

**Proof that**  $\exp([A, -])X = \exp(A)X \exp(-A)$ .

**Lemma 1**

$$\exp([A, -])X = \exp(A)X \exp(-A)$$

**Proof** Expanding the r.h.s.,

$$\begin{aligned} \exp(A)X \exp(-A) &= \left[ \sum_{n=0}^{\infty} \frac{1}{n!} A^n \right] X \left[ \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} A^n \right] \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^n \frac{(-1)^k}{k!(n-k)!} A^{n-k} X A^k \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} [A, -]^n X \quad \text{via Lemma 2} \\ &= \exp([A, -])X \end{aligned}$$

**Lemma 2**

$$[A, -]^n X = \sum_{k=0}^n (-1)^k \binom{n}{k} A^{n-k} X A^k$$

**Proof** by induction. Assuming true for  $n$ ,

$$[A, -]^n AX = \sum_{k=0}^{n+1} (-1)^k \binom{n+1}{k} \frac{n+1-k}{n+1} A^{(n+1)-k} X A^k$$

via Lemma 3, and

$$\begin{aligned} [A, -]^n XA &= \sum_{k=0}^n (-1)^k \binom{n+1}{k+1} \frac{k+1}{n+1} A^{(n+1)-(k+1)} X A^{k+1} \\ &= - \sum_{\bar{k}=0}^{n+1} (-1)^{\bar{k}} \binom{n+1}{\bar{k}} \frac{\bar{k}}{n+1} A^{(n+1)-\bar{k}} X A^{\bar{k}} \end{aligned}$$

via Lemma 4. Taking both together,

$$\begin{aligned} [A, -]^{n+1} X &= [A, -](AX - XA) \\ &= \sum_{k=0}^{n+1} (-1)^k \binom{n+1}{k} A^{(n+1)-k} X A^k \end{aligned}$$

which shows that the  $n+1$  case holds, and hence  $\forall n$ .

**Lemma 3**

$$\binom{n}{k} = \binom{n+1}{k} \frac{n+1-k}{n+1}$$

**Proof**

$$\frac{n!}{k!(n-k)!} = \frac{(n+1)!}{k!(n+1-k)!} \frac{n+1-k}{n+1}$$

**Lemma 4**

$$\binom{n}{k} = \binom{n+1}{k+1} \frac{k+1}{n+1}$$

**Proof**

$$\frac{n!}{k!(n-k)!} = \frac{(n+1)!}{(k+1)!(n-k)!} \frac{k+1}{n+1}$$