Proof that $\det \circ \exp = \exp \circ \operatorname{tr}$

> **Ingredient.** *Triangular matrices are closed under multiplication.* Let A and B be upper triangular matrices, so that $A_{ij} = B_{ij} = 0$ for $i > j$. From
>
> $$(AB)_{ij} = \sum_k A_{ik}B_{kj} = \sum_k \begin{cases} A_{ik}B_{kj} & \text{if } i \le k \le j \\ 0 & \text{otherwise} \end{cases}$$
>
> it follows that AB is also upper triangular. In particular, $(AB)_{ii} = A_{ii}B_{ii}$.

> **Ingredient.** *Only diagonal elements of triangular matrices affect the trace.*
>
> $$\operatorname{tr}(AB) = \sum_k (AB)_{kk} = \sum_k A_{kk}B_{kk} = \operatorname{diag}(A) \cdot \operatorname{diag}(B)$$

> **Ingredient.** *Only diagonal elements of triangular matrices affect the determinant.* If A is triangular, then
>
> $$\det(A) = \sum_{\sigma \in S_n} (-1)^\sigma \prod_{i=1}^{n} A_{i\sigma(i)} = \prod_{i=1}^{n} A_{ii}$$
>
> because all the permutations σ except the identity have some $1 \le k \le n$ such that $\sigma(k) < k$.

> **Ingredient.** *Any square matrix A can be put in Jordan normal form $A = PJP^{-1}$, where J is upper triangular.*

$$\begin{aligned}
\det(\exp(A)) &= \det\left(\exp\left(PJP^{-1}\right)\right) \\
&= \det\left(P\exp(J)P^{-1}\right) \\
&= \det(P)\det(\exp(J))\det\left(P^{-1}\right) \\
&= \det(\exp(J)) \\
&= \prod_{i=1}^{n} \exp(J)_{ii} \\
&= \prod_{i=1}^{n} \sum_{n=0}^{\infty} \frac{1}{n!}(J^n)_{ii} \\
&= \prod_{i=1}^{n} \sum_{n=0}^{\infty} \frac{1}{n!}(J_{ii})^n \\
&= \prod_{i=1}^{n} \exp(J_{ii}) \\
&= \exp\left(\sum_{i=1}^{n} J_{ii}\right) \\
&= \exp(\operatorname{tr}(J)) \\
&= \exp\left(\operatorname{tr}\left(PP^{-1}J\right)\right) \\
&= \exp\left(\operatorname{tr}\left(PJP^{-1}\right)\right) \\
&= \exp(\operatorname{tr}(A))
\end{aligned}$$