This lesson plan was modified from Yifan Gu's lesson plan.

1 Learning Objectives

(1) Solutions to hard midterm problem

(2) By the end of the section, students will understand each component in the definition of the Riemann integral: tagged partition, mesh, integrability, etc.

2 Notes

5.2: The Definite Integral

Definition 2.1 (Partition). Let $[a, b]$ be an interval. A **partition** of $[a, b]$ is a collection of points $x_0, \ldots, x_n$ satisfying:

$$a = x_0 < x_1 \cdots < x_n = b$$

The **mesh** of a partition P is the largest subinterval in P:

$$\text{mesh}(P) = \max_{1 \leq i \leq n} x_i - x_{i-1}$$

Definition 2.2 (Refinement of a partition). Let P, Q be two partitions of an interval $[a, b]$. We say Q is a **refinement** of P if $P \subset Q$. We also say P is coarser than Q.

Definition 2.3 (Sample points and tagged partition). Given a partition $P : a = x_0 < \cdots < x_n = b$, a set of **sample points** is a collection of points $c_1, \ldots, c_n$ such that

$$c_i \in [x_{i-1}, x_i]$$

for each i.

A partition along with a set of sample points is called a **tagged partition**.

Definition 2.4 (Riemann sum). Given a function $f : [a, b] \to \mathbb{R}$ and a tagged partition (P, C) of $[a, b]$, the **Riemann sum** of f with tagged partition (P, C) is

$$R(f, P, C) = \sum_{i=1}^{n} f(c_i)(x_i - x_{i-1})$$

Definition 2.5 (Integrable function/Definite integral). We say a function $f : [a, b] \to \mathbb{R}$ is **integrable** if

$$L := \lim_{\substack{\text{mesh}(P) \to 0 \\ (P,C) \text{ tagged partition}}} R(f, P, C)$$

exists. In this case, we call this limit the **definite integral** of f over $[a, b]$, write

$$L = \int_a^b f(x) dx$$

Theorem 2.1 (Continuous functions are Riemann integrable). Functions which are continuous except at finitely many jump discontinuities on and interval $[a, b]$ are integrable over $[a, b]$.

Theorem 2.2 (Properties of definite integrals). (1) Linearity

(2) reversing limits of integration

(3) integral over point

(4) breaking into two integrals via subinterval

(5) monotonicity

5.3: The Indefinite Integral

Definition 2.6 (Antiderivative). A function F is the **antiderivative** of f on an open interval (a, b) if $F'(x) = f(x)$ for all $x \in (a, b)$.

Remark 2.1. Antiderivatives are unique up to a constant.

Definition 2.7. The **indefinite integral** is a just general antiderivative:

$$\int f(x)dx = F(x) + C$$

5.4: Fundamental Theorem of Calculus

Theorem 2.3 (The Fundamental Theorem of Calculus). Assume $a < b$ and f is continuous on $[a, b]$. If F is an antiderivative of f over (a, b), then

$$\int_a^b f(x)dx = F(b) - F(a)$$

Remark 2.2. The FTC connects antiderivatives and definite integration.

3 Exercises

Exercise 3.1 (Midterm Question). A person with height H walks away from a 2-m lamppost such that the tip of their shadow moves twice as fast as they do. What is their height?

Exercise 3.2 (Coarsest and finest partitions). What is the coarsest partition of $[0, 1]$? Is there a finest partition?

Exercise 3.3 (Mesh and refinements). If Q is a refinement of P, how are their meshes related?

Exercise 3.4 (Integrating a constant). Show that $\int_a^b cdx = c(b - a)$

Proof. Show that any Riemann sum is $c(b - a)$. $\hspace{8cm}$ $\square$

Exercise 3.5 (Power Rule for integrals). Prove the power rule for integrals:

$$\int x^n dx = \frac{x^{n+1}}{n + 1} + C$$

Exercise 3.6 (FTC). Evaluate the integral using FTC:

$$\int_1^8 x^{4/3}dx$$