Poincaré half-plane model

Define a metric on the upper half-plane $\mathbb{R} \times (0, \infty)$ by

$$g = \frac{\mathrm{d}x^2 + \mathrm{d}y^2}{y^2}$$

Transformations in the half-plane which preserve this metric (the isometries) are given by

$$z \overset{\psi}{\mapsto} \frac{az + b}{cz + d}$$

where $z = x + iy$, for any real numbers a, b, c, d. By differentiating this mapping, we find that tangent vectors transform as

$$\delta z \mapsto J \delta z \quad \text{where} \quad J := \frac{ad - bc}{(cz + d)^2}$$

where δz is a tangent vector at z. By showing that

$$g(u, v)|_z = g(Ju, Jv)|_{\psi(z)}$$

we prove that the metric is preserved by this family of transformations.

See `[hyperbolic-isometries]` for numerical proofs.

Example transformations

An upward ray transforms as:

$$ie^t \mapsto \frac{aie^t + b}{cie^t + d}$$

which, written without reference to complex numbers, is

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \frac{1}{c^2 e^{2t} + d^2} \begin{pmatrix} ace^{2t} + bd \\ ade^t - bce^t \end{pmatrix}$$