1 Learning Objectives

(1) Midterm review: implicit differentiation, related rates, linear approximations, EVT, MVT, First and Second Derivative Test.

2 Notes

3.8 Implicit Differentiation

Implicit differentiation is used to compute $\frac{dy}{dx}$ when x and y are related by an equation.

(1) Take the derivative of both sides of the equation with respect to x.

(2) Solve for $\frac{dy}{dx}$

Remark 1. Remember to apply Chain Rule when differentiating expressions involving y. For example

$$\frac{d}{dx}y^2 = 2y\frac{dy}{dx}$$

3.9: Related Rates

Related-rates step by step:

(1) Identify variables and the rates that are related

(2) Find an equation relating the variables (usually we draw a picture here)

(3) Differentiate both sides of the equation with respect to the desired variable

(4) Solve for quantity of interest

4.1: Linear Approximation

The tangent line of a graph of a function f at the point $(a, f(a))$ can be used to approximate f near $x = a$. In fact, we call the tangent line the **linearization** of f at $x = a$:

$$L(x) = f(a) + f'(a)(x - a)$$

For points near $x = a$, we have

$$f(x) \approx L(x)$$

Re-arranging this approximation gives

$$\underbrace{f(x) - f(a)}_{\Delta f} \approx f'(a)\underbrace{(x - a)}_{\Delta x}$$

Note that $\Delta f = f(a + \Delta x) - f(a)$.

The differentials dy and dx are related by

$$dy = f'(a)dx$$

The differential form of Linear Approximation is

$$\Delta y \approx dy = f'(a)dx$$

4.2: Extreme Values

Theorem 1 (Extreme Value Theorem)**.** If f is continuous on a closed interval $[a, b]$, then f has both a min and a max on $[a, b]$.

Definition 1 (Critical points)**.** A critical point of a function f is a point c such that $f'(c) = 0$ or $f'(c)$ does not exist.

Theorem 2 (Local extrema are critical points)**.** If c is a local extrema of f, then c is a critical point of f.

Remark 2 (Finding extrema)**.** These two theorems allow us to find extreme values of a continuous function f over a closed interval $[a, b]$. In particular, each extrema will either be at an endpoint or in (a, b). In the latter case, the extrema is local and thus a critical point. Thus all extrema will be at endpoints or critical points

Theorem 3 (Rolle's Theorem)**.** If f is continuous on $[a, b]$ and differentiable on (a, b) with $f(a) = f(b)$, then there exists $c \in (a, b)$ such that $f'(c) = 0$.

4.3: Mean Value Theorem and Monotonicity

Theorem 4 (Mean Value Theorem)**.** If f is continuous on $[a, b]$ and differentiable on (a, b), then there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Proof. Define $g(x) = f(x) - \frac{f(b)-f(a)}{b-a}(x - a)$. Then apply Rolle's Theorem. □

Corollary 1. If $f'(x) = 0$ for all $x \in (a, b)$, then f is constant on (a, b).

Remark 3 (First Derivative Test)**.** From MVT, we can get

$$f'(x) > 0 \text{ for } x \in (a, b) \implies f \text{ is increasing on } (a, b)$$

and

$$f'(x) < 0 \text{ for } x \in (a, b) \implies f \text{ is decreasing on } (a, b)$$

Moreover, if $x = c$ is a critical point of f such that

$$f'(x) > 0 \text{ to the left of } x = c$$

and

$$f'(x) < 0 \text{ to the right of } x = c$$

then we have that $x = c$ is a local maximum.

Similarly, if $x = c$ is a critical point of f such that

$$f'(x) < 0 \text{ to the left of } x = c$$

and

$$f'(x) > 0 \text{ to the right of } x = c$$

then we have that $x = c$ is a local minimum.

4.4: The Second Derivative and Concavity

Definition 2 (Concavity)**.** A differentiable function f is **concave up** if f' is increasing and **concave down** if f' is decreasing.

Definition 3 (Inflection point)**.** A **point of inflection** is a point $(c, f(c))$ where the concavity of f changes.

Remark 4. If $(c, f(c))$ is an inflection point, then $f''(c) = 0$.

Remark 5 (Second Derivative Test)**.** A function f is concave up if and only if f' is increasing if and only if $f'' > 0$ and concave down if and only if f' is decreasing if and only if $f'' < 0$.

Using this fact, we suppose we have a critical point $x = c$ of f. Then if $f''(c) > 0$ then f' is increasing at c f' goes from negative to positive at $x = c$, so by First Derivative Test, $x = c$ is a local minimum.

Similarly, if $x = c$ is a critical point of f with $f''(c) < 0$, then $x = c$ is a local maximum.

3 Exercises

Exercise 1 (3.8.36)**.** Find an equation of the tangent line at the given point:

$$\sin(x - y) = x\cos(y + \tfrac{\pi}{4}), \quad (\tfrac{\pi}{4}, \tfrac{\pi}{4})$$

Solution. Differentiation of both sides of the equation w.r.t to x yields:

$$\cos(x - y) \cdot (1 - \tfrac{dy}{dx}) = \cos(y + \tfrac{\pi}{4}) - x\sin(y + \tfrac{\pi}{4}) \cdot \tfrac{dy}{dx}$$

Rearrangement yields:

$$\cos(x - y) - \cos(y + \tfrac{\pi}{4}) = \tfrac{dy}{dx} \cdot \big(\cos(x - y) - x\sin(y + \tfrac{\pi}{4})\big)$$

Evaluation at the point $(x, y) = (\frac{\pi}{4}, \frac{\pi}{4})$ gives

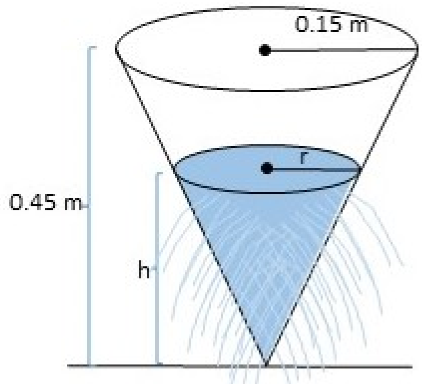
$$1 = \tfrac{dy}{dx} \cdot (1 - \tfrac{\pi}{4})$$

The tangent line thus goes through the point $(\frac{\pi}{4}, \frac{\pi}{4})$ with slope $\frac{dy}{dx} = \frac{1}{1-\frac{\pi}{4}}$:

$$y - \tfrac{\pi}{4} = \frac{1}{1 - \frac{\pi}{4}}(x - \tfrac{\pi}{4})$$

□

Exercise 2 (Challenge: Holey Cone)**.** Consider a conical watering pail with base radius 0.15m and height 0.45m. Suppose the pail is filled with water up to a height h and radius r. Now imagine that there are a bunch of holes in the cone, so that water leaks out at a rate of kA m^3/min, where $k = 0.25$ m/min and $A = \pi r\sqrt{h^2 + r^2}$ is the surface area of the part of the cone in contact with the water. What is the rate of change of the water level when the water level is 0.3 m?



Solution. The figure is Exercise 3.9.39 in the textbook.

(1) The variables are volume V, surface area of water and cone interface $A = \pi r\sqrt{h^2 + r^2}$, radius of water r, and height of water h. We are also told that

$$\frac{dV}{dt} = -kA$$

Why is it negative?

(2) Using similar triangles, we relate r and h:

$$\frac{r}{h} = \frac{1}{3}$$

Using geometry, we get

$$V = \frac{1}{3}\pi r^2 h$$

(3) Differentiating the two equations gives

$$\frac{dr}{dt} = \frac{1}{3}\frac{dh}{dt}$$

and

$$\frac{dV}{dt} = \frac{1}{3}\pi\left(2hr\frac{dr}{dt} + r^2\frac{dh}{dt}\right)$$

(4) Plugging in quantities gives us

$$\begin{aligned} -k\pi r\sqrt{h^2 + r^2} &= \frac{1}{3}\pi(2hr \cdot \frac{1}{3}\frac{dh}{dt} + r^2\frac{dh}{dt}) \\ &= \left(\frac{2\pi hr}{9} + \frac{r^2\pi}{3}\right)\frac{dh}{dt} \end{aligned}$$

(5) We are interested in $\frac{dh}{dt}$ when $h = 0.3$ (and thus $r = 0.1$). Plugging in values gives us

$$\frac{dh}{dt} = \frac{-k\pi r\sqrt{h^2 + r^2}}{\left(\frac{2\pi hr}{9} + \frac{r^2\pi}{3}\right)} \approx -0.79$$

What are the units?

$\square$

Exercise 3 (4.1.26)**.** Find the linearization at $x = a$ and then use it to approximate $f(b)$:

$$y = \frac{\sin x}{x}, \quad a = \frac{\pi}{4}, \quad b = \frac{\pi}{5}$$

Solution. We have by quotient rule:

$$\frac{dy}{dx} = \frac{x\cos x - \sin x}{x^2}$$

The linearization is at $x = a$ is

$$\begin{aligned} L(x) &= \frac{\sin(\frac{\pi}{4})}{\frac{\pi}{4}} + \frac{\frac{\pi}{4}\cos(\frac{\pi}{4}) - \sin(\frac{\pi}{4})}{(\frac{\pi}{4})^2}(x - \tfrac{\pi}{4}) \\ &= \frac{4}{\pi}\frac{\sqrt{2}}{2} + \frac{16}{\pi^2}\frac{\sqrt{2}}{2}\left(\frac{\pi}{4} - 1\right)(x - \tfrac{\pi}{4}) \\ &= \frac{2\sqrt{2}}{\pi} + \frac{8\sqrt{2}}{\pi^2}\left(\frac{\pi}{4} - 1\right)(x - \tfrac{\pi}{4}) \end{aligned}$$

Then we have

$$f(b) \approx L(b) \approx 0.94$$

$\square$

Exercise 4 (4.1.30)**.** Estimate Δy using differentials:

$$y = \frac{3 - \sqrt{x}}{\sqrt{x+3}}, \quad a = 1, \quad dx = -0.1$$

Solution. We have

$$\Delta y \approx dy = \left.\frac{dy}{dx}\right|_{x=a} dx$$

where

$$\frac{dy}{dx} = \frac{-\frac{1}{2\sqrt{x}}\sqrt{x+3} - (3 - \sqrt{x})\frac{1}{2\sqrt{x+3}}}{x+3}$$

so

$$\left.\frac{dy}{dx}\right|_{x=a} = \frac{-1 - \frac{1}{2}}{4} = -\frac{3}{8}$$

Thus

$$\Delta y \approx \frac{3}{80}$$

$\square$

Exercise 5 (4.2.61)**.** Verify Rolle's Theorem:

$$f(x) = x + \frac{1}{x}, \quad [\tfrac{1}{2}, 2]$$

Solution. To verify Rolle's Theorem, we need to check:

(1) f is continuous on $[\frac{1}{2}, 2]$

(2) f is differentiable on $(\frac{1}{2}, 2)$

(3) $f(\frac{1}{2}) = f(2)$

(4) There exists $c \in (\frac{1}{2}, 2)$ such that $f'(c) = 0$

Showing (1)-(3) are easy. To see (4), differentiate f to get:

$$f'(x) = 1 - \frac{1}{x^2}$$

Note that $f'(1) = 0$ and $1 \in (\frac{1}{2}, 1)$. □

Exercise 6 (4.3.53)**.** Find conditions on a and b that ensure $f(x) = x^3 + ax + b$ is increasing on $(-\infty, +\infty)$.

Solution. We need to find conditions that ensure $f'(x) > 0$. We have

$$f'(x) = 3x^2 + a$$

Since $x^2 \geq 0$, $f'(x) > 0$ for all x if and only if $a > 0$. □

Exercise 7 (4.3.58)**.** Which values of c satisfy the conclusion of MVT on the interval $[a, b]$ if f is a linear function.

Solution. If f is linear, then $f(x) = mx$. Note that

$$\frac{f(b) - f(a)}{b - a} = m$$

and $f'(x) = m$ so any c works. □

Exercise 8 (4.4.52)**.** Find the intervals on which f is concave up or concave down, the points of inflection, and the local extrema:

$$f(x) = \frac{x}{x^6 + 5}$$

Solution. First, we have

$$f'(x) = \frac{x^6 + 5 - x(6x^5)}{(x^6 + 5)^2} = \frac{-5x^6 + 5}{(x^6 + 5)^2} = -5\frac{x^6 - 1}{(x^6 + 5)^2}$$

and

$$\begin{aligned} f''(x) &= -5\frac{6x^5(x^6 + 5)^2 - 2(x^6 + 5) \cdot 6x^5 \cdot (x^6 - 1)}{(x^6 + 5)^4} \\ &= -5\frac{6x^5(x^6 + 5) - 12x^5(x^6 - 1)}{(x^6 + 5)^3} \\ &= \frac{-30x^5}{(x^6 + 5)^3}(x^6 + 5 - 2x^6 + 2) \\ &= \frac{-30x^5}{(x^6 + 5)^3}(-x^6 + 7) \\ &= \frac{30x^5(x^6 - 7)}{(x^6 + 5)^3} \end{aligned}$$

Points of inflection occur at zeros of f'' (but not all such points are inflection points). Between any two roots of f'', the sign is unchanging. The (real) roots of f'' are $x = 0, \pm 7^{\frac{1}{6}}$.

Then we have

$$\begin{aligned} f''(1) &< 0 \\ f''(-1) &> 0 \\ f''(7) &> 0 \\ f''(-7) &< 0 \end{aligned}$$

So f is concave up on

$$(-7^{\frac{1}{6}}, 0) \cup (7^{\frac{1}{6}}, \infty)$$

and concave down on

$$(-\infty, -7^{\frac{1}{6}}) \cup (0, 7^{\frac{1}{6}})$$

Moreover, $x = 0, \pm 7^{\frac{1}{6}}$ are points of inflection.

The critical points are $x = \pm 1$, so by second derivative test, $x = -1$ is a local minimum and $x = 1$ is a local maximum. $\square$