

Week 1: Spot and Discount Curve

HW1

Discount Curve

- For **bootstrapping**, we need a perfect defined matrix $n \times n$:

```
RESTRICT_YLD = True
```

```
RESTRICT_TIPS = True
```

```
RESTRICT_DTS_MATURING = True
```

```
RESTRICT_REDUNDANT = True
```

```
data = filter_treasuries(quotes, t_date=t_current, filter_yld = RESTRICT_YLD, filter_tips = RES`  
CF = filter_treasury_cashflows(calc_cashflows(data),filter_maturity_dates=RESTRICT_DTS_MATURING`
```

- For **OLS**: We have more than one bond in a specific maturity (overdefined system), but we avoid having dates with no payments (if not, the regression is going to be close unidenfied due to multicollinearity, so it will have extrem values on those maturities)

```
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```

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```

```
RESTRICT_DTS_MATURING = True
```

```
RESTRICT_REDUNDANT = False
```

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data = filter_treasuries(quotes, t_date=t_current, filter_yld = RESTRICT_YLD, filter_tips = RES`  
CF = filter_treasury_cashflows(calc_cashflows(data),filter_maturity_dates=RESTRICT_DTS_MATURING`
```

- Nelson-Siegel Model**: In this case, we could have both, a system where in an specific maturity have more than one bond and dates with not payments, N-S create a more smooth curve

```
RESTRICT_YLD = True
RESTRICT_TIPS = True
```

```
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```

```
data = filter_treasuries(quotes, t_date=t_current, filter_yld = RESTRICT_YLD, filter_tips = RES
CF = filter_treasury_cashflows(calc_cashflows(data),filter_maturity_dates=RESTRICT_DTS_MATURING,
```

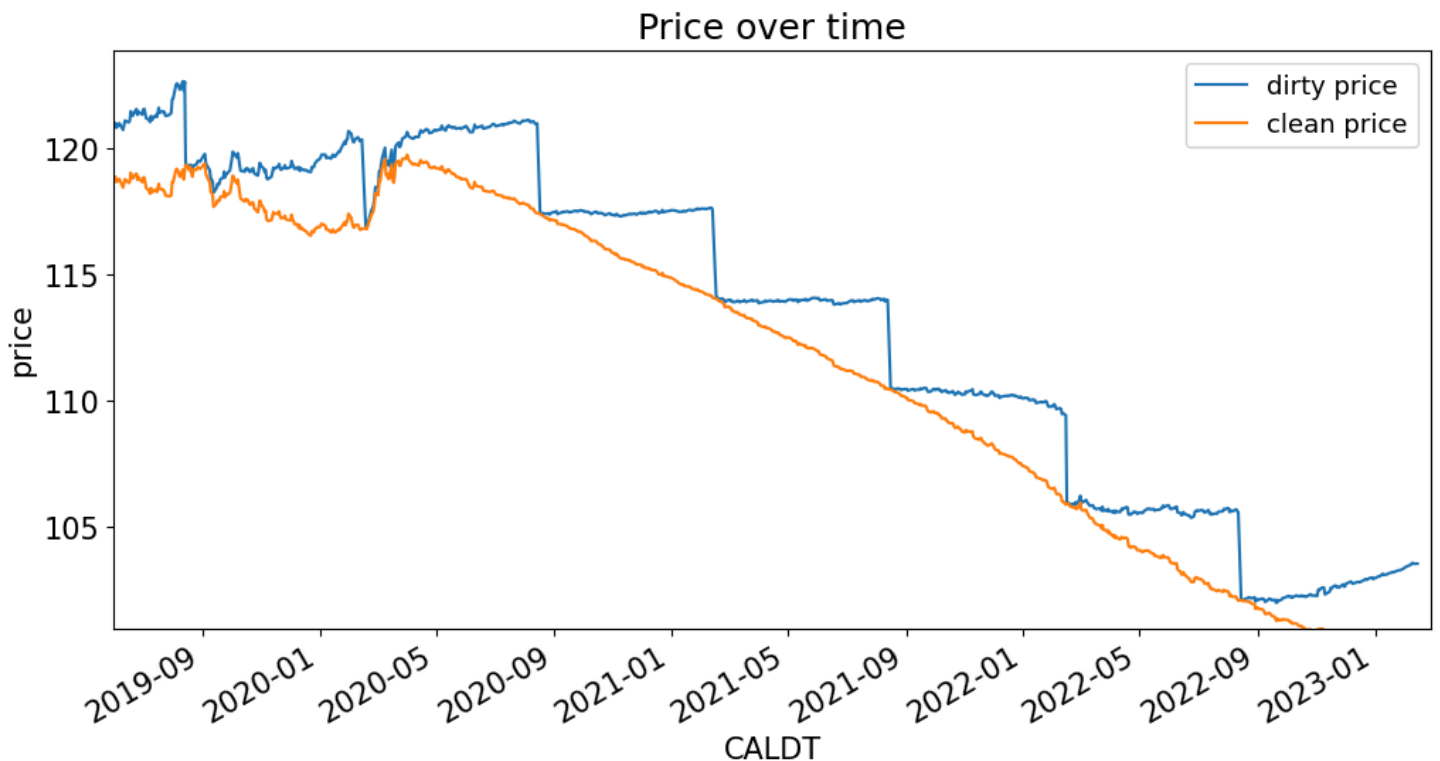
- **Comments: Bootstrap** will price perfectly in-sample and **N-S** can easily extend to arbitrary maturities and generate a smoother forward rates
- When the interest rate change, impacting the discount curve and coupons, the effect are nearly offsetting (example fo floating rate notes)
- The months with more payments (in Treasury Notes (2-10y) and Bond (20,30y)) are in Feb, May, Aug and Nov.

Discussions

- Accrued interest: Based on the days until the next coupon

$$AccrInt = \frac{\text{days counted}}{\text{days in ref. period}} * \text{Interest on ref. period}$$

- Dirty price = Clean price + Accrued interest. In the market (Bid/Ask), the published price is the clean price. For **Spot curve construction we should use the dirty price!**



- Return vs YTM:
They are equivalent iif:
 - We are discussing about a zero-coupon bond (T-Bills)
 - The investor holds until maturity
 If not:
 - It isn't a return, as there is not guarantee you can re-invest intermedite cash-flows (coupons) at the YTM

Curves

- **Yield to Matirity**: Pricing a particular bond (Treasury) using the Yield to Maturity(YTM), for a particular security (Plot: YTM/ Time to Maturity (TTM)) -> It can be used as a discount rate **only** that specific bond
- Zero coupon bond: Usually there is not in the market, it is useful to create the discount rate directly using its prices (STRIS)
- **Spot curve**: Approach using bonds in the market to get the spot discount curve
- **Discount curve**: It does not depend on the compounding frequency of the rate, is unique

$$Z(t, T) = \frac{1}{\left(1 + \frac{r_n}{n}\right)^{n(T-t)}} = e^{-r(T-t)}$$

For continous compounding from the discount curve

$$r(t, T) = -\frac{\ln(Z(t, T))}{T - t}$$

Yield-Curve Factors

- **Level:** mean of the different maturities
- **Slope:** Long-term yield (10y or 30y) minus short-term yield
- **Curvature:** Mid-range yield (5y or 10y) minus short-term yield minus long-term yield.

PCA

- PCA gives linear combinations of interest rates along the curve which have maximum explanatory power in its variation
- Generate a linear combination of the different maturities of the yield curve that are orthogonal to each other
- The first component explains almost all the variance (~97%) and the first 3 components around the 99% of the variance
- The first three PCs are interpretable as Level, Slope and Curvature. This is an empirical result!

Week 2: Duration and Convexity

HW2

Arbitrage two bonds:

- We take a **long position** on the bond with **max(YTM)** ; **Short position min(YTM)**
- The bond with the **largest duration/Lower coupon**, will have **higher convexity**, so the approximation using only duration will be less accurate
- Risks:
 - **Long-term:** By taking a long (max(YTM))/short position (min(YTM)), if we held to maturity, the return will be guaranteed giving the difference of YTM
 - **Short-term:** Given the fluctuation of the spot curve (mkt movements), the YTM spread could diverge and generate losses, given the leverage (repo) this could generate a pressure to close the position (unwind)
- **Hedge ratio**

- n_i : number of bonds purchased (or sold)
- D_i : duration of bond i
- $D_{\$,i}$: dollar duration of bond i , equal to $p_i D_i$

$$0 = n_i D_{\$,i} + n_j D_{\$,j}$$

$$n_j = -n_i \frac{D_{\$,i}}{D_{\$,j}}$$

Discussions

- We **don't use regressions** to estimate the sensitivities of the bond prices to the change of the rate, given:
 - Over timeseries, maturities changes [ie, 12M move to 11M]
 - We have a clear mathematical formula to estimate it
- Taylor approximation:

$$\frac{dP}{P} \approx -D \times dr + \frac{1}{2} C \times (dr)^2 \quad (1)$$

where

- dr is a small change in the level of the spot curve
- D is the **duration**
- C is the **convexity**

Duration

$$D \equiv -\frac{1}{P} \frac{dP}{dr} \quad (2)$$

$$\frac{dP}{P} \approx -D \times dr$$

- Sensitivity to the **level** of the interest rate curve
- **Parallel shift** of the spot curve
- For zero-coupon bonds, Duration = Maturity ; For coupon bonds, Duration < Time-to-Maturity
- A higher coupon decrease the duration
- Linear approximation, good for small changes on the rates. **Duration always underestimate the actual price**

$$D_{\text{zero}} \equiv -\frac{1}{P(t, T, 0)} \frac{dP(t, T, 0)}{dr} = T - t \quad (3)$$

- The **duration of the coupon bond** is the weighted average of the cashflow maturities:

$$D_{\text{coupon bond}} \equiv \frac{1}{P(t, T, c)} \frac{dP(t, T, c)}{dr} \quad (4)$$

$$= \sum_{i=1}^n w_i D_i \quad (5)$$

$$= \sum_{i=1}^n w_i T_i \quad (6)$$

Others durations

- **Dollar duration:** Dollar change in the price, instead of a percentage change

$$\text{dollar duration} = D_{\$} \equiv -\frac{dP}{dr} \quad (7)$$

- **Macauley Duration:** Sensitivity to a parallel shift in the **spot curve**, it's the one defined above
- **Modified Duration:** It is used to calculate the change in the price given a change in the **YTM** of the bond

$$\text{modified duration} \equiv D_{\text{ytm}} \equiv \frac{1}{1 + y/2} D \quad (8)$$

$$= -\frac{1}{1 + y/2} \frac{1}{P} \frac{dP}{dr} \quad (9)$$

Convexity

Second order approximation, refer to the change of the convexity of the spot curve. It is alwaysd positive.

$$\frac{dP}{P} \approx -D \times dr + \frac{1}{2} C \times (dr)^2 \quad (10)$$

where C denotes the convexity of the bond,

$$C \equiv \frac{1}{P} \frac{d^2 P}{dr^2} \quad (11)$$

- For a zero coupon bond, it is the squared of the time to maturity

$$C = \frac{1}{P(t, T, 0)} \frac{d^2 P(t, T, 0)}{dr^2} \quad (12)$$

$$= (T - t)^2 \quad (13)$$

$$C = \sum_{i=1}^n w_i C_i = \sum_{i=1}^n w_i (T - t)^2 \quad (14)$$

Week 3: Floating-Rates Notes and Swaps

Discussions

Floating-Rates Notes (FRN)

- Pay coupons based on a specific variable (floating) short-term rate (r) and pre-specified spread (s). And it pays a Face Value + coupon at the maturity
- On the coupons dates (or **reset dates**), the Note pays the coupon and the following rate (for the next period) is set

$$cpn_{T_i} = 100(r(T_{i-1}, T_i) + s) \quad (15)$$

Note that

- This coupon is annualized, so the payment is $cpn_{T_i} / freq$, similar to how the vanilla treasury coupon was divided by 2.
- This coupon is based on the **rate in arrears**. Thus the coupon paid at T_i depends on the rate at T_{i-1} .
- The spread, s , is fixed over the life of the bond. The coupon only "floats" due to r , not s .
- Return FRN vs Fixed rate bonds:
 - $R_{FRN} > R_{Fixed}$ (rising rates periods)
 - $R_{FRN} < R_{Fixed}$ (falling rates periods)
- The price of a FRN with zero spread at reset date is equals to its face value

$$price_{float}(T_i, T; 0) = 100$$

Swaps

- Components: Notional, fixed rate (C_{swap}), floating rate (short-term variable rate: SOFR or FED), frequency (k), maturity
- CashFlow at each period (Paying fixed) (where $\tau = 1/k$):

$$C(T_i) = K \tau [r_{\kappa}(T_{i-1}, T_i) - c_{swap}]$$

- No cost to initiate
- Replication with bonds (paying-fixed swap): Short fixed-rate bond and Long FRN
- Swap Value:

$$V_{swap}(t, T, c_{swap}) = P_{float}(t, T; 0) - P_{fixed}(t, T; c_{swap})$$

At the beginning, floating bond = 100

$$\begin{aligned} V_{swap}(0, T; c_{swap}) &= 100 - 100 \left[\sum_{i=1}^M Z(0, T_i) \frac{c_{swap}}{\kappa} + Z(0, T) \right] \\ &= 100 \left[1 - Z(0, T) - \frac{c_{swap}}{\kappa} \sum_{i=1}^M Z(0, T_i) \right] \end{aligned}$$

Solving for the fixed rate:

$$c_{swap}(0, T; \kappa) = \kappa \frac{1 - Z(0, T)}{\sum_{i=1}^M Z(0, T_i)}$$

- **Negative Swap Spreads:** (Swap Spread = Swap rate – fixed rate bond)
Nowadays (after the financial crisis of 2008), the swap spread in the US Market is negative, giving that the floating rate (SOFR) is a collateralized lending rate, the regulations to higher capital requirements, Swap now (usually) goes through a Central Clearing House, eliminating the counterparty risk.

Week 4: Forwards

Fwd Formula

Interest rate in a future time (from T_1 to T_2)

- Forward Discount Factor:

$$t \leq T_1 \leq T_2$$

$$F(t, T_1, T_2) = \frac{Z(t, T_2)}{Z(t, T_1)} \quad (16)$$

- Forward Discount Rate:

$$f(t, T_1, T_2) \equiv - \frac{\ln \{F(t, T_1, T_2)\}}{T_2 - T_1} \quad (17)$$

$$F(t, T_1, T_2) = e^{-f(t, T_1, T_2)(T_2 - T_1)} \quad (18)$$

- With a N-S spot curve, we can obtain a more smooth fwd curve
- Spot curve vs Fwd Curve: The fwd curve is above (below) the spot curve iff the spot curve is sloping upward (down)

$$f(0, T, T + \Delta) = r(0, T) + (T + \Delta) \underbrace{\frac{r(0, T + \Delta) - r(0, T)}{\Delta}}_{\text{slope of spot curve}} \quad (19)$$

- From Fwd Rates to Spot Rates:

$$r(0, T_n) = \frac{1}{T_n} \sum_{i=1}^n f(0, T_{i-1}, T_i) \Delta \quad (20)$$

where

- $T_0 = 0$,
- $f(0, 0, T_1) = r(0, T_1)$.

```
QUOTE_DATE_ALT = '2006-12-29'
```

```
display(f'Quote Date: {QUOTE_DATE_ALT}')
```

```
curves_imputed = curves_set[QUOTE_DATE_ALT][['forward rate']]
```

```
curves_imputed.columns.name = QUOTE_DATE_ALT
```

```
mat_grid = curves_imputed.index.to_series()
```

```
delta_grid = mat_grid.diff()
```

```
# assuming first delta is same as second delta
```

```
delta_grid.iloc[0] = delta_grid.iloc[1]
```

```
curves_imputed['spot rate'] = (curves_imputed['forward rate'] * delta_grid).cumsum(skipna=True)
```

```
curves_imputed['spot rate'].iloc[0] = curves_imputed['forward rate'].iloc[0]
```

Forward Rate Agreements (FRA)

- Key characteristics:
 - Not capital exchange
 - One party pays a fixed interest from T1 to T2 (fwd rate)
 - The other part pays a spot rate (floating, defined at T1)

$$N \Delta [f_n(0, T_1, T_2) - r_n(T_1, T_2)] \quad (21)$$

$$\Delta \equiv T_2 - T_1 \quad (22)$$

$$N \equiv \text{Notional Investment} \quad (23)$$

Yield curve

- Term spread: The slope of the yield curve, long-term minus short-term (typically 10y - 3m, but given that they are highly correlated (~90%), the results wouldn't be substantively different)
- A negative term spread has preceded every recession since the mid 20th century. It has had only one "false alarm"
- According to the paper, the absolute level of the term spread is more important than its change. However, changes in the spread become significant when the level is already low, regardless of whether the shift is driven by short or long-term rates.
- Sacks equations for **Term premium**

$$i_t^{10} = \frac{1}{10} \sum_{j=0} \mathbb{E}_t[i_{t+j}^1] + \text{tp}_t$$

$$i_t^{10} - i_t^1 = \sum_{j=0} \frac{10-j}{10} \mathbb{E}_t[\Delta i_{t+j}^1] + \text{tp}_t$$

- Term premium (tp): risk premium that the investors demand for taking a long-term interest instead of rolling over the short-term rate
- Changes in the term spread may be driven by shifts in the term premium rather than changes in expected short-term rates