Silhouette Score and Coefficient: Explanation, Math, and Example

The **Silhouette Score** is a metric used to evaluate the quality of a clustering. It measures how similar an object is to its own cluster compared to other clusters. The Silhouette Coefficient provides an indication of how well each data point has been clustered.

1. **Explanation**

- **Silhouette Score**: It is a measure of how close each sample in one cluster is to the samples in the neighboring clusters. A higher score indicates that the samples are well-clustered, while a lower score suggests that the clustering might be poor.
- The **Silhouette Coefficient** for a point can range from **-1 to +1**:
  - **+1**: indicates that the point is far away from the neighboring clusters, meaning it is well-clustered.
  - **0**: indicates that the point is on or very close to the decision boundary between two neighboring clusters.
  - **-1**: indicates that the point is incorrectly clustered and would be better off in another cluster.

2. **Mathematics of Silhouette Score**

The Silhouette Coefficient for a single sample i is defined as:

$$s(i) = \frac{b(i) - a(i)}{\max(a(i), b(i))}$$

Where:

- $a(i)$ = The average distance between the sample i and all other points in the same cluster (cohesion).
- $b(i)$ = The average distance between the sample i and all points in the nearest cluster (separation).

The **Silhouette Score** for a clustering solution is the average silhouette score of all samples in the dataset:

$$S = \frac{1}{n} \sum_{i=1}^{n} s(i)$$

Where:

- n = total number of samples.
- $s(i)$ = Silhouette Coefficient for sample i.

3. **Example**

Let's consider a simple example with a set of points belonging to two clusters:

- **Cluster 1**: Points A, B, and C
- **Cluster 2**: Points D, E, and F

Now, let's calculate the Silhouette Score for a point, say A, in Cluster 1.

1. **Cohesion**: Calculate the average distance from point A to other points in its own cluster (Cluster 1). Suppose the distances from A to B and A to C are 1.2 and 1.5, respectively.

$$a(A) = \frac{1.2 + 1.5}{2} = 1.35$$

2. **Separation**: Calculate the average distance from point A to points in the nearest cluster (Cluster 2). Suppose the distances from A to D, E, and F are 4.5, 4.0, and 4.3, respectively.

$$b(A) = \frac{4.5 + 4.0 + 4.3}{3} = 4.27$$

3. **Silhouette Coefficient for A**:

$$s(A) = \frac{b(A) - a(A)}{\max(a(A), b(A))} = \frac{4.27 - 1.35}{\max(1.35, 4.27)} = \frac{2.92}{4.27} \approx 0.68$$

This score (0.68) indicates that point A is fairly well-clustered with respect to the other points, but not perfectly.

4. **Silhouette Score for the Clustering**: To calculate the overall Silhouette Score for the clustering, we would repeat this process for each point and take the average of the individual scores.

4. Interpretation

- A **higher** Silhouette Score (close to +1) indicates that the clusters are **well-separated** and **compact**.
- A **score close to 0** indicates that the clusters are **overlapping**, and the points are near the decision boundary.
- A **negative score** indicates that the points may be assigned to the **wrong clusters**.

In summary:

- Silhouette Score is a useful method for **evaluating clustering performance**.
- It provides insight into both **cohesion** (how close points within a cluster are) and **separation** (how distinct clusters are).