Geometric Algebra for Special and General Relativity

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Part I.

Special Relativity and Geometric Algebra

Chapter 3.

The Geometric Algebra

In chapter 2, we defined the metric-independent exterior algebra of multivectors over a vector space V. While metrical operations can be achieved by introducing the Hodge dual (of section 2.3.1), tacking it onto $\wedge V$, the geometric algebra is a generalisation of $\wedge V$ which has the metric (and concomitant notions of orientation and duality) built-in.

Geometric algebras are also known as real *Clifford algebras* $Cl(V, q)$ after their first inventor [4]. Especially in mathematics, Clifford algebras are defined in terms of a quadratic form q, and the vector space V is usually complex. However, in physics, where V is taken to be real and a metric η is usually supplied instead of q, the name "geometric algebra" is preferred.[19]

[19] The newer name was coined by David Hestenes in the 1970s, who popularised Clifford algebra for physics [9, 10].

Construction as a quotient algebra

Informally put, the geometric algebra is obtained by enforcing the single rule

$$\boldsymbol{u}^2 = \langle \boldsymbol{u}, \boldsymbol{u} \rangle \tag{3.1}$$

for any vector $\boldsymbol{u}$, along with the associative algebra axioms of definition 1. The rich algebraic structure which follows from this is remarkable. Formally, we may give the geometric algebra as a quotient, just like our presentation of $\wedge V$.

Definition 16. *Let V be a finite-dimensional real vector space with metric. The* GEOMETRIC ALGEBRA *over V is*

$$\mathcal{G}(V,\eta) \coloneqq V^{\otimes}/\{\{\boldsymbol{u}\otimes\boldsymbol{u} - \langle\boldsymbol{u},\boldsymbol{u}\rangle\}\} \ .$$

The ideal defines the congruence generated by $\boldsymbol{u}\otimes\boldsymbol{u} \sim \langle\boldsymbol{u},\boldsymbol{u}\rangle$, encoding eq. (3.1). This uniquely defines the associative (but not generally commutative) *geometric product* which we denote by juxtaposition.

As 2^n-dimensional vector spaces, $\mathcal{G}(V,\eta)$ and $\wedge V$ are isomorphic, each with a $\binom{n}{k}$-dimensional subspace for each grade k. Denoting the k-grade subspace $\mathcal{G}_k(V,\eta)$, we have the vector space decomposition

$$\mathcal{G}(V,\eta) = \bigoplus_{k=0}^{\infty} \mathcal{G}_k(V,\eta).$$

Note that this is not a $\mathbb{Z}$ grading of the geometric algebra: the quotient is by *inhomogeneous* elements $\boldsymbol{u}\otimes\boldsymbol{u} - \langle\boldsymbol{u},\boldsymbol{u}\rangle \in V^{\otimes 2}\oplus V^{\otimes 0}$, and therefore the geometric product of a p-vector and a q-vector is not generally a $(p+q)$-vector. However, the congruence *is* homogeneous with respect to the $\mathbb{Z}_2$-grading, so $\mathcal{G}(V,\eta)$ is $\mathbb{Z}_2$-graded. This shows that the algebra separates into 'even' and 'odd' subspaces

$$\mathcal{G}(V,\eta) = \mathcal{G}_+(V,\eta)\oplus\mathcal{G}_-(V,\eta) \quad \text{where} \quad \begin{cases} \mathcal{G}_+(V,\eta) = \bigoplus_{k=0}^{\infty}\mathcal{G}_{2k}(V,\eta) \\ \mathcal{G}_+(V,\eta) = \bigoplus_{k=0}^{\infty}\mathcal{G}_{2k+1}(V,\eta) \end{cases}$$

where $\mathcal{G}_+(V,\eta)$ is closed under the geometric product, forming the EVEN SUBALGEBRA.

The geometric product of vectors

By expanding $(\boldsymbol{u}+\boldsymbol{v})^2 = \langle\boldsymbol{u}+\boldsymbol{v},\boldsymbol{u}+\boldsymbol{v}\rangle$, it follows[20] that

$$\langle\boldsymbol{u},\boldsymbol{v}\rangle = \frac{1}{2}(\boldsymbol{u}\boldsymbol{v}+\boldsymbol{v}\boldsymbol{u}).$$

[20] $\boldsymbol{u}^2+\boldsymbol{v}\boldsymbol{u}+\boldsymbol{u}\boldsymbol{v}+\boldsymbol{v}^2 = \langle\boldsymbol{u},\boldsymbol{u}\rangle+2\langle\boldsymbol{u},\boldsymbol{v}\rangle+\langle\boldsymbol{v},\boldsymbol{v}\rangle$

We recognise this as the symmetrised product of two vectors. The remaining antisymmetric part coincides with the *alternating* or *wedge* product familiar from exterior algebra

$$\boldsymbol{u} \wedge \boldsymbol{v} = \frac{1}{2}(\boldsymbol{uv} - \boldsymbol{vu}).$$

This is a 2-vector, or bivector, in $\mathcal{G}_2(V, \eta)$. Thus, the geometric product on vectors is

$$\boldsymbol{uv} = \langle \boldsymbol{u}, \boldsymbol{v} \rangle + \boldsymbol{u} \wedge \boldsymbol{v},$$

and some important features are immediate:

- *Parallel vectors commute, and vice versa:* If $\boldsymbol{u} = \lambda \boldsymbol{v}$, then $\boldsymbol{u} \wedge \boldsymbol{v} = 0$ and $\boldsymbol{uv} = \langle \boldsymbol{u}, \boldsymbol{v} \rangle = \langle \boldsymbol{v}, \boldsymbol{u} \rangle = \boldsymbol{vu}$.

- *Orthogonal vectors anti-commute, and vice versa:* If $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = 0$, then $\boldsymbol{uv} = \boldsymbol{u} \wedge \boldsymbol{v} = -\boldsymbol{v} \wedge \boldsymbol{u} = -\boldsymbol{vu}$.

In particular, if $\{\boldsymbol{e}_i\} \subset V$ is an orthonormal basis, then we have $\boldsymbol{e}_i^2 = \langle \boldsymbol{e}_i, \boldsymbol{e}_i \rangle$ and $\boldsymbol{e}_i \boldsymbol{e}_j = -\boldsymbol{e}_j \boldsymbol{e}_i$, which can be summarised by the anticommutation relation $\boldsymbol{e}_i \boldsymbol{e}_j + \boldsymbol{e}_j \boldsymbol{e}_i = 2\eta_{ij}$.

- *Vectors are invertible under the geometric product:* If $\boldsymbol{u}$ is a vector for which the scalar $\boldsymbol{u}^2$ is non-zero, then $\boldsymbol{u}^{-1} = \boldsymbol{u}/\boldsymbol{u}^2$.

- *Geometric multiplication produces objects of mixed grade:* The product $\boldsymbol{uv}$ has a scalar part $\langle \boldsymbol{u}, \boldsymbol{v} \rangle$ and a bivector part $\boldsymbol{u} \wedge \boldsymbol{v}$.

3.1. Higher-grade elements

As with two vectors, the geometric product of two homogeneous multivectors is generally inhomogeneous. We can gain insight by separating geometric products into grades and studying each part.

Definition 17. *The* GRADE k PROJECTION *of a multivector* $A \in \mathcal{G}(V, \eta)$ *is*

$$\langle A \rangle_k = \begin{cases} A & \text{if } A \in \mathcal{G}_k(V, \eta) \\ 0 & \text{otherwise.} \end{cases}$$

We can generalise the definition of the wedge product of vectors $\boldsymbol{u} \wedge \boldsymbol{v} = \langle \boldsymbol{u}\boldsymbol{v} \rangle_2$ to arbitrary homogeneous multivectors by taking the highest-grade part of their product,

$$A \wedge B = \langle AB \rangle_{p+q},$$

where $A \in \mathcal{G}_p(V, \eta)$ and $B \in \mathcal{G}_q(V, \eta)$. Dually, we can define an inner product on homogeneous multivectors by taking the lowest-grade part, $|p - q|$. These can be extended by linearity to inhomogeneous elements.

Definition 18. *Let $A, B \in \mathcal{G}(V, \eta)$ be possibly inhomogeneous multivectors. The* WEDGE PRODUCT *is*

$$A \wedge B := \sum_{p,q} \left\langle \langle A \rangle_p \langle B \rangle_q \right\rangle_{p+q},$$

and the GENERALISED INNER PRODUCT, *or "fat dot" product, is*

$$A \bullet B := \sum_{p,q} \left\langle \langle A \rangle_p \langle B \rangle_q \right\rangle_{|p-q|}.$$

With the wedge product defined on all of $\mathcal{G}(V, \eta)$, we use language of multivectors as we did with the exterior algebra, so that $\boldsymbol{u}_1 \wedge \cdots \wedge \boldsymbol{u}_k \in \mathcal{G}_k(V, \eta)$ is a k-blade, and a sum of k-blades is a k-multivector, etcetea. The products in definition 18 work together nicely, and extend the notion of a dual vector basis to a dual basis of blades.

Lemma 4. *If $\{\boldsymbol{e}_i\} \subset V$ is a basis with dual $\boldsymbol{e}^i \bullet \boldsymbol{e}_j = \delta^i_j$, then*

$$(\boldsymbol{e}^{i_1} \wedge \cdots \wedge \boldsymbol{e}^{i_k}) \bullet (\boldsymbol{e}_{j_k} \wedge \cdots \wedge \boldsymbol{e}_{j_1}) = \varepsilon^i_j$$

where $\varepsilon^i_j = (-1)^\sigma$ is the sign of the permutation sending $\sigma(i_p) = j_p$ for $1 \leq p \leq k$, or zero if there is no such permutation or if i or j contain repeated indices.

Note the reverse order of the j indices.

Proof. If i or j contain repeated indices, then the left-hand side vanishes by antisymmetry of the wedge product, and the right-hand side by definition. If i contains no repeated indices, and the j indices are some permutation $j_p = \sigma(i_p)$, then $\boldsymbol{e}^{i_1} \wedge \cdots \wedge \boldsymbol{e}^{i_k} = \boldsymbol{e}^{i_1} \cdots \boldsymbol{e}^{i_k}$ by orthogonality.

Rewriting the left-hand side,

$$\left\langle e^{i_1} \cdots e^{i_k} e_{j_k} \cdots e_{j_1} \right\rangle = (-1)^\sigma \langle e^{i_1} \cdots \underbrace{e^{i_k} e_{i_k}}_{1} \cdots e_{i_1} \rangle = (-1)^\sigma.$$

Finally, if i contains no repeated indices, but j is not a permutation of i, then there is at least one pair of indices in the symmetric difference of $\{i_p\}$ and $\{j_p\}$, say i_r and j_s. Commuting this pair e^{i_r} and e_{j_s} together shows that the left-hand side vanishes, since $e^{i_r} e_{j_s} = 0$. □

3.2. Relations to Other Algebras

{TO DO: into}

Fundamental algebra automorphisms

Operations such complex conjugation $\overline{AB} = \overline{A}\,\overline{B}$ or matrix transposition $(AB)^\mathsf{T} = B^\mathsf{T} A^\mathsf{T}$ are useful because they preserve or reverse multiplication. Linear functions with this property are called algebra automorphisms or antiautomorphisms, respectively. The geometric algebra possesses this (anti)automorphism operations.

Isometries of (V, η) are linear functions $f : V \to V$ which preserve the metric, so that $\langle f(\boldsymbol{u}), f(\boldsymbol{v}) \rangle = \langle \boldsymbol{u}, \boldsymbol{v} \rangle$ for any $\boldsymbol{u}, \boldsymbol{v} \in V$. Vector spaces always possess the involution isometry $\iota(\boldsymbol{u}) = -\boldsymbol{u}$, as well as the trivial isometry. An isometry extends uniquely to an algebra (anti)automorphism by defining $f(AB) = f(A)f(B)$ or $f(AB) = f(B)f(A)$. Thus, by extending the two fundamental isometries of (V, η) in the two possible ways, we obtain four fundamental (anti)automorphisms of $\mathcal{G}(V, \eta)$.

Definition 19.

- *Reversion* $\dagger$ *is the identity map on vectors* $\boldsymbol{u}^\dagger = \boldsymbol{u}$ *extended to general multivectors by the rule* $(AB)^\dagger = B^\dagger A^\dagger$.
- *Grade involution* ι *is the extension of the involution* $\iota(\boldsymbol{u}) = -\boldsymbol{u}$ *to general multivectors by the rule* $\iota(AB) = \iota(A)\iota(B)$.

If $A \in \mathcal{G}_k(V,\eta)$ is a k-vector, then $\iota(A) = (-1)^k A$ and $A^\dagger = s_k A$ where

$$s_k = (-1)^{\frac{(k-1)k}{2}} \tag{3.2}$$

is the sign of the reverse permutation on k symbols.

Reversion and grade involution together generate the four fundamental automorphisms

id	ι	automorphisms
$\dagger$	$\iota \circ \dagger$	anti-automorphisms

$\iota \circ \dagger$ is sometimes referred to as the Clifford conjugate

which form a group isomorphic to $\mathbb{Z}_2^2$ under composition.

These operations are very useful in practice. In particular, the following result follows easily from reasoning about grades.

Lemma 5. *If $A \in \mathcal{G}_k(V,\eta)$ is a k-vector, then A^2 is a $4\mathbb{N}$-multivector, i.e., a sum of blades of grade $\{0, 4, 8, \dots\}$ only.*

Proof. The multivector A^2 is its own reverse, since $(A^2)^\dagger = (A^\dagger)^2 = (\pm A)^2 = A^2$, and hence has parts of grade $\{4n, 4n+1 \mid n \in \mathbb{N}\}$. Similarly, A^2 is self-involutive, since $\iota(A^2) = \iota(A)^2 = (\pm A)^2 = A^2$. It is thus of even grade, leaving the possible grades $\{0, 4, 8, \dots\}$. □

Common algebra isomorphisms

An efficient way to become familiar with geometric algebras is to study their relations to other common algebras encountered in physics.

- **Complex numbers:** $\mathcal{G}_+(2) \cong \mathbb{C}$

  The complex plane is contained within $\mathcal{G}(2)$ as the even subalgebra, with the isomorphism

  $$\mathbb{C} \ni x + iy \leftrightarrow x + y\boldsymbol{e}_1\boldsymbol{e}_2 \in \mathcal{G}_+(2)$$

  Complex conjugation in $\mathbb{C}$ coincides with reversion in $\mathcal{G}(2)$.

- **Quaternions:** $\mathcal{G}_+(3) \cong \mathbb{H}$

  Similarly, the quaternions are the even subalgebra $\mathcal{G}_+(3)$, with the isomorphism[21]

  $$q_0 + q_1\hat{\boldsymbol{\imath}} + q_2\hat{\boldsymbol{\jmath}} + q_3\hat{\boldsymbol{k}} \longleftrightarrow q_0 + q_1\boldsymbol{e}_2\boldsymbol{e}_3 - q_2\boldsymbol{e}_3\boldsymbol{e}_1 + q_3\boldsymbol{e}_1\boldsymbol{e}_2.$$

  Again, quaternion conjugation corresponds to reversion in $\mathcal{G}(3)$.

[21] Note the minus sign. Viewed as rotations through respective normal planes, $(\hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}}, \hat{\boldsymbol{k}})$ form a *left*-handed basis. This is because Hamilton chose $\hat{\boldsymbol{\imath}}\hat{\boldsymbol{\jmath}}\hat{\boldsymbol{k}} = -1$, not $+1$.

- **Complexified quaternions:** $\mathcal{G}_+(1,3) \cong \mathbb{C} \otimes \mathbb{H}$

  The complexified quaternion algebra, which has been applied to special relativity [11–13], is isomorphic to the subalgebra $\mathcal{G}_+(1,3)$. The isomorphism

  $$\begin{aligned}\mathbb{C} \otimes \mathbb{H} \ni (x + yi) \otimes (q_0 + q_1\hat{\boldsymbol{\imath}} + q_2\hat{\boldsymbol{\jmath}} + q_3\hat{\boldsymbol{k}}) \longleftrightarrow \\ (x + y\boldsymbol{e}_{0123})(q_0 + q_1\boldsymbol{e}_{23} - q_2\boldsymbol{e}_{31} + q_3\boldsymbol{e}_{12}) \in \mathcal{G}_+(1,3)\end{aligned}$$

  associates quaternion units with bivectors, and the complex plane with the scalar–pseudoscalar plane. Reversion in $\mathcal{G}(1,3)$ corresponds to quaternion conjugation (preserving the complex i).

- **The Pauli algebra:** $\mathcal{G}(3) \cong \{\sigma_i\}_{i=1}^3$

  The *algebra of physical space*, $\mathcal{G}(3)$, admits a complex representation $\boldsymbol{e}_i \longleftrightarrow \sigma_i$ via the Pauli spin matrices

  $$\sigma_1 = \begin{pmatrix} 0 & +1 \\ +1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix}.$$

  Reversion in $\mathcal{G}(3)$ corresponds to the adjoint (Hermitian conjugate), and the volume element $\mathbb{I} := \boldsymbol{e}_{123} \longleftrightarrow \sigma_1\sigma_2\sigma_3 = i$ corresponds to the unit imaginary.

- **The Dirac algebra:** $\mathcal{G}(1,3) \cong \{\gamma_\mu\}_{\mu=0}^3$

  The relativistic analogue to the Pauli algebra is the Dirac algebra, generated by the 4×4 complex Dirac matrices

$$\gamma_0 = \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma_1 = \begin{pmatrix} 0 & +\sigma_1 \\ -\sigma_1 & 0 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0 & -i\sigma_2 \\ +i\sigma_2 & 0 \end{pmatrix}, \quad \gamma_3 = \begin{pmatrix} 0 & +\sigma_3 \\ -\sigma_3 & 0 \end{pmatrix}.$$

These form a complex representation of the *algebra of spacetime*, $\mathcal{G}(1,3)$, via $\boldsymbol{e}_\mu \longleftrightarrow \gamma_\mu$. Again, reversion corresponds to the adjoint, and $\mathbb{I} := \boldsymbol{e}_0\boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3 \longleftrightarrow \gamma_0\gamma_1\gamma_2\gamma_3 = -i\gamma_5$.

Relation to Exterior Forms

The geometric algebra is a generalisation of the exterior algeba. If the inner product is completely degenerate (i.e., $\langle \boldsymbol{u}, \boldsymbol{v}\rangle_0 = 0$ for all vectors), then there is an algebra isomorphism $\mathcal{G}(V,0) \cong \wedge V$.

A qualitative difference between $\mathcal{G}(V,\eta)$ and $\wedge V$, however, is that while inhomogeneous multivectors find little use in exterior algebra,[22] these have a significant geometrical describing reflections and rotations in $\mathcal{G}(V,\eta)$.

[22] In fact, some authors [14] leave sums of terms of differing grade undefined.

Exterior forms can be mimicked in the geometric algebra by making use of a dual basis V, as in the following lemma. Note that the dual space V^* does not make an appearance — all elements belong to $\mathcal{G}(V,\eta)$.

Lemma 6. *If $A \in \mathcal{G}_k(V,\eta)$ is a k-vector and $\varphi \in \Omega^k(V)$ is a k-form whose components coincide (i.e., $A_{i_1\cdots i_k} = \varphi_{i_1\cdots i_k}$ given a common basis of V) then*

$$A \bullet (\boldsymbol{u}_k \wedge \cdots \wedge \boldsymbol{u}_1) = k!\,\varphi(\boldsymbol{u}_1, \dots, \boldsymbol{u}_k).$$

Note the reversed order of the wedge products on the left-hand side. The factor of $k!$ is due to the Spivak convention for exterior forms (replace $k! \mapsto 1$ for the Kobayashi–Nomizu convention).

Proof. Fix an orthonormal basis $\{\boldsymbol{e}_i\} \subset V$ and a dual basis $\boldsymbol{e}^i \bullet \boldsymbol{e}_j = \delta^i_j$. Expanding the right-hand side with respect to his basis,

$$A \bullet (\boldsymbol{u}_k \wedge \cdots \wedge \boldsymbol{u}_1) = A_{i_1\cdots i_k}(\boldsymbol{e}^{i_1} \wedge \cdots \wedge \boldsymbol{e}^{i_k}) \bullet (\boldsymbol{e}_{j_k} \wedge \cdots \wedge \boldsymbol{e}_{j_1}) u_k^{j_k} \cdots u_1^{j_1}.$$

By lemma 4, the dot product of k-blades is $(-1)^\sigma$ is the sign of the permutation $\sigma(i_p) = j_p$, and zero for all non-permutation terms in the sum.

Thus, for each (non-zero) term in the sum we have

$$u_1^{j_1} \cdots u_k^{j_k} = u_1^{\sigma^{-1}(j_1)} \cdots u_k^{\sigma^{-1}(j_k)} = u_{\sigma(1)}^{i_1} \cdots u_{\sigma(k)}^{i_k},$$

where the last equality is obtained by permuting the scalar components $u_{\sigma(p)}^{i_p}$ by σ. Putting this together,

$$A \bullet (\boldsymbol{u}_k \wedge \cdots \wedge \boldsymbol{u}_1) = \sum_{\sigma \in S_k} (-1)^\sigma A_{i_1 \cdots i_k} u_{\sigma(1)}^{i_1} \cdots u_{\sigma(k)}^{i_k},$$

which by $A_{i_1 \cdots i_k} = \varphi_{i_1 \cdots i_k}$ is equal to

$$\cdots = \sum_{\sigma \in S_k} (-1)^\sigma \varphi(\boldsymbol{u}_{\sigma(1)}, \ldots, \boldsymbol{u}_{\sigma(k)}) = k!\, \varphi(\boldsymbol{u}_{\sigma(1)}, \ldots, \boldsymbol{u}_{\sigma(k)})$$

where all $k!$ terms are equal due to the alternating property of φ. □

Even subalgebra isomorphisms

As noted above, multivectors of even grade are closed under the geometric product, and form the even subalgebra $\mathcal{G}_+(p, q)$. There is an isomorphism $\mathcal{G}_+(p, q) \cong \mathcal{G}_+(q, p)$ given by $\bar{\boldsymbol{e}}_i := \boldsymbol{e}_i$ with opposite signature $\bar{\boldsymbol{e}}_i{}^2 := -\boldsymbol{e}_i^2$, since the factor of -1 occurs only an even number of times for even elements.

The even subalgebras are also isomorphic to full geometric algebras of one dimension less:

Lemma 7. *There are isomorphisms*

$$\mathcal{G}_+(p, q) \cong \mathcal{G}(p, q-1) \quad \textit{and} \quad \mathcal{G}_+(p, q) \cong \mathcal{G}(q, p-1)$$

when $q \geq 1$ and $p \geq 1$, respectively.

Proof. Select a unit vector $\boldsymbol{u} \in \mathcal{G}(p, q)$ with $\boldsymbol{u}^2 = -1$, and define a linear map $\Psi_{\boldsymbol{u}} : \mathcal{G}(p, q-1) \to \mathcal{G}_+(p, q)$ by

$$\Psi_{\boldsymbol{u}}(A) = \begin{cases} A & \text{if } A \text{ is even} \\ A \wedge \boldsymbol{u} & \text{if } A \text{ is odd} \end{cases}.$$

Note we are taking $\mathcal{G}(p,q-1) \subset \mathcal{G}(p,q)$ to be the subalgebra obtained by removing $\boldsymbol{u}$ (i.e., restricting V to $\boldsymbol{u}^\perp$) so there is a canonical inclusion from the domain of $\Psi_{\boldsymbol{u}}$ to the codomain. Let $A \in \mathcal{G}(p,q-1)$ be a k-vector. Note that $A \wedge \boldsymbol{u} = A\boldsymbol{u}$ since $\boldsymbol{u} \perp \mathcal{G}(p,q-1)$, and that A commutes with $\boldsymbol{u}$ if k is even and anticommutes if k is odd.

To so $\Psi_{\boldsymbol{u}}$ is a homomorphism, suppose $A, B \in \mathcal{G}(p,q-1)$ are both even; then $\Psi_{\boldsymbol{u}}(AB) = AB = \Psi_{\boldsymbol{u}}(A)\Psi_{\boldsymbol{u}}(B)$. If both are odd, then AB is even and $\Psi_{\boldsymbol{u}}(AB) = AB = -AB\boldsymbol{u}^2 = A\boldsymbol{u}B\boldsymbol{u} = \Psi_{\boldsymbol{u}}(A)\Psi_{\boldsymbol{u}}(B)$. If A is odd and B even, then $\Psi_{\boldsymbol{u}}(AB) = AB\boldsymbol{u} = A\boldsymbol{u}B = \Psi_{\boldsymbol{u}}(A)\Psi_{\boldsymbol{u}}(B)$ and similarly for A even and B odd. Injectivity and surjectivity are clear, so $\Psi_{\boldsymbol{u}}$ is an algebra isomorphism. □

The special case $\mathcal{G}_+(1,3) \cong \mathcal{G}(3)$ is of great relevance to special relativity, and is discussed in detail in **??**. Here the isomorphism $\Psi_{\boldsymbol{u}}$ is called a *space/time split* with respect to an observer of velocity $\boldsymbol{u}$. This provides an impressively efficient algebraic method for transforming relativistic quantities between inertial frames.

3.3. Higher notions of orthogonality

As discussed in section 3.1, the lack of a $\mathbb{Z}$-grading means that a geometric product of blades is generally an inhomogeneous multivector. Geometrically, the grade k part of product of blades reveals the degree to which the two blades are 'orthogonal' or 'parallel', in a certain k-dimensional sense.

To see this, first consider the special case where the product of blades a and b is a homogeneous k-blade. This occurs when there exists a common orthonormal basis $\{\boldsymbol{e}_i\}$ such that

$$a = \alpha \boldsymbol{e}_{i_1}\boldsymbol{e}_{i_2}\cdots\boldsymbol{e}_{i_p} \qquad \text{and} \qquad b = \beta \boldsymbol{e}_{j_1}\boldsymbol{e}_{j_2}\cdots\boldsymbol{e}_{j_q}$$

simultaneously, for scalars α, β. The resulting grade k is the number of basis vectors $\boldsymbol{e}_{h_i}$ which are not common to both a and b; i.e., $\{h_i\}$ is the symmetric difference of i and j.

$$ab = \pm\alpha\beta \boldsymbol{e}_{h_1}\boldsymbol{e}_{h_2}\cdots\boldsymbol{e}_{h_k}$$

Each pair of common basis vectors contribute an overall factor of $\boldsymbol{e}_i^2 = \pm 1$, and each transposition required to bring each pair together flips the overall sign. Thus, the possible values of k are separated by steps of two, with the maximum $k = p + q$ attained when no basis vectors are in common. In terms of the spans of the blades, we have

$$\begin{aligned} k &= \underbrace{\dim \operatorname{span}\{a\}}_{p} + \underbrace{\dim \operatorname{span}\{b\}}_{q} - \underbrace{2 \dim(\operatorname{span}\{a\} \cap \operatorname{span}\{b\})}_{2m} \\ &\in \{|p-q|, |p-q|+2, ..., p+q-2, p+q\}. \end{aligned}$$

Solving for the dimension of the intersection, we have

$$m = \frac{1}{2}(p + q - k).$$

Thus, the higher the grade k of the product ab, the lower the dimension m of the intersection of their spans.

We are used to the geometric meaning of two vectors being parallel or orthogonal. In terms of vector spans, they imply that the intersection is one or zero dimensional, respectively. Similarly, blades of higher grade can be 'parallel' or 'orthogonal' to varying degrees, depending on the dimension m of their intersection.

For example, the intersection of two 2-blades may be of dimension two, one or (in four or more dimensions) zero. The notion of parallel (i.e., being a scalar multiple) remains clear ($m = 2$), but there are now two different types of orthogonality for 2-blades ($m = 1$ and $m = 0$). An example of the first type can be pictured as two planes meeting at right-angles along a line; the second type requires at least four dimensions.

Definition 20. *A p-blade a and q-blade b satisfying $ab = \langle ab \rangle_k$ are called $\mathit{\Delta}$-ORTHOGONAL where $\Delta = k - |p - q| = n - m$.*

Informally, Δ-orthogonality of a and b means that ab is of the Δth grade above the minimum possible grade $|p - q|$. The higher Δ, the fewer linearly independent directions are shared by a and b. Different cases are exemplified in table 3.1.

p	q	k	$\langle ab \rangle_k$	Δ	m	*commutativity*	*geometric interpretation of* $ab = \langle ab \rangle_k$
1	1	0	$a \bullet b$	0	1	commuting	vectors are parallel; $a \parallel b \iff a = \lambda b$
1	1	2	$a \wedge b$	1	0	anticommuting	vectors are orthogonal $a \perp b$
2	2	0	$a \bullet b$	0	2	commuting	bivectors are parallel $a = \lambda b$
2	2	2	$a \times b$	1	1	anticommuting	bivectors are at right-angles to each other
2	2	4	$a \wedge b$	2	0	commuting	—
1	2	1	$a \bullet b$	0	1	anticommuting	vector a lies in plane of bivector b
1	2	3	$a \wedge b$	1	0	commuting	vector a is normal to plane of bivector b
2	3	1	$a \bullet b$	0	2	commuting	bivector a lies in span of trivector b
2	3	3	$\langle ab \rangle_3$	1	1	anticommuting	—
2	3	5	$a \wedge b$	2	0	commuting	—

Table 3.1.: Geometric interpretation of the k-blade $ab = \langle ab \rangle_k$ where a and b are of grades p and q respectively, and where $m = \dim(\mathrm{span}\{a\} \cap \mathrm{span}\{b\})$.

Familiarity with some special cases may aid intuition when considering general products of blades. For instance, if the product of two bivectors is $\sigma_1\sigma_2 = \sigma_1 \cdot \sigma_2 + \sigma_1 \times \sigma_2$, then it is understood that σ_1 has a component parallel to σ_2, and a component which meets σ_2 at right-angles along a line of intersection. In other words, σ_1 and σ_2 are planes that intersect along a line with some angle between them. On the other hand, if $\sigma_1\sigma_2 = \sigma_1 \wedge \sigma_2$, then the bivectors share no common direction, existing in orthogonal planes (a scenario requiring at least four dimensions).

3.4. Rotors and the Associated Lie Groups

There is a consistent pattern in the algebra isomorphisms listed above. Note how the complex numbers $\mathbb{C}$ are fit for describing $\mathrm{SO}(2)$ rotations in the plane, and the quaternions $\mathbb{H}$ describe $\mathrm{SO}(3)$ rotations in $\mathbb{R}^3$. Common to both the respective isomorphisms with $\mathcal{G}_+(2)$ and $\mathcal{G}_+(3)$ is the identification of each "imaginary unit" in $\mathbb{C}$ or $\mathbb{H}$ with a *unit bivector* in $\mathcal{G}(n)$.

- In 2d, there is one linearly independent bivector, $\boldsymbol{e}_1\boldsymbol{e}_2$, and one imaginary unit, i.

- In 3d, there are $\dim \mathcal{G}_2(3) = \binom{3}{2} = 3$ such bivectors, and so three imaginary units $\{\hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}}, \hat{\boldsymbol{k}}\}$ are needed.

- In $(1+3)$d, we have $\dim \mathcal{G}_2(1,3) = \binom{4}{2} = 6$, corresponding to three 'spacelike' $\{\hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}}, \hat{\boldsymbol{k}}\}$ and three 'timelike' $\{i\hat{\boldsymbol{\imath}}, i\hat{\boldsymbol{\jmath}}, i\hat{\boldsymbol{k}}\}$ units of $\mathbb{C} \otimes \mathbb{H}$.

The interpretation of a bivector is clear: it takes the role of an 'imaginary unit', generating a rotation through the oriented plane which it spans.

To see how bivectors act as rotations, observe that rotations in the $\mathbb{C}$-plane may be described as mappings $z \mapsto e^{\theta i} z$, while $\mathbb{R}^3$ rotations are described in $\mathbb{H}$ using a double-sided transformation law, $u \mapsto e^{\theta \hat{n}/2} u e^{-\theta \hat{n}/2}$, where $\hat{n} \in \operatorname{span}\{\hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}}, \hat{\boldsymbol{k}}\}$ is a unit quaternion defining the plane of rotation. Due to the commutativity of $\mathbb{C}$, the double-sided transformation law is actually general to both $\mathbb{C}$ and $\mathbb{H}$.

Similarly, rotations in a geometric algebra are described as

$$u \mapsto e^{-\theta \hat{b}/2} u e^{\theta \hat{b}/2},$$

where $\hat{b} \in \mathcal{G}_2(V, \eta)$ is a unit bivector. Multivectors of the form $R = e^{\sigma}$ for $\sigma \in \mathcal{G}_2(V, \eta)$ are called *rotors*. Immediate advantages to geometric algebra's rotor formalism are clear:

- *It is general to n dimensions, and to any metric signature.* Rotors describe generalised rotations,[23] depending on the metric and algebraic properties of the generating unitbivector σ. If $\sigma^2 < 0$, then e^{σ} describes a Euclidean rotation; if $\sigma^2 > 0$, then e^{σ} is a hyperbolic rotation or *Lorentz boost.*

[23] a.k.a., proper orthogonal transformations

- *Vectors are distinguished from bivectors.* One of the subtler points about quaternions is their transformation properties under reflection. A quaternion 'vector' $v = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}}$ reflects through the origin as $v \mapsto -v$, but a quaternion 'rotor' of the same value is invariant — vectors and pseudovectors are confused with the same kind of object. Not so in the geometric algebra: vectors are vectors in $\mathcal{G}_1$, and $\mathbb{R}^3$ pseudovectors are bivectors in $\mathcal{G}_2$. The price to pay for the introduction of more objects is not a price but a benefit: the generalisation to arbitrary dimensions is immediate and the geometric role of objects becomes clear. [24]

[24] See [5, 9, 15] for similarly impassioned testaments to the elegance of geometric algebra.

3.4.1. The rotor groups

We will now see more rigorously how the rotor formalism arises. An orthogonal transformation in n dimensions may be achieved by the composition of at most n reflections.[25] A reflection may be described in the geometric algebra by conjugation with an invertible vector. For instance, the linear map

[25] This is the Cartan–Dieudonné theorem [16].

$$A \mapsto -\boldsymbol{v}A\boldsymbol{v}^{-1} \tag{3.3}$$

reflects the multivector A along the vector $\boldsymbol{v}$, that is, across the hyperplane with normal $\boldsymbol{v}$. By composing reflections of this form, any orthogonal transformation may be built, acting on multivectors as

$$A \mapsto \pm RAR^{-1} \tag{3.4}$$

for some $R = \boldsymbol{v}_1\boldsymbol{v}_2 \cdots \boldsymbol{v}_3$, where the sign is positive for an even number of reflections, and negative for odd.

Scaling the axis of reflection $\boldsymbol{v}$ by a non-zero scalar λ does not affect the reflection map (3.3), since $\boldsymbol{v} \mapsto \lambda\boldsymbol{v}$ is cancelled out by $\boldsymbol{v}^{-1} \mapsto \lambda^{-1}\boldsymbol{v}^{-1}$. Therefore, a more direct correspondence exists between reflections and normalised vectors $\hat{\boldsymbol{v}}^2 = \pm 1$ (although there still remains an overall ambiguity in sign). For an orthogonal transformation built using normalised vectors,

$$R^{-1} = \hat{\boldsymbol{v}}_3^{-1} \cdots \hat{\boldsymbol{v}}_2^{-1}\hat{\boldsymbol{v}}_1^{-1} = \pm R^\dagger$$

since $\hat{\boldsymbol{v}}^{-1} = \pm\hat{\boldsymbol{v}}$, and hence eq. (3.4) may be written with the reversion instead of inversion:

$$A \mapsto \pm RAR^\dagger \tag{3.5}$$

All such elements $R^{-1} = \pm R^\dagger$ taken together form a group under the geometric product. This is called the *pin* group:

$$\mathrm{Pin}(p,q) := \{R \in \mathcal{G}(p,q) \mid RR^\dagger = \pm 1\}$$

There are two "pinors" for each orthogonal transformation, since $+R$ and $-R$ give the same map (3.5). Thus, the pin group forms a double cover of the orthogonal group $\mathrm{O}(p, q)$.

Furthermore, the even-grade elements of $\mathrm{Pin}(p, q)$ form a subgroup, called the *spin* group:

$$\mathrm{Spin}(p, q) := \{R \in \mathcal{G}_+(p, q) \mid RR^\dagger = \pm 1\}$$

This forms a double cover of $\mathrm{SO}(p, q)$.

Finally, the additional requirement that $RR^\dagger = 1$ defines the restricted spinor group, or the *rotor* group:

$$\mathrm{Spin}^+(p, q) := \{R \in \mathcal{G}_+(p, q) \mid RR^\dagger = 1\}$$

$$\begin{tikzcd}
\mathrm{Spin}^+ \arrow[r, phantom, "\subseteq"] \arrow[d, two heads] & \mathrm{Spin} \arrow[r, phantom, "\subset"] \arrow[d, two heads] & \mathrm{Pin} \arrow[d, two heads] \\
\mathrm{SO}^+ \arrow[r, phantom, "\subseteq"] & \mathrm{SO} \arrow[r, phantom, "\subset"] & \mathrm{O}
\end{tikzcd}$$

Figure 3.1.: Relationships between Lie groups associated with a geometric algebra. An arrow $a \twoheadrightarrow b$ signifies that a is a double-cover of b.

The rotor group is a double cover of the restricted special orthogonal group $\mathrm{SO}^+(p, q)$. Except for the degenerate case of $\mathrm{Spin}^+(1, 1)$, the rotor group is simply connected to the identity.

3.4.2. The bivector subalgebra

The multivector commutator product

$$A \times B := \frac{1}{2}(AB - BA) \tag{3.6}$$

forms a Lie bracket on the space of bivectors $\mathcal{G}_2$.

Proof. The commutator product $A \mapsto A \times \sigma$ with a bivector σ is a grade-preserving operation. If $A = \langle A \rangle_k$ then $A\sigma$ and σA are $\{k-2, k, k+2\}$-multivectors. The $k \pm 2$ parts are

$$\langle A \times \sigma \rangle_{k\pm 2} = \frac{1}{2}\big(\langle A\sigma \rangle_{k\pm 2} - \langle \sigma A \rangle_{k\pm 2}\big).$$

However, $\langle \sigma A \rangle_{k\pm 2} = s_{k\pm 2}\langle A^\dagger \sigma^\dagger \rangle_{k\pm 2} = -s_{k\pm 2}s_k\langle A\sigma \rangle_{k\pm 2}$ and the reversion signs[26] satisfy $s_{k\pm 2}s_k = -1$ for any k. Hence, $\langle A \times \sigma \rangle_{k\pm 2} = 0$, leaving only the grade k part, $A \times \sigma = \langle A \times \sigma \rangle_k$. Clearly eq. (3.6) is bilinear and satisfies the Jacobi identity, so $(\mathcal{G}_2, \times)$ is closed and forms a Lie algebra. □

[26] Recall from eq. (3.2) that $A^\dagger = s_k A$ for a k-vector where $s_k = (-1)^{\frac{(k-1)k}{2}}$.

Because the even subalgebra $\mathcal{G}_+ \supset \mathcal{G}_2$ is closed under the geometric product, the exponential $e^\sigma = 1 + \sigma + \frac{1}{2}\sigma^2 + \cdots$ of a bivector is an even multivector. Furthermore, note that the reverse $(e^\sigma)^\dagger = e^{\sigma^\dagger} = e^{-\sigma}$ is the inverse, and also that e^σ is continuously connected to the identity by the path $e^{\lambda\sigma}$ for $\lambda \in [0,1]$. Therefore, $e^\sigma \in \mathrm{Spin}^+$ is a rotor, and we have a Lie algebra–Lie group correspondence shown in fig. 3.2. Thus, both the rotor groups and their Lie algebras are directly represented within the mother algebra $\mathcal{G}(p,q)$.

$$\begin{tikzcd}
\mathrm{Spin}^+(p,q) \arrow[r, two heads] & \mathrm{SO}^+(p,q) \\
\mathcal{G}_2(p,q) \arrow[u, mapsto, "\exp"] \arrow[r, phantom, "\cong"] & \mathfrak{so}(p,q) \arrow[u, mapsto, "\exp"]
\end{tikzcd}$$

Figure 3.2.: The Lie algebras $\mathfrak{so}(p,q)$ and $\mathcal{G}_2(p,q)$ under $\times$ are isomorphic, and are associated respectively to $\mathrm{SO}^+(p,q)$ and its universal double cover $\mathrm{Spin}^+(p,q)$.

Part II.

General Relativity and Manifold Geometry

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