Multivector Conjugation

Lemma. Conjugation by a 1-vector u is a reflection. In terms of the projections and rejections,

$$uAu^{-1} = \left(A^{\perp u}\right)^{\star} - \left(A^{\| u}\right)^{\star}$$

for any multivector A.

Proof. Assume A is a k-vector and then use linearity to extend to general multivectors. Using the projection and rejection to write $A = A^{\perp u} + A^{\| u}$, we have

$$uA^{\perp u} = u \wedge A^{\perp u} = \left(A^{\perp u}\right)^{\star} \wedge u = \left(A^{\perp u}\right)^{\star} u$$

and similarly

$$uA^{\| u} = u \rfloor A^{\| u} = \widetilde{\widetilde{A^{\| u}} \lfloor u} = \mathfrak{s}_{k-1}\mathfrak{s}_k A^{\| u} \lfloor u = -(-1)^k A^{\| u} \lfloor u = -\left(A^{\| u}\right)^{\star} \lfloor u = -\left(A^{\| u}\right)^{\star} u$$

where $\mathfrak{s}_k$ is the reversion sign. Summing and left-multiplying by u^{-1} gives the result. ∎

Lemma. Conjugation by an invertible multivector s is an automorphism.

Proof. Let $a, b \in G$ be general multivectors. then $sabs^{-1} = \left(sas^{-1}\right)\left(sbs^{-1}\right)$. ∎

In particular, this means multivector conjugation is grade-preserving.