Homogeneous Covers of Finite Groups

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1 Opening Remarks for Grad Applications

This paper is a transcription of our final presentation given at the end of the 2018 *Summer Program for Undergraduate Research* (SPUR) hosted by Cornell's Mathematics department. The talk was mainly targeted towards undergraduates participating in separate projects of the program. I (Jon Miles) was responsible for writing and presenting section 5 on universal algebra and its relationship to our problem. It is only necessary to read section 3 in order to follow section 5.

2 Motivation

2.1 K-Theory

Algebraic K-theory is the "linear algebra" of general rings. It associates to any ring a sequence of abelian groups K_i. Classical K-functors (K_0 and K_1) are not too difficult to construct while the higher ones are more mysterious.

2.2 K_0

Theorem 2.1. *For any vector space V over the base field F, there exists a basis consisting of n vectors and every basis has cardinality n. And n is called the dimension. Also, any two vector spaces of dimension n are isomorphic.*

Definition 2.1 ($K_0(F)$)**.** $K_0(F)$ is defined as the abelian group of integers, as the Grothendieck group of the abelian semigroup N, the natural numbers, each representing an isomorphism class of finite dimensional vector space over F.

We wish to recover the behavior of a basis for modules over general rings. Unfortunately, they behave pathologically in general. For example, two bases may have different cardinality.

Example 2.2.1. Let V be an infinite dimensional k-vector space. Take R to be $End_{\mathrm{k}}(V)$. $R^2 \cong R$.

Definition 2.2 (Projective Modules)**.** A finitely-generated module P over R is **projective** if it is a direct summand of a free module. $P \bigoplus M \cong R^n$.

Definition 2.3 ($K_0(R)$)**.** We can form the abelian semigroup of isomorphism classes of finitely-generated projective modules under direct sum. $K_0(R)$ is defined as the group of formal differences of that semigroup (Grothendieck construction).

2.3 K_1

Definition 2.4 (The General Linear Group)**.** Given a ring with identity, the group $GL_n(R)$ is the group of n by n invertible matrices with entries in R. The group $E_n(R)$ is the subgroup generated by all elementary n by n matrices (of form $e_{ij}(a)$)

We can embed $GL_n(R)$ into $GL_{\text{n+1}}(R)$ by sending $A \mapsto \begin{pmatrix} \text{A} & 0 \\ 0 & 1 \end{pmatrix}$

We then have a series of embeddings of $GL_n(R)$ and $E_n(R)$.

$$\cdots \hookrightarrow \mathrm{GL}_{n-1}(R) \hookrightarrow \mathrm{GL}_n(R) \hookrightarrow \mathrm{GL}_{n+1}(R) \hookrightarrow \cdots$$

$$\cdots \hookrightarrow \mathrm{E}_{n-1}(R) \hookrightarrow \mathrm{E}_n(R) \hookrightarrow \mathrm{E}_{n+1}(R) \hookrightarrow \cdots$$

We can take the direct limit of the above diagrams. The resulting group are denoted by $GL(R)$ and $E(R)$, called the general linear group and the group of elementary matrices respectively.

$$GL(R) := \varinjlim GL_n(R)$$

$$E(R) := \varinjlim E_n(R)$$

Those matrices can be seen as the infinite square matrices with a square matrix at the upper left corner and 1 in the remaining diagonals.

$$\begin{pmatrix} A & 0 & \cdots & \cdots \\ 0 & 1 & 0 & \cdots \\ \vdots & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Theorem 2.2 (Whitehead)**.** *The commutator subgroup of* $\mathrm{GL}(R)$ *is* $\mathrm{E}(R)$*. In particular,* $\mathrm{GL}(R)/\,\mathrm{E}(R)$ *is the Abelianization of* $\mathrm{GL}(R)$*.*

Definition 2.5 ($K_1(R)$)**.** $K_1(R)$ of a ring with unit is defined to be $\mathrm{GL}_{\mathrm{ab}}(R)$.

One could interpret $K_1(R)$ as a generalization of the group of units of a ring. For example, $K_1(F)$ with F a field is isomorphic to F^* (Any matrix can be row reduced to a diagonal matrix)

Question (Persi Diaconis)**.** Would it be possible to develop a K-theory for finite groups where the vectors are generating sequences?

Idea: Consider action of subgroups of $Aut(G)$ on generating sequence. One could look at the analogue of elementary matrices in $Aut(G)$, defined by elementary operations on generating sequences.

Direction: In the category of R-modules, we have free objects that have a particular basis mapping property. Now we are searching for the correct category related to G so that the free objects satisfy the same basis property.

3 Homogeneous Covers

3.1 Homogeneous Groups

Definition 3.1 (F_n)**. The free group on** n **generators**, denoted F_n, can be defined in terms of its group presentation, $\langle\ x_1,\ x_2,\ ...,\ x_n\ \rangle$ (a group on n generators and no relations).

For instance, $F_1 = \langle a\rangle \cong \mathbb{Z}$, and $F_2 = \langle a, b\rangle \cong \pi_1$(one-point union of two circles)

Definition 3.2 ($\Gamma_n(G)$)**.** We denote the set of all **generating sequences of length** n **of** G by $\Gamma_n(G)$. Let $r(G)$ denote the smallest integer such that $\Gamma_{r(G)} \neq \emptyset$.

Given a finite group G, we can define a set of of surjective homomorphisms $\pi_s : F_n \to G$ for each $s \in \Gamma_n$ by sending the set of generators $\{x_1, ..., x_n\}$ to the generating seqence s. For each map π_s, we'll denote $\ker(\pi_s) = \mathrm{K}_s$ and $\bigcap\limits_{s\in\Gamma_n} \mathrm{K}_s = \mathrm{K}$.

Definition 3.3 (nth homogeneous cover)**.** Given a finite group G, **the** n**th homogeneous cover of** G is equal to the quotient F_n/K for any $n \geq r(G)$ (where $r(G)$ is the smallest size of a generating sequence of G). We denote the homogeneous cover by $H(n, G)$.

Homogeneous covers are sub-direct products of $|\Gamma_n|$ copies of G, meaning that they inject into $G^{|\Gamma_n|}$ and surject onto every component subgroup isomorphic to G.

$$\begin{tikzcd} F_n \arrow[d, "\pi_n"'] \arrow[dr, "\prod_{s\in\Gamma_n} \pi_s"] & \\ H(n,G) \arrow[r, dashed] & \prod_{s\in\Gamma_n} F_n/K_s \cong \prod G \end{tikzcd}$$

We note that the kernels of both maps out of F_n are the intersection of all kernels K_s, inducing an injective map $H(n,G) \hookrightarrow \prod G$.

Since $H(n,G)$ is a subdirect product of G, we know some nice things about it in relation to G. To list a few, we have the following:

1. G is Abelian $\iff$ $H(n,G)$ is Abelian
2. G is solvable $\iff$ $H(n,G)$ is solvable
3. G is nilpotent $\iff$ $H(n,G)$ is nilpotent
4. $\exp(G) = \exp(H(n,G))$

Explicitly, for some word $w \in F_n$, we can write an element $h \in H(n,G)$ as

$$h = (w(g_1^{s_1}, ..., g_n^{s_1}), ..., w(g_1^{s_{|\Gamma_n|}}, ..., g_n^{s_{|\Gamma_n|}})),$$

where each $w(g_1^s, ..., g_n^s)$ is the image of the word under the projection $F_n \to G$ given by the generating sequence s.

Definition 3.4 (n-homogeneous)**.** A group G is **homogeneous of rank** n if $G \cong H(n,G)$. As a consequence, any nth homogeneous cover of a group is a homogeneous group of rank n. In fact, $H(n, H(n,G)) = H(n,G)$.

We can give an equivalent characterization of homogeneous groups in terms of the following mapping property.

3.2 Basis Mapping Property

Lemma 3.1 (Collins)**.** *A group G is homogeneous of rank n if and only if given any two generating sequences s and t of length n, there is a unique $\alpha \in Aut(G)$ such that $\alpha(s) = t$.*

A group being homogeneous is equivalent to having all K_s equal, since $G \cong F_n/K_s$ for each $s \in \Gamma_n$ by the first isomorphism theorem, and this gives us an automorphism between each generating sequence:

$$\begin{tikzcd} F_n \arrow[r, "\pi_t"] \arrow[d, "\pi_s"] & G \arrow[dl, "\exists!\alpha"] \\ G & \end{tikzcd}$$

Intuitively, a group is n-homogeneous if all of its length n generating sequences are symmetric via a unique automorphism of G, and the homogeneous covers $H(n,G)$ are the symmetrizations of G.

Corollary 3.1.1 (Collins)**.** *The minimum length of a generating sequence of H(n,G) is n.*

Proof. Assume for the sake of contradiction that we have a generating sequence $s = (g_1, ..., g_{n-1})$ for $H(n,G)$. Then, for any non-trivial $h \in H(n,G)$ both $s' = (g_1, ..., g_{n-1}, h) \in \Gamma_n$ and $s" = (g_1, ..., g_{n-1}, 1) \in \Gamma_n$, so $\exists!\alpha \in Aut(G)$ such that $\alpha(s') = s''$, but this automorphism would then be taking h to 1, contradicting our choice of generating sequence. □

In particular, this shows that the order of the covers is strictly increasing.

3.3 Mappings

We can define a natural surjective homomorphism $\varphi : H(m,G) \to H(n,G)$ for any n and m such that $r(G) \leq n \leq m$. The map is constructed from the induced mappings of F_n into F_m ($x_i \mapsto x_i,\ 1 \leq i \leq n$ where the x_i are generators) and F_m onto F_n ($x_i \mapsto 1,\ i > n$).

$$\begin{tikzcd}
F_n * \langle x_{n+1}, ..., x_m \rangle = F_m \arrow[d, "\pi_m"'] \arrow[drr, "\pi_n'"] & & F_n \arrow[ll, "\iota"'] \arrow[d, "\pi_n"] \\
H(m,G) \arrow[rr, dashed] & & H(n,G)
\end{tikzcd}$$

Defining the following notation of kernels, $\ker(\pi_m) =: K_m,\ \ker(\pi_n) =: K_n$, we see that $\ker(\pi_n') = K_n * \langle x_{n+1}, ..., x_m \rangle =: K_n'$, where $\pi_n'(x_i) = \begin{cases} x_i \text{ for } x_i \in Fn, \\ 1 \text{ otherwise} \end{cases}$

Definition 3.5 (Notation)**.** We summarize the above definitions for brevity.

- $K_n = \ker(\pi_n)$ the kernel of the projection of F_n onto $H(n,G)$ for any $n \geq r(G)$.
- $K_n' = \ker(\pi_n) * \langle x_{n+1}, ..., x_m \rangle = \ker(\pi_n')$ the kernel of the projection F_m onto $H(m,G)$ for fixed $m \geq n$.

We see that K_n' is the intersection of kernels corresponding to generating sequences $s = (g_1, ..., g_n) \in \Gamma_n$ thought of in Γ_m, as $s' = (g_1, ..., g_n, 1, ..., 1)$, so we have that $K_m = \bigcap_{s \in \Gamma_m(G)} K_s \leq \bigcap_{s \in \Gamma_n(G)} K_{s'} = K_n'$. This gives us the desired induced surjection $H(m,G) \to H(n,G)$, since π_n factors through the quotient map.

4 Calculations

4.1 Calculations of $H(n,G)$ for G Abelian

We want to compute $H(n,G)$ for G finite abelian. By the classification theorem of finite abelian groups, assume G $= \mathbb{Z}_{k_1} \times \cdots \times \mathbb{Z}_{k_m}$ is the decomposition by invariant factors, i.e. there are positive integers $k_1 \leq \cdots \leq k_m,\ k_i | k_j$ for $i < j$.

Lemma 4.1. *With notation as above, $r(G) = m$.*

Claim. $H(n,G) = (\mathbb{Z}_{k_m})^n$ for all $n \geq m$.

Proof of claim. $G = \mathbb{Z}_{k_1} \times \cdots \times \mathbb{Z}_{k_m}$
(By definition) $\phi : F_n \to H(n,G)$ with kernel $K = \bigcap_{s \in \Gamma_n} \mathrm{K}_s$.
Let C be the commutator subgroup of F_n.
G abelian $\Rightarrow H(n,G)$ abelian $\Rightarrow C \leq K$.
$\exp(G) = \exp(H(n,G)) = k_m \Rightarrow x_i^{k_m} \in F_n$ should be in the kernel.
Hence, K should contain $K' =$ the normal closure of $\langle C, x_1^{k_m}, ..., x_n^{k_m} \rangle$ since it's normal.
Therefore, $H(n,G) = F_n/K$ is a quotient of $F_n/K' \cong \mathbb{Z}_{k_m}^n$.
In fact, K and K' are equal, completing the proof. □

4.2 $H(n,G)$ for G simple

Claim. Let S be a finite nonabelian simple group. Then $H(n,S) = S^{h_n(S)}$ for $n \geq r(S)$, where $h_n(S) = |\Gamma_n(S)|/|Aut(S)|$.

Lemma 4.2. *Let $s_1, ..., s_k$ be generating sequences of length n for a finite simple group S; then $s_1 \times \cdots \times s_k$ generates S^k if and only if the s_i are pairwise inequivalent under the action of Aut(S).*

The idea is that if two of the generating sequences are in the same orbit, then there is loss of information when looking at the subgroup they generate in the product.

Proof of claim. We always have the embedding $H(n,S) \hookrightarrow \prod_{s \in \Gamma_n/Aut(G)} F_n/K_s \cong \prod_{1 \leq i \leq h_n(s)} S$ as a subdirect product. By the lemma above, this map is also a surjection, and thus an isomorphism. □

4.3 Other Groups

Let T_p be the group of 3×3 upper triangular matrices with entries in F_p ($p \geq 3$) and diagonal entries 1's.

Theorem 4.3 (Collins)**.** $H(n, T_p) = \langle x_1, ..., x_n | x_i^p; [x_i, x_j]^p; [x_i, x_j] \text{ central } \rangle$.

Example 4.3.1 (Homogeneous Covers of A_4)**.** In practice, these objects get large very quickly, as demonstrated by the order of the covers for A_4:
$|H(2, A_4)| = 2304$
$|H(3, A_4)| = 121597189939003392$
$|H(4, A_4)| = 1431146122470491306962470975601763997820261956469251356078501023047022 01856$

5 Varieties

We find inspiration from universal algebra in order to get closer to finding sufficient conditions for canonical embeddings $H(n, G) \hookrightarrow H(m, G)$.

5.1 Definitions

Definition 5.1 (Verbal Subgroup)**.** Let F_∞ be the free group on the set $\{x_i\}_{i \in \mathbb{N}}$, and let $\mathfrak{w}$ be a collection of words in F_∞. If G is a group, define the **verbal subgroup of** G to be $\mathfrak{w}(G) = \langle \{\alpha(w) | w \in \mathfrak{w}, \alpha \in \mathrm{Hom}(F_\infty, G)\} \rangle$

In other words, $\mathfrak{w}(G)$ is the subgroup of G generated by the images of words in $\mathfrak{w}$ by replacing the variables with elements of G.

Definition 5.2. We say $w \in F_\infty$ is a **law of** G if $\forall \alpha \in \mathrm{Hom}(F_\infty, G)$, we have $\alpha(w) = 1 \in G$. Let $\mathfrak{w}$ be the set of all words that are laws of G. **The variety generated by a group G** is defined to be the class of groups $\mathcal{V}(G)$ such that $A \in \mathcal{V}(G) \iff \mathfrak{w}(A) = 1$.

That is, A is in the Variety generated by G if and only if every law of G is a law of A.
We can actually define a full subcategory of Groups with the groups in $\mathcal{V}(G)$ as objects. The free objects in this category are called **relatively free groups** and can be realized as a quotient $F/\mathfrak{w}(F)$ where F is a free group.

5.2 Examples of Varieties

- The variety of abelian groups is generated by the law $x_1 x_2 x_1^{-1} x_2^{-1}$
- The variety of groups with exponent dividing $m \in \mathbb{N}$ is generated by the law x_1^m

Theorem 5.1 (B.H. Neumann)**.** *Every word w is equivalent (can be obtained by relabelling variables) to a pair of words, one of the form x^m, $m \neq 0$, the other a commutator word.*

So, up to equivalence, every law is of the same flavor as those above.

5.3 Relationship with Homogeneous Covers

Lemma 5.2. $\mathcal{V}(H(n, G)) = \mathcal{V}(G)$.

Equivalently, the laws of $H(n, G)$ are precisely the laws of G, so in particular the homogeneous covers of G belong to the variety defined by G.

Proof. Suppose w is a law of G. Then since $H(n, G)$ is a subdirect product, any mapping $F \to H(n, G) \subseteq \prod G$ projects to a map $F \to G$ in each coordinate, so w is killed in each coordinate and is thus a law for $H(n, G)$. The converse follows similarly. $\square$

Lemma 5.3 (MacDonald and Powell)**.** *If G is any finite group, then the variety defined by the laws of G can be defined by a finite subset of F_∞.*

So there is reason to believe that the laws of G begin to grow "predictably" after some sufficiently large cover.
Identify F_{n-r} contained in F_n by the subgroup generated by $\{x_1, ..., x_{n-r}\}$ and recall that K_n is the kernel of the sequence $1 \to K_n \to F_n \to H(n, G) \to 1$.

Lemma 5.4. *Let G be a finite group and let n be greater than $r(G)$. Then the intersection $K_n \bigcap F_{n-r}$ is equal to the laws of G involving $n - r(G)$ or fewer variables.*

Remark. This lemma gives a way to relate the kernels defining each homogeneous cover to the laws of G, which may be useful in defining embeddings of covers of sufficiently large rank. We denote the laws involving i or fewer variables by $\mathfrak{w}_i = \bigcap_{\varphi \in \mathrm{Hom}(F_i, G)} \ker \varphi$. Contrast this with our previous definition $K_i = \bigcap_{\varphi \in \mathrm{Epi}(F_i, G)} \ker \varphi$. Another paraphrase of our stability criterion is that, since G is finite, the intersection of kernels of *surjections* $F_n \twoheadrightarrow G$ will begin to look like the intersection of kernels of *arbitrary* maps $F_n \to G$ for $n >> 0$.

Proof. $\mathfrak{w}_{n-r} \subset K_n \bigcap F_{n-r}$: Suppose $w \in F_{n-r}$ is a law of G. Then for any generating sequence $s = (g_1, ..., g_{n-r}, ..., g_n) \in \Gamma_n(G)$, $w(g_1, ..., g_{n-r}) = 1$, implying $w \in K_s$. Since this is true for all generating sequences, we get $w \in K_n = \bigcap_{s \in \Gamma_n(G)} K_s$.
$K_n \bigcap F_{n-r} \subset \mathfrak{w}_{n-r}$: Fix a generating sequence $(h_1, ..., h_r) \in \Gamma_r(G)$, and suppose $w \in K_n \bigcap F_{n-r}$ (That is, w is a word in $\{x_1, ..., x_{n-r}\}$ and is killed by every surjection $F_n \to G$). Then *any* sequence $(g_1, ..., g_{n-r})$ of $n - r$ elements of G can be extended to a generating sequence $(g_1, ..., g_{n-r}, h_1, ..., h_r) \in \Gamma_n(G)$. The image of w via the projection onto G corresponding to that sequence is $w(g_1, ..., g_{n-r})$, which is 1 since w is in the kernel of all projections. Thus, w is a law for G. □

5.4 Embeddings

Fix a finite group G and integers $m > n \geq r(G)$.

Lemma 5.5. $K_m \bigcap F_n = K_n \implies$ *there is a canonical embedding*

$$H(n, G) \to H(m, G)$$

Remark: Recall that $K_m \leq K_n' := K_n \ * \langle \{x_{n+1}, ..., x_m\} \rangle$ since K_n' is the intersection of kernels corresponding to generating sequences $(g_1, ..., g_n, 1, ..., 1)$, which are contained in Γ_m. This implies that $K_m \bigcap F_n \leq K_n' \bigcap F_n = K_n$.
The nontrivial condition is that $K_m \bigcap F_n \geq K_n$, which in a certain sense is to say that the kernels grow in a "predictable" way. Putting the three earlier lemmas together suggests that it is reasonable to believe that this criterion holds for infinitely many pairs (n, m).

Proof. The idea of the proof is to induce an injection via the universal mapping property of the quotient $H(n, G) \cong F_n / K_n$.

$$\begin{tikzcd}
1 \arrow[r] & K_m \arrow[r] & F_m \arrow[r, "\pi_m"] & H(m,G) \arrow[r] & 1 \\
1 \arrow[r] & K_n \arrow[r] & F_n \arrow[r, "\pi_n"] \arrow[u, "\iota_{n,m}"] & H(n,G) \arrow[r] \arrow[u, dashed] & 1
\end{tikzcd}$$

$\ker \pi_m \circ \iota_{n,m} = K_m \bigcap F_n = K_n$ (by assumption), so since K_n is contained in the kernel of the composition, there is an induced map $H(n, G) \dashrightarrow H(m, G)$. Furthermore, this map has trivial kernel since $\ker \pi_n = K_n = \ker \pi_m \circ \iota_{n,m}$, and is thus an injection. □

6 Next Steps, References, and Acknowledgements

6.1 Next Steps

- Prove exactly when there are natural embeddings $H(n, G) \to H(m, G)$.
  Our method essentially tries to find sections of the exact sequences $H(m, G) \to H(n, G) \to 1$; perhaps group cohomology may shed some light on when these exist.

- Find properties of the profinite group $H(G) = \varprojlim H(n, G)$.
  These may be interesting objects in their own right, capturing some sort of invariants of finite groups that can be studied systematically.

- Is the class of field extensions with Galois group $H(G)$ for some finite group G special?
  This question is an extension of the second bullet point, and is more for fun than rigorously motivated.

- Compute covers of Extra-Special p-Groups.
  These groups seem to be a natural next-choice in expanding our corpus of computations of homogeneous covers.

- Find the right category in analogy to R-Modules (i.e. in what category are the $H(n, G)$ free/projective objects?)
  This goes hand-in-hand with the first bullet point; we suspect that whatever category this may be is somehow related to the variety generated by G, $\mathcal{V}(G)$.

6.2 References

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