

Variance of a sum of random variables

Let X and Y be random variables.

$$\text{Var}[X] := \langle (X - \langle X \rangle)^2 \rangle = \langle X^2 \rangle - \langle X \rangle^2$$

Then:

$$\begin{aligned}\text{Var}[X + Y] &= \langle (X + Y)^2 \rangle - \langle X + Y \rangle^2 \\ &= \langle X^2 + 2XY + Y^2 \rangle - (\langle X \rangle^2 + 2\langle X \rangle \langle Y \rangle + \langle Y \rangle^2) \\ &= \text{Var}[X] + \text{Var}[Y] + 2(\langle XY \rangle - \langle X \rangle \langle Y \rangle)\end{aligned}$$

Now, if the random variables X and Y are independent in the sense that $\rho(X, Y) = \rho(X)\rho(Y)$, then the expected value of their product is the product of their expected values.

$$\langle XY \rangle = \int XY \rho(X, Y) \, dX \, dY = \int X \rho(X) \, dX \int Y \rho(Y) \, dY = \langle X \rangle \langle Y \rangle$$

Hence

$$X \perp Y \implies \text{Var}[X + Y] = \text{Var}[X] + \text{Var}[Y]$$

where $X \perp Y \iff \rho(X, Y) = \rho(X)\rho(Y)$.