Product of Gaussian probability density functions

The product of two Gaussian PDFs is an unnormalised Gaussian:

$$\mathcal{N}(\mu_1, \Sigma_1)\mathcal{N}(\mu_2, \Sigma_2) \propto \mathcal{N}\left((\Sigma_1 + \Sigma_2)^{-1}(\Sigma_1\mu_2 + \Sigma_2\mu_1), \left(\Sigma_1^{-1} + \Sigma_2^{-1}\right)^{-1}\right)$$

For many Gaussians:

$$\mathcal{N}(\mu_1, \Sigma_1)\mathcal{N}(\mu_2, \Sigma_2)\cdots\mathcal{N}(\mu_N, \Sigma_N) \propto \mathcal{N}\left(\Sigma\left(\Sigma_1^{-1}\mu_1 + \cdots + \Sigma_N^{-1}\mu_N\right), \Sigma\right)$$
$$\text{where } \Sigma = \left(\Sigma_1^{-1} + \cdots + \Sigma_N^{-1}\right)^{-1}$$

In other words, precisions are summed and means weighted by precisions are summed.

$$\Sigma^{-1} = \Sigma_1^{-1} + \Sigma_2^{-1} \quad \text{and} \quad \Sigma^{-1}\mu = \Sigma_1^{-1}\mu_1 + \Sigma_2^{-1}\mu_2.$$

The D-dimensional Gaussian pdf with mean vector μ and covariance matrix Σ is

$$\mathcal{N}(\mu, \Sigma) := \frac{1}{\sqrt{\tau^D \det(\Sigma)}} \exp\left(-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)\right)$$

Consider the product of two D-dimensional Gaussians:

$$\mathcal{N}_{12} := \mathcal{N}(\mu_1, \Sigma_1)\mathcal{N}(\mu_2, \Sigma_2) \propto \exp\left(-\frac{1}{2}(x-\mu_1)^T\Sigma_1^{-1}(x-\mu_1) - \frac{1}{2}(x-\mu_2)^T\Sigma_2^{-1}(x-\mu_2)\right)$$

The result is also a Gaussian. To find the resulting mean μ and covariance matrix Σ, rearrange the exponent as

$$\begin{aligned}
-2\log(\mathcal{N}_{12}) &= (x-\mu_1)^T\Sigma_1^{-1}(x-\mu_1) + (x-\mu_2)^T\Sigma_2^{-1}(x-\mu_2) + c \\
&= x^T\left(\Sigma_1^{-1} + \Sigma_2^{-1}\right)x - x^T\left[\Sigma_1^{-1}\mu_1 + \Sigma_2^{-1}\mu_2\right] - \left[\mu_1^T\Sigma_1^{-1} + \mu_2^T\Sigma_2^{-1}\right]x + c
\end{aligned}$$

where c is constant with respect to x. Then compare this to the exponent of a single Gaussian:

$$\begin{aligned}
-2\log(\mathcal{N}(\mu, \Sigma)) &= (x-\mu)^T\Sigma^{-1}(x-\mu) \\
&= x^T\Sigma^{-1}x - x^T\Sigma^{-1}\mu + \mu^T\Sigma^{-1}x + c
\end{aligned}$$

By equating coefficients of x, we see that the resulting mean and covariance matrix are

$$\begin{aligned}
\Sigma &= \left(\Sigma_1^{-1} + \Sigma_2^{-1}\right)^{-1} \\
\mu &= \Sigma\left(\Sigma_1^{-1}\mu_1 + \Sigma_2^{-1}\mu_2\right) \\
&= (\Sigma_1 + \Sigma_2)^{-1}(\Sigma_1\mu_2 + \Sigma_2\mu_1)
\end{aligned}$$

The last line follows from the fact that covariance matrices are symmetric, so $\Sigma_1\Sigma_2\Sigma_1^{-1} = \Sigma_2$ and similarly for $1 \leftrightarrow 2$.