Orthogonal splits of geometric algebras

Let G be a geometric algebra. For any 1-vector u, we can define the subspaces

$$\begin{aligned} G^{\parallel u} &:= \left\{A^{\parallel u} \mid A \in G\right\} = \{A \in G \mid u \wedge A = 0\} \\ G^{\perp u} &:= \left\{A^{\perp u} \mid A \in G\right\} = \{A \in G \mid u \rfloor A = 0\} \end{aligned}$$

where we use the multivector projections and rejections $A = A^{\parallel u} + A^{\perp u}$.

Lemma. $G = G^{\perp u} \oplus G^{\parallel u}$

Proof. Any $A \in G$ is a sum of its projections and rejections $A^{\parallel u} \in G^{\parallel u}$ and $A^{\perp u} \in G^{\perp u}$.

Lemma. $G^{\perp u}$ is a subalgebra.

Proof. Let $A, B \in G^{\perp u}$ which means $u \rfloor A = u \rfloor B = 0$. Since $u \rfloor$ is linear, $G^{\perp u}$ is a vector space. To show that $G^{\perp u}$ is closed under the geometric product, observe that $u \rfloor (AB) = (u \rfloor A)B + A^\star(u \rfloor B) = 0$ from the anti-derivation identity. ∎

Lemma. Any element $A \in G^{\parallel u}$ is of the form $A = uA_\perp = u \wedge A_\perp$ where $A_\perp \in G^{\perp u}$.

Proof. For any $A \in G^{\parallel u}$ take $A_\perp := u^{-1}A$. To show $A_\perp \in G^{\perp u}$, write

$$u \rfloor A_\perp = u \rfloor \left(u^{-1}A\right) = \left(u \rfloor u^{-1}\right)A - u^{-1}(u \rfloor A) = A - u^{-1}(uA) = 0$$

using the anti-derivation identity. Hence, $A = uA_\perp$. ∎

Corollary. $G^{\parallel u} = uG^{\perp u} = \left\{uA \mid A \in G^{\perp u}\right\}$

Proof. By the lemma above, the map $f(uA) = A$ is well defined for $f : G^{\parallel u} \to G^{\perp u}$. Since it has inverse $f^{-1}(A) = uA$ it is bijective so $f^{-1}\left(G^{\perp u}\right) = G^{\parallel u}$ and hence $uG^{\perp u} = G^{\parallel u}$.

Lemma. $G = G^{\perp u} \oplus G^{\parallel u}$ forms a $\mathbb{Z}_2$-grading: elements multiply under the geometric product according to the multiplication table:

$$\begin{array}{c|cc} & G^{\perp u} & G^{\parallel u} \\ \hline G^{\perp u} & G^{\perp u} & G^{\parallel u} \\ G^{\parallel u} & G^{\parallel u} & G^{\perp u} \end{array}$$

Proof. We have already shown that $G^{\perp u}$ is a subalgebra, so $G^{\perp u} \times G^{\perp u} \to G^{\perp u}$ under the geometric product.

To show that $G^{\perp u} \times G^{\parallel u} \to G^{\parallel u}$, take $A \in G^{\perp u}$ and $B \in G^{\parallel u}$. By the previous lemma, $B = uB_\perp$ where $B_\perp \in G^{\perp u}$. We show $u \wedge (AB) = 0$ and hence $AB \in G^{\parallel u}$ by writing

$$u \rfloor (AB) = \cancel{(u \rfloor A)}uB_\perp + A^\star(u \rfloor (uB_\perp)) = A^\star u^2 B_\perp - A^\star u\cancel{(u \rfloor B_\perp)} = uAuB_\perp = uAB$$

where we used $u \rfloor A = 0 \iff uA = u \wedge A = A^\star \wedge u = A^\star u$. Same for $G^{\parallel u} \times G^{\perp u} \to G^{\parallel u}$.

To complete the table, pick $A, B \in G^{\parallel u}$ and let $A = uA_\perp$ and $B = uB_\perp$ where $A_\perp, B_\perp \in G^{\perp u}$ as before. We can show $u \rfloor (AB) = 0$ since

$$\begin{aligned} u \rfloor (uA_\perp uB_\perp) &= (u \rfloor u)A_\perp uB_\perp - u\cancel{(u \rfloor A_\perp)}uB_\perp - uA_\perp^\star(u \rfloor u)B_\perp + uA_\perp^\star u\cancel{(u \rfloor B_\perp)} \\ &= A_\perp uB_\perp - uA_\perp^\star B_\perp = A_\perp uB_\perp - A_\perp uB_\perp = 0 \end{aligned}$$

which means $AB \in G^{\perp u}$. ∎