Spacetime algebra

Define:

- The spacetime basis vectors $\gamma_0^2 = -\gamma_i^2 = \pm 1$ for $i \in \{1, 2, 3\}$ and $\gamma_i \gamma_j = -\gamma_i \gamma_j$ for $i \neq j$.
- $\gamma^\mu = \pm \gamma_\mu$ such that $\gamma^\mu \gamma_\mu = 1$ for each of $\mu \in \{0, 1, 2, 3\}$.
- The relative vectors $\vec{\sigma}_i := \gamma_i \gamma^0$ and $\vec{\sigma}^i := \gamma_0 \gamma^i$.
- The pseudoscalar $\mathbb{I} := \gamma_0 \gamma_1 \gamma_2 \gamma_3$.

Note:

- The relative vectors $\vec{\sigma}_i$ for $i \in \{1, 2, 3\}$ form a basis for the geometric algebra of 3d space.
- $\vec{\sigma}_i^2 = \left(\vec{\sigma}^i\right)^2 = 1$ for each $i \in \{1, 2, 3\}$.
- $\mathbb{I} = \gamma_0 \gamma_1 \gamma_2 \gamma_3 = \vec{\sigma}_1 \vec{\sigma}_2 \vec{\sigma}_3$ and $-\mathbb{I} = \gamma^0 \gamma^1 \gamma^2 \gamma^3 = \vec{\sigma}^1 \vec{\sigma}^2 \vec{\sigma}^3$

Define:

$$\partial := \gamma^\mu \partial_\mu \equiv \sum_{\mu=0}^{3} \gamma^\mu \frac{\partial}{\partial x_\mu}$$

$$\vec{\nabla} := \vec{\sigma}^i \partial_i = \sum_{i=1}^{3} \vec{\sigma}^i \frac{\partial}{\partial x_i}$$

Note:

$$\gamma_0 \partial = \partial_0 + \vec{\nabla} = \frac{1}{c} \frac{\partial}{\partial t} + \vec{\nabla}$$

$$\partial \gamma_0 = \partial_0 - \vec{\nabla} = \frac{1}{c} \frac{\partial}{\partial t} - \vec{\nabla}$$

Maxwell's equations

Define:

- The Faraday bivector $F = \vec{E} + c\mathbb{I}\vec{B}$ where $\vec{E} = E^i \vec{\sigma}_i$ and $\vec{B} = B^i \vec{\sigma}_i$.
- The 4-current $J = J_\mu \gamma^\mu$ where $J_0 = \frac{\rho}{\varepsilon_0}$ and $J_i \vec{\sigma}^i = -c\mu_0 \vec{j}$.
- Maxwell's equation $\partial F = J$.

Derive:

Perform a spacetime split by left-multiplying by γ_0.

$$\gamma_0 \partial F = \gamma_0 J$$

$$= \left(\frac{1}{c}\frac{\partial}{\partial t} + \vec{\nabla}\right)\left(\vec{E} + c\mathbb{I}\vec{B}\right) = J_0 + \vec{\sigma}^i J_i$$

$$= \frac{1}{c}\frac{\partial \vec{E}}{\partial t} + \vec{\nabla}\vec{E} + \mathbb{I}\frac{\partial \vec{B}}{\partial t} + c\vec{\nabla}\mathbb{I}\vec{B} = \frac{\rho}{\varepsilon_0} - c\mu_0 \vec{j}$$

Using $\cdot$ and $\wedge$ in the sense of the 3d algebra, note that $\vec{\nabla}\vec{E} = \underbrace{\vec{\nabla} \cdot \vec{E}}_{(0)} + \underbrace{\vec{\nabla} \wedge \vec{E}}_{(2)}$

and

$$\begin{aligned}
\vec{\nabla}\mathbb{I}\vec{B} &= \underbrace{\vec{\nabla} \cdot \mathbb{I}\vec{B}}_{(1)} + \underbrace{\vec{\nabla} \wedge \mathbb{I}\vec{B}}_{(3)} \\
&= \langle \mathbb{I}\vec{\nabla}\vec{B} \rangle_1 + \langle \mathbb{I}\vec{\nabla}\vec{B} \rangle_3 \\
&= \mathbb{I}\langle \vec{\nabla}\vec{B} \rangle_2 + \mathbb{I}\langle \vec{\nabla}\vec{B} \rangle_0 \\
&= \underbrace{\mathbb{I}\vec{\nabla} \wedge \vec{B}}_{(1)} + \underbrace{\mathbb{I}\vec{\nabla} \cdot \vec{B}}_{(3)}
\end{aligned}$$

Separate the spacetime split Maxwell equation into grades:

Grade	Projection
0	$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$
1	$\frac{1}{c}\frac{\partial \vec{E}}{\partial t} + c\mathbb{I}\vec{\nabla} \wedge \vec{B} = -c\mu_0 \vec{j}$
2	$\vec{\nabla} \wedge \vec{E} + \mathbb{I}\frac{\partial \vec{B}}{\partial t} = 0$
3	$\mathbb{I}\vec{\nabla} \cdot \vec{B} = 0$

Using the relation $\vec{u} \wedge \vec{v} = \mathbb{I}\left(\vec{u} \times \vec{b}\right)$ with the vector cross product, these take the traditional form:

$$\begin{aligned}
&\text{Gauß's law} & \vec{\nabla} \cdot \vec{E} &= \frac{\rho}{\varepsilon_0} \\
&\text{Ampère's law} & \vec{\nabla} \times \vec{B} &= \mu_0 \vec{j} + \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} = \mu_0\left(\vec{j} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t}\right) \\
&\text{Faraday's law} & \vec{\nabla} \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} \\
& & \vec{\nabla} \cdot \vec{B} &= 0
\end{aligned}$$

Summary:

If $\vec{E}$ and $\vec{B}$ are the electric and magnetic fields, ρ is charge density, and $\vec{j}$ is current density,

- $F = \vec{E} + c\mathbb{I}\vec{B}$ is the Faraday bivector, and
- $J = \frac{\rho}{\varepsilon_0}\gamma^0 - c\mu_0 j_i \gamma^i$ is the charge 4-current,

then Maxwell's equations are $\partial F = J$.

Electromagnetic plane waves

A solution to $\partial F = 0$ is

$$F = A\sin(\omega t - kx)\big(\vec{\sigma}_y + \mathbb{I}\vec{\sigma}_z\big)$$

which is a plane wave moving in the $+x$ direction, with $\vec{E}$ oscillating along $+y$ and $\vec{B}$ along $+z$.

Derive:

$$\partial F = \left\{\begin{array}{cc} (3) & 0 \\ (2) & 0 \\ (1) & \vec{\nabla} \\ (0) & \frac{1}{c}\frac{\partial}{\partial t} \end{array}\right\} A\sin(\omega t - kx) \left\{\begin{array}{cc} (3) & 0 \\ (2) & \mathbb{I}\vec{\sigma}_z \\ (1) & \vec{\sigma}_y \\ (0) & 0 \end{array}\right\}$$

$$= A\sin(\omega t - kx) \left\{\begin{array}{cc} (3) & 0 \\ (2) & 0 \\ (1) & -k\vec{\sigma}_x \\ (0) & \frac{1}{c}\omega \end{array}\right\} \left\{\begin{array}{cc} (3) & 0 \\ (2) & \mathbb{I}\vec{\sigma}_z \\ (1) & \vec{\sigma}_y \\ (0) & 0 \end{array}\right\}$$

$$= A\sin(\omega t - kx) \left\{\begin{array}{cc} (3) & -k\vec{\sigma}_x \wedge \mathbb{I}\vec{\sigma}_z \\ (2) & \frac{\omega}{c}\mathbb{I}\vec{\sigma}_z - k\vec{\sigma}_x \wedge \vec{\sigma}_y \\ (1) & \frac{\omega}{c}\vec{\sigma}_y - k\vec{\sigma}_x \cdot (\mathbb{I}\vec{\sigma}_z) \\ (0) & -k\vec{\sigma}_x \cdot \vec{\sigma}_y \end{array}\right\}$$

$$= A\sin(\omega t - kx) \left\{\begin{array}{cc} (3) & 0 \\ (2) & \big(\frac{\omega}{c} - k\big)\vec{\sigma}_x\vec{\sigma}_y \\ (1) & \big(\frac{\omega}{c} - k\big)\vec{\sigma}_y \\ (0) & 0 \end{array}\right\}$$

This vanishes if and only if $k = \frac{\omega}{c}$.