Geometric Algebra for Special and General Relativity

Joseph Wilson

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Special Relativity and Geometric Algebra

Chapter 1.

Introduction

The Special Theory of Relativity is a model of *spacetime* — the geometry in which physical events take place. Spacetime comprises the Euclidean dimensions of space and time, but only in a way relative to each observer moving through it: there exists no single 'universal' ruler or clock. Instead, two observers in relative motion define different decompositions of spacetime, and their respective clocks and rulers are found to disagree according to the Lorentz transformation laws. The insight of special relativity is that one should focus not on the observer-dependent notions of space and time, but on the Lorentzian geometry of spacetime itself.

Seven years after Albert Einstein introduced this theory,[1] he succeeded in formulating a relativistic picture which included gravity. In this General Theory of Relativity, gravitation is identified with the curvature of spacetime over astronomical distances. Both theories coincide locally when confined to sufficiently small extents of spacetime, over which the effects of curvature are negligible. In part I, we will focus on special relativity, leaving gravity and curvature to part II.

In reference to the Erlangen programme,[2] the study of local spacetime geometry amounts to the study of its intrinsic symmetries. These symmetries form the Poincaré group, and consist of spacetime translations and Lorentz transformations, the latter being the extension of the rotation group for Euclidean space to relativistic rotations of spacetime. The standard matrix representation of the Lorentz group, $\mathrm{SO}^+(1,3)$, is the connected component of the orthogonal group

$$\mathrm{O}(1,3) = \{\Lambda \in \mathrm{GL}(\mathbb{R}^4) \mid \Lambda^\mathsf{T} \upeta \Lambda = \upeta\}$$

with respect to the bilinear form $\eta = \pm\mathrm{diag}(-1,+1,+1,+1)$. The rudimentary tools of matrix algebra are sufficient for an analys the Lorentz group, and are familiar to any physicist.

[1] Einstein's paper [1] was published in 1905, the so-called *Annus Mirabilis* or "miracle year" during which he also published on the photoelectric effect, Brownian motion and the mass-energy equivalence. Each of the four papers was a monumental contribution to modern physics.

[2] Introduced by Felix Klein in 1872 [2], the Erlangen program is the characterisation of geometries (Euclidean, hyperbolic, projective, etc.) by their symmetry groups and the properties invariant under those groups. E.g., Euclidean geometry studies the invariants of rigid transformations.

However, the last century has seen many other mathematical tools be applied to the study of generalised rotation groups such as $\mathrm{SO}^+(1,3)$ or the rotation group $\mathrm{SO}(3)$ of $\mathbb{R}^3$. Among these tools is the *geometric algebra*, invented[3] by William Clifford in 1878 [4]. Geometric algebra remains largely unknown in the physics community, despite arguably being far superior for the description of rotations than traditional matrix techniques. It is good to glean some of the history that led to this (perhaps unfortunate) state of the field.

[3] Clifford algebra was independently discovered by Rudolf Lipschitz two years later [3]. He was the first to use them to the study the orthogonal groups.

The quest for an optimal formalism for rotations

Mathematics has seen the invention of a variety of vector formalisms since the 1800s, and the question of which is best suited to physics has a long contentious history.

Complex numbers had been known for a long time[4] to be useful descriptions of planar rotations. William Hamilton's efforts to extend the same ideas into three dimensions by inventing a "multiplication of triples" bore fruition in 1843, when the quaternion algebra $\hat{\boldsymbol{\imath}}^2 = \hat{\boldsymbol{\jmath}}^2 = \hat{\boldsymbol{k}}^2 = \hat{\boldsymbol{\imath}}\hat{\boldsymbol{\jmath}}\hat{\boldsymbol{k}} = -1$ came to him in revelation. In following decades, William Gibbs developed the vector calculus of $\mathbb{R}^3$ with the usual vector cross and dot products. The ensuing vector algebra "war" of 1890–1945 saw Hamilton's prized[5] quaternion algebra $\mathbb{H}$, hailed as the optimal tool for describing 3d rotations, struggle for popularity against Gibbs' easier-to-visualise vector calculus. Today, quaternions are generally regarded in physics as an old-fashioned mathematical curiosity.

[4] Since Wessel, Argand and Gauss in the 1700s [5].

[5] Hamilton had dedicated following in the time that quaternions were in fasion: the *Quaternion Society* existed from 1895 to 1913.

Chapter 2.

Preliminary Theory

Many of the tools we will develop for the study of spacetime take place in various associative algebras. As well as the geometric algebra of spacetime, we will encounter tensors, exterior forms, quaternions, and other structures in this category. Instead of defining each algebra axiomatically as needed, it is easier to develop the general theory and then define each algebra succinctly as a particular quotient of the free algebra. This enables the use of the same tools and the same terminology thoughout.

Therefore, this section is an overview of the abstract theory of associative algebras, which more generally belongs to *ring theory*.[6] Algebras, quotients, gradings, homogeneous and inhomogeneous tensor and multivectors are defined, as well as standard operations on exterior forms. Most definitions in this chapter can be readily generalised by replacing the field $\mathbb{F}$ with a ring. The excitable reader may skip this chapter and refer back as needed.

[6] A RING is a field without the requirement that multiplicative inverses exist nor that multiplication commutes.

2.1. Associative Algebras

Throughout, $\mathbb{F}$ denotes the underlying field of some vector space. (Eventually, $\mathbb{F}$ will always be taken to be $\mathbb{R}$, but we may begin in generality.)

Definition 1. *An* ASSOCIATIVE ALGEBRA A *is a vector space equipped with a product* $\circledast : A \times A \to A$ *which is associative and bilinear.*

Associativity means $(\boldsymbol{u} \circledast \boldsymbol{v}) \circledast \boldsymbol{w} = \boldsymbol{u} \circledast (\boldsymbol{v} \circledast \boldsymbol{w})$, while bilinearity means the product is:

- compatible with scalars: $(\lambda \boldsymbol{u}) \circledast \boldsymbol{v} = \boldsymbol{u} \circledast (\lambda \boldsymbol{v}) = \lambda(\boldsymbol{u} \circledast \boldsymbol{v})$ for $\lambda \in \mathbb{F}$; and

- distributive over addition: $(\boldsymbol{u}+\boldsymbol{v}) \circledast \boldsymbol{w} = \boldsymbol{u} \circledast \boldsymbol{w} + \boldsymbol{v} \circledast \boldsymbol{w}$, and similarly for $\boldsymbol{u} \circledast (\boldsymbol{v}+\boldsymbol{w})$.

This definition can be generalised by relaxing associativity or by letting $\mathbb{F}$ be a ring. However, we will use "algebra" exclusively to mean an associative algebra over a field (usually $\mathbb{R}$).

Any ring forms an associative algebra when considered as a one-dimensional vector space. The complex numbers can be viewed as a real 2-dimensional algebra by defining $\circledast$ to be complex multiplication; $(x_1, y_1) \circledast (x_2, y_2) := (x_1x_2 - y_1y_2, x_1y_2 + y_1x_2)$.

The free tensor algebra

The most general (associative) algebra containing a given vector space V is the TENSOR ALGEBRA $V^{\otimes}$. The tensor product $\otimes$ satisfies exactly the relations of definition 1 with no others. Thus, the tensor algebra associative, bilinear and *free* in the sense that no further information is required in its definition.

As a vector space, the tensor algebra is equal to the infinite direct sum

$$V^{\otimes} \cong \bigoplus_{k=0}^{\infty} V^{\otimes k} \equiv \mathbb{F} \oplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \cdots \tag{2.1}$$

where each $V^{\otimes k}$ is the subspace of TENSORS OF GRADE k.

2.1.1. Quotient algebras

Owing to the maximal generality of the free tensor algebra, any other associative algebras may be constructed as a *quotient* of $V^{\otimes}$. In order for a quotient $V^{\otimes}/\sim$ by an equivalence relation $\sim$ to itself form an algebra, the relation must preserve the associative algebra structure:

Definition 2. *A CONGRUENCE on an algebra A is an equivalence relation $\sim$ which is compatible with the algebraic relations, so that if $a \sim a'$ and $b \sim b'$ then $a + b \sim a' + b'$ and $a \circledast b \sim a' \circledast b'$.*

The quotient of an algebra by a congruence naturally has the structure of an algebra, and so is called a QUOTIENT ALGEBRA.

Lemma 1. *The QUOTIENT $A/\sim$ of an algebra A by a congruence $\sim$, consisting of equivalence classes $[a] \in A/\sim$ as elements, forms an algebra with the naturally inherited operations $[a]+[b] := [a+b]$ and $[a]\circledast[b] := [a \circledast b]$.*

Proof. The fact that the operations $+$ and $\circledast$ of the quotient are well-defined follows from the structure-preserving properties of the congruence. Addition is well-defined if $[a] + [b]$ does not depend on the choice of representatives: if $a' \in [a]$ then $[a'] + [b]$ should be $[a] + [b]$. By congruence, we have from $a \sim a'$ so that $[a + b] = [a' + b]$ and indeed $[a] + [b] = [a'] + [b]$. Likewise for $\circledast$. $\square$

Instead of presenting an equivalence relation, it is often easier to define a congruence by specifying the set of elements which are equivalent to zero, from which all other equivalences follow from the algebra axioms. Such a set of all 'zeroed' elements is called an ideal.

Definition 3. *A (TWO-SIDED) IDEAL of an algebra A is a subset $I \subseteq A$ which is closed under addition and invariant under multiplication, so that*

- *if $a, b \in I$ then $a + b \in I$; and*
- *if $r \in A$ and $a \in I$ then $r \circledast a \in I \ni a \circledast r$.*

We will use the notation $\{\{A\}\}$ to mean the ideal generated by setting $a \sim 0$ for all a in A. For example, $\{\{a\}\} = \operatorname{span}\{r \circledast a \circledast r' \mid r, r' \in A\}$ is the ideal consisting of sums and products involving the specified element a, and $\{\{\boldsymbol{u} \otimes \boldsymbol{u} \mid \boldsymbol{u} \in V\}\}$, or simply $\{\{\boldsymbol{u} \otimes \boldsymbol{u}\}\}$, is the ideal in $V^{\otimes}$ consisting of sums of terms of the form $a \otimes \boldsymbol{u} \otimes \boldsymbol{u} \otimes b$ for vectors $\boldsymbol{u}$ and arbitrary $a, b \in V^{\otimes}$.

Lemma 2. *An ideal uniquely defines a congruence, and vice versa, by the identification of I as the set of elements equivalent to zero; $a \sim 0 \iff a \in I$.*

Proof. The set $I := \{a \mid a \sim 0\}$ is indeed an ideal because it is closed under addition (for $a, b \in I$ we have $\implies a + b \sim 0 + 0 = 0$ so $a + b \in I$) and invariant under multiplication (for any $a \in I$ and $r \in A$, we have $r \circledast a \sim r \circledast 0 = 0 = 0 \circledast r \sim a \circledast r$). Conversely, let $a \sim a'$ and $b \sim b'$. Since $\sim$ respects addition:

$$\left.\begin{aligned} a - a' \in I \\ b - b' \in I \end{aligned}\right\} \implies (a + b) - (a' + b') \in I \iff a + b \sim a' + b',$$

and multiplication:

$$\left.\begin{aligned} (a - a') \circledast b \in I \\ a' \circledast (b - b') \in I \end{aligned}\right\} \implies a \circledast b - a' \circledast b' \in I \iff a \circledast b \sim a' \circledast b',$$

the equivalence defined by $a \sim b \iff a - b \in I$ is a congruence. □

The equivalence of ideals and congruences is a general feature of abstract algebra.[7] Furthermore, both can be given in terms of a homomorphism between algebras,[8] and this is often the most convenient way to define a quotient.

[7] For example, in group theory, ideals are *normal subgroups* and a congruence is an equivalence relation satisfying $gag^{-1} \sim \mathrm{id}$ whenever $a \sim \mathrm{id}$. A group modulo a normal subgroup forms a quotient group.

[8] A *homomorphism* is a structure-preserving map; in the case of algebras, a linear map $\Psi : A \to A'$ which satisfies $\Psi(a \circledast b) = \Psi(a) \circledast' \Psi(b)$.

Theorem 1 (first isomorphism theorem). *If $\Psi : A \to B$ is a homomorphism, between algebras, then*

1. *the relation $a \sim b$ defined by $\Psi(a) = \Psi(b)$ is a congruence;*
2. *the kernel $I := \ker \Psi$ is an ideal; and*
3. *the quotients $A/{\sim} \equiv A/I \cong \Psi(A)$ are all isomorphic.*

Proof. We assume A and B associative algebras. (For a proof in universal algebra, see [6, § 15].)

To verify item 1, suppose that $\Psi(a) = \Psi(a')$ and $\Psi(b) = \Psi(b')$ and note that $\Psi(a + a') = \Psi(b + b')$ by linearity and $\Psi(a \circledast b) = \Psi(a' \circledast b')$ from $\Psi(a \circledast b) = \Psi(a) \circledast \Psi(b)$, so the congruence properties of definition 2 are satisfied.

For item 2, note that $\ker \Psi$ is a vector subspace, and that $a \in \ker \Psi$ implies $a \circledast r \in \ker \Psi$ for any $r \in A$ since $\Psi(a \circledast r) = \Psi(a) \circledast \Psi(r) = 0$. Thus, $\ker \Psi$ is an ideal by definition 3.

The first equivalence in item 3 follows from lemma 2. For an isomorphism $\Phi : A/\ker \Psi \to \Omega(A)$, pick $\Phi([a]) = \Psi(a)$. This is well-defined because the choice of representative of the equivalence class $[a]$ does not matter; $a \sim a'$ if and only if $\Psi(a) = \Psi(a')$ by definition of $\sim$, which simultaneously shows that Φ is injective. Surjectivity follows since any element of $\Psi(A)$ is of the form $\Psi(a)$ which is the image of $[a]$. □

With the free tensor algebra and theorem 1 in hand, we are able to describe any associative algebra as a quotient of the form $V^{\otimes}/I$.

Definition 4. *The* DIMENSION $\dim A$ *of a quotient algebra $A = V^{\otimes}/I$ is its dimension as a vector space. The* BASE DIMENSION *of A is the dimension of the underlying vector space V.*

Algebras may be infinite-dimensional, as is the case for the tensor algebra itself (which is a quotient by the trivial ideal).

2.1.2. Graded algebras

Associative algebras may possess another layer of useful structure: a grading. The grading of the tensor algebra has already been exhibited in eq. (2.1). A grading is a generalisation of the degree or rank of tensors or forms, and of the notion of parity for objects functions or polynomials.

Definition 5. *An algebra A is R-GRADED for $(R, +)$ a monoid*[9] *if there exists a decomposition*

$$A = \bigoplus_{k \in R} A_k$$

such that $A_i \circledast A_j \subseteq A_{i+j}$, i.e., $a \in A_i, b \in A_j \implies a \circledast b \in A_{i+j}$.

[9] A MONOID is a group without the requirement of inverses; i.e., a set with an associative binary operation for which there is an identity element.

The monoid is usually taken to be $\mathbb{N}$ or $\mathbb{Z}$ with addition, possibly modulo some integer. The tensor algebra $V^{\otimes}$ is $\mathbb{N}$-graded, since if $a \in V^{\otimes p}$ and $b \in V^{\otimes q}$ then $a \otimes b \in V^{\otimes p+q}$. Indeed, $V^{\otimes}$ is also $\mathbb{Z}$-graded if for $k < 0$ we understand $V^{\otimes k} := \{\mathbf{0}\}$ to be the trivial vector space. The tensor algebra is also $\mathbb{Z}_p$-graded, where $\mathbb{Z}_p \equiv \mathbb{Z}/p\mathbb{Z}$ is addition modulo any $p > 0$, since the decomposition

$$V^{\otimes} = \bigoplus_{k=0}^{p-1} Z_k \quad \text{where} \quad Z_k = \bigoplus_{n=0}^{\infty} V^{\otimes k+np} = V^{\otimes k} \oplus V^{\otimes (k+p)} \oplus \cdots$$

satisfies $Z_i \otimes Z_j \subseteq Z_k$ when $k \equiv i + j \mod p$. In particular, $V^{\otimes}$ is $\mathbb{Z}_2$-graded,[10] its elements admit a notion of *parity*: elements of $Z_0 = \mathbb{F} \otimes V^{\otimes 2} \otimes \cdots$ are even, while elements of $Z_1 = V \otimes V^{\otimes 3} \otimes \cdots$ are odd, and parity is respected by $\otimes$ as it is for integers.

Importantly, just as not all functions $f : \mathbb{R} \to \mathbb{R}$ are even or odd, not all elements of a $\mathbb{Z}_2$-graded algebra are even or odd; and more generally not all elements of a graded algebra belong to a single graded subspace.

[10] Algebras which are $\mathbb{Z}_2$-graded are sometimes called *superalgebras*, with the prefix 'super-' originating from supersymmetry theory.

Definition 6. *If $A = \bigoplus_{k \in R} A_k$ is an R-graded algerba, then an element $a \in A$ is HOMOGENEOUS if it belongs to some A_k, in which case it is said to be a k-VECTOR. If $a \in A_{k_1} \oplus \cdots \oplus A_{k_n}$ is inhomogeneous, we may call it a $\{k_1, \ldots, k_n\}$-multivector.*

All elements of a graded algebra are either inhomogeneous or a k-vector for some k; and each k-vector is either a k-*blade* or a sum of k-blades.

Definition 7. *A k-BLADE is a k-vector $a \in A_k$ of the form $a = \boldsymbol{u}_1 \circledast \cdots \circledast \boldsymbol{u}_k$ where each $\boldsymbol{u}_i \in A_1$ is a 1-vector.*

Note that not all k-vectors are blades. The simplest counterexample requires at least four dimensions: the bivector $\boldsymbol{e}_1 \otimes \boldsymbol{e}_2 + \boldsymbol{e}_3 \otimes \boldsymbol{e}_4 \in (\mathbb{R}^4)^{\otimes 2}$, where $\{\boldsymbol{e}_i\}$ are the standard basis of $\mathbb{R}^4$, cannot be factored into a blade of the form $\boldsymbol{u} \otimes \boldsymbol{v}$ for any $\boldsymbol{u}, \boldsymbol{v} \in V$.

{TO DO: Does this even make sense for a general graded algebra??}

Graded quotient algebras

A grading structure may or may not be inherited by a quotient — in particular, not all quotients of $V^{\otimes}$ inherit its $\mathbb{Z}$-grading. When reasoning about quotients of graded algebras, the following fact is useful.

Lemma 3. *Quotients commute with direct sums, so if*

$$A = \bigoplus_{k \in R} A_k \quad \text{and} \quad I = \bigoplus_{k \in R} I_k \quad \text{then} \quad A/I = \bigoplus_{k \in R} (A_k/I_k)$$

where R is some index set.

Proof. It is sufficient to prove the case for direct sums of length two. We then seek an isomorphism $\Phi : (A \oplus B)/(I \oplus J) \to (A/I) \oplus (B/J)$. Elements of the domain are equivalence classes of pairs $[(a, b)]$ with respect to the ideal $I \oplus J$. The direct sum ideal $I \oplus J$ corresponds to the congruence defined by $(a, b) \sim (a', b') \iff a \sim a'$ and $b \sim b'$. Therefore, the assignment $\Phi = [(a, b)] \mapsto ([a], [b])$ is well-defined. Injectivity and surjectivity follow immediately. □

This motivates the following strengthening to the notion of an ideal:

Definition 8. *An ideal I of an R-graded algebra $A = \bigoplus_{k \in R} A_k$ is HOMOGENEOUS if $I = \bigoplus_{k \in R} I_k$ where $I_k = I \cap A_k$.*

Not all ideals are homogeneous.[11] The additional requirement that an ideal be homogeneous ensures that the associated equivalence relation, as well as respecting the basic algebraic relations of definition 2, also preserves the grading structure. And so, we have a graded analogue to lemma 1:

[11] For example, the ideal $I = \{\{\boldsymbol{e}_1 + \boldsymbol{e}_2 \otimes \boldsymbol{e}_3\}\}$ is distinct from $\bigoplus_{k=0}^{\infty} (I \cap V^{\otimes k}) = \{\{\boldsymbol{e}_1, \boldsymbol{e}_2 \otimes \boldsymbol{e}_3\}\}$ because the former does not contain span$\{\boldsymbol{e}_1\}$, while the latter does.

Theorem 2. *If A is an R-graded algebra and I a homogeneous ideal, then the quotient A/I is also R-graded.*

Proof. By lemma 3 and the homogeneity of I, we have $A/I = \bigoplus_{k\in R}(A_k/I_k)$. Elements of A_k/I_k are equivalence classes $[a_k]$ where the representative is of grade k. Thus, $(A_p/I_p)\circledast(A_q/I_q) \subseteq A_{p+q}/I_{p+q}$ since $[a_p]\circledast[a_q] = [a_p \circledast a_q] = [b]$ for some $b \in A_{p+q}$. Hence, A/I is R-graded. □

2.2. The Wedge Product: Multivectors

One of the simplest algebras to construct as a quotient of the tensor algebra, yet still one of the most useful, is the *exterior algebra*, first introduced by Hermann Grassmann in 1844.

Definition 9. *The* EXTERIOR ALGEBRA *over a vector space V is*

$$\wedge V := V^{\otimes}/\{\{\boldsymbol{u}\otimes\boldsymbol{u}\}\} .$$

The product in $\wedge V$ is denoted $\wedge$ and called the WEDGE PRODUCT.

The ideal $\{\{\boldsymbol{u}\otimes\boldsymbol{u}\}\} \equiv \{\{\boldsymbol{u}\otimes\boldsymbol{u} \mid \boldsymbol{u}\in V\}\}$ corresponds to the congruence defined by $\boldsymbol{u}\otimes\boldsymbol{u} \sim 0$ for any vectors $\boldsymbol{u}\in V$. The wedge product is also called the *exterior*, *alternating* or *antisymmetric* product. The property suggested by its various names is easily seen by expanding the square of a sum;

$$(\boldsymbol{u}+\boldsymbol{v})\wedge(\boldsymbol{u}+\boldsymbol{v}) = \boldsymbol{u}\wedge\boldsymbol{u} + \boldsymbol{u}\wedge\boldsymbol{v} + \boldsymbol{v}\wedge\boldsymbol{u} + \boldsymbol{v}\wedge\boldsymbol{v}.$$

Since all terms of the form $\boldsymbol{w}\wedge\boldsymbol{w} = 0$ are definitionally zero, we have

$$\boldsymbol{u}\wedge\boldsymbol{v} = -\boldsymbol{v}\wedge\boldsymbol{u}$$

for all vectors $\boldsymbol{u},\boldsymbol{v}\in V$. By associativity, it follows that $\boldsymbol{v}_1\wedge\boldsymbol{v}_2\wedge\cdots\wedge\boldsymbol{v}_k$ vanishes exactly when the $\boldsymbol{v}_i$ are linearly dependent.[12]

The ideal $\{\{\boldsymbol{u}\otimes\boldsymbol{u}\}\}$ is homogeneous with respect to the $\mathbb{Z}$-grading of the parent tensor algebra; hence $\wedge V$ is itself $\mathbb{Z}$-graded. In particular, it is the direct sum of fixed-grade subspaces

$$\wedge V = \bigoplus_{k=0}^{\dim V} \wedge^k V \quad \text{where} \quad \wedge^k V = \operatorname{span}\{\boldsymbol{v}_1\wedge\boldsymbol{v}_2\wedge\cdots\wedge\boldsymbol{v}_k \mid \boldsymbol{v}_i \in V\},$$

and the wedge product respects grade, $(\wedge^p V)\wedge(\wedge^q V) \subseteq \wedge^{p+q}V$. By counting the number of possible linearly independent sets of k vectors in $\dim V$ dimensions, it follows that in base dimension $\dim V = n$,

$$\dim \wedge^k V = \binom{n}{k}, \quad \text{and hence} \quad \dim \wedge V = 2^n.$$

[12] *Proof.* Blades of the form $a = \boldsymbol{u}_1\wedge\cdots\wedge\boldsymbol{u}_k$ vanish when two or more vectors are repeated. If $\{\boldsymbol{u}_i\}$ is linearly dependent, then any one $\boldsymbol{u}_i$ can be written in terms of the others, and thus a can be expanded into a sum of such vanishing terms. □

In particular, note that $\dim \wedge^k V = \dim \wedge^{n-k} V$. Elements of the one-dimensional subspace $\wedge^n V$ are called PSEUDOSCALARS.[13]

[13] The prefix 'pseudo' means $k \mapsto n-k$. Hence, a pseudovector is an $(n-1)$-vector, etc.

2.2.1. As antisymmetric tensors

The exterior algebra may equivalently be viewed as the space of antisymmetric tensors equipped with an antisymmetrising product. Consider the map

$$\mathrm{Sym}^{\pm}(\boldsymbol{u}_1 \otimes \cdots \otimes \boldsymbol{u}_k) = \frac{1}{k!} \sum_{\sigma \in S_k} (\pm 1)^{\sigma} \boldsymbol{u}_{\sigma(1)} \otimes \cdots \otimes \boldsymbol{u}_{\sigma(k)} \tag{2.2}$$

where $(-1)^\sigma$ denotes the sign of the permutation σ in the symmetric group of k elements, S_k. By enforcing linearly, $\mathrm{Sym}^{\pm} : V^{\otimes} \to V^{\otimes}$ is defined on all tensors. A tensor A is called SYMMETRIC if $\mathrm{Sym}^{+}(A) = A$ and ANTISYMMETRIC if $\mathrm{Sym}^{-}(A) = A$.

Denote the image $\mathrm{Sym}^{-}(V^{\otimes})$ by S. The linear map $\mathrm{Sym}^{-} : V^{\otimes} \to S$ is not an algebra homomorphism with respect to the tensor product on S, since, e.g., $\mathrm{Sym}^{-}(\boldsymbol{u} \otimes \boldsymbol{v}) \neq \mathrm{Sym}^{-}(\boldsymbol{u}) \otimes \mathrm{Sym}^{-}(\boldsymbol{v}) = \boldsymbol{u} \otimes \boldsymbol{v}$. However, it *is* if we instead equip S with the antisymmetrising product $\wedge : S \times S \to S$ defined by

$$A \wedge B := \mathrm{Sym}^{-}(A \otimes B). \tag{2.3}$$

This makes $\mathrm{Sym}^{-} : V^{\otimes} \to S$ an algebra homomorphism, and by theorem 1, we have

$$S \cong V^{\otimes} / \ker \mathrm{Sym}^{-} . \tag{2.4}$$

Furthermore, note that the kernel of Sym^{-} consists of tensor products of linearly dependent vectors, and sums thereof,[14]

$$\ker \mathrm{Sym}^{-} = \mathrm{span}\{\boldsymbol{u}_1 \otimes \cdots \otimes \boldsymbol{u}_k \mid k \in \mathbb{N}, \{\boldsymbol{u}_i\} \text{ linearly dependent}\},$$

which is exactly the ideal $\{\{\boldsymbol{u} \otimes \boldsymbol{u}\}\}$. Therefore, the right-hand side of eq. (2.4) is identically the exterior algebra of definition 9. Hence, we have an algebra isomorphism $\mathrm{Sym}^{-}(V^{\otimes}) \cong \wedge V$, where the left-hand side is equipped with the product (2.3). This gives an alternative construction of the exterior algebra.

[14]

Proof. If $A = \boldsymbol{u}_1 \otimes \cdots \otimes \boldsymbol{u}_k$ where two vectors $\boldsymbol{u}_i = \boldsymbol{u}_j$ are equal, then $\mathrm{Sym}^{-}(A) = 0$ since each term in the sum in eq. (2.2) is paired with an equal and opposite term with $i \leftrightarrow j$ swapped. If $\{\boldsymbol{u}_i\}$ is linearly dependent, any one vector is a sum of the others, so A is a sum of blades with at least two vectors repeated. □

Note on conventions

The factor of $\frac{1}{k!}$ present in eq. (2.2) is not necessary for the above fact that $\mathrm{Sym}^-(V^\otimes) \cong \wedge V$ Indeed, some authors omit the normalisation factor, which has the effect of changing eq. (2.3) to

$$A \wedge B = \frac{(p+q)!}{p!q!}\mathrm{Sym}^-(A \otimes B)$$

for A and B of respective grades p and q, written in terms of our convention (2.2). The different normalisations of $\wedge$ as an antisymmetrising product lead to distinct identifications of multivectors in $\wedge V$ with tensors in $S \subset V^\otimes$, as clarified in table 2.1.

Kobayashi–Nomizu [7]	*Spivak* [8]
$A \wedge B := \mathrm{Sym}^-(A \otimes B)$	$A \wedge B := \frac{(p+q)!}{p!q!}\mathrm{Sym}^-(A \otimes B)$
$\boldsymbol{u} \wedge \boldsymbol{v} = \frac{1}{2}(\boldsymbol{u} \otimes \boldsymbol{v} - \boldsymbol{v} \otimes \boldsymbol{u})$	$\boldsymbol{u} \wedge \boldsymbol{v} = \boldsymbol{u} \otimes \boldsymbol{v} - \boldsymbol{v} \otimes \boldsymbol{u}$

Table 2.1.: Different embeddings of $\wedge V$ into $V^\otimes$. We employ the Kobayashi–Nomizu convention as this is coincides with the wedge product of geometric algebra. The Spivak convention is dominant for exterior differential forms in physics.

2.2.2. Exterior forms

The exterior algebra is most frequently encountered by physicists as an operation on *exterior (differential) forms*, which are alternating[15] multilinear maps.

We may wish to use the exterior algebra $\wedge V^*$ over the dual space of linear maps $V \to \mathbb{R}$ as a model for exterior forms. Using a basis $\{\boldsymbol{e}^i\} \subset V^*$, any element $f \in \wedge^k V^*$ has the form $f = f_{i_1 \cdots i_k} \boldsymbol{e}^{i_1} \wedge \cdots \wedge \boldsymbol{e}^{i_k}$, and each component acts on $\boldsymbol{u}_1 \otimes \cdots \otimes \boldsymbol{u}_k \in V^{\otimes k}$ as

$$\begin{aligned}(\boldsymbol{e}^{i_1} \wedge \cdots \wedge \boldsymbol{e}^{i_k})(\boldsymbol{u}_1 \otimes \cdots \otimes \boldsymbol{u}_k) &= \frac{1}{k!}\sum_{\sigma \in S_k} (-1)^\sigma \boldsymbol{e}^{i_{\sigma(1)}}(\boldsymbol{u}_1) \cdots \boldsymbol{e}^{i_{\sigma(k)}}(\boldsymbol{u}_k) \\ &= \frac{1}{k!}\det[\boldsymbol{e}^{i_m}(\boldsymbol{u}_n)]_{mn}. \qquad (2.5)\end{aligned}$$

However, this differs from the standard definition of exterior forms in two important ways:

15 An ALTERNATING linear map is one which changes sign upon transposition of any pair of arguments.

1. In eq. (2.5), the dual vectors $\boldsymbol{e}^i \in V^*$ are permuted while the order of the arguments $\boldsymbol{u}_i$ are preserved; but for standard exterior forms, the opposite is true. This prevents the proper extension of $\wedge V^*$ to non-Abelian vector-valued forms, where the values $\boldsymbol{e}^i(\boldsymbol{u}_j)$ may not commute.

2. Trivially, we insist on the the Kobayashi—Nomizu convention of normalisation factor for $\wedge V^*$; but the Spivak convention for exterior forms is much more standard in physics.

Thus, we define exterior forms separately from the exterior algebra.

Definition 10. *For a vector space V over $\mathbb{F}$, a k-FORM $\varphi \in \Omega^k(V)$ is an alternating multilinear map $\varphi : V^{\otimes k} \to \mathbb{F}$. For another vector space A, an A-VALUED k-FORM $\varphi \in \Omega^k(V, A)$ is such a map $\varphi : V^{\otimes k} \to A$ with codomain A.*

The evaluation of a form is denoted $\varphi(\boldsymbol{u}_1 \otimes \cdots \otimes \boldsymbol{u}_k)$ or $\varphi(\boldsymbol{u}_1, \dots, \boldsymbol{u}_k)$, and the wedge product of a p-form φ and q-form ϕ is defined (in the Spivak convention)

$$\varphi \wedge \phi = \frac{(p+q)!}{p!q!}(\varphi \otimes \phi) \circ \mathrm{Sym}^- . \tag{2.6}$$

Explicitly, eq. (2.6) acts to antisymmetrise arguments. To see this, choose a basis $\{\mathrm{d}x^\mu\}$ of $\Omega(V)$, and compare to eq. (2.5),

$$\begin{aligned}(\mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_k})(\boldsymbol{u}_1 \otimes \cdots \otimes \boldsymbol{u}_k) &= \sum_{\sigma \in S_k} (-1)^\sigma \mathrm{d}x^{\mu_1}(\boldsymbol{u}_{\sigma(1)}) \cdots \mathrm{d}x^{\mu_k}(\boldsymbol{u}_{\sigma(k)}) \\ &= \det[\mathrm{d}x^{\mu_m}(\boldsymbol{u}_n)]_{mn}.\end{aligned}$$

If $\varphi, \phi \in \Omega(V, A)$ are A-valued forms, where A is equipped with a bilinear product $\circledast : A \times A \to A$, then scalar multiplication may be replaced by $\circledast$ so that

$$(\varphi \wedge \phi)(\boldsymbol{u}_1 \otimes \cdots \otimes \boldsymbol{u}_k) = \sum_{\sigma \in S_k} (-1)^\sigma \varphi(\boldsymbol{u}_1 \otimes \cdots \otimes \boldsymbol{u}_p) \circledast \phi(\boldsymbol{u}_1 \otimes \cdots \otimes \boldsymbol{u}_q).$$

The product $\circledast$ need not be commutative nor associative; in particular, we may have Lie algebra–valued forms. For example, if $\varphi, \phi \in \Omega^1(V, \mathfrak{g})$ are Lie algebra–valued, then

$$(\varphi \wedge \phi)(\boldsymbol{u}, \boldsymbol{v}) = [\varphi(\boldsymbol{u}), \phi(\boldsymbol{v})] - [\varphi(\boldsymbol{v}), \phi(\boldsymbol{u})],$$

where $[\ \ ,\ \ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is the Lie bracket. Note that this implies that $\varphi \wedge \varphi$ does not necessarily vanish for non-Abelian forms.[16]

[16] E.g., in the case above we have $(\varphi \wedge \varphi)(\boldsymbol{u}, \boldsymbol{v}) = 2[\varphi(\boldsymbol{u}), \varphi(\boldsymbol{v})]$.

2.3. The Metric: Length and Angle

The tensor and exterior algebras considered so far are build from a vector space V alone. Notions of length and angle are central to geometry, but are not intrinsic to a vector space — additional structure must be provided.

Definition 11. *A* METRIC[17] *is a function $\eta : V \times V \to \mathbb{F}$, often written $\eta(\boldsymbol{u}, \boldsymbol{v}) \equiv \langle \boldsymbol{u}, \boldsymbol{v} \rangle$ which satisfies*

- *symmetry, $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \langle \boldsymbol{v}, \boldsymbol{u} \rangle$; and*
- *linearity, $\langle \alpha\boldsymbol{u} + \beta\boldsymbol{v}, \boldsymbol{w} \rangle = \alpha \langle \boldsymbol{u}, \boldsymbol{w} \rangle + \beta \langle \boldsymbol{v}, \boldsymbol{w} \rangle$ for $\alpha, \beta \in \mathbb{F}$.*

[17] a.k.a. an *inner product*, or *symmetric bilinear form*

Linearity in either argument implies linearity in the other by symmetry, so η is bilinear. A metric is NON-DEGENERATE if $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = 0$ for all $\boldsymbol{u}$ implies that $\boldsymbol{v}$ is zero. With respect to a basis $\{\boldsymbol{e}_i\}$ of V, the metric components $\eta_{ij} = \langle \boldsymbol{e}_i, \boldsymbol{e}_j \rangle$ are defined. Non-degeneracy means that $\det \eta \neq 0$ when viewing $\eta = [\eta_{ij}]$ as a matrix, and in this case the matrix inverse η^{ij} is also defined and satisfies $\eta^{ik}\eta_{kj} = \delta^i_j$.

A vector space V together with a metric η is called an INNER PRODUCT SPACE (V, η). Alternatively, instead of a metric, an inner product space may be constructed with a quadratic form:

Definition 12. *A* QUADRATIC FORM *is a function $q : V \to \mathbb{F}$ satisfying*

- *$q(\lambda\boldsymbol{v}) = \lambda^2 q(\boldsymbol{v})$ for all $\lambda \in \mathbb{F}$; and*
- *the requirement that the* POLARIZATION OF *q,*

$$(\boldsymbol{u}, \boldsymbol{v}) \mapsto q(\boldsymbol{u} + \boldsymbol{v}) - q(\boldsymbol{u}) - q(\boldsymbol{v}),$$

  is bilinear.

To any quadratic form q there is a unique associated bilinear form, which is *compatible* in the sense that $q(\boldsymbol{u}) = \langle \boldsymbol{u}, \boldsymbol{u} \rangle$. It is recovered[18] by the *polarization identity*

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \frac{1}{2}\big(q(\boldsymbol{u} + \boldsymbol{v}) - q(\boldsymbol{u}) - q(\boldsymbol{v})\big).$$

[18] Except, of course, if the characteristic of $\mathbb{F}$ is two. We only consider fields of characteristic zero.

The prescription of either η or q is therefore equivalent — but the notion of a metric is more common in physics, whereas the mathematical viewpoint often starts with a quadratic form.

Covectors and dual bases

The dual space $V^* := \{f : V \to \mathbb{F} \mid f \text{ linear}\}$ of a vector space consists of DUAL VECTORS OR COVECTORS, which are linear maps from V into its underlying field. Convention dictates that components of vectors be written superscript, $\boldsymbol{u} = u^i \boldsymbol{e}_i \in V$, and covectors subscript, $\varphi = \varphi_i \boldsymbol{e}^i \in V^*$, for bases $\{\boldsymbol{e}_i\} \subset V$ and $\{\boldsymbol{e}^i\} \subset V^*$.

A metric η on V defines an isomorphism between itself and its dual space. Collectively known as the MUSICAL ISOMORPHISMS, the map $\flat : V \to V^*$ and its inverse $\sharp : V^* \to V$ are defined by

$$\boldsymbol{u}^\flat(\boldsymbol{v}) = \langle \boldsymbol{u}, \boldsymbol{v} \rangle \quad \text{and} \quad \left\langle \varphi^\sharp, \boldsymbol{u} \right\rangle = \varphi(\boldsymbol{u})$$

for $\boldsymbol{u}, \boldsymbol{v} \in V$ and $\varphi \in V^*$. The names become justified when working with a basis; the relations

$$(\boldsymbol{u}^\flat)_i = \eta_{ij} \boldsymbol{u}^j \quad \text{and} \quad (\varphi^\sharp)^i = \eta^{ij} \varphi_j$$

show that $\flat$ lowers indices, while $\sharp$ raises them.

The basis $\{\boldsymbol{e}_i\}$ also defines a DUAL BASIS $\{\boldsymbol{e}^i\} \subset V^*$ OF V via the metric by $\boldsymbol{e}^i := \eta^{ij} \boldsymbol{e}_j^\flat$. Note that basis vectors and covectors defined in this way do not exist in the same vector space, but are related by their evaluation on one another via $\boldsymbol{e}^i(\boldsymbol{e}_j) = \delta^i_j$. In some contexts, given a basis $\{\boldsymbol{e}_i\}$ of V, we will define a dual basis $\{\boldsymbol{e}^i\} \subset V$ (not in V^*). By this we mean the dual basis vectors are instead defined as $\boldsymbol{e}^i := \eta^{ij} \boldsymbol{e}_j$, relating to the non-dual basis vectors via $\left\langle \boldsymbol{e}^i, \boldsymbol{e}_j \right\rangle = \delta^i_j$. We use both senses of dual basis, but the distinction can often be safely ignored.

2.3.1. Metrical exterior algebra

In an exterior algebra $\bigwedge V$ with a metric defined on V, there is an induced metric on k-vectors defined by

$$\begin{aligned}\langle \boldsymbol{u}_1 \wedge \cdots \wedge \boldsymbol{u}_k, \boldsymbol{v}_1 \wedge \cdots \wedge \boldsymbol{v}_k \rangle &= \sum_{\sigma \in S_k} (-1)^\sigma \left\langle \boldsymbol{u}_1, \boldsymbol{v}_{\sigma(1)} \right\rangle \cdots \left\langle \boldsymbol{u}_k, \boldsymbol{v}_{\sigma(k)} \right\rangle \\ &= \det[\langle \boldsymbol{u}_m, \boldsymbol{u}_n \rangle]_{mn}.\end{aligned}$$

In particular, a metric on $\bigwedge V$ defines a magnitude for pseudoscalars.

Definition 13. *Let V be an n-dimensional vector space with a metric. The* VOLUME ELEMENT $\mathbb{I}$ *of the metrical exterior algebra $\bigwedge V$ is the unique (up to sign) n-vector satisfying $\langle \mathbb{I}, \mathbb{I} \rangle = 1$.*

A choice of sign for the volume element defines an ORIENTATION. Given an ordered orthonormal basis $\{\boldsymbol{e}_i\}$ with $\langle \boldsymbol{e}_i, \boldsymbol{e}_i \rangle = \pm 1$, the basis is called right-handed if $\boldsymbol{e}_1 \wedge \cdots \wedge \boldsymbol{e}_n = \mathbb{I}$, and left-handed otherwise.

Hodge duality

A useful duality operation can be defined in an exterior algebra $\wedge V$ with a metric on V.

Definition 14. *Let $\wedge V$ be a metrical exterior algebra with base dimension n and volume element $\mathbb{I}$. The HODGE DUAL is the unique operator satisfying*

$$A \wedge \star B = \langle A, B \rangle \, \mathbb{I}$$

for any k-vectors $A, B \in \wedge^k V$.

The Hodge dual $\star : \wedge^k V \to \wedge^{n-k} V$ associates fixed-grade subspaces of the same dimension; in particular, $\star 1 = \mathbb{I}$.

Chapter 3.

The Geometric Algebra

In chapter 2, we defined the metric-independent exterior algebra of multivectors over a vector space V. Metrical operations can be achieved by introducing the Hodge dual (of section 2.3.1) and tacking onto $\wedge V$. The geometric algebra is a generalisation of $\wedge V$ which has the metric (and its concomitant notions of orientation and duality) directly built-in.

Geometric algebras are also known as real *Clifford algebras* $Cl(V, q)$ after their first inventor. Especially in mathematics, Clifford algebras are defined in terms of a quadratic form q, and the vector space V is usually complex. However, in physics, where V is taken to be real and a metric η is usually supplied instead of q, the name "geometric algebra" is preferred.[19]

[19] The newer name was coined by David Hestenes in the 1970s, who popularised Clifford algebra for physics [9, 10].

Construction as a quotient algebra

Informally put, the geometric algebra is obtained by enforcing the single rule

$$\boldsymbol{u}^2 = \langle \boldsymbol{u}, \boldsymbol{u} \rangle \tag{3.1}$$

for any vector $\boldsymbol{u}$, along with the associative algebra axioms of definition 1. The rich algebraic structure which follows from this is remarkable. Formally, we may give the geometric algebra as a quotient, just like our presentation of $\wedge V$.

Definition 15. *Let V be a finite-dimensional real vector space with metric. The* GEOMETRIC ALGEBRA *over V is*

$$\mathcal{G}(V, \eta) := V^{\otimes} / \{\{\boldsymbol{u} \otimes \boldsymbol{u} - \langle \boldsymbol{u}, \boldsymbol{u} \rangle\}\} .$$

The ideal defines the congruence generated by $\boldsymbol{u} \otimes \boldsymbol{u} \sim \langle \boldsymbol{u}, \boldsymbol{u} \rangle$ for any vector $\boldsymbol{u} \in V$, encoding eq. (3.1). This uniquely defines the associative (but not generally commutative) *geometric product* which we denote by juxtaposition.

As 2^n-dimensional vector spaces, $\mathcal{G}(V, \eta)$ and $\wedge V$ are isomorphic, each with a $\binom{n}{k}$-dimensional subspace for each grade k. Denoting the k-grade subspace $\mathcal{G}_k(V, \eta)$,

$$\mathcal{G}(V, \eta) = \bigoplus_{k=0}^{\infty} \mathcal{G}_k(V, \eta).$$

If the inner product is completely degenerate (i.e., $\langle \boldsymbol{u}, \boldsymbol{v} \rangle_0 = 0$ for all vectors), then there is an algebra isomorphism $\mathcal{G}(V, 0) \cong \wedge V$ — so the geometric algebra is more general. A qualitative difference between $\mathcal{G}(V, \eta)$ and $\wedge V$ is that, while inhomogeneous multivectors find little use in exterior algebra,[20] such elements have the significant geometrical role as reflections and rotations in $\mathcal{G}(V, \eta)$.

[20] In fact, some authors [11] leave sums of terms of differing grade undefined.

Note that $\mathcal{G}(V, \eta)$ is not $\mathbb{Z}$-graded, since it is a quotient by *inhomogeneous* elements $\boldsymbol{u} \otimes \boldsymbol{u} - \langle \boldsymbol{u}, \boldsymbol{u} \rangle \in V^{\otimes 2} \oplus V^{\otimes 0}$; therefore the geometric product of a p-vector and a q-vector is not generally a $(p+q)$-vector. However, the congruence *is* homogeneous with respect to the $\mathbb{Z}_2$-grading, so $\mathcal{G}(V, \eta)$ is $\mathbb{Z}_2$-graded. This shows that the algebra separates into 'even' and 'odd' subspaces

$$\mathcal{G}(V, \eta) = \mathcal{G}_+(V, \eta) \oplus \mathcal{G}_-(V, \eta) \quad \text{where} \quad \begin{cases} \mathcal{G}_+(V, \eta) = \bigoplus_{k=0}^{\infty} \mathcal{G}_{2k}(V, \eta) \\ \mathcal{G}_+(V, \eta) = \bigoplus_{k=0}^{\infty} \mathcal{G}_{2k+1}(V, \eta) \end{cases}$$

and also implies that $\mathcal{G}_+(V, \eta)$ is closed under the geometric product, forming the EVEN SUBALGEBRA.

The geometric product of vectors

By expanding $(\boldsymbol{u} + \boldsymbol{v})^2 = \langle \boldsymbol{u} + \boldsymbol{v}, \boldsymbol{u} + \boldsymbol{v} \rangle$, it follows[21] that

[21] $\boldsymbol{u}^2 + \boldsymbol{v}\boldsymbol{u} + \boldsymbol{u}\boldsymbol{v} + \boldsymbol{v}^2 = \langle \boldsymbol{u}, \boldsymbol{u} \rangle + 2\langle \boldsymbol{u}, \boldsymbol{v} \rangle + \langle \boldsymbol{v}, \boldsymbol{v} \rangle$

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \frac{1}{2}(\boldsymbol{u}\boldsymbol{v} + \boldsymbol{v}\boldsymbol{u}).$$

We recognise this as the symmetrised product of two vectors. The remaining antisymmetric part coincides with the *alternating* or *wedge* product familiar from exterior algebra

$$\boldsymbol{u} \wedge \boldsymbol{v} = \frac{1}{2}(\boldsymbol{u}\boldsymbol{v} - \boldsymbol{v}\boldsymbol{u}).$$

This is a 2-vector, or bivector, in $\mathcal{G}_2(V, \eta)$. Thus, the geometric product on vectors is

$$\boldsymbol{uv} = \langle \boldsymbol{u}, \boldsymbol{v} \rangle + \boldsymbol{u} \wedge \boldsymbol{v},$$

and some important features are immediate:

- *Parallel vectors commute, and vice versa.* If $\boldsymbol{u} = \lambda \boldsymbol{v}$, then $\boldsymbol{u} \wedge \boldsymbol{v} = 0$ and $\boldsymbol{uv} = \langle \boldsymbol{u}, \boldsymbol{v} \rangle = \langle \boldsymbol{v}, \boldsymbol{u} \rangle = \boldsymbol{vu}$.
- *Orthogonal vectors anti-commute, and vice versa.* If $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = 0$, then $\boldsymbol{uv} = \boldsymbol{u} \wedge \boldsymbol{v} = -\boldsymbol{v} \wedge \boldsymbol{u} = -\boldsymbol{vu}$.
- *Vectors are invertible under the geometric product.* If $\boldsymbol{u}$ is a vector for which the scalar $\boldsymbol{u}^2$ is non-zero, then $\boldsymbol{u}^{-1} = \boldsymbol{u}/\boldsymbol{u}^2$.
- *Geometric multiplication produces objects of mixed grade.* The product $\boldsymbol{uv}$ has a scalar part $\langle \boldsymbol{u}, \boldsymbol{v} \rangle$ and a bivector part $\boldsymbol{u} \wedge \boldsymbol{v}$.

Higher–grade elements

{TO DO: }

Fundamental algebra automorphisms

Operations such complex conjugation $\overline{AB} = \overline{A}\,\overline{B}$ or matrix transposition $(AB)^\mathsf{T} = B^\mathsf{T} A^\mathsf{T}$ are useful because they preserve or reverse multiplication. Linear functions with this property are called algebra automorphisms or antiautomorphisms, respectively. The geometric algebra possesses this (anti)automorphism operations.

Isometries of (V, η) are linear functions $f : V \to V$ which preserve the metric, so that $\langle f(\boldsymbol{u}), f(\boldsymbol{v}) \rangle = \langle \boldsymbol{u}, \boldsymbol{v} \rangle$ for any $\boldsymbol{u}, \boldsymbol{v} \in V$. Vector spaces always possess the involution isometry $\iota(\boldsymbol{u}) = -\boldsymbol{u}$, as well as the trivial isometry. An isometry extends uniquely to an algebra (anti)automorphism by defining $f(AB) = f(A)f(B)$ or $f(AB) = f(B)f(A)$. Thus, by extending the two fundamental isometries of (V, η) in the two possible ways, we obtain four fundamental (anti)automorphisms of $\mathcal{G}(V, \eta)$.

Definition 16.

- *Reversion* $\dagger$ *is the identity map on vectors* $\boldsymbol{u}^\dagger = \boldsymbol{u}$ *extended to general multivectors by the rule* $(AB)^\dagger = B^\dagger A^\dagger$.

- *Grade involution ι is the extension of the involution $\iota(\boldsymbol{u}) = -\boldsymbol{u}$ to general multivectors by the rule $\iota(AB) = \iota(A)\iota(B)$.*

If $A \in \mathcal{G}_k(V, \eta)$ is a k-vector, then $\iota(A) = (-1)^k A$ and $A^\dagger = s_k A$ where

$$s_k = (-1)^{\frac{(k-1)k}{2}} \tag{3.2}$$

is the sign of the reverse permutation on k symbols.

Reversion and grade involution together generate the four fundamental automorphisms

id	ι	automorphisms
$\dagger$	$\iota \circ \dagger$	anti-automorphisms

$\iota \circ \dagger$ is sometimes referred to as the Clifford conjugate

which form a group isomorphic to $\mathbb{Z}_2^2$ under composition.

These operations are very useful in practice. In particular, the following result follows easily from reasoning about grades.

Lemma 4. *If $A \in \mathcal{G}_k(V, \eta)$ is a k-vector, then A^2 is a $4\mathbb{N}$-multivector, i.e., a sum of blades of grade $\{0, 4, 8, \dots\}$ only.*

Proof. The multivector A^2 is its own reverse, since $(A^2)^\dagger = (A^\dagger)^2 = (\pm A)^2 = A^2$, and hence has parts of grade $\{4n, 4n+1 \mid n \in \mathbb{N}\}$. Similarly, A^2 is self-involutive, since $\iota(A^2) = \iota(A)^2 = (\pm A)^2 = A^2$. It is thus of even grade, leaving the possible grades $\{0, 4, 8, \dots\}$. □

3.1. Relations to Other Algebras

An efficient way to become familiar with geometric algebras is to study their relations to other common algebras encountered in physics.

{To do: Include $\mathbb{C}$, $\mathbb{H}$ and exterior forms!}

- **Complex numbers:** $\mathcal{G}_+(2) \cong \mathbb{C}$ The complex plane is contained within $\mathcal{G}(2)$ as the even subalgebra, with the isomorphism

  $$\mathbb{C} \ni x + iy \leftrightarrow x + y\boldsymbol{e}_1\boldsymbol{e}_2 \in \mathcal{G}_+(2)$$

  Complex conjugation in $\mathbb{C}$ coincides with reversion in $\mathcal{G}(2)$.

- **Quaternions:** $\mathcal{G}_+(3) \cong \mathbb{H}$

  Similarly, the quaternions are the even subalgebra $\mathcal{G}_+(3)$, with the isomorphism[22]

  $$q_0 + q_1\hat{\boldsymbol{\imath}} + q_2\hat{\boldsymbol{\jmath}} + q_3\hat{\boldsymbol{k}} \longleftrightarrow q_0 + q_1\boldsymbol{e}_2\boldsymbol{e}_3 - q_2\boldsymbol{e}_3\boldsymbol{e}_1 + q_3\boldsymbol{e}_1\boldsymbol{e}_2.$$

  Again, quaternion conjugation corresponds to reversion in $\mathcal{G}(3)$.

[22] Note the minus sign. Viewed as rotations through respective normal planes, $(\hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}}, \hat{\boldsymbol{k}})$ form a *left*-handed basis. This is because Hamilton chose $\hat{\boldsymbol{\imath}}\hat{\boldsymbol{\jmath}}\hat{\boldsymbol{k}} = -1$, not $+1$.

- **Complexified quaternions:** $\mathcal{G}_+(1,3) \cong \mathbb{C} \otimes \mathbb{H}$

  The complexified quaternion algebra, which has been applied to special relativity [12–14], is isomorphic to the subalgebra $\mathcal{G}_+(1,3)$. The isomorphism

  $$\begin{gathered}(x + yi) \otimes (q_0 + q_1\hat{\boldsymbol{\imath}} + q_2\hat{\boldsymbol{\jmath}} + q_3\hat{\boldsymbol{k}}) \longleftrightarrow \\ (x + y\boldsymbol{e}_{0123})(q_0 + q_1\boldsymbol{e}_{23} - q_2\boldsymbol{e}_{31} + q_3\boldsymbol{e}_{12})\end{gathered}$$

  associates quaternion units with bivectors, and the complex plane with the scalar–pseudoscalar plane. Reversion in $\mathcal{G}(1,3)$ corresponds to quaternion conjugation (preserving the complex i).

- **The Pauli algebra:** $\mathcal{G}(3) \cong \{\sigma_i\}_{i=1}^3$

  The *algebra of physical space*, $\mathcal{G}(3)$, admits a complex representation $\boldsymbol{e}_i \longleftrightarrow \sigma_i$ via the Pauli spin matrices

  $$\sigma_1 = \begin{pmatrix} 0 & +1 \\ +1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix}.$$

  Reversion in $\mathcal{G}(3)$ corresponds to Hermitian conjugation, and the volume element $\boldsymbol{e}_{123} \longleftrightarrow \sigma_1\sigma_2\sigma_3 = i$ corresponds to the unit imaginary.

- **The Dirac algebra:** $\mathcal{G}(1,3) \cong \{\gamma_\mu\}_{\mu=0}^3$

  The relativistic analogue to the Pauli algebra is the Dirac algebra, generated by the 4×4 complex Dirac matrices

$$\gamma_0 = \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma_1 = \begin{pmatrix} 0 & +\sigma_1 \\ -\sigma_1 & 0 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0 & -i\sigma_2 \\ +i\sigma_2 & 0 \end{pmatrix}, \quad \gamma_3 = \begin{pmatrix} 0 & +\sigma_3 \\ -\sigma_3 & 0 \end{pmatrix}.$$

  These form a complex representation of the *algebra of spacetime*, $\mathcal{G}(1,3)$, via $\boldsymbol{e}_\mu \longleftrightarrow \gamma_\mu$.

3.2. Rotors and the Associated Lie Groups

There is a consistent pattern in the algebra isomorphisms listed above. Note how the complex numbers $\mathbb{C}$ are fit for describing $\mathrm{SO}(2)$ rotations in the plane, and the quaternions $\mathbb{H}$ describe $\mathrm{SO}(3)$ rotations in $\mathbb{R}^3$. Common to both the respective isomorphisms with $\mathcal{G}_+(2)$ and $\mathcal{G}_+(3)$ is the identification of each "imaginary unit" in $\mathbb{C}$ or $\mathbb{H}$ with a *unit bivector* in $\mathcal{G}(n)$.

- In 2d, there is one linearly independent bivector, $\boldsymbol{e}_1\boldsymbol{e}_2$, and one imaginary unit, i.
- In 3d, there are $\dim \mathcal{G}_2(3) = \binom{3}{2} = 3$ such bivectors, and so three imaginary units $\left\{\hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}}, \hat{\boldsymbol{k}}\right\}$ are needed.
- In $(1+3)$d, we have $\dim \mathcal{G}_2(1,3) = \binom{4}{2} = 6$, corresponding to three 'spacelike' $\left\{\hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}}, \hat{\boldsymbol{k}}\right\}$ and three 'timelike' $\left\{i\hat{\boldsymbol{\imath}}, i\hat{\boldsymbol{\jmath}}, i\hat{\boldsymbol{k}}\right\}$ units of $\mathbb{C} \otimes \mathbb{H}$.

The interpretation of a bivector is clear: it generates a rotation through the oriented plane which it spans.

To see how bivectors act as rotations, observe that rotations in the $\mathbb{C}$-plane may be described as mappings $z \mapsto e^{\theta i} z$, while $\mathbb{R}^3$ rotations are described in $\mathbb{H}$ using a double-sided transformation law, $u \mapsto e^{\theta\hat{n}/2} u e^{-\theta\hat{n}/2}$, where $\hat{n} \in \mathrm{span}\left\{\hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}}, \hat{\boldsymbol{k}}\right\}$ is a unit quaternion defining the plane of rotation. Due to the commutativity of $\mathbb{C}$, the double-sided transformation law is actually general to both $\mathbb{C}$ and $\mathbb{H}$.

Similarly, rotations in a geometric algebra are described as

$$u \mapsto e^{-\theta\hat{b}/2} u e^{\theta\hat{b}/2},$$

where $\hat{b} \in \mathcal{G}_2(V, \eta)$ is a unit bivector. Multivectors of the form $R = e^{\sigma}$ for $\sigma \in \mathcal{G}_2(V, \eta)$ are called *rotors*. Immediate advantages to geometric algebra's rotor formalism are clear:

- *It is general to n dimensions, and to any metric signature.* Rotors describe generalised rotations,[23] depending on the metric and algebraic properties of the generating unitbivector σ. If $\sigma^2 < 0$, then e^{σ} describes a Euclidean rotation; if $\sigma^2 > 0$, then e^{σ} is a hyperbolic rotation or *Lorentz boost*.

[23] a.k.a., proper orthogonal transformations

- *Vectors are distinguished from bivectors.* One of the subtler points about quaternions is their transformation properties under reflection. A quaternion 'vector' $v = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}}$ reflects through the origin as $v \mapsto -v$, but a quaternion 'rotor' of the same value is invariant — vectors and pseudovectors are confused with the same kind of object. Not so in the geometric algebra: vectors are vectors in $\mathcal{G}_1$, and $\mathbb{R}^3$ pseudovectors are bivectors in $\mathcal{G}_2$. The price to pay for the introduction of more objects is not a price but a benefit: the generalisation to arbitrary dimensions is immediate and the geometric role of objects becomes clear. [24]

[24] See [5, 9, 15] for similarly impassioned testaments to the elegance of geometric algebra.

3.2.1. The rotor groups

We will now see more rigorously how the rotor formalism arises. An orthogonal transformation in n dimensions may be achieved by the composition of at most n reflections.[25] A reflection may be described in the geometric algebra by conjugation with an invertible vector. For instance, the linear map

[25] This is the Cartan–Dieudonné theorem [16].

$$A \mapsto -\boldsymbol{v}A\boldsymbol{v}^{-1} \tag{3.3}$$

reflects the multivector A along the vector $\boldsymbol{v}$, that is, across the hyperplane with normal $\boldsymbol{v}$. By composing reflections of this form, any orthogonal transformation may be built, acting on multivectors as

$$A \mapsto \pm RAR^{-1} \tag{3.4}$$

for some $R = \boldsymbol{v}_1\boldsymbol{v}_2 \cdots \boldsymbol{v}_3$, where the sign is positive for an even number of reflections, and negative for odd.

Scaling the axis of reflection $\boldsymbol{v}$ by a non-zero scalar λ does not affect the reflection map (3.3), since $\boldsymbol{v} \mapsto \lambda\boldsymbol{v}$ is cancelled out by $\boldsymbol{v}^{-1} \mapsto \lambda^{-1}\boldsymbol{v}^{-1}$. Therefore, a more direct correspondence exists between reflections and normalised vectors $\hat{\boldsymbol{v}}^2 = \pm 1$ (although there still remains an overall ambiguity in sign). For an orthogonal transformation built using normalised vectors,

$$R^{-1} = \hat{\boldsymbol{v}}_3^{-1} \cdots \hat{\boldsymbol{v}}_2^{-1}\hat{\boldsymbol{v}}_1^{-1} = \pm R^\dagger$$

since $\hat{\boldsymbol{v}}^{-1} = \pm\hat{\boldsymbol{v}}$, and hence eq. (3.4) may be written with the reversion instead of inversion:

$$A \mapsto \pm RAR^\dagger \tag{3.5}$$

All such elements $R^{-1} = \pm R^\dagger$ taken together form a group under the geometric product. This is called the *pin* group:

$$\mathrm{Pin}(p,q) := \{R \in \mathcal{G}(p,q) \mid RR^\dagger = \pm 1\}$$

There are two "pinors" for each orthogonal transformation, since $+R$ and $-R$ give the same map (3.5). Thus, the pin group forms a double cover of the orthogonal group $\mathrm{O}(p,q)$.

Furthermore, the even-grade elements of $\mathrm{Pin}(p,q)$ form a subgroup, called the *spin* group:

$$\mathrm{Spin}(p,q) := \{R \in \mathcal{G}_+(p,q) \mid RR^\dagger = \pm 1\}$$

This forms a double cover of $\mathrm{SO}(p,q)$.

Finally, the additional requirement that $RR^\dagger = 1$ defines the restricted spinor group, or the *rotor* group:

$$\mathrm{Spin}^+(p,q) := \{R \in \mathcal{G}_+(p,q) \mid RR^\dagger = 1\}$$

The rotor group is a double cover of the restricted special orthogonal group $\mathrm{SO}^+(p,q)$. Except for the degenerate case of $\mathrm{Spin}^+(1,1)$, the rotor group is simply connected to the identity.

$$\begin{array}{ccccc} \mathrm{Spin}^+ & \subseteq & \mathrm{Spin} & \subset & \mathrm{Pin} \\ \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow \\ \mathrm{SO}^+ & \subseteq & \mathrm{SO} & \subset & \mathrm{O} \end{array}$$

Figure 3.1.: Relationships between Lie groups associated with a geometric algebra. An arrow $a \twoheadrightarrow b$ signifies that a is a double-cover of b.

3.2.2. The bivector subalgebra

The multivector commutator product

$$A \times B := \frac{1}{2}(AB - BA) \tag{3.6}$$

forms a Lie bracket on the space of bivectors $\mathcal{G}_2$.

Proof. The commutator product $A \mapsto A \times \sigma$ with a bivector σ is a grade-preserving operation. If $A = \langle A \rangle_k$ then $A\sigma$ and σA are $\{k-2, k, k+2\}$-multivectors. The $k \pm 2$ parts are

$$\langle A \times \sigma \rangle_{k\pm2} = \frac{1}{2}\big(\langle A\sigma \rangle_{k\pm2} - \langle \sigma A \rangle_{k\pm2}\big).$$

However, $\langle \sigma A \rangle_{k\pm2} = s_{k\pm2} \langle A^\dagger \sigma^\dagger \rangle_{k\pm2} = -s_{k\pm2} s_k \langle A\sigma \rangle_{k\pm2}$ and the reversion signs[26] satisfy $s_{k\pm2}s_k = -1$ for any k. Hence, $\langle A \times \sigma \rangle_{k\pm2} = 0$, leaving only the grade k part, $A \times \sigma = \langle A \times \sigma \rangle_k$. Clearly eq. (3.6) is bilinear and satisfies the Jacobi identity, so $(\mathcal{G}_2, \times)$ is closed and forms a Lie algebra. □

[26] Recall from eq. (3.2) that $A^\dagger = s_k A$ for a k-vector where $s_k = (-1)^{\frac{(k-1)k}{2}}$.

Because the even subalgebra $\mathcal{G}_+ \supset \mathcal{G}_2$ is closed under the geometric product, the exponential $e^\sigma = 1 + \sigma + \frac{1}{2}\sigma^2 + \cdots$ of a bivector is an even multivector. Furthermore, note that the reverse $(e^\sigma)^\dagger = e^{\sigma^\dagger} = e^{-\sigma}$ is the inverse, and also that e^σ is continuously connected to the identity by the path $e^{\lambda\sigma}$ for $\lambda \in [0, 1]$. Therefore, $e^\sigma \in \mathrm{Spin}^+$ is a rotor, and we have a Lie algebra–Lie group correspondence shown in fig. 3.2. Thus, both the rotor groups and their Lie algebras are directly represented within the mother algebra $\mathcal{G}(p, q)$.

$$\begin{tikzcd}
\mathrm{Spin}^+(p,q) \arrow[r, two heads] & \mathrm{SO}^+(p,q) \\
\mathcal{G}_2(p,q) \arrow[u, "\exp"] \arrow[r, phantom, "\cong"] & \mathfrak{so}(p,q) \arrow[u, "\exp"]
\end{tikzcd}$$

Figure 3.2.: The Lie algebras $\mathfrak{so}(p, q)$ and $\mathcal{G}_2(p, q)$ under $\times$ are isomorphic, and are associated respectively to $\mathrm{SO}^+(p, q)$ and its universal double cover $\mathrm{Spin}^+(p, q)$.

Chapter 4.

The Algebra of Spacetime

Special relativity is geometry with a Lorentzian signature. The SPACE-TIME ALGEBRA (STA) is the name given to the geometric algebra of a Minkowski vector space, $\mathcal{G}(\mathbb{R}^4, \eta) \equiv \mathcal{G}(1, 3)$, where $\eta = \pm\mathrm{diag}(-1, +1, +1, +1)$. Other introductory material on the STA can be found in [17–19].

We denote the standard vector basis by $\{\gamma_\mu\}$, where Greek indices run over $\{0, 1, 2, 3\}$. This is a deliberate allusion to the Dirac γ-matrices, whose algebra is isomorphic to the STA — however, the $\gamma_\mu \in \mathbb{R}^{1+3}$ of STA are real, genuine spacetime vectors. A basis for the entire 2^4-dimensional STA is then

$$\overset{\text{1 scalar}}{\{1\}} \cup \overset{\text{4 vectors}}{\{\gamma_0, \gamma_i\}} \cup \overset{\text{6 bivectors}}{\{\gamma_0\gamma_i, \gamma_j\gamma_k\}} \cup \overset{\text{4 trivectors}}{\{\gamma_0\gamma_j\gamma_k, \gamma_1\gamma_2\gamma_3\}} \cup \overset{\text{1 pseudoscalar}}{\{\mathbb{I} := \gamma_0\gamma_1\gamma_2\gamma_3\}}$$

where Latin indices range over spacelike components, $\{1, 2, 3\}$. Blades shown on the left-hand side of $\{\quad,\quad\}$ are called TIMELIKE, and those in on right-hand side SPACELIKE.

Chapter 5.

Calculus in Flat Space(time)

{TO DO:

1. Exterior derivative and vector derivative
2. Stokes' theorem and the GA equivalent
3. Maxwell's equations

}

5.1. Differentiation

5.1.1. The Exterior Derivative

5.1.2. The Vector Derivative

5.1.3. Case Study: Maxwell's Equations

5.2. Integration

5.2.1. Stokes' Theorem for Exterior Calculus

Theorem 3 (Stokes' theorem in $\mathbb{R}^n$). *If $R \subseteq \mathbb{R}^n$ is a compact k-dimensional hypersurface with boundary ∂R, then a smooth differential form $\omega \in \Omega^{k-1}(R)$ satisfies*

$$\int_R \mathrm{d}\omega = \int_{\partial R} \omega. \tag{5.1}$$

Proof. Since R is a k-dimensional region with boundary, every point $x \in R$ has a neighbourhood diffeomorphic to a neighbourhood of the origin in either $\mathbb{R}^k$ or $H^k := [0,\infty) \oplus \mathbb{R}^{k-1}$, depending on whether x is an interior point or a boundary point, respectively.

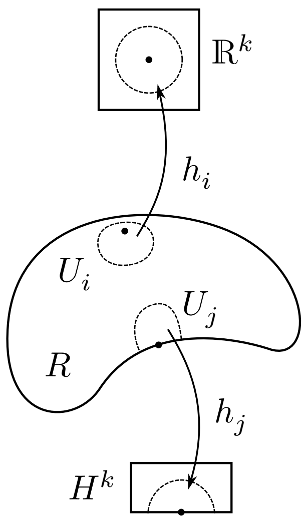

Figure 5.1.: Neighbourhoods in R are diffeomorphic either to interior balls or boundary half-balls.

Let $\{U_i\}$ be a cover of R consisting of such neighbourhoods. Since R is compact, we may assume $\bigcup_{i=1}^N \{U_i\} = R$ to be a finite covering. Thus, we have finitely maps $h_i : U_i \to X$ where X is either $\mathbb{R}^k$ or the half-space H^k, where $U_i \cong h_i(U_i)$ are diffeomorphic (see fig. 5.1).

Finally, let $\{\phi_i : R \to [0,1]\}$ be a partition of unity subordinate to $\{U_i\}$, so that $\{x \in R \mid \phi_i(x) > 0\} \subseteq U_i$ and $\omega = \sum_{i=1}^N \phi_i\omega$. We need only prove the equality (5.1) for each $\omega_i := \phi_i\omega$, and the full result follows be linearity.

The form $h_i^*\omega_i \in \Omega^{k-1}(X)$ can be written with respect to canonical coordinates of X as

$$h_i^*\omega_i = \sum_{j=1}^k f_j(-1)^{j-1}\mathrm{d}x^{1\cdots\hat{j}\cdots k}$$

using the multi-index notation $\mathrm{d}x^{i_1\cdots i_k} \equiv \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}$, where the hat denotes an omitted term. The factor of $(-1)^{j-1}$ gives the $(k-1)$-form the boundary orientation induced by the volume form $\mathrm{d}x^{1\cdots k}$ for convenience. Since pullbacks commute with d,

$$h^*\mathrm{d}\omega_i = \mathrm{d}(h_i^*\omega_i) = \sum_{j=1}^k \frac{\partial f_j}{\partial x^j}\mathrm{d}x^{1\cdots n}.$$

There are then two cases to consider.

- *Interior case.* If $h_i : U_i \to \mathbb{R}^k$, then the right-hand side of eq. (5.1) vanishes because ω_i is zero outside the neighbourhood $U_i \subset R$ which nowhere meets the boundary ∂R.

$$\int_{\partial R} \omega_i = \int_{\partial U_i} \omega_i = \int_{\varnothing} \omega_i = 0$$

The left-hand side evaluates to

$$\begin{aligned}\int_R \mathrm{d}\omega_i = \int_X \mathrm{d}(h_i^* \omega_i) &= \int_{\mathbb{R}^k} \sum_{j=1}^{k} \frac{\partial f_j}{\partial x^j} \mathrm{d}x^{1\cdots n} \\ &= \underbrace{\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty}}_{k} \sum_{j=1}^{k} \frac{\partial f_j}{\partial x^j} dx^1 \cdots dx^k \\ &= \underbrace{\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty}}_{k-1} \sum_{j=1}^{k} f_j \Big|_{x^j=-\infty}^{+\infty} (-1)^{j-1} dx^1 \cdots \widehat{dx^j} \cdots dx^k = 0,\end{aligned}$$

which vanishes because $h_i^* \omega_i$, and hence the f_j, vanish outside the neighbourhood $h_i(U_i) \subset \mathbb{R}^k$.

- *Boundary case.* If $h_i : U_i \to H^k$, then the boundary $\partial U_i \subset \partial R$ is mapped onto the hyperplane $\partial H^k = \{(0, x^2, \dots, x^k) \mid x^j \in \mathbb{R}\}$. Thus, $dx^1 = 0$ on this boundary, and the right-hand side of eq. (5.1) becomes

$$\begin{aligned}\int_{\partial R} \omega_i = \int_{\partial U_i} h_i^* \omega_i &= - \int_{\mathbb{R}^{k-1}} f_1 dx^2 \cdots dx^k \\ &= - \underbrace{\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty}}_{k-1} f_1(0, x^2, \dots, x^k) dx^2 \cdots dx^k.\end{aligned}$$

The factor of -1 comes from the induced orientation of the boundary ∂H^k, which is outward-facing, so in the *negative* x^1 direction. For the left-hand side of eq. (5.1),

$$\int_R \mathrm{d}\omega_i = \int_{H^k} h_i^* \mathrm{d}\omega_i = \int_0^{\infty} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \sum_{j=1}^{k} \frac{\partial f_j}{\partial x^j} dx^1 \cdots dx^k$$

All terms $\frac{\partial f_j}{\partial x^j} dx^j$ in the sum for $j > 1$ integrate to boundary terms $x_j \to \pm\infty$ where f_j vanishes. This leaves the single term from the integration of dx^1,

$$= - \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} f_1 \Big|_{x^1=0}^{\infty} dx^2 k \cdots k dx^{\cdot}$$

Thus, we have equality for all ω_i, so

$$\int_R \mathrm{d}\omega = \sum_{i=1}^{N} \int_R \mathrm{d}\omega_i = \sum_{i=1}^{N} \int_{\partial R} \omega_i = \int_{\partial R} \omega$$

by linearity. □

5.2.2. Fundamental Theorem of Geometric Calculus

Part II.

General Relativity and Manifold Geometry

Chapter 6.

The Problem: Curvature

Here we only give a pragmatic definition of a manifold as a space which locally looks like $\mathbb{R}^n$ upon which one can do calculus. (More rigour can be found in the first chapter of [20].)

Definition 17. *A* MANIFOLD $\mathcal{M}$ OF DIMENSION n *is a nice*[27] *topological space which is locally Euclidean, meaning for every* $x \in \mathcal{M}$ *there exist neighbourhoods* $x \in \mathcal{U} \subseteq \mathcal{M}$ *and subsets* $U \subseteq \mathbb{R}^n$ *with a homeomorphism (continuous bijection)* $\mathcal{U} \hookrightarrow\!\!\!\!\rightarrow U$ *between them.*

A SMOOTH MANIFOLD *is a manifold with the stricter requirement that* $\mathcal{U} \hookrightarrow\!\!\!\!\rightarrow U$ *be a diffeomorphism (differentiable bijection).*

[27] Here, a 'nice' topological space is:

1. *Hausdorff*, meaning each distinct pair of points have mutually disjoint neighbourhoods (so it is "not too small"); and
2. *second-countable*, meaning there exists a countable base (so it is "not too large").

The definitions that follow take place in the category of manifolds. Furthermore, if the qualifier "smooth" is present, then the objects exist in the category of *smooth manifolds* in which all maps are smooth (i.e., infinitely differentiable). This means all maps between manifolds are *assumed to be continuous.*

Essentially, definition 17 is designed to guarantee that well-behaved local coordinates always exist.

Definition 18. *Let* $\mathcal{M}$ *be an* n*-dimensional manifold. A* (GLOBAL) COORDINATE CHART $\{x^i\} \equiv \{x^1, \dots, x^n\}$ *of* $\mathcal{M}$ *is a set of scalar fields* $x^i : \mathcal{M} \to \mathbb{R}$ *such that each point in* $\mathcal{M}$ *is specified uniquely by the coordinate values* $(x^1, \dots, x^n) \in \mathbb{R}^n$*. A* LOCAL COORDINATE CHART *about a point* $x \in \mathcal{M}$ *is a coordinate chart of a neighbourhood of* x*.*

We will often call a point $x \in \mathcal{M}$ by the same symbol as the local coordinates $x^i : \mathcal{M} \to \mathbb{R}$ without the index — but these objects are not interchangeable.

Chapter 7.

The Answer: Fibre Bundles

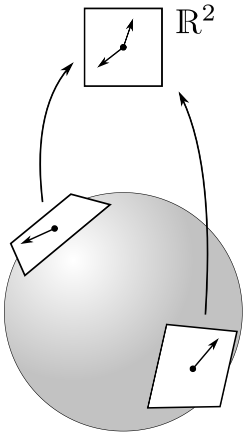

Figure 7.1.: Vectors in different tangent spaces, and their basis-dependent representation as an $\mathbb{R}^2$-valued field.

{TO DO: Treating physical things as fields} would suggest that all values are directly comparable, making expressions like $f(x) + f(y) \in A$ geometrically meaningful for different points $x, y \in \mathcal{M}$. However, an important lesson from physical theories like general relativity is that it is very often beneficial to distinguish between codomains *at each point in the domain.*

This can be motivated with the simple example of a fluid flowing on a sphere: The instantaneous fluid velocity at a point is a vector lying in the sphere's tangent plane at that point. If the fluid flow is given as a field $f : \mathcal{S}^2 \to \mathbb{R}^2$, then any two velocity vectors exist in the "same" space, even when *geometrically* they do not (fig. 7.1). This is more than a purely philosophical point: the fluid flow's representation as a field $f : \mathcal{S}^2 \to \mathbb{R}^2$ is dependent on the choice of basis, i.e., the way in which the single codomain $\mathbb{R}^2$ is identified with each tangent plane on the sphere. We would do better with a more geometrical representation of the vector field which is independent of any choice of basis, viewing the fluid velocities at different points as existing in different spaces.

This leads to the formulation of a tangent *bundle* $\mathrm{T}\,\mathcal{S}^2$, where all the tangent planes of $\mathcal{S}^2$ are collected in a disjoint union forming a *bulk manifold.* The vector field on the sphere now becomes a *section* of $\mathrm{T}\,\mathcal{S}^2$, which is a map $f : \mathcal{S}^2 \to \mathrm{T}\,\mathcal{S}^2$ satisfying some conditions. The tangent bundle is a special case of a *fibre bundle*, which is a manifold consisting of disjoint copies of a space (called the *fibre*) taken at every point in a base manifold.

Definition 19. *A* FIBRE BUNDLE $F \hookrightarrow \mathcal{F} \overset{\pi}{\twoheadrightarrow} \mathcal{M}$ *consists of*

- *a* BULK MANIFOLD $\mathcal{F}$*;*

- *a* BASE MANIFOLD $\mathcal{M}$*; and*
- *a surjection* $\pi : \mathcal{F} \to \mathcal{M}$*, the* PROJECTION*, such that*
- *the inverse image* $F_x := \pi^{-1}(x)$ *of a base point* $x \in \mathcal{M}$ *is homeomorphic to the* FIBRE F*.*

Definition 19 takes place in the category of manifolds, so the projection $\pi : \mathcal{F} \to \mathcal{M}$ is continuous. In a SMOOTH FIBRE BUNDLE, the projection π is differentiable and F, $\mathcal{F}$ and $\mathcal{M}$ are all smooth manifolds.

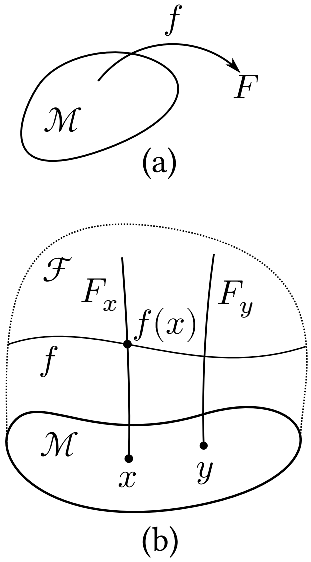

Figure 7.2.: (a) A field $f : \mathcal{M} \to F$, where values at any point can be compared. (b) A fibre bundle $F \hookrightarrow \mathcal{F} \twoheadrightarrow \mathcal{M}$ with a section $f \in \Gamma(\mathcal{F})$ whose individual fibres F are labelled by base point in $\mathcal{M}$.

Trivialisations and coordinates

The bulk $\mathcal{F}$ of a fibre bundle $F \hookrightarrow \mathcal{F} \twoheadrightarrow \mathcal{M}$ is itself a manifold (of dimension $\dim \mathcal{F} = \dim \mathcal{M} + \dim F$) so we may always prescribe local coordinates on $\mathcal{F}$. If we already have coordinates $\{x^\mu\}$ on the base $\mathcal{M}$ and $\{x^a\}$ on a fibre F, then we often want to use the same coordinates $\{x^\mu, x^a\}$ to describe the bulk $\mathcal{F}$. This first requires a way of continuously splitting the bulk $\mathcal{F} \to \mathcal{M} \times F$ into its base and fibre "components", in a way which respects the fibred structure of the bundle. This splitting is known as a *trivialisation* of the bundle.

Definition 20. *A* TRIVIALISATION *of a fibre bundle* $F \hookrightarrow \mathcal{F} \overset{\pi}{\twoheadrightarrow} \mathcal{M}$ *is a homeomorphism* $\varphi : \mathcal{F} \to \mathcal{M} \times F$ *such that* $\mathrm{pr}_1 \circ \varphi = \pi$*.*

It is not always possible to find a trivialisation of a fibre bundle, and if it is, the bundle is called a TRIVIAL FIBRE BUNDLE and there may be different possible trivialisations.[28]

However, it is always possible trivialise *locally*. That is, for any base point $x \in \mathcal{M}$, there exists a neighbourhood $x \in U \subseteq \mathcal{M}$ for which the subbundle $F \hookrightarrow \pi^{-1}(U) \overset{\pi}{\twoheadrightarrow} U$ admits a trivialisation. Hence, it is always possible to assign *local* coordinates $\{x^\mu, x^a\}$ to the bulk of a fibre bundle, where x^μ are coordinates on the base and x^a are coordinates on the fibres, such that x^μ do not vary along the fibres.

[28] A simple non-trivial fibre bundle is the Möbius strip, viewed as a bundle over the circle $\mathcal{S}^1$ with fibre $[0, 1]$. The trivial bundle $\mathcal{S}^1 \times [0, 1]$ describes a strip without a twist.

Sections of fibre bundles

In the language of fibre bundles, a field $f : \mathcal{M} \to F$ becomes a *section*, which is a "vertical" map $f : \mathcal{M} \to \mathcal{F}$ into the bulk $\mathcal{F}$ such that $f(x) \in F_x$.

Definition 21. *A* SECTION f *of a fibre bundle* $F \hookrightarrow \mathcal{F} \stackrel{\pi}{\twoheadrightarrow} \mathcal{M}$ *is a right-inverse of* π*. The space of sections is denoted*

$$\Gamma(\mathcal{F}) = \{f : \mathcal{M} \to \mathcal{F} \mid \pi \circ f = \mathrm{id}\}.$$

(Again, sections $f \in \Gamma(\mathcal{F})$ are assumed continuous, and SMOOTH SECTIONS are sections of smooth fibre bundles for which f is smooth.)

For example, the instantaneous fluid velocity $\boldsymbol{u}$ on a sphere $\mathcal{S}^2$ is a section $\boldsymbol{u} \in \Gamma(\mathrm{T}\,\mathcal{S}^2)$ of the tangent bundle, with a single vector at $x \in \mathcal{S}^2$ is denoted $\boldsymbol{u}|_x \in \mathrm{T}_x\,\mathcal{S}^2$.

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