1 Learning Objectives

(1) By the end of the section, students will know the steps for implicit differentiation and related rates problems through practice.

2 Notes

3.8: Implicit Differentiation

Implicit differentiation is used to compute $\frac{dy}{dx}$ when x and y are related by an equation.

(1) Take the derivative of both sides of the equation with respect to x.

(2) Solve for $\frac{dy}{dx}$

Remark 2.1. Remember to apply Chain Rule when differentiating expressions involving y. For example

$$\frac{d}{dx}y^2 = 2y\frac{dy}{dx}$$

3.9: Related Rates

Related-rates step by step:

(1) Identify variables and the rates that are related

(2) Find an equation relating the variables (usually we draw a picture here)

(3) Differentiate both sides of the equation with respect to the desired variable

(4) Solve for quantity of interest

3 Exercises

Excercise 3.1 (Tangent lines and Implicit Differentiation)**.** Find an equation of the tangent line at the given point:

$$\sin(2x - y) = \frac{x^2}{y}, \quad (0, \pi)$$

Solution. Recall that a line requires a point and a slope. The point is specified; the slope comes from the derivative at the specified x-coordinate. Differentiation with respect to x results in

$$\cos(2x - y) \cdot \left(2 - \frac{dy}{dx}\right) = \frac{2xy - x^2\frac{dy}{dx}}{y^2}$$

Rearranging results in

$$\frac{dy}{dx} = \frac{2y^2\cos(2x - y) - 2xy}{y^2\cos(2x - y) - x^2}$$

Thus

$$\left.\frac{dy}{dx}\right|_{(0,\pi)} = 2$$

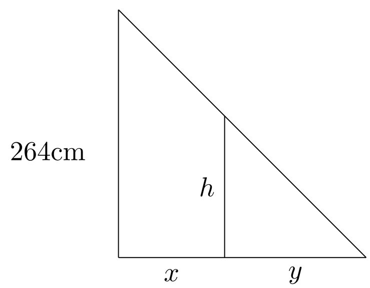
So the desired tangent line is

$$y - \pi = 2x$$

I encourage you to go on Desmos and plot these curves. □

Excercise 3.2 (Growing Shadows). As Claudia walks away from a 264-cm lamppost, the tip of her shadow moves twice as fast as she does. What is Claudia's height?

Solution. Consider the sketch:



(1) In the figure, x is the distance from Claudia to the pole, y is the length of Claudia's shadow. Claudia's height h is a constant.

The related rate is

$$\frac{d}{dt}(x+y) = 2\frac{dx}{dx}$$

Rearranging gives

$$\frac{dy}{dt} = \frac{dx}{dt}$$

(2) An equation relating the variables is from similar triangles:

$$\frac{x+y}{264} = \frac{y}{h}$$

(3) Differentiating both sides gives

$$\frac{\frac{dx}{dt}+\frac{dy}{dt}}{264} = \frac{\frac{dy}{dt}}{h}$$

(4) Rearranging and using the related rate gives

$$2h\frac{dx}{dt} = 264\frac{dx}{dt}$$

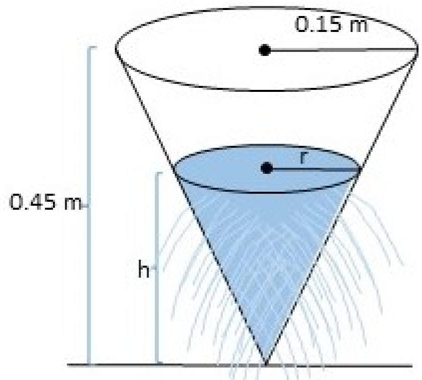
Since Claudia is walking away, $\frac{dx}{dt} \neq 0$, so we can cancel it to get

$$h = 132$$

What are the units?

□

Excercise 3.3 (Challenge: Holey Cone). Consider a conical watering pail with base radius 0.15m and height 0.45m. Suppose the pail is filled with water up to a height h and radius r. Now imagine that there are a bunch of holes in the cone, so that water leaks out at a rate of kA m^3/min, where $k = 0.25$ m/min and $A = \pi r\sqrt{h^2+r^2}$ is the surface area of the part of the cone in contact with the water. What is the rate of change of the water level when the water level is 0.3 m?



Solution. The figure is Exercise 3.9.39 in the textbook.

(1) The variables are volume V, surface area of water and cone interface $A = \pi r\sqrt{h^2 + r^2}$, radius of water r, and height of water h. We are also told that

$$\frac{dV}{dt} = -kA$$

Why is it negative?

(2) Using similar triangles, we relate r and h:

$$\frac{r}{h} = \frac{1}{3}$$

Using geometry, we get

$$V = \frac{1}{3}\pi r^2 h$$

(3) Differentiating the two equations gives

$$\frac{dr}{dt} = \frac{1}{3}\frac{dh}{dt}$$

and

$$\frac{dV}{dt} = \frac{1}{3}\pi\left(2hr\frac{dr}{dt} + r^2\frac{dh}{dt}\right)$$

(4) Plugging in quantatites gives us

$$\begin{aligned} -k\pi r\sqrt{h^2 + r^2} &= \frac{1}{3}\pi(2hr \cdot \frac{1}{3}\frac{dh}{dt} + r^2\frac{dh}{dt}) \\ &= \left(\frac{2\pi hr}{9} + \frac{r^2\pi}{3}\right)\frac{dh}{dt} \end{aligned}$$

(5) We are interested in $\frac{dh}{dt}$ when $h = 0.3$ (and thus $r = 0.1$). Plugging in values gives us

$$\frac{dh}{dt} = \frac{-k\pi r\sqrt{h^2 + r^2}}{\left(\frac{2\pi hr}{9} + \frac{r^2\pi}{3}\right)} \approx -0.79$$

What are the units?

□