Time Series 2

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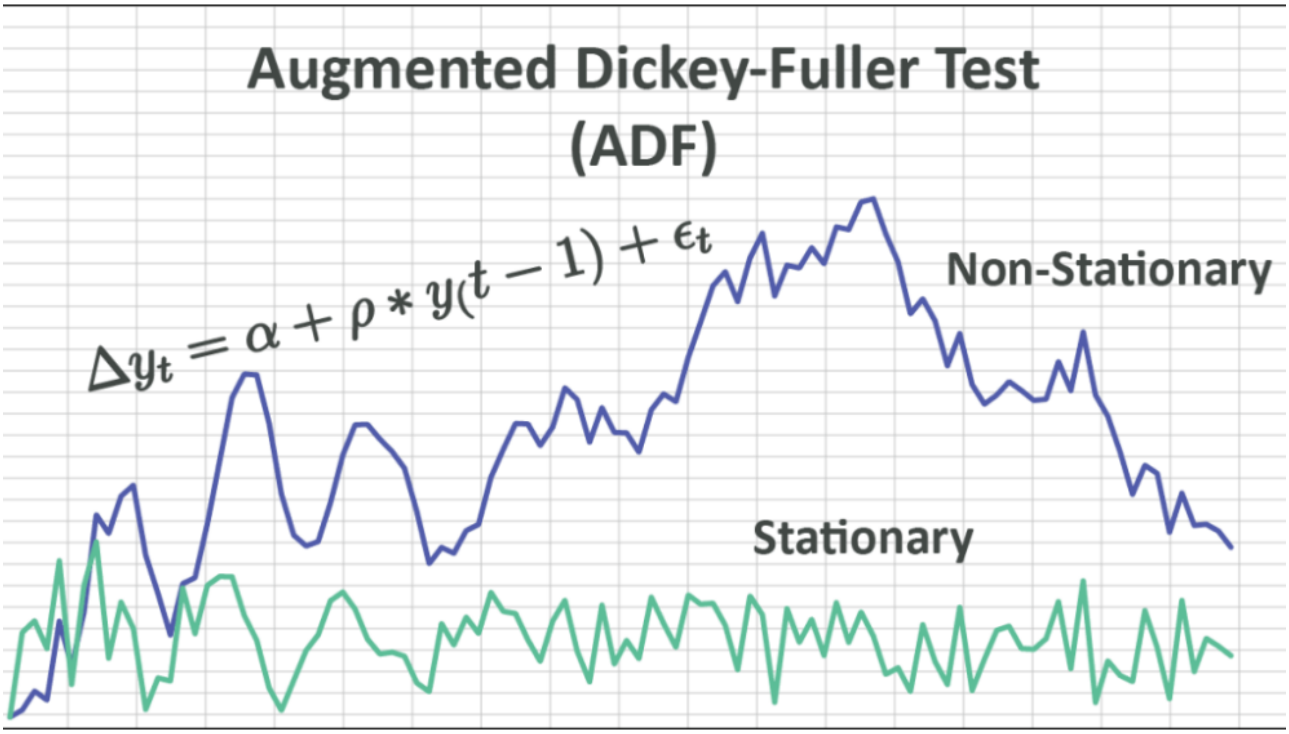
Augmented Dickey-Fuller (ADF) test

- The Augmented Dickey-Fuller (ADF) test is one of the most commonly used statistical tests to determine whether a time series is stationary or non-stationary.

- Stationarity is crucial in many time series models, especially in models like ARIMA (Autoregressive Integrated Moving Average), which assume that the statistical properties of a time series—such as its mean, variance, and autocorrelation structure—remain constant over time.
- The ADF test helps assess whether this assumption holds true for a given time series.

What is Stationarity?

- Before diving into the ADF test, it's essential to understand stationarity. A time series is considered stationary if its statistical properties do not change over time. Specifically, the series must meet the following criteria:
    - Constant Mean: The mean value of the series should remain the same over time.
    - Constant Variance: The variability (spread) of the series around the mean should be consistent over time.
    - Constant Autocorrelation: The correlation between values of the time series at different time points should depend only on the time lag between them and not on the actual time at which the correlation is calculated.
- Non-stationary time series often exhibit trends, seasonal effects, or varying variances over time, which can lead to misleading results if used in traditional time series models without appropriate transformations, such as differencing.



Key Concepts

- The ADF test is a unit root test designed to check for the presence of a unit root in a time series.

- A unit root is a feature of some stochastic processes that can cause non-stationarity. If a time series has a unit root, it means the time series exhibits a random walk and is non-stationary.
    - Null Hypothesis (H_0): The time series has a unit root, meaning it is non-stationary.
    - Alternative Hypothesis (H_1): The time series does not have a unit root, meaning it is stationary.
- If the test rejects the null hypothesis, the series is stationary. If the null hypothesis is not rejected, the series is non-stationary and may require differencing or other transformations to become stationary.

The ADF Test Equation

- The ADF test expands on the basic Dickey-Fuller test by including lagged differences of the time series in the regression equation to account for higher-order correlation.
- The ADF test equation is generally of the form:

$$\Delta Y_t = \alpha + \beta t + \gamma Y_{t-1} + \sum_{i=1}^{p} \delta_i \Delta Y_{t-i} + \epsilon_t$$

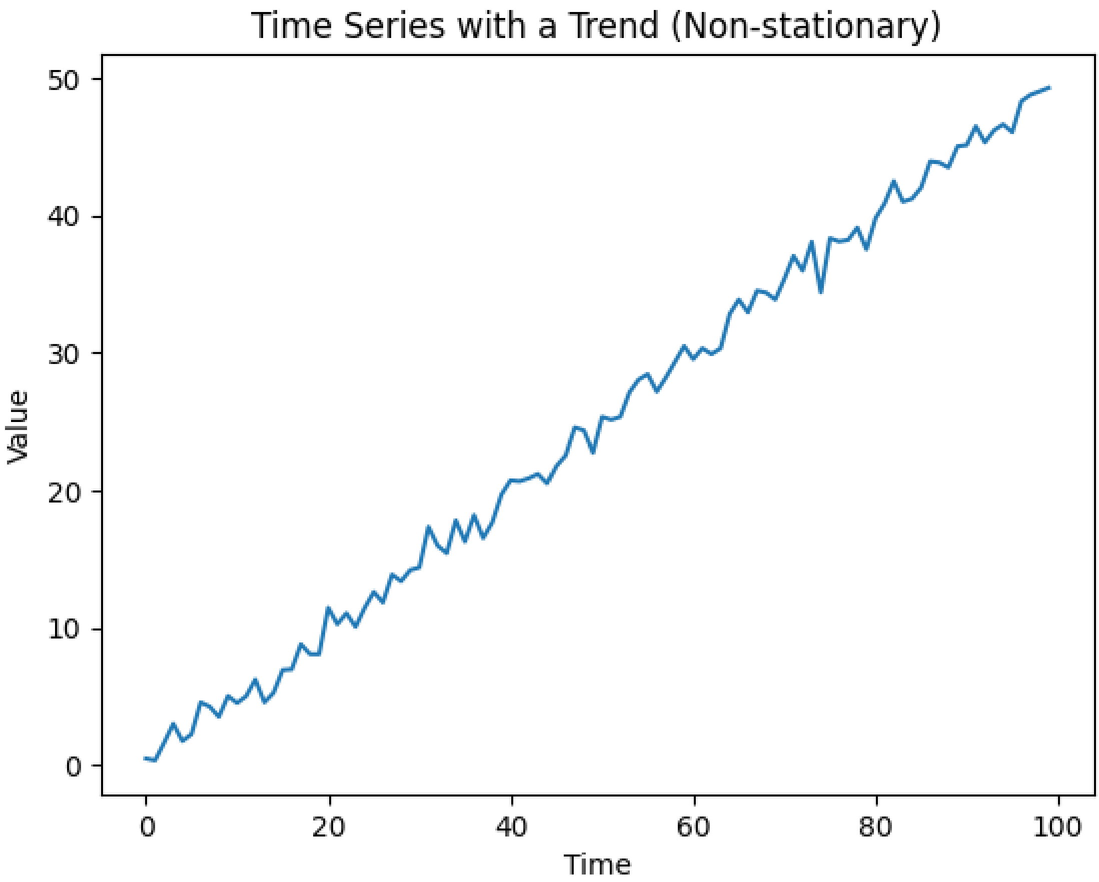
where

- Y_t : Value of the time series at time t
- $\Delta Y_t = Y_t - Y_{t-1}$: The first difference of the time series
- α: Constant (drift term)
- β: Coefficient on a time trend
- γ: Coefficient of
- Y_{t-1} , which tests for the presence of a unit root
- δ_i : Coefficients for lagged differences of the time series
- p: Number of lagged differences included in the model
- ϵ_t : Error term (white noise)

- In this equation: γ tests for a unit root. If γ=0, the series has a unit root (non-stationary). If γ < 0 , the series is stationary.

Example

```
import pandas as pd
import numpy as np
import matplotlib.pyplot as plt
from statsmodels.tsa.stattools import adfuller

# Example: Generating a simple time series with a trend
np.random.seed(42)

# Generate a time series with a trend (non-stationary)
time = np.arange(100)
series = 0.5 * time + np.random.normal(size=100)

# Convert the series into a pandas DataFrame
df = pd.DataFrame({'Time': time, 'Value': series})

# Plot the time series to visualize the trend
plt.plot(df['Time'], df['Value'])
plt.title('Time Series with a Trend (Non-stationary)')
plt.xlabel('Time')
plt.ylabel('Value')
plt.show()

# Performing the Augmented Dickey-Fuller test
adf_result = adfuller(df['Value'])

# Printing the results
print("ADF Statistic: {:.4f}".format(adf_result[0]))
print("p-value: {:.4f}".format(adf_result[1]))
print("Critical Values:")

for key, value in adf_result[4].items():
    print("\t{}: {:.4f}".format(key, value))

# Interpret the result
if adf_result[1] < 0.05:
    print("The time series is stationary (reject the null hypothesis of unit root
else:
    print("The time series is non-stationary (fail to reject the null hypothesis
```



```
ADF Statistic: 0.2481
p-value: 0.9748
Critical Values:
        1%: -3.5011
        5%: -2.8925
        10%: -2.5833
The time series is non-stationary (fail to reject the null hypothesis of unit
```

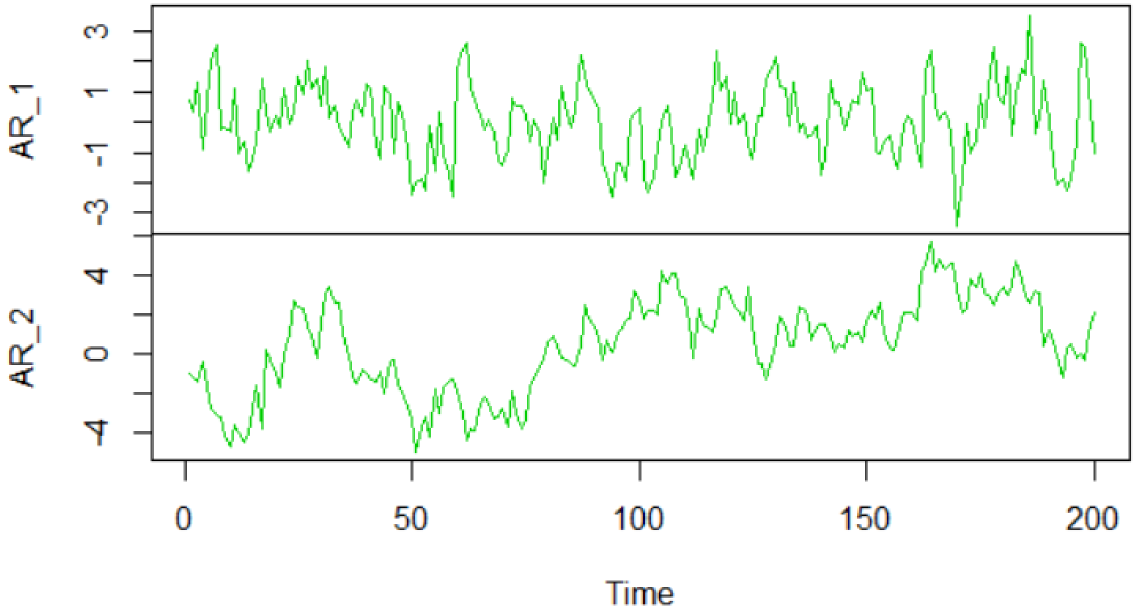
- In this example, since the p-value is greater than 0.05, we fail to reject the null hypothesis, meaning that the time series is non-stationary. You can apply transformations like differencing to make the time series stationary for further analysis.

- The Augmented Dickey-Fuller (ADF) test is a key tool in time series analysis to test for stationarity, which is a crucial assumption in many models like ARIMA. By using lagged differences, the ADF test provides a robust method for detecting unit roots, allowing analysts to determine whether a series needs to be transformed before further modeling. Proper stationarity testing helps ensure more accurate forecasting and model performance, making the ADF test an essential component in time series analysis.

Autoregressive (AR) Model

- The Autoregressive (AR) model is a fundamental model in time series analysis that leverages the relationship between an observation and its previous values (or lags) to predict future values.

- The term "autoregressive" itself reflects the idea that the model is based on regression of the time series onto its past values.
- By capturing the dependency of the current data point on its previous observations, the AR model is widely used in fields like economics, finance, and meteorology to forecast time-dependent data.



Key Concept of the AR Model

- The core idea of an AR model is that the value of the time series at any time t can be expressed as a linear combination of its previous values, also known as lagged observations.
- The model is "autoregressive" because the variable is regressed against itself, using the historical data to make predictions about future values.
- For a time series Y_t , the AR model of order p (denoted as AR(p)) can be written as:

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \cdots + \phi_p Y_{t-p} + \epsilon_t$$

Where:

- Y_t is the value of the time series at time t.
- $\phi_1, \phi_2 ,..., \phi_p$ are the autoregressive parameters (coefficients).
- p is the order of the AR model, representing how many previous time steps are included.
- ϵ_t is the white noise or error term at time t (assumed to be normally distributed with zero mean and constant variance).

The AR model captures the relationship between a value in a time series and its previous values by weighting these previous values with coefficients.

- **Autoregressive Order (p):** The order of the AR model refers to the number of lagged terms included in the model. For example:
    - AR(1): Only the previous time step (t−1) influences the current value.
    - AR(2): The previous two time steps (t−1 and t−2) influence the current value, and so on.
- **Stationarity:** For the AR model to be valid, the time series must be stationary, meaning that its statistical properties (like mean and variance) do not change over time. A stationary time series has no clear trend or seasonality and exhibits constant fluctuation around a stable mean. If the time series is non-stationary, techniques like differencing can be applied to achieve stationarity before fitting an AR model.
- **Lag Operator:** The AR model can be rewritten using the lag operator (L) to make the notation more compact. Using this operator:

$$L^k Y_t = Y_{t-k}$$

The AR(1) model can then be written as:

$$Y_t = \phi_1 L Y_t + \epsilon_t$$

- **Autocorrelation:** The strength of an AR process depends on how the past values (lags) correlate with the current values. In an AR(p) process, the Autocorrelation Function (ACF) decreases gradually as the lag increases, meaning the ACF will have significant values up to the lag order p, then drop off.
- **Error Term:** The random error term (ϵ_t) represents the unpredictable components of the time series that are not captured by the lagged values. The error term is assumed to be normally distributed with zero mean and constant variance, and it plays a crucial role in the accuracy of the model.

Steps to Build an AR Model

- **Data Preparation:**
    - The first step in time series analysis is to ensure the series is stationary, meaning its statistical properties (mean, variance, autocovariance) remain constant over time.
    - If the series is non-stationary, techniques like differencing (subtracting the previous observation from the current one), detrending (removing long-term trends), or log transformations (to stabilize variance) can be applied.

  - Achieving stationarity is crucial as many time series models, such as ARIMA, assume the data to be stationary for accurate forecasting and analysis.

- **Model Identification:**
  - The order of an AutoRegressive (AR) model, denoted by p, refers to the number of lagged observations used as predictors in the model.
  - To determine p, statistical tools like the Autocorrelation Function (ACF) and the Partial Autocorrelation Function (PACF) are utilized.
  - The PACF is particularly valuable for identifying significant lags, as it shows the direct correlation between an observation and a lagged value, controlling for the values of all the shorter lags. - The point where the PACF cuts off is typically where the appropriate value of p is determined.

- **Parameter Estimation:**
  - After identifying the order p of the AR model, the parameters ϕ_1 , ϕ_2 , ... , ϕ_p (which represent the weights of the lagged observations) must be estimated.
  - Two common methods for this are Least Squares and Maximum Likelihood Estimation (MLE).
    - Least Squares minimizes the sum of squared differences between the observed and predicted values, providing estimates of the parameters.
    - MLE finds parameter values that maximize the likelihood of observing the given data, assuming a probabilistic model for the errors.

- **Model Diagnostics:**
  - Once the AR model is fitted, residual diagnostics are essential to assess model adequacy. This involves checking if the residuals (errors) are uncorrelated and resemble white noise.
  - Tools like the Autocorrelation Function (ACF) of the residuals and the Ljung-Box test are commonly used to detect any remaining autocorrelation.
  - If patterns or significant autocorrelations are observed in the residuals, this indicates that the model is not capturing all the dependencies in the data, and further adjustments, such as increasing the order p or incorporating other model terms, may be necessary.

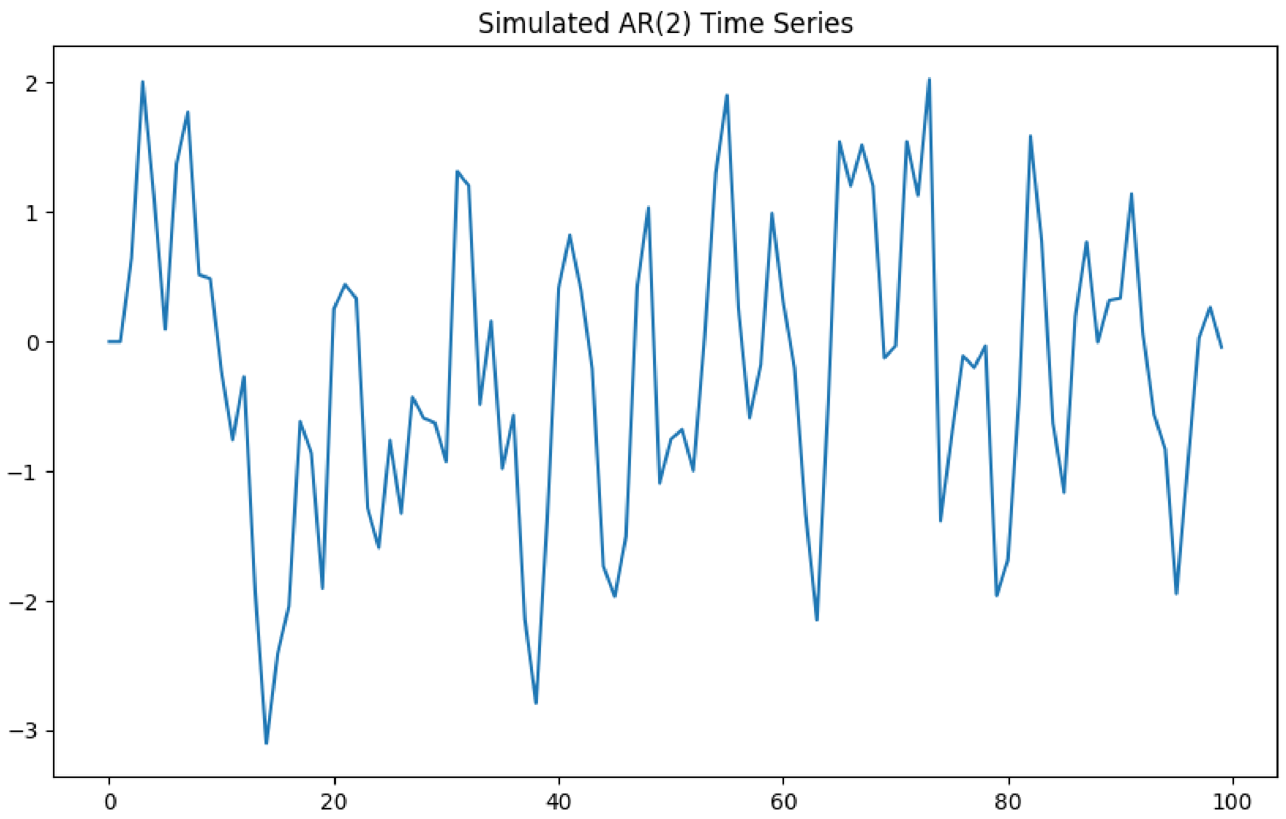
- **Forecasting:** After validating the AR model, it can be used to forecast future values by substituting past observations into the model along with the estimated coefficients ϕ_1 , ϕ_2 , ... , ϕ_p. The forecasted value for the next time step is calculated using the formula:

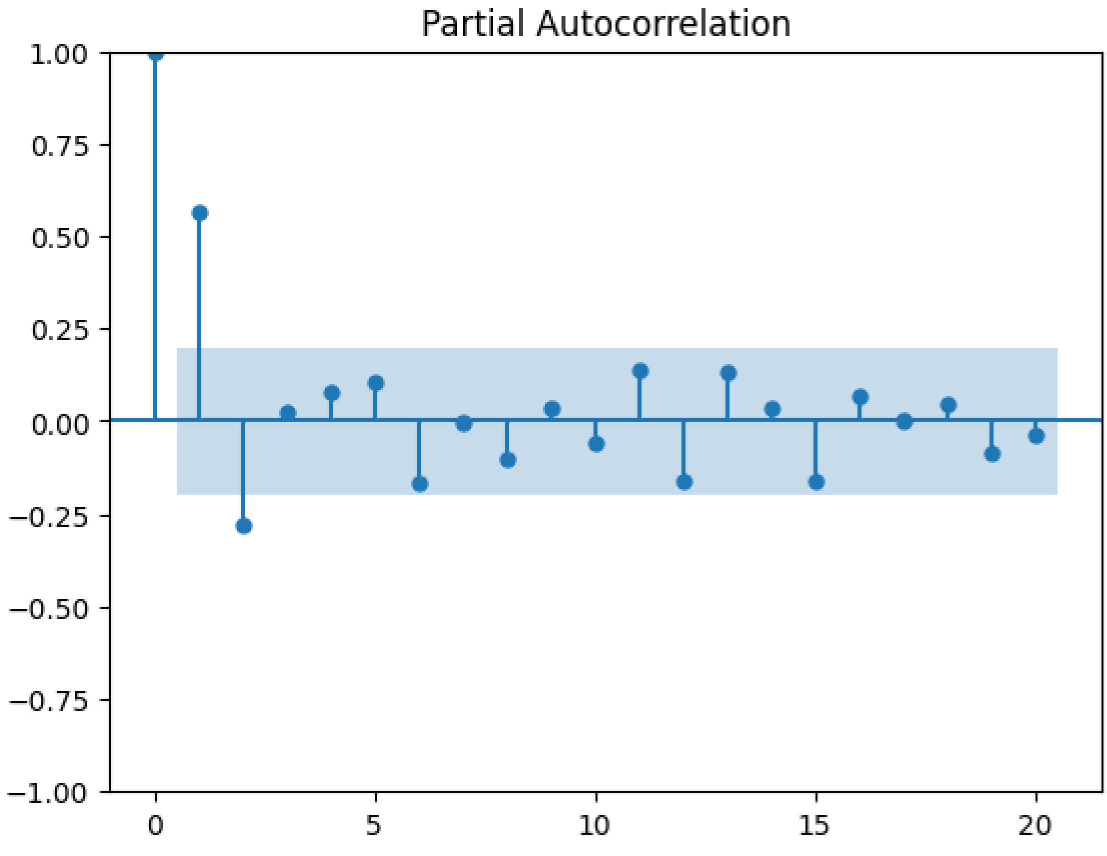
$$\hat{y}_{t+1} = \phi_1 y_t + \phi_2 y_{t-1} + \ldots + \phi_p y_{t-p+1} + \epsilon_t$$

where $y^{\wedge}_{t+1}$ is the predicted value, y_t ,y_{t-1},…,y_{t-p+1} are the most recent observations, and ϵ_t represents the error term (often assumed to be zero in forecasting). This process can be repeated for subsequent time steps to generate a series of forecasts.

Example: Fitting an AR Model in Python

```
import pandas as pd
import numpy as np
import matplotlib.pyplot as plt
from statsmodels.tsa.ar_model import AutoReg
from statsmodels.graphics.tsaplots import plot_pacf

# Generate synthetic AR(2) time series data
np.random.seed(42)
n = 100
phi1, phi2 = 0.75, -0.25
time_series = np.zeros(n)
error_terms = np.random.normal(0, 1, n)
for t in range(2, n):
    time_series[t] = phi1 * time_series[t-1] + phi2 * time_series[t-2] + error_te

# Create a DataFrame
df = pd.DataFrame(time_series, columns=['Values'])

# Plot the generated time series
plt.figure(figsize=(10, 6))
plt.plot(df['Values'], label='AR(2) Time Series')
plt.title('Simulated AR(2) Time Series')
plt.show()

# Plot the Partial Autocorrelation Function (PACF)
plot_pacf(df['Values'], lags=20)
plt.show()

# Fit an AR model (Order 2)
model = AutoReg(df['Values'], lags=2)
model_fit = model.fit()

# Summary of the model
print(model_fit.summary())

# Forecast the next 5 values
forecast = model_fit.predict(start=len(df), end=len(df)+5, dynamic=False)
print(forecast)
```





```
                            AutoReg Model Results
==============================================================================
Dep. Variable:                 Values   No. Observations:                  10
Model:                     AutoReg(2)   Log Likelihood                -129.69
Method:               Conditional MLE   S.D. of innovations              0.90
Date:                Tue, 15 Oct 2024   AIC                            267.39
Time:                        06:07:18   BIC                            277.73
Sample:                             2   HQIC                           271.57
                                  100
==============================================================================
                 coef    std err          z      P>|z|      [0.025      0.975
```

```
------------------------------------------------------------------------------
const         -0.1220      0.094     -1.299      0.194      -0.306      0.06
Values.L1      0.7202      0.097      7.420      0.000       0.530      0.91
Values.L2     -0.2771      0.097     -2.853      0.004      -0.468     -0.08
                                    Roots
==============================================================================
                  Real          Imaginary           Modulus         Frequency
------------------------------------------------------------------------------
AR.1            1.2995           -1.3856j            1.8996           -0.1301
AR.2            1.2995           +1.3856j            1.8996            0.1301
------------------------------------------------------------------------------
100   -0.216270
101   -0.273384
102   -0.256239
103   -0.230835
104   -0.217289
105   -0.214574
dtype: float64
```

- Synthetic Data Generation: Creates a synthetic AR(2) time series with 100 observations using specified coefficients $\phi_1 = 0.75$ and $\phi_2 = -0.25$, along with random noise.
- Time Series Plotting: Visualizes the generated time series data to identify any trends or patterns.
- PACF Plotting: Displays the Partial Autocorrelation Function (PACF) to determine the order of the AR model, helping to confirm the appropriate lag structure.
- AR Model Fitting: Fits an AR(2) model to the time series data using the AutoReg function from the statsmodels library, estimating the coefficients.
- Model Summary: Outputs a summary of the fitted AR model, including parameter estimates and statistical significance.
- Forecasting: Uses the fitted model to predict the next 5 values in the time series, providing future projections based on the AR(2) process.

Example in Practice

- Stock Prices: Stock price movements are often modeled using autoregressive models. For example, the price of a stock today could depend on its price from the previous day or the previous week.
- Weather Forecasting: The temperature on a given day can be modeled based on temperatures from previous days. If the temperature today is influenced by temperatures from the past week, an AR(7) model could be appropriate.
- Economic Data: Economic indicators like GDP, inflation rates, or interest rates are often modeled using autoregressive models to forecast future trends.

Advantages of AR Model

- **Simplicity and Interpretability**
  - AutoRegressive (AR) models are celebrated for their conceptual simplicity and ease of interpretation. The foundation of an AR model lies in its reliance solely on past observations of the same time series, making it straightforward to understand the relationship between past and present values.
  - Each parameter in the model represents the influence of a specific lagged observation, allowing analysts to easily explain how previous values impact current predictions.
  - This intuitive framework enables users, even those without extensive statistical training, to grasp the underlying mechanics of the model, facilitating effective communication of results and insights.

- **Good for Stationary Data**
    - AR models are particularly effective when applied to stationary data, which exhibits consistent statistical properties over time, such as constant mean and variance.
    - These models excel in capturing both short-term and long-term dependencies within a time series.
    - By leveraging the inherent autocorrelations—where current values are correlated with their past values—AR models can adeptly identify and model the underlying structure of the data. This ability to account for temporal patterns makes AR models a powerful tool for analyzing time series that do not exhibit trends or seasonality.
- **Efficient for Forecasting**
    - When properly tuned and equipped with the appropriate order selection, AR models can deliver highly accurate forecasts, especially for univariate time series data.
    - The forecasting capability stems from the model's reliance on historical data to predict future values, making it a useful approach in various applications, such as economic forecasting, stock price predictions, and resource consumption projections.
    - The effectiveness of AR models in providing reliable forecasts is enhanced through methods like cross-validation and the use of information criteria (e.g., AIC or BIC) for optimal parameter selection.
    - By aligning the model closely with the underlying data patterns, analysts can achieve robust predictive performance, empowering decision-makers with timely and accurate insights.

Disadvantages of AR Model

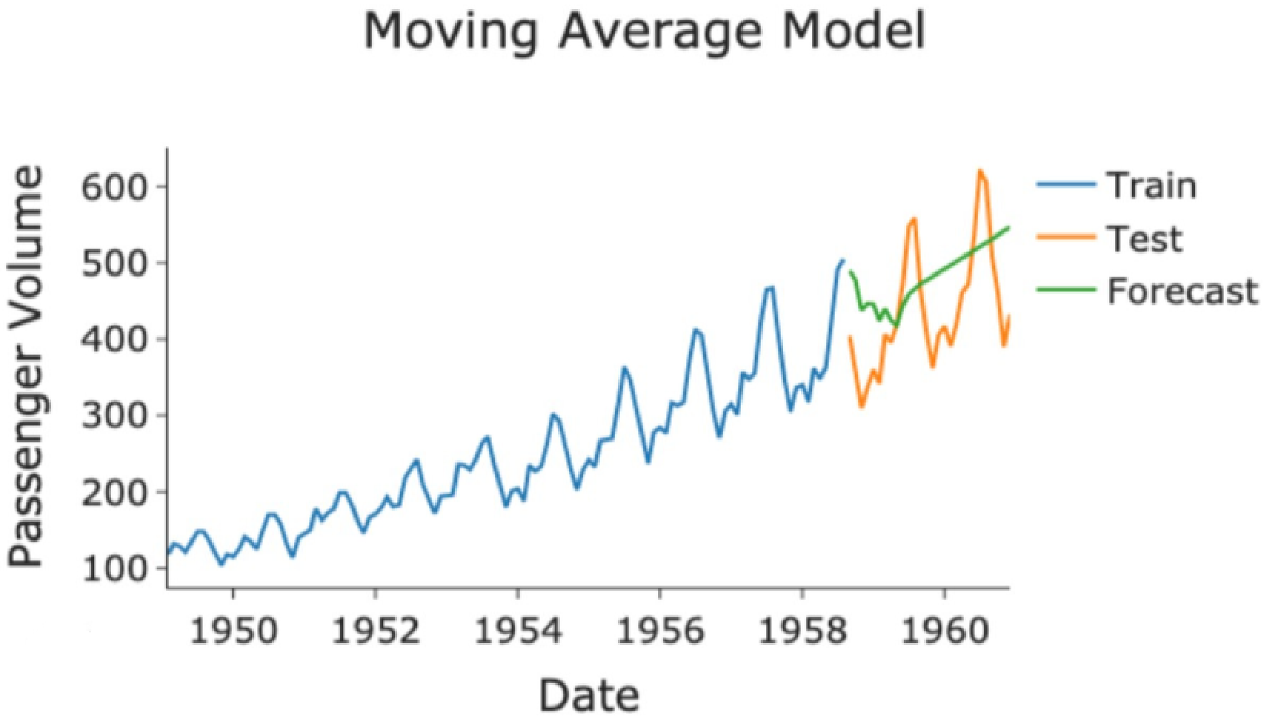
- **Stationarity Requirement**
    - One of the fundamental assumptions of AutoRegressive (AR) models is that the time series data must be stationary. A stationary series exhibits constant statistical properties over time, such as a constant mean and variance.
    - If the data is non-stationary—showing trends, seasonal patterns, or changing variances—applying an AR model directly can lead to misleading results.
    - Consequently, preprocessing steps like differencing (subtracting previous observations) or applying transformations (such as logarithmic or square root transformations) may be necessary to achieve stationarity.
    - These steps can complicate the modeling process, as they introduce additional considerations for determining the appropriate transformations and their effects on the interpretability of the model.
- **Limited for Complex Patterns**

  - AR models are primarily designed to capture linear relationships between observations and their lagged values. While this linear approach is effective for many time series, it may struggle with more complex patterns such as seasonality, cyclical trends, or non-linear dynamics.
  - For instance, if a time series exhibits strong seasonal effects, an AR model alone may not adequately account for those variations without additional modifications.
  - Analysts often need to enhance the AR model by combining it with moving averages (as in ARMA models) or incorporating seasonal components (as in SARIMA models).
  - This necessity to adapt the model for capturing complex relationships can add complexity to the modeling process and require a deeper understanding of the underlying data characteristics.

- **Sensitivity to Lag Selection**
  - Selecting the correct order p for the AR model is crucial for its performance. If p is too small, the model may underfit, failing to capture significant autocorrelations and leading to inaccurate forecasts.
  - Conversely, if p is too large, the model may overfit the noise in the data, capturing random fluctuations rather than the underlying pattern, which can degrade predictive performance on unseen data.
  - Thus, determining the optimal lag order involves a careful balance and often relies on statistical techniques such as examining PACF plots or using information criteria like the Akaike Information Criterion (AIC) or Bayesian Information Criterion (BIC).
  - The sensitivity of AR models to lag selection emphasizes the importance of thorough exploratory analysis and validation in the modeling process to achieve the best results.

Moving Average (MA) Model

- The Moving Average (MA) model is a critical component of time series analysis, particularly within the context of the broader Autoregressive Moving Average (ARMA) framework.
- The MA model provides an effective way to capture the relationship between a current observation and past error terms, allowing for improved forecasting accuracy in time series data.
- By focusing on the residual errors from previous observations, the MA model helps to smooth out the noise in the data and identify underlying patterns.



Key Concepts of Moving Average (MA) Model

- In an MA model, the value of the time series at time t, denoted as Y_t , is expressed as a linear combination of past error terms (shocks or white noise).
  - For an MA(q) model, the formula is:

$$Y_t = \mu + \epsilon_t + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2} + \cdots + \theta_q \epsilon_{t-q}$$

Where:

- μ is the mean of the series.
- ϵ_t is the white noise at time t (assumed to be normally distributed with zero mean and constant variance).
- θ_1 ,θ_2 ,…,θ_q are the model parameters (coefficients) for past errors.
- q is the order of the MA model, representing how many lagged error terms are included in the model.

How the MA Model Works

1. **Order of the Model:** The order of a Moving Average (MA) model, represented by q, defines the number of past error terms included in the model to predict the current value of the time series. This specification is crucial because it determines how much of the past

information — specifically, the noise or error from previous observations—will influence the current forecast.

- MA(1) Model: In this model, only the most recent error term is utilized. This means that the current value is influenced by the error from the immediately preceding observation. The MA(1) model is particularly effective for time series where the impact of recent shocks is significant but diminishes quickly.
- MA(2) Model: An MA(2) model incorporates the last two error terms. This allows for a more nuanced representation of the time series by capturing the influence of the previous two shocks. By considering additional lagged error terms, the model can account for a more extended effect of past disturbances on the current value.
- MA(q) Model: In a general MA(q) model, the model employs the q most recent error terms to forecast the current value. This provides flexibility in capturing longer-range dependencies on past errors, making it suitable for time series where multiple previous shocks can significantly impact the present value.

Choosing the appropriate order q for an MA model is essential for achieving accurate forecasts. If the order is too low, the model may fail to capture important patterns in the error terms, leading to underfitting. Conversely, a high q could result in overfitting, where the model becomes too complex and starts to capture noise rather than the underlying signal. Therefore, statistical techniques such as examining the autocorrelation function (ACF) and employing information criteria are often used to guide the selection of the optimal order for the MA model.

2. **Coefficients (Parameters):** In a Moving Average (MA) model, the coefficients θ_1 ,θ_2 ,…,θ_q play a critical role by quantifying the influence of each lagged error term on the current value of the time series. These coefficients indicate how much of the past errors contribute to predicting the present observation, thereby providing insights into the underlying dynamics of the data.
   - Influence of Lagged Error Terms: Each coefficient θ_i corresponds to a specific lagged error term from the previous q observations. For example, θ_1 reflects the impact of the most recent error, while θ_2 represents the influence of the error from two periods ago. A positive coefficient indicates that an increase in the past error leads to an increase in the current value, whereas a negative coefficient suggests a reverse relationship. The magnitude of these coefficients indicates the strength of the relationship between the past errors and the current observation.
   - Estimation Techniques: The coefficients in an MA model are typically estimated using statistical techniques such as Maximum Likelihood Estimation (MLE). MLE seeks to find the parameter values that maximize the likelihood of observing the given data, thereby ensuring that the fitted model best represents the underlying process generating the time series. This method provides estimates that are

statistically efficient and yield desirable properties, such as consistency and asymptotic normality.

- Model Interpretation: Once estimated, the coefficients can be interpreted in the context of the specific time series being analyzed. They can reveal important information about how past shocks or disturbances influence current values, helping analysts understand the behavior of the series over time. Moreover, examining the significance of these coefficients through hypothesis testing can inform whether the included lags are statistically meaningful in predicting the current observations.

3. **Error Term:** In a Moving Average (MA) model, the error term ϵ_t represents the unpredictable variations or noise that cannot be explained by the past observations or lagged error terms. This component is crucial for understanding the behavior of the time series, as it captures the random fluctuations and external influences that affect the observed values but are not accounted for by the model.

Steps to Build an MA Model

1. **Data Preparation**
   - The first step in building a Moving Average (MA) model is to ensure that the time series data is stationary. A stationary time series has constant mean and variance over time, which is a critical assumption for the validity of MA models. If the data exhibits trends, seasonality, or other non-stationary characteristics, it may require transformations. Common techniques include:
     - Differencing: Subtracting the previous observation from the current observation to eliminate trends.
     - Detrending: Removing a deterministic trend from the data, which can help stabilize the mean.
     - Transformation: Applying logarithmic or square root transformations to stabilize the variance.
2. **Model Identification**
   - Once the data is stationary, the next step is to determine the appropriate order q for the MA model. This is typically accomplished using the Autocorrelation Function (ACF), which measures the correlation between observations at different lags. The key steps include:
     - Plotting the ACF to visualize the correlation coefficients.
     - Identifying significant spikes in the ACF plot. A significant spike at lag q indicates that the past error terms at that lag have a meaningful relationship with the current value, suggesting that an MA(q) model may be appropriate.
3. **Parameter Estimation**

- After identifying the order q, the parameters θ_1 ,θ_2 ,…,$\theta_q$are estimated. Common methods for estimating these parameters include:
  - Maximum Likelihood Estimation (MLE): This technique finds the parameter values that maximize the likelihood of observing the data given the model.
  - Least Squares: An alternative approach that minimizes the sum of the squared differences between the observed values and the predicted values from the model.

Both methods yield estimates of the coefficients that define the influence of past error terms on the current observation.

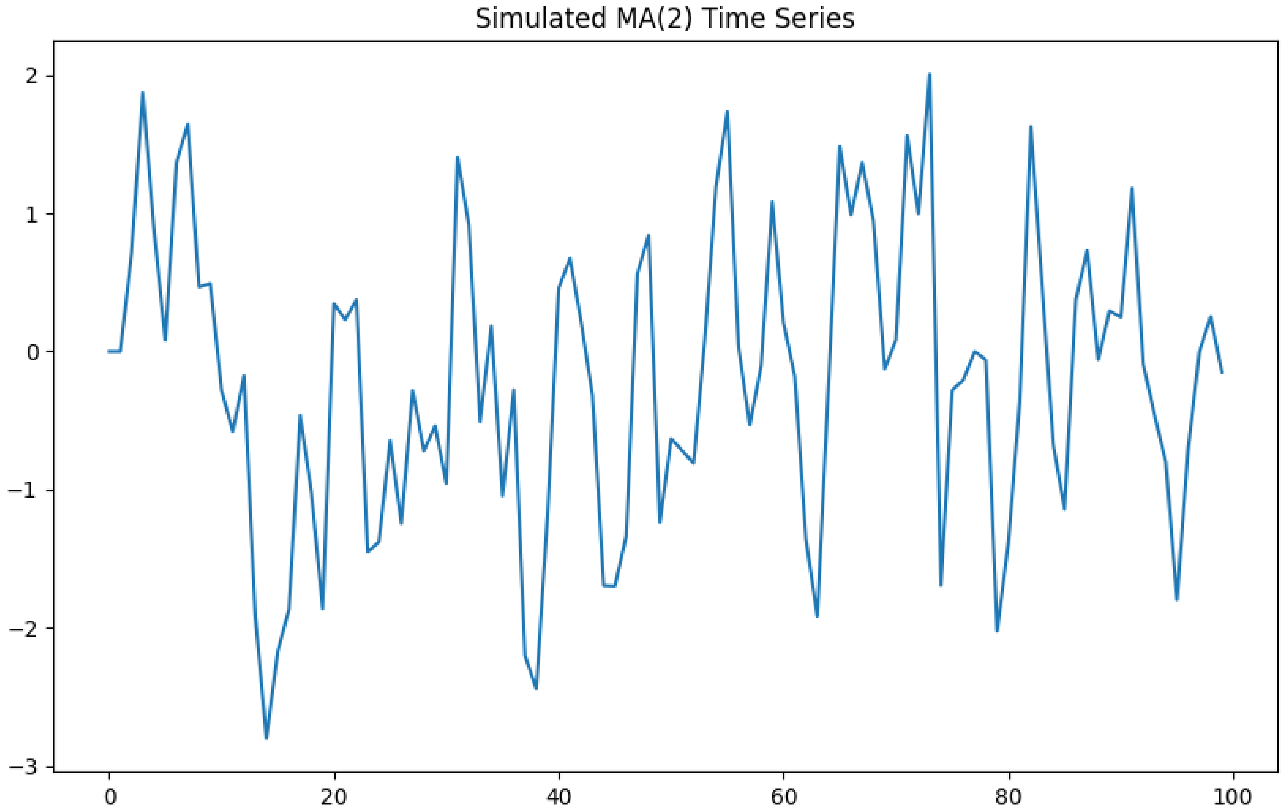
4. **Model Diagnostics**
   - After fitting the MA model, it's essential to perform diagnostic checks on the residuals (the differences between observed values and predicted values). The goal is to verify that the residuals resemble white noise, meaning:
     - They should exhibit no autocorrelation.
     - The residuals should have a mean of zero and a constant variance.

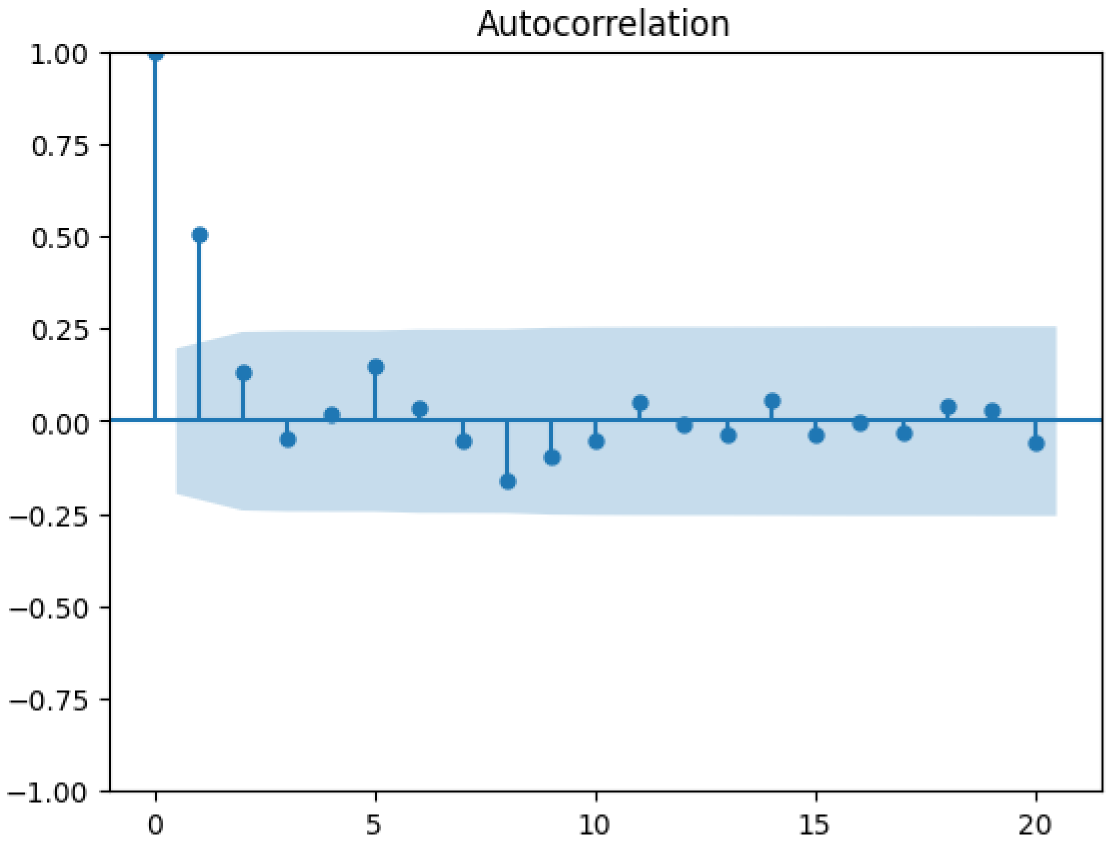
If the residuals show patterns or significant autocorrelation, this suggests that the model may not adequately capture the underlying data dynamics, necessitating adjustments. This could involve re-evaluating the order of the model or considering additional factors.

5. **Forecasting**
   - Once the MA model has been validated and the residuals are acceptable, it can be used for forecasting future values. The forecasting process involves:
     - Applying the estimated parameters θ_1 ,θ_2 ,…,$\theta_q$to predict future observations based on the most recent error terms.
     - Utilizing the model to generate forecasts for the desired number of future time points, incorporating the random nature of the error terms to provide a range of possible outcomes.

Example: Fitting an MA Model in Python

```
import pandas as pd
import numpy as np
import matplotlib.pyplot as plt
from statsmodels.tsa.arima.model import ARIMA
from statsmodels.graphics.tsaplots import plot_acf

# Generate synthetic time series data with an MA(2) process
np.random.seed(42)
n = 100
error_terms = np.random.normal(0, 1, n)
theta1, theta2 = 0.6, 0.3
time_series = np.zeros(n)
for t in range(2, n):
    time_series[t] = error_terms[t] + theta1 * error_terms[t-1] + theta2 * error_

# Create a DataFrame
df = pd.DataFrame(time_series, columns=['Values'])

# Plot the generated time series
plt.figure(figsize=(10, 6))
plt.plot(df['Values'], label='Time Series')
plt.title('Simulated MA(2) Time Series')
plt.show()

# Plot the Autocorrelation Function (ACF)
plot_acf(df['Values'], lags=20)
plt.show()

# Fit an MA model (Order 2)
model = ARIMA(df['Values'], order=(0, 0, 2))  # ARIMA(p=0, d=0, q=2) is equivalen
model_fit = model.fit()

# Summary of the model
print(model_fit.summary())

# Forecast the next 5 values
forecast = model_fit.forecast(steps=5)
print(forecast)
```





```
                               SARIMAX Results
==============================================================================
Dep. Variable:                 Values   No. Observations:                   10
Model:                 ARIMA(0, 0, 2)   Log Likelihood                -131.62
Date:                Tue, 15 Oct 2024   AIC                            271.24
Time:                        06:53:16   BIC                            281.66
Sample:                             0   HQIC                           275.46
                                - 100
Covariance Type:                  opg
==============================================================================
                 coef    std err          z      P>|z|      [0.025      0.975
```

- ~~Coefficients: The output will provide the estimates of the $\theta_1, \theta_2, ..., \theta_q$ parameters.~~

```
--------------------------------------------------------------------------------------------
const         -0.1999      0.171     -1.168      0.243      -0.535       0.13
ma.L1          0.5779      0.103      5.634      0.000       0.377       0.77
ma.L2          0.2905      0.120      2.428      0.015       0.056       0.52
sigma2         0.8111      0.123      6.570      0.000       0.569       1.05
============================================================================================
Ljung-Box (L1) (Q):                   0.00   Jarque-Bera (JB):
Prob(Q):                              0.97   Prob(JB):
Heteroskedasticity (H):               1.00   Skew:
Prob(H) (two-sided):                  1.00   Kurtosis:
============================================================================================

Warnings:
[1] Covariance matrix calculated using the outer product of gradients (comple
100   -0.242078
101   -0.239111
102   -0.199923
103   -0.199923
104   -0.199923
Name: predicted_mean, dtype: float64
```

- ACF Plot: The autocorrelation plot will show the significant correlation up to q-lags, confirming the MA(q) structure.
- Forecasting: The model can forecast future values based on the learned relationships between the current value and past residuals.

Advantages of MA Model

- **Simplicity:**
  - The MA model is conceptually straightforward and simpler compared to Autoregressive (AR) and Autoregressive Integrated Moving Average (ARMA) models.
  - This simplicity makes it easier to understand and implement, particularly in cases where the data exhibits short-term dependencies.
  - By focusing on past error terms rather than past observations, practitioners can quickly grasp the model's mechanics and apply it to time series forecasting.
- **Stationarity:**
  - One of the significant advantages of the MA model is that it is inherently stationary.
  - This means that the assumptions of constant mean and variance are automatically satisfied without requiring additional transformations.
  - Unlike AR models, which often necessitate differencing or detrending to achieve stationarity, MA models allow analysts to work directly with the original data, simplifying the modeling process and saving time in data preparation.
- **Short-Term Impact:**
  - MA models are particularly effective in scenarios where shocks or errors have a temporary effect on the time series.
  - They are designed to capture the immediate impact of recent error terms, making them suitable for data characterized by short-lived disturbances.
  - For example, in financial markets or demand forecasting, the influence of unexpected events often diminishes quickly, and MA models can effectively capture these transient effects.
  - This property allows for accurate short-term forecasting without the complexities associated with long-term dependencies.

Disadvantages of MA Model

- Lagged Residuals:
  - One of the challenges with MA models is their reliance on past residuals (error terms) rather than actual past observations.
  - This dependence can complicate the interpretation of the model's parameters.

  - In contrast, Autoregressive (AR) models use direct past values of the time series, making it easier to understand the influence of previous observations on the current value.
  - In MA models, the interpretation of coefficients becomes less intuitive, as they represent the impact of past errors rather than past values.

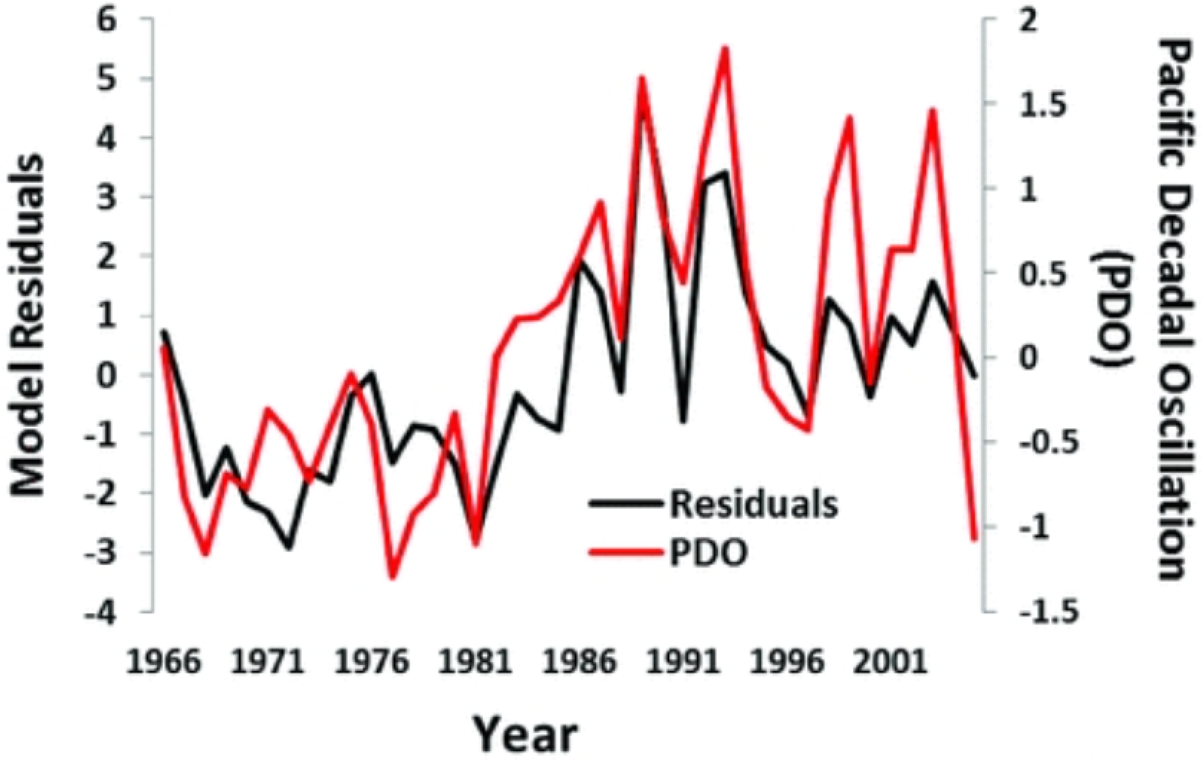
- Limited Applicability:
  - MA models are primarily effective for time series data exhibiting short-term dependencies and are not well-suited for capturing long-term trends or patterns.
  - When the data demonstrates long-term dependencies or persistent trends, AR models or a combination of AR and MA (ARMA) models are often more appropriate.
  - Relying solely on MA models in such situations may lead to poor forecasts and misinterpretation of the underlying data dynamics.

- Parameter Estimation Complexity:
  - Estimating parameters in MA models can become increasingly complex, particularly for higher-order models (MA(q)).
  - As the order of the model increases, the number of parameters to estimate also grows, which can lead to challenges in fitting the model accurately.
  - This complexity may necessitate advanced statistical techniques and additional computational resources.
  - Moreover, if the model order is chosen poorly, it can result in overfitting or underfitting, further complicating the modeling process and potentially degrading forecast accuracy.

AutoRegressive Moving Average (ARMA) Model

- The Autoregressive Moving Average (ARMA) model is fundamental in time series analysis, serving as a powerful tool for modeling and forecasting stationary time series data.
- It combines two key components: the Autoregressive (AR) part, which uses past observations to predict current values, and the Moving Average (MA) part, which employs past error terms to refine predictions.
- This integration allows ARMA to effectively capture various patterns, trends, and correlations within the data, making it versatile for a wide range of applications.
- ARMA models are particularly beneficial for datasets exhibiting stationary behavior, as they can accurately represent short-term dependencies.
- Additionally, they provide a framework for understanding the relationship between past values and errors, facilitating better forecasting accuracy.

- Overall, the ARMA model is a cornerstone in time series modeling, offering a robust approach to analyzing temporal data.



Components of the ARMA Model

1. **Autoregressive (AR) Component:**
   - The AR component of the ARMA model represents the relationship between an observation and a number of its previous observations (lags).
   - In mathematical terms, an AR model of order p (denoted as AR(p)) can be expressed as:

$$Y_t = c + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \cdots + \phi_p Y_{t-p} + \epsilon_t$$

2. **Moving Average (MA) Component:**
   - The MA component captures the relationship between an observation and a number of lagged error terms. It models how current observations are influenced by past shocks or unexpected changes.
   - An MA model of order q (denoted as MA(q)) can be represented as:

$$Y_t = \mu + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2} + \cdots + \theta_q \epsilon_{t-q} + \epsilon_t$$

3. **Combining AR and MA:**

- The ARMA model combines these two components into a single equation for a stationary time series. An ARMA model of order p and q (denoted as ARMA(p, q)) is expressed as:

$$Y_t = c + \phi_1 Y_{t-1} + \cdots + \phi_p Y_{t-p} + \theta_1 \epsilon_{t-1} + \cdots + \theta_q \epsilon_{t-q} + \epsilon_t$$

This equation allows for both the influence of past values (through the AR part) and the impact of past errors (through the MA part) on the current observation.

Properties of ARMA Models

1. **Stationarity:**
   - ARMA models are specifically designed for stationary time series data. A stationary time series has constant mean and variance over time, and its autocovariance function only depends on the lag between observations.
   - Before applying an ARMA model, it is crucial to check for stationarity, often using techniques such as the Augmented Dickey-Fuller (ADF) test. If the data is non-stationary, it may require differencing or transformation to achieve stationarity.
2. **Model Order Selection:** The order of the AR and MA components (p and q) is determined through techniques such as:
3. **Autocorrelation Function (ACF):**
   - Used to identify the appropriate order of the MA component by examining significant lags.
   - Partial Autocorrelation Function (PACF): Used to determine the order of the AR component by analyzing the correlation between the series and its lags, removing the effects of intermediate lags.
   - Parameter Estimation: Once the orders p and q are established, the parameters (coefficients) of the ARMA model can be estimated using methods such as Maximum Likelihood Estimation (MLE) or Least Squares.

Steps to Build an ARMA Model

1. **Data Preparation:**
   - The first step in building an ARMA model is to ensure that the time series data is stationary.
   - Stationarity means that the statistical properties of the series (mean, variance, and autocorrelation) do not change over time. To achieve this, you may need to apply

differencing, detrending, or transformations (such as logarithmic or square root) to stabilize the variance.
  - Visual inspections, such as plotting the time series and performing statistical tests (like the Augmented Dickey-Fuller test), can help verify stationarity.

2. **Identify Model Orders:**
    - To determine the appropriate orders p and q for the AR and MA components of the model, you can use Autocorrelation Function (ACF) and Partial Autocorrelation Function (PACF) plots.
    - The ACF helps identify the order q (the number of lagged error terms), while the PACF indicates the order p (the number of lagged observations). Significant spikes in these plots suggest the appropriate lags to include in the model.

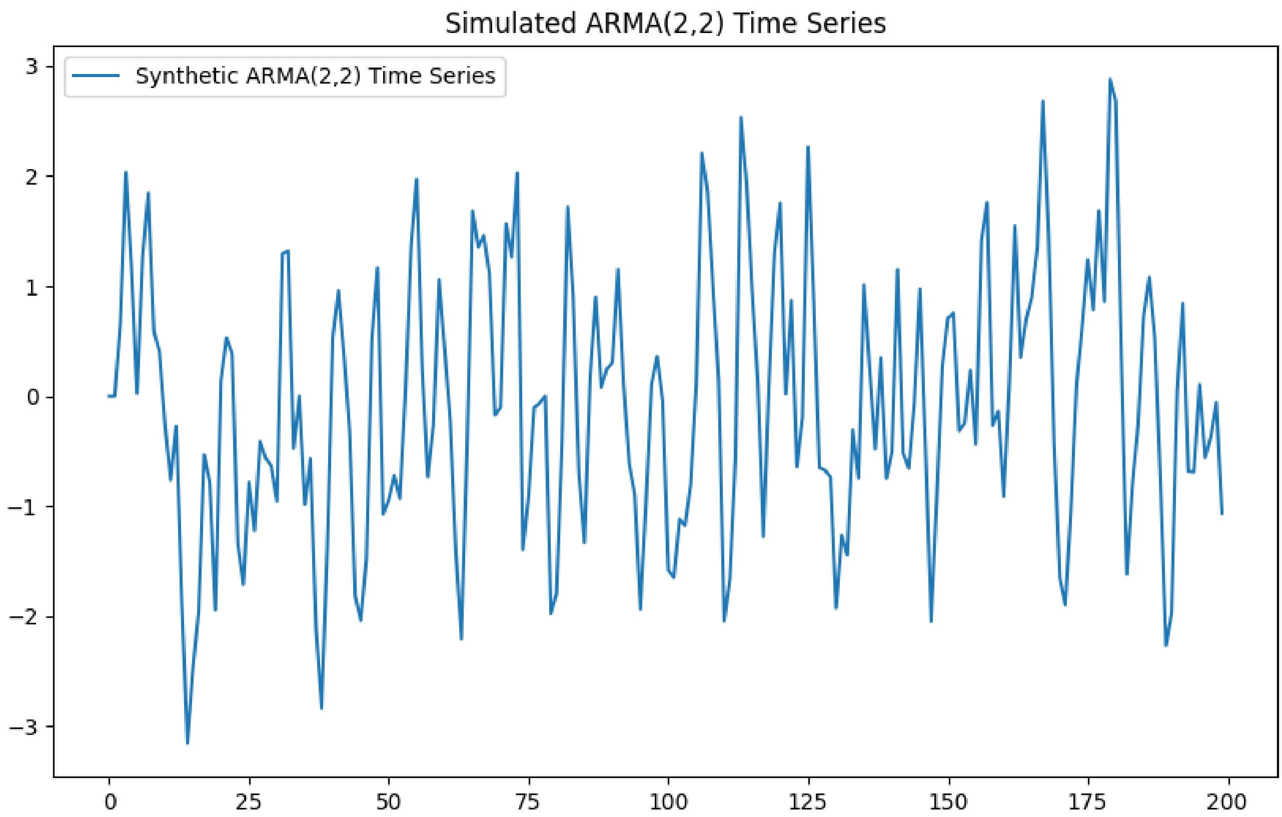
3. **Estimate Parameters:**
    - Once the model orders are identified, you can fit the ARMA model to the data using statistical software or programming libraries, such as Statsmodels in Python.
    - This library provides functions to automatically select the best parameters based on information criteria like the Akaike Information Criterion (AIC) or the Bayesian Information Criterion (BIC), which help to balance model fit and complexity.

4. **Diagnostic Checking:**
    - After fitting the model, it is essential to check the residuals (the differences between observed and predicted values) to ensure they resemble white noise.
    - This indicates that the model has adequately captured the structure in the data. You can perform diagnostic tests, such as the Ljung-Box test, to evaluate whether there is significant autocorrelation in the residuals, or visually inspect the ACF plot of the residuals.

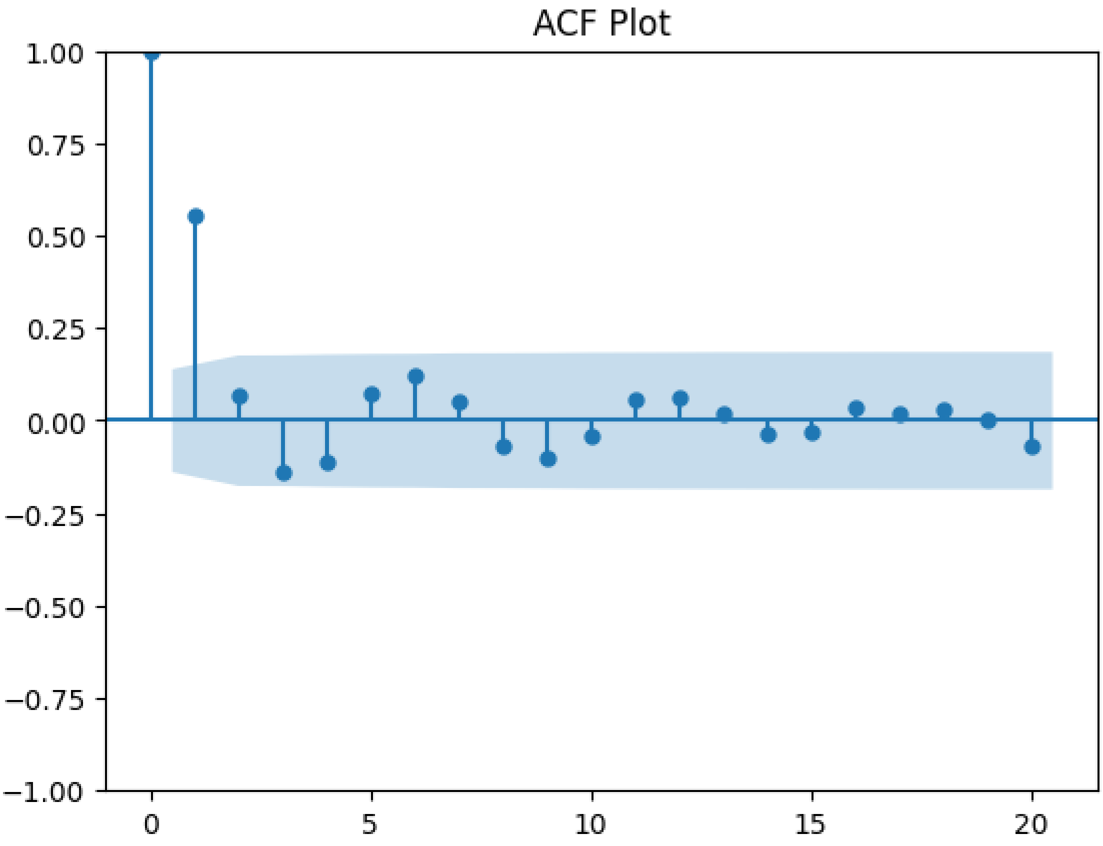
5. Forecasting:
    - Finally, use the fitted ARMA model to make predictions for future values. This involves leveraging the estimated parameters and the past observations of the time series.
    - The model can be used for short-term forecasts, allowing analysts to make informed decisions based on the predicted trends and patterns derived from the historical data

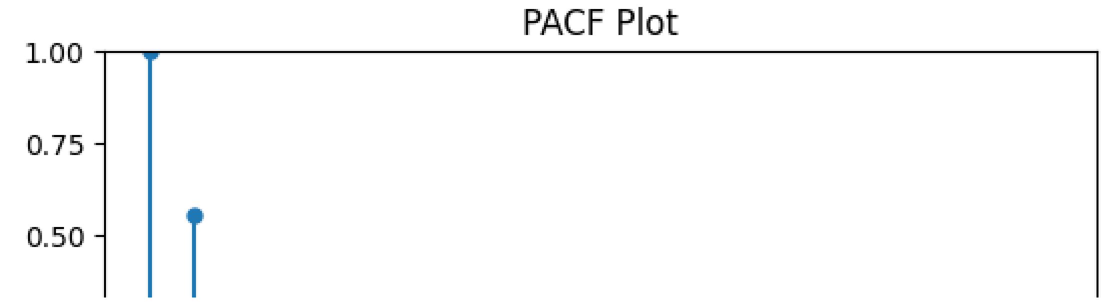
Example of Fitting an ARMA Model in Python

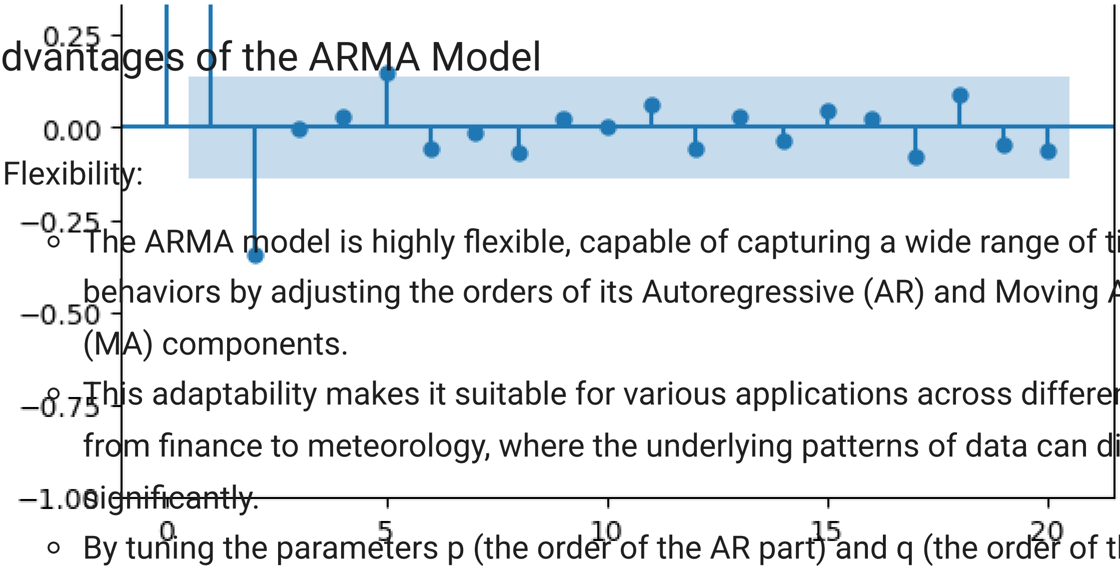
```
 import numpy as np
import pandas as pd
import matplotlib.pyplot as plt
from statsmodels.tsa.stattools import adfuller
from statsmodels.tsa.arima.model import ARIMA
from statsmodels.graphics.tsaplots import plot_acf, plot_pacf

# Generate synthetic ARMA(2,2) time series data
np.random.seed(42)
n = 200
phi1, phi2 = 0.5, -0.2
theta1, theta2 = 0.3, 0.1
error_terms = np.random.normal(0, 1, n)
time_series = np.zeros(n)

for t in range(2, n):
    time_series[t] = (phi1 * time_series[t-1] + phi2 * time_series[t-2] +
                      theta1 * error_terms[t-1] + theta2 * error_terms[t-2] +
                      error_terms[t])

# Create a DataFrame
df = pd.DataFrame(time_series, columns=['Values'])

# Plot the generated time series
plt.figure(figsize=(10, 6))
plt.plot(df['Values'], label='Synthetic ARMA(2,2) Time Series')
plt.title('Simulated ARMA(2,2) Time Series')
plt.legend()
plt.show()

# Perform Augmented Dickey-Fuller test for stationarity
adf_result = adfuller(df['Values'])
print('ADF Statistic:', adf_result[0])
print('p-value:', adf_result[1])

# Plot ACF and PACF
plot_acf(df['Values'], lags=20)
plt.title('ACF Plot')
plt.show()

plot_pacf(df['Values'], lags=20)
plt.title('PACF Plot')
plt.show()

# Fit ARMA(2,2) model
model = ARIMA(df['Values'], order=(2, 0, 2))
model_fit = model.fit()

# Summary of the model
print(model_fit.summary())

# Forecast the next 5 values
forecast = model_fit.forecast(steps=5)
print('Forecasted Values:', forecast)
```



```
ADF Statistic: -9.431958715745374
p-value: 5.161430723857462e-16
```







```
                               SARIMAX Results
==============================================================================
Dep. Variable:                 Values   No. Observations:                  20
Model:                 ARIMA(2, 0, 2)   Log Likelihood                 -268.07
Date:                Tue, 15 Oct 2024   AIC                             548.15
Time:                        08:55:22   BIC                             567.94
Sample:                             0   HQIC                            556.16
                                - 200
Covariance Type:                  opg
==============================================================================
                 coef    std err          z      P>|z|      [0.025      0.975
------------------------------------------------------------------------------
const         -0.0828      0.110     -0.750      0.453      -0.299       0.13
ar.L1          0.6932      0.367      1.892      0.059      -0.025       1.41
ar.L2         -0.3534      0.138     -2.568      0.010      -0.623      -0.08
ma.L1          0.0436      0.370      0.118      0.906      -0.682       0.76
ma.L2          0.0610      0.207      0.294      0.769      -0.345       0.46
sigma2         0.8520      0.088      9.683      0.000       0.680       1.02
==============================================================================
Ljung-Box (L1) (Q):                   0.01   Jarque-Bera (JB):
Prob(Q):                              0.92   Prob(JB):
Heteroskedasticity (H):               0.97   Skew:
Prob(H) (two-sided):                  0.91   Kurtosis:
==============================================================================

Warnings:
[1] Covariance matrix calculated using the outer product of gradients (comple
Forecasted Values: 200   -0.814194
201   -0.310646
202    0.017731
203    0.067408
204   -0.014211
Name: predicted_mean, dtype: float64
```

Advantages of the ARMA Model

- Flexibility:
  - The ARMA model is highly flexible, capable of capturing a wide range of time series behaviors by adjusting the orders of its Autoregressive (AR) and Moving Average (MA) components.
  - This adaptability makes it suitable for various applications across different fields, from finance to meteorology, where the underlying patterns of data can differ significantly.
  - By tuning the parameters p (the order of the AR part) and q (the order of the MA part), analysts can tailor the model to fit the specific characteristics of the time series being studied.
- Interpretability:
  - One of the key strengths of the ARMA model lies in its interpretability. The parameters derived from the model—specifically the AR coefficients and MA coefficients—provide valuable insights into the relationships between past values and the influence of past errors (shocks) on the current observation.
  - This understanding helps analysts and decision-makers comprehend the underlying processes driving the time series, facilitating more informed interpretations and conclusions.
- Forecasting Power:
  - The combination of AR and MA components enhances the model's ability to effectively capture complex time series dynamics, often leading to accurate short-term forecasts.
  - By utilizing both past observations and past error terms, the ARMA model can account for various temporal dependencies, resulting in improved predictive performance.
  - This capability is particularly advantageous in contexts where precise forecasting is crucial, such as stock price predictions or demand forecasting in supply chain management.

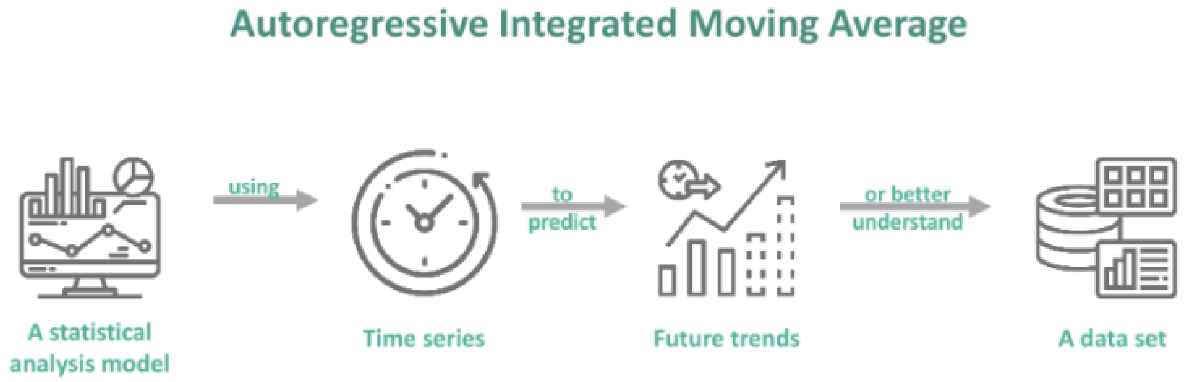
Disadvantages of ARMA Model

- Stationarity Requirement: ARMA models assume that the time series is stationary, meaning its statistical properties, such as mean and variance, do not change over time. If the data is non-stationary, the model will produce unreliable forecasts and misleading interpretations. As a result, it is often necessary to transform the data through techniques such as differencing or detrending before fitting an ARMA model. This requirement can add complexity to the modeling process, as it may involve multiple steps to achieve stationarity.

- Sensitivity to Order Selection:
    - Identifying the correct orders p (for the AR component) and q (for the MA component) is crucial for the ARMA model's performance.
    - Selecting incorrect orders can lead to underfitting (too small p or q) or overfitting (too large p or q), resulting in poor model performance and misleading forecasts.
    - The process of determining the appropriate orders often requires careful analysis of autocorrelation and partial autocorrelation plots, along with validation techniques, which can be time-consuming and complex.
- Limited for Complex Patterns:
    - While ARMA models excel at capturing linear relationships within time series data, they may struggle with more complex patterns, such as seasonality, nonlinear trends, or irregular cycles.
    - For time series exhibiting these characteristics, ARMA models may not provide adequate representations.
    - In such cases, more sophisticated models, such as Seasonal ARIMA (SARIMA) or nonlinear models, may be necessary to achieve better accuracy and capture the intricacies of the data.
    - This limitation can restrict the applicability of ARMA models in certain domains where complex dynamics are prevalent.

AutoRegressive Integrated Moving Average (ARIMA) Model

- The AutoRegressive Integrated Moving Average (ARIMA) model is a powerful and versatile statistical method employed for analyzing and forecasting time series data.
- It enhances the capabilities of the Autoregressive Moving Average (ARMA) model by incorporating mechanisms to handle non-stationary data, making it suitable for a wider range of applications.
- This model is particularly effective in forecasting future values based on historical observations, allowing for insightful analysis in various fields, including finance, economics, and environmental science.



Components of the ARIMA Model

The ARIMA model is characterized by three key components, denoted as (p,d,q):

1. Autoregressive (AR) Component (p):
   - This component represents the relationship between an observation and a specified number of lagged observations (previous time points).
   - An AR model of order p can be expressed as:

$$Y_t = c + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \ldots + \phi_p Y_{t-p} + \epsilon_t$$

   - Here, Y_t is the current value, c is a constant, ϕ are the AR coefficients, and ϵ_t represents the error term.
2. Integrated (I) Component (d):
   - The integrated component refers to the number of times the data needs to be differenced to achieve stationarity. Differencing is the process of subtracting the previous observation from the current observation, which can help stabilize the mean of a time series by removing changes in the level of a time series.
   - For example, if d=1, the first difference is calculated as:

$$Y_t' = Y_t - Y_{t-1}$$

   - If the time series is already stationary, then d=0.
3. Moving Average (MA) Component (q):
   - This component captures the relationship between an observation and a specified number of lagged forecast errors (the residuals from previous time points).
   - An MA model of order q can be expressed as:

$$Y_t = \mu + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2} + \ldots + \theta_q \epsilon_{t-q} + \epsilon_t$$

- Here, μ is the mean of the series, θ are the MA coefficients, and ϵ_t represents the error term.

4. Combining the Components:
   - The complete ARIMA model, therefore, combines these three components into a single framework. An ARIMA model of order (p,d,q) is represented as:

$$Y_t' = c + \phi_1 Y_{t-1}' + \ldots + \phi_p Y_{t-p}' + \theta_1 \epsilon_{t-1} + \ldots + \theta_q \epsilon_{t-q} + \epsilon_t$$

Steps to Build an ARIMA Model

1. Check for Stationarity:
   - Before fitting the ARIMA model, it's crucial to assess whether the time series data is stationary. Stationarity means that the statistical properties of the series remain constant over time.
   - You can evaluate stationarity visually using plots or employ statistical tests such as the Augmented Dickey-Fuller (ADF) test.
   - If the data is found to be non-stationary, apply differencing until the time series achieves stationarity. The number of times differencing is applied is denoted by d, which represents the order of differencing.
2. Identify Model Orders p and q:
   - The next step involves determining the appropriate orders for the AR and MA components of the ARIMA model. This can be accomplished using the Autocorrelation Function (ACF) and Partial Autocorrelation Function (PACF) plots:
   - ACF: This plot helps identify the order of the MA component, q. Significant spikes in the ACF indicate the number of lagged error terms needed.
   - PACF: This plot aids in determining the order of the AR component, p. Significant spikes in the PACF suggest the number of lagged values of the series that should be included.
3. Fit the ARIMA Model:
   - Once the parameters p, d, and q are established, you can fit the ARIMA model to the time series data. This is typically done using statistical software or programming libraries, such as Statsmodels in Python, which allow for straightforward implementation and fitting of ARIMA models.
4. Diagnostic Checking:

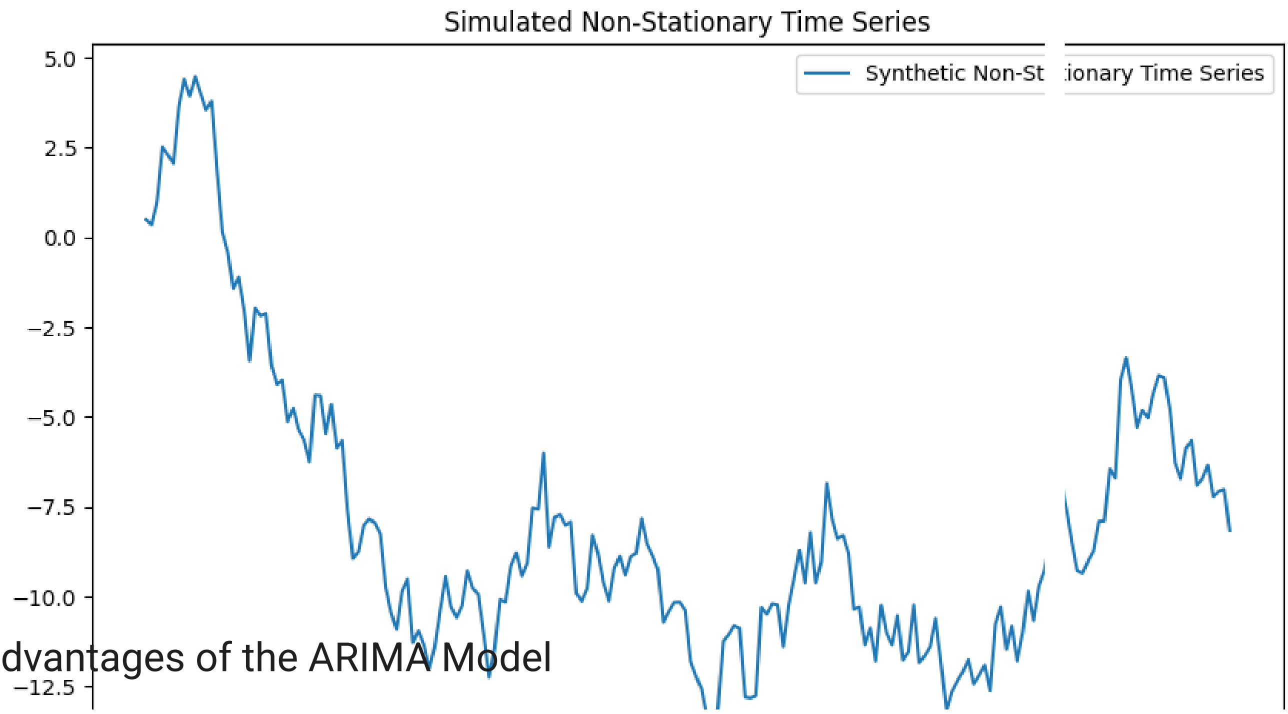
- After fitting the model, it’s essential to evaluate the residuals to ensure that they behave like white noise, indicating that the model adequately captures the structure of the data.
- This evaluation can be done by analyzing the ACF of the residuals. Additionally, statistical tests such as the Ljung-Box test can be performed to validate the model's adequacy by checking if residuals exhibit any autocorrelation.

5. Forecasting:
   - With a well-fitted ARIMA model, you can now make future predictions based on historical data. Use the fitted model and the estimated parameters to project future values, allowing for informed decision-making based on these forecasts.

Example of Fitting an ARIMA Model in Python

```
import numpy as np
import pandas as pd
import matplotlib.pyplot as plt
from statsmodels.tsa.stattools import adfuller
from statsmodels.tsa.arima.model import ARIMA
from statsmodels.graphics.tsaplots import plot_acf, plot_pacf

# Generate synthetic time series data (non-stationary)
np.random.seed(42)
n = 200
time_series = np.cumsum(np.random.normal(0, 1, n))  # Random walk

# Create a DataFrame
df = pd.DataFrame(time_series, columns=['Values'])

# Plot the generated time series
plt.figure(figsize=(10, 6))
plt.plot(df['Values'], label='Synthetic Non-Stationary Time Series'
plt.title('Simulated Non-Stationary Time Series')
plt.legend()
plt.show()

# Perform Augmented Dickey-Fuller test for stationarity
adf_result = adfuller(df['Values'])
print('ADF Statistic:', adf_result[0])
print('p-value:', adf_result[1])

# Differencing to achieve stationarity
df['Differenced'] = df['Values'].diff().dropna()

# Plot the differenced series
plt.figure(figsize=(10, 6))
plt.plot(df['Differenced'], label='Differenced Time Series')
plt.title('Differenced Time Series')
plt.legend()
plt.show()

# Check for stationarity again
adf_result_diff = adfuller(df['Differenced'].dropna())
print('ADF Statistic after differencing:', adf_result_diff[0])
print('p-value after differencing:', adf_result_diff[1])

# Plot ACF and PACF of the differenced series
plot_acf(df['Differenced'].dropna(), lags=20)
plt.title('ACF Plot of Differenced Series')
plt.show()

plot_pacf(df['Differenced'].dropna(), lags=20)
plt.title('PACF Plot of Differenced Series')
plt.show()

# Fit ARIMA(1,1,1) model
model = ARIMA(df['Values'], order=(1, 1, 1))
model_fit = model.fit()
```

```
# Summary of the model
print(model_fit.summary())

# Forecast the next 5 values
forecast = model_fit.forecast(steps=5)
print('Forecasted Values:', forecast)
```



Advantages of the ARIMA Model