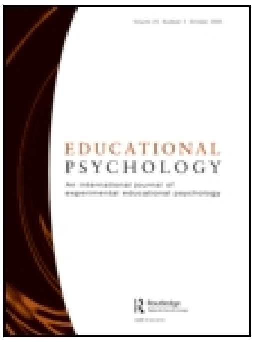

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Students' Improper Proportional Reasoning: A result of the epistemological obstacle of "linearity"

Modestina Modestou and Athanasios Gagatsis*
University of Cyprus

The aim of the present study is to provide further evidence that the errors that arise from improper application of the linear model are not random and not easy to overcome. Using three different types of test, we attempt to show that the errors referred to in the literature as "pseudo-analogous" are the result of an epistemological obstacle—"linearity". The results of this study reveal that students' erroneous application of proportional reasoning in non-proportional problem situations is not a random phenomenon. On the contrary, this phenomenon resists, persists, and reappears regardless of students' grade and of the tests' settings. This is evidence that pseudo-analogous errors result from the epistemological obstacle of linearity.

Students of any age, any country, and any era, irrespective of their performance in mathematics, have experienced getting mathematics wrong (Gagatsis & Kyriakides, 2000). It was therefore natural for educators and psychologists to show very early an interest in this topic. This interest resulted in the formation of many theories about the nature of mathematical errors, their interpretation, and ways of overcoming them.

Some mathematical errors, in various fields of everyday life, are based on our tendency to see the linear function everywhere. This is mainly due to the importance of the linear function as a mathematical tool for explaining and mastering phenomena in different fields of human activity (De Bock, Verschaffel, & Janssens, 1998; Freudenthal, 1973). Proportionality, ratio preservation, and linearity appear to be universal models – a view that is reinforced by their frequent use.

Linear or proportional relations refer to the function of the form $f(x) = ax$ (with $a \neq 0$) and are represented graphically by a straight line passing through the origin

*Corresponding author. Department of Education, University of Cyprus, P.O. Box 20537, 1678 Nicosia, Cyprus. Email: gagatsis@ucy.ac.cy

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(De Bock, Van Dooren, Janssens, & Verschaffel, 2004). The basic linguistic structure for problems involving proportionality includes four quantities (a, b, c, and d), of which, in most cases, three are known and one unknown, and an implication that the same multiplicative relationship links a with b and c with d. "Two shirts cost 30 pounds. How much money do five similar shirts cost?" In this case, of true proportionality, the relationship is a fixed ratio ($2 \times 15 = 30$, $5 \times 15 = \ldots$; Behr, Harel, Post, & Lesh, 1992).

If a problem matches this general structure, the tendency to evoke direct proportionality can be extremely strong—even if direct proportionality does not fit the problem (Verschaffel, Greer, & De Corte, 2000). In the case of the constant problem "Two shirts need 30 minutes to dry out on a sunny day. How much time do four shirts need to dry out under the same weather conditions?" students spontaneously answer that four shirts need 60 minutes to dry out, falling victim to the pseudo-proportionality error. It is obvious that proportionality is so embedded in student thought that it tends to be applied to problem situations without any consideration of the realistic constraints (Verschaffel et al., 2000). The linear growth of height and weight, or even athletes' capacity to run with differential speeds the 100m and 400m, are characteristic examples of this lack of consideration of realistic constrains (Verschaffel, De Corte, & Lasure, 1994).

Research into students' tendency to apply proportional reasoning to problem situations for which it is not suited (De Bock, Verschaffel, & Janssens, 2002; Van Dooren, De Bock, Hessels, Janssens, & Verschaffel, 2005) has provided clear indications that this phenomenon is partially caused by the characteristics of the problem formulation with which students have learned to associate proportional reasoning throughout their school life. In particular, Greer (1997) points out that multiplicative missing-value structures, notably those that on a superficial reading may create an illusion of proportionality, provide an example of inappropriate invocation of proportionality, as a result of an unconscious reaction to linguistic form. In addition, Van Dooren et al. (2005) argue that students' tendency to use proportional strategies for non-proportional tasks increases considerably from grade to grade (second to fifth), together with students' self-assurance in solving word problems based on superficial cues.

However, it is our belief that this line of interpretation does not provide a sufficient and coherent framework within which the phenomenon of the illusion of linearity could be analysed. This phenomenon occurs not just as a reaction to a stereotyped problem formulation: Verschaffel et al. (2000) have indicated that individual students throughout history have built up a non-reflective link built up between the mathematical structure of proportional relationships and a stereotyped linguistic formulation. Linearity is the cultural way to conquer the concept of functions, and therefore it has been given a special status, which leads to the overgeneralisation of its validity to situations where linearity is not applicable. Thus, repeated confirmation of the validity of linearity can cause a deeply entrenched belief that every relationship between two quantities is linear, and that proportions are a panacea for nearly all problems (Van Dooren, De Bock, Hessels, Janssens, & Verschaffel, 2004).

Linearity, Area, and Volume

Freudenthal (1983) focuses on the appropriateness of the linear relation as a phenomenal tool of description and indicates that there are cases in which this primitive phenomenology fails. One of these cases, which is the focus of the present study, is the case of the non-linear behaviour of area and volume under linear multiplication.

This subject is diachronic. The first written evidence of its existence can be found in Plato's *Menon* where Socrates, using his maieutic method, attempts to "awaken" a slave's memory, in order for the slave to be able to double the area of a square:

Socrates:	If the length of this side is two feet and the length of this one two feet, how many feet would the area of the square be?
Slave:	Four, Socrates.
Socrates:	Could it be another ruled surface, double of this one, with all its sides equal?
Slave:	Yes.
Socrates:	How many feet would it be?
Slave:	Eight.
Socrates:	Now try to tell me how large would its side be? The side of this square is two feet. What would the side of the doubled be?
Slave:	It's obvious Socrates, that it would be doubled. (Fragkos, 1983, p.70)

As De Bock, Verschaffel, et al. (2002) point out, students' former real-life practices with enlarging and reducing operations do not necessarily make them aware of the different growth rates of lengths, areas, and volumes. Therefore, students strongly tend to see the relations between length and area or between length and volume as linear instead of quadratic and cubic. Consequently, they apply the linear scale factor instead of its square or cube to determine the area or volume of an enlarged or reduced figure.

Understanding that multiplication of lengths by d, of areas by d^2, and of volumes by d^3 is highly associated with the geometrical multiplication by d is mathematically so fundamental, that phenomenologically and didactically it should be put first and foremost (Freudenthal, 1983). Students should be able to distinguish that, for instance, volume is proportional to length only when width and height are held constant—and, similarly, to width (or height) only when the other two variables are held constant. It is essential for students to understand the difference between the product of two variables in double proportion tasks and the product of one variable by a constant in simple proportion problems (Vergnaud, 1997). Students have to break the pattern of linearity and become aware of the multi-dimensional impact of increase.

Students' tendency to apply proportional reasoning to non-proportional tasks could be construed as a special case of intuitive rule theory (Stavy & Tirosh, 2000), which suggests that a change in a quantity A causes the same change in a quantity B (same A–same B). According to this theory, students' answers to non-proportional tasks could be explained in terms of the application of common intuitive rules. However, Van Dooren, De Bock, Weyers, and Verschaffel (2004) have shown that students are considerably less affected by the intuitive rules than their answers suggest, as essentially different errors can be found leading to the same answer.

In recent years, researchers (De Bock, Van Dooren, Janssens, & Verschaffel, 2002; De Bock et al., 1998; De Bock, Verschaffel, et al., 2002; De Bock, Verschaffel, Janssens, Van Dooren, & Claes, 2003; Van Dooren, De Bock, Hessels, et al., 2004, Van Dooren, et al., 2005) have examined students' tendency to deal linearly with non-proportional tasks, and have suggested ways of overcoming it. In particular, De Bock et al. (1998) and De Bock, Verschaffel, et al. (2002) showed an alarmingly strong tendency among 12–16-year-old students to apply proportional reasoning to problems to which it was not suited.

Furthermore, the use of a number of different experimental scaffoldings did not yield the expected results. The inclusion of visual support for non-proportional problems, like self-made or given drawings, did not have a beneficial effect on students' performance, as students most often relied on formal strategies such as using formulae (De Bock et al., 1998). Worse, increasing the authenticity of the problem context (De Bock et al., 2003), with students being asked to visit the island of Lilliput and find the dimensions of different figures there, given that all the lengths are 12 times smaller than ours, yielded a negative effect on students' performance on non-proportional items.

In a real problem situation with real materials and authentic actions—the calculation of the tiles needed for a dollhouse—students avoided the linear model, and therefore showed high performance at the task (Van Dooren, De Bock, Janssens, & Verschaffel, in press-a). However, the results were only temporary, as students failed a post-test with non-proportional tasks. We think that the reason for this failure lies in the fact that in the authentic situation the students had immediate feedback for the correctness of their answer—which allowed them to revise their answers, if wrong, at any time without been affected by the implications of the length–area relation.

The inclusion of an introductory warning, before the actual test, that informed students of the non-routine character of the tests, yielded small but significant effects on students' performance (De Bock, Verschaffel, et al., 2002). In the same study, the rephrasing of the usual missing value problems into comparison problems proved to be a substantial help for many students. However, in both cases student success rates on proportional items decreased, as some students started to apply non-proportional methods to these problems.

Similar drawbacks were observed in a series of 10 experimental lessons aiming for students' conceptual change (Van Dooren, De Bock, Weyers, et al., 2004). In particular, while some students' automatic use of proportional strategies for solving non-proportional problems decreased, others started to apply non-proportional strategies to proportional tasks. Therefore, it appears that the whole experiment did not result in a deep conceptual understanding of proportional and non-proportional relations (Van Dooren, De Bock, Weyers, et al., 2004).

The actual processes and mechanisms used by students while solving non-proportional problems were unravelled by means of interviews (De Bock, Van Dooren, et al., 2002). It appears that the illusion of linearity is not the only factor responsible for inappropriate proportional responses. Other factors include intuitive reasoning, shortcomings in geometrical knowledge, and inadequate habits and beliefs about solving word problems. Van Dooren, De Bock, Janssens, and Verschafell (in press-b), in a follow-up study, attempt to clarify further the psychological and educational factors underlying this tendency to respond linearly to non-linear problems, and in addition to increase our understanding of the roots of the phenomenon in general. In particular, they argue that the explanatory elements of the phenomenon of the illusion of linearity can be found in: (1) students' experiences in the mathematics classroom; (2) the intuitive, heuristic nature of the linear model; and (3) elements related to the specific mathematical problem situation in which linear errors occur.

From the literature review it becomes evident that the illusion of linearity is not a result of a particular experimental setting. In contrast, it is a recurrent phenomenon that seems to be quite universal and resistant to a variety of forms of support aimed at overcoming it (De Bock et al., 2003). Proportions appear to be deeply rooted in students' intuitive knowledge and are used in a spontaneous and even unconscious way, which makes the linear approach quite natural, unquestionable, and to certain extent inaccessible to introspection or reflection (De Bock, Van Dooren, et al., 2002). Therefore, as Verschaffel et al. (2000) argue, it takes a radical conceptual shift to move from the uncritical application of this simple and neat mathematical formula to the modeling perspective that takes into account the reality of the situation being described.

Linearity as an Epistemological Obstacle?

Errors are not always the effect of ignorance, uncertainty, or chance; they can result from the application of a previous piece of knowledge which was interesting and successful, but which in another context is revealed as false or simply unadapted (Brousseau, 1997). Errors of this type are not erratic and unexpected, but are reproducible and persistent.

Errors of this kind, made by the same subject, are interconnected by a common source:

> a way of knowing, a characteristic conception, coherent if not correct, an ancient "knowing" that has been successful throughout an action-domain. They do not completely disappear all at once; they resist, they persist, then they reappear, and manifest themselves long after the subject has rejected the defective model from the conscious cognitive system. (Brousseau, 1997, p. 84)

These errors are indicative of epistemological obstacles. An obstacle of epistemological origin manifests in errors that are not made by chance and are reproducible and persistent. An epistemological obstacle is characterised by its appearance in both the history of mathematics and in everyday mathematical activity (Radford, Boero, & Vasco, 2000)

Duroux (1982) proposes a list of necessary conditions for the use of the term "epistemological obstacle" and not just "difficulty":

1) An obstacle is a piece of knowledge or a conception, not a difficulty or a lack of knowledge.
2) This piece of knowledge produces responses which are appropriate within a particular, frequently experienced, context.
3) It generates false responses outside this context. A correct, universal response requires a notably different point of view.
4) This piece of knowledge withstands both occasional contradictions and the establishment of a better piece of knowledge. Possession of a better piece of knowledge is not sufficient for the preceding one to disappear. It is therefore essential to identify it and to incorporate the reason for its rejection into the new piece of knowledge.
5) After its inaccuracy has been recognised, it continues to crop up in an untimely, persistent way.

Therefore, in the case of the typical errors $6 \div {}^1/_2 = 3$, $.2 \times .3 = .6$, and $(-5) \times (+3) = 15$, we can argue that knowledge of natural numbers becomes an obstacle to the conception of rational numbers, decimals, and negative numbers (Gagatsis & Kyriakides, 2002). In particular, the conception that an increase is obtained by multiplication and a decrease by division is well established in the field of natural numbers. However, this conception does not apply to rational numbers and decimals. As a result, false responses are generated outside the context of natural numbers—responses that continue to appear sporadically for a long time after instruction.

It is our belief that pseudo-proportionality errors—which are caused by the improper application of proportionality—are a result of the epistemological obstacle of linearity. Linearity is a knowledge which is successful in a particular context and for a particular set of situations. However, its application outside that context results in false responses accompanied by a strong belief in their correctness. These responses are recurrent and seem to be quite universal and resistant to a variety of forms of support aimed at overcoming the problem (De Bock et al., 2003). Therefore, we argue that linearity constitutes an epistemological obstacle to the acquisition of non-linear functions, and consequently that errors occur because of the obstacle and not just because of stereotyped linguistic formulations.

Research Aims and Questions

The main aim of the present study is to provide further evidence that the errors that occur from the improper application of the linear model are not random, and not easy to overcome. The fundamental claim of this study is that the errors referred to in the literature as "pseudo-analogous" are the result of the epistemological obstacle of linearity. Accordingly, the following research questions were formed:

1) How consistent is students' behaviour with respect to non-proportional tasks regarding area and volume between and within different experimental settings?
2) To what degree is the appearance of this illusion of linearity independent of students' grade and the settings of the given tasks?
3) What are the possible connections between this study's findings and Brousseau's (1997) theory of epistemological obstacles? Is linearity an epistemological obstacle to the acquisition of non-linear functions?

Method

Participants

The sample consisted of 307 students from the seventh and eighth grades of six different secondary schools of Cyprus. Specifically, 163 of the students were seventh graders (12-year-olds) and 144 students were eighth graders (13-year-olds).

Instruments: The written tasks

The study was completed in two phases. In each phase, a different written test was administered. At the first phase of the study all students were given Test A. The second phase followed 15 days later with the administration of Tests B and C. Students were given about 30 minutes for the solution of each test.

The first test (Test A) was administered to all 307 students and consisted of five different word problems, two of which concerned volume (problems A1 and A4), two area (A3 and A5), and one length (A2). All problems were in a multiplicative comparison form: the area or volume of a rectangular figure was given, as well as the relation that connected it with the area or volume of the new figure. The first task of Test A (A1) is given below:

> A gym's swimming pool has a rectangular shape and 70m^3 of water capacity. What is the water capacity of Nicosia's public swimming pool if its dimensions are two times the dimensions of the gym's swimming pool?

The purpose of this test was to examine the extent to which students would apply proportional reasoning to the non-proportional word tasks involving the area and volume of rectangular figures.

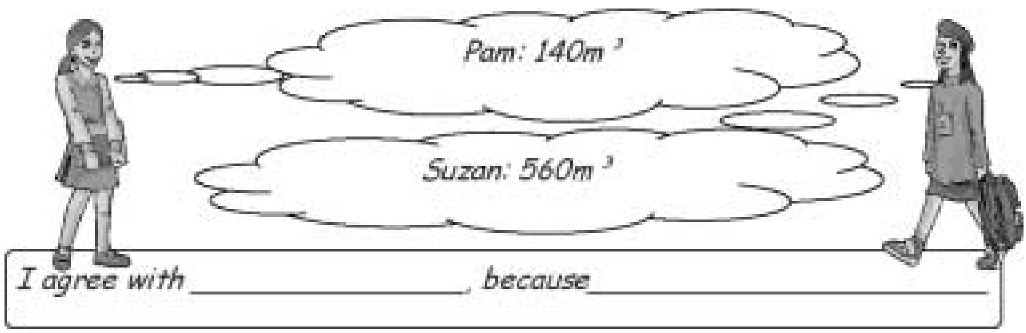
The second test (Test B) was administered to only 157 of the students that participated in the first phase and consisted of the same five word problems as Test A, but with more data given for each problem. In particular, for each problem the dimensions of the first rectangular figure were given. Therefore, task B1, corresponding to A1, was formulated as follows:

> A gym's swimming pool has a rectangular shape with 7m length, 5m width, 2m depth and 70m^3 of water capacity. What is the water capacity of Nicosia's public swimming pool if its dimensions are two times the dimensions of the gym's swimming pool?

The purpose of this test was to examine whether inclusion of the dimensions would induce students to execute a multiplicative comparison using the dimensions of each figure and then find out the area or volume of the new figure, instead of applying direct proportionality between only one dimension (length, width or height) and the area or volume of the figure.

The third test (Test C) was administered to the remaining 150 students that were not administered Test B and consisted of the same five word problems as Test A, but with a different presentation. Each task was accompanied by two alternative answers given by two fictitious peers. One expressed the dominant conception that the area and volume are directly proportional to length, whereas the other expressed the correct answer:

> A gym's swimming pool has a rectangular shape and $70m^3$ of water capacity. What is the water capacity of Nicosia's public swimming pool if its dimensions are two times the dimensions of the gym's swimming pool?



The students were asked to find the solution strategy each peer used to find the given answer, and then to choose the correct reasoning, justifying their choice. The purpose of this scaffolding was to give us clear indications of the magnitude and persistence of the problem in a clear multiple choice environment, where the right answer had a 50% chance of being picked by each student.

Finally, it should be mentioned that all three tests were accompanied by the formulae for finding the area and volume of a rectangular figure, and by pictures of a plane and a solid figure. This information was given to ensure that students would not be unable to solve tasks through lack of this knowledge.

Data Analysis

All answers were classified into three distinct categories. All correct answers were assigned one point. All erroneous answers were not given any points. In the cases where the mathematical expression of the problem was correct but the answer false, a score of 0.5 was given.

Two separate data analyses were conducted. A factor analysis (principal components extraction and varimax rotation) and pairwise *t* tests for means comparison were conducted using the SPSS software package, as well as an implicative statistical analysis using the CHIC software (Bodin, Coutourier, & Gras, 2000). The latter data analysis (CHIC) enables the distribution and classification of variables, as well as implicative identification among the variables or variable categories (Gagatsis, Sriraman, Elia & Modestou, 2006). The implicative statistical analysis generated a similarity diagram (Lerman, 1981) of children's responses to Tests A–B and B–C. The similarity diagram, which is analogous to the results of the more common method of cluster analysis, allows the arrangement of tasks into groups according to the homogeneity with which they were handled by students.

Results

Analysis of the data revealed the true magnitude of the illusion of linearity in 12- and 13-year-old Cypriot students. Even though a statistically significant improvement in relation to Test A was observed for students' achievement on Tests B and C, where a differentiation of the settings was applied (Table 1), almost 60% of the students persisted in applying proportional reasoning in problem situations concerning area and volume, for which it was not suited (Modestou, Gagatsis, & Pitta-Pantazi, 2004).

A more detailed analysis of students' errors, presented in each test separately, indicated that 78% of these errors were the result of the improper application of proportional reasoning in Test A, 56.5% in Test B, and 68% in Test C. It is worth mentioning that 30% of students' non-proportional errors in Test B, where all the dimensions of the figures were given, resulted after students had correctly calculated the sides of the new figure and had only to use the given volume formula. For example, in task B1 students calculated the length ($2 \times 7m = 14m$), width ($2 \times 5m = 10m$), and depth ($2 \times 2m = 4m$) of the new swimming pool and concluded that its

Table 1. Percentages achieved on Tests A, B, and C

	Problem	Test A	Test B	Test C	Test	Sd	*t*	*df*	*p*
Area	3	7%	32%	19%	A–B	.46	−7.38	145	.00*
					A–C	.39	−3.51	141	.01*
	5	13%	34%	24%	A–B	.51	−4.86	141	.00*
					A–C	.42	−3.07	139	.00*
Volume	1	6%	30%	19%	A–B	.43	−6.78	148	.00*
					A–C	.34	−4.17	144	.00*
	4	7%	31%	21%	A–B	.45	−6.43	144	.00*
					A–C	.37	−4.51	140	.00*
Length	2	90%	90%	93%	A–B	.29	−.43	147	.67
					A–C	.27	−1.32	146	.19

*$p < .01$

water capacity would be twice the water capacity of the previous pool ($2 \times 70\ m^3 = 140\ m^3$).

The great differences in student achievement on the non-linear tasks (area and volume) in relation to the linear task (length) are also shown in Table 1. This difference, even though more prominent in Test A, is statistically significant in all three tests (A = 40.9, $p < .01$; B = 15.29, $p < .01$; C = 21.92, $p < .01$).

Figure 1 details students' achievement in Tests A, B, and C in relation to grade. Students' mean achievement scores on Test A indicate no difference between seventh and eighth graders in terms of ability to solve non-proportional problems of area and volume. This observation is associated with the concept of epistemological obstacles, since it indicates that the illusion of linearity persists and has great duration.

However, students from both grades showed a statistically significant improvement (grade 7: $t = -5.9$, $p < .01$; grade 8: $t = -6.7$, $p < .01$) on the non-proportional area and volume tasks of Tests B and C over the corresponding tasks of Test A. This improvement was greater for eighth grade than for seventh grade students—a difference of progression that was statistically significant ($t = -2.3$, $p < .05$).

Figure 2 provides further information concerning the observed improvement in eighth graders' achievement in the non-proportional tasks of Tests B and C, when compared to seventh graders. More precisely, Figure 2 makes clear that this improvement is due to the settings of Test B, which possibly favoured eighth graders' ability to apply mathematical formulae and execute mathematical operations. Contrary to Test B, no differentiation between the two grades can be observed for Test C, even though with its multiple choice settings there was a 50% chance of giving the correct answer. This indicates that students' answers at both grades are not given randomly.

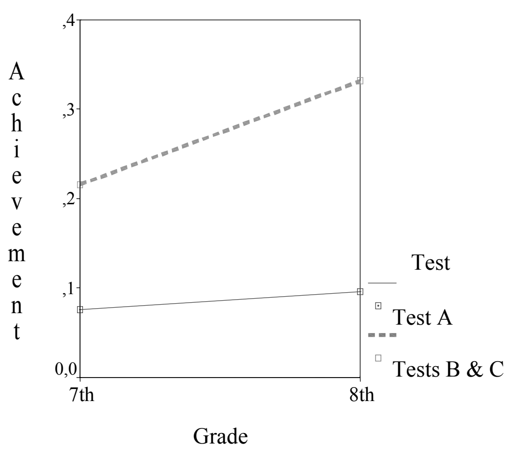


Figure 1. Students' achievement means in Tests A, B, and C in relation to grade

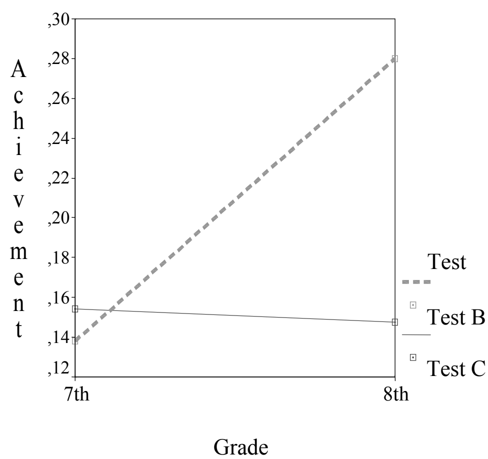


Figure 2. Students' achievement means in Tests B and C in relation to grade

Tables 2 and 3 present the factor loadings from analysis of responses to Tests A and B, and A and C, respectively. Both loadings suggest the grouping of tasks into three distinct factors. In Table 2, the first factor is formatted by the area and volume tasks of Test A, the second factor by the same tasks of Test B, and the third by the linear tasks of both tests. A similar grouping of tasks is presented in Table 3 concerning the Tests A and C. The first factor is formatted, as in Table 2, by the area and volume tasks of Test A, the second by the same tasks of Test C, and the third by the linear tasks of both tests.

It must be noted that none of the tasks have high factor loadings onto more than one factor, and therefore the factors in both cases are not interrelated. This suggests that students have dealt with the non-proportional tasks within each test in a similar way. However, a differentiation of their behaviour by test is indicated by the fact that the non-proportional tasks of each test are separated from the tasks of the other test, in distinct factors (Modestou & Gagatsis, 2004).

A different analysis by means of the computer software CHIC (Bodin et al., 2000) suggests the same grouping of tasks as the factor analysis. It also provides insights into the formulation of these groups and the relations between the tasks themselves. Figure 3 shows the similarity diagram for all tasks from Tests A and B. Students' responses to each task form the different variables.

Students' responses to the tasks form two large clusters (i.e., groups of variables) of similarity; the linear and non-linear tasks are separated. More specifically, the first cluster of similarity is formed by two groups (Groups 1 and 2) that consist of the same non-linear tasks (1, 3, 4, and 5) of Tests A and B, respectively. The third group consists of the linear tasks (2) of Tests A and B, something quite natural since both tasks are the same.

The first similarity group is formed by two distinct sub-groups of tasks that correspond to the volume (1 and 4) and area (3 and 5) tasks of Test A. The fact that the area and volume tasks are separated in Test A means that the students deal with

Table 2. Factor loadings of the tasks in Tests A and B

	Factors		
Task	I	II	III
A4 (v)	.90	.16	.00
A1 (v)	.88	.19	.00
A3 (a)	.87	.00	.00
A5 (a)	.83	.16	.00
B4 (v)	.19	.91	.00
B3 (a)	.00	.85	.13
B5 (a)	.16	.82	.00
B1 (v)	.16	.81	.00
B2 (l)	.00	.00	.86
A2 (l)	.00	.00	.86

Table 3. Factor loadings of the tasks in Tests A and C

	Factors		
Task	I	II	III
A4 (v)	.92	.18	.00
A1 (v)	.89	.22	.00
A3 (a)	.82	.16	.00
A5 (a)	.80	.30	.00
C4 (v)	.26	.89	.00
C5 (a)	.12	.88	.00
C3 (a)	.18	.84	.00
C1 (v)	.31	.80	.00
C2 (l)	.00	.00	.85
A2 (l)	.00	.00	.83

these problems in a different way, without realising their common non linear character.

This tendency does not seem to apply to the non-proportional tasks of Test B, which constitute the second similarity sub-cluster. Here, all the area and volume tasks are mingled together, which shows that there is no differentiation in the intensity of the phenomenon. This is mainly due to the provision of the figures' dimensions, which made it easier for students to apply the given mathematical formulae for volume

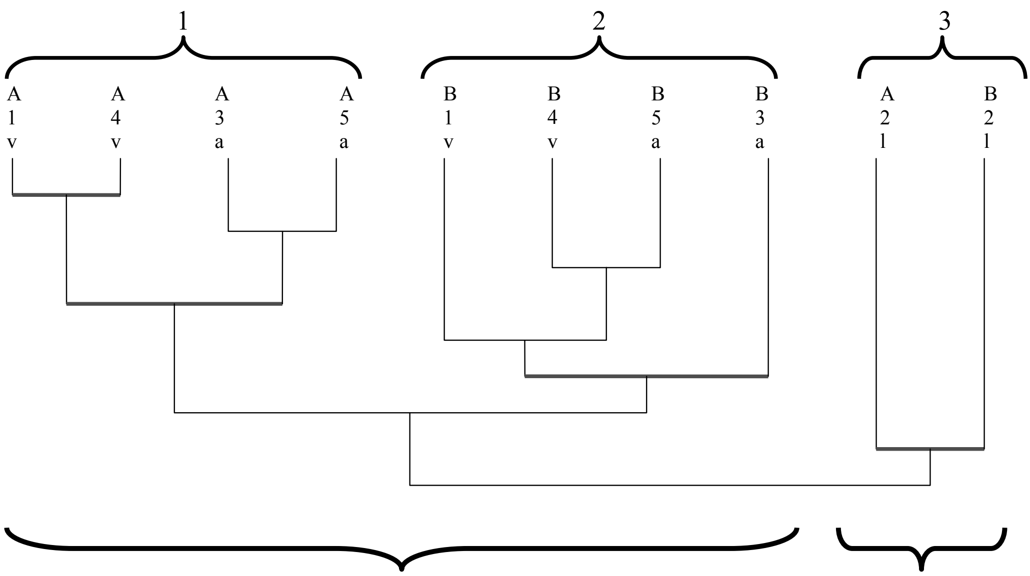


Figure 3. Similarity diagram for the variables in Tests A and B (similarities presented with bold lines significant at the level of 99%)

and area and to find the respective answers. Therefore, it may be possible that students realised that the tasks they were asked to deal with were not just problems with different mathematical content, but also problems that were characterised by the same phenomenon—the non-linear increase of area and volume in relation to length.

In Figure 4, all the similarity relations of the tasks from Tests A and C are illustrated. As in Figure 3, three distinct similarity clusters are formed in addition to the two large ones. The first two groups consist of the non-linear area and volume tasks (1, 3, 4, and 5) of Tests A and C, whereas the third group consists of the linear problems of Tests A and C.

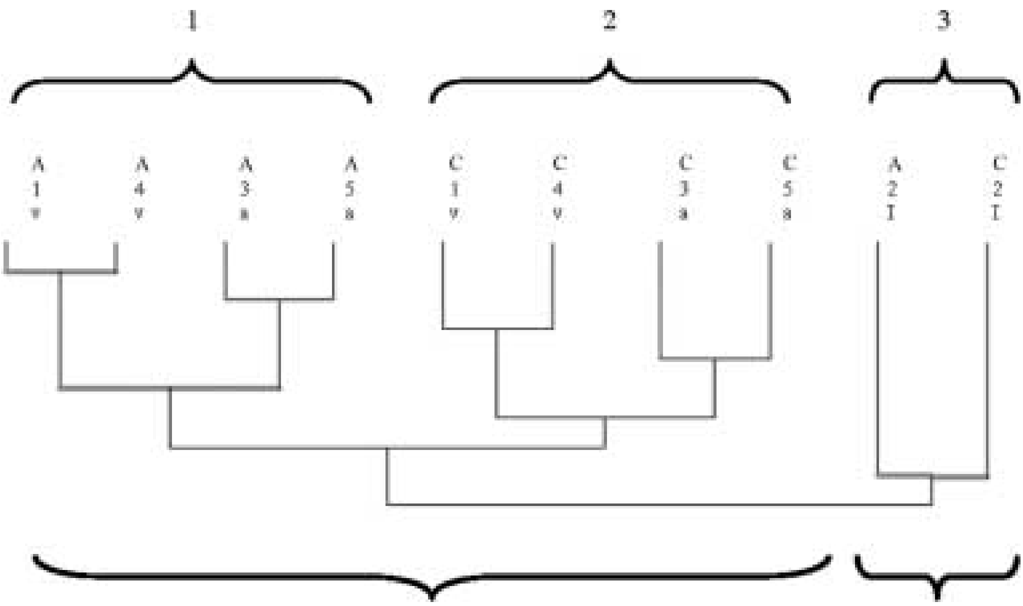


Figure 4. Similarity diagram for the variables in Tests A and C

In this figure, in contrast to Figure 3, the first and the second similarity sub-clusters are formed by two distinct sub-groups of tasks that correspond to the non-proportional problems of volume (1 and 4) and area (3 and 5) from Tests A and C . The existence of the same task separation within both Test A and Test C indicates that in fact all the tasks are handled in a similar way and therefore students' answers refer to the same phenomenon. The fact that the volume and area tasks are separated suggests that students are misguided by the pseudo-proportionality phenomenon to a different extent for the area and volume tasks.

Discussion

The results of our study reveal the great discrepancy in students' performance when dealing with linear and non-linear geometrical tasks, irrespective of the settings of each test. This difference can be investigated if we pay attention to the mathematical errors students made when dealing with the non-linear problems of area and volume, mainly because of their tendency to see the linear function everywhere (Gagatsis & Kyriakides, 2000).

The errors caused by this tendency are not random; they persist and reappear in different circumstances. Almost 60% of the students of this study behaved in a consistent manner; that is, they applied proportional methods when dealing with non-proportional tasks, irrespective of the three different settings. This provides further evidence for the deep rooted and resistant character of the illusion. Moreover, the fact that in the similarity diagrams the non-proportional tasks from Tests A and B, and from Tests A and C, form a group of similarity distinct from the corresponding proportional tasks verifies the consistency of students' linear behaviour between the three tests (addressing the first research question).

The separation of the non-proportional tasks into distinct factors within each test, as revealed by factor analysis, indicates students' consistent fall into the linearity trap for both area and volume tasks. However, students seem to be affected to different extents in Tests A and C by the pseudo-proportionality phenomenon, as the volume and area tasks are separated in the relevant similarity diagram.

Students' behaviour exhibits a systemisation independent of their age and mathematical level. In Tests A and C, the mean achievements of students of both grades were not statistically significantly different. This is evidence for the non-random and resistant-to-instruction character of the pseudo-proportionality phenomenon (addressing the second research question). However, differences between the mean achievement scores of the two grades were observed in Test B. This was rather expected, as the settings of Test B possibly favoured eighth graders' ability to apply mathematical formulae and execute mathematical operations.

Based on the above observations and Duroux's (1982) five necessary conditions for an epistemological obstacle, this study adduces evidence in support of our claim that linearity constitutes an epistemological obstacle to non-linear functions (addressing the third research question). The five conditions are point by point discussed below.

1) The errors that result from the inappropriate application of linearity are not an outcome of a lack of knowledge. On the contrary, the fact that many students (18%) who double the dimensions in Test B and are given the volume formula for their calculations continue to use the improper linearity model indicates that they have all the necessary knowledge to solve the problem. However, linearity creates an obstacle that makes it impossible for them to find the correct solution.
2) Linearity produces responses that are appropriate within a particular context. This explains students' high performance on the linear task in all three tests (90%, 90%, and 93%).
3) Linearity generates false responses outside this context. Due to the improper application of proportionality, students failed at the area and volume tasks of all three tests, with success percentages barely reaching 8.5% for Test A, 32% for Test B, and 21% for Test C.

4) Linearity resists occasional contradictions or the establishment of a better piece of knowledge. This is indicated by students' low success rates on the non-linear tasks in Tests B and C, where the settings were differentiated. In particular, the fact that the student success rate in Test C was just 21% while the possibility of a correct answer by chance was 50%—question right or wrong with two possible answers—suggests that students answer by deliberately applying the inappropriate (in this context) linear model. Therefore, Test C confirms that the phenomenon observed in Test A is not random. In addition, students' behaviour in Test B suggests that knowledge of the dimension values does not suffice to overcome the obstacle of linearity. In particular, as mentioned above in point (1), some students doubled the dimensions of the figures in Test B yet did not reject the linear model and used it to conclude in B1 that 2×7m = 14m, 2×5m = 10m, 2×2m = 4m, and therefore, $2 \times 70\ m^3 = 140\ m^3$.
5) Finally, the fact that these errors occur both in secondary and primary education, as reported in other studies (De Bock et al., 1998; Gagatsis & Kyriakides, 2000), indicates that linearity is an epistemological obstacle that endures throughout schooling. It continues to appear in an untimely persistent way and does not disappear after a simple instruction.

Further Research

The cases described in this study belong to those known situations in the area of psychology and the didactics of mathematics in which significant errors occur on account of the implementation of improper mathematical models. It is clear from the literature review that a radical conceptual shift is needed to overcome the errors caused by the illusion of linearity. It is obvious that the experimental conditions described above are not sufficient, and were not designed to effectuate such a conceptual shift. However, a more systematic didactic intervention concerning the non-proportional nature of different tasks, which enhances students' metacognitive control and metaconceptual awareness, has not yielded long lasting effects (Van Dooren, De Bock, Hessels, et al., 2004). In contrast, it was accompanied by the presence of the opposite phenomenon—the application of non-proportional strategies to proportional tasks.

These results can be explained by Sierpinska's (1987) reference to "dual obstacles". A dual obstacle is any attempt to overcome an epistemological obstacle by simply replacing the conviction with which it is linked with the opposite conviction. In the case of the illusion of linearity, the dual obstacle consists of uncritically replacing the conviction that all relations are linear with the opposite conviction that all relations are not linear. To deal with this problem, the student will have to rise above these convictions, analyse the means used to solve problems in order to make explicit the hypotheses tacitly admitted so far, and become aware of possible rival hypotheses (Sierpinska, 1987). How does this radical and permanent change occur?

This question belongs to a wider field of research into mathematics education, which deals with the organisation of proper didactical situations that take into consideration the need to construct a mathematical concept and therefore deal with the existence of possible epistemological obstacles that hinder such construction. In particular, a didactical situation is organised in such a way that the mathematical concept in question arises spontaneously as a necessary tool for solving a particular problem. A didactical situation determines the set of circumstances with which the student interacts in a specific environment and which designate his/her actions (Brousseau, 1997). Therefore, with this approach the emphasis is on the conditions of learning (Rouchier, 1999).

A didactical situation is based on specific games in which the concepts that have to be constructed are brought into play. These games have adidactical character in the sense that the teacher does not intervene in their progress and that the students get feedback from the settings of the situation itself. A teacher's role in these situations is to present at the beginning the rules of these games and then devolve responsibility for learning to students.

A didactical situation is characterised by three games of adidactical character: (1) the game of interaction with an environment, emphasising students' actions (situation of action); (2) the game of interaction with an environment especially structured for communication, emphasising students' formulations (situation of formulation); and (3) the game of interaction with an opponent to sort out statements and distinguish the true from the false (situation of validation) (Rouchier, 1999). At the end of the three games the teacher intervenes, having a crucial role to play in the final stage of the didactical situation: institutionalisation. Here, the students give meaning to the results of their games in relation to formal and socially accepted knowledge.

The question is whether such a didactical situation exists for the concept of linearity. In what follows, we attempt to propose a didactical situation for the confrontation of the epistemological obstacle of linearity, through three games for the elementary school. In the first game, which constitutes the situation of action, we present students with a 2cm × 3cm × 4cm box. We ask students first to calculate how many 1cm × 1cm cubes are needed to fill the box and then to verify their answer. Then the game begins, with students taking twice as many cubes (2 × 24 = 48) and choosing from a set a box into which the cubes fit exactly. The students will probably choose the box with double dimensions (4cm × 6cm × 8cm). Then they verify their answer and compare the initial box with the new one. Students get a point for each right answer. The game continues for three rounds with students doubling the number of cubes each time (96 and 192).

During the second game, which constitutes the situation of formulation, the students form groups of two. The two students, A and B, do not make eye contact. They each have the same boxes in front of them but in different positions. Student A is given an action situation, with different numbers from the first game 1 (12 and 72) and has to work out which box the cubes fit into. Then Student A is asked to formulate a message to Student B, who does not know the situation, in order to help him/her choose the right box. In this message Student A is not allowed to use numbers to indicate the dimensions of the box. A pair of students wins the game when Student

B finds the right box and the right number of cubes. Then the roles are interchanged.

In the third game, which constitutes the situation of validation, the class forms two groups which compete to find the relationship between the different numbers of cubes and the different lengths of the boxes. At this point the panacea of linearity must be rejected. When the three games end, the institutionalisation of this new knowledge must be pursued.

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