Multiple View Geometry in Computer Vision

- Projective Geometry and Transformation of 2D -

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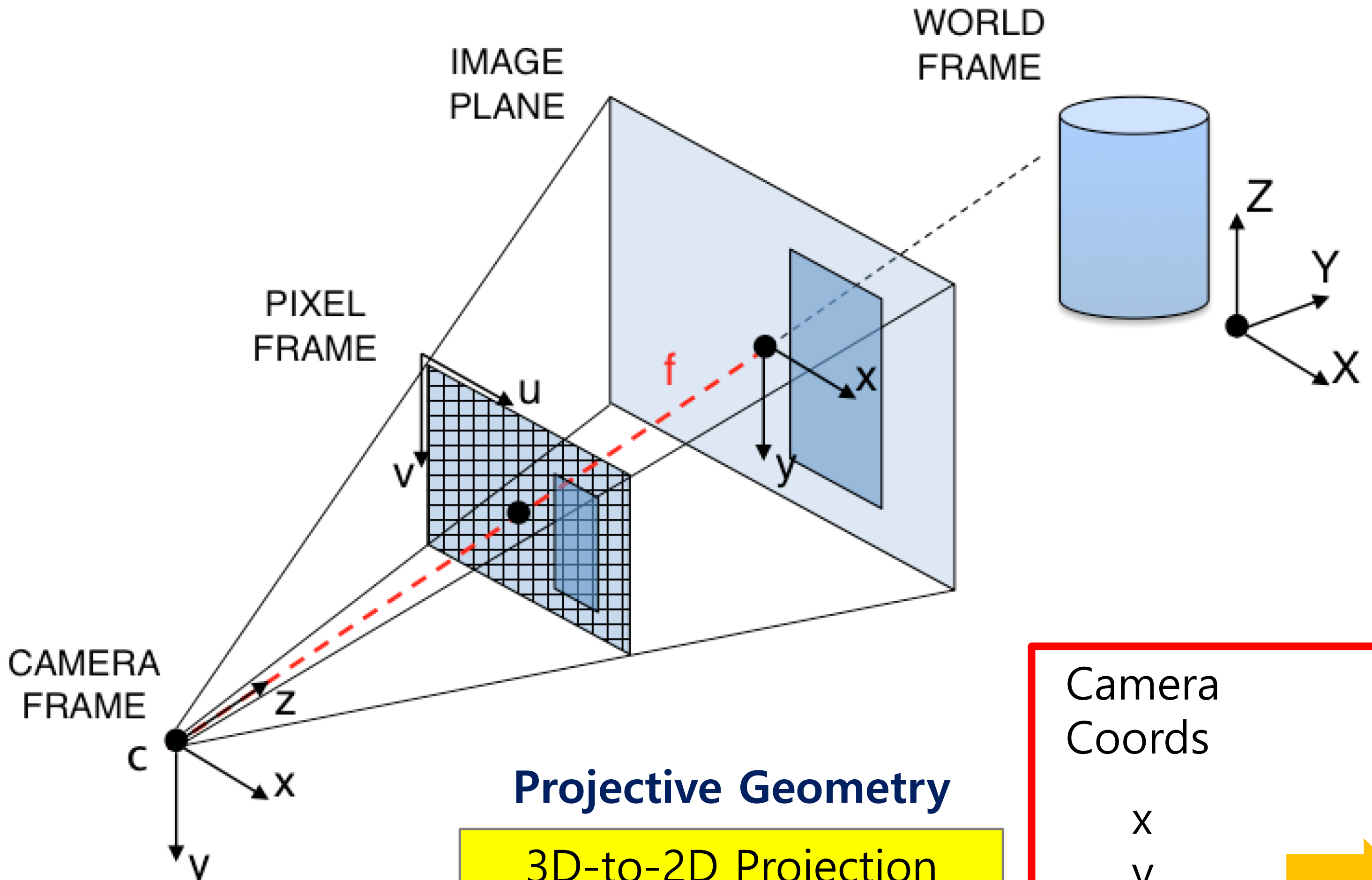
Homogeneous Coordinates



Projective Geometry

3D-to-2D Projection
Perspective projection

Camera Coords		Image Coords
x y z	→	x y

Homogeneous Coordinates

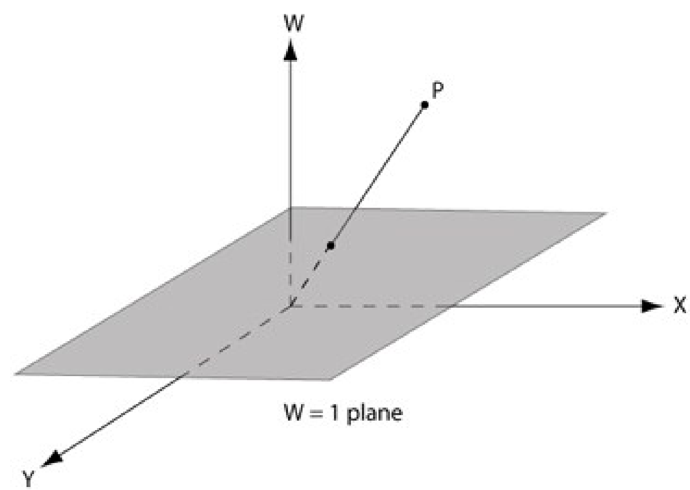
Cartesian **Homogeneous**

$$(x, y) \longrightarrow \begin{bmatrix} wx \\ wy \\ w \end{bmatrix} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

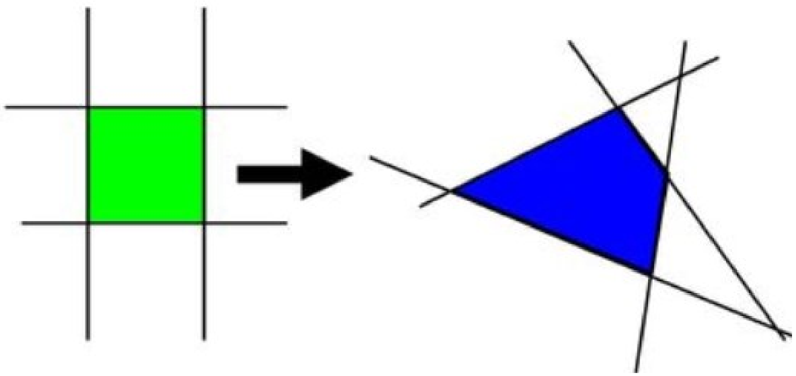
$$\mathrm{p} = \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix} \qquad \vec{\mathrm{v}} = \begin{bmatrix} v_x \\ v_y \\ v_z \\ 0 \end{bmatrix}$$

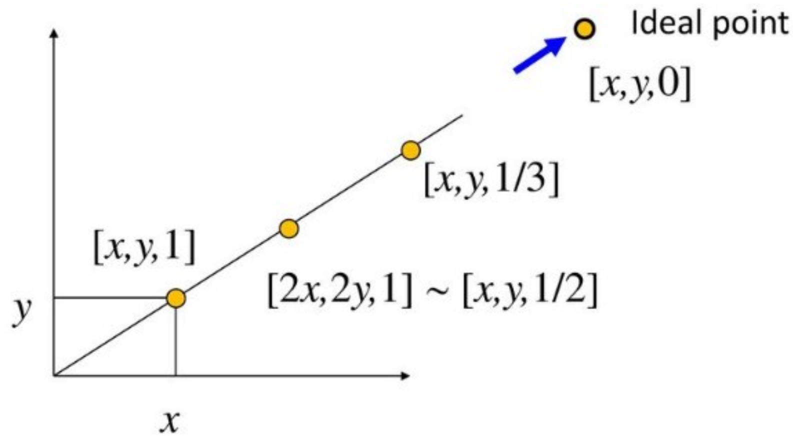
scale factor $\begin{cases} 0 : \text{vector} \\ 1 : \text{point} \end{cases}$

$$(x/w, y/w) \longleftarrow \begin{bmatrix} x \\ y \\ w \end{bmatrix} = \begin{bmatrix} x/w \\ y/w \\ 1 \end{bmatrix}$$



Point at infinity can be represented using finite coordinates





$$\text{a point } (x,\ y) \ \rightarrow \ \left(\frac{x}{w}, \frac{y}{w}, \frac{1}{w}\right) \xrightarrow[w \to 0]{} \left(\frac{x}{w}, \frac{y}{w}\right) \ \rightarrow \ (\infty,\ \infty)$$

homogeneous coordinates

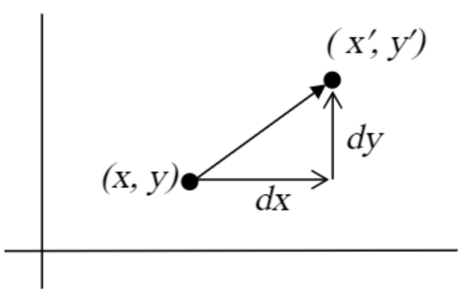
A single matrix can represent affine & projective transformations

Combine rotation and translation into a single transformation matrix

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} dx \\ dy \end{bmatrix} = \begin{bmatrix} x+dx \\ y+dy \end{bmatrix}$$

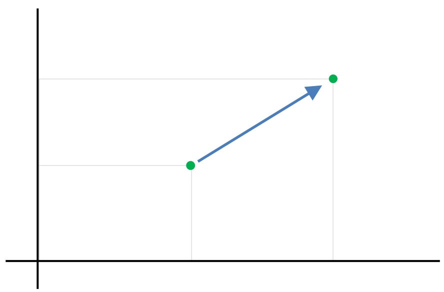
$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} m_{xx} & m_{xy} & dx \\ m_{yx} & m_{yy} & dy \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

- Translation



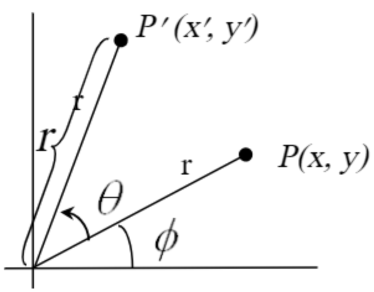
$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & dx \\ 0 & 1 & dy \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

- Scaling



$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

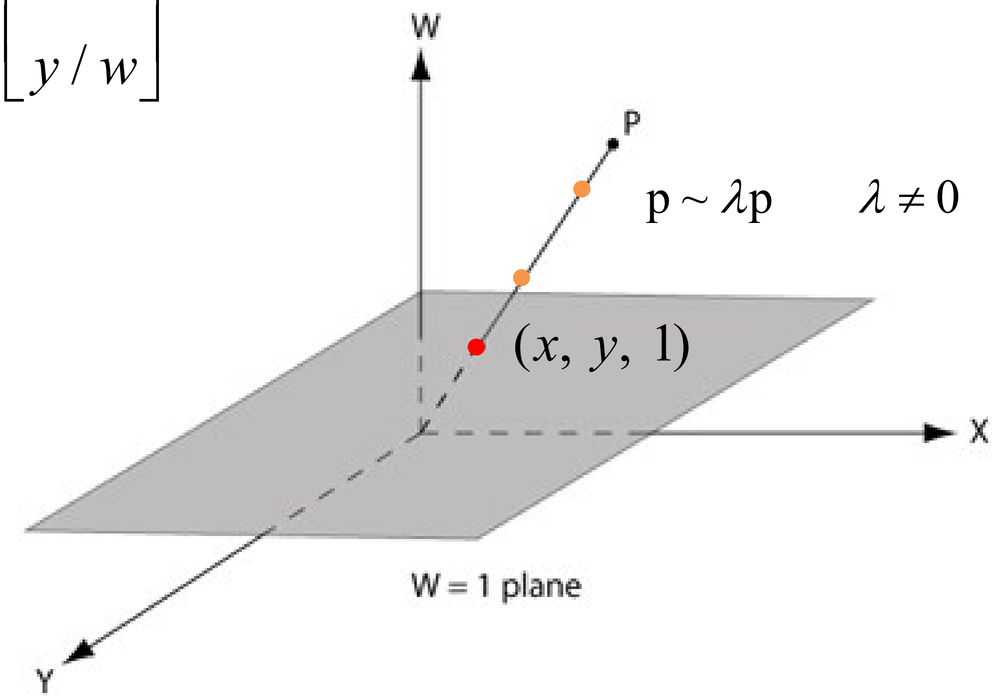
- Rotation



$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Invariant to scaling

$$k\begin{bmatrix} x \\ y \\ w \end{bmatrix} = \begin{bmatrix} kx \\ ky \\ kw \end{bmatrix} \Rightarrow \begin{bmatrix} kx/kw \\ ky/kw \end{bmatrix} = \begin{bmatrix} x/w \\ y/w \end{bmatrix}$$



$p \sim \lambda p \qquad \lambda \neq 0$

$(x,\ y,\ 1)$

Equivalence Class

Equivalence Relation

Let R be the relation on X. If R satisfies three condition,

(1) Reflexive : $\forall\, x \in \mathrm{X},\ x \sim x$

(2) Symmetry : if $x \sim y$, then $y \sim x$

(3) Transisitive : if $x \sim y$ and $y \sim z$ then $x \sim z$

Then the relation R is called "Equivalence relation on X"

Equivalence Class

Let R($\neq \varnothing$) be the equivalence relation on X. For each $x \in \mathrm{X}$, the set

$[x] = \{y \in \mathrm{X} \mid y \sim x\}$ then $[x]$ is called the equivalence class

Quotient Set

Collection of equivalence class is called 'Quotient Set'

$$X / \sim := \{[x] \mid x \in X\}$$

$= \;$ The partition of X induced by R

$= \;$ A modulo R

Partitions

If the relation $'\sim'$ is an equivalence on X.

$X/\sim$ is partition of X

Line and points

Homogeneous representation

$$\mathrm{x} = (x,\ y,\ 1)^{\mathrm{T}}$$

$$\mathrm{l}\ :\ (a,\ b\ ,c)^{\mathrm{T}} \sim k(a,\ b\ ,c)^{\mathrm{T}} \quad k \neq 0$$

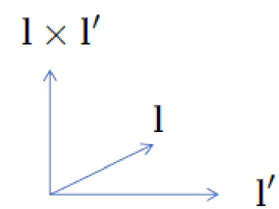
A point x lies on the line l

$$ax + by + c = 0 \qquad (x,\ y,\ 1)(a,\ b\ ,c)^{\mathrm{T}} = 0$$

$$\mathrm{x}^{\mathrm{T}}\mathrm{l} = \mathrm{l}^{\mathrm{T}}\mathrm{x} = 0$$

The intersection of 2 lines l and l

$$\mathrm{x} = \mathrm{l} \times \mathrm{l}'$$



The line through 2 points x and x′

$$\mathrm{l} = \mathrm{x} \times \mathrm{x}'$$

Ideal points and the line at infinity

Intersection of parallel lines

$\mathbf{l} = (a,\ b\ ,c)^{\mathrm{T}}$ and $\mathbf{l}' = (a,\ b\ ,c')^{\mathrm{T}}$ are parallel

$$\mathbf{l} \times \mathbf{l}' = \begin{vmatrix} \mathrm{i} & \mathrm{j} & \mathrm{k} \\ a & b & c \\ a & b & c' \end{vmatrix} = (c'-c)(b,\ -a,\ 0)^{\mathrm{T}} \underset{\text{ignoring}}{\overset{\text{Scale}}{\sim}} (b,\ -a,\ 0)^{\mathrm{T}}$$

$\underset{\text{Homogenoeus coordinates}}{\text{point } (b,\ -a,\ 0)^{\mathrm{T}}} \rightarrow \underset{\text{Inhomogeneous coordinates}}{\cancel{(b/0,\ -a/0)^{\mathrm{T}}}} \rightarrow (\infty,\ \infty)$

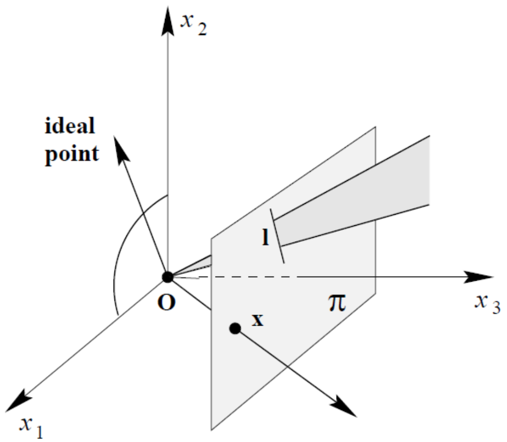
Ideal points $(x_1,\ x_2,\ 0)^{\mathrm{T}}$ Line at infinity $\mathbf{l}_\infty = (0,\ 0,\ 1)^{\mathrm{T}}$

Ideal points and the line at infinity

$$\mathbf{l}\times\mathbf{l}'=\begin{vmatrix} \mathrm{i} & \mathrm{j} & \mathrm{k} \\ a & b & c \\ a & b & c' \end{vmatrix}=(b,\ -a,\ 0)^{\mathrm{T}} \quad \rightarrow \text{intersection} \quad \Rightarrow \quad \text{ideal point}$$

Ideal point set lies on a single line, the line at infinity

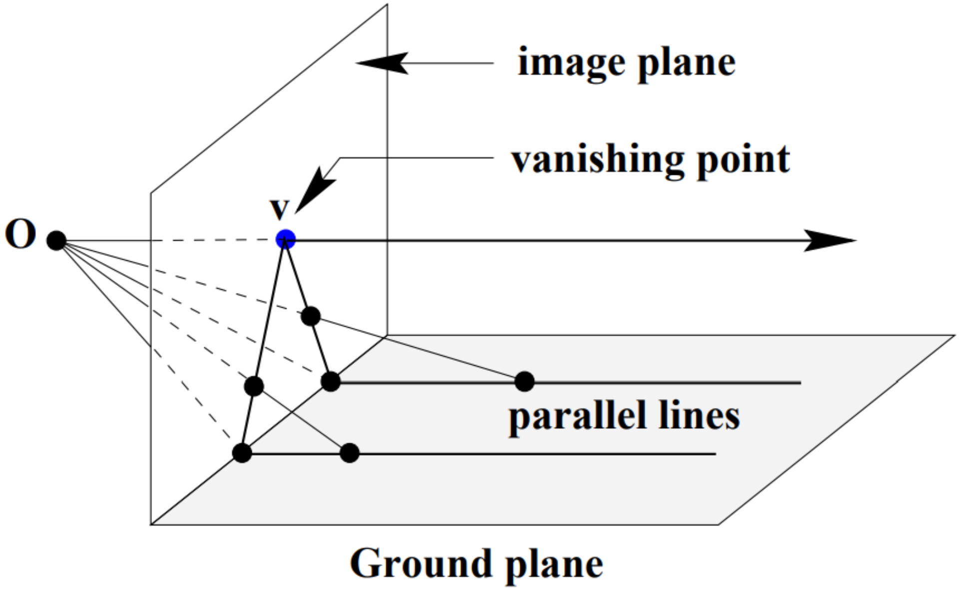
$$\mathbf{l}_{\infty}=(0,\ 0,\ 1)^{\mathrm{T}} \qquad (0,\ 0,\ 1)(x_1,\ x_2,\ 0)^{\mathrm{T}}=0$$

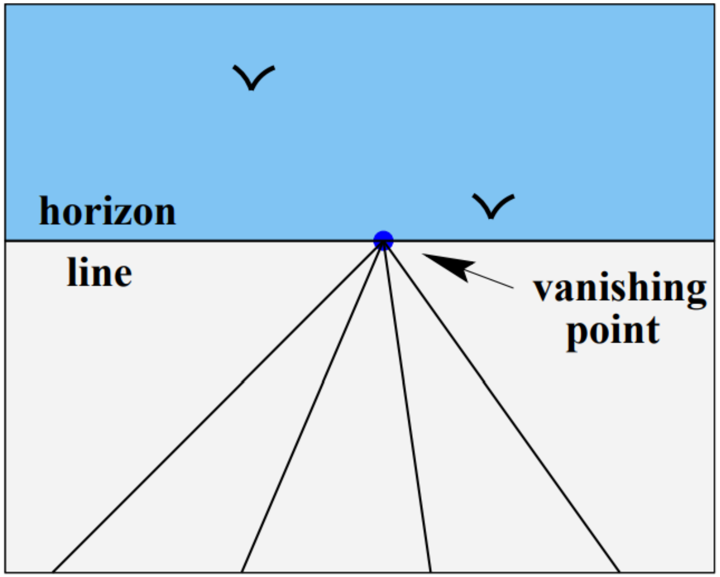


A model of the projective plane

R^3	→	P^2
Points	→	Rays
Lines	→	Planes
$x_1\,x_2$-plane	→	line at infinity l_∞
Lines in $x_1\,x_2$-plane	→	ideal points

Ideal points and the line at infinity





Duality

There is symmetry between points and lines

$$\mathbf{l}^{T}\mathbf{x} = \mathbf{x}^{T}\mathbf{l} = 0$$

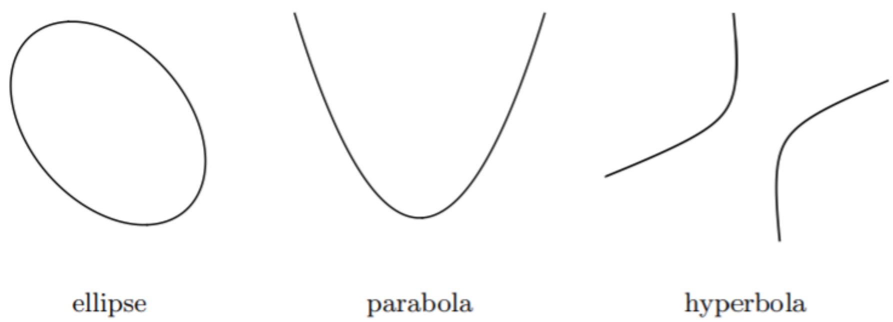
Intersection of lines $\quad \mathbf{x} = \mathbf{l} \times \mathbf{l}'$

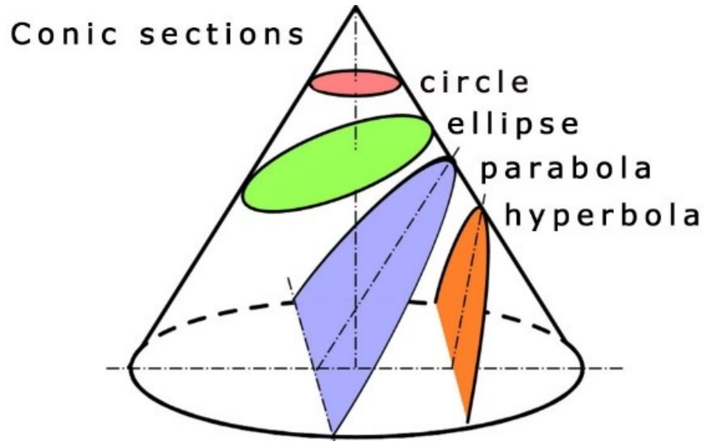
$\Leftrightarrow$ **A line passing through 2 points** $\quad \mathbf{l} = \mathbf{x} \times \mathbf{x}'$

Conics

Non-degenerate conic

A curve obtained as the intersection of a cone with a plane





2nd Equation

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

$$\Downarrow \quad x \leftarrow x_1 / x_3 \qquad y \leftarrow x_2 / x_3$$

$$ax_1^2 + bx_1x_2 + cx_2^2 + dx_1x_3 + fx_3^2 = 0$$

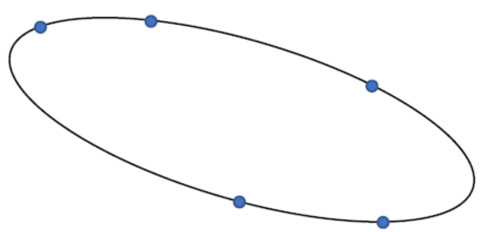
Matrix form

$$\mathrm{x}^{\mathrm{T}} \mathrm{C}\, \mathrm{x} = 0$$

$$\{a : b : c : d : e : f\} \quad 5\ \mathrm{DOF}$$

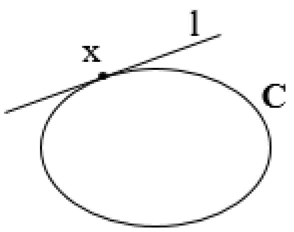
$$\begin{pmatrix} x & y & 1 \end{pmatrix} \begin{pmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = 0$$

symmetric matrix



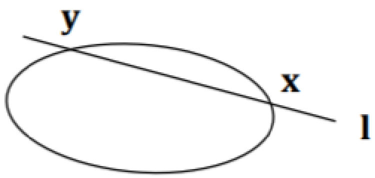
Tangent lines to conics

The line l tangent to a conic C at a point x is given by $\mathrm{l} = \mathrm{C}\mathrm{x}$



proof : x lies on l, since $\mathrm{x}^{\mathrm{T}}\mathrm{l} = \mathrm{x}^{\mathrm{T}}\mathrm{C}\mathrm{x} = 0$

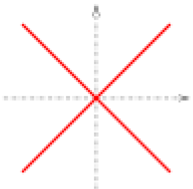
Assume) There exists another point y lying on C and l



$$(\mathrm{x}+\alpha\mathrm{y})^{\mathrm{T}}\,\mathrm{C}\,(\mathrm{x}+\alpha\mathrm{y}) = 0 \quad \forall\alpha$$

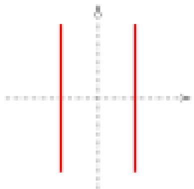
any point $\mathrm{x}+\alpha\mathrm{y}$ should lie on C

This is possible only if C is a line $\Rightarrow$ Degenerate conics



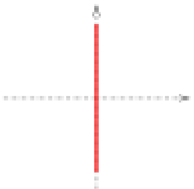
$x^2 - y^2 = 0$

$(x+y)(x-y) = 0$

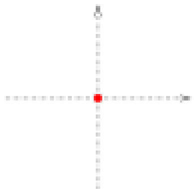


$x^2 - 1 = 0$

$(x+1)(x-1) = 0$



$x^2 = 0$



$x^2 + 1 = 0$

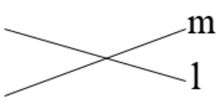
$(x+i)(x-i) = 0$

Degenerate conics

A conic is degenerate if the plane goes through the vertex of the cones.

If C is not of full rank, then the conic is termed degenerate

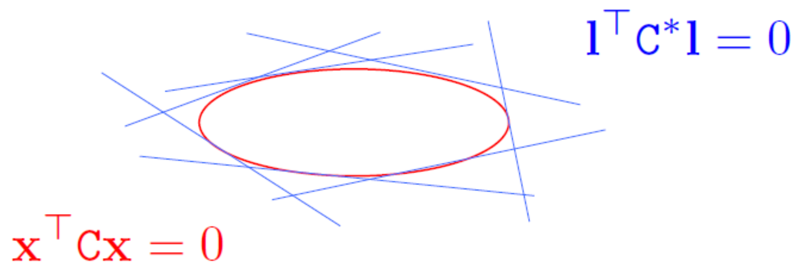
rank 2 : two lines $\quad C = lm^T + ml^T$



rank 1 : repeated lines $\quad C \sim ll^T$

Dual conics = Line conics = Conic envelope

A conic C gives a set of point
A dual conic C* gives a set of tangent lines to conic



If C is full rank, then $C^{*} = C^{-1}$

$$l = Cx \quad \rightarrow \quad x = C^{-1} l$$

$$l^{T} C^{*}\, l = 0 \quad \rightarrow \quad (Cx)^{T} C^{*}\, (Cx) = x^{T} C\, x \qquad \text{where } C^{*} = C^{-1} \; (C : \text{symmetric})$$

$$x^{T} C\, x = 0 \quad \rightarrow \quad (C^{-1} l)^{T} C\, (C^{-1} l) = l^{T} C^{-1} l = 0 \qquad \text{where } C^{-T} = C^{-1} \; (C : \text{symmetric})$$


Q&A
Thank you