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Interest rate models for the Icelandic bond market

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1 Introduction

1.1 Interest rates

The main reason why money will be worth less in the future then the same amount today is that money due in the future or that is locked in a fixed-term account cannot be spent right away. In addition, prices may rise in the meantime and the amount will not have the same purchasing power as it would have at the present time, that's why interest rates are important.[1]

Interest rate is the amount charged on top of the principal by a lender to a borrower for the use of assets.[2] The interest rates generally depend on the term, i.e. the length of loan or deposit, where different terms need to be discounted at different interest rates and different methods are used for discounting at different scenarios. Continuous compounding is typically used in the analysis and the pricing of interest rate derivatives, annual compounding is typically used for the pricing of bonds and simple compounding is mainly used in modern LIBOR market models and generally in the money markets for short term maturities. This fact leads to the concept of a "term structure of interest rates".[3]

1.2 Bonds

A bond is a fixed income instrument that is based as a type of loan from a lender to the borrower that represents a fixed contract between said individual entities. Bonds are used when entities like government and corporate need to raise capital to finance new and ongoing projects or if needed to refinance existing debt. This is done by the issuer where he gives out the terms of the loan where the interest rates are set up and require the loan to be paid back at maturity with interest.

Bonds can be set up in a variety of ways, the most common being a coupon paying and a non coupon paying bond (*Zero Coupon Bond*). a coupon paying bond bears an interest rate called *coupon rate* that is a periodic payment promised to the bondholder which is the product of the coupon rate times the face value of the bond. these bonds can be both publicly traded meaning that at any given moment the bond can be sold in the market for its market price, it can also be traded in an *Over The counter* (*OTC*) manner meaning that only certain entities can participate in the bonds trading. A bond's yield to maturity (*YTM*) is the internal rate of return of the bond, assuming that it is held to maturity and that it does not default.[4] A zero coupon bond, on the other hand, is set up with its face value at maturity and has no coupon payments, it only pays the face value at maturity. For these type of bonds, the face value is discounted with respect to the lifetime of the bond and the interest rate for the time period.[1]

1.3 The Icelandic Market

The Icelandic banking sector was hit hard during the 2008 financial crisis but the transformation and restructuring of the banks laid a solid foundation for the continuation of highly developed banking services. The commercial banking sector now consists of three universal banks, one investment bank, and four small savings banks. The banks are predominantly funded with domestic deposits that are around ISK 1,800 billion or almost 75% of GDP. Bond issuance has been increasing over the last few years, first and foremost, the issuance of contingent and covered bonds. Total loans in the banking sector amount to ISK 3,492 billion. The asset base is predominately domestic, where total domestic assets are ISK 3,112 billion. Icelandic banks have also sold bonds in the international market in recent years, with total foreign liabilities of ISK 631.5 billion and total domestic liabilities of ISK 2,256 billion. All the major banks have been increasing their funding in European bond markets and that trend has been strengthened with significant improvements in their credit ratings in 2016. In March 2017 authorities removed the last remaining hurdles of the capital controls that were introduced in 2008 to stabilize the currency during the financial crisis. The lifting of the capital controls on individuals, firms, and pension funds marks the completion of Iceland's return to the international financial markets.[5]

Nominal Treasury bonds in Iceland pay an annual coupon with maturity up to twenty years which are issued by the Central Bank of Iceland - Government Debt Management, on behalf of the Republic of Iceland. The treasury instruments have a name convention where the first four letters in the symbol indicate whether it is a bill, nominal bond or inflation-linked bond. This is followed by two numbers which indicate the year and the last four numbers indicate the month and day of final maturity, see *Table 1.* Treasuries are issued at the *NASDAQ CSD* Iceland hf. Nominal Treasury bonds are auctioned in bi-monthly auctions at preset dates according to an annual auction calendar. These auctions take place through *Bloomberg* auction system and direct access is limited to primary dealers, they can though offer to submit bids on behalf of the other investors.[6]

Inflation-linked Treasury bonds were first issued in 1963, currently there are three inflation-linked Treasury bonds outstanding, shown in *Table 1.* These bonds are long-term bonds with annual interest payment with inflation-calculated principal payable at maturity. Treasury bonds are inflation-linked using the consumer price Index.[7]

Type	Symbol	Maturity
Nominal T-bond	RIKB 20 0205	February 5th 2020
Indexed T-bond	RIKS 21 0414	April 4th 2021
Nominal T-bond	RIKB 22 1026	October 26th 2022
Nominal T-bond	RIKB 25 0612	June 12th 2025
Indexed T-bond	RIKS 26 0216	February 16th 2026
Nominal T-bond	RIKB 28 1115	November 15th 2028
Indexed T-bond	RIKS 30 0701	July 1st 2030
Nominal T-bond	RIKB 31 0124	January 24th 2031

Table 1: Icelandic Treasury Bonds

Yields on indexed Treasury and Housing Financing Fund (HFF) bonds have been falling since the second half of 2016 but started to climb again in late April. Nominal Treasury bond yields began to rise late in 2017, followed by a rise in the break-even inflation rate on the bond market, see *Figure 1.1.*[8]

GDP in Iceland is expected to rise by 1.7% in 2019. Growth is substantially lower than the last five years' average growth rate of 4.4%. This year's deteriorating economic outlook is mostly derived from lower than expected exports, with gross national expenditure rising by 3%. In the coming years, GDP growth is expected to range from 2.5-2.8%.

Recently, investment has slowed down and is expected to rise by 1% this year, but after that, the pace will pick up. Investment levels are expected to be slightly above the 20-year average of 22% as a share of GDP. Investment in the business sector is expected to decline this year by

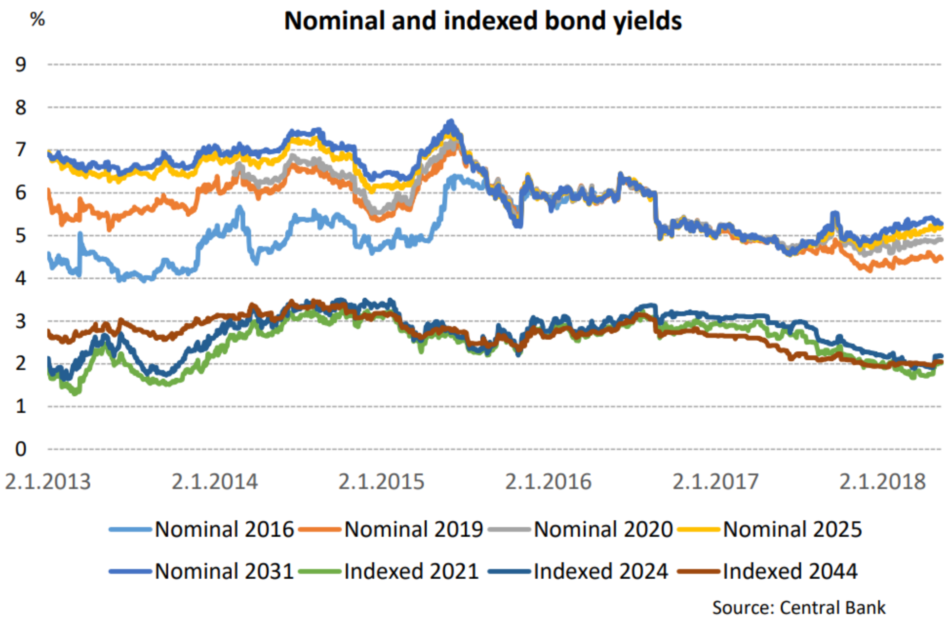


Figure 1.1: Nominal and indexed bond yields from 2013 to June 2018

4.5%, partly due to a slowdown in investments in the heavy industry and energy sector. In the first three quarters of 2018, growth in residential investment was 14% and is expected to grow by 20% this year.

Last fall's depreciation of the Icelandic Króna resulted in a more negative outlook for inflation. This year, inflation is expected to be 3.8%, greatly influenced by the exchange rate. Next year's inflation is projected to rise to 3.3% and then gradually level off toward the central bank's inflation target. However, there is a great deal of uncertainty involving the outlook for inflation due to changes in tourism and the outcome of the wage contract negotiations, which covers the major part of the labour market.[9]

Although the market has been rising in recent years, the yield curve is now inverted, see *Figure 1.2*. The shape of the yield curve changes in accordance with the state of the economy. The normal or up-sloped yield curve may persist when the economy is growing and conversely the inverted yield curve suggests yields on longer-term bonds may continue to fall, considered to be a predictor to an economic recession because historically inversions of the yield curve have preceded many recessions. When investors expect longer-maturity bond yields to become even lower in the future, many would purchase longer-maturity bonds to lock in yields before the decrease further. The increase onset of demand for longer-maturity bonds and the lack of demand for shorter-term securities lead to higher prices but lower yields on shorter-term securities, further inverting a down-sloped yield curve. An inverse yield curve predicts lower interest rates in the future as longer-term bonds are demanded, sending yields down.[10][11]

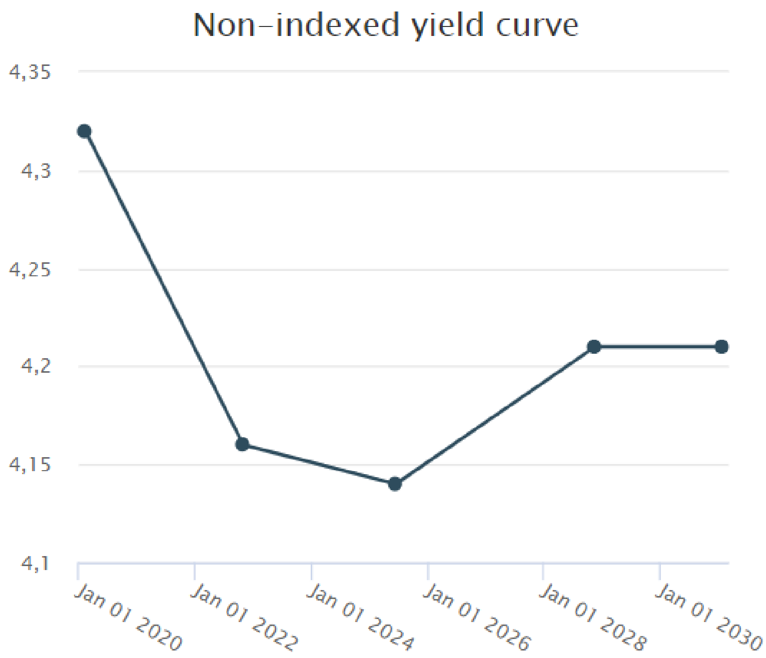


Figure 1.2: Yield curve of Icelandic Nominal Treasury bonds at 08.05.19

2 Constructing curves with bootstraping

2.1 Zero curve

Bootstrapping method is most commonly used to calculate zero rates. The method involves starting with short-term instruments and moving to longer-term instruments, so the zero rates calculated at each stage are consistent with the prices of the instruments. The zero curve is linear between the zero rates, determined when applying the bootstrap method.

Par value of a bond $= V$
Current price of a bond $= P$
Coupon payments $= \{c_1, c_2, c_3 \dots c_n\}$
Time to maturity $= \{t_1, t_2, t_3 \dots t_n\}$
Rate $= \{r_1, r_2, r_3 \dots r_n\}$

For non coupon paying bonds, we have:

$$V = P \exp\left(r_1 t_1\right) \tag{2.1}$$

We isolate r_1 from the equation above to find the zero rate at time t_1 and we get:

$$r_1 = \frac{\ln \frac{V}{P}}{t_1} \tag{2.2}$$

Then this step is repeated for time t_2 for the next bond to find r_2 and the process is repeated until we have calculated zero rates of all the bonds. For coupon paying bonds the process is a little bit different. For example, if the first two bonds are zero coupon bonds and we have calculated r_1 and r_2 for the maturity time t_1 and t_2, we find the rate r_3 of coupon paying bond by calculating:

$$P = c_1 \exp\big(-r_1 t_1\big) + c_2 \exp\big(-r_2 t_2\big) + \big(c_3 V\big) \exp\big(-r_3 t_3\big) \tag{2.3}$$

Then we isolate r_3 and we get:

$$r_3 = \frac{-\ln \frac{P - c_1 \exp\left(-r_1 t_1\right) - c_2 \exp\left(-r_2 t_2\right)}{\left(c_3 V\right)}}{t_3} \tag{2.4}$$

Those steps are then repeated until all the rates have been calculated for all bonds.[12]

The *Matlab* function *Zbtprice* uses the bootstrap method to return a zero curve for the coupon paying bonds in the portfolio. The zero curve consists of the yields to maturity for the portfolio of theoretical zero-coupon bonds that are derived from the input bonds portfolio. The function uses theoretical par bond arbitrage and yield interpolation to calculate the zero rates and the interest rates for the cash flow are calculated using linear interpolation. However, the function does not require alignment of the cash flow dates of the bonds in the portfolio.[13]

There are two main limitations to the bootstrapping method. This method does not perform optimization, therefore it computes zero-coupon yields that exactly fit the bond prices which leads to over-fitting since bond prices often contain idiosyncratic errors due to lack of liquidity, bid-ask spreads, special tax effects, etc., hence, the term structure will not necessarily be smooth. This method requires ad-hoc adjustments when the number of bonds is not the same as the bootstrapping maturities and when cash flows of different bonds do not fall on the same bootstrapping dates.[14]

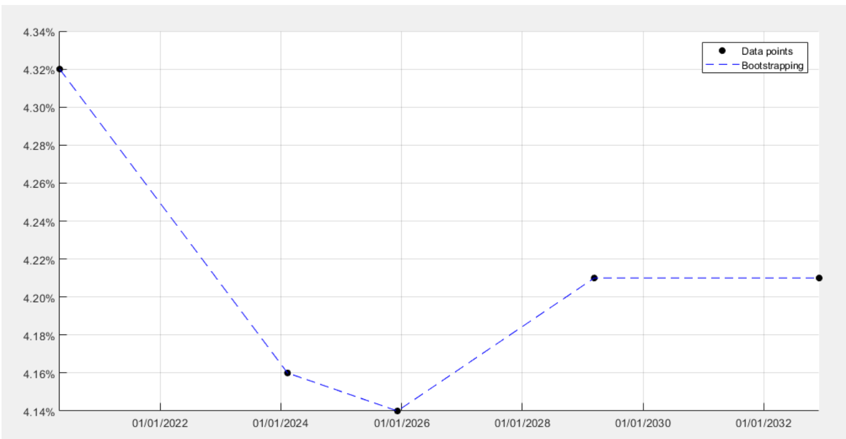


Figure 2.1: Zero curve of Icelandic Nominal Treasury bonds at 15.05.19

2.2 Forward curve

Forward interest rates are the future rates of interest implied by current zero rates for periods of time in the future.[12] We used these forward rates, calculated from the zero rates to construct the forward curve. The forward curve is a key component in swap contracts, which will be discussed later in this report.

The forward rate $F(t, T_k, T_{k+1})$, at time t, for the future time from T_k to T_{k+1} needs to satisfy,

$$\exp(R(t, T_{k+1})(T_{k+1} - t)) = \exp(R(t, T_k)(T_k - t)) \exp(F(t, T_k, T_{k+1})(T_{k+1} - T_k)) \tag{2.5}$$

After taking the logarithm and solving for $F(t, T_k, T_{k+1})$ we get.[3]

$$F(t, T_k, T_{k+1}) = \frac{R(t, T_{k+1})(T_{k+1} - t) - R(t, T_k)(T_k - t)}{T_{k+1} - T_k} \tag{2.6}$$

We used the *MATLAB* function *zero2fwd* to return an implied forward rate curve given a zero curve and its maturity dates.[15]

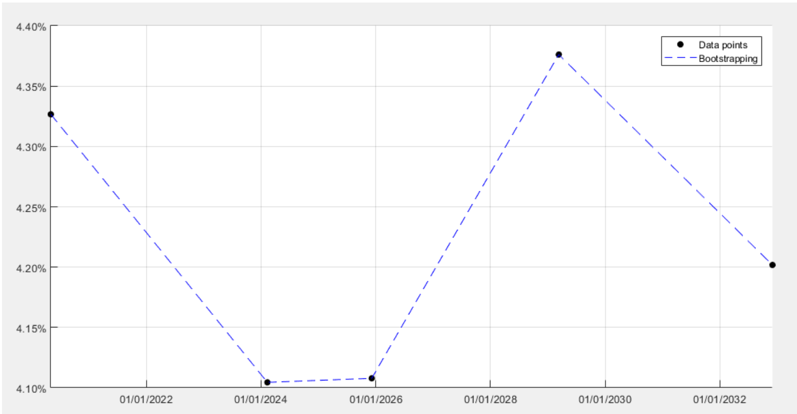


Figure 2.2: Forward curve of Icelandic Nominal Treasury bonds at 15.05.19

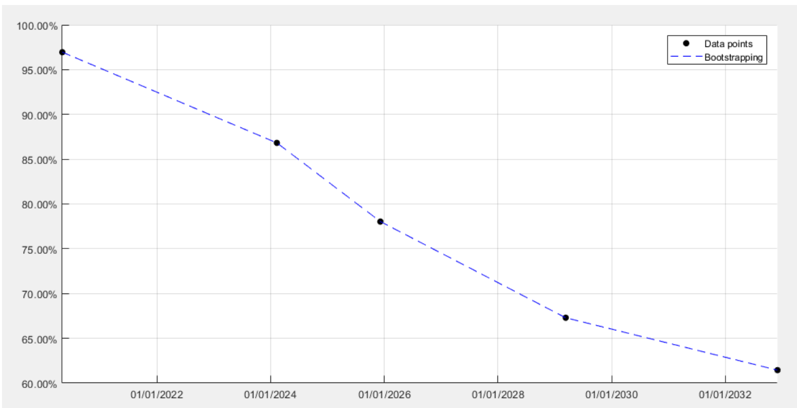


Figure 2.3: Discount factor curve of Icelandic Nominal Treasury bonds at 15.05.19

3 Fitting the term structure

3.1 Nelson-Siegel and Svensson

The Nelson-Siegel method is often used for developing yield curves. Nelson and Siegel introduced in 1987 a simple, parsimonious model, which can adapt to the range of shapes of yield curves. The Nelson-Siegel method makes the following assumption for the instantaneous forward rates.

$$f(T) = \beta_0 + \beta_1 \exp\Big(-\frac{T}{\tau_1}\Big) + \beta_2\Big(\frac{T}{\tau_1}\Big)\exp\Big(-\frac{T}{\tau_1}\Big) \tag{3.1}$$

Then we can find the term structure for spot rates:

$$R(t,T) = \frac{1}{T-t}\int_t^T f(t,\tau)d\tau = \beta_0 + \beta_1\Bigg(\frac{1-\exp(-\frac{T}{\tau_1})}{\frac{T}{\tau_1}}\Bigg) + \beta_2\Bigg(\frac{1-\exp(-\frac{T}{\tau_1})}{\frac{T}{\tau_1}} - \exp(-\frac{T}{\tau_1})\Bigg) \tag{3.2}$$

The Svensson method for term-structure estimation is most frequently used by central banks. It was introduced in 1994 by Svensson, he proposed to increase the flexibility and fit of the Nelson-Siegel method by adding a second hump-shape factor with a separate decay parameter. This technique is called the Svensson four-factor model which is an extension on Nelson-Siegel from 1987. This model constructs the four-factor instantaneous forward rates as follows:

$$f(T) = \beta_0 + \beta_1 \exp\Big(-\frac{T}{\tau_1}\Big) + \beta_2\Big(\frac{T}{\tau_1}\Big)\exp\Big(-\frac{T}{\tau_1}\Big) + \beta_3\Big(\frac{T}{\tau_2}\Big)\exp\Big(-\frac{T}{\tau_2}\Big) \tag{3.3}$$

Then we can find the term structure for the spot rates with :

$$\begin{aligned} R(t,T) = \frac{1}{T-t}\int_t^T f(t,\tau)d\tau &= \beta_0 + \beta_1\Bigg(\frac{1-\exp(-\frac{T}{\tau_1})}{\frac{T}{\tau_1}}\Bigg) \\ &+ \beta_2\Bigg(\frac{1-\exp(-\frac{T}{\tau_1})}{\frac{T}{\tau_1}} - \exp(-\frac{T}{\tau_1})\Bigg) + \beta_3\Bigg(\frac{1-\exp(-\frac{T}{\tau_2})}{\frac{T}{\tau_2}} - \exp(-\frac{T}{\tau_2})\Bigg) \end{aligned}$$

Where t is fixed as the present time, T the difference of time to maturity and the payment of the bond and τ time to payment, all measured in years.[16]

By using the Nelson-Siegel method or the Svensson method, we have unknown parameters which need to be estimated, they are $\beta_0, \beta_1, \beta_2, \beta_3, \tau_1$ and τ_2. We used the *MATLAB* functions *fitNelsonSiegel* and *fitSvensson* to estimate these parameters for both methods.

The Nelson-Siegel is highly nonlinear which causes many to report estimation problems. This method uses a single exponential function for the entire term structure, in most cases this gives a reasonable fit, but for more complex shapes of the yield curve it is unsatisfactory.[17] Two problems can be obtained when using Svensson's extension. First, different parameter values may lead to similar yield curves. Such a result is possible because the model is estimated from coupon bonds with finite maturities. Over the observed maturities, different parameters can generate almost identical yield curve estimates. These yield curves would, however, be different if longer maturities were considered. The second problem comes from the optimization procedure. The function to optimize is not guaranteed to be convex and may show several local optima.[18]

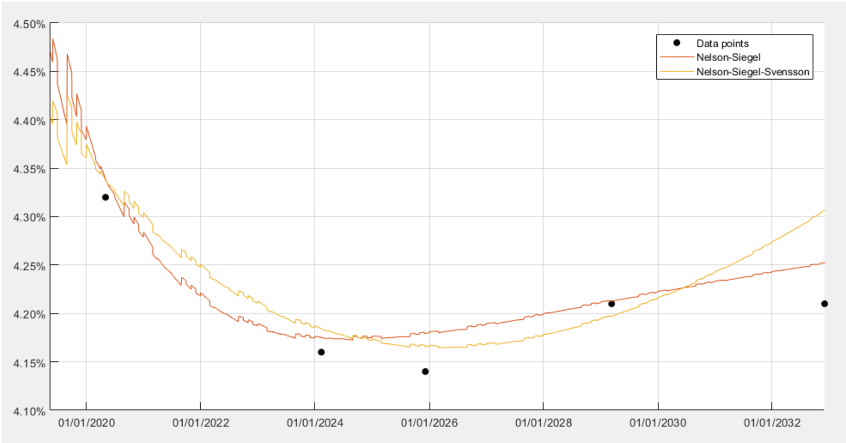


Figure 3.1: The yield curve of Icelandic Nominal Treasury bonds at 15.05.19 fitted with Nelson-Siegel and Nelson-Siegel Svensson

3.2 Polynomials

The polynomial fitting is a method that consists of minimizing the error function:

$$E(\bar{a}) = \sum_{i=1}^{N} \Big(y(x_i) - P_n(x_i) \Big)^2; \quad 1 \le n \le N; \quad P_n(x) = \sum_{k=0}^{n} a_k x^k \tag{3.4}$$

where N is the number of data pairs $(x, y(x))$, $P_n(x_i)$ is polynomial of the n-th degree and $\bar{a}$ are the coefficients of the polynomial. More detailed explanation on how the coefficients for the polynomial are determined can be seen in the appendix section (A.1).

Polynomial fitting is one of the easiest ways to model a yield curve. High degree polynomials would fit the data well but the result can have unrealistic shapes. Polynomial yield curve estimations lead to quick and good solutions in some rare cases but they are not the best choice most of the time.

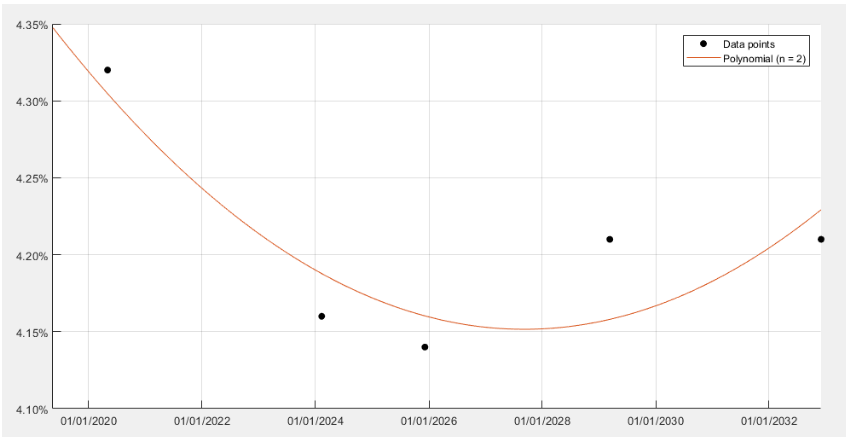


Figure 3.2: THe yield curve of Icelandic Nominal Treasury bonds at 15/05/19 fitted with 2nd degree polynomial

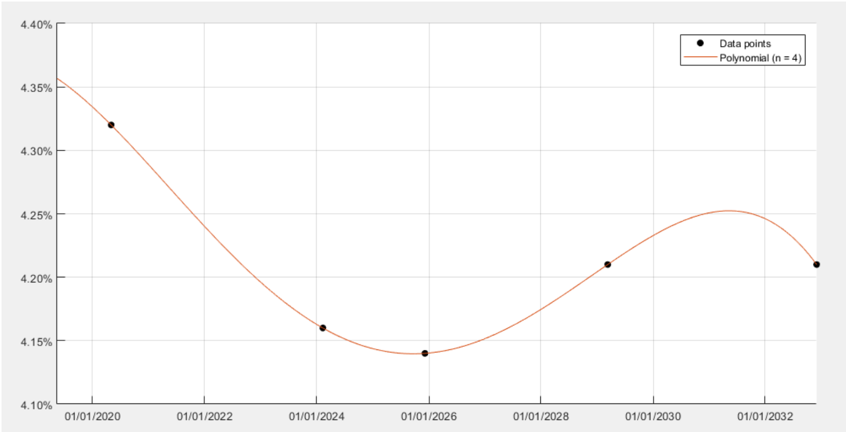


Figure 3.3: The yield curve of Icelandic Nominal Treasury bonds at 15/05/19 fitted with 4th degree polynomial

3.3 Lagrange interpolation

The Lagrange interpolation polynomial is a fitting method which constructs a polynomial that goes through all desired data points. Let's assume that the given Data points are:

$$F_i = (x_i, f(x_i)) \tag{3.5}$$

Then the assumption is that the measurements for the data points are gathered by some function where the only known values are:

$$f_i = f(x_i), i = 1, 2, \ldots, N \tag{3.6}$$

Considering an interpolation approximation function $p(x)$ that should pass through all the data points with a polynomial of N^{th} degree:

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_N x^N \tag{3.7}$$

And therefore:

$$p(x_1) = a_0 + a_1 x + a_2 x^2 + \cdots + a_N x^N = f(x_i) = f_i \tag{3.8}$$

Which gives the linear system for the collected data as

$$\begin{array}{c} a_0 + a_1 x_0 + a_2 x_0^2 + \cdots + a_N x_0^N = f_0 \\ a_0 + a_1 x_1 + a_2 x_1^2 + \cdots + a_N x_1^N = f_1 \\ . \\ . \\ a_0 + a_1 x_N + a_2 x_N^2 + \cdots + a_N x_N^N = f_N \end{array} \Rightarrow \begin{pmatrix} 1 & x_0 & x_0^2 & \ldots & x_0^N \\ 1 & x_1 & x_1^2 & \ldots & x_1^N \\ . & & & & \\ . & & & & \\ 1 & x_N & x_N^2 & \ldots & x_N^N \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ . \\ . \\ a_N \end{pmatrix} = \begin{pmatrix} f_0 \\ f_1 \\ . \\ . \\ f_N \end{pmatrix}$$

If there can be found a polynomial φ_k of the N^{th} degree such that:

$$\varphi_i(x_k) = \delta_{i,k} = \begin{cases} 1 \ if \ i = k \\ 0 \ if \ i \neq k \end{cases}$$

Then, the polynomial:

$$p(x) = \sum_{k=0}^{N} f(x_i)\varphi_i(x) \tag{3.9}$$

is of the N^{th} degree and thus clearly satisfies the condition:

$$p(x_k) = \sum_{i=0^N} f(x_i)\varphi_i(x_k) = \sum_{i=0}^{N} f(x_i)\delta_{i,k} = f(x_k) = f_k \tag{3.10}$$

To be able to find an explicit construction for φ_k the fact that $\varphi_k(x_i) = 0$ when $i \neq k$ is used. This implies that $\varphi_k(x)$ must contain factors from $(x - x_i)$ when $i \neq k$. So the assumption is made that:

$$\varphi_k(x) = \alpha(x - x_i)(x - x_1)\dots(x - x_k)\dots(x - x_N) = \alpha \prod_{\substack{i=0 \\ i\neq k}}^{N} (x - x_i) \tag{3.11}$$

so that $\varphi_k(x)$ satisfies the normalization condition needed to choose α as:

$$\varphi_k(x_i) = \delta_{k,i}; \varphi_k(x_k) = 1 \Rightarrow \ \alpha = \left(\prod_{\substack{i=0 \\ i\neq k}}^{N} (x_k - x_i) \right)^{-1} \tag{3.12}$$

and therefore, the basic functions take the form:

$$\varphi_k(x) = \prod_{\substack{i=0 \\ i\neq k}}^{N} \frac{(x - x_i)}{(x_k - x_i)} \tag{3.13}$$

[19]

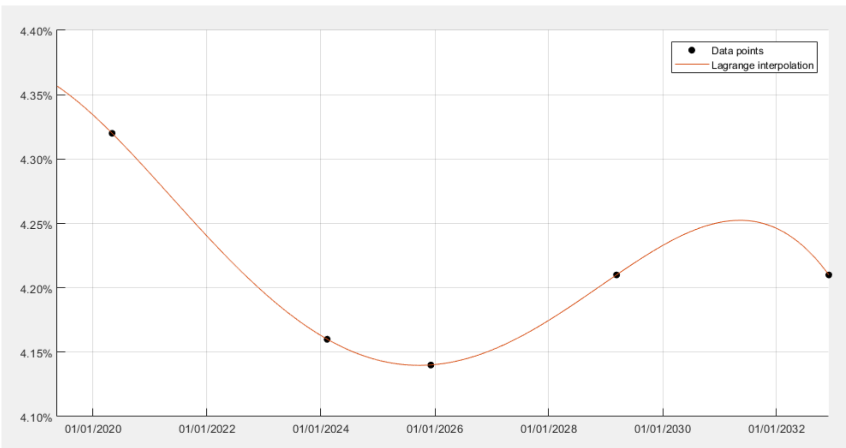


Figure 3.4: The yield curve of Icelandic Nominal Treasury bonds at 15/05/19 fitted with Lagrange interpolation

3.4 Splines

Rather than using one polynomial for an approximation of data points the method of splines uses a number of polynomials, each defined over a sub-range. Lets define a interval [a,b] where,

$$a = x_0 < x_1 < ... < x_n = b \tag{3.14}$$

Spline, s, is a continuous function of degree m on [a,b] with knots, $x_0, x_1, ..., x_n$ where:
s is a piece-wise polynomial of degree m on each sub-interval $[x_k, x_{k+1}]$ and is differentiable $m-1$ times on each sub-interval $[x_k, x_{k+1}]$.

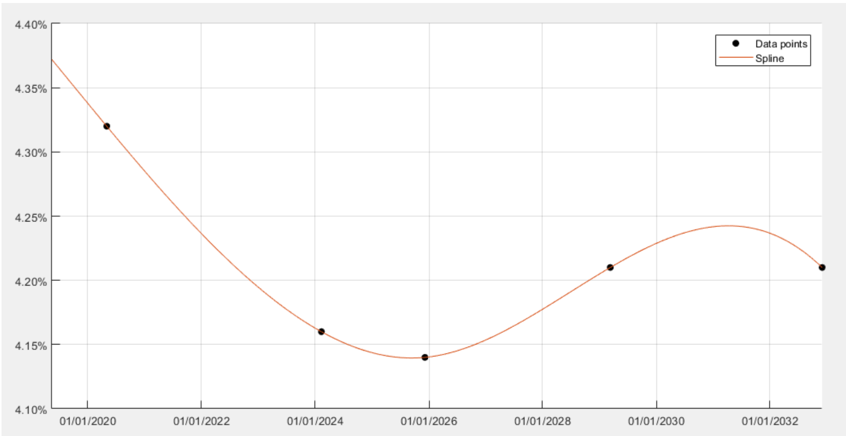


Figure 3.5: The yield curve of Icelandic Nominal Treasury bonds at 15/05/19 fitted with spline

3.5 Cubic splines

A cubic spline is a function of order three, and a piece-wise cubic polynomial that is twice differentiable at each knot point. At each knot point, the slope and curvature of the curve on either side must match.[20]

Let $f(x)$ be a function defined on an interval [a,b], and let x_0, x_1,...,x_n be $n+1$ distinct points in [a,b], where a $= x_0 < x_1 < ... < x_n =$ b. A cubic spline, is a piece-wise polynomial s(x) that satisfies the following conditions:

1. On each interval [x_i-1,x_i], for i = 1,...,n, s(x) = s_i(x), where s_i(x) is a cubic polynomial.
2. $s(x_i) = f(x_i)$ for i = 0,1,...,n.
3. $s(x)$ is twice continuously differentiable on (a,b).
4. Either of the following boundary conditions are satisfied:
   (a) $s'(a) = s''(b) = 0$,
   (b) $s'(a) = f'(a)$ and $s'(b) = f'(b)$

The cubic spline is a piece-wise polynomial of the form:

$$s(x) = s_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3,\ i = 1, 2, .., n,\ x_i - 1 \le x \le x_i \tag{3.15}$$

That is, the value of s(x) is obtained by evaluating a different cubic polynomial for each subinterval $[x_{i-1}]$, for i = 1,2,...,n. A further explanation on how each coefficient is determined can be found in the appendix section (A.2). [21]

The *MATLAB* function *csaps* does these calculations for us and returns the cubic smoothing spline interpolation to the given data in ppform. There is some limitation to this technique, here we can mention, different knot points result in variations in the discount function, which can sometimes be significant, so, one must be careful in the selection of both the number and the placing of the knot points. Also, too many knot points may lead to overfitting of the discount function, which we don't have to worry about in the Icelandic market. A shortcoming of cubic-splines is that they give unreasonably curved shapes for the term structure at the long end of the maturity spectrum, a region where the term structure must have very little curvature. The discount function estimated using cubic splines is usually reasonable up to the maturity of the longest bond in the dataset but tend to be positive or negative infinity when extrapolated to longer terms. Although the use of polynomials splines moderates the wavy shape of simple polynomials around the curve to be fitted, this shape might not disappear completely and hence, the fitted discount function might wave around the real discount function introducing a significant variability in both spot and forward rates. Despite these shortcomings, polynomial splines are widely used in the financial industry.[14]

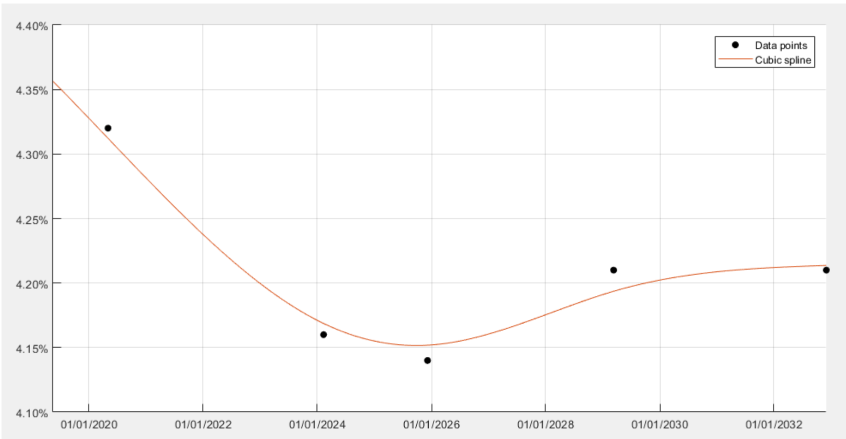


Figure 3.6: The yield curve of Icelandic Nominal Treasury bonds at 15.05.19 fitted with cubic spline

3.6 Constrained cubic splines

Compared to the cubic splines method it is possible to prevent over and undershooting by using the constrained cubic splines method. This is possible by replacing the requirement from the equal second order derivative for every point with specific first order derivatives. like in the traditional cubic splines the constraint cubic splines are constructed from the following equations where $i = 1, ..., n$,

$$f_i(x_1) = f_{i+1} = y_i \tag{3.16}$$

$$f_i^{'}(x_i) = f_{i+1}^{'}(x_i) \tag{3.17}$$

$$f_1^{''}(x_0) = f_n^{''}(x_n) \tag{3.18}$$

and where $f_i^{''}(x_i) = f_{i+1}^{''}(x_i)$ is replaced with the first order derivative:

$$f_i^{'}(x_i) = f_{i+1}^{'}(x_i) \tag{3.19}$$

with this at hand it becomes possible to to calculate the slope at intermediate points with the following equation:

$$f^{'}(x_i) = \frac{2}{\frac{x_{i-1}-x_i}{y_{i+1}-y_i} + \frac{x_i-x_{i-1}}{y_i-y_{i-1}}} \tag{3.20}$$

and for the end points

$$f_n^{'}(x_n) = \frac{3(y_n - y_{n-1})}{2((x_n - x_{n-1})} - \frac{f^{'}(x_{n-1})}{2} \tag{3.21}$$

$$f_1^{'}(x_0) = \frac{3(y_1 - y_0)}{2((x_1 - x_0)} - \frac{f^{'}(x_1)}{2} \tag{3.22}$$

with the determined slope for each data point each spline function can be given as 3. order polynomial, and can thus be calculated from the its adjacent data points (*for example* x_i *and* x_{i+1}). the ability to find the spline function for the knot point intervals these following second order derivatives are first calculated.

$$f_i''(x_{x_{i-1}}) = -\frac{2[f_i'(x_i) + 2f_i'(x_{i-1})]}{(s_1 - x_{i-1})} + \frac{6(y_i - y_{i-1})}{(x_i - x_{i-1})^2} \tag{3.23}$$

$$f_i''(x_i) = \frac{2[2f_i'(x_i) + f_i'(x_{i-1})]}{(x_i - x_{i-1})} + \frac{6(y_i - y_{i-1})}{(x_i - x_{i-1})^2} \tag{3.24}$$

with this at hand the spline function is given as the result of the following equation's in the form of a 3rd order polynomial.

$$d_i = \frac{f_i''(x_i) - f_i''(x_{i-1})}{6(x_i - x_{i-1})} \tag{3.25}$$

$$c_i = \frac{x_i f_i''(x_{i-1}) - x_{i-1} f_i''(x_i)}{2(x_i - x_{i-1})} \tag{3.26}$$

$$b_i = \frac{(y_i - y_{i-1}) - c_i(x_i^2 - x_{i-1}^2 - d_i(x_i^3 - x_{i-1}^3)}{(x_1 - x_{i-1})} \tag{3.27}$$

$$a_i = y_{i-1} - b_i x_{i-1} - c_i x_{i-1}^2 - d_i x_{i-1}^3 \tag{3.28}$$

$$f_i(x_i) = a_i + b_i x_i + c_i x_i^2 + d_i x_i^3 \tag{3.29}$$

Constrained cubic spline method produces a yield curve that scarifies some smoothness in return for a more stable shape that does not overshoot. This is done by slightly modifying the mathematical constraints placed upon the cubic spline method.[22]

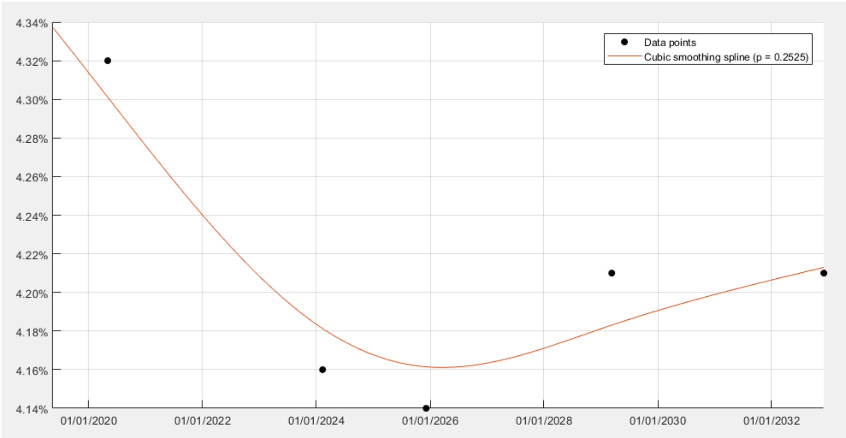


Figure 3.7: Contstrained cubic spline with 0.25 smoothing factor

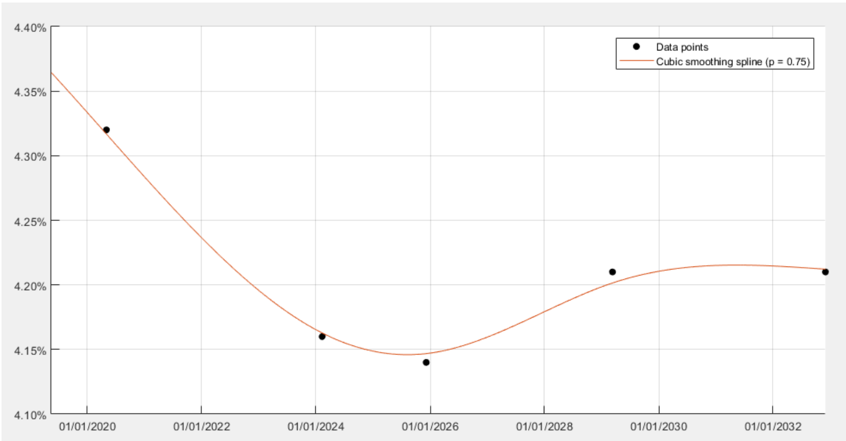


Figure 3.8: Constrained cubic spline with 0.75 smoothing factor

4 Swap

4.1 Swap contract

Swap contracts is an over-the-counter agreement between two parties which consists of exchanging payments where one party pays floating rate and the other one pays the swap rate, which is the fixed rate on some nominal sum. The floating rate is usually based on Libor rates or similar benchmark interest rates on the same nominal amount. Then both parties agree to swap cash-flow over some agreed period of time.

In beginning the swap contract value is,

$$S_{value}(0) = PV_{fix}(0) - PV_{flt}(0) = 0; \quad t = 0 \tag{4.1}$$

and the fixed rate is determined by making the value at beginning equal to zero.[3] In the section below we will go over the swap rates and the calculations. However, the swap contract value will change with time because of changes in interest rates, then one party will pay more/less than the other party, all depending on the floating rate.[12]

The advantage of using swap contract is to change interest rate exposure to outstanding loan, where one party pays floating and the other one fixed.

For example, if party A takes a loan on a floating rate contract it can transform it to a fixed rate loan by entering a swap contract. Party A receives floating from party B, then pays floating rate to the loan provider and pays at fixed rate to party B. Party B will then transform the fixed rate loan to a floating rate loan by using swap contract. Then party B pays floating rate to party A, fixed rate to the loan provider and receive a fixed rate from party A.

4.2 Swap rate

Like mentioned above, the swap rate is determined by making the value of the swap contract at $t_0 = 0$ equal to zero. To find the swap rate we first need to evaluate the present values of both cash flow, fixed and floating.

$$PV_{flt}(t_0) = \sum_k \alpha_{flt}(T_{k-1}, T_k) F(t_0, T_{k-1}, T_k) D(T_0, T_k) \tag{4.2}$$

$$PV_{fix}(t_0) = \sum_k \alpha_{fix}(T_{k-1}, T_k) S(t_0, T_0, T_N) D(T_0, T_k) \tag{4.3}$$

By putting the two present values equal we get,

$$PV_{fix}(t_0) = PV_{flt}(t_0) \tag{4.4}$$

and by solving for the swap rate we get the general swap rate equation

$$S(t_0, T_0, T_N) = \frac{\sum_k \alpha_{flt}(T_{k-1}, T_k) F(t_0, T_{k-1}, T_k) D(T_0, T_k)}{\sum_k \alpha_{fix}(T_{k-1}, T_k) D(T_0, T_k)} \tag{4.5}$$

we can also write the swap rates as

$$S(t_0, T_0, T_N) = \sum_k w_k F(t_0, T_{k-1}, T_k) \tag{4.6}$$

where,

$$W_k = \frac{\alpha_{flt}(T_{k-1}, T_k) D(T_0, T_k)}{\sum_k \alpha_{fix}(T_{k-1}, T_k) D(T_0, T_k)} \tag{4.7}$$

and

$$\sum_k W_k = 1 \tag{4.8}$$

4.3 Forward swap contract in simple compounding

To calculate forward swap contract in simple compounding, the forward rates are:

$$F(t_0, T_{k-1}, T_k) = \frac{1}{\alpha_{flt}(T_{k-1}, T_k)} \left(\frac{D(T_0, T_{k-1})}{D(T_0, T_k)} - 1 \right) \tag{4.9}$$

if we put the forward rates in the general swap rate equation

$$
\begin{aligned}
S(t_0, T_0, T_N) &= \frac{\displaystyle\sum_k \alpha_{flt}(T_{k-1}, T_k) \frac{1}{\alpha_{flt}(T_{k-1}, T_k)} \left(\frac{D(T_0, T_{k-1})}{D(T_0, T_k)} - 1 \right) D(T_0, T_k)}{\displaystyle\sum_k \alpha_{fix}(T_{k-1}, T_k) D(T_0, T_k)} \\
&= \frac{\displaystyle\sum_k \left(\frac{D(T_0, T_{k-1})}{D(T_0, T_k)} - 1 \right) D(T_0, T_k)}{\displaystyle\sum_k \alpha_{fix}(T_{k-1}, T_k) D(T_0, T_k)} \\
&= \frac{\displaystyle\sum_k D(T_0, T_{k-1}) - D(T_0, T_k)}{\displaystyle\sum_k \alpha_{fix}(T_{k-1}, T_k) D(T_0, T_k)} \\
&= \frac{D(t_0, T_0) - D(T_0, T_N)}{\sum \alpha_{fix}(T_{k-1}, T_k) D(T_0, T_k)}
\end{aligned} \tag{4.10}
$$

where $t_0 < T_0$.

4.4 Spot swap contract

From the forward swap rate equation, we can easily find the spot swap rate. Forward swap rate formula in simple compounding is

$$
S(t_0, T_0, T_N) = \frac{D(t_0, T_0) - D(T_0, T_N)}{\sum \alpha_{fix}(T_{k-1}, T_k) D(T_0, T_k)} \tag{4.11}
$$

and spot swap rate is found be setting $t_0 = T_0$, therefore, $D(t_0, T_0) = 1$ by inserting that into the equation we get [3]

$$
S(t_0, T_0, T_N) = \frac{1 - D(T_0, T_N)}{\sum \alpha_{fix}(T_{k-1}, T_k) D(T_0, T_k)} \tag{4.12}
$$

To find the spot swap rate in continuous compounding the forward rates are

$$
F(t_0, T_{k-1}, T_k) = \frac{1}{\alpha_{flt}(T_{k-1}, T_k)} \ln \frac{D(T_0, T_{k-1})}{D(T_0, T_k)} \tag{4.13}
$$

4.5 Forward rate agreement

Consist of only one exchange of payment while swap agreement can be viewed as a sequence of FRA agreements because swaps are agreements between two parties to exchange a future stream of fixed rate payments for floating rate payments. [12]

5 Estimating parameters

5.1 Maximum Likelihood Estimation

The estimated values of the three parameters κ, θ and σ can be calculated using historical data and the maximum likelihood estimation. Assuming the rates r_t evolves according to the following stochastic differential equation

$$dr_t = \kappa(\theta - r_t)dt + \sigma dW_t \tag{5.1}$$

Now assume that we have at times $T = (t, t + \Delta t, t + \Delta t, ..., t + N\Delta t)$ made the following rate observations

$$r_t, r_{t+\Delta t}, r_{t+2\Delta t}, ..., r_{t+N\Delta t} \tag{5.2}$$

By using Itô's lemma on the stochastic differential equation we have

$$r_{t+\Delta t} = r_t e^{-\kappa\Delta t} + \theta(1 - e^{-\kappa\Delta t}) + \int_t^{t+\Delta t} \sigma e^{-\kappa(t+\Delta t-u)} dW_u \tag{5.3}$$

which can be used to find the likelihood function

$$L(\kappa, \theta, s) = \prod_{i=1}^{N} \frac{1}{s\sqrt{2\pi}} \exp\left\{ -\frac{1}{2}\left\{ \frac{r_{t+i\Delta t} - r_{t+(i-1)\Delta t}e^{-\kappa\Delta t} - \theta(1 - e^{-\kappa\Delta t})}{s} \right\}^2 \right\} \tag{5.4}$$

where,

$$s = \frac{\sigma^2}{2\kappa}(1 - e^{-2\kappa\Delta t})$$

The log-likelihood function is derived by taking the natural logarithm of the likelihood function

$$\log L(\kappa, \theta, s) = -N\log(s) - \frac{N}{2}\log(2\pi) - \frac{1}{2s^2}\sum_{i=1}^{N}\left[r_{t+i\Delta t} - r_{t+(i-1)\Delta t}e^{\kappa\Delta t} - \theta(1 - e - \kappa\Delta t) \right]^2 \tag{5.5}$$

The parameters κ, θ and σ can be derived by setting the derivatives of the log-likelihood function with respect to κ, θ and s equal to 0. Solving the three equations yields the estimates for the parameters:

$$\hat{\theta} = \frac{S_y S_{xx} - S_x S_{xy}}{N(S_{xx} - S_{xy}) - S_x^2 + S_x S_y} \tag{5.6}$$

$$\hat{\kappa} = -\frac{1}{\Delta t}\log\left\{ \frac{S_{xy} - \hat{\theta}(S_x + S_y) + N\hat{\theta}^2}{S_{xx} - 2\hat{\theta}S_x + N\hat{\theta}^2} \right\} \tag{5.7}$$

$$\hat{\sigma} = \sqrt{\frac{2\hat{\kappa}\hat{s}^2}{1 - e^{-2\hat{\kappa}\Delta t}}} \tag{5.8}$$

Where,

$$S_x = \sum_{i=1}^{N} r_{t+(i-1)\Delta t} \tag{5.9}$$

$$S_{xx} = \sum_{i=1}^{N} r^2_{t+(i-1)\Delta t} \tag{5.10}$$

$$S_y = \sum_{i=1}^{N} r_{t+i\Delta t} \tag{5.11}$$

$$S_{yy} = \sum_{i=1}^{N} r^2_{t+i\Delta t} \tag{5.12}$$

$$S_{xy} = \sum_{i=1}^{N} r_{t+(i-1)\Delta t} r_{t+i\Delta t} \tag{5.13}$$

$$\hat{s}^2 = \frac{1}{N}\{S_{yy} - 2S_x e^{-\kappa\Delta t} + S_{xx} e^{-2\hat{\kappa}\Delta t} - 2\hat{\theta}(S_y - S_x e^{-\hat{\kappa}\Delta t})(1 - e^{-\hat{\kappa}\Delta t}) + N\hat{\theta}^2(1 - e^{-\hat{\kappa}\Delta t})^2\} \tag{5.14}$$

Estimation of these parameters can be seen here below on *Figure 5.1*, where we see the parameter estimation with a different number of iterations.

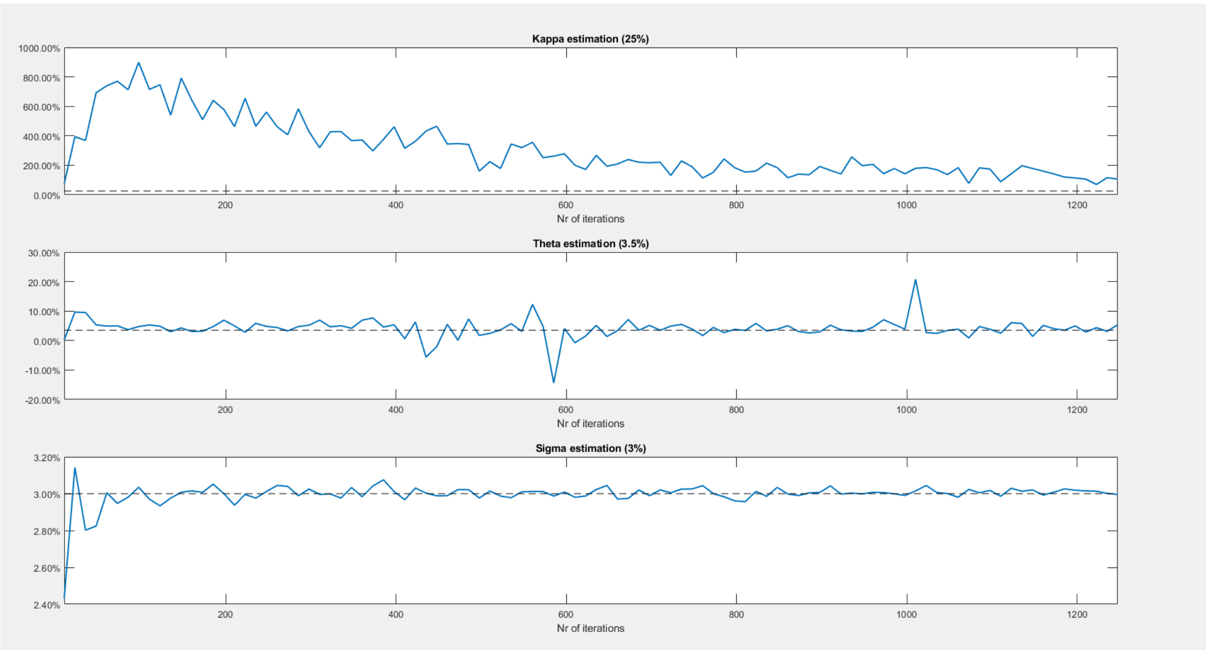


Figure 5.1: κ, θ and σ estimation using Maximum Likelihood Estimation

5.2 Ordinary least square for Ornstein-Uhlenback

Ordinary least square method, henceforth referred to as OLS. Is a generalized linear modeling technique that is used to model a single response variable which has been recorded on the least interval scale. The technique can be applied to a single or multi explanatory variables.[23] Considering the implementation of the OLS method. The following relationship between consecutive interest rates is assumed,

$$r_{t+\Delta t} = mr_t + c + \epsilon_t, \epsilon_t \sim N(0, \sigma_\epsilon^2) \tag{5.15}$$

and on the other hand, the solution for Ornstein-Uhlenbeck, henceforth referred to as OU. is given as,[19]

$$r_{t+\Delta t} = r_t e^{-\kappa\Delta t} + \theta\Big(1 - e^{-\kappa\Delta t}\Big) + \int_t^{t+t} \sigma e^{-\kappa(t+\Delta t-u} dW_u \tag{5.16}$$

To be able to estimate the parameters for m and c. the following least square objective function is defined as,

$$\Omega(c, m) = \sum_{i=1}^{N}(r_{t+i\Delta t} - mr_{t+(i-1)\Delta t} - c)^2 \tag{5.17}$$

and estimate the parameters from the first two order conditions,

$$\frac{\delta\Omega(c,m)}{\delta c} = 0 \quad \frac{\delta\Omega(c,m)}{\delta m} = 0 \tag{5.18}$$

so that we can find the following expressions,*(see MLE for S_x S_y and S_{xy})*

$$\hat{m} = \frac{NS_{xy} - S_x S_y}{NS_{xx} - s_x^2} \tag{5.19}$$

$$\hat{c} = \frac{S_y - \hat{m}S_x}{N} \tag{5.20}$$

$$\hat{\sigma_\epsilon} = \sqrt{\frac{\sum_{i=1}^{N}(r_{t+i\Delta t} - \hat{m}r_{t+(i-1)\Delta t} - \hat{c}^2)}{N-2}} \tag{5.21}$$

Now by comparing OLS and OU solutions we find the following relationships between the parameters

$$m = e^{-\kappa\Delta t} \Rightarrow \kappa = -\frac{log(m)}{\Delta t} \tag{5.22}$$

$$c = \theta(1 - e^{-\kappa\Delta t}) \Rightarrow \theta = \frac{c}{(1 - e^{-\kappa\Delta t})} \tag{5.23}$$

$$\theta = \frac{c}{\Big(1 - e^{\frac{log(m)\Delta t}{\Delta t}}\Big)} \Rightarrow \frac{c}{1-m} \tag{5.24}$$

and for σ we know that,

$$\mathrm{Var}\Bigg[\int_t^{t+\Delta t} \sigma e^{\kappa(t+\Delta t-u)} dW_u\Bigg] \tag{5.25}$$

Which we can use to solve for σ by evaluating the right and side

$$\sigma^2 e^{-2\kappa(t+\Delta t} \int_t^{t+\Delta t} e^{2\kappa u} du \Rightarrow \sigma^2 e^{-22\kappa(t+\Delta t)}\Bigg(\frac{1}{2\kappa}\Big(e^{2\kappa(t+\Delta t) - e^{2\kappa t}} = \frac{\sigma^2}{2\kappa}\Big(1 - e^{-2\kappa\Delta t}\Big) = \sigma_\epsilon^2 \tag{5.26}$$

That gives the equation for σ as follows,

$$\sigma = \sigma_\epsilon \sqrt{\frac{2\kappa}{1 - e^{-2\kappa \Delta t}}} \tag{5.27}$$

Estimation of these parameters using the OLS method can be seen here below on *Figure 5.2*, where we see the parameter estimation with a different number of iterations.

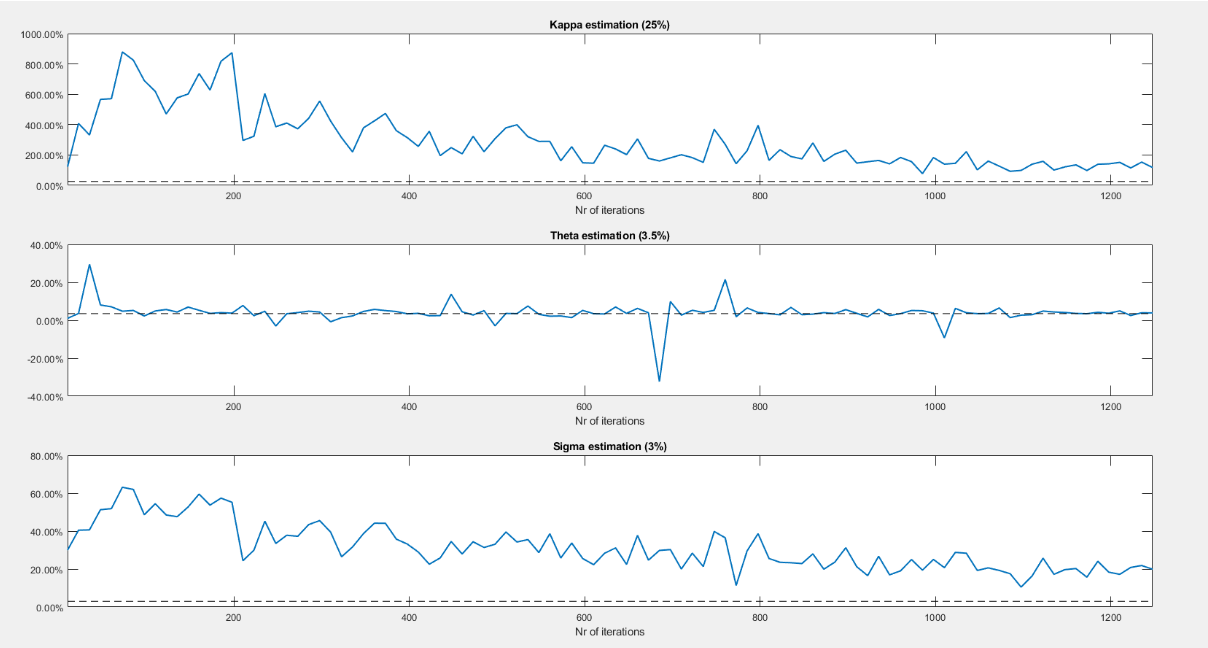


Figure 5.2: κ, θ and σ estimation using Ordinary least square

6 Stochastic models for interest rates

Wiener process, also called Brownian motion is a simple continuous stochastic process that is widely used in physics and finance for modeling random behavior that evolves over time. Intuitively, we may think of a Wiener process as a limiting case of some random walk as its time increment goes to zero.[24]

The Wiener process($W(t)$) has the following properties:

1. $W(0) = 0$
2. $W(t)$ has independent increments. For any $t > 0$, the future increment $W(t+dt) - W(t)$, $dt \geq 0$, is independent of the past values $W(s)$ where $s < t$.
3. $W(t)$ has Gaussian increments: $W(t+dt) - W(t) \sim N(0, dt)$
4. $W(t) = W(\omega, t)$ is a continuous function of t for every $\omega \in \Omega$.

6.1 Models

6.1.1 Simple Wiener model

For this model we assume that the short interest rates follow a simple Wiener process,

$$dr(t) = \sigma dW(t) \tag{6.1}$$

Integrating both sides of (6.1) yields

$$\int_s^t dr(\tau) = \int_s^t \sigma dW(\tau)$$
$$r(t) = r(s) + \sigma \int_s^t W(\tau) d\tau$$

which gives

$$r(t) = r(s) + \sigma(W(t) - W(s)) \tag{6.2}$$

Using the third property of the wiener process and calculating both the expected value and variance of $r(t)$

$$\mathbb{E}[r(t)] = \mathbb{E}[r(s)] + \sigma\, \mathbb{E}[W(t) - W(s)] = r(s) \tag{6.3}$$
$$\mathrm{Var}[r(t)] = \mathrm{Var}[r(s)] + \sigma^2 \mathrm{Var}[W(t) - W(s)] = \sigma^2(t-s) \tag{6.4}$$

Since $W(t+dt) - W(t) \sim N(0, dt)$ it now follows that

$$r(t) \sim N(r(s),\ \sigma^2(t-s)) \tag{6.5}$$

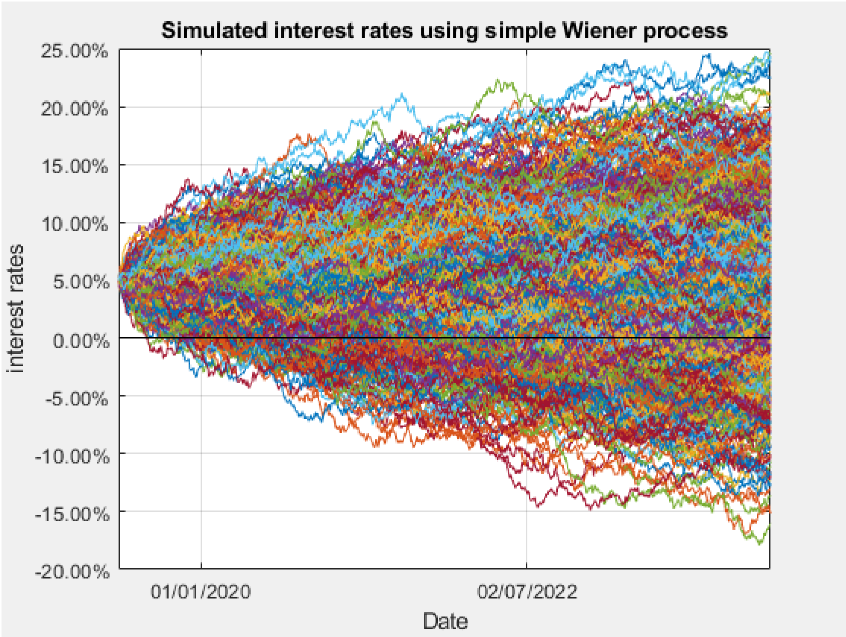


Figure 6.1: 1000 simulations with 5.0% initial rate and 3.0% volatility using simple Wiener process

6.1.2 Brownian motion model

Another simple model for zero rates is the Brownian motion model, given by the stochastic formula,

$$dr(t) = \alpha dt + \sigma dW(t) \tag{6.6}$$

where α and σ are constants. The solution is given by,

$$r(t) = r(s) + \alpha(t - s) + \sigma(W(t) - W(s)) \tag{6.7}$$

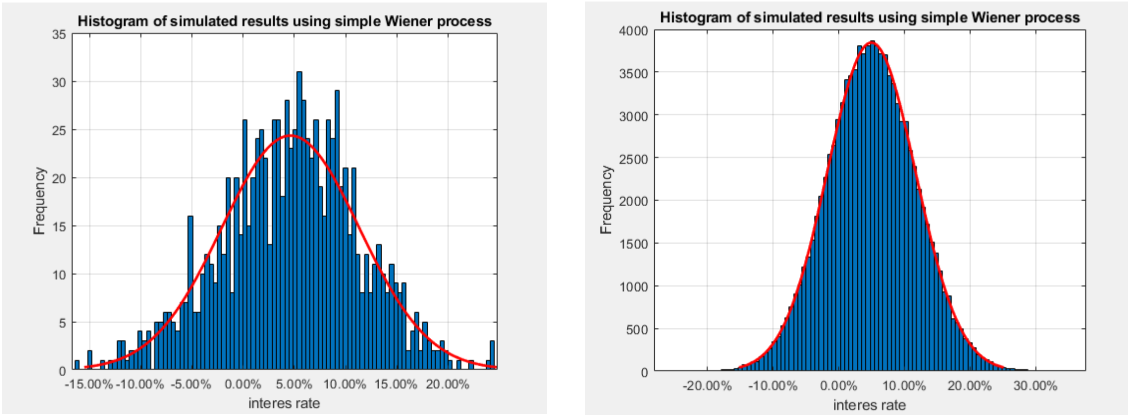


Figure 6.2: Histograms with 1.000 and 100.000 simulations using simple Wiener process

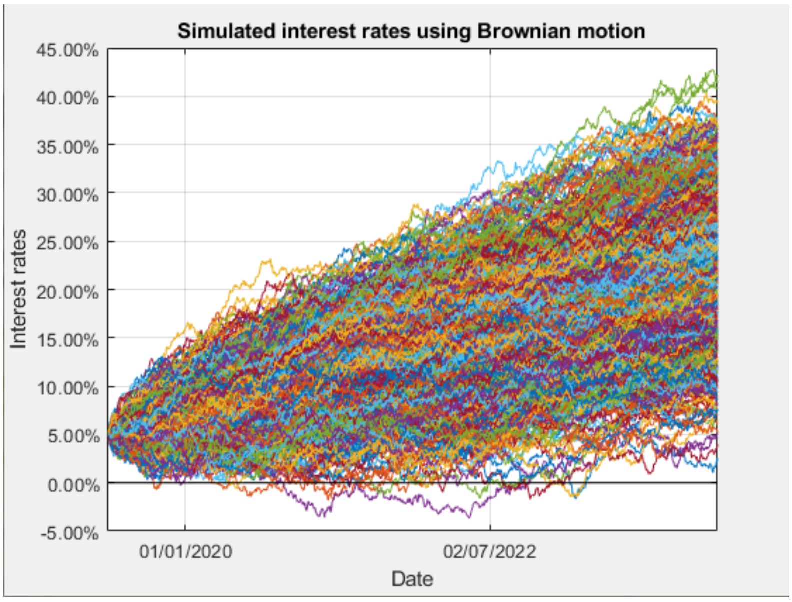


Figure 6.3: 1000 simulations with 5.0% initial rate and 3.0% volatility using Brownian process

6.1.3 Vasicek model

The Vasicek is a mathematical model describing the evolution of interest rates. It is the first model to capture the mean reversion property of interest rates

$$dr(t) = \kappa(\theta - r(t))dt + \sigma dW \tag{6.8}$$

where κ is speed of reversion, θ is the long term mean level, σ is volatility and dW is a standard Brownian motion, with solution.[25]

$$r(t) = e^{-\kappa(t-s)}r(s) + \theta\left(1 - e^{-\kappa(t-s)}\right) + \sigma \int_s^t e^{-\kappa(t-\tau)}dW(\tau) \tag{6.9}$$

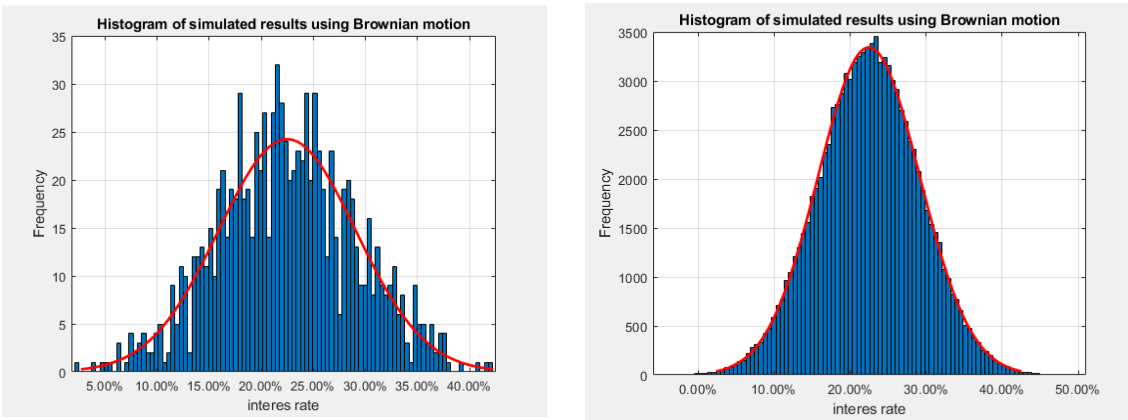


Figure 6.4: Histograms with 1.000 and 100.000 simulations using Brownian process

The expected value and the variance is

$$\mathbb{E}_s[r(t)|r(s)] = r(s)e^{-\kappa(t-s)} + \theta\left(1 - e^{\kappa(t-s)}\right) \tag{6.10}$$

$$var_s[r(t)|r(s)] = \frac{\sigma^2}{2\kappa}\left(1 - e^{-2\kappa(t-s)}\right) \tag{6.11}$$

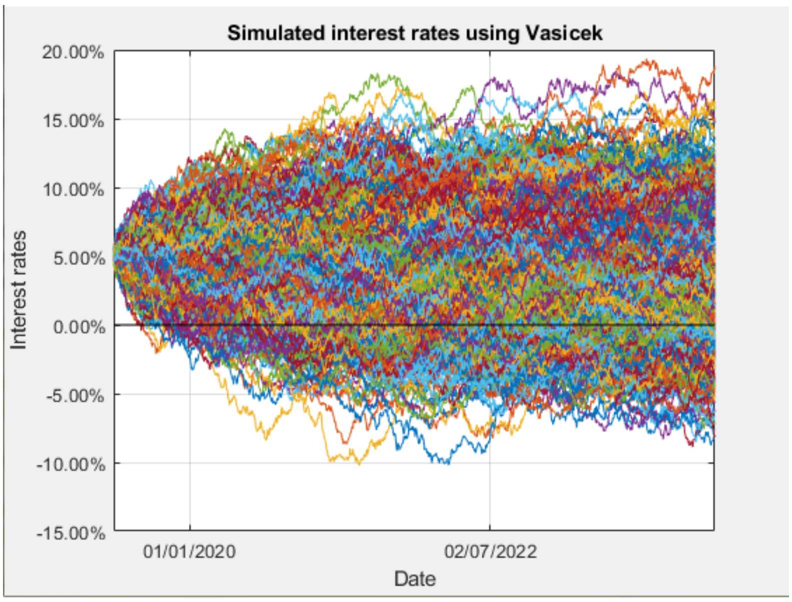
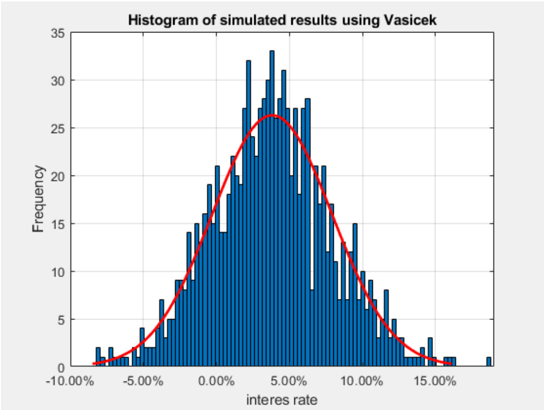


Figure 6.5: 1000 simulations with 5.0% initial rate and 3.0% volatility using Vasicek process



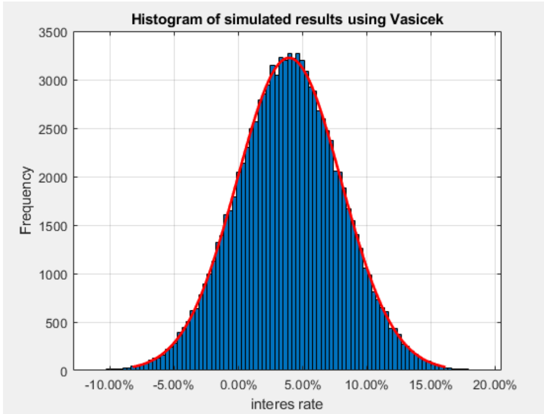


Figure 6.6: Histograms with 1.000 and 100.000 simulations using Vasicek process

6.2 Pricing zero coupon bonds

To determine the price of the zero coupon bond and it's term rate, we consider a zero coupon bond which pays \$1 at time T and has option maturity at 4 years for our simulation example.

6.2.1 Simple Wiener and Brownian motion model

The price of the zero coupon bond at time t, $0 \leq t \leq T$, is given by,

$$P(t,T) = \mathbb{E}\left[\exp\left(-\int_t^T r(\tau)d\tau\right)\right] \tag{6.12}$$

which equals to

$$P(t,T) = \exp\left(-r(t)(T-t) + \frac{\sigma^2}{6}(T-t)^3\right) \tag{6.13}$$

For Brownian motion model. The price of a zero coupon bond in this model is given by,

$$P(t,T) = \mathbb{E}_t\left[exp\Big(-\int_t^T r(s)ds\Big)\right] = exp\Big(-r(t)(T-t) - \frac{1}{2}\alpha(T-t)^2 + \frac{1}{6}\sigma^2(T-t)^3\Big) \tag{6.14}$$

A derivation of these pricing formulas can be found in the appendix section (A.3 and A.4). [26]

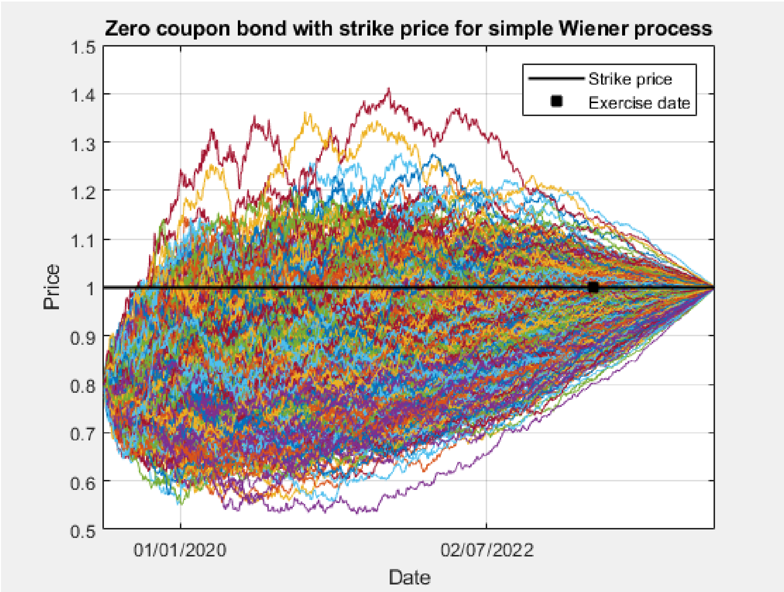


Figure 6.7: 1.000 simulation using simple Wiener process

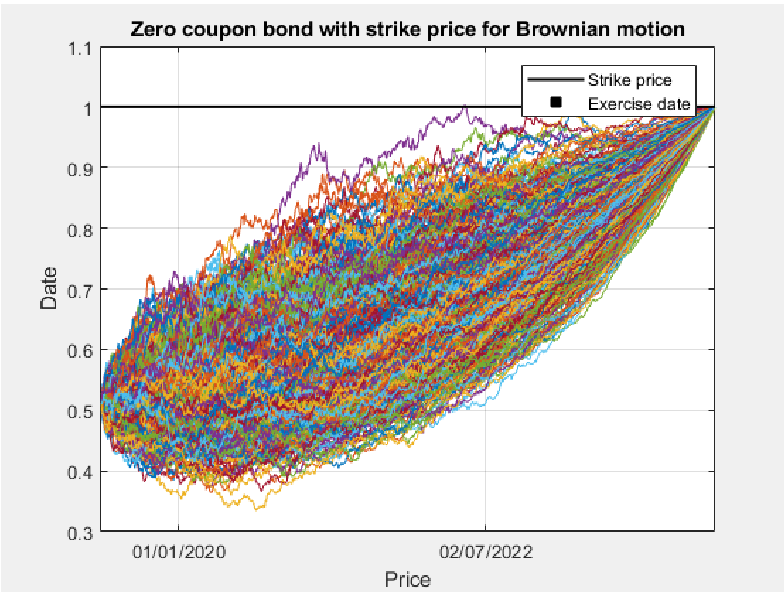


Figure 6.8: 1.000 simulations using Brownian process

6.2.2 Vasicek model

The price of a zero coupon bond in the Vasicek model can be written as,

$$P(t,T) = A(t,T)e^{-B(t,T)r(t)} \tag{6.15}$$

where,

$$B(t,T) = \frac{1 - e^{\kappa(T-t)}}{\kappa} \tag{6.16}$$

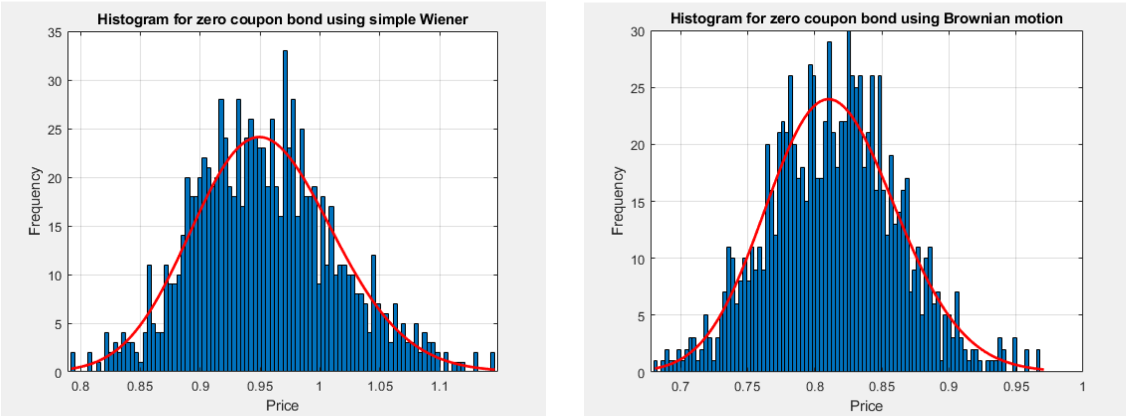


Figure 6.9: Histograms with 1.000 simulations for simple Wiener and Brownian process

$$A(t,T) = exp\left(\frac{(B(t,T) - T + t)(\kappa^2\Theta - \sigma^2/2)}{\kappa^2} - \frac{\sigma^2 B(t,T)^2}{4\kappa}\right) \tag{6.17}$$

then,

$$R(t,T) = -\frac{1}{T-t}\log A(t,T) + \frac{1}{T-t}B(t,T)r(t) \tag{6.18}$$

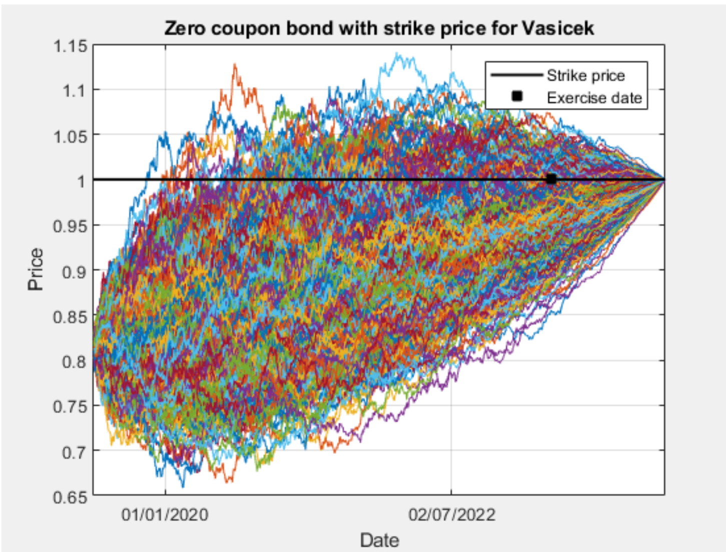


Figure 6.10: 1.000 simulations using Vasicek process

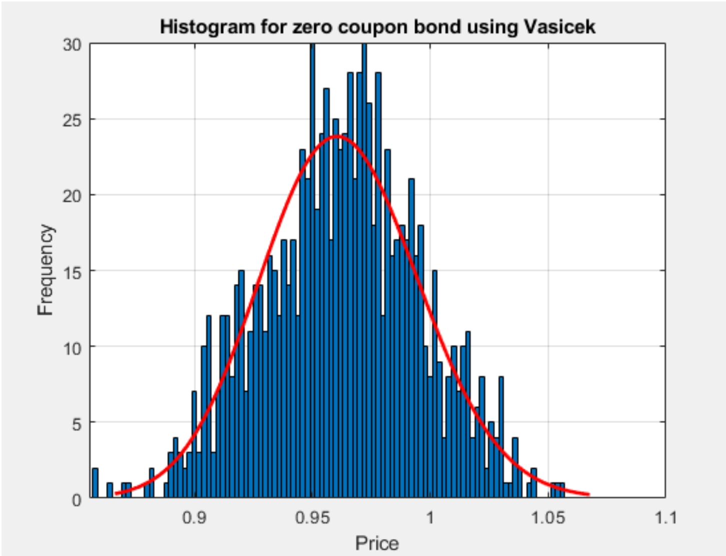


Figure 6.11: Histogram with 1.000 simulations using Vasicek process

6.3 Pricing call and put options

6.3.1 Simple Wiener and Brownian motion model

Black's model is a popular tool for pricing European options in terms of the forward or future price of the underlying asset when interest rates are constant. Black model can be used to price European options in terms of the forward price of the underlying asset when interest rates are stochastic.

We consider a European call option on a bond with strike price K that lasts until time T. The call option's price is given by the formula:

$$c = P(0,T)\,\mathbb{E}_T[max(S_T - K, 0)] \tag{6.19}$$

Where S_T is the bonds price at time T and $\mathbb{E}_T$ denotes expectations in a world that is forward risk neutral with respect to P(t,T). Define F_0 and F_T as the forward price of the bond at time 0 and time T for a contract maturing at time T. Because $S_T = F_T$,

$$c = P(0,T)\,\mathbb{E}_T[max(F_T - K, 0)] \tag{6.20}$$

Assuming F_T is log-normal in the world being considered, with the standard deviation of $ln(F_T)$ equal to $\sigma_F\sqrt{T}$. This could be because the forward price follows a stochastic process with volatility σ_F.Then:

$$c = P(0,T)[F_0 N(d_1) - KN(d_2)] \tag{6.21}$$

where,

$$d_1 = \frac{ln[\frac{F_0}{K} + \sigma_F^2 \frac{T}{2}]}{\sigma_F\sqrt{T}} \tag{6.22}$$

$$d_2 = \frac{ln[\frac{F_0}{K} - \sigma_F^2 \frac{T}{2}]}{\sigma_F \sqrt{T}} \tag{6.23}$$

Similarly,

$$p = P(0,T)[KN(-d_2) - F_0 N(d_1)] \tag{6.24}$$

where p is the price of a European put option on the bond with strike price K and time to maturity T. This is true when interest rates are stochastic provided that F_0 is the forward bond price. The variable σ_F can be interpreted as the volatility of the forward bond price.[12]
Both calculated and simulated prices for the Wiener- and the Brownian process can be seen in the tables below:

	1.000 Simulations	100.000 Simulations
Call price simulated	0.0068	0.0058
Call price calculated	5.9e-07	5.9e-07
Put price simulated	0.0438	0.0441
Put price calculated	0.1703	0.1693

Table 2: Wiener process

	1.000 Simulations	100.000 Simulations
Call price simulated	0.0058	2.4e-06
Call price calculated	5.9e-07	6.2e-30
Put price simulated	0.0441	0.1525
Put price calculated	0.1693	0.1743

Table 3: Brownian process

6.3.2 Vasicek model

This model allows options on zero-coupon bonds to be valued analytically. The price at time zero of a call option that matures at time T on a zero-coupon bond maturing at time S is,

$$c(t) = QP(t,S)N(d) - KP(t,T)N(d - \sigma_p) \tag{6.25}$$

and the price of a put option is,

$$p(t) = KP(t,T)N(-d + \sigma_p) - QP(t,S)N(-d) \tag{6.26}$$

where Q is the principal of the bond, K is the strike price, S > T, d is:

$$d = \frac{1}{\sigma_p} \log\left(\frac{QP(t,S)}{KP(t,T)}\right) + \frac{\sigma_p}{2} \tag{6.27}$$

and σ_p is:

$$\sigma_p = \frac{\sigma}{\kappa}\left(1 - e^{-\kappa(S-T)}\right)\sqrt{\frac{1 - e^{-2\kappa(T-t)}}{2\kappa}} \tag{6.28}$$

The calculated and simulated prices for the Vasicek process can be seen in the table below.

	1.000 Simulations	100.000 Simulations
Call price simulated	0.0017	0.0019
Call price calculated	0.0019	0.0334
Put price simulated	0.0326	0.0022
Put price calculated	0.0333	0.0323

Table 4: Vasicek process

6.4 Pricing caps and floors

Let $L_k = L(t_k, t_{k+1})$ be the Libor rate set at time t_k for the time interval $[t_k, t_{k+1}]$, let L_c be the interest rate cap. Assuming Q is the principal then the payoff from a caplet at the time t_{k+1} is

$$C(T, L_k, L_c) = \kappa(t_k, t_{k+1})Q \times \max(L(t_k, t_{k+1}) - L_c, 0) \tag{6.29}$$

Similarly, payoff for flooret is

$$P(T, L_k, L_c) = \kappa(t_k, t_{k+1})Q \times \max(L_c - L(t_k, t_{k+1}), 0) \tag{6.30}$$

Assuming Libor rates follow the stochastic differential equation

$$dL_k(t) = \sigma_k(t)L_k(t)dW^{k+1} \tag{6.31}$$

The prices of a caplet and a flooret can be given by

$$C(t, L_k, L_c) = \kappa_k Q D(t, t_{k+1})F_k(t)N(d_1) - L_c N(d_2) \tag{6.32}$$

$$P(t, L_k, L_c) = \kappa_k Q D(t, t_{k+1})L_c N(-d_2) - F_k N(-d_1) \tag{6.33}$$

Where,

$$\kappa_k = \kappa(t_k, t_{k+1}); \quad F_k(t) = F(t, t_k, t_{k+1}) \tag{6.34}$$

and,

$$d_1 = \frac{\log(\frac{F_k(t)}{L_c}) + \frac{\sigma_k^2}{2}t_k}{\sigma_k\sqrt{t_k}}; \quad d_2 = \frac{\log(\frac{F_k(t)}{L_c}) - \frac{\sigma_k^2}{2}t_k}{\sigma_k\sqrt{t_k}} = d_1 - \sigma_k\sqrt{t_k}; \tag{6.35}$$

A cap consists of series of caplets where payments are made at the time $T = t_1, t_2, ..., t_N$
Similarly, a floor consist of series of floorlets where payments are made at the times $T = t_1, t_2, ..., t_N$
So the price of a cap and a floor is

$$C(t_1, t_2, ..., t_N, L_c) = \sum_{k=1}^{N} C(t, L_k, L_c) \tag{6.36}$$

and,

$$P(t_1, t_2, ..., t_N, L_c) = \sum_{k=1}^{N} P(t, L_k, L_c) \tag{6.37}$$

[27]

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A Appendix

A.1 Polynomials

The polynomial fitting is a method that consists of minimizing the error function:

$$E(\bar{a}) = \sum_{i=1}^{N} \Big(y(x_i) - P_n(x_i) \Big)^2; \quad 1 \leq n \leq N; \quad P_n(x) = \sum_{k=0}^{n} a_k x^k \tag{A.1}$$

where N is the number of data pairs $(x, y(x))$, $P_n(x_i)$ is polynomial of the n-th degree and $\bar{a}$ are the coefficients of the polynomial. The error function is minimized by taking the derivative with regards to each coefficient, a, and setting it to zero.

$$\begin{aligned}
\frac{\delta E}{\delta a_j} &= \frac{\delta}{\delta a_j} \sum_{i=1}^{N} \Big(y(x_i) - \sum_{k=0}^{n} a_k x_i^k \Big)^2 \\
&= -2 \sum_{i=1}^{N} \Big(y(x_i) - \sum_{k=0}^{n} a_k x_i^k \Big) x_i^j \\
&= -2 \sum_{i=1}^{N} y(x_i) x_i^j + 2 \sum_{i=1}^{N} \sum_{k=0}^{n} a_k x_i^j x_i^k \stackrel{!}{=} 0
\end{aligned} \tag{A.2}$$

This yields the result:

$$\sum_{i=1}^{N} y(x_i) x_i^j = \sum_{i=1}^{N} \sum_{k=0}^{n} a_k x_i^j x_i^k \tag{A.3}$$

by making the following definitions:

$$X_{j,k} = \sum_{i=1}^{N} x_i^j x_i^k; \quad Y_j = \sum_{i=1} y_i x_i^j \tag{A.4}$$

$$\tag{A.5}$$

the result can be written in matrix notation:

$$\sum_{i=1}^{N} y(x_i) x_i^j = \sum_{i=1}^{N} \sum_{k=0}^{n} a_k x_i^j x_i^k \tag{A.6}$$

$$Y_j = \sum_{k=0}^{n} X_{j,k} a_k \tag{A.7}$$

$$\mathbf{Y} = \mathbf{X}\mathbf{a} \tag{A.8}$$

$$\mathbf{a} = \mathbf{X}^{-1}\mathbf{Y} \tag{A.9}$$

Since **X** is symmetric it is only necessary to look at the diagonal and the upper diagonal elements when writing out the **X** matrix.[19]

$$\mathbf{X} = \begin{pmatrix} N & \sum_{i=1}^{N} x_i^0 x_i^1 & \cdots & \sum_{i=1}^{N} x_i^0 x_i^N \\ & \sum_{i=1}^{N} x_i^1 x_i^1 & \cdots & \sum_{i=1}^{N} x_i^1 x_i^N \\ & & \ddots & \vdots \\ & & & \sum_{i=1}^{N} x_i^n x_i^N \end{pmatrix}; \quad \mathbf{Y} = \begin{pmatrix} \sum_{i=1}^{N} y_i x_i^0 \\ \sum_{i=1}^{N} y_i x_i^1 \\ \vdots \\ \sum_{i=1}^{N} y_i x_i^n \end{pmatrix} \tag{A.10}$$

A.2 Cubic splines

Let $f(x)$ be a function defined on an interval [a,b], and let x_0, x_1,...,x_n be $n+1$ distinct points in [a,b], where a $=$ x_0 $<$ x_1 $<$ ... $<$ x_n $=$ b. A cubic spline, is a piece-wise polynomial s(x) that satisfies the following conditions:

1. On each interval [x_i-1,x_i], for i $=$ 1,...,n, s(x) $=$ s_i(x), where s_i(x) is a cubic polynomial.
2. $s(x_i) = f(x_i)$ for i $=$ 0,1,...,n.
3. $s(x)$ is twice continuously differentiable on (a,b).
4. Either of the following boundary conditions are satisfied:
   (a) $s'(a) = s''(b) = 0$,
   (b) $s'(a) = f'(a)$ and $s'(b) = f'(b)$

The cubic spline is a piece-wise polynomial of the form:

$$s(x) = s_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3, \; i = 1, 2, .., n, \; x_i - 1 \leq x \leq x_i \tag{A.11}$$

That is, the value of s(x) is obtained by evaluating a different cubic polynomial for each sub-interval [x_{i-1}], for i $=$ 1,2,...,n. Now we can construct a system of equations that must be satisfied by the coefficients a_i, b_i, c_i and d_i for i $=$ 1,2,...,n. We can compute these coefficients by solving the system. $s(x)$ must fit the given date, therefore we have:

$$a_i = y_{i-1}, i = 1, 2, ..., n. \tag{A.12}$$

We define h_i $=$ x_i - x_{i-1}, for i $=$ 1,2,...,n, and define a_{n+1} $=$ y_n, then we must have $s_i(x_i)$ $=$ $s_{i+1}(x_i)$ for i $=$ 1,2,...,n-1. We also must have s(x_n) $=$ $s_n(x_n)$ $=$ y_n, because s(x) must fit the data. Therefore we have:

$$d_i h_i^3 + c_i h_i^2 + b_i h_i + a_i = a_{i+1}, i = 1, 2, ..., n. \tag{A.13}$$

To ensure that $s(x)$ has a continuous first derivative and second derivative at the interior nodes, we must have $s'_i(x_i) = s'_{i+1}(x_i)$ and $s''_i(x_i) = s''_{i+1}(x_i)$ for i = 1,2,...,n-1, therefore we have respectively:

$$3d_i h_i^2 + 2c_i h_i + b_i h_i = b_{i+1}, i = 1, 2, ..., n-1. \tag{A.14}$$

and

$$3d_i h_i + c_i = c_{i+1}, i = 1, 2, ..., n-1. \tag{A.15}$$

There are 4n coefficients to determine, since there are n cubic polynomials, each with 4 coefficients. However, we have only prescribed 4n-2 constraints, so we must specify 2 more in order to determine a unique solution. If condition 4a is satisfied, we have:

$$c_0 = 0 \tag{A.16}$$

$$3d_n h_n + c_n = 0 \tag{A.17}$$

On the other hand, if condition 4b is satisfied, we have:

$$b_0 = z_0 \tag{A.18}$$

$$3d_n h_n^2 + 2c_n h_n + b_n = z_n \tag{A.19}$$

where z_i = f'(x_i) for i = 0,1,...,n. Having determined the constraints that are satisfied by s(x), we can set up a system of linear equations based on these constraints, Ax = b, and solve to determine the coefficients a_i, b_i, c_i, d_i for i = 1,2,...,n. If condition 4a is satisfied, A is an $(n+1) \times (n+1)$ matrix defined by:

$$A = \begin{bmatrix} 1 & 0 & 0 & ... & ... & 0 \\ h_1 & 2(h_1+h_2) & h_2 & \ddots & & \vdots \\ 0 & h_2 & 2(h_2+h_3) & h_3 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & h_{n-1} & 2(h_{n-1}+h_n) & h_n \\ 0 & \dots & \dots & 0 & 0 & 1 \end{bmatrix}$$

and the (n + 1)-vectors x and b are

$$x = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_{n+1} \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ \frac{3}{h_2}(a_3 - a_2) - \frac{3}{h_1}(a_2 - a_1) \\ \vdots \\ \frac{3}{h_n}(a_{n+1} - a_n) - \frac{3}{h_{n-1}}(a_n - a_{n-1} \\ 0 \end{bmatrix}$$

where c_{n+1} = s"$(x_n)/2$.

If condition 4b is satisfied, we have:

$$A = \begin{bmatrix} 2h_1 & h_1 & 0 & ... & ... & 0 \\ h_1 & 2(h_1+h_2) & h_2 & \ddots & & \vdots \\ 0 & h_2 & 2(h_2+h_3) & h_3 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & h_{n-1} & 2(h_{n-1}+h_n) & h_n \\ 0 & \dots & \dots & 0 & h_n & 2h_n \end{bmatrix}$$

and the $(n+1)$-vectors, x and b are

$$x = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_{n+1} \end{bmatrix}, \quad b = \begin{bmatrix} \frac{3}{h_1}(a_2 - a_1) - 3z_0 \\ \frac{3}{h_2}(a_3 - a_2) - \frac{3}{h_1}(a_2 - a_1) \\ \vdots \\ \frac{3}{h_n}(a_{n+1} - a_n) - \frac{3}{h_{n-1}}(a_n - a_{n-1} \\ 3z_n - \ h_n(a_{n+1} - a_n) \end{bmatrix}$$

Once the coefficients c_1, c_2,...,c_{n+1} have been determined, the remaining coefficients can be found as follows:
1. The coeffecients a_1, a_2,...,a_{n+1} have already been defined by the relation $a_i = y_{i-1}$, $i = 0, 1, ..., n$.
2. The coefficients b_1, b_2,...,b_n are given by

$$b_i = \frac{1}{h_i}(a_{i+1} - a_i) - \frac{h_i}{3}(2c_i + c_{i+1}), \quad i = 1, 2, ..., n. \tag{A.20}$$

3. The coefficients d_1, d_2,...,d_n can be obtained by using the constraints

$$3d_i h_i + c_i = c_{i+1}, \quad i = 1, 2, ..., n. \tag{A.21}$$

A.3 Pricing zero coupon bonds

The price of the zero coupon bond at time t, $0 \leq t \leq T$, is given by,

$$P(t,T) = \mathbb{E}\left[\exp\left(-\int_t^T r(\tau)d\tau\right)\right] \tag{A.22}$$

where,

$$\begin{aligned} \int_t^T r(\tau)d\tau &= \int_t^T r(t)ds + \sigma\int_t^T\left(\int_t^s dW(v)\right)ds \\ &= r(t)(T-t) + \sigma\int_t^T\left(\int_v^T ds\right)dW(v) \quad \text{(Fubini's theorem)} \\ &= r(t)(T-t) + \sigma\int_t^T (T-v)dW(v) \end{aligned} \tag{A.23}$$

For a normally distributed variable, $X \sim N(\mu, \sigma^2)$, which can be written as $X = \mu + \sigma Z$, where $Z = \dfrac{X-\mu}{\sigma}$ and $Z \sim N(0,1)$, We can write,

$$\begin{aligned}
\mathbb{E}[e^X] &= \mathbb{E}[\exp(\mu + \sigma Z])\\
&= \mathbb{E}[\exp(\mu)\exp(\sigma Z)]\\
&= \exp(\mu)\,\mathbb{E}[\exp(\sigma Z)]\\
&= \exp(\mu)\exp\left(\frac{\sigma^2}{2}\right)\\
&= \exp\left(\mu + \frac{\sigma^2}{2}\right)
\end{aligned} \tag{A.24}$$

Therefore, defining $X(t,T) = \int_t^T r(\tau)d\tau$ would give the following formula for the price of the zero coupon bond,

$$\begin{aligned}
P(t,T) &= \mathbb{E}[\exp(-X(t,T))]\\
&= \exp\Big(-\mathbb{E}[X(t,T)] - \frac{1}{2}\text{Var}[X(t,T)]\Big)
\end{aligned} \tag{A.25}$$

where,

$$\begin{aligned}
\mathbb{E}[X(t,T)] &= \mathbb{E}\left[\int_t^T r(\tau)d\tau\right]\\
&= \mathbb{E}\left[r(t)(T-t) + \sigma\int_t^T (T-v)dW(v)\right]\\
&= r(t)(T-t)
\end{aligned} \tag{A.26}$$

and

$$\begin{aligned}
\text{Var}[X(t,T)] &= \text{Var}\left[\int_t^T r(\tau)d\tau\right]\\
&= \text{Var}\left[r(t)(T-t) + \sigma\int_t^T (T-t)dW(v)\right]\\
&= \sigma^2\text{Var}\left[\int_t^T (T-t)dW(v)\right]\\
&= \sigma^2\int_t^T (T-v)^2 dv\\
&= \frac{\sigma^2}{3}(T-t)^3
\end{aligned} \tag{A.27}$$

Finally, we have

$$P(t,T) = \exp\Big(-r(t)(T-t) + \frac{\sigma^2}{6}(T-t)^3\Big) \tag{A.28}$$

Comparing this result to the price of the bond, given that we know the term rate $R(t,T)$, gives

$$P(t,T) = \exp(-R(t,T)(T-t)) = \exp\Big(-r(t)(T-t) + \frac{\sigma^2}{6}(T-t)^3\Big) \tag{A.29}$$

which yields a formula for the term rate

$$R(t,T) = r(t) - \frac{\sigma^2}{6}(T-t)^2 \tag{A.30}$$

We have previously shown that the price of a zero coupon bond which follows the drift less stochastic differential equation, $dr(t) = \sigma dW(t)$, can be written as

$$P(t,T) = \exp\left(-r(t)(T-t) + \frac{\sigma^2}{6}(T-t)^3 \right) \tag{A.31}$$

Applying Itô's lemma yields the change in the bond's price

$$\begin{aligned}
dP(t,T) &= \frac{\delta P(t,T)}{\delta t}dt + \frac{\delta P(t,T)}{\delta r}dr + \frac{1}{2}\frac{\delta^2 P(t,T)}{\delta r^2}dr^2 \\
&= P(t,T)(r(t) - \frac{1}{2}\sigma^2(T-t)^2)dt + P(t,T)(-(T-t))dr + \frac{1}{2}P(t,T)(T-t)^2 dr^2 \\
&= P(t,T)(r(t)dt - \frac{1}{2}\sigma^2(T-t)^2 dt - (T-t)dr + \frac{1}{2}(T-t)^2 dr^2) \\
&= P(t,T)(r(t)dt + \frac{1}{2}\sigma^2(T-t)^2 dt - \sigma(T-t)dW(t) + \frac{1}{2}\sigma^2(T-t)^2(dW(t)^2)) \\
&= P(t,T)(r(t)dt - \frac{1}{2}\sigma^2(T-t)^2 dt - \sigma(T-t)dW(t) + \frac{1}{2}\sigma^2(T-t)^2 dt); \quad (dW_t^2 = dt) \\
&= P(t,T)(r(t)dt - \sigma(T-t)dW(t))
\end{aligned} \tag{A.32}$$

By taking the natural logarithm of the bond price, we observe the bond price distribution.

$$\begin{aligned}
\ln[P(t,T)] &= -r(t)(T-t) + \frac{\sigma^2}{6}(T-t)^3 \\
&= -(r(s) + \sigma(W(t) - W(s)))(T-t) + \frac{\sigma^2}{6}(T-t)^3
\end{aligned} \tag{A.33}$$

It therefore follows, that $\ln[P(t,T)] \sim N(\mathbb{E}[\ln[P(t,T)], \mathrm{Var}[P(t,T)])$ where,

$$\begin{aligned}
\mathbb{E}[\ln[P(t,T)]] &= \mathbb{E}[-r(s)(T-t) - \sigma(W(T) - W(t))(T-t) + \frac{\sigma^2}{6}(T-t)^3] \\
&= -r(s)(T-t) + \frac{\sigma^2}{6}(T-t)^3
\end{aligned} \tag{A.34}$$

and

$$\begin{aligned}
\mathrm{Var}[\ln[P(t,T)]] &= \mathrm{Var}[-r(s)(T-t) - \sigma(W(t) - W(s))(T-t) + \frac{\sigma^2}{6}(T-t)^3] \\
&= \sigma^2(T-t)^2 \mathrm{Var}[W(t) - W(s)] \\
&= \sigma^2(T-t)^2(t-s)
\end{aligned} \tag{A.35}$$

To conclude,

$$\ln[P(t,T)] \sim N(-r(s)(T-t) + \frac{\sigma^2}{6}(T-t)^3,\ \sigma^2(T-t)^2(t-s)) \tag{A.36}$$

A.4 Pricing bond with Brownian motion model

The formal solution to the Brownian motion process is,

$$r(s) = r(u) + \int_u^s m dv + \int_u^s \sigma dW(v) \tag{A.37}$$

To evaluate the price equation for zero coupon bond,

$$P(t,T) = E_t\Big(\exp\Big(-\int_t^T r(s)ds\Big)\Big) \tag{A.38}$$

we need to workout the integral,

$$\begin{aligned}\int_t^T r(s)ds &= \int_t^T \Big(r(t) + \int_t^s m d\upsilon + \int_t^s \sigma dW(\upsilon)\Big)ds \\ &= \int_t^T r(t)ds + m\int_t^T \Big(\int_t^s d\upsilon\Big)ds + \sigma\int_t^T \Big(\int_t^s dW(\upsilon)\Big)ds \\ &= r(t)(T-t) + m\int_t^T (s-t)ds + \sigma\int_t^T \Big(\int_t^s dW(\upsilon)\Big)ds \\ &= r(t)(T-t) + \frac{m}{2}(T-t)^2 + \sigma\int_t^T \Big(\int_t^s dW(\upsilon)\Big)ds\end{aligned} \tag{A.39}$$

Applying Fubini's Therorem to the last integral gives,

$$\sigma\int_t^T \Big(\int_t^s dW(\upsilon)\Big)ds = \sigma\int_t^T \Big(\int_\upsilon^T ds\Big)dW(\upsilon) = \sigma\int_t^T (T-\upsilon)dW(\upsilon) \tag{A.40}$$

We know that the stochastic process $X(t,T) = -\sigma\int_t^T (T-\upsilon)dW(\upsilon)$ is normally distributed. Therefore, the process, $Y(t,T) = e^{X(t,T)}$ is log-normally distributed with,

$$E_tY(t,T) = e^{E_tX(t,T)+\frac{1}{2}\sigma^2 X(t,T)} \tag{A.41}$$

We have

$$E_tX(t,T) = 0 \tag{A.42}$$

and

$$\sigma^s_{X(t,T)} = \Big(-\sigma\int_t^T (T-\upsilon)dW(\upsilon)\Big) = \sigma^2\int_t^T (T-\upsilon)^2 = -\frac{\sigma^2}{3}(T-\upsilon)^3\Big|_t^T = \frac{\sigma^2}{3}(T-t)^3 \tag{A.43}$$

Therefore,

$$P(t,T) = \exp\Big(-r(t)(T-t) - \frac{m}{2}(T-t)^2 + \frac{\sigma^2}{6}(T-t)^3\Big) \tag{A.44}$$