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A model of students' combinatorial thinking

Elise Lockwood*

Oregon State University, Kidder Hall 368, Corvallis, OR 97331-4605, United States

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ABSTRACT

Combinatorial topics have become increasingly prevalent in K-12 and undergraduate curricula, yet research on combinatorics education indicates that students face difficulties when solving counting problems. The research community has not yet addressed students' ways of thinking at a level that facilitates deeper understanding of how students conceptualize counting problems. To this end, a model of students' combinatorial thinking was empirically and theoretically developed; it represents a conceptual analysis of students' thinking related to counting and has been refined through analyzing students' counting activity. In this paper, the model is presented, and relationships between formulas/expressions, counting processes, and sets of outcomes are elaborated. Additionally, the usefulness and potential explanatory power of the model are demonstrated through examining data both from a study the author conducted, and from existing literature on combinatorics education.

1. Introduction and motivation

The importance of combinatorics in K-12 and undergraduate mathematics curricula is well-established in the mathematics education literature (e.g., Batanero, Navarro-Pelayo, & Godino, 1997; English, 1991; Kapur, 1970; NCTM, 2000), both for its rich potential as a problem solving context, and for its applications in probability and computer science. As such, knowledge and pedagogy related to combinatorics is of great importance. As students advance mathematically, however, they tend to experience a great deal of difficulty as they face increasingly complex counting problems; these difficulties are well documented (e.g., Batanero et al., 1997; English, 2005; Kavousian, 2008) and are even noted by authors of combinatorics textbooks (e.g., Martin, 2001; Tucker, 2002). Attempts have been made to improve the implementation of combinatorial topics in the classroom (e.g., Kenney & Hirsch, 1991; NCTM, 2000), but in spite of such efforts, students struggle with understanding the concepts that underpin this topic. Batanero et al. (1997) note the need for an improvement in this area and make the following claim:

> . . .Combinatorics is a field that most pupils find very difficult. Two fundamental steps for making the learning of this subject easier are understanding the nature of pupils' mistakes when solving combinatorial problems and identifying the variables that might influence this difficulty (p. 182).

This call by Batanero et al. acknowledges the difficulties described above, and it also highlights a need for a deeper look at students' mistakes that will help researchers comprehend the nature of students' difficulties. In addition to (and perhaps prior to) examining student errors, there is a need for researchers to better understand the *ways of thinking* that students bring to combinatorial tasks – not only the reasons behind student mistakes, but how students think about combinatorial activity in the first place.

* Tel.: +1 503 502 7217; fax: +1 541 737 0517.
E-mail address: elise314@gmail.com

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The literature on combinatorics education has not yet addressed such ways of thinking at a level that enables researchers and educators to understand how students conceptualize counting problems. Researchers have noted pitfalls that students face (e.g., Hadar & Hadass, 1981), suggested variables that may contribute to student difficulties (e.g., Batanero et al., 1997; Kavousian, 2008), addressed combinatorial strategies among young students (e.g., English, 1991; Piaget & Inhelder, 1975), examined students' verification strategies (Eizenberg & Zaslavsky, 2004), explored student-generated connections among counting problems (Lockwood, 2011b), and emphasized students' potential for powerful combinatorial reasoning (e.g., Maher & Speiser, 1997). However, studies do not identify, trace, describe, or explain students' conceptualizations of the solving of combinatorial tasks. In order to help students develop robust combinatorial thinking (and ultimately succeed as they solve combinatorial tasks), researchers need a deeper understanding of students' conceptualizations of the mathematical activity of solving counting problems.

To this end, the notion of a *conceptual analysis* (Von Glasersveld, 1995) is considered, which entails a detailed description of what is involved in knowing a particular (mathematical) concept. Thompson (2008) elaborates this idea and makes a case for the importance of developing such conceptual analyses. Thompson points out that Von Glasersfeld utilized conceptual analyses in part "to generate models of knowing that help us think about how others might know particular ideas" (p. 57). In this paper, I present such a model of students' combinatorial thinking.[1] The model is primarily a first-order model, in that it is based on my perspective as a researcher and was, in this sense, theoretically developed.[2] However, the model was also empirically refined, and data from interviews with post-secondary students contributed to the refinement of the model. Specifically, the act of conducting and analyzing interview data solidified the particular components and relationships in the model, and while aspects of the model might have existed prior to gathering empirical data, I had not explicitly detailed the model prior to data analysis.

The model thus represents a conceptual analysis of students' activities related to combinatorial enumeration (counting) and has been refined and elaborated through analysis of students' counting activity. The purpose of this model is to shed light on relevant facets of students' counting and to provide language by which to describe and explain aspects of such counting activity, with the end goal of elaborating ways in which students might think about combinatorial ideas.

The term *model* here refers to a particular system for identifying, describing, and explaining certain phenomena related to a particular mathematical topic – in this case combinatorial thinking. Regarding models, Lesh and Doerr (2000) note that,

> Not just any old system functions as a model. To be a model, a system must be used to describe some other system, or to think about it, or to make sense of it, or to explain it, or to make predictions about it (p. 362).

In this sense, the "other system" that the model in this paper is meant to explain is students' combinatorial thinking. With this model, the attempt is to be explanatory, and not merely descriptive, in its discussion of significant phenomena related to students' combinatorial thinking. Additionally, in speaking about conceptual models in the context of problem solving, Lesh and Zawojewski (2007) point out that "the conceptual models researchers use to study and understand mathematical problem solving are expected to be continually under development" (p. 779). In the same way, I suggest that the conceptual model discussed in this paper is an initial attempt at explaining students' combinatorial thinking; it will likely develop and progress over time.

In the remainder of the paper I set up and present the model. I begin by elaborating the specifics of the model, drawing upon illustrative mathematical examples to discuss the components of the model and to examine relevant relationships in the model. Then, I further exemplify the model and exhibit its applicability by discussing particular examples from the study in which the model was developed.[3] I then demonstrate the explanatory power of the model by using it to discuss existing data in combinatorics education literature; this discussion is an attempt at generality beyond the single study from which the model emerged. I conclude by further addressing the rationale for such a model and describing a handful of potential uses.

2. The model

In this section, I present the model of students' combinatorial thinking (Fig. 1), highlighting relationships between formulas/expressions, counting processes, and sets of outcomes. In what follows, I explicate the components of the model and describe the ways in which these components interact with one another.

I begin by explaining each of the components of the model: formulas/expressions, counting processes, and sets of outcomes. *Formulas/expressions*[4] refer to mathematical expressions that yield some numerical value. The formula could have

[1] While I cannot know with certainty what is going on in students' minds, I can examine their observable external activity (including language, gestures, and inscriptions) and make interpretations about their thinking. By student thinking, then, I mean my interpretation of student thinking via their observable mathematical language and activity.

[2] As a first-order model, the model may not be sufficient for addressing some students' thinking. While I suggest below that the model can be used for examining younger students' combinatorial thinking, further work would have to be done to determine the model's effectiveness in explaining less sophisticated student reasoning.

[3] The model emerged from a study in which twenty-two post-secondary students were interviewed in 60–90 min individual, videotaped interviews as they solved five combinatorial tasks. Details of the study can be found in Lockwood (2011a).

[4] It could be argued that there is a distinction between formulas and expressions, perhaps on the basis of an expression's function (a formula could be seen as a certain kind of expression that serves a particular purpose). However, there is currently no such differentiation between formulas and expressions in this model.

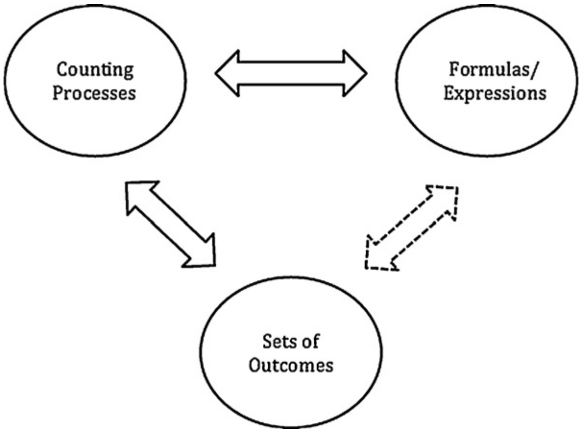

Fig. 1. A model of students' combinatorial thinking.

some inherent combinatorial meaning (such as a binomial coefficient $\binom{8}{3}$), or it could be some combination of numerical operations (such as a sum of products $9 \times 13 + 3 \times 12$). It may be the case that two expressions may be mathematically equivalent (in the sense that one expression could be simplified into the other), but they differ in form (that is, the expressions themselves appear different on the page). I consider two expressions to be different if they differ in form. *Counting processes* refer to the enumeration process (or series of processes) in which a counter engages (either mentally or physically[5]) as they solve a counting problem. These processes consist of the steps or procedures the counter does, or imagines doing, in order to complete a combinatorial task. As examples, the implementation of a case breakdown or successive applications of the multiplication principle (which states that the number of options for a pair of independent choices is the product of the number of options for each choice (Tucker, 2002)) could represent counting processes that a counter might enact as they solve a problem. *Sets of outcomes* refer to the collection of objects being counted – those sets of elements that one can imagine being generated or enumerated by a counting process. In the context of a counting problem, this may be the set whose cardinality represents the answer to that counting problem, but sets of outcomes could also refer to any set that can be associated with a counting process (even if that set is not the answer to the counting problem at hand). For example, in a counting problem asking for the number of 10-letter sequences (repetition of letters allowed) that contain exactly two consecutive As, the desirable set of outcomes is all such 10-letter sequences that satisfy the constraint; those sequences could be conceived of as a set of outcomes. That set could be considered in light of another set – the set of all possible 10-letter sequences.

For a given counting problem, a student may work with one or more of these components and may explicitly or implicitly coordinate them. The key relationships between these components are now elaborated.

2.1. Key relationships between components of the model

2.1.1. Counting processes and formulas/expressions

I first discuss the relationship between counting processes and expressions/formulas, which addresses the relationship shaded in Fig. 2. Note that as the arrow is bidirectional, I discuss both directions of this relationship.

In the context of a counting problem, a given mathematical expression can often naturally be associated with a counting process. To be clear, in the discussion that follows, I am interested in students' constructions of the relationship between counting processes and formulas/expressions (and vice versa) and not the objective reality of this relationship, if there is one. The discussion is meant to elaborate the relationship between formulas/expressions and counting processes that arise for students in solving combinatorial enumeration problems, not to claim that there is a particular process that necessarily and universally underlies a given formula/expression.

2.1.1.1. Formulas/expressions → counting processes. A given formula/expression may elicit a counting process. As an example that highlights this direction of the relationship, we may consider the expression $\binom{5}{2} \cdot \binom{5}{3}$. This product of binomial coefficients can represent a number of possible options. From one perspective, it simply represents a numerical value; we could calculate the product to arrive at 100. However, in the context of counting, this same product tends to signify a

[5] At this point, the distinction between whether the counting process is actually carried out or is mentally conceived is not made in this model. APOS theory (Dubinsky, 1994) could provide further insight into this distinction, where the activity of physically engaging with the process is considered an *action*, but the mental consideration is a *process*, and this may warrant further investigation and subsequent refinement.

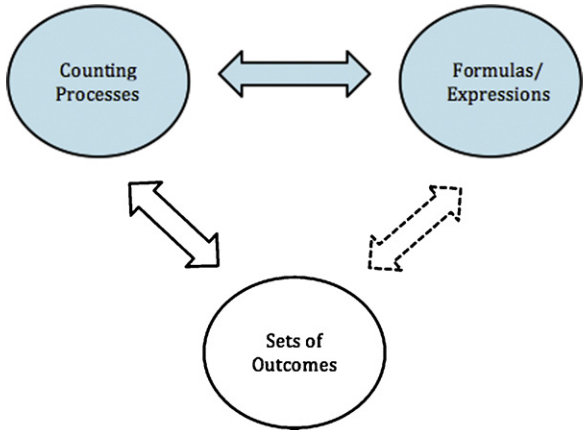

Fig. 2. The relationship between counting processes and formulas/expressions.

particular process. Specifically, it commonly represents an instance of the multiplication principle in which a typical element that is being counted is constructed in two stages. In the first stage, two objects are chosen from five distinct objects, and in the second three objects are chosen from five distinct objects; the multiplication indicates that the two stages are performed independently. We can further specify a context, such as a problem that states, "suppose an exam consists of 10 questions, and you must answer 5 questions. In how many ways can you choose exactly 5 questions to answer if you must answer exactly 2 of the first 5 questions?" In the context of such a problem, the expression can represent an even more specific process – choosing two of the first five questions and then choosing three of the second five questions. In another situation, this process might involve choosing committee members or books. Regardless of the context, however, counters can attribute combinatorial meaning to a mathematical expression in the form of a counting process.

2.1.1.2. Counting processes → formulas/expressions. In the opposite direction, we could conceptualize a counting process that generates an appropriate formula. If we wanted to count the number of ways of arranging 5 objects from a set of 10 distinct objects, there is a counting process that would allow us to do that, and this counting process could be conveyed through an expression. We could consider the number of options for which object could go in the first position (10), then consider the number of options for the second position (9), etc., and using the multiplication principle we could arrive at an answer of $10\cdot 9\cdot 8\cdot 7\cdot 6$. There are thus formulas and mathematical expressions that can be generated by a particular counting process with which we might engage. In fact, this particular act of producing a formula from a counting process is often the end goal of solving a counting problem. In counting problems, it is common for an expression or a formula (and not a numerical value) to be the more desirable and meaningful solution to a problem; this can be particularly true of solutions with very large numerical answers.

2.1.1.3. Further comments on the relationship between counting processes and formulas/expressions. There may be more than one counting process associated with a single formula or expression, and there may be more than one formula associated with a given counting process. As an example of the former, we consider the expression $\binom{10}{5}$. This is a numerical expression with a numerical value; it is equivalent to $\frac{10!}{5!5!}$, or 252. If we consider the question "How many ways are there to choose a committee of 5 people from a faculty of size 10?," the answer is $\binom{10}{5} = 252$, but there are two different counting processes that could get us there, each represented by the same expression of $\binom{10}{5}$. We could have first arrived at the answer by choosing 5 of 10 people to be in the committee, yielding $\binom{10}{5}$. We also could have arrived at the answer by choosing 5 people *not* to be on the committee, also done in $\binom{10}{5}$ ways. So, while the expressions are the same in form, the processes by which we arrived at the expressions differ. The particular counting process that was implemented would depend on a given person's way of thinking about the problem.

It also may be the case that there could be more than one expression associated with some counting process. An example is that two students may have learned different expressions for the process of choosing a set of k objects from n distinct objects. For one student, an expression associated with that process may be $\binom{n}{k}$, for another it may be $\frac{n!}{(n-k)!k!}$. These are externally distinct expressions that may be associated with the same counting process.

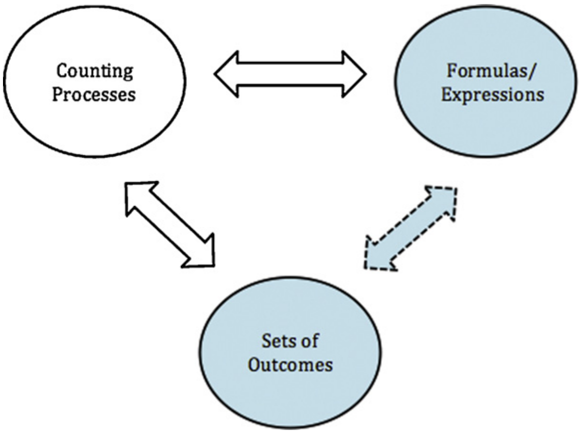

Fig. 3. The relationship between sets of outcomes and formulas/expressions.

There also may be distinct processes that arrive at different expressions, which accomplish the same counting result. For example, if we wanted to arrange 5 objects in 10 slots, we could use the multiplication principle to place objects successively in positions, arriving at $10{\cdot}9{\cdot}8{\cdot}7{\cdot}6$. This represents one process-to-expression pair. However, instead of directly arranging 5 of 10 objects in slots, we could first choose 5 of the 10 objects that we will arrange, done in $\binom{10}{5}$ ways, and then arrange them in 5! ways. This yields an answer of $\binom{10}{5} \cdot 5!$, which represents another process-to-expression pair. The expressions $10{\cdot}9{\cdot}8{\cdot}7{\cdot}6$ and $\binom{10}{5} \cdot 5!$ are equivalent, but they differ in form, and each represents a unique counting process. Ultimately, the same end result is achieved (the number of ways of arranging 5 of 10 objects is calculated), but two different processes led to two different expressions.

There are many possible combinations for how counting processes and formulas/expressions may interact. As such, it may be beneficial for counters to be able to move back and forth between counting processes and formulas/expressions, and to recognize not only that a counting process can yield an expression or a formula, but also that a given formula can represent some counting process. Being able to make sense of both directions of this relationship is an important aspect of evaluating alternative expressions, for example.

2.1.2. Sets of outcomes and formulas/expressions

In the diagram of the model in Fig. 3, the arrow representing this relationship is dotted because in the data this relationship was less clearly linked than the other two. My discussion of this relationship is much briefer than the other relationships, although I include it because I conjecture that it may be a relevant relationship for some counters. Specifically, I suspect perhaps for experienced counters there may be certain sets of outcomes that could be directly connected to particular formulas or expressions without having to consider a counting process. A possible example of this is a formula for a binomial coefficient, $\binom{n}{k}$. While there is an underlying counting process that it represents (choosing a subset of k objects from a set of n distinct objects), for some counters it may become an expression with encapsulated set-theoretic meaning. Specifically, it can be seen as the set of all possible k-element subsets whose elements come from some larger n-element set.

I did not find evidence in the data that would help to flesh out the relationship, and as such it is a theoretical rather than empirical aspect of the model. I mention the relationship here primarily for the sake of completeness and to highlight it as an aspect of the model that could be examined more closely in subsequent research. I suspect that it may be the case that this particular relationship does not commonly arise directly, but rather that sets of outcomes and formulas/expressions tend to be connected *through* counting processes.

2.1.3. Counting processes and sets of outcomes

As with the relationships between counting processes and formulas/expressions, the relationship shaded in Fig. 4 is bi-directional. Counting processes may generate some set of outcomes, and conversely, a given set of outcomes may be enumerated (or its size may be determined) via some counting process. The answer to a given counting problem may be conceptualized as a counting process that yields an appropriate expression, but it also may be conceptualized as the cardinality of an appropriate set of outcomes.

I elaborate the following example to explore the relationship between counting processes and sets of outcomes: "How many 3-letter 'words' are there using the letters A, B, and C (repetition of letters allowed)?" The set of outcomes associated

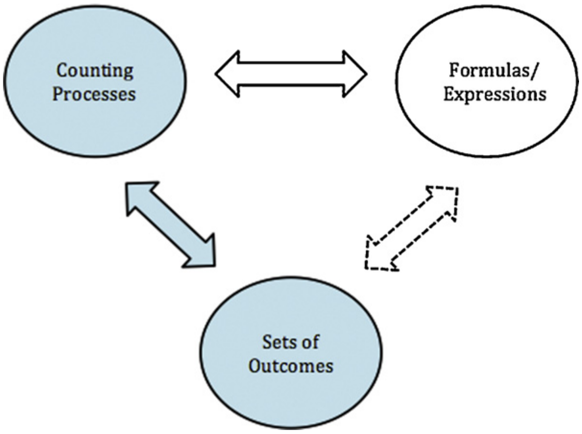

Fig. 4. The relationship between counting processes and sets of outcomes.

with this problem consists of the 3-letter words that satisfy the constraint, of which there are 27. There are multiple counting processes that could be used to answer the counting problem correctly, and I discuss two such processes for this example. One possible counting process is first to apply the multiplication principle to consider the number of choices for the first letter, second letter, and third letter of the word. The choices are independent, and, per the discussion of counting processes and formulas/expressions above, this process can be represented by the expression 3·3·3, which gives an answer of 27. A second process breaks the problem into cases, organizing the words according to the number of distinct letters that appear in a particular outcome. The counting process involves enumerating each type of word and adding the cases. That is, we first consider the solution with only one letter appearing (all As, all Bs, or all Cs), then we consider solutions consisting of exactly two letters appearing (only As and Bs, only As and Cs, or only Cs and Bs), and finally we consider solutions consisting of all three letters appearing. The three parts of the case breakdown have sizes 3, 18, and 6, respectively, which gives a total answer of 3 + 18 + 6 = 27.

2.1.3.1. Counting processes → sets of outcomes. In this direction of the relationship between the counting processes and the set of outcomes, a counting process can be seen as generating some set of outcomes. Staying with the example of 3-letter words above, the process of considering choices for the three respective positions in the word produces a particular listing of the set of outcomes. That is, by first considering that the first letter can be A, B, or C, and then noting that for each of those choices, the second letter can be A, B, or C, and so on, the set of outcomes can be generated. The tree diagram in Fig. 5 makes the generation of outcomes more apparent; the structure of the diagram (the three points of branching) highlights the three-stage process of the multiplication principle, and the resulting list of the set of outcomes is in the rightmost column.

In addition to generating a set of outcomes, a counting process can impose a structure onto a set of outcomes (and, in fact, different counting processes can result in different structures). In Fig. 5, the counting process in the tree diagram actually organizes the set of outcomes into an alphabetical list, and, given the counting process of considering letter options for the respective positions, this makes sense. Alternatively, the second process of breaking the problem into cases and counting words based on the number of letters that appear organizes the set of outcomes in a different way. In Fig. 6, the alphabetical list of outcomes that was generated by the multiplication principle process is on the left, and on the right is the list of outcomes based on the number of repeated letters. This diagram shows the two ways in which the different counting methods structured the set of solutions; there is a one-to-one correspondence between the set on the left and the set on the right. The set of outcomes is represented in both lists, and the cardinalities are the same, but the processes that yielded the set of outcomes differed. This example of two different counting processes illustrates the fact that a given counting process can impose a particular structure on the set of objects being counted.

2.1.3.2. Sets of outcomes → counting processes. The discussion above has focused on one direction of the relationship – how a student can generate (and organize) a set of outcomes from a given counting process. I now discuss the other direction, in which a student can arrive at a counting process from a set of outcomes. For both of the processes in the 3-letter word example, we could also think about starting with the set of outcomes, deciding to organize that set in a particular way, and then coming up with a formula to enumerate the set that is consistent with that specific organization of the set. For instance, we could have decided to start the problem by imagining listing (or actually listing) the outcomes alphabetically. This could have led to the consideration of choices for each letter, and ultimately to the process of implementing the multiplication principle. Or, we could have realized that the set of outcomes could be partitioned according to the number of distinct letters in each word, and we might have decided that we wanted to break up the outcomes accordingly. We could have then implemented a counting process that determined how many passwords were in each possibility and then added to

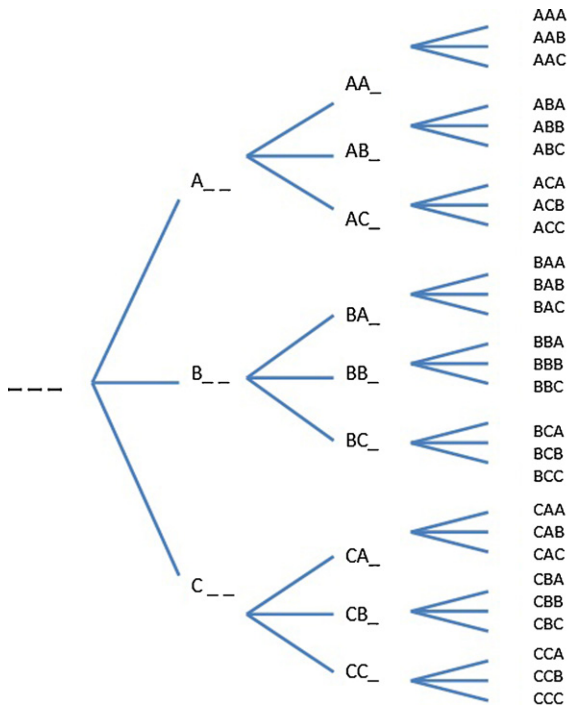

Fig. 5. A tree diagram for 3-letter words.

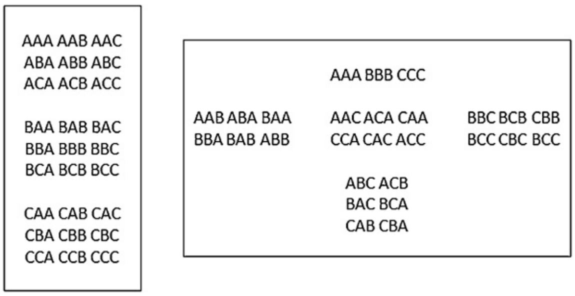

Fig. 6. Two ways to structure the set of outcomes.

find the total. It is noteworthy that such consideration of the set of outcomes may be possible regardless of whether one can conceptualize every element of the set.[6]

2.1.3.3. Further comments on the relationship between counting processes and sets of outcomes. I propose that there could additionally be a back and forth association between counting processes and sets of outcomes. That is, a student could start a counting problem by making an initial attempt at a correct solution – he or she might choose a particular counting process that generates a set of outcomes. The student could then consider that set of outcomes and evaluate whether that set correctly answers the counting problem. In a case in which the generated set of outcomes does not align with the desired set of outcomes, the student might compare those sets and then return to the counting process to try to engineer a process that correctly enumerates the desirable set of outcomes. Such activity would involve movement back and forth between counting processes and sets of outcomes.

[6] While the set of outcomes may exist as the set whose cardinality is the answer to a counting problem, a student may not necessarily want to (or be able to) consider the entire set of outcomes. In some cases, the size of the set of outcomes might be quite manageable, and it might not be difficult physically to list out the set of outcomes or to conceive of the set of outcomes in its entirety. For sets that are larger, though, students may not be able to list out all the outcomes, and they may not even be able to conceive of the set all at once as some finite collection of elements. It is possible to consider and even organize very large sets of outcomes, however, perhaps by thinking abstractly or by considering representative elements of the set of outcomes. Therefore, students might be able to think about sets of outcomes conceptually (rather than concretely) or may theoretically organize sets of outcomes even when cardinalities are very large.

In light of such activity, I contend that the link between counting processes and sets of outcomes can be (and should be) a very flexible relationship, in which students fluidly move from one component to another. If students can easily coordinate a counting process and a set of outcomes, this affords them tractability in their counting. Counting can thus be seen as an activity that relates counting processes to an underlying set of outcomes. An important point to make about the relationship between counting processes and sets of outcomes is that a set of outcomes can provide a way for students to ground their combinatorial activity and can ultimately help to determine whether a counting process is correct. Part of what makes counting fascinating is that counting processes can seem logically sound, but they can yield incorrect answers. When this happens, it can be difficult to determine *why* a given counting process is incorrect. I suggest that students can gain much traction in determining whether a counting process is correct by fundamentally basing their work in a set of outcomes.

3. The model in action

In this section, I use the model to talk through specific examples from the study from which the model emerged (Lockwood, 2011a). While I have mostly presented the model thus far as a first-order model, it was refined through the process of analyzing interviews with post-secondary students. That is, as I considered how to describe what I was seeing in the data, the model took shape and served as an effective lens through which to interpret the data. The examples presented below should help to elaborate the components and relationships within the model in further detail, but even more they should demonstrate the model "in action." That is, I have spent the preceding pages outlining details of the model, and now I illustrate how the model was used to analyze data from my study in which post-secondary students solved counting problems. In so doing, I hope to demonstrate its utility.

3.1. Description of relevant tasks

In this section, at times I refer to student work on two particular tasks from the study. In order to facilitate subsequent discussion and to make the examples clearer, I present two tasks[7] below. I give a brief explanation of common solutions to these tasks, and for one of the tasks, I also present a common incorrect answer that arose during the interviews.

3.1.1. The Passwords problem

The Passwords problem states, "A password consists of 8 upper case-letters. How many such 8-letter passwords contain at least 3 Es?" The *at least* constraint is noteworthy. With this constraint, the problem can be broken down into cases, in which the passwords containing exactly three, four, five, six, seven, or eight Es are counted separately and then summed. For any of those cases, the number of passwords containing exactly k Es is found by choosing k spots for those Es to go (done in $\binom{8}{k}$ ways), and then filling in the remaining $8-k$ spots with any of the 25 letters that are not E. Therefore, a correct expression for the answer is

$$\binom{8}{3}\cdot 25^5+\binom{8}{4}\cdot 25^4+\binom{8}{5}\cdot 25^3+\binom{8}{6}\cdot 25^2+\binom{8}{7}\cdot 25^1+\binom{8}{8}\cdot 25^0.$$

Another correct approach is to subtract the "bad" cases from the total number of passwords – that is, to subtract from the number of all 8-letter passwords the number of those that contain either 0, 1, or 2 Es. This answer yields the expression

$$26^8-\left[\binom{8}{2}\cdot 25^6+\binom{8}{1}\cdot 25^7+\binom{8}{0}\cdot 25^8\right].$$

There is also a tempting approach that does not involve a case breakdown that reflects a subtle error. Specifically, we could arrive at an answer by arguing that by first choosing where to put three Es (done in $\binom{8}{3}$ ways), we are guaranteed to have at least three Es in the password. Therefore, the remaining five letters could be any letter, including an E (done in 26^5 ways). This yields an expression of the form

$$\binom{8}{3}\cdot 26^5.$$

However, the problem with this answer is that some particular passwords get counted more than once. For example, the password $\underline{E}\,\underline{E}\,\underline{E}\,\underline{A}\,\underline{B}\,\underline{E}\,\underline{E}\,\underline{E}$ gets counted multiple times, both when the first three Es were chosen in the $\binom{8}{3}$ step ($\underline{E}\,\underline{E}\,\underline{E}$ _ _ _ _ _)

[7] The Passwords problem was adapted from Tucker (2002), and the Apples and Oranges problem was found in Martin (2001).

and the rest of the password was filled in with ABEEE in the 26^5 step, and then again when the last three Es were chosen in the $\binom{8}{3}$ step (_ _ _ _ _ $\underline{E}\,\underline{E}\,\underline{E}$) and the rest of the password was filled in with EEEAB.

3.1.2. The Apples and Oranges problem

The Apples and Oranges problem asks, "How many different nonempty collections can be formed from five (identical) apples and eight (identical) oranges?" To solve this problem, we can choose zero to five apples (six possibilities) and zero to eight oranges (nine possibilities); we cannot choose zero of each, so there are $6 \times 9 - 1 = 53$ collections. Another common solution to the problem is to consider as cases the different options for the number of pieces of fruit in the collections. There cannot be a collection of size zero, but there can be collections of size 1 through 13. These can be counted directly and summed, yielding the expression $2+3+4+5+6+6+6+6+5+4+3+2+1=53$ as the total number of collections.

3.2. Examples from the data

In this section I offer examples from the study from which the model emerged. These examples are organized according to the relationships in the model. First I discuss the relationship between counting processes and expressions/formulas, and second I address the relationship between counting processes and sets of outcomes (as mentioned above, the relationship between sets of outcomes and formulas/expressions is a theoretical, and not empirically demonstrated, aspect of the model).

3.2.1. Examples from the data – counting processes and expressions/formulas

The first example from the data highlights the relationship between counting processes and formulas/expressions. This example involves a student for whom it was natural to associate a counting process with a given combinatorial expression; it is an instance in which the relationship is intact, even well established. We consider part of Marcus' work on the Passwords problem. He had initially gotten the problem incorrect, yielding $\binom{8}{3} \cdot 26^5$. When we revisited the problem, I asked him to evaluate the correct expression

$$\binom{8}{3} \cdot 25^5 + \binom{8}{4} \cdot 25^4 + \binom{8}{5} \cdot 25^3 + \binom{8}{6} \cdot 25^2 + \binom{8}{7} \cdot 25^1 + \binom{8}{8} \cdot 25^0 \tag{1}$$

We see below that he analyzed the alternative answer and described a process that he associated with the expression.

Marcus: Mmm, . . . I think they're, this answer (Expression 1) . . . is taking it more case by case where I mean, . . . whether you have 3 Es, 4 Es, 5 Es, or 6 Es, or, all the way up to 8 Es.

Interviewer: And what tips you off to that?

Marcus: Just from the first term I see that this is the case where, 8 choose 3, well that's, of 8 of your spots, choose 3 of them to be Es, and there's 8 choose 3 combinations of that, and then for your 5 remaining spots, choose out of the remaining alphabet, which is 25, a letter to take that spot.

In this example, I want to emphasize that Marcus talked fairly easily about the fact that the numerical expression in front of him represented a counting process. Although I presented him only with the sum of a product involving binomial coefficients (with no explanation of a particular process), he associated the expression with a counting process. Specifically, he saw that the expression could entail a number of cases for the number of Es in a given password, and he recognized that the 8 choose 3 could represent picking 3 spots in which to place Es. In terms of the model, I took Marcus' discussion here to indicate that for him, the relationship between this expression and an associated counting process was intact. Throughout the interviews, for the most part, students naturally made that connection between counting processes and formulas; in fact, the notion that an expression represented a counting process appeared to be so engrained that it rarely came up as something they felt they had to discuss or explain explicitly.

3.2.2. Examples from the data – counting processes and sets of outcomes

To demonstrate the model being used to discuss the relationship between counting processes and sets of outcomes, I provide four brief examples of student work – two of students' work on the Apples and Oranges problem, and two of students' work on the Passwords problem. Both of the students working on the Apples and Oranges problem solved the problem correctly, and I discuss their work here because they employed different counting processes that resulted in different structures on the set of outcomes.

First, Brandon's counting process on the Apples and Oranges problem was to utilize a case breakdown by considering cases for the size of each collection. His work is seen in Fig. 7, and he arrived at a total of 53 by adding up the thirteen cases he had found. Here, Brandon's process yielded an organization of the set of outcomes according to the size of each collection. He took advantage of a specific partition of the set of outcomes, and his process thus structured the set of outcomes into thirteen subsets, each of which he enumerated, summing all of their cardinalities to find the total.

Another student, Zach, implemented a different counting process. As we see in the excerpt below, Zach used the multiplication principle to arrive at the total number of collections he could make; doing so helped him compute that there

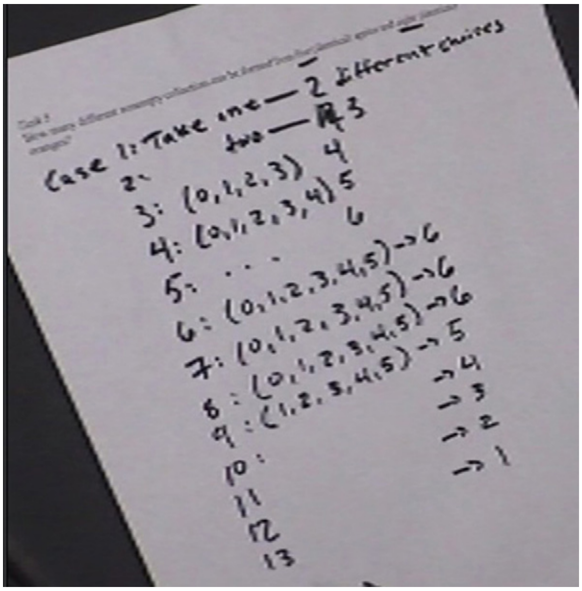

Fig. 7. Brandon's work on the Apples and Oranges problem.

were 54 total outcomes. From this total he could subtract the combinations that had zero of each kind of fruit (there is one), ultimately resulting in the correct answer to the problem, 53.

Zach: I'm just going to start worrying about collections, and then I'll decide whether I've accidentally counted something that was empty. So, I think if I'm going to form a collection, that collection would have anywhere from 0 to 5 apples. . .I've got 6 different things going on there, and then 9 different things that can happen here, and those are totally independent notions. So, 6 different amounts of apple. Um, 9 different amounts of orange.
Those are independent, so I would think that I could do 54 different collections, however, um, one of those collections that I just counted would be the option of grabbing none of the apples, paired with the option of grabbing none of the oranges. So I would say, since there's no other way to do an empty collection other than not grab anything, there is exactly one thing I shouldn't have just counted there.

These examples of student work show how different counting processes that students employed resulted in different organizations of the same set of outcomes. For Brandon, the structure of the set of outcomes involved counting up different subsets, and Zach subtracted the undesirable outcome from the total. For both of these students, their solutions are intimately linked with the set of outcomes. These examples highlight the ways in which a counting process can generate and structure a set of outcomes; the language of the model was used to describe phenomena that happened for students in their work on the Apples and Oranges problem.

I now contrast two examples of students' work on the Passwords problem. In Kim's work on the Passwords problem, we see a student who solely talks about counting processes and formulas/expressions, but for whom sets of outcomes do not arise. In her solution, she wrote out 5 dashes, wrote 26 in each of the dashes, and then wrote $\binom{8}{3}$. Her final answer was incorrect; she arrived at $26^5 \cdot \binom{8}{3}$, as she explains below.

Kim: Alright well I always start these problems[8] by putting the dash things in it,
Interviewer: Okay.
Kim: So 1, 2, 3, 4, so we need at least 3 Es, so the option of having exactly 3 Es, so the rest of them will be open, 26 options for all the letters in the other 5 slots, and it's an 'and,' so you multiply, and then I'm thinking that you have 3 Es are there are 3 spots out of the 8 areas where they could fall into.

Her process was first to pick any letter to place in the five spots that were not designated Es, and then to select 3 of the 8 spots which had to contain Es. This process yielded an expression, which she presented as her final answer to the problem. Kim did not indicate that she considered the set of outcomes at all (when asked if she had pictured a particular password, she said that she had not). On this problem, Kim's counting process was not correct, and, as discussed in the description of the problem above, the reason for the error is quite subtle. What is noteworthy about this example is not that she arrived at a common incorrect answer, but that her work on the problem seemed to be entirely about her counting process, which

[8] Kim did not give further indication of what she meant by "these problems."

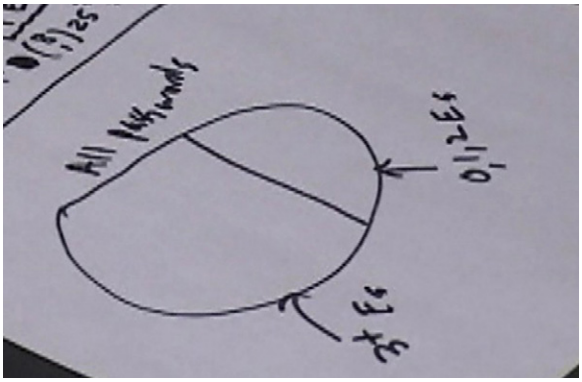

Fig. 8. Ruben's diagram for the Passwords Problem.

yielded an expression. I maintain that for Kim, there was not a well-established relationship between her counting process and the set of outcomes, and in this way the model provides insight into why this might have been problematic for her.

In contrast, in his work on the Passwords problem, Ruben initially arrived at the correct answer

$$26^8 - \left[\binom{8}{2}\cdot 25^6 + \binom{8}{1}\cdot 25^7 + \binom{8}{0}\cdot 25^8\right],$$

and his reasoning is seen in the excerpt below.

Ruben: I want to know how many contain at least 3 Es . . . just because the counting is easier, I'm going to probably turn that around, and say I want to know how many contain 2 or 1 or 0 Es, and then subtract that from the total.

Ruben went on to draw the diagram in Fig. 8 as well, which reflects the strategy he described in his language above. Ruben's language and his diagram reflect what might be called a *total-minus-bad* approach, because he subtracted the "bad" outcomes from the total number of outcomes. In the excerpt above Ruben did not explicitly mention the set of outcomes, but I interpret that he implicitly utilized the notion of sets of outcomes by organizing the set of all 8-letter passwords into two parts – those that contain 3 or more Es and those that contain fewer than 3 Es. In terms of the model, I take Ruben's work here as an instance of the relationship from the set of outcomes to the counting processes. That is, Ruben made a decision about how he would organize the set of outcomes, and that organization led him to implement a particular counting process, which ultimately yielded a correct expression.

The examples presented in this section are intended not only to describe further some of the components and relationships within the model, but they are also meant to illustrate ways in which the model might be used as an analytic lens. These were some specific instances of the ways in which I used the model to make sense of phenomena in my data. The model gave me useful language to describe and explain phenomena I observed, and ultimately helped to shed light on how students in my study thought about and approached counting problems.

In an effort to illustrate the potential explanatory power of the model beyond my specific study, I now demonstrate the applicability of the model beyond my particular data set. In particular, I use the model to examine some data within the existing literature on combinatorics education.

3.3. *Using the model to gain additional insights by examining existing data*

While the model was developed in the context of my study on post-secondary students' combinatorial thinking, I now present specific examples from other studies of how the model might be used to analyze data. The goal in this section is not to suggest that my analysis is in any way better than the authors' own analyses, or that their data is in need of an alternative analysis. Nor do I mean to presume that I can meaningfully analyze the relatively small amount of data presented in a paper in the same way that these authors could analyze their entire set of data. Rather, the purpose of this discussion is to demonstrate the utility of the model as a lens through which to examine a variety of data on combinatorics education research. Given that the model was developed during one particular study, this exercise is an attempt to make a stronger case for the generality of the model. I have chosen brief examples of data from four studies in the literature,[9] and below I offer my analysis of this data in terms of the model. In this discussion, my purpose is to use the model as a lens to make sense of data, not as a re-analysis; I hope to show that the model could be viable as a way to provide some additional insight for combinatorics education researchers and educators.

[9] The authors cited in this section were consulted and graciously contributed to the consideration of how the model might apply.

Firstly, A group needs 3 people. So the solution is $C(8,3)$. Then I need to find the solution for B group. In this question we are not given the information of what kind of groups they are. So we could think same person could be in the different group. We could choose 3 people from 8 also for B group. The ways of choosing people for B group are $C(8,3)$ as A group.

So the answer can be $C(8,3) + C(8,3) = 2 \times C(8,3)$.

Fig. 9. A student excerpt (Kavousian, 2008, p. 120).

3.3.1. Example 1 – Kavousian (2008)

Kavousian (2008) conducted two studies on undergraduate combinatorics students. In her first study, her analysis focuses on examining the role of structures, formulae, and representations as students encounter new combinatorial structures or concepts; she investigates this through providing students with a novel combinatorial definition and answering questions related to this new concept. In the second study, she discusses a refining cycle in which students can interact with other students or the teacher in order to "reexamine their thoughts on a particular question or structure" (p. 114). In one example from this second study, Kavousian's students answered the question, "In how many ways can you choose two groups of 3 people from 8 people to serve on two different committees?" With the assumption that a person could be in two groups, the correct answer is $\binom{8}{3} \cdot \binom{8}{3}$. In a student's answer given in Fig. 9, he seems to identify the counting process correctly – that he wants to choose 3 people from 8 for group A (he arrives at a correct formula for that process), and then he wants to do the same for group B (he again finds the correct formula).

The problem arises when the student goes to combine those individual processes, and he incorrectly adds (instead of multiplies) the binomial coefficients. This incorrect answer was common among Kavousian's students on this problem, and she discusses this phenomenon in detail according to her analytical framework.

In terms of the model, I interpret that in this example there was a breakdown in the relationship between counting processes and formulas/expressions. The student arrived at an incorrect answer not because he identified a counting process incorrectly, but because he was unable to associate a correct expression with his desired counting process (indeed he generated an incorrect formula). The model allows for an overarching description of where the problem broke down for the student.

I would argue, too, that this example reveals something deeper that is happening for students regarding sets of outcomes. The student uses the operation of addition instead of multiplication, and while this may seem like a mere conflation of operations, I suggest that he failed to clearly identify what the operation of addition does (versus what the operation of multiplication might have done) in the given problem. This is particularly noteworthy because the student's language suggests that he did not draw upon the set of outcomes. His work above is situated within the relationship between counting processes and formulas/expressions, but sets of outcomes do not come into play. He does not seem to see that drawing upon sets of outcomes could possibly provide an explanation for why that expression is incorrect – to articulate what outcomes the operation of addition would yield instead of the operation of multiplication. Without drawing upon the sets of outcomes, it can be difficult for a student to see how the relationship between his counting processes (choosing 3 people to be in group A and then choosing 3 people to be in group B) and his expression $\binom{8}{3} + \binom{8}{3}$ is incongruous.

3.3.2. Example 2 – Eizenberg and Zaslavsky (2004)

Eizenberg and Zaslavsky (2004) studied students' combinatorial verification strategies, which can be particularly difficult for students to utilize in the field of combinatorics. They observed five strategies in their work with undergraduate students. I highlight how the model relates to two of these strategies in particular, making the case that the model might be a useful lens through which to talk about combinatorial verification strategies more generally.

First, Eizenberg and Zaslavsky demonstrate a strategy that they call "Verification by adding justification to the solution" (p. 23). In the excerpt in Fig. 10, we see a student talking through his partner's initial solution, convincing himself (and his partner) that the initial proposed answer is correct.

The model provides language for how to discuss this verification strategy. I maintain that the student's argument (and justification) is situated entirely within the component of the counting process. That is, the student's verification above is based primarily in talking back through the process and making sure the constraints of the problem are met in the solution. The student does not ground the verification in the set of outcomes, nor does the student draw upon expressions/formulas. With very few accessible verification strategies for complex counting problems, this method of talking back through a solution and adding justification is one of the primary checking options available to students. This method of verification

Problem 8: A teacher has 8 pupils and 4 different pieces of candy. In how many ways can the teacher distribute the candies to the pupils, if each pupil may get more than one piece, and not all pieces need to be distributed?

Yuval: If you want, we can check this [the answer].

Gal: In my way? It's a very long way. You have the possibility that 1 gets them, 2 gets them, 3 gets them, and 4 gets them. You will get confused with all the numbers. We leave it. This is right. It is clear that it is right. You satisfied all the conditions. This is the checking: Each pupil can get more than 1 piece of candy. You covered this limitation. [According to our solution] you choose for a piece of candy to go to a pupil, and then it could be that 2 pieces go to the same pupil. So you covered the possibility that each pupil can get more than 1 piece of candy. Also, not all pieces have to be distributed. You took this as a possibility. This is definitely right.

Fig. 10. A student excerpt (Eizenberg & Zaslavsky, 2004, p. 24).

can be effective, but unfortunately mistakes in counting problems are sometimes very difficult to detect (see Lockwood (2011a)), and even re-checking through a counting process may not reveal subtle errors.

Second, there is another relatively common way in which verification can occur in combinatorial problems, and that is through solving the problem in a completely different way and comparing the results. Eizenberg and Zaslavsky call this "Verification using a different solution method and comparing answers" (p. 28). The model offers an additional way of conceptualizing this approach. In particular, when solving a problem a student may have used counting process A to arrive at expression X, which has some numerical value. Not knowing whether expression X is correct, the student may try to solve the same problem by employing counting process B, which yields expression Y. The numerical values of X and Y can then be compared. If they are different, this might alert students that they need to re-check their work; if they are the same, students can have more confidence in their original answer.[10] Even more, I suggest that there could be an additional step to such a verification strategy in terms of the model. That is, when comparing two particular counting processes A and B, students could relate each process with what they count in terms of the set of outcomes. If those sets of outcomes that the respective counting processes enumerate are the same (there is a one-to-one correspondence between them), it could serve as confirmation that the processes are also the same. Eizenberg and Zaslavsky offered examples of students attempting alternative counting processes, but the model could potentially suggest an additional step in which students might attempt to relate their processes to sets of outcomes. In this way the model could extend work on verification strategies for combinatorial problems. Thus the model can provide supplementary insights not only on the findings from Eizenberg and Zaslavsky, but also more broadly on the notion of verification within the domain of combinatorial problems.

3.3.3. *Example 3 – Maher and Martino (1996)*

In a well-known longitudinal study, Maher and her colleagues (e.g., Maher, Powell, & Uptegrove, 2010; Maher & Speiser, 1997) show many instances in which students make significant mathematical progress on a variety of topics, including combinatorial tasks. In their 1996 paper, Maher and Martino emphasize students' development of proof in the combinatorial context of determining the number of block towers (of varying height, with varying numbers of color possibilities). In one specific instance of student work, they highlight a fourth grader's justification of why she had counted all towers made of two different colors that were three blocks high. In this justification, the student wrote out a systematic list of particular towers, and she utilized that list to make her argument. The researchers go on to extend the discussion and show how this student developed a "proof by cases" (p. 194) method as she encountered similar (but more complex) problems in subsequent years. The student's work on this problem was undeniably strong, and she displayed the ability to develop proof and to engage in sophisticated mathematical reasoning. Maher and her colleagues have shown significant developments of such thinking over time in a variety of studies (e.g., Maher et al., 2010), but I highlight this particular example in order to discuss the model.

Using the model, I examine this data not from a framework of proof, but from a combinatorial perspective. It is noteworthy that this student was successful in solving counting problems involving block towers (and many other combinatorial problems in addition). Using the model described in this paper, I interpret that the student's success could be due, in part, to her facility with the set of outcomes. Indeed, the fact that she wrote down and reasoned about particular outcomes indicated that she seemed to have a well-connected relationship between her counting process and the sets of outcomes, and this relationship could have contributed to the student's combinatorial success. This example not only gives another perspective and potential additional insight into the data, but it also indicates that the model could also apply to data involving students at the middle school level.

[10] I agree with Eizenberg and Zaslavsky (2004) as they point out that this is not a foolproof verification strategy, as two erroneous processes could be used to yield equivalent expressions, thus giving students undue confidence. Nonetheless, it can, on the whole, be quite a beneficial strategy.

3.3.4. Example 4 – English (1991)

English (1991) identified and categorized combinatorial strategies in young children from ages four to nine. The children were given problems in the context of dressing bears; these problems increased in complexity, and more advanced tasks involved various constraints. English observed five strategies that increased in sophistication, and her data revealed that students' levels of sophistication tended to correspond to their ages. The strategies began with random selection of items (with no rejection of inappropriate items) and ultimately developed into complete odometer strategies (p. 458, 461).

While the combinatorial tasks the children solved were perhaps less traditionally advanced than some of the counting problems given to undergrads in Kavousian's (2008) or Eizenberg and Zaslavsky's (2004) work described above, they remain combinatorial in nature and are worthy of examination. Indeed, unlike undergraduate students, the children counted without having had access to strategies like the multiplication principle or expressions like binomial coefficients. Their strategies thus involved activities such as listing elements (randomly or systematically) and determining useful patterns among outcomes. In terms of the model, then, English's work is situated within the relationship between counting processes and sets of outcomes.

In examining English's work through this lens, it seems that in spite of the fact that at some point counting can become something formulaic and procedural, this is not necessarily how combinatorics begins. For the children in English's study, counting involved manipulating particular outcomes (within a relatively small, finite set of elements) that they could carefully list out and enumerate. Thus, the children seemed to be primarily working in the set of outcomes and were perhaps just starting to extend into considering counting processes. They had processes in the sense that they were engaging in activity, but they were just beginning to develop counting processes and procedures that are more complex than a simple random generation of elements or a guess and check strategy. On the contrary, in Kavousian's (2008) student's work described above (and in Kim's work on the Passwords problem), we see examples of students who worked almost entirely within the realm of expressions and formulas as connected with processes. At some point, it seems that there is a shift from a focus on sets of outcomes among young children to an emphasis on procedures and formulas among undergraduates (a pedagogical discussion of this phenomenon can be found in Lockwood (2012)). I propose that the model can be used to give an additional perspective on middle or high school students in order to capture that transition between English's (1991) findings and Kavousian's (2008) findings – where students go from enumerating small finite sets of outcomes to implementing counting processes to arrive at formulas/expressions. While we see this very activity in Maher and Martino's (1996) student's work, there is the potential to learn more about this specific transition by analyzing additional studies.

To summarize this section, I do not suggest that the model is the only analytical lens through which to examine combinatorial activity, nor do I claim that it is sufficiently fine-grained to address every issue in combinatorics education. Rather, I argue that it is a broad model that gives a sense of what might be happening conceptually for students as they count. There is in fact much that could be done to flesh out the particulars of the relationships, and avenues for further research are discussed below. Nonetheless, as an overarching model, it appears to fit as an appropriate way to view a variety of data, including studies that involve students ranging from young children to undergraduate students.

4. Conclusion

In this paper, I presented a model of students' combinatorial thinking, with the goal of providing language to characterize useful components and relationships involved in the solving of combinatorial tasks. As discussed in the introduction to this paper, there is a dearth of studies that address students' underlying conceptualizations of combinatorial ideas. The model offers a first attempt at addressing what kinds of concepts might be underlying students' combinatorial thinking, and in doing so it satisfies a gap in the mathematics education research on the area of combinatorics. I now propose reasons for the significance of the model and relate it back to Thompson's (2008) notion of a conceptual analysis.

Thompson (2008) describes several uses of conceptual analyses in mathematics education research. First, he says that conceptual analyses can be used "to generate models of knowing" that can shed light on ways in which other people might think about certain ideas (p. 57). Second, conceptual analyses can be used to consider those ways of understanding that might foster effective learning in students; teaching experiment studies (e.g., Steffe & Thompson, 2000) are an example of such a use of conceptual analyses. Third, conceptual analyses can allow researchers to make conjectures about why students struggle with particular ideas and whether there are certain ways of knowing "that might be deleterious to students' understanding of important ideas" (p. 59).

As it was presented above, the model serves the first use of conceptual analyses that Thompson (2008) describes – to generate a model of knowing that helps mathematics education researchers "think about how others might know particular ideas" (p. 57). Given that relatively little has been done to uncover students' conceptualizations of counting problems, this is an appropriate purpose. Researchers could use the model as a lens through which to describe and analyze student work on counting problems, using the components of the model (and the relationships between the components) to describe and evaluate students' counting activity. By facilitating the common articulation of phenomena that researchers encounter, the model can ultimately lead to researchers (and the mathematics education community) improving their understanding of students' conceptualizations of combinatorial ideas. In particular, I demonstrated above that the model could be used to address work with younger students' combinatorial thinking and activity. Using the model in studies on grade 6–12 student populations might give further insight into how students conceptualize the transition of counting problems from being relatively straightforward (involving physically listing out elements, such as those studied by English, 1991) to fairly complex (involving particular processes and formulas, such as those used by Kavousian, 2008). I thus propose that researchers at a variety of levels of investigation related to combinatorics education could use the model effectively to gain insight into students' knowing of combinatorial ideas.

Additionally, related to Thompson's (2008) second use described above, designers can use the components and the relationships in the model as a starting point to design teaching experiments that examine students' learning of combinatorial ideas. As an example, I have emphasized the relationship between counting processes and sets of outcomes and proposed it as a relevant aspect of combinatorial thinking. The importance of this relationship on student learning could be further explored through a teaching experiment, and the effects of other relationships could be investigated as well.

Related to Thompson's third point above, just as researchers could use the model to better know what might be required for students to learn particular combinatorial concepts, so too could the model be used to identify particular reasons behind students' struggles. This again relates to the statement by Batanero et al. (1997) cited previously – that there is a need for mathematics education researchers to better understand the *nature* of students' mistakes in the area of combinatorics. The model provides a first articulation of students' conceptualizations, and it could shed light on and provide language to discuss particular ways in which students struggle, ultimately helping researchers to uncover ways to mitigate students' difficulties. For example, researchers could look at where and how certain relationships in the model break down, they could examine relationships that are unidirectional, and they could examine components that are used in isolation.

In summary, there is a need in mathematics education research for further insight into ways in which students conceptualize combinatorial ideas. The model elaborated in this paper is meant to be an initial attempt at a model of combinatorial thinking, providing ideas and common language that researchers can utilize in evaluating their own students' combinatorial thinking and activity. With this model I offer the mathematics education community a starting point for the deeper investigation of students' combinatorial thinking.

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