The minimum number of steps required for the Faddeev–LeVerrier multivector inverse algorithm

The Faddeev–LeVerrier inverse algorithm may be used to find the inverse of an $n \times n$ matrix A in exactly n steps (with one matrix multiplication per step). As input, the algorithm takes the matrix A and the dimension n. The algorithm succeeds if and only if an inverse exists.

This method can be used to invert elements in a geometric algebra $\mathcal{A}$ by considering a linear representation $\rho : \mathcal{A} \to \mathrm{GL}(n)$.

For best performance, we would like to know the minimum number of steps required by the algorithm for a given multivector.

Trivial cases

Trivially, real scalars admit a one-dimensional linear representation, and the method takes a single step.

If an element A has a scalar square $A^2 \in \mathbb{R}$, then its inverse is trivially $A^{-1} = A/A^2$. Additionally, since $A^{2n} \in \mathbb{R}$ and $A^{2n+1} \propto A$, the algebraic closure of A is $[\![A]\!] = \mathbb{R} \oplus \operatorname{span}\{A\}$ and hence there exists a two dimensional representation $\rho : [\![A]\!] \to \mathrm{GL}(2)$. Any k-vector with $k \in \{0, 1, d-1, d\}$ has scalar square and hence can be inverted in two steps.

General cases

Any geometric algebra over a d-dimensional vector space is itself 2^d-dimensional, so by the existence of the standard linear representation we know the algorithm works in 2^d steps. However, this can be reduced to $2^{\lceil d/2 \rceil}$ for a general d-dimensional multivector (Prodanov, 2024).

From numerical simulations

We have “seen” that even multivectors may be inverted in $2^{\lfloor d/2 \rfloor}$ steps (tested for non-degenerate geometric algebras in ≤ 13 dimensions). Furthermore, for homogeneous multivectors of a given grade, the minimum number of steps is shown in Table 1.

$d \backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	full	even
0	1														1	1
1	1	2													2	1
2	1	2	2												2	2
3	1	2	2	2											4	2
4	1	2	4	2	2										4	4
5	1	2	4	4	2	2									8	4
6	1	2	8	8	4	2	2								8	8
7	1	2	8	16	8	8	2	2							16	8
8	1	2	16	16	16	16	16	2	2						16	16
9	1	2	16	16	16	32	16	16	2	2					32	16
10	1	2	32	16	16	32	32	16	16	2	2				32	32
11	1	2	32	32	16	32	32	32	16	32	2	2			64	32
12	1	2	64	32	32	32	64	32	32	32	64	2	2		64	64
13	1	2	64	64	32	64	64	64	32	64	64	64	2	2	128	64

Table 1: Minimum number of steps required to invert a d-dimensional k-vector (for any non-degenerate metric).

References

Prodanov, D. (2024). Algorithmic Computation of Multivector Inverses and Characteristic Polynomials in Non-Degenerate Clifford Algebras. In B. Sheng, L. Bi, J. Kim, N. Magnenat-Thalmann, & D. Thalmann (Eds.), *Advances in Computer Graphics* (pp. 379–390). Springer Nature Switzerland. https://doi.org/10.1007/978-3-031-50078-7_30