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An Overview of the Strong CP Problem and Axion Cosmology

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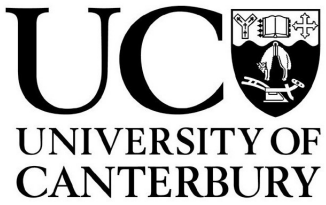

Literature Review

Abstract

The standard model of particle physics suffers from the strong CP problem—the fine tuning of the QCD $\bar{\theta}$-parameter to the experimental value $|\bar{\theta}| \lesssim 1 \times 10^{-10}$. This review discusses the theoretical background of QCD and of the strong CP problem, and describes its most famous resolution: the Peccei–Quinn axion. The axion has wide-ranging implications for cosmology and astrophysics, and up-to-date constraints of its properties from various laboratory and cosmological experiments are reviewed.

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0 Introduction

The Physics of Nature at the smallest and largest scales underwent a spectacular revolution in the twentieth century. However, the last two decades have been troubled by a few especially stubborn mysteries, and fundamental physics seems to have stagnated. The last century bore the standard model of particle physics and, in parallel, the general relativistic theory of gravity. The essential incompatibility between the two calls for a major breakthrough which still remains out of sight. Aside from this foundational issue, key unresolved problems include the apparent need for dark matter in the standard model of cosmology, and the apparently unnatural fine-tuning of the standard model of particle physics. One such issue of fine-tuning is named the strong CP problem, pertaining to the unexplained time-reversal symmetry in quantum chromodynamics (QCD), and the concomitant non-observation of the neutron's electric dipole moment. One such resolution to this problem is named the Peccei–Quinn axion solution.

The axion solution predicts the existence of a light, weakly interacting particle which naturally restores time-reversal symmetry in QCD. Undoubtedly, extensions to the standard model are more appealing when they resolve many existing issues with the same underpinning theoretical supposition. In this sense, the axion is an alluring solution to the strong CP problem, potentially serving as a dark matter candidate, providing a mechanism for cosmic inflation or even explaining baryogenesis [1]. The shortcoming is that it has never admitted any sign of existence, despite extensive searches. Nevertheless, "physics thrives on crisis" as Weinberg put it [2], and in the last two decades the axion has regained popularity.[1] The axion has a remarkable range of implications for particle physics, cosmology and astrophysics. Should it ever turn out an accurate reflection of nature, the axion would certainly reshape more than one area of physics.

The focus of this pedagogical review is part theoretical and part phenomenological. The first half is an overview of the theory underlying quantum chromodynamics, the strong CP problem and its axion solution. It begins assuming minimal familiarity with the standard model, giving a high-level description of the mathematical structures involved, and ends with a basic exposition of axion phenomenology. The second half is devoted to the implications of axions in cosmology and astrophysics, and the tests and constraints they offer for axion models.

[1] The publication count on Inspire-HEP for the term "axion" has been exponentially growing since the new millennium, doubling every 6.5 years.

1 QCD and the Strong CP Problem

The Standard Model of particle physics is a quantum field theory which describes all known fundamental particles and interactions to very high precision—with notable exceptions including the absence of gravity, dark matter, and the experimentally incorrect prediction that neutrinos are massless. Alongside these shortcomings, the standard model also exhibits issues of a more philosophical nature, such as the reasons for the values of the theory's many free parameters. (For example; why are there three generations of particles? Why do the particle masses and coupling constants have the values they do?) Among these mysteries is the *strong CP problem*, a fine-tuning problem regarding the apparent symmetry of the strong force under time reversal.[1] This chapter summarises the theory necessary to understand the statement and origin of the strong CP problem.

The standard model was born amid the explosion of phenomenological particle physics in the early 1960s, in an effort to explain the patterns in the rapidly growing catalogue of known particles. The introduction of the quark model efficiently explained the properties of the numerous hadrons in terms of six kinds of quarks possessing *colour symmetry*, and the theory was given the name *quantum chromodynamics* (QCD). This, together with the well-established quantum theory of electromagnetism, *quantum electrodynamics* (QED), became the first approximation to the standard model [3]. The modern statement of the standard model is that it is a Lorentz-invariant quantum field theory whose *gauge group* is the compact connected Lie group

$$\big(\mathrm{SU}(3) \oplus \mathrm{SU}(2) \oplus \mathrm{U}(1)\big)\big/\mathbb{Z}_6, \tag{1.1}$$

equipped with a particular action on matter fields. A quantum field theory is a quantised *gauge field theory*, which is a field theory whose Lagrangian is symmetric under the action of the gauge group (explained more in § 1.1). Each term in the gauge group of the standard model (1.1) corresponds to a fundamental force of nature: The factor $\mathrm{SU}(3)$ is the gauge group of quantum chromodynamics, the sector of the standard model describing the interactions of colour-charged quarks and gluons via the strong force. The factor $\mathrm{SU}(2) \oplus \mathrm{U}(1)$ corresponds to the unified electroweak force, and contains another $\mathrm{U}(1)$ sub-factor corresponding to the electromagnetic force of QED.

Quantum chromodynamics was formulated quite some time after quantum electrodynamics was understood, which is a reflection that QCD is more intricate than QED. Unlike QED, quantum chromodynamics exhibits *asymptotic freedom*, meaning that the effective strength of the strong interaction between colour-changed particles *increases* with increasing separation.

[1] Time reversal T and charge–parity CP symmetry are equivalent assuming the combined charge–parity–time symmetry; i.e., $T = CP \mod CPT$.

Thus, QCD is severely non-perturbative and evades even modern analytical treatment, except at low energies. Asymptotic freedom also results in quark confinement, which contributed to the confusion of particle physicists in the 1960s—quarks could never be observed in isolation. The dissimilarity of QCD to QED can be largely credited to the fact that the gauge group $\mathrm{SU}(3)$ is non-Abelian, implying that the gluon force carriers of QCD are *themselves* colour-charged, leading to asymptotic freedom [4]. This is in contrast to the Abelian $\mathrm{U}(1)$ theory of QED, in which the force carriers are uncharged photons. Despite the technical challenges which come with its non-perturbative nature, quantum chromodynamics remains a successfully predictive and elegant theory.

However, QCD comes with a challenge of theoretical concern: the *strong CP problem*, or "why does QCD fail to forbid CP violation?" Since we do not observe CP symmetry violation in QCD [5], we expect the theory to prohibit it—but a careful inspection of QCD reveals that it does not. This problem proves to be of the fine-tuning variety, but it lacks even an anthropic solution: it does not change the universe in dramatic ways whether CP symmetry is broken in the QCD sector or not (as we so observe). Instead, the disparity indicates a deeper theoretical shortcoming, and provides good reason to pursue physics beyond the standard model.

This chapter fills in the background relevant to the formal statement of the strong CP problem, and assumes surface-level familiarity with differential geometry, Lagrangian mechanics, quantum mechanics and the notation of exterior calculus. The significance and role of the gauge group is outlined in the next section, where a geometrical overview of gauge field theory is presented (primary sources are [6], [7] and [8]). This overview aims to provide a basic understanding of the main mathematical objects—or 'moving parts'—of a gauge field theory, and of their physical interpretations. The chapter later introduces further concepts specific to QCD and the strong CP problem as they become relevant. Finally, QCD is defined and the origin of the so-called *strong CP problem* is highlighted.

1.1 Overview of Classical and Quantum Gauge Theory

Gauge theories take place on a spacetime manifold and consist of two mathematically distinct dynamical entities: a *matter field* and a *gauge field*. A classical gauge theory is completely specified by the prescription of four ingredients: the base manifold $\mathcal{M}$; the vector space V in which the matter field ψ takes its values; a *gauge group* G equipped with an action on the matter field; and equations of motion, usually supplied via a Lagrangian density $\underset{\sim}{\mathcal{L}}$. The dynamical gauge field is not prescribed at the outset—it arises naturally as a consequence of the matter field's symmetry under the action of the gauge group.

The underlying premise of gauge theory is that there may be physical redundancy in the mathematical description of a matter field at each point in spacetime. (For example, the complex phase of a total wavefunction is physically irrelevant.) This redundancy results in non-physical degrees of freedoms of the matter field at every point in spacetime: these are called *local gauge freedoms*. (Keeping with our example: a local rotation of phase is a gauge freedom.) Crucially, local gauge freedoms introduce ambiguity in the notion of the physical rate of change of a matter field about a point. This is because the point-wise independence of the matter field's gauge freedom means that differences in the field's value between nearby points is gauge-dependent. In other words, there is no preferred directional derivative of a matter field if a local gauge

Notations used in this chapter

$\mathsf{A}, \mathsf{B}, \ldots$	matrices
$\mathcal{A}, \mathcal{B}, \ldots$	objects with manifold structure
$\rightarrowtail; \twoheadrightarrow; \leftrightarrow$	injection; surjection; bijection
$F \rightarrowtail \mathcal{V} \overset{\pi}{\twoheadrightarrow} \mathcal{M}$	fibre bundle $\mathcal{V}$ over base space $\mathcal{M}$ with fibre F and projection $\pi : \mathcal{V} \twoheadrightarrow \mathcal{M}$
$\mathrm{T}_p\mathcal{M}; \mathrm{T}\mathcal{M}$	tangent space at $p \in \mathcal{M}$; tangent bundle of $\mathcal{M}$
$\mathrm{T}^r_s\mathcal{M}$	type $\binom{r}{s}$ tensors over $\mathcal{M}$, equal to $\left(\bigotimes_{i=1}^{r} \mathrm{T}\mathcal{M}\right) \otimes \left(\bigotimes_{i=1}^{s} \mathrm{T}^*\mathcal{M}\right)$
$\mathbb{T}\mathcal{M}$	tensor bundle of $\mathcal{M}$, equal to $\bigoplus_{r,s=0}^{\infty} \mathrm{T}^r_s\mathcal{M}$
$\underset{\sim}{A}$	spacetime tensor field; i.e., a section of $\mathbb{T}\mathcal{M}$
$\boldsymbol{A}$	vector field of some abstract vector space not contained in $\mathbb{T}\mathcal{M}$
$\underset{\sim}{\boldsymbol{A}}$	spacetime tensor field with values in some other abstract vector space
$\wedge^p V$	pth exterior power of the vector space V; i.e., the space of V-valued p-forms
$\Gamma(\mathcal{V})$	smooth sections $\mathcal{M} \to F$ of fibre bundle $F \rightarrowtail \mathcal{V} \twoheadrightarrow \mathcal{M}$
$\Omega^p(\mathcal{M})$	space of p-forms on $\mathcal{M}$, equal to $\Gamma(\wedge^p \mathrm{T}^*\mathcal{M})$
$\Omega^p(\mathcal{M}, V)$	space of V-valued p-forms, equal to $\Gamma(\mathcal{V} \otimes \wedge^p \mathrm{T}^*\mathcal{M})$ where $V \rightarrowtail \mathcal{V} \twoheadrightarrow \mathcal{M}$

freedom is present—until the choice of a *connection* is made. The triumph of gauge theory is that, by introducing the *gauge field* to act as a connection, this ambiguity is recast as a separate set of physical degrees of freedom. The implications of this are twofold: Firstly, the gauge field isolates a choice of derivative (namely, the covariant derivative with respect to the gauge field), allowing the inclusion of well-defined derivatives of the matter field in the theory's equations of motion. Secondly, the gauge field has a dynamical role in the theory, and it describes a new kind of field: force fields (and, after quantisation, force carrier bosons).

In geometrical language, the matter field $\boldsymbol{\psi} \in \Gamma(\mathcal{V})$ is a section of a vector bundle $V \rightarrowtail \mathcal{V} \twoheadrightarrow \mathcal{M}$. In other words, at any point $p \in \mathcal{M}$ in spacetime, the field continuously assigns a vector, $\boldsymbol{\psi}|_p \cong V$.[2] The gauge transformations of $\boldsymbol{\psi}|_p$ at a point p form a group G under composition—but this group does not describe local gauge transformations which act on the *entire* field $\boldsymbol{\psi}$. Rather, the total space of local gauge transformations is a principle G-bundle $G \rightarrowtail \mathcal{G} \twoheadrightarrow \mathcal{M}$, consisting of smooth maps $\mathcal{M} \to G$.[3] The action of $\mathcal{G}$ on the space of matter fields $\mathcal{V}$ may be denoted

$$\boldsymbol{\psi} \overset{g}{\mapsto} \boldsymbol{\psi}' = g \cdot \boldsymbol{\psi}$$

for $\boldsymbol{\psi} \in \Gamma(\mathcal{V})$ and $g \in \Gamma(\mathcal{G})$ (i.e., $\boldsymbol{\psi} : \mathcal{M} \to V$ and $g : \mathcal{M} \to G$). This specification of an action "$\cdot$" of $\mathcal{G}$ on the vector bundle $\mathcal{V}$ is equivalent to the choice of a linear representation $\rho : G \to \mathrm{End}(V)$ of G on V, applied globally on $\mathcal{M}$. Thus, we may write the action as a matrix product, $g \cdot \boldsymbol{\psi} = \rho(g)\boldsymbol{\psi} \equiv g_\rho \boldsymbol{\psi}$, remembering that both $\boldsymbol{\psi}$ and g (but not ρ) vary across $\mathcal{M}$.

It is worth emphasising that all gauge theories are isomorphic to a gauge theory possessing only one matter field $\boldsymbol{\psi}_{\text{total}}$ with one gauge group (G, ρ). For instance, if a theory consists of different fundamental particles and forces, represented by separate fields $\boldsymbol{\varphi}$ and $\boldsymbol{\phi}$ with gauge

[2] The reason many objects in gauge theory (such as fields $\boldsymbol{\psi}$) are defined as sections of fibre bundles (and not simply as smooth maps $\boldsymbol{\psi} : \mathcal{M} \to V$) is because fibre bundles are themselves smooth manifolds which admit the construction of connections (whereas the space of smooth functions lacks a manifold topology).

[3] We sometimes loosely refer to the fibre group G as the gauge group, but strictly we mean the fibre bundle $\mathcal{G}$ (following [7]).

groups $(G_\varphi, \rho_\varphi)$ and (G_ϕ, ρ_ϕ), then we may take the total matter field $\boldsymbol{\psi}_{\text{total}} = \boldsymbol{\varphi} \oplus \boldsymbol{\phi}$ to be the direct sum of the fields in the theory, and similarly equip it with the gauge group $G_{\text{total}} = G_\varphi \oplus G_\phi$ with representation $\rho_{\text{total}} = \rho_\varphi \oplus \rho_\phi$. This new theory, inheriting the original equations of motion, is physically identical to the original. In full generality, therefore, we treat our theory as possessing one matter and one gauge field, while freely speaking of its composite parts as separate fields where convenient.

A *connection* on $\mathcal{V}$ is a derivation[4] $\underset{\sim}{\nabla} : \Gamma(\mathcal{V}) \to \Omega^1(\mathcal{M}, \mathcal{V})$ from vector fields to vector-valued 1-forms. The interpretation is that $(\underset{\sim}{\nabla}\boldsymbol{\psi})(\underset{\sim}{X})$—that is, the action of the V-valued 1-form $\underset{\sim}{\nabla}\boldsymbol{\psi}$ on a vector $\underset{\sim}{X} \in \mathrm{T}\mathcal{M}$—gives the directional derivative of $\boldsymbol{\psi}$ along $\underset{\sim}{X}$ (with respect to the connection $\underset{\sim}{\nabla}$). Employing a basis $\{\boldsymbol{e}_a\}$ of $\mathcal{V}$ and local coordinates $\{x^\mu\}$ of $\mathcal{M}$, *any* connection is of the form

$$\begin{aligned}\underset{\sim}{\nabla}\boldsymbol{\psi} &= \underset{\sim}{\mathrm{d}}\boldsymbol{\psi} + \underset{\sim}{\boldsymbol{A}}\boldsymbol{\psi} \\ &= \left(\partial_\mu \psi^a{}_b + A_\mu{}^a{}_b \psi^b\right) \boldsymbol{e}_a \otimes \underset{\sim}{\mathrm{d}}x^\mu\end{aligned}$$

for some matrix-valued 1-form $\underset{\sim}{\boldsymbol{A}}$, where $\underset{\sim}{\mathrm{d}}$ is the exterior derivative. (More precisely, $\underset{\sim}{\boldsymbol{A}} \in \Omega^1(\mathcal{M}, \mathfrak{g})$ is a $\mathfrak{g}$-valued 1-form, where $\mathfrak{g}$ is the Lie algebra of the gauge group G.) If $\underset{\sim}{\nabla}\boldsymbol{\psi}$ is required to transform like the matter field $\boldsymbol{\psi}$ under local gauge transformations, that is, as

$$\underset{\sim}{\nabla}\boldsymbol{\psi} \overset{g}{\mapsto} g \cdot (\underset{\sim}{\nabla}\boldsymbol{\psi}) = (g \cdot \underset{\sim}{\nabla})(g_\rho \boldsymbol{\psi}) \overset{!}{=} g_\rho \underset{\sim}{\nabla}\boldsymbol{\psi},$$

then $\underset{\sim}{\nabla}$ is called a *covariant derivative* and the connection 1-form $\underset{\sim}{\boldsymbol{A}}$ consequently obeys

$$\underset{\sim}{\boldsymbol{A}} \overset{g}{\mapsto} g \cdot \underset{\sim}{\boldsymbol{A}} = g_\rho \underset{\sim}{\boldsymbol{A}} g_\rho^{-1} - \left(\underset{\sim}{\mathrm{d}} g_\rho\right) g_\rho^{-1}. \tag{1.2}$$

Such a connection $\underset{\sim}{\nabla}_A$ is not unique; it depends on the choice of the 1-form field $\underset{\sim}{\boldsymbol{A}}$. It is exactly this connection 1-form which is promoted to a dynamical object in a gauge theory and given the name *the gauge field*. At each point in spacetime, and the gauge field $\underset{\sim}{\boldsymbol{A}}$ linearly assigns to each direction in spacetime an infinitesimal transformation of the matter field (this is why $\underset{\sim}{\boldsymbol{A}}$ is Lie algebra valued).

In order for the gauge field to be incorporated in the theory's equations of motion, we would like to have a notion of its derivative. However, the covariant derivative $\underset{\sim}{\nabla} : \Gamma(\mathcal{V}) \cong \Omega^0(\mathcal{M}, \mathcal{V}) \to \Omega^1(\mathcal{M}, \mathcal{V})$ is not readily defined on 1-forms such as $\underset{\sim}{\boldsymbol{A}}$ until we canonically extend it to a *covariant exterior derivative*[5] $\underset{\sim}{\mathrm{d}}_\nabla : \Omega^p(\mathcal{M}, \mathcal{V}) \to \Omega^{p+1}(\mathcal{M}, \mathcal{V})$, given by

$$\underset{\sim}{\mathrm{d}}_\nabla \underset{\sim}{\boldsymbol{\varphi}} := \underset{\sim}{\mathrm{d}}\underset{\sim}{\boldsymbol{\varphi}} + \underset{\sim}{\boldsymbol{A}} \wedge \underset{\sim}{\boldsymbol{\varphi}}.$$

This enables the construction of, among other things, the curvature 2-form or *gauge field strength*

$$\begin{aligned}\underset{\sim}{\boldsymbol{F}} &:= \underset{\sim}{\mathrm{d}}_\nabla \underset{\sim}{\boldsymbol{A}} \\ &= \underset{\sim}{\mathrm{d}}\underset{\sim}{\boldsymbol{A}} + \underset{\sim}{\boldsymbol{A}} \wedge \underset{\sim}{\boldsymbol{A}} \equiv \left(\underset{\sim}{\mathrm{d}}\underset{\sim}{A}^a{}_b + \underset{\sim}{A}^a{}_c \wedge \underset{\sim}{A}^c{}_b\right) \boldsymbol{e}_a \otimes \boldsymbol{e}^b,\end{aligned}$$

[4] More precisely, $\underset{\sim}{\nabla}$ is a $\mathcal{C}^\infty(\mathcal{M})$-linear derivation, meaning $\underset{\sim}{\nabla}(f\boldsymbol{u} + g\boldsymbol{v}) = f\underset{\sim}{\nabla}\boldsymbol{u} + g\underset{\sim}{\nabla}\boldsymbol{v}$ for scalar fields $f, g \in \mathcal{C}^\infty(\mathcal{M})$ and $\underset{\sim}{\nabla}(\boldsymbol{u} \otimes \boldsymbol{v}) = \underset{\sim}{\nabla}(\boldsymbol{u}) \otimes \boldsymbol{v} + \boldsymbol{u} \otimes \underset{\sim}{\nabla}(\boldsymbol{v})$.

[5] The extension is uniquely defined by requiring the graded Leibniz property $\underset{\sim}{\mathrm{d}}_\nabla(\underset{\sim}{\boldsymbol{\varphi}} \otimes \boldsymbol{\psi}) = \underset{\sim}{\mathrm{d}}\underset{\sim}{\boldsymbol{\varphi}} \otimes \boldsymbol{\psi} + (-1)^p \underset{\sim}{\boldsymbol{\varphi}} \wedge \underset{\sim}{\nabla}\boldsymbol{\psi}$ for $\underset{\sim}{\boldsymbol{\varphi}} \in \Omega^p(\mathcal{M})$ and $\boldsymbol{\psi} \in \Gamma(\mathcal{V})$, analogous to the usual exterior derivative.

where $\{\boldsymbol{e}_a\}$ and $\{\boldsymbol{e}^a\}$ form a basis and dual basis of $\mathcal{V}$. The field strength is also equivalent to

$$\underset{\sim}{\boldsymbol{F}} = \underset{\sim}{\nabla} \wedge \underset{\sim}{\nabla} \equiv [\nabla_\mu, \nabla_\nu]\, \underset{\sim}{\mathrm{d}}x^\mu \wedge \underset{\sim}{\mathrm{d}}x^\nu.$$

The field strength $\underset{\sim}{\boldsymbol{F}}$ is useful because it is tensorial, in the sense that it transforms like the matter field; $\underset{\sim}{\boldsymbol{F}} \mapsto g \cdot \underset{\sim}{\boldsymbol{F}} = g_\rho \underset{\sim}{\boldsymbol{F}} g_\rho^{-1}$ under gauge transformations, even though $\underset{\sim}{\boldsymbol{A}}$ does not. The gauge field strength automatically satisfies the *Bianchi identity*, $\underset{\sim}{\mathrm{d}}_\nabla \underset{\sim}{\boldsymbol{F}} = 0$.

We have seen how the gauge field $\underset{\sim}{\boldsymbol{A}}$ and its strength $\underset{\sim}{\boldsymbol{F}}$ arise when we require the notion of a spacetime derivative for a matter field which possesses local gauge freedom, and are now acquainted with the dynamical objects of a gauge theory. The matter field $\boldsymbol{\psi}$, its derivative $\underset{\sim}{\nabla}_A \boldsymbol{\psi}$ and the strength of the gauge field $\underset{\sim}{\boldsymbol{F}}$ all transform regularly under gauge transformations. All that remains to be specified in our theory are the equations of motion, which are to be expressed in terms of these three geometrical objects in a gauge invariant manner.

1.1.1 Lagrangians in Field Theories

For classical gauge theories, the equations of motion may be specified as the extremisers of an action

$$S[\boldsymbol{\psi}, \underset{\sim}{\nabla}_A \boldsymbol{\psi}, \underset{\sim}{\boldsymbol{F}}] = \int_{\mathcal{M}} \underset{\sim}{\mathcal{L}}[\boldsymbol{\psi}, \underset{\sim}{\nabla}_A \boldsymbol{\psi}, \underset{\sim}{\boldsymbol{F}}],$$

where $\underset{\sim}{\mathcal{L}}$ is a local Lagrangian density. In order that the equations of motion are physically well-defined, we require the Lagrangian to possess the relevant symmetries: gauge symmetry, so that the equations of motion are gauge invariant; and Lorentz symmetry (which is automatic if $\underset{\sim}{\mathcal{L}} = \mathcal{L}\, \underset{\sim}{\mathrm{vol}}$ is expressed as a volume form) [6, § 7.1]. The equations of motion are invariant under adjustments to the Lagrangian density by a total derivative $\underset{\sim}{\mathrm{d}}\underset{\sim}{K}$, since by Stokes' theorem these contribute only to terms on the boundary, where the fields are fixed by assumption. Therefore, a Lagrangian which possesses gauge symmetry is still generally permitted to transform as

$$\underset{\sim}{\mathcal{L}} \mapsto \underset{\sim}{\mathcal{L}} + \underset{\sim}{\mathrm{d}}\underset{\sim}{K} \tag{1.3}$$

under gauge transformations. If a Lagrangian transforms as (1.3) under a continuous gauge symmetry parametrised by n parameters α_i, then Noether's theorem implies the existence of n conserved current densities (viz. 3-forms[6]), one for each α_i,

$$\underset{\sim}{J}_{(i)} = \frac{\partial \underset{\sim}{\mathcal{L}}}{\partial \underset{\sim}{\nabla} \boldsymbol{\psi}} \frac{\partial \boldsymbol{\psi}}{\partial \alpha_i}\bigg|_{\mathrm{id}} - \frac{\partial \underset{\sim}{K}}{\partial \alpha_i}, \tag{1.4}$$

whose continuity equations read $\underset{\sim}{\mathrm{d}}\underset{\sim}{J}_{(i)} = 0$.

On the other hand, the Lagrangian of a *quantum* field theory enjoys an enlarged criterion of gauge symmetry: it is also permitted to transform as

$$\int_{\mathcal{M}} \underset{\sim}{\mathcal{L}} \mapsto \int_{\mathcal{M}} \underset{\sim}{\mathcal{L}} + n \cdot 2\pi\hbar, \tag{1.5}$$

[6] A *current density* naturally transforms as a 3-form. The partial derivative $\partial \underset{\sim}{\mathcal{L}} / \partial \underset{\sim}{\nabla} \boldsymbol{\psi}$ of a volume form $\underset{\sim}{\mathcal{L}}$ with respect to a 1-form $\underset{\sim}{\nabla} \boldsymbol{\psi}$ is itself a 3-form, and is defined by canonically extending the scalar partial derivative to be an anti-derivation on differential forms. For details, see [8].

where $n \in \mathbb{Z}$ may vary discretely under different gauges. This is because of the origin of a QFT's equations of motion in the Feynman path integral. Explicitly, the quantum mechanical amplitude that the fields $\boldsymbol{\psi}$ and $\underset{\sim}{\boldsymbol{A}}$ satisfy prescribed boundary conditions on $\partial\Omega$ surrounding some region of spacetime $\Omega \subseteq \mathscr{M}$ is given by the path integral

$$\mathscr{A} = \int_{\partial\Omega} \mathcal{D}[\boldsymbol{\psi}, \underset{\sim}{\boldsymbol{A}}] \exp\left\{\frac{i}{\hbar} S[\boldsymbol{\psi}, \underset{\sim}{\nabla}_A \boldsymbol{\psi}, \underset{\sim}{\boldsymbol{A}}]\right\}, \tag{1.6}$$

where the intended meaning of $\mathcal{D}[\cdots]$ is an integration over all field configurations on Ω. If the Lagrangian were to undergo a discrete gauge transformation (1.5), the amplitude $\mathscr{A} \mapsto \mathscr{A} \exp(n \cdot 2\pi i)$ would be left invariant. In other words, physical consistency of a QFT does not require the single-valuedness of the action S, but only of $\exp(iS/\hbar)$. This leads to an enlargement of the space of possible Lagrangian densities to include, in particular, *topological terms*. These prove to be especially relevant to QFTs and to the strong CP problem itself.

1.1.2 The Yang–Mills Lagrangian and the Topological θ-Term

An important component of a gauge theory's equations of motion are the terms in the Lagrangian which describe the dynamics of the gauge field $\underset{\sim}{\boldsymbol{A}}$. These specify how the gauge field behaves in the vacuum (e.g., describing the classical electromagnetic field, or the quantum theory of photons, in the absence of matter). Thus, we are interested in the possible *consistent* Lagrangians which may be constructed from the gauge field $\underset{\sim}{\boldsymbol{A}}$ without the matter field $\underset{\sim}{\psi}$.

Along with requiring Lorentz and gauge invariance, consistency also requires that the quantum field theory associated to a Lagrangian be *renormaliseble*. Loosely speaking, a classical field theory is renormalisable if it can be "quantised without introducing irrecoverable infinities". (This restricts the form of the Lagrangian considerably, but how this happens is beyond this review's scope.) Under these constraints, the only admissible QFT Lagrangians which may be constructed from the gauge field $\underset{\sim}{\boldsymbol{A}}$ alone are linear combinations of

$$\langle \underset{\sim}{\boldsymbol{F}} \wedge \star \underset{\sim}{\boldsymbol{F}} \rangle \equiv \frac{1}{2} \left\langle \boldsymbol{F}_{\mu\nu}, \boldsymbol{F}_{\rho\sigma} \right\rangle_{\mathrm{Ad}} g^{\mu\rho} g^{\nu\sigma} \underset{\sim}{\mathrm{vol}} \quad \text{and} \quad \langle \underset{\sim}{\boldsymbol{F}} \wedge \underset{\sim}{\boldsymbol{F}} \rangle \equiv \frac{1}{4} \left\langle \boldsymbol{F}_{\mu\nu}, \boldsymbol{F}_{\rho\sigma} \right\rangle_{\mathrm{Ad}} \epsilon^{\mu\nu\rho\sigma} \underset{\sim}{\mathrm{vol}},$$

where $\star$ is the Hodge dual [6, § 7.1.2]. The inner product $\langle\ ,\ \rangle_{\mathrm{Ad}}$ on the Lie algebra $\mathfrak{g}$ of the gauge group G is chosen such that it is gauge-invariant.[7] Such an inner product $\langle\ ,\ \rangle_{\mathrm{Ad}}$ on $\mathfrak{g}$ is not unique; it depends on a choice of *coupling constants*. In particular, if the gauge group G is the direct sum of n simple Lie groups,[8] then $\langle\ ,\ \rangle_{\mathrm{Ad}}$ is specified by the choice of exactly n coupling constants, one corresponding to each factor of G [6, § 2.5]. Physically, the coupling constants determine the relative interaction strengths of the forces associated to each factor of G, and must enter the theory as free parameters determined experimentally. (For instance, the gauge group of the standard model (1.1) has three such coupling constants for the strong $\mathrm{SU}(3)$, weak $\mathrm{SU}(2)$, and electromagnetic $\mathrm{U}(1)$ interactions.)

The term $\langle \underset{\sim}{\boldsymbol{F}} \wedge \star \underset{\sim}{\boldsymbol{F}} \rangle$ is known as the *Yang–Mills Lagrangian*, and is a major component in the standard model, describing boson force carrier propagation and self-interaction (such as

[7] Recall that $\underset{\sim}{\boldsymbol{A}}$, and hence $\underset{\sim}{\boldsymbol{F}}$, are $\mathfrak{g}$-valued forms, so an inner product on $\mathfrak{g}$ is needed to produce a scalar. For this scalar to be gauge invariant, the inner product must additionally be "Ad-invariant" [6, § 7.3].

[8] Any compact connected Lie group is either of this form, or is a finite quotient of such a group, if $\mathrm{U}(1)$ is counted as "simple". [6, § 2.4.3].

gluon self-interactions). The Yang–Mills Lagrangian yields the equation of motion $\underset{\sim}{\mathrm{d}}_\nabla \star \underset{\sim}{\boldsymbol{F}} = 0$. For the Abelian gauge group $G = \mathrm{U}(1)$ of electromagnetism, this equation, together with the Bianchi identity $\underset{\sim}{\mathrm{d}}_\nabla \underset{\sim}{F} = 0$, are the source-free Maxwell equations. (Since the Lie algebra of $\mathrm{U}(1)$ is $\mathbb{R}$, the 2-form $\underset{\sim}{F}$ of QED is a scalar-valued.) Expressing $\underset{\sim}{F} = \underset{\sim}{\mathrm{d}}ct \wedge \vec{E} + \star(\underset{\sim}{\mathrm{d}}ct \wedge \vec{B})$ in terms of the 3-component non-relativistic electric and magnetic field 1-forms by choosing a spacetime split, the Yang–Mills term is the familiar electromagnetic energy density $\underset{\sim}{F} \wedge \star \underset{\sim}{F} = \vec{B}^2 + \vec{E}^2/c^2$.

The other term $\langle \underset{\sim}{\boldsymbol{F}} \wedge \underset{\sim}{\boldsymbol{F}} \rangle$ is known as the *Chern–Simons term* or the *topological θ-term*, for reasons which will become apparent after a survey of its properties.

- The Chern–Simons term is odd under both time-reversal symmetry T and parity P (notice $\epsilon^{\mu\nu\rho\sigma} \mapsto -\epsilon^{\mu\nu\rho\sigma}$ under T or P) but not under charge conjugation C. This may also be seen by introducing a spacetime split, whereby $\langle \underset{\sim}{\boldsymbol{F}} \wedge \underset{\sim}{\boldsymbol{F}} \rangle = \mathrm{tr}(\boldsymbol{E} \cdot \boldsymbol{B})\, \underset{\sim}{\mathrm{vol}}$, since $\underset{\sim}{\boldsymbol{E}}$ is a vector of odd-parity and $\underset{\sim}{\boldsymbol{B}}$ is a pseudovector of even-parity. Hence, it may give rise to CP-violating dynamics.

- It is *topological* because it does not depend on the geometry of spacetime via the metric $g^{\mu\nu}$ (instead, all spacetime indices are contracted with $\epsilon^{\mu\nu\rho\sigma}$). Hence, an action $\int_{\mathcal{M}} \langle \underset{\sim}{\boldsymbol{F}} \wedge \underset{\sim}{\boldsymbol{F}} \rangle$ depends only on the integrand's topology over $\mathcal{M}$. (Recall that $\underset{\sim}{\boldsymbol{F}}$ is $\mathfrak{g}$-valued, so for sufficiently interesting gauge groups, the space of gauge fields may have non-trivial topology.)

- Furthermore, $\langle \underset{\sim}{\boldsymbol{F}} \wedge \underset{\sim}{\boldsymbol{F}} \rangle$ is a total derivative of the *Chern–Simons 3-form* $\underset{\sim}{\omega}_3$,

$$\langle \underset{\sim}{\boldsymbol{F}} \wedge \underset{\sim}{\boldsymbol{F}} \rangle = \underset{\sim}{\mathrm{d}}\underset{\sim}{\omega}_3 = \underset{\sim}{\mathrm{d}}\,\mathrm{tr}\left(\underset{\sim}{\boldsymbol{A}} \wedge \underset{\sim}{\mathrm{d}}\underset{\sim}{\boldsymbol{A}} + \frac{2}{3} \underset{\sim}{\boldsymbol{A}} \wedge \underset{\sim}{\boldsymbol{A}} \wedge \underset{\sim}{\boldsymbol{A}} \right),$$

  meaning that the action $\int_{\mathcal{M}} \langle \underset{\sim}{\boldsymbol{F}} \wedge \underset{\sim}{\boldsymbol{F}} \rangle$ depends only on the topology of $\underset{\sim}{\boldsymbol{A}}$ on the spacetime boundary $\partial\mathcal{M}$. As such, it does not affect the classical equations of motion. However, it has important implications in the quantum theory.

- The integral is a discrete topological invariant

$$n = \frac{1}{8\pi^2} \int_{\mathcal{M}} \langle \underset{\sim}{\boldsymbol{F}} \wedge \underset{\sim}{\boldsymbol{F}} \rangle = \frac{1}{8\pi^2} \int_{\partial\mathcal{M}} \underset{\sim}{\omega}_3 \in \mathbb{Z}, \tag{1.7}$$

  known as the *Pontryagin number*, the *second Chern class* [9, § 1] or simply the 'winding number' [7, § 2.2] of the gauge field configuration $\underset{\sim}{\boldsymbol{A}}$. Importantly, this means that the Chern–Simons term is *not* totally gauge invariant if there are topologically distinct gauge fields $\underset{\sim}{\boldsymbol{A}}$ with varying winding number.

The choice of the symbol θ in the name "θ-term" reflects the angular nature of any coefficient θ attached to the Chern–Simons term, as in

$$\underset{\sim}{\mathcal{L}}_\theta[\underset{\sim}{\boldsymbol{A}}] = \frac{\theta}{8\pi^2} \langle \underset{\sim}{\boldsymbol{F}} \wedge \underset{\sim}{\boldsymbol{F}} \rangle\, \hbar. \tag{1.8}$$

The action of this Lagrangian is $\theta n \hbar$ whenever $\underset{\sim}{\boldsymbol{A}}$ has winding number n. Since this enters the path integral as $e^{i\theta n} = e^{i(\theta + 2\pi)n}$, the coefficient θ, henceforth the *θ-parameter*, is only distinguishable modulo 2π and is hence an angular quantity. As a whole, (1.8) is referred to as the θ-term.

Glossary of technical terms

- *degenerate* — quantum eigenstates which share the same eigenvalues (usually energy) but which are physically distinguishable
- *axial, chiral* — a transformation acting differently on left- and right-handed fermions.
- *anomalous symmetry* — a symmetry of the classical Lagrangian, but not of the measure $\mathcal{D}[\psi, \underset{\sim}{\boldsymbol{F}}]$ in the path integral (1.6), and hence not a symmetry of the associated quantum theory.
- *spontaneously broken symmetry* — a symmetry of the Lagrangian which fails to manifest in the ground state solutions.
- *Nambu–Goldstone boson* — a scalar boson which arises due to a spontaneously broken symmetry. The dimension of the broken symmetry group is the number of resulting Nambu–Goldstone bosons.

The standard model employs the Yang–Mills Lagrangian $\langle \underset{\sim}{\boldsymbol{F}} \wedge \star \underset{\sim}{\boldsymbol{F}} \rangle$, but does not find use for the other possible θ-term. Historically, the θ-term was dismissed as unphysical, since at first sight it appears to be a gauge-dependent boundary term. It was only with the discovery of *instantons* and *the non-trivial vacuum* of QCD in the mid 1970s [10] that it was realised that the θ-term should be considered, and does not necessarily vanish [11].

1.1.3 The θ-Term as a Consequence of the Non-trivial Vacuum

The θ-term is more than just a mathematical possibility which lacks physical reason for its inclusion in the Lagrangian. It is in fact central to non-Abelian gauge theories as a direct consequence of their *non-trivial vacuum structure*, which is responsible for interesting non-perturbative dynamical effects. The non-trivial vacuum is not an obvious feature of non-Abelian theories, as it is absent in Abelian theories such as QED, out of which QCD emerged.

For an Abelian gauge group $G = \mathrm{U}(1)$ with total gauge group bundle $\mathcal{G} = \mathcal{U}(1)$, the space of asymptotically-identity gauge transformations $\Gamma^{\mathrm{id}}(\mathcal{U}(1))$ is continuously connected to the identity.[9] This means that any two gauge-equivalent gauge field configurations can be *continuously* gauge transformed into each other. Hence, all gauge transformations preserve the topology of the gauge field, and the winding number is zero for all $\underset{\sim}{\boldsymbol{A}} \in \Omega^1(\mathcal{M}, \mathfrak{u}(1))$. Consequently, the θ-term (with constant θ) is neither relevant in classical electromagnetism nor in QED, because it vanishes identically.

However, for non-Abelian gauge groups, the space of gauge transformations $\Gamma^{\mathrm{id}}(\mathcal{G})$ may have a non-trivial topology, containing gauge transformations which are not diffeomorphic. This allows for the possibility of gauge-equivalent field configurations which cannot be *continuously* transformed into one another. In the case where $\mathcal{M}$ is $(1+3)$-dimensional spacetime and $\mathcal{G} = \mathcal{SU}(3)$ is the total gauge group of QCD, the space $\Gamma^{\mathrm{id}}(\mathcal{SU}(3))$ consists of path-disconnected regions labelled by some $\nu \in \mathbb{Z}$. (In topological language, the third homotopy group $\pi_3(\Gamma^{\mathrm{id}}(\mathcal{SU}(3))) \cong \mathbb{Z}$ is the group of integers.) This means that there are topologically inequivalent gauge transformations of any given $\underset{\sim}{\boldsymbol{A}}$, with the winding number labelling each

[9] We consider the space of asymptotically-identity gauge transformations $\Gamma^{\mathrm{id}}(\mathcal{G})$, whose elements are maps $g : \mathcal{M} \to G$ with $g|_{\partial\mathcal{M}} = \mathrm{id}$ (or equivalently with $g(x) \to \mathrm{id}$ as $x \to \infty$), because $\underset{\sim}{\boldsymbol{A}}$ must be fixed on the boundary in order to prescribe boundary conditions.

distinct topological class. Elements $g_0 : \mathcal{M} \to G$ in the identity-connected component of $\Gamma^{\mathrm{id}}(\mathcal{G})$ are named *small* gauge transformations, and all others *large*. A large gauge transformation $g_\nu \in \Gamma^{\mathrm{id}}(\mathcal{G})$ shifts the winding number n of a gauge field $\underset{\sim}{\boldsymbol{A}}$ to $n+\nu$, giving rise to $\mathbb{Z}$-many gauge-equivalent fields $\{g_\nu \cdot \underset{\sim}{\boldsymbol{A}}\}_{\nu\in\mathbb{Z}}$ which belong to distinct homotopy classes. Gauge field configurations with nonzero winding number are known as *instantons* [11, 12].

Despite the name, large gauge transformations are not truly gauge symmetries, in the sense that they are not genuine automorphisms of the equations of motion. Indeed, large gauge transformations may transition between states which can be distinguished by physical measurement. For instance, if the gauge field $\underset{\sim}{\boldsymbol{A}}$ has winding number n, then the action of the θ-term

$$S[\underset{\sim}{\boldsymbol{A}}] = \int_{\mathcal{M}} \underset{\sim}{\mathcal{L}}_\theta[\underset{\sim}{\boldsymbol{A}}] = \theta n \hbar$$

is proportional to n. A large gauge transformation g_ν then shifts this action $S[\underset{\sim}{\boldsymbol{A}}] \to S[g_\nu \cdot \underset{\sim}{\boldsymbol{A}}] = S[\underset{\sim}{\boldsymbol{A}}] + \theta\nu\hbar$. From the path integral (1.6), this induces relative phases $e^{i\theta\nu}$ varying across the domain of integration which may interfere and alter the amplitude. In other words, instantons are measurable.

The θ-Vacuum

Of particular interest are the implications of instantons for the QCD vacuum. In a Yang–Mills theory whose Lagrangian includes the term $\underset{\sim}{\mathcal{L}}_{\mathrm{YM}} = \langle \underset{\sim}{\boldsymbol{F}} \wedge \star \underset{\sim}{\boldsymbol{F}} \rangle$, a *vacuum state* is one in which the field strength (and all other fields) vanish; $\underset{\sim}{\boldsymbol{F}} = \underset{\sim}{\boldsymbol{0}}$. Not only is this consistent with an identically vanishing gauge field $\underset{\sim}{\boldsymbol{A}}_0 = \underset{\sim}{\boldsymbol{0}}$, but also with gauge transformations (1.2) of $\underset{\sim}{\boldsymbol{A}}_0$,

$$g \cdot \underset{\sim}{\boldsymbol{A}}_0 = (\underset{\sim}{\mathrm{d}} g_\rho)(g_\rho)^{-1},$$

which are called "pure gauge" configurations. Small gauge transformations are not measurable and describe the same vacuum state, whereas large ones $g_n \cdot \underset{\sim}{\boldsymbol{A}}_0$ are distinguishable, hence describing *distinct vacua* $|n\rangle$ labelled by winding number. Since $g_\nu \cdot |n\rangle = |n+\nu\rangle$, these states are not gauge-invariant. The only vacuum states which are invariant under the gauge (up to an overall phase) are those of the form

$$|\theta\rangle = \sum_{n\in\mathbb{Z}} e^{i\theta n} |n\rangle ,$$

for some constant parameter θ. Thus, the theory possesses a topological circle of distinct gauge-invariant vacua $|\theta\rangle$ labelled by the *vacuum angle* $\theta \in [0, 2\pi)$.

The existence of the enriched vacuum $|\theta\rangle$ is equivalent to the inclusion of the θ-term in an effective Lagrangian, in the following way: Denote by ${}_+\langle n|m\rangle_-$ the quantum amplitude that the vacuum state $|n\rangle$ at $t \to -\infty$ evolves to $|m\rangle$ at $t \to \infty$. The vacuum-to-vacuum amplitude is

$${}_+\langle\theta|\theta\rangle_- = \sum_{n,m} e^{i\theta(n-m)} {}_+\langle m|n\rangle_- = \sum_\nu e^{i\theta\nu} \sum_n {}_+\langle n|n+\nu\rangle_-, \tag{1.9}$$

with all summations over $\mathbb{Z}$. In the path integral formulation (1.6), the amplitude ${}_+\langle n|n+\nu\rangle_-$ can be expressed explicitly as

$${}_+\langle n|n+\nu\rangle_- = \int_{\partial\Omega} \mathcal{D}[\underset{\sim}{\boldsymbol{A}}; \nu] \exp\left\{ \frac{i}{\hbar} \int_\Omega \underset{\sim}{\mathcal{L}} \right\}, \tag{1.10}$$

where $\mathcal{D}[\underset{\sim}{\boldsymbol{A}};\nu]$ means that the path integral is over all instanton gauge fields with winding number ν, since only those induce a transition $|n\rangle \mapsto |n+\nu\rangle$. Combining (1.9) and (1.10), the amplitude of evolution from the gauge invariant vacuum to itself is

$$\begin{aligned} {}_+\langle\theta|\theta\rangle_- &= \sum_\nu e^{i\theta\nu} \int_{\partial\Omega} \mathcal{D}[\underset{\sim}{\boldsymbol{A}};\nu] \exp\left\{\frac{i}{\hbar}\int_\Omega \underset{\sim}{\mathcal{L}}\right\} \\ &= \sum_\nu \int_{\partial\Omega} \mathcal{D}[\underset{\sim}{\boldsymbol{A}};\nu] \exp\left\{\frac{i}{\hbar}\int_\Omega \underset{\sim}{\mathcal{L}} + i\theta\nu\right\}, \end{aligned}$$

which, using (1.7) to write $i\theta\nu$ as the θ-term in the Lagrangian, is

$$= \int_{\partial\Omega} \mathcal{D}[\underset{\sim}{\boldsymbol{A}}] \exp\left\{\frac{i}{\hbar}\int_\Omega \Big[\underbrace{\underset{\sim}{\mathcal{L}} + \frac{\theta}{8\pi^2}\langle \underset{\sim}{\boldsymbol{F}} \wedge \underset{\sim}{\boldsymbol{F}}\rangle \hbar}_{\underset{\sim}{\mathcal{L}}_{\text{eff}}}\Big]\right\}.$$

The effects of the non-trivial vacuum are thus encapsulated in the effective Lagrangian

$$\underset{\sim}{\mathcal{L}}_{\text{eff}} = \underset{\sim}{\mathcal{L}} + \frac{\theta}{8\pi^2}\langle \underset{\sim}{\boldsymbol{F}} \wedge \underset{\sim}{\boldsymbol{F}}\rangle \hbar.$$

It is in this sense that the θ-term *arises* in the Lagrangian due to the non-trivial QCD vacuum.

1.1.4 Dirac Fermion Fields and the Chiral Anomaly

An important kind of matter field in the standard model is the *Dirac fermion field* $\boldsymbol{\varphi}$, which transforms under the spin-$\frac{1}{2}$ representation of the Lorentz group. Dirac fermions take their values in a 4-component complex space, denoted $\mathbb{C}^4_{\text{D}}$. The entire matter content of the standard model, including quarks and leptons, is comprised purely of such fermion fields.

The Dirac matrices $\boldsymbol{\gamma}^\mu$ form a basis for the algebra of spacetime, the Clifford algebra $\mathcal{C}l_{1,3}(\mathbb{C})$, satisfying $\boldsymbol{\gamma}^{(\mu}\boldsymbol{\gamma}^{\nu)} = \eta^{\mu\nu}$, and are used to write 4-component Dirac fermions in the spin-$\frac{1}{2}$ representation. An inner product on fermions $\langle\boldsymbol{\psi},\boldsymbol{\varphi}\rangle \equiv \bar{\boldsymbol{\psi}}\boldsymbol{\varphi} \in \mathbb{R}$ is provided by the Dirac adjoint $\bar{\boldsymbol{\psi}} := \boldsymbol{\psi}^\dagger\boldsymbol{\gamma}^0$, so that Lorentz-invariant quantities may be naturally constructed. Finally, fermions may be separated into left-handed $\boldsymbol{\varphi}_+$ and right-handed $\boldsymbol{\varphi}_-$ components with the projection operators $\boldsymbol{\varphi}_\pm = \frac{1}{2}(1 \pm \boldsymbol{\gamma}^5)$ where $\boldsymbol{\gamma}^5 := i\boldsymbol{\gamma}^0\boldsymbol{\gamma}^1\boldsymbol{\gamma}^2\boldsymbol{\gamma}^3$. In theories which violate parity, left- and right-handed fermions may experience different interactions. (For instance, left-handed neutrinos interact in the standard model, while right-handed neutrinos are completely inert.)

The simplest fermion equation of motion, the Dirac equation, derives from the *Dirac Lagrangian density*

$$\underset{\sim}{\mathcal{L}}_{\text{Dirac}} = \bar{\boldsymbol{\varphi}}\big(i\hbar c\boldsymbol{\gamma}^\mu\partial_\mu - mc^2\big)\boldsymbol{\varphi}\ \underset{\sim}{\text{vol}},$$

which describes a single non-interacting spin-$\frac{1}{2}$ fermion of mass m. The Dirac Lagrangian may be localised in the presence of a gauge symmetry, giving

$$\underset{\sim}{\mathcal{L}}_{\text{Dirac}} = \bar{\boldsymbol{\psi}}\big(i\boldsymbol{\gamma}^\mu\nabla_\mu - m\big)\boldsymbol{\psi}\ \underset{\sim}{\text{vol}}, \tag{1.11}$$

where $\underset{\sim}{\nabla}\boldsymbol{\psi} \equiv (\nabla_\mu\boldsymbol{\psi})\,\underset{\sim}{\mathrm{d}}x^\mu$ is the covariant derivative with respect to the gauge field, and where we have now begun to employ units in which $\hbar = c = 1$ for brevity. The matter field $\boldsymbol{\psi} =$

$\boldsymbol{\varphi}_{(1)} \oplus \cdots \oplus \boldsymbol{\varphi}_{(n)}$ may now more generally be comprised of multiple fermion fields, which may be rotated into one another by the gauge group (an example being *flavour symmetry* in QCD). The localised Dirac Lagrangian describes multiple fermion types and their interactions with the gauge bosons. The electromagnetic interactions of charged fermionic matter are described by (1.11) in the case of a $\mathrm{U}(1)$ gauge symmetry, where the gauge field $\underset{\sim}{\boldsymbol{A}} \equiv \underset{\sim}{A}$ is the electromagnetic vector potential.

The Anomalous Axial Symmetry

In quantum theories of fermions, the θ-term makes another important appearance, arising in the context of the *chiral anomaly*. The Dirac Lagrangian classically possesses two fundamental $\mathrm{U}(1)$ symmetries which rotate fermion phases *vectorially* or *axially*. The classical axial symmetry is violated upon quantisation—an effect known as the chiral anomaly.

To illustrate, the vectorial symmetry $\mathrm{U}(1)_V$ is the invariance of the Dirac Lagrangian under global Abelian transformations known as *vector fermion rotations*. The action of $\mathrm{U}(1)_V$ is a simple phase rotation $\boldsymbol{\psi} \mapsto e^{i\alpha}\boldsymbol{\psi}$, where α is the parameter. The conserved Noether current density associated to this symmetry is

$$\underset{\sim}{J}_V = \hbar c\, \bar{\boldsymbol{\psi}} \star \underset{\sim}{\boldsymbol{\gamma}}\, \boldsymbol{\psi}, \qquad\qquad \text{i.e.,} \quad j^\mu_V = \hbar c\, \bar{\boldsymbol{\psi}}\, \boldsymbol{\gamma}^\mu \boldsymbol{\psi},$$

where $\underset{\sim}{\boldsymbol{\gamma}} = \boldsymbol{\gamma}_\mu\, \underset{\sim}{\mathrm{d}}x^\mu$. (These 3-form and vector representations are related by $\underset{\sim}{J}_V = \star \underset{\sim}{j}_V$.) The associated continuity equation reads $\underset{\sim}{\mathrm{d}}\underset{\sim}{J}_V = 0$ (i.e., $\partial_\mu j^\mu_V = 0$), corresponding to the conservation of charge.

On the other hand, the axial symmetry $\mathrm{U}(1)_A$ is invariance under *axial fermion rotations*, which transform left- $\boldsymbol{\psi}_+$ and right-handed $\boldsymbol{\psi}_-$ fermion components oppositely:

$$\boldsymbol{\psi} \overset{\mathrm{U}(1)_A}{\mapsto} e^{i\boldsymbol{\gamma}^5\theta}\boldsymbol{\psi}, \qquad\qquad \text{i.e.,} \quad \boldsymbol{\psi}_\pm \overset{\mathrm{U}(1)_A}{\mapsto} e^{\pm i\theta}\boldsymbol{\psi}_\pm. \tag{1.12}$$

The associated Noether current density,

$$\underset{\sim}{J}_A = \hbar c\, \bar{\boldsymbol{\psi}} \star \underset{\sim}{\boldsymbol{\gamma}}\, \boldsymbol{\gamma}_5\, \boldsymbol{\psi}, \qquad\qquad \text{i.e.,} \quad j^\mu_A = \hbar c\, \bar{\boldsymbol{\psi}}\, \boldsymbol{\gamma}^\mu \boldsymbol{\gamma}_5\, \boldsymbol{\psi},$$

is conserved classically. The charge associated to $\underset{\sim}{J}_A$ is the number of left-handed particles minus the number of right-handed, named the *baryon number* in QCD. However, the $\mathrm{U}(1)_A$ symmetry is *anomalous*, meaning it does not survive the quantisation procedure, and is not an exact symmetry of the quantum theory. Specifically, an anomalous symmetry does not leave invariant the integral measure $\mathcal{D}[\cdots]$ in (1.6) of the path integral in the quantum theory [7]. Instead, the continuity equation $\underset{\sim}{\mathrm{d}}\underset{\sim}{J} = 0$ fails by the presence of none-other than the θ-term,

$$\underset{\sim}{\mathrm{d}}\underset{\sim}{J}_A \propto \langle \underset{\sim}{\boldsymbol{F}} \wedge \underset{\sim}{\boldsymbol{F}} \rangle\,,$$

with the constant of proportionality depending on the details of the theory (the number of fermion species, etc). This means that the axial current $\underset{\sim}{J}_A$ is not conserved—and hence CP symmetry is violated—in the presence of instantons, where $\int \langle \underset{\sim}{\boldsymbol{F}} \wedge \underset{\sim}{\boldsymbol{F}} \rangle$ is nonzero.

By Noether's theorem (1.4), this is equivalent to the addition of a total derivative to the Lagrangian $\underset{\sim}{\mathcal{L}}_{\text{eff}} = \underset{\sim}{\mathcal{L}} + \underset{\sim}{\mathrm{d}}\underset{\sim}{K}$. This total derivative is precisely the θ-term, because

$$\underset{\sim}{\mathrm{d}}\underset{\sim}{J}_A = \langle \underset{\sim}{\boldsymbol{F}} \wedge \underset{\sim}{\boldsymbol{F}} \rangle = \frac{\partial \underset{\sim}{\mathrm{d}}\underset{\sim}{K}}{\partial\theta} \qquad \Longrightarrow \qquad \underset{\sim}{\mathrm{d}}\underset{\sim}{K} = \theta\, \langle \underset{\sim}{\boldsymbol{F}} \wedge \underset{\sim}{\boldsymbol{F}} \rangle\,,$$

where here θ is the axial rotation parameter appearing in (1.12). The axial current $\underset{\sim}{J}'_A$ of this new Lagrangian $\underset{\sim}{\mathcal{L}}_{\text{eff}}$ is indeed conserved. In other words, an axial rotation does not leave the Lagrangian invariant, but instead generates an effective θ-term [13, § 8].

1.2 QCD and the Strong CP Problem

We are almost prepared to express the theory of quantum chromodynamics so that the strong CP problem manifests itself. All that remains is to introduce the final piece of QCD—the fermion mass terms, and the implications of the chiral anomaly.

Quantum chromodynamics describes the strong interactions among hadronic matter. Fields which interact via the strong force are called *colour-charged*, and in QCD, the colour-charged fields are the *quarks*.

In QCD with N_c colours, a quark is a colour-charged Dirac fermion, represented by a matter field $\boldsymbol{\psi} = \boldsymbol{\varphi} \otimes \boldsymbol{c}$ with values in $\mathbb{C}^4_{\text{D}} \otimes \mathbb{C}^{N_c}_{\text{C}}$, where $\boldsymbol{\varphi}(x) \in \mathbb{C}^4_{\text{D}}$ is a plain Dirac fermion and $\boldsymbol{c}(x) \in \mathbb{C}^{N_c}_{\text{C}}$ is a N_c-component vector in *colour space*. A quark field $\boldsymbol{\psi}$ can either be viewed as a fermion with components in colour space, or equivalently as a N_c-tuple of fermions. The gauge group is $\mathrm{SU}(N_c)$, and its action on $\boldsymbol{\psi}$ is to transform colour space under the fundamental representation; i.e., $\boldsymbol{\psi} \mapsto \boldsymbol{\psi}' = \boldsymbol{\varphi} \otimes \mathsf{U}\boldsymbol{c}$ with $\mathsf{U} \in \mathrm{SU}(N_c)$. The gauge field $\underset{\sim}{\boldsymbol{A}}$, named the *gluon field*, is an $\mathfrak{su}(N_c)$-valued 1-form, which can equivalently be viewed as a collection of $\dim \mathfrak{su}(N_c) = N_c^2 - 1$ independent 1-form fields, or eight distinct gluon types in the case $N_c = 3$ as in the standard model. QCD can be constructed with N_f quark types—or *flavours*—by taking the matter field to be a direct product of N_f quark fields, each sharing the same $\mathrm{SU}(N_c)$ gauge action.

The Lagrangian of pure QCD is a sum of the Dirac and Yang–Mills Lagrangians,

$$\underset{\sim}{\mathcal{L}}_{\text{QCD}} = \bar{\boldsymbol{\psi}}\big(i\boldsymbol{\gamma}^\mu \nabla_\mu - \mathsf{m}\big)\boldsymbol{\psi}\,\underset{\sim}{\text{vol}} - \langle \underset{\sim}{\boldsymbol{F}} \wedge \star \underset{\sim}{\boldsymbol{F}} \rangle + \frac{\theta}{8\pi^2}\,\langle \underset{\sim}{\boldsymbol{F}} \wedge \underset{\sim}{\boldsymbol{F}} \rangle\,,$$

where $\boldsymbol{\psi} \equiv \boldsymbol{\psi}^{(1)} \oplus \cdots \oplus \boldsymbol{\psi}^{(N_f)}$ is the matter field separated into its quark flavours, and $\mathsf{m} = m_1 \oplus \cdots \oplus m_{N_f}$ is a diagonal matrix of quark masses. With indices written explicitly, and $\hbar$ and c temporarily reinstated for completeness, the Lagrangian density may be spelled out as

$$\mathcal{L}_{\text{QCD}} = \sum_{q=1}^{N_f} \bar{\psi}^{(q)}_{a\mathfrak{c}}\big(i\hbar c\gamma^{\mu a}{}_b \nabla_\mu - m_q c^2 \delta^a{}_b\big)\psi^{b\mathfrak{c}}_{(q)} - \frac{\hbar}{4} F^{\mathfrak{a}}{}_{\mathfrak{b}\mu\nu} F_{\mathfrak{a}}{}^{\mathfrak{b}\mu\nu} + \frac{\theta\hbar}{8\pi^2} F^{\mathfrak{a}}{}_{\mathfrak{b}\mu\nu} F_{\mathfrak{a}}{}^{\mathfrak{b}}{}_{\rho\sigma} \epsilon^{\mu\nu\rho\sigma},$$

where Latin and Fraktur indices denote $\mathbb{C}^4_{\text{D}}$ fermion components and $\mathbb{C}^3_{\text{C}}$ colour components, respectively.

Yukawa Couplings and the Measurable $\bar{\theta}$-parameter

The QCD sector of the standard model is an extension of pure QCD with three colours and six quarks. The quarks are partitioned into *up type* and *down type*, and again into three *generations*, each with varying masses and charges under the other components of the standard model gauge group (1.1).

generation	I	II	III
up type	u 2.2 MeV/c^2	c 1.3 GeV/c^2	t 170 GeV/c^2
down type	d 4.7 MeV/c^2	s 0.1 GeV/c^2	b 4.2 GeV/c^2

Figure 1.1: Quark masses in the standard model.

Pure QCD contains a mass term $\bar{\boldsymbol{\psi}}\mathsf{m}\boldsymbol{\psi}$, giving each quark $\boldsymbol{\psi}^{(q)}$ an intrinsic mass m_q. This is *not* the mechanism by which quarks exhibit mass in the standard model—it cannot be, since $\bar{\boldsymbol{\psi}}\mathsf{m}\boldsymbol{\psi}$ is not invariant under axial rotations. Instead, quarks obtain mass via the *Higgs mechanism*, whereby $\boldsymbol{\psi}$ is coupled to the Higgs field H in *Yukawa interaction terms* in the Lagrangian

$$\underset{\sim}{\mathcal{L}}_{\text{mass}} = \Re\big(H\bar{\boldsymbol{\psi}}_+\mathsf{m}\,\boldsymbol{\psi}_-\big) = \frac{1}{2}\big(H\bar{\boldsymbol{\psi}}_+\mathsf{m}\,\boldsymbol{\psi}_- + H^\dagger\bar{\boldsymbol{\psi}}_-\mathsf{m}^\dagger\boldsymbol{\psi}_+\big).$$

The Lagrangian of the standard model QCD sector is thus [6, § 7.6.6]

$$\underset{\sim}{\mathcal{L}}^{\text{SM}}_{\text{QCD}} = \big[\bar{\boldsymbol{\psi}}i\boldsymbol{\gamma}^\mu\nabla_\mu\boldsymbol{\psi} + \Re\big(H\bar{\boldsymbol{\psi}}_+\mathsf{m}\,\boldsymbol{\psi}_-\big)\big]\,\underset{\sim}{\text{vol}} - \langle\underset{\sim}{\boldsymbol{F}}\wedge\star\underset{\sim}{\boldsymbol{F}}\rangle + \frac{\theta}{8\pi^2}\langle\underset{\sim}{\boldsymbol{F}}\wedge\underset{\sim}{\boldsymbol{F}}\rangle\,. \tag{1.13}$$

Under independent axial rotations of each of the quark fields, the Yukawa mass term is invariant provided the quark masses also shift phase. In addition, each independent $\mathrm{U}(1)_A$ rotation generates a corresponding θ-term, due to the chiral anomaly (§ 1.1.4). Thus, the full QCD Lagrangian (1.13) is invariant under transformations of the form

$$\boldsymbol{\psi}_{(q)} \mapsto e^{i\boldsymbol{\gamma}^5 a_q/2}\boldsymbol{\psi}_{(q)}, \qquad m_q \mapsto e^{-ia_q}m_q, \qquad \theta \mapsto \theta + \sum_{q=1}^{N_f} a_q, \tag{1.14}$$

where α_q parametrise the N_f independent $\mathrm{U}(1)_A$ rotations. To aid physical interpretation, these $\mathrm{U}(1)_A$ freedoms are exploited in order to *normalise* the Yukawa mass terms by making the mass phases real.

The fact that the θ-parameter may be redefined by axially rotating the quark fields means that θ is not directly observable. However, this gauge freedom can be fixed by defining

$$\bar{\theta} = \theta + \arg\det\mathsf{m} = \theta + \arg\prod_{q=1}^{N_f} m_q,$$

which is invariant under (1.14), as the two right-hand terms transform by $\pm\sum a_q$. The Lagrangian of the QCD sector of the standard model, complete with the $\bar{\theta}$-term, is thus

$$\underset{\sim}{\mathcal{L}}^{\text{SM}}_{\text{QCD}} = \big[\bar{\boldsymbol{\psi}}i\boldsymbol{\gamma}^\mu\nabla_\mu\boldsymbol{\psi} + \Re\big(H\bar{\boldsymbol{\psi}}_+\mathsf{m}\,\boldsymbol{\psi}_-\big)\big]\,\underset{\sim}{\text{vol}} - \langle\underset{\sim}{\boldsymbol{F}}\wedge\star\underset{\sim}{\boldsymbol{F}}\rangle + \frac{\bar{\theta}}{8\pi^2}\langle\underset{\sim}{\boldsymbol{F}}\wedge\underset{\sim}{\boldsymbol{F}}\rangle\,. \tag{1.15}$$

The Statement of the Strong CP Problem

Proceeding with the assumption that $\bar{\theta} \neq 0$, one finds that the strong force now violates CP symmetry. A physical prediction of the standard model modified with a CP-violating QCD

sector (1.15) is that the neutron is expected to possess an electric dipole moment of approximate magnitude $|d_n| \approx 10^{-18}\, e\,\text{cm}$. In reality, current measurements [14, 15] of the neutron's electric dipole moment yield a tight upper bound of $|d_n| \lesssim 10^{-26}\, e\,\text{cm}$, which in turn implies a stringent constraint on the $\bar{\theta}$-parameter, $|\bar{\theta}| \lesssim 10^{-10}$ [5]. Thus, the $\bar{\theta}$-term is not considered to be part of the standard model.

However, $\bar{\theta}$ can only be zero if apparently unrelated parameters of the standard model perfectly cancel each other: the vacuum angle θ, a QCD parameter; and the quark mass phases $\arg\det \mathsf{m}$, deriving from multiple electroweak parameters. One therefore expects $\bar{\theta}$ to be $\mathcal{O}(1)$ in Nature, and its extremely small value is hence a problem of fine-tuning. The strong CP problem is then the question, "why is $\bar{\theta}$ so small?"

At first sight, the strong CP problem may not appear to be a problem at all. After all, QCD is a theory whose Lagrangian possibly—but not necessarily—admits a term $\propto \langle \underset{\sim}{\boldsymbol{F}} \wedge \underset{\sim}{\boldsymbol{F}} \rangle$ which gives rise to CP-violating interactions in the strong force, predicting an electric dipole moment of the neutron. Empirical data is consistent with the neutron's electric dipole moment (and hence the CP-violating term) being zero. From a phenomenological perspective, it is satisfactory to simply leave the $\bar{\theta}$-term out of the theory's Lagrangian and end the story there. Indeed, a tautological way to 'resolve' the strong CP problem is to simply require that CP be a symmetry of the strong force. However, this only begs the question of why CP symmetry appears to be preserved in some sectors of the standard model while it is broken in others.

Furthermore, there is a strong argument that the inclusion of the CP-violating term is "natural." That is, we lack reason to exclude it on a theoretical basis: it is Lorentz and gauge invariant, etc.; it is an implication of the non-trivial vacuum structure of QCD (instantons); and it arises via the chiral anomaly for fermions. From an empirical perspective, if no CP-violating interactions were observed in Nature, then $\bar{\theta}$ could be justifiably set to zero on the basis of symmetry. However, the weak interaction is explicitly parity-violating.[10] Hence, the fact that the $\bar{\theta}$-term violates CP is not theoretically satisfactory reason for its exclusion.

The strong CP problem differers from other fine-tuning problems in the standard model in the sense that it is of almost no consequence to everyday physics. Variation of the θ-parameter hardly affects nuclear physics at all because its effects are suppressed by the quark masses [16]. On the other hand, variations of the cosmological constant, for example, predict universes drastically different to our own, and similarly for the value of the weak scale, or the quark and lepton masses. Such fine-tuning problems at least have anthropic solutions—but the strong CP problem does not.[11] The strong CP problem is therefore a compelling theoretical indication that the standard model remains incomplete.

1.3 The Massless Quark Solution

The simplest resolution to the strong CP problem is to stipulate that at least one quark is in fact massless. If this were true, then $\det \mathsf{m}$ would vanish, and the parameter $\bar{\theta} = \theta + \arg\det \mathsf{m}$ would be rendered unphysical. The *massless quark solution* is the claim that $\bar{\theta} \approx 0$ because the

[10] In fact, the standard model is asymmetric under all combinations of charge conjugation, C; parity P; and time-reversal T modulo the prevailing combined CPT symmetry.

[11] Given a theory linking the presence of dark matter to the smallness of θ, an anthropic solution may exist if it turns out that dark matter is necessary for, e.g., galaxy formation (investigated in [16]).

up quark is massless, $m_u = 0$.

At first sight, this economical resolution to the strong CP problem appears to be in contradiction with the experimentally determined nonzero masses of the quarks (particularly $m_u \approx 2.2\,\text{MeV}$). However, it was realised in the mid-1980s that the mass of the up quark has *two* contributions in the standard model Lagrangian: not only the Yukawa mass m_u (the 'bare mass') as introduced above, but also a non-perturbative contribution m_{eff} from topological effects (i.e., instantons) [17]. Only the bare quark masses contribute to the value of $\bar{\theta}$ via the quark mass matrix m. Importantly, it was plausible that this secondary source of the up quark's mass could be of order $m_{\text{eff}} \approx 2.2\,\text{MeV}$, allowing m_u to vanish while still preserving the up quark's overall mass.

The massless up quark hypothesis remained controversial until *lattice gauge theory* had advanced sufficiently to made numerical simulations of non-perturbative effects in QCD possible. Strong consensus that the instanton contribution m_{eff} is not sufficiently large was reached in late 2019 [17–19]. Instead, another mechanism is required to explain the smallness of the $\bar{\theta}$-parameter.

2 The Peccei–Quinn Axion Solution

The most famous resolution to the strong CP problem is the Peccei–Quinn theory of the *axion*, first proposed in 1977 [20]. In essence, the axion solution involves extending the standard model in order to promote the original $\bar{\theta}$-parameter to a field $\bar{\theta}(x) = \bar{\theta}_{\text{SM}} + a(x)$ in such a way that it is dynamically relaxed to zero, $\bar{\theta} \to 0$. In doing so, a new massive boson described by the pseudoscalar *axion field* $a(x)$ is necessarily introduced. There are different inequivalent ways to extend the standard model to realise the axion solution, but all Peccei–Quinn axion models share the same necessary features:

- The extended Lagrangian $\underset{\sim}{\mathcal{L}}_{\text{PQ}}$ possesses an additional global chiral symmetry $\text{U}(1)_{\text{PQ}}$. The exact action of this *Peccei–Quinn symmetry* depends on the particular axion model, and is not of central importance. The defining feature of $\text{U}(1)_{\text{PQ}}$ is that it is chiral, so that a $\text{U}(1)_{\text{PQ}}$ transformation by α radians anomalously induces a θ-term $\alpha \left\langle \underset{\sim}{\boldsymbol{F}} \wedge \underset{\sim}{\boldsymbol{F}} \right\rangle$ in the effective Lagrangian (via the chiral anomaly).
- The Peccei–Quinn symmetry $\text{U}(1)_{\text{PQ}}$ is *spontaneously broken*, and the single resulting Nambu–Goldstone boson is named *the axion field*, $a(x)$. Being a Nambu–Goldstone boson, the axion transforms as $a(x) \mapsto a(x) + \alpha$ under a $\text{U}(1)_{\text{PQ}}$ rotation of α radians. The chiral anomaly results in a potential for the axion $a(x)$ (giving rise to axion mass) with a potential minimum occurring where $a(x) = -\bar{\theta}_{\text{SM}}$.

If the extra $\text{U}(1)_{\text{PQ}}$ symmetry was indeed an exact gauge symmetry, then the strong CP problem is trivially solved, because a $\text{U}(1)_{\text{PQ}}$ rotation by $\bar{\theta}$ radians cancels the $\bar{\theta}$-term in the effective Lagrangian, meaning the dynamics of the theory are equivalent to one in which $\bar{\theta} = 0$. However, the main result of Peccei and Quinn [20] is that $\text{U}(1)_{\text{PQ}}$ need not be an exact symmetry of the theory: if $\text{U}(1)_{\text{PQ}}$ is spontaneously broken, then $\bar{\theta}$ is still driven to zero because it obtains a potential from the chiral anomaly with a minimum at $\bar{\theta} = 0$, [21].

2.1 Axion Models

Different extensions to the standard model which fulfil the requirements of a $\text{U}(1)_{\text{PQ}}$ symmetry have been proposed, resulting in phenomenologically distinct classes of axion, with varying masses and couplings to various standard model particles. When the Peccei–Quinn symmetry was first described [20], the proposed model predicted strongly interacting, visible axions which were experimentally refuted within a decade. Since then, the axion has remained a possibility through models compatible with very weakly interacting, 'invisible' particles, in particular as dark matter candidates [22].

A Toy Axion Model

To illustrate the Peccei–Quinn mechanism explicitly, consider a minimal toy axion model, which involves the addition of two fields to the standard model: a complex scalar field Φ known as the *parent field* (so named since its phase, after spontaneous breaking, is the axion field); and an additional fermion $\boldsymbol{q}$. The extended Lagrangian takes the form

$$\underset{\sim}{\mathcal{L}}_{\mathrm{PQ}} = \underset{\sim}{\mathcal{L}}^{\bar{\theta}}_{\mathrm{SM}} + [\langle \underset{\sim}{\mathrm{d}}\Phi, \underset{\sim}{\mathrm{d}}\Phi\rangle + \mathbb{R}(\bar{\boldsymbol{q}}_+ \Phi \boldsymbol{q}_-)]\underset{\sim}{\mathrm{vol}} + \underset{\sim}{\mathcal{L}}_q, \tag{2.1}$$

where $\underset{\sim}{\mathcal{L}}^{\bar{\theta}}_{\mathrm{SM}}$ is the standard model Lagrangian (including the $\bar{\theta}$-term); $\langle \underset{\sim}{\mathrm{d}}\Phi, \underset{\sim}{\mathrm{d}}\Phi\rangle$ is a kinetic term for the parent field; $\mathbb{R}(\bar{\boldsymbol{q}}_+ \Phi \boldsymbol{q}_-)$ is a Yukawa coupling term; and $\underset{\sim}{\mathcal{L}}_q$ stands for any other terms involving the new fermion $\boldsymbol{q}$. The action of $\mathrm{U}(1)_{\mathrm{PQ}}$ on these fields is

$$\Phi \mapsto e^{i2\alpha}\Phi, \qquad \boldsymbol{q} \mapsto e^{i\gamma^5\alpha}\boldsymbol{q} \quad \text{or} \quad \begin{cases} \bar{\boldsymbol{q}}_+ \mapsto e^{i\alpha}\bar{\boldsymbol{q}}_+ \\ \boldsymbol{q}_- \mapsto e^{i\alpha}\boldsymbol{q}_- \end{cases},$$

which indeed leaves $\langle \underset{\sim}{\mathrm{d}}\Phi, \underset{\sim}{\mathrm{d}}\Phi\rangle$ and $\mathbb{R}(\bar{\boldsymbol{q}}_+ \Phi \boldsymbol{q}_-)$ invariant. However, since $\boldsymbol{q}$ undergoes an axial rotation, the entire (effective) Lagrangian $\underset{\sim}{\mathcal{L}}_{\mathrm{PQ}}$ is only invariant with the simultaneous subtraction of a term $\alpha \langle \underset{\sim}{\boldsymbol{F}} \wedge \underset{\sim}{\boldsymbol{F}}\rangle$ arising from the chiral anomaly. Therefore, the entire action of $\mathrm{U}(1)_{\mathrm{PQ}}$ is to transform $\bar{\theta} \mapsto \bar{\theta} - \alpha$, as well as the fields. At this stage, the theory predicts CP violation in the strong sector in areas where the axions and instantons interact such that $\bar{\theta}(x) \neq 0$.

The final component of the axion solution is to make the parent field Φ spontaneously break. This may be done by adding a "Mexican hat" potential to $\underset{\sim}{\mathcal{L}}_{\mathrm{PQ}}$ of the form

$$V(|\Phi|) = \lambda(|\Phi|^2 - f_a^2)^2,$$

where the parameter f_a is interpreted as the *axion scale*: the energy below which axion dynamics are relevant. Below this energy scale, the parent field Φ relaxes to some non-unique minimum of the form

$$\Phi = |\Phi| e^{i \arg \Phi} = f_a e^{ia/f_a},$$

where a varies across space. At sufficiently low energies, the effective degree of freedom is the phase of Φ, not Φ itself.[1] The resulting phase $a(x)$ is named the Nambu–Goldstone boson associated with the spontaneous breaking of $\mathrm{U}(1)_{\mathrm{PQ}}$ by the parent field Φ, and is identified as the axion field. After spontaneous breaking, the Yukawa term involves a complex mass, which can be normalised by a $\mathrm{U}(1)_{\mathrm{PQ}}$ rotation by $-a/f_a$

$$\mathbb{R}(\bar{\boldsymbol{q}}_+ \Phi \boldsymbol{q}_-) = f_a \mathbb{R}(\bar{\boldsymbol{q}}_+ e^{ia/f_a} \boldsymbol{q}_-) \mapsto f_a \mathbb{R}(\bar{\boldsymbol{q}}_+ \boldsymbol{q}_-).$$

This axial rotation of $\boldsymbol{q}$ induces a corresponding rotation of $\theta \mapsto \theta - a/f_a$, so that the (effective, normalised) Lagrangian (2.1) becomes

$$\underset{\sim}{\mathcal{L}}_{\mathrm{PQ}} = \underset{\sim}{\mathcal{L}}_{\mathrm{SM}} + [f_a^2 \langle \underset{\sim}{\mathrm{d}}a, \underset{\sim}{\mathrm{d}}a\rangle + f_a \mathbb{R}(\bar{\boldsymbol{q}}_+ \boldsymbol{q}_-)]\underset{\sim}{\mathrm{vol}} + \underset{\sim}{\mathcal{L}}_q + \left(\bar{\theta} - \frac{a}{f_a}\right) \frac{1}{8\pi^2} \langle \underset{\sim}{\boldsymbol{F}} \wedge \underset{\sim}{\boldsymbol{F}}\rangle, \tag{2.2}$$

[1] We assume that λ is sufficiently large that the radial degree of freedom ρ in $\Phi = (f_a + \rho)e^{ia/f_a}$ can be neglected at the energy scale f_a.

after spontaneous symmetry breaking of Φ.

Peccei and Quinn showed that the last term in (2.2) provides an effective potential $V(a)$ for the axion whose minimum occurs at $V(\bar{\theta} f_a) = 0$, giving the axion a vacuum expectation value $\langle a \rangle = \bar{\theta} f_a$ and a mass $m_a = \partial^2 V / \partial a^2 \,|_{\langle a \rangle}$. The axion field a is not physical, since $a \mapsto a + \alpha$ under a $\mathrm{U}(1)_{\mathrm{PQ}}$ rotation; however, the deviation from the expectation value $a_{\mathrm{phys}} := a - \langle a \rangle$ is physical. Expressing the Lagrangian in terms of the physical axion field reveals that the $\bar{\theta}$-term vanishes, thus solving the strong CP problem [21]. Focusing on terms involving a_{phys}, the effective Lagrangian is

$$\underset{\sim}{\mathcal{L}}_{\mathrm{PQ}} = \underset{\sim}{\mathcal{L}}_{\mathrm{SM}} + \underset{\sim}{\mathcal{L}}'_q + \left[f_a^2 \left\langle \underset{\sim}{\mathrm{d}} a_{\mathrm{phys}}, \underset{\sim}{\mathrm{d}} a_{\mathrm{phys}} \right\rangle + \frac{1}{2} m_a^2 a_{\mathrm{phys}}^2 \right] \underset{\sim}{\mathrm{vol}} + \frac{a_{\mathrm{phys}}}{f_a} \frac{1}{8\pi^2} \left\langle \underset{\sim}{\boldsymbol{F}} \wedge \underset{\sim}{\boldsymbol{F}} \right\rangle , \tag{2.3}$$

where $\underset{\sim}{\mathcal{L}}'_q$ includes all terms involving $\boldsymbol{q}$. Different axion models give rise to different $\underset{\sim}{\mathcal{L}}'_q$, but otherwise share the Lagrangian (2.3). The axion mass m_a depends on the axion scale f_a as

$$m_a \approx 6\,\mathrm{eV} \left(\frac{10^6 \mathrm{GeV}}{f_a} \right)$$

and is otherwise independent of the axion model, using accepted values of standard model parameters [23].

Figure 2.1: Axion–photon and axion-gluon interaction vertices.

The precise axion–matter interactions entering through the term $\underset{\sim}{\mathcal{L}}'_q$ are model dependent, but generally have coupling strengths inversely proportional to the axion scale, f_a [22]. However, all Peccei–Quinn axions interact with the gauge field though the last term in the Lagrangian (2.3). The last term is proportional to $\langle \underset{\sim}{\boldsymbol{F}} \wedge \underset{\sim}{\boldsymbol{F}} \rangle = \underset{\sim}{F} \wedge \underset{\sim}{F} + \langle \underset{\sim}{\boldsymbol{G}} \wedge \underset{\sim}{\boldsymbol{G}} \rangle$, where $\underset{\sim}{\boldsymbol{F}} = \underset{\sim}{F} \oplus \underset{\sim}{\boldsymbol{G}}$ is the total gauge field split into the electromagnetic $\underset{\sim}{F}$ and gluonic $\underset{\sim}{\boldsymbol{G}}$ sectors. In perturbation theory, this corresponds to a Feynman vertex in which an axion and two photons $a\gamma\gamma$, or an axion and two gluons agg meet, as in figure 2.1a.

The $a\gamma\gamma$ interaction is strong where $\underset{\sim}{F} \wedge \underset{\sim}{F} = (\vec{E} \cdot \vec{B}) \underset{\sim}{\mathrm{vol}}$ is large. This implies that axions may be generated from photons and vice-versa in the presence of strong electromagnetic fields. Near a charged particle such as an electron, where the field is concentrated in a Coulomb potential, the $\gamma \leftrightarrow a$ conversion is best viewed as a scattering process, $\gamma + e^{\pm} \to e^{\pm} + a$, and is named the *Primakoff effect* [5, § 93.1.3]. The agg vertex gives rise to interactions between axions and strongly-interacting hadronic matter (particularly pions and kaons) [23]. These interactions are universal to all axion models. For *leptonic* axion models with an electron interaction term $\underset{\sim}{\mathcal{L}}_q$, there is also an axion–electron vertex. This enables a Compton scattering process in addition to Primakoff scattering, both of which are shown in figure 2.2.

(a) Primakoff

(b) Compton

Figure 2.2: Dominant axion processes with electrons (or positrons with arrows reversed) [23]. Compton scattering only occurs for leptonic axions.

2.1.1 The Original Peccei–Quinn–Weinberg–Wilczek Axion

The first proposed axion model, the Peccei–Quinn–Weinberg–Wilczek (PQWW) axion, [24] implements the $\mathrm{U}(1)_{\mathrm{PQ}}$ symmetry by supposing that the standard model possesses two Higgs fields H_1 and H_2 which couple differently to up and down quarks, instead of just one which couples to all quarks. Denoting by $\boldsymbol{\psi}_{\pm} = \boldsymbol{\psi}_{\pm}^{(u)} \oplus \boldsymbol{\psi}_{\pm}^{(d)}$ the left- and right-handed quarks arranged into up-type and down-type parts, the two Higgs fields

$$\mathcal{L}_{\text{Yukawa}} = \mathfrak{R}\big(H_1 \bar{\boldsymbol{\psi}}_{+} \mathsf{m}_u \boldsymbol{\psi}_{-}^{(u)} + H_2 \bar{\boldsymbol{\psi}}_{+} \mathsf{m}_d \boldsymbol{\psi}_{-}^{(d)}\big),$$

where m_u and m_d are (non-square) matrices of Yukawa coupling constants. The first Higgs field H_1 gives mass to the up-type quarks, and H_2 to down-type quarks. The presence of the two Higgs fields lets $\mathcal{L}_{\text{Yukawa}}$ be invariant under two independent chiral rotations of the up and down quarks, hence accomplishing the additional $\mathrm{U}(1)_{\mathrm{PQ}}$ symmetry.

In this model, the axion scale f_a is necessarily on the order of the electroweak scale, $f_{\mathrm{EW}} \approx 246\,\mathrm{GeV}$ (which is the vacuum expectation value of the Higgs field). The resulting axions are too massive ($m_a \approx 25\,\mathrm{keV}$) and too strongly interacting to agree with experiment. In particular, the PQWW axion is ruled out by the non-observation of the kaon decay $K^+ \to \pi^+ + a$ in electron beam-dump experiments[2] [21, 25]. Any successful axion model must have a higher energy scale f_a (i.e., lighter mass m_a) to be compatible with the constraints which excluded the PQWW model [24].

2.1.2 Light Invisible Axion Models

Axions with a larger energy scale $f_a \gg f_{\mathrm{EW}}$ are light ($m_a \sim 1/f_a$), long lived (e.g., the rate of $a \to 2\gamma$ goes as $(f_a)^5$) and weakly interacting (couplings generally are suppressed by $1/f_a$). In particular, their electromagnetic interactions are weak, rendering them invisible [21, 24]. Such axion models generally fall into two classes:

- *The Kim–Shifman–Vainshtein–Zakharov (KSVZ) Axion*
  The KSVZ model introduces an additional massive quark $\boldsymbol{q}$ as well as the parent field Φ.

- *The Dine–Fischler–Srednicki–Zhitnitsky (DFSZ) Axion*
  The DFSZ model contains two Higgs doubles like the PQWW model, but also contains a

[2] Beam-dump experiments involve firing high-energy protons into a high-absorption material in order to isolate neutral particles which are created from the decelerating protons and which propagate through the absorber.

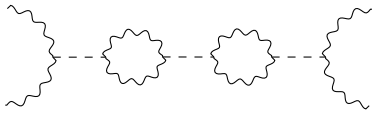

Figure 2.3: Example axion–photon oscillation diagram.

separate scalar parent field Φ. The DFSZ model contains the additional mass term

$$\mathfrak{R}\big(H_2 \bar{\boldsymbol{\varphi}}_+ \mathsf{m}_e \boldsymbol{\varphi}_-^{(e)}\big),$$

which is a Yukawa coupling between one of the Higgs doublets H_2, the left-handed leptons $\boldsymbol{\varphi}_+$, and the right-handed electron $\boldsymbol{\varphi}_-^{(e)}$, where m_e is the (non-square) matrix of Yukawa couplings. The $(H_2, \boldsymbol{\varphi}_-^{(e)})$ coupling gives rise to an axion–electron vertex, and hence DFSZ axions are leptonic and undergo Compton scattering (figure 2.2b).

2.2 Laboratory Bounds on Axions

Axions have never been observed [5, § 91]. Since the dynamics of the axion can be essentially parametrised by the mass m_a and coupling $g_{a\gamma\gamma}$ (and also the axion–electron coupling g_{aee} for leptonic axions), negative axion detection experiments serve to exclude specific regions of $(m_a, g_{a\gamma\gamma})$ parameter space. An exclusion plot of constraints from major laboratory experiments discussed in this section is shown in figure 2.4.

Direct Axion Production

In the presence of a strong, uniform electromagnetic field with $\vec{E} \parallel \vec{B}$ so that $\underset{\sim}{F} \wedge \underset{\sim}{F} \propto \vec{E} \cdot \vec{B}$ is large, the $a\gamma\gamma$ interaction is best viewed as an axion–photon oscillation, analogous to neutrino flavour oscillation (see figure 2.3) [5, § 91.3.1]. This suggests a scheme for detecting axions by 'shining light through walls,' whereby a laser is beamed at an optical barrier in a strong magnetic field, and any generated axions (freely passing through the barrier) which oscillate back into photons on the other side are detected. The first such experiment was performed in 1992 with a 3.7 T superconducting magnet over a length of 4.4 m, finding that $|g_{a\gamma\gamma}| < 6.7 \times 10^{-7}\,\mathrm{GeV}^{-1}$ for axions lighter than 1 meV [26]. The current best limit from light-shining-though-walls (LSW) experiments, $|g_{a\gamma\gamma}| < 3.5 \times 10^{-8}\,\mathrm{GeV}^{-1}$, was obtained in 2015 with two 9 T Large Hadron Collider dipole magnets [27], with

Another consequence of axion–photon oscillation is that light suffers from dichroism[3] and birefringence in a strong, uniform magnetic field $\vec{B}$. The dichroism arises since the polarisation of light parallel to the magnetic field, $\vec{E}_\parallel$, where $\vec{E} \cdot \vec{B}$ is large, undergoes $a\gamma\gamma$ oscillation and is depleted while $\vec{E}_\perp$ remains unaffected [5, § 91.3.2]. A similar process results in birefringence, where linearly polarised light becomes elliptically polarised, but the experimental limits from dichroism experiments are stronger: $|g_{a\gamma\gamma}| < 3.6 \times 10^{-7}\,\mathrm{GeV}^{-1}$ for sub-meV axions [28]. In 2006, a collaboration reported a false positive in a vacuum dichroism experiment, detecting axions with $m_a \approx 1.3\,\mathrm{meV}$ and $g_{a\gamma\gamma} = 3 \times 10^{-6}\,\mathrm{GeV}^{-1}$, but the detection was attributed to instrumental artefacts two years later [29].

[3] Generally, *dichroism* is the dependence of a medium's optical absorption on the polarization of light. In this

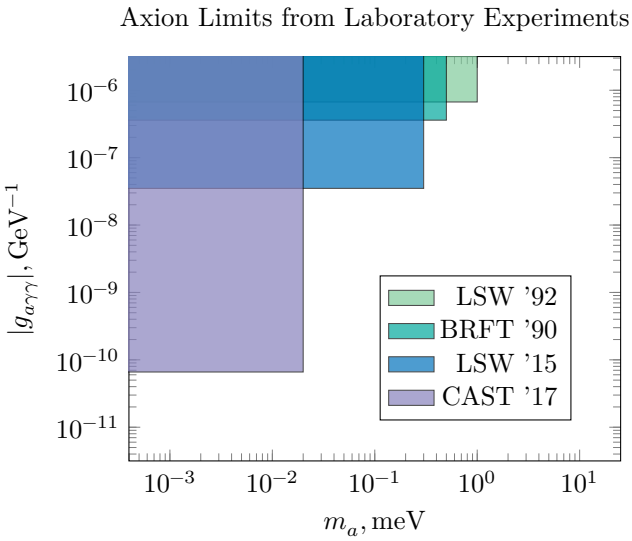

Figure 2.4: Limits on axion mass m_a and photon coupling $g_{a\gamma\gamma}$ from laboratory experiments. Shaded regions are experimentally excluded. (LSW '92 [26], BRFT '90 [28], LSW '15 [27], CAST '17 [32])

Detection of Solar Axions

We also expect low-mass, weakly interacting particles to be produced in the astrophysical plasmas found inside stars. Efforts have therefore been made to detect axions of solar origin. In a strong macroscopic $\vec{B}$ field, axions (or indeed, any axion-like particles with a two-photon vertex) may be converted into x-rays via a reverse Primakoff process, $a + e^{\pm} \to e^{\pm} + \gamma$. With standard solar models, the expected axion flux on Earth is

$$\varphi_{\odot} \approx 3.8 \times 10^{11} \, (10^{10} \, \mathrm{GeV} \, |g_{a\gamma\gamma}|)^2 \, \mathrm{cm}^{-2} \, \mathrm{s}^{-1},$$

which is significant and detectable for axions with $|g_{a\gamma\gamma}| \gtrsim 10^{-11} \, \mathrm{GeV}^{-1}$ [30].

The Tokyo axion helioscope, initially constructed in 1995 and then continuously upgraded, utilised a 4 T superconducting magnet on a sun-tracking mount to detect solar axions. In 2008, it was reported that its negative results implied a limit $|g_{a\gamma\gamma}| < 6 \times 10^{-10} \, \mathrm{GeV}^{-1}$ for sub-meV axions [31]. More recently, the CERN Axion Solar Telescope used a decommissioned 9 T LHC dipole magnet in a similar helioscope apparatus to establish a stronger limit: $|g_{a\gamma\gamma}| < 6.6 \times 10^{-11} \, \mathrm{GeV}^{-1}$, though only for axions of mass $m_a < 0.03 \, \mathrm{eV}$ [32].

Further laboratory experiments to detect axions are under way (e.g., [33], [34, § 5.3]), but direct detection on Earth is not the only experimental probe available. Axions may have significant roles in stellar evolution, the universe's early history and as dark matter, placing them in the domain of cosmology.

case, the medium is the vacuum in regions of large $\underset{\sim}{F} \wedge \underset{\sim}{F}$.

3 Axions in Cosmology

With the advent of precision cosmology in the 21st century, particle physicists can use the entire universe as a laboratory for ever more sensitive experiments. Depending on the strength and variety of their interactions with other matter, axions may have significant implications for the evolution of the universe and of astrophysical structures, leaving behind detectable signatures. Of special interest is the axion's promise as a dark matter candidate. This section is a review of the current status of the axion in cosmology; its astrophysical predictions and observational constraints. Primary sources are Cadamuro, Hannestad *et al.*, [23, 2011] and Irastorza and Redondo [34, 2018], along with sections of the Particle Data Group's *Review of Particle Physics* [5, 2020].

3.1 The Standard Cosmological Picture

Many cosmological tests of axion-like particles involve predicting relative particle abundances in the universe at various epochs, such as the baryon-to-photon or neutrino-to-photon ratios. Such arguments begin with a minimal 'thermodynamical' model of the universe as a homogeneous, isotropic, expanding background upon which different particle species exist in uniform thermal equilibrium. The universe's macrostate is characterised by each species' abundance and energy distribution, and is characterised by a cosmological temperature, T, quantifying average energy density. Interactions and processes between species, which at any time may depend on present particle abundances and energies, define differential relations which can be solved to determine the abundance of each species at any point in the universe's evolution. In general, this is expressed by the *Boltzmann transport equation*, describing the statistical behaviour of a thermodynamic system out of equilibrium [34, § 3]

The harsh assumptions of isotropy and homogeneity specify a spacetime with a Friedmann–Lemaître–Robertson–Walker (FLRW) metric,

$$\underset{\sim}{g} = -c^2 \underset{\sim}{\mathrm{d}}t^2 + a(t)^2 \left(\frac{\underset{\sim}{\mathrm{d}}r^2}{1 - kr^2} + r^2 \underset{\sim}{\Theta} \right)$$

where $k \in \{+1, 0, -1\}$ reflects the type of spatial curvature, $\underset{\sim}{\Theta} = \underset{\sim}{\mathrm{d}}\theta^2 + \sin\theta \, \underset{\sim}{\mathrm{d}}\varphi^2$ is the metric of the unit sphere, and $\underset{\sim}{\mathrm{d}}x^2 \equiv \underset{\sim}{\mathrm{d}}x \otimes \underset{\sim}{\mathrm{d}}x$. Standard cosmological models are spatially flat with $k = 0$. The scale factor $a(t)$ describes the cosmological evolution of the universe and defines the *Hubble parameter,* $H := \dot{a}/a$, or cosmic expansion rate. The equations of general relativity determine $a(t)$ uniquely in terms of the mass–energy content of the universe.

Terminology from cosmology

- *thermalisation* — the process of a particle species reaching thermal equilibrium (i.e., uniform energy and abundance) over cosmological scales, mitigated by energy-diffusing self-interactions or processes with other species.
- *freeze-out* — the point beyond which the rates of thermalising processes become negligible due to cooling cosmological temperature or accelerating cosmic expansion. Freeze-out results in persistent non-equilibrium distributions of a particle species, analogous to a change of phase from a gas to a cooler condensate.

If the rate Γ of a particle interaction is large (corresponding to large probability per unit spacetime volume for the interaction to occur), then it will provide a mechanism for thermalisation of the species involved—or in the case of production and decay processes, will drive the species to abundance or extinction. If the rate is smaller than the rate of cosmic expansion, $\Gamma \ll H$, then the interaction or process will freeze-out and become negligible. Freeze-out may also result from the cosmological temperature T being lower than a processes' threshold energy. If a species' most dominant interactions freeze-out, then it becomes thermally isolated from other matter and its abundance remains fixed.

3.2 Axion Interactions and Processes

Axions spontaneously decay into photons via the axion-photon vertex at a well-known rate

$$\Gamma_{a\to 2\gamma} = \frac{g_{a\gamma\gamma}^2 m_a^3}{64\pi} \approx 10^{-24}\,\mathrm{s}^{-1}\left(\frac{m_a}{\mathrm{eV}}\right)^5,$$

where the coupling strength $g_{a\gamma\gamma}$ can be approximately written in terms of the mass m_a (which introduces an overall $\mathcal{O}(1)$ dependence on the particular axion model) [23]. Thus, for axions to exist in significant abundance in the present epoch, they must be sufficiently light, or else the decay process dominates. If they are too light, $m_a < 18\,\mathrm{eV}$, than the axion half-life exceeds the age of the universe. An inverse decay process $2\gamma \to a$ is also possible, with a rate $\Gamma_{2\gamma\to a} \propto 1/T$ increasing as the universe cools. This implies that axions may recouple to photons at late stages of the universe's evolution [24].

Axions also interact by the strong force with hadronic matter via the gluon-axion vertex, giving rise to the Primakoff (and, for leptonic axions, Compton) electron scattering processes shown in figure 2.2. When the universe is sufficiently hot, $T \gg m_a$, the rate of Primakoff scattering $\Gamma_\mathrm{P} \propto g_{a\gamma\gamma}^2 n_e$ is proportional to the number density of electrons-plus-positions, n_e [23]. For leptonic (e.g., DFSZ) axions, Cadamuro *et al.* [23] approximate the rate of Compton scattering as

$$\Gamma_\mathrm{C} \propto \frac{g_{aee}^2 n_e}{\max\{T^2, m_a^2\}}.$$

These scattering processes, along with photon decay and inverse decay, are the dominant interactions relevant to the cosmological arguments employed by Cadamuro *et al.* The freeze-out temperatures of these processes vary non-linearly with axion mass, summarised in figure 3.2.

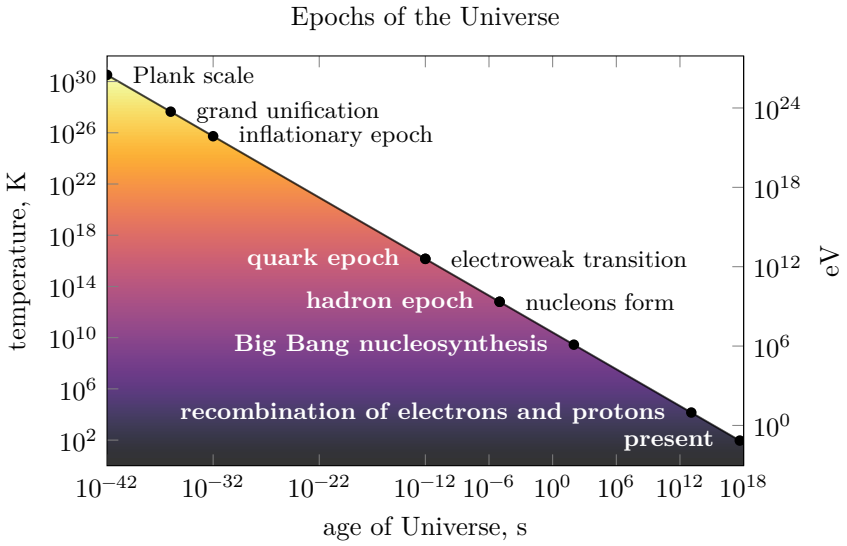

Figure 3.1: Temperature T of the universe from the Big Bang to the present, with major epochs indicated.

By the analysis of Cadamuro *et al.*, axions remain in thermal equilibrium at all temperatures for $m_a \gtrsim 20\,\text{keV}$ ($m_a \gtrsim 10\,\text{keV}$ for leptonic axions), while lighter axions freeze-out around $T \sim 10^2\,\text{keV}$ before eventually recoupling by inverse decay when the universe cools enough. Extremely light axions, $m_a \lesssim 200\,\text{eV}$, remain thermally isolated from all matter forever after recombination where $T \lesssim 10^5\,\text{K}$.

3.2.1 Constraints from Big Bang Nucleosynthesis

A few minutes after the beginning of time, when the universe cooled to $\sim 10^{10}\,\text{K}$ during *Big Bang nucleosynthesis* (BBN), protons and neutrons began to bind together to form light atomic nuclei. The primary fusion reactions that occurred are:

$$\begin{aligned} \mathrm{p} + \mathrm{n} &\to {}^2\mathrm{H} + \gamma \\ \mathrm{p} + {}^2\mathrm{H} &\to {}^3\mathrm{He} + \gamma \\ {}^2\mathrm{H} + {}^2\mathrm{H} &\to {}^3\mathrm{He} + \mathrm{n}, {}^3\mathrm{H} + \mathrm{p} \\ {}^3\mathrm{He} + {}^2\mathrm{H} &\to {}^4\mathrm{He} + \mathrm{p} \\ &\vdots \end{aligned}$$

Photons are involved in the production of deuterium ${}^2\mathrm{H}$ and the lightest isotopes of helium, but not in the fusion of heavier nuclei. An analysis of the reaction rates provides a relationship between the photon abundance n_γ and the abundance of light nuclei (i.e., baryons, n_B). A larger initial baryon–photon ratio $\eta = n_B/n_\gamma$ corresponds to more efficient production of deuterium and ultimately to a larger helium ${}^4\mathrm{He}$ abundance in the present. Relative element abundances in the present epoch constrain the value of the initial baryon–photon ratio to $\eta = (6.2 \pm 0.4) \times 10^{-10}$ [5, § 24.4].

Equipped with a cosmological model of axion interactions, Cadamuro *et al.* solve the associated Boltzmann equation for the axion abundance n_a over time as a function of mass m_a. At later epochs, the axions eventually decay by $a \to 2\gamma$, increasing the entropy and abundance

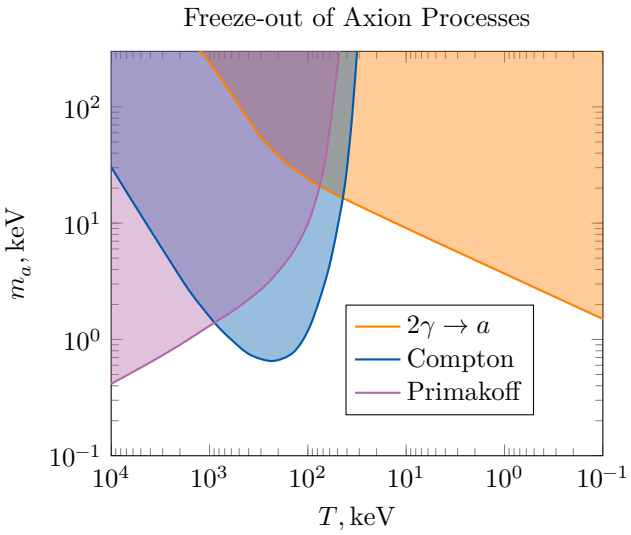

Figure 3.2: Axion decoupling and recoupling with varying cosmological temperature, as reported in [23]. The evolution of the universe progresses from left-to-right, in the direction of *decreasing* temperature. Shaded regions indicate periods where each processes occurs spontaneously and axions are thermalised. Each horizontal line represents a possible timeline of axion freeze-out and recombination. The upper edge of the plot is the line $m_a = 300\,\text{keV}$.

of photons in proportion to n_a at late times. This is measurable as an increase of the temperature of the cosmic microwave background (CMB), or equivalently, as a decrease in abundance of baryons and neutrinos relative to the CMB. This can be compared to observation and used to limit the photon excess due to axion decay, in turn limiting n_a, which constrains the axion mass m_a. Cadamuro *et al.* find that the predicted deuterium abundance is reduced two standard deviations below its observed value for axion masses less than 300 keV, thus obtaining the limit

$$m_a \gtrsim 300\,\text{keV} \tag{3.1}$$

at $2\sigma = 97\,\%$ confidence [23]. Axions within this bound decay into photons sufficiently quickly so as not to deviate the outcome of BBN from observation. Such heavy axions are thermalised at all epochs, even purely hadronic axions which do not undergo Compton scattering (see figure 3.2). The limit (3.1) therefore applies to all axion types, and is far more restrictive than laboratory constraints.

3.2.2 Constraints from Stellar Evolution

Galaxies consist of *globular clusters* (GCs), each typically consisting of between 10^5 and 10^7 stars. A typical galaxy like our own hosts hundreds of GCs, each one a gravitationally self-contained island of stars. The population of a GC can be partitioned into different branches by stage of stellar evolution, as in figure 3.3. Young stars exist in the main sequence and approach the *red giant branch* (RGB) as they burn hydrogen fuel and produce helium. Red giants approach the helium fusion phase as their cores become He-dense and H-depleted. The ignition of He-fusion in late-stage red giants causes an immediate rise in temperature and rate of He-fusion, transitioning the star into a new equilibrium state in the *horizontal branch* (HB) [34, § 3.2]. The parameter $R := N_{\text{RGB}}/N_{\text{HB}}$ is defined as the population ratio between the horizontal and red giant branches in a given GC, and is a useful observable for testing models of stellar evolution.

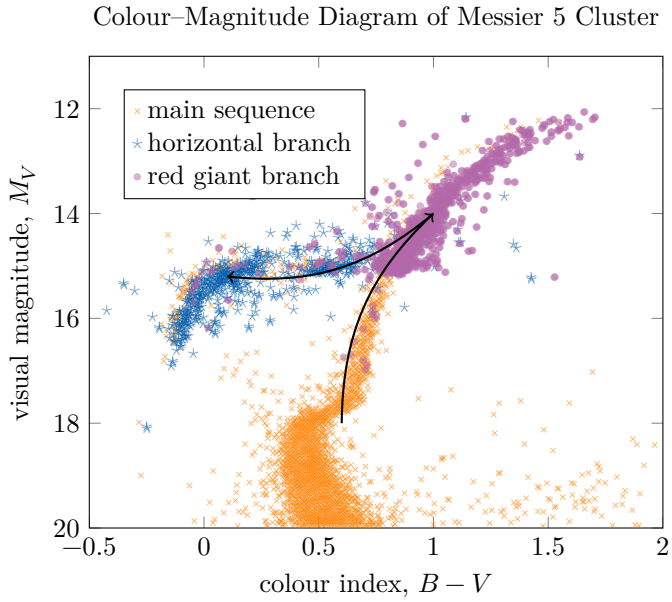

Figure 3.3: An example colour-magnitude plot for the Messier 5 globular cluster, showing main branches of stellar evolution. The visual magnitude encodes luminosity (increasing upwards) and the colour index encodes surface temperature (increasing right-to-left). Stellar evolution flows in the direction of the arrows. (Data from [37].)

The alleged production of axions in stars has implications for stellar evolution. Specifically, axion production via the Primakoff process $\gamma + A \to A + a$ in stellar plasma results in a hotter core temperature and faster burning of helium fuel in red giants. This affects the relative durations of the RGB and HB stages. The observed population of stars in each stage of the stellar sequence is statistically proportional to the stage's average lifetime, and thus the observable R parameter measures relative stage lifetimes, $R = N_{\text{RGB}}/N_{\text{HB}} = T_{\text{RGB}}/T_{\text{HB}}$. Thus, the axion–photon interaction strength may be constrained by comparing the predicted R-parameter to those observed in local GCs. A recent survey [35, 36] of 39 of GCs reports the constraint $|g_{a\gamma\gamma}| < 6.5 \times 10^{-11}\,\text{GeV}^{-1}$ at 95 % confidence [34, § 3.2]. Stronger interacting axions cause horizontal branch stars to burn too quickly, reducing the R-parameter below the observed window.

3.3 Outlook and Conclusion

It is only a possibility that axions exist, and if they do, they remain in the ever-shrinking areas of parameter space (a modern exclusion plot shown in figure 3.4). Despite this, axion phenomenology is earning increasingly more attention. Many more cosmological tests other than those mentioned in this report exist, and experiments are currently under way to further explore the axion parameter space [1, 33, 34]. Arguably, the reason for its popularity is that the axion conceivably plays a central role in open problems in more than one area of cosmology.

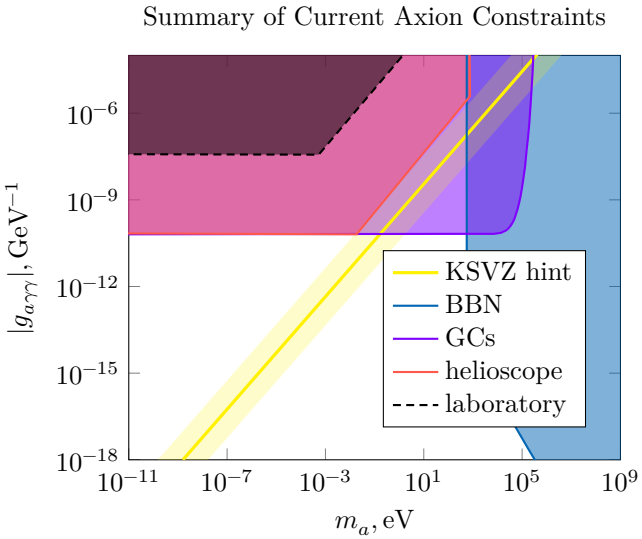

Figure 3.4: Summary of up-to-date (m_a, $g_{a\gamma\gamma}$)-space constraints for the axion. Shaded regions are excluded experimentally. The "hint" line indicates where KSVZ axions would be found. Many more experimental constraints exist; see [1, 5, 34] for more detailed exclusion plots.

A Solution to Dark Matter, Inflation and Baryogenesis?

Most notably, the axion exhibits all the necessary features for a cold dark matter candidate. There are several proposed theoretical frameworks by which the axion comes to contribute (in some cases, totally) to the present dark matter density, estimated to be $\rho_{CDM} \approx 0.45\,\text{GeV}\,\text{cm}^{-3}$ [1, § 5.3]. One such framework is the *misalignment mechanism*, whereby the present day energy density ρ_{CDM} is stored in the zero modes of the $\bar{\theta}$-vacuum angle as it continues to undergo lightly damped oscillation about the potential minimum at $\bar{\theta} = 0$ after acquiring a large nonzero angle after inflation [1, § 5.3]. Assuming axions as solely responsible for dark matter, the axion number density would be $n_a \approx 4.5 \times 10^{14} \left(\frac{\mu\text{eV}}{m_a}\right) \text{cm}^{-3}$. There are numerous active *haloscope experiments* designed to detect ambient axions, making use of coherence effects to overcome their incredibly weak interactions: the *Axion Dark Matter eXperiment* (ADMX) [38] is the longest standing [34, § 7].

Inflation is the hypothesized period of rapid cosmic expansion in the early universe, invoked to explain certain observed features of the cosmos. Degrees of freedom known as *inflatons* drive inflation until they reach a potential minimum. The inflaton potential must be very flat compared to the Hubble scale c/H at the time of inflation. Axions with a suitable energy scale f_a and mass m_a (incidentally, very different to CDM axions) provide a appropriately flat potential $V(\bar{\theta})$, making them especially good inflaton candidates [24, § 7].

If that were not enough, the problem of baryogenesis—the evident matter–antimatter asymmetry in the universe—may plausibly be explained by a cosmological evolution of the axion field. The $\bar{\theta}$-angle must be very small today, as constrained by the neutron electric dipole moment [14], but it is conceivable that $\bar{\theta} \sim \mathcal{O}(1)$ in the very early universe, which may have resulted in preferential generation of matter over antimatter [1, § 7.1].

All these features earn the axion a status of special interest in particle physics and cosmology, even with a pessimistic view of its existence. It has been claimed [34] that the rapidly improving experimental landscape makes it likely that the axion, if it exists, will be detected in the near future. In that case, a broad subset of physics would enjoy a revitalising breakthrough, and half a century's worth of theoretical discovery would become tangible progress.

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