Conditional Gaussian

Let $\boldsymbol{x} \sim \mathcal{N}(\mu, \Sigma)$ be a normally distributed vector in $\mathbb{R}^D$. Suppose the space $\mathbb{R}^D = \mathbb{R}^m \oplus \mathbb{R}^n$ is split in two and write:

$$\boldsymbol{x} = \begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \end{bmatrix}, \quad \mu = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} \Sigma_1 & R \\ R^T & \Sigma_2 \end{bmatrix}$$

Then, $\boldsymbol{x}_1$ given $\boldsymbol{x}_2$ is distributed as:

$$\boxed{\boldsymbol{x}_1 \mid \boldsymbol{x}_2 \sim \mathcal{N}\left(\mu_1 + R\Sigma_2^{-1}(\boldsymbol{x}_2 - \mu_2), \Sigma_1 - R\Sigma_2 R^T\right)}$$

Proof. The conditional distribution $P(\boldsymbol{x}_1 \mid \boldsymbol{x}_2) = P(\boldsymbol{x}_1, \boldsymbol{x}_2)/P(\boldsymbol{x}_2)$ is also Gaussian, as it is the product of two exponentials of quadratic forms. To fully specify a Gaussian distribution, we need only find the leading coefficients of x that appear in the exponent (as mentioned in `[gaussian]`).

$$\begin{aligned} Q(\boldsymbol{x}_1) &\coloneqq -2 \ln P(\boldsymbol{x}_1 \mid \boldsymbol{x}_2) \\ &= -2 \ln P(\boldsymbol{x}_1, \boldsymbol{x}_2) + 2 \ln P(\boldsymbol{x}_2) \\ &= \begin{bmatrix} \boldsymbol{x}_1 - \mu_1 \\ \boldsymbol{x}_2 - \mu_2 \end{bmatrix}^T \begin{bmatrix} \Sigma_1 & R \\ R^T & \Sigma_2 \end{bmatrix}^{-1} \begin{bmatrix} \boldsymbol{x}_1 - \mu_1 \\ \boldsymbol{x}_2 - \mu_2 \end{bmatrix} + \text{constant} \end{aligned}$$

The constant is independent on $\boldsymbol{x}_1$, but may depend on $\boldsymbol{x}_2$. Using results from (Rasmussen & Williams, 2008, A.3), the inverse of the block matrix is of the form

$$\begin{bmatrix} \Sigma_1 & R \\ R^T & \Sigma_2 \end{bmatrix}^{-1} = \begin{bmatrix} \tilde{\Sigma}_1 & \tilde{R} \\ \tilde{R}^T & \tilde{\Sigma}_2 \end{bmatrix} \quad \text{where} \begin{cases} \tilde{\Sigma}_1 &= (\Sigma_1 - R\Sigma_2 R^T)^{-1} \\ \tilde{R} &= -\tilde{\Sigma}_1 R \Sigma_2^{-1} \\ \tilde{R}^T &= -\Sigma_2^{-1} R^T \tilde{\Sigma}_1 \\ \tilde{\Sigma}_2 &= \text{not important} \end{cases}$$

Using this, we may expand the quadratic form in $\boldsymbol{x}_1$ as

$$\begin{aligned} Q(\boldsymbol{x}_1) &= (\boldsymbol{x}_1 - \mu_1)^T \tilde{\Sigma}_1 (\boldsymbol{x}_1 - \mu_1) + \boldsymbol{x}_1^T \tilde{R}^T (\boldsymbol{x}_2 - \mu_2) + (\boldsymbol{x}_2 - \mu_2)^T \tilde{R} \boldsymbol{x}_1 + \text{constant} \\ &= \boldsymbol{x}_1^T \underbrace{\tilde{\Sigma}_1}_{\Sigma^{-1}} \boldsymbol{x}_1 - \boldsymbol{x}^T \underbrace{\left[\tilde{\Sigma}_1 \mu_1 - \tilde{R}^T (\boldsymbol{x}_2 - \mu_2)\right]}_{\Sigma^{-1}\mu} - \underbrace{\left[\mu_1^T \tilde{\Sigma}_1 - (\boldsymbol{x}_2 - \mu_2)^T \tilde{R}\right]}_{\mu^T \Sigma^{-1}} \boldsymbol{x}_1 \end{aligned}$$

The underbraces show the corresponding coefficients for a standard Gaussian, $\mathcal{N}(\mu, \Sigma)$. The resulting mean and covariance matrix are therefore

$$\begin{aligned} \Sigma &= \tilde{\Sigma}_1^{-1} \\ \mu &= \tilde{\Sigma}_1^{-1} (\tilde{\Sigma}_1 \mu_1 - \tilde{R}^T (\boldsymbol{x}_2 - \mu_2)) \end{aligned}$$

which, using the fact that $\Sigma_2^{-1} R^T = R \Sigma_2^{-1}$ is symmetric, can be expressed in terms of the original block matrix components as:

$$\begin{aligned} \Sigma &= \Sigma_1 - R\Sigma_2 R^T \\ \mu &= \mu_1 + R\Sigma_2^{-1}(\boldsymbol{x}_2 - \mu_2) \end{aligned}$$

References

Rasmussen, C. E., & Williams, C. K. I. (2008). *Gaussian Processes for Machine Learning* (3. print). MIT Press.