the move operation that returns true are before returning. We should also point out that there's only one *op* lead to the linearization point for each operation.

Lemma 38. *For the insert operation, the remove operation, and the move operation that returns true, op is created at the last iteration in the while loop.*

Proof. Based on Lemma 23, all successful insert, remove and move operations follow the $flag \rightarrow replace \rightarrow unflag$ transition. $flag$ is the first successful flag operation after reading $op_i.parent$, so there is no other flag operation further. This establishes the claim. □

Lemma 39. *If the insert operation, the remove operation, and the move operation that return false, successful $replace$ happens before returning.*

Proof. As Lemma 38 points out that *op* is created at the last iteration. Moreover, before returning true, $mflag$, $iflag$ and $rflag$ must succeed according to the program order. As the first replace operation that belongs to op_i must succeed (Lemma 5, $replace$ happens before returning. □

Lemma 40. *If the insert operation, the remove operation, and the move operation that return false, there's no successful $replace$ ever happened.*

Proof. Consider the execution of the insert operation, it returns false at line 108 where *op* created are not flagged on nodes. Consider the remove operation, it returns false at line 165. Also, *op* created are not flagged on nodes. For the move operation, it returns false by calling the findCommon function or the continueFindCommon function so that the helpMove function must fail or it is not called. If the helpMove function is not called, there is no replace operations. If the helpMove function returns false, by the ordering of $mflag$ and $unflag$ (Lemma 23 and Lemma 25), $op.iParent$ and $op.rParent$ are different. Moreover, the flag operation on the latter parent will fail such that $doCAS$ is false and no replace operation will perform. □

In the next step, we show that how the algorithm could be ordered as a sequential execution. First we linearize the insert operation, the remove operation and the move operation that returns true, where $replace$ results in changing the key set. Let s_i be a set of *Leaf* nodes that are active in $snapshot_{T_i}$ after performing operations o_0, o_1, ..., o_n, ordered by their linearization points $replace_0$, $repalce_1$, ..., $replace_n$ sequentially.

Observation 6. *If a Leaf node is moved, its op is set before the replace operation on $op.iParent$, which is before the replace operation on $op.rParent$.*

Lemma 41. *For the contain operation that returns true, there is a corresponding snapshot that l^k is active in $snapshot_{T_i}$. For the contain operation that returns false, there is a corresponding snapshot that l^k is inactive in $snapshot_{T_i}$.*

Proof. For the first part, we prove that in a snapshot, the end node l^k contains $keys^k$ and is not *moved*. Lemma 36 shows that there is a snapshot consists of $physical\_path(keys^k)$ after calling, thus l^k that contains $keys^k$ is in $snapshot_{T_i}$ before judging whether the node is *moved*. We only have to discuss $mreplace$ in the next cases because other replace operations will not affect $l^k.op$. If $l^k.op$ is null at verifying, $mreplace$ must have not been done according to Observation 6. In this case, we could linearize it at $snapshot_{T_i}$. If $l^k.op$ is not null, but $mreplace$ occurs after verifying whether $l.op.iParent$ has $l.op.oldIChild$, then it is also before the second $mreplace$ on $l.op.rParent$ (Observation 6). We could also linearize it $snapshot_{T_i}$.

For the second part, Lemma 36 shows that there is a snapshot consists $physical\_path(keys^k)$. By Lemma 35, in every snapshot our quadtree's property maintains. Hence, if l^k does not contain $keys^k$, we could linearize it at $snapshot_{T_i}$. Or else, if l^k is in $physical\_path$, but l^k is *moved* at verifying, we could linearize at $snapshot_{T_{i1}}$ after $mreplace$. □

Lemma 42.
1) *For the insert operation, it returns true when $key \notin s_{i-1}$, $key \in s_i$.*
2) *For the remove operation, it returns true and $key \in s_{i-1}$, $key \notin s_i$.*
3) *For the move operation, it returns true and $oldKey \in s_{i-1}$, $newKey \notin s_{i-1}$, $oldKey \notin s_i$, and $newKey \in s_i$.*

Proof. Lemma 39 shows that there is successful $replace$ occurs before the insert operation, the remove operation, and the move operation that returns true.
If a replace operation succeeds, $op.parent$ has been flagged with op other than *Compress* before $replace$ (Lemma 24 and Lemma 23). Lemma 29 implies that $op.parent$ is reachable from the root. Therefore, it is active. Besides, Corollary 31 shows that the path from the root to $op.parent$ is an active. Therefore a snapshot exists such that the path from the root to $op.oldChild$ is a $physical\_path(key)$.
First we prove that $key \notin s_{i-1}$. By Lemma 41, in the snapshot l is inactive. Otherwise, replace operations will not execute. Hence, $key \notin s_{i-1}$.
Second, by the post-conditions of the createNode function 17, it creates a node or a sub-tree that contains $newKey$. According to Corollary 35, after $ireplace$ the node contains $newKey$ is inserted and quadtree's properties maintained. Hence, $key \in s_i$.

2) First we prove that $key \in s_i$. By Lemma 41, in the snapshot l is active. Otherwise, replace operations will not execute. Hence, $key \notin s_{i-1}$.
Second, the remove operation creates an *Empty* node. According to Corollary 35, after $rreplace$ the node contains $oldKey$ is removed so that $key \in s_i$.

3) First we prove that $oldKey \in s_{i-1}$, $newKey \notin s_{i-1}$. By Lemma 41, rl is active at $snapshot_{T_i}$, il is inactive at $snapshot_{T_{i1}}, T_i < T_{i1}$. We have to demonstrate that rl is still active at $snapshot_{T_{i1}}$ so that we can use $snapshot_{T_{i1}}$ as s_{i-1}.
We prove it by contradiction. Assuming that at $snapshot_{T_{i1}}$ rl is inactive, hence there's some replace operation happens before T_{i1} and after T_i. However, based on Lemma 23, $replace_i$ is the first successful replace operation that after reading rl. Therefore, it derives a contradiction. Hence we prove that in $snapshot_{T_{i1}}$ rl is still active.

□

Finally we order linearization points for the insert operation, the remove operation and the move operation that returns false. We also order the contain operation. We consider they are linearized between $replace_{i-1}$ and $replace_i$ if exists.

Lemma 43.
1) *For the insert operation returns false, $key \in s_{i-1}$.*
2) *For the remove operation returns false, $key \notin s_{i-1}$.*
3) *For the move operation returns false, either $oldKey \notin s_{i-1}$ or $newKey \in s_{i-1}$.*

Proof. The first two parts are equivalent as the case in Lemma 41. □

Lemma 44. *Our quadboost is a linearizable implementation.*

Proof. By lemma 42 and Lemma 43, our algorithm returns the same result as they are finished in the linearized order. Therefore we prove our algorithm is linearizable and our linearization points are correct. □

We say an algorithm is non-blocking if the system as a whole is making progress even if some threads are starving.

We prove our quadboost is non-blocking by following set of lemmas. We assume that there are finite number of *basic operations* invocations.

Observation 7. *There are finite number of basic operations.*

Lemma 45. *$path(key^k)$ returned by $find(keys)$ consists of finite number of keys.*

Proof. Observation 7 shows that the number of successful insert operation and move operation is limited. Lemma 40 illustrates that for *basic operations* return false, there is no successful replace operation. Hence, only successful insert and move operations add nodes into quadtree. The post-conditions of the createNode function (Lemma 17) and the effects of $replace$ (Lemma 35) demonstrate that they will add finite number of nodes into quadtree. We then prove that $path^k$ is terminable. By Lemma 34, $find(keys)$ returns $path(key^k)$ which is a subpath of $physical\_path(keys^k)$ in $snapshot_{T_i}$ which is terminable. Thus, there are finite number of nodes in $path^k$. □

Corollary 6. *The compress function must terminate.*

Proof. By Lemma 45, $path^k$ is terminable. Since other functions called by the compress function do not consist loops, the compress function must terminate. □

The we shall demonstrate that there are finite number of three different CAS transitions.

Lemma 46. *There is an unique spatial order among nodes in quadtree in $snapshpt_{T_i}$*

Proof. As illustrated by the algorithm at line 223, the getSpatialOrder function compares ip with rp by the order: $x \rightarrow y \rightarrow w$. In our quadtree, we only consider square partitions on two-dimensional space. Hence, w is always equal to h. We prove the lemma by contradiction. Assuming there are two different *Internal* nodes with the same $\langle x, y, w\rangle$, they represent the same square starting with left coroner $\langle x, y\rangle$ with width w. By Lemma 35, in every snapshot quadtree's properties remain. There cannot be two squares with the same left corner and w. Hence, in $snapshpt_{T_i}$, a *Internal* node consists of a unique $\langle x, y, w\rangle$ tuple that can be ordered correctly. □

Lemma 47.
1) *There are finite number of successful $flag \rightarrow replace \rightarrow unflag$ transitions.*
2) *There are finite number of successful $flag \rightarrow replace$ transitions.*
3) *There are finite number of successful $flag \rightarrow unflag$ transitions.*

Proof.
1) By Lemma 39, replace operations only occur in *basic operations* that return true. By Lemma 38, there's a unique *op* created for operations that return true. Hence, for the move operation, the insert operation and the remove operation, there are finite number of $flag \rightarrow replace \rightarrow$ circles.
2) By Lemma 39, replace operations only occur in *basic operations* that return true. Lemma 24 shows that the $flag \rightarrow replace$ transition happens only in the compress function which is called by the successful remove operation or the move operation. Hence, there are a finite number of $flag \rightarrow replace$ transitions, as Corollary 6 shows that the compress function is terminable.
3) The $flag \rightarrow unflag$ transition executes only in the helpMove function where $op.iParent$ cannot be flagged according to Lemma 25 (if $op.iFirst$ is true). The case that $op.iFirst$ is false can be proved symmetrically.
From first two parts of the lemma, there are finite number of replace operations that change quadtree's structure. If the $flag \rightarrow replace \rightarrow$ transition happens simultaneously, it will reset a node's *op* to *Clean*. If the $flag \rightarrow replace \rightarrow unflag$ transition happens in the meantime. Lemma 33 indicates that a node with *Compress op* will not be pushed twice. Consider if the helpMove function detects $op.iParent.op$ as *Compress*, then $op.iParent$ is not reachable in the next time. Hence, quadtree is stabilized and only when $flag \rightarrow unflag$ transitions execute infinitely and never set $doCAS$ as true.
According to Lemma 46, we order *spatial order* on both $op.iParent$ and $op.rParent$ as $ip_0 > ip_1 ... > ip_n$ and $rp_0 > rp_1 ... > rp_n$. We assume that there's a ring such that $move_i$ which flags rp_i but fails on ip_i, and $move'_i$ which flags ip_i but fails on rp_i. By assumption we order rp_i and ip_i as $rp_i > ip_i$ by $move_i$'s order, and $ip_i > rp_i$ by $move'_i$'s order. But by Lemma 46, all *Internal* nodes in a snapshot can be ordered uniquely. Therefore it derives a contradiction.
Then, we prove that there's no ring among all $move_i$ in the same snapshot, so at least one of $move_i$ could set $doCAS$ to true, resulting in the $flag \rightarrow replace \rightarrow unflag$ transition. Let's consider the dependency among all $move_i$. If $move_i$ which fails on p_i has been flagged by $move'_i$, then there is a directed edge from $move_i$ to $move'_i$ by order. In this way, the head node in the graph must have no out edge as proved. It will set $doCAS$ to true, all move operations that are directly connected to it will help it finish. After erase the former head node from the graph, there will be other nodes that have no out edge. Finally, all dependency edges are removed. Therefore, there are a finite number of $flag \rightarrow unflag$ transitions.

□

Observation 8. *The help function, the helpCompress function, the helpMove function, and the helpSubstitute function are not called in a mutual way. (If method A calls method B, and method B calls method A reciprocally, we say method A and B are called in a mutual way.)*

Lemma 48. *If help function returned at T_i, and $find(keys)$ at the prior iteration reads $p^0.op$ at $T_{i1} < T_i$, Leaf nodes in $snapshot_{T_i}$ and $snapshot_{T_{i1}}$ are different.*

Proof. We prove the lemma by contradiction, there are two possible scenarios. First, the help function might be interminable. But based on Observation 8, the help function must terminate. Second, the help function might execute without changing quadtree. According to Lemma 47, there are finite number of CAS transitions. So if some help function does not change quadtree, the structure might be changed before T_{i1}. For the move function, if ip and rp are the same but their *op*s are different, the help function might not change the structure. Nevertheless, $snapshot_{T_i}$ and $snapshot_{T_{i1}}$ are different. Or else, there are some interval that all transitions are $flag \rightarrow unflag$. However, Lemma 47 also shows that all transitions form a acyclic graph that there is no interval that all transitions are $flag \rightarrow unflag$. Therefore, a $flag \rightarrow replace \rightarrow unflag$ transition or a $flag \rightarrow unflag$ transition is executed to change the snapshot. □

Lemma 49. *Our quadboost algorithms are non-blocking.*

Proof. We have to prove that no process will execute loops infinitely without changing keys in quadtree. First, we shall prove that $path$ is terminable. Next, we shall prove that $find(keys^k)$ starts from an active node in $physical\_path(keys^k)$ in $snapshot_{T_i}$ between i_{th} iteration and $i+1_{th}$ iteration. Finally, *Leaf* nodes in $snapshot_{T_{i1}}$ at the returning of $find(keys)$ at i_{th} iteration is different from $snapshot_{T_{i2}}$ that at the returning of $find(keys)$ at the $i+1_{th}$ iteration.

For the first part, initially we start from the root node. Therefore, $path$ is empty. Moreover, as Lemma 45 shows that $path^k$ consists of finite number of keys, we establish this part.

For the second part, the continueFind function and the continueFindCommon function pops all nodes with *Compress op* from $path$. For the insert operation and the remove operation, since Lemma 29 shows that an *Internal* nodes whose *op* is not *Compress* is active and Lemma 31 shows that nodes above the active node are also active, there is a snapshot such that the top node of $path$ is still in $physical\_path(keys^k)$. For the move operation, if either $rFail$ or $iFail$ is true, it is equivalent with the prior case. Or if $cFail$ is true, Lemma 32 illustrates that if the LCA node is active, it is in $physical\_path$ for both $oldKey$ and $newKey$. Thus, there is also a snapshot that the start node is in $physical\_path$.

For the third part, we prove it by contradiction. Assuming that quadtree is stabilized at T_i, and all invocations after T_i are looping infinitely without changing *Leaf* nodes.

For the insert operation and the remove operation, before the invocation of $find(keys)$ at the next iteration, they must execute the help function at line 118 and line 175 accordingly. In both cases, the help function changes the snapshot (Lemma 48).

For the move operation, consider different situations of the continueFindCommon function. If $rFail$ or $iFail$ is true, there are two situations: (1) iOp or rOp is *Clean* but $mflag$ fails. (2) iOp or rOp is not *Clean*. Consider the first case that $mflag$ fails, iOp and rOp are updated respectively at line 236 and line 235. If ip and rp are different, before its invocation of $find(keys^k)$, the help function is performed at line 311 and line 287. Thus by Lemma 48, the snapshot is changed between two iterations. Now consider if $cFail$ is true. If ip and rp are the same, it could result from the difference between rOp and iOp. In this case, the snapshot might be changed between reading rOp and iOp. It could also result from the failure of $mflag$. For above cases, the help function at line 332 would change quadtree. If ip and rp are different, it results from that either $iPath$ or $rPath$ has popped the LCA node. We have proved the case in the former paragraph. Thus, we derives a contradiction.

From above discussions, we prove that quadboost is non-blocking. □

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