Reciprocal basis

Let $\{e_i\} \subset V$ be a basis and $\cdot : V \times V \to \mathbb{R}$ a bilinear form.

A *reciprocal basis* $\{\overline{e}_i\} \subset V$ with respect to $\{e_i\}$ and $\cdot$ satisfies $e_i \cdot \overline{e}_i = \delta_{ij}$.

The basis $\{e_i\}$ need not be orthogonal.

Solving for the reciprocal basis

There exists some matrix A such that $u \cdot v = u^T A v$.

Using matrix notation, $E = (e_1 \ \cdots \ e_n)$ is an $n \times n$ matrix.

Define:

$$G = E^T A E \Longleftrightarrow G_{ij} = e_i^T A e_j = e_i \cdot e_j$$

We want to find $\overline{E} = (\overline{e}_1 \ \cdots \ \overline{e}_n)$ so that:

$$I = E^T A \overline{E} \Longleftrightarrow \delta_{ij} = e_i^T A \overline{e}_j = e_i \cdot \overline{e}_j$$

Solve this as $\overline{E} = \left(E^T A\right)^{-1} I = \left(E^T A E E^{-1}\right)^{-1} = \left(G E^{-1}\right)^{-1} = E G^{-1} \Leftrightarrow \overline{e}_i = e_i G^{-1}$.

$$\overline{E} = E G^{-1} \Longleftrightarrow \overline{e}_k = e_k \left[e_i \cdot e_j\right]_{ij}^{-1}$$

See `[reciprocal-basis-test]` for Julia example.