1 Learning Objectives

(1) Students will be able to find minimum and maximum values of a function on a closed interval

(2) Students will be able to use the first derivative test to determine if a critical point is a local min or max.

2 Notes

4.2: Extreme Values

Definition 1 (Critical point). $x = c$ is a critical point of f if $f'(x) = 0$ or $f'(x)$ does not exists

To find the extreme values (min or max) of a continuous function f on a closed interval $[a, b]$, we do the following:

(1) Find the critical points of f.

(2) The max and min of f over $[a, b]$ will be among the critical points and endpoints a, b.

Theorem 1 (Rolle's Theorem). If f is continuous on $[a, b]$ and differentiable on (a, b), and $f(a) = f(b)$, then there exists $c \in [a, b]$ such that $f'(c) = 0$

4.3: The Mean Value Theorem

Theorem 2 (Mean Value Theorem). If f is continuous on $[a, b]$ and differentiable on (a, b), then there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

The **First Derivative Test** is used to determine whether a critical point is a local min or max: if f is differentiable and c is a critical point, then c is a local maximum if $f'(x)$ goes from positive to negative at c and a local minimum if $f'(x)$ goes from negative to positive at c.

3 Exercises

Exercise 1 (4.2.37). Find the minimum and maximum values the function on the given interval:

$$f(x) = x - \frac{4x}{x+1}, \quad [0, 3]$$

Solution. First, note that f is continuous on $[0, 3]$. Then we have

$$\begin{aligned}
\frac{df}{dx}(x) &= 1 - \frac{d}{dx}\left(\frac{4x}{x+1}\right) \\
&= 1 - \frac{4(x+1) - 4x}{(x+1)^2} \\
&= \frac{(x+1)^2 - 4}{(x+1)^2}
\end{aligned}$$

The critical points of f are $x = 1, -3$. We only care about $x = 1$ (why?). We have

$$f(0) = 0, f(1) = -1, f(3) = 0$$

Thus the minimum is

$$f(1) = -1$$

and the max is

$$f(0) = f(3) = 0$$

$\square$

Exercise 2 (Corollary of MVT)**.** Assume f is continuous on $[a, b]$ and differentiable on (a, b). Show that if $f'(x) = 0$ for all $x \in (a, b)$, then f is constant on (a, b).

Solution. Note that f is constant on (a, b) if and only if $f(x) = f(y)$ for all $x, y \in (a, b)$. So if we have $x, y \in (a, b)$, we can assume that $x < y$. Then by MVT, we have

$$f(y) - f(x) = 0$$

$\square$

Exercise 3 (4.3.42)**.** Find the critical points and intervals on which the function is increasing or decreasing. Use the First Derivative Test to determine whether the critical point is a local min or max (or neither):

$$f(x) = \frac{2x + 1}{x^2 + 1}$$

Solution. We have

$$\begin{aligned} f'(x) &= \frac{2(x^2 + 1) - (2x)(2x + 1)}{(x^2 + 1)^2} \\ &= \frac{2x^2 + 2 - 4x^2 - 2x}{(x^2 + 1)^2} \\ &= \frac{-2x^2 - 2x + 2}{(x^2 + 1)^2} \\ &= \frac{-2(x^2 + x - 1)}{(x^2 + 1)^2} \end{aligned}$$

The critical points are

$$x = \frac{-1 \pm \sqrt{5}}{2}$$

The function f is increasing when f' is positive. These intervals are

$$\left(\frac{-1 - \sqrt{5}}{2}, \frac{-1 + \sqrt{5}}{2} \right)$$

The function f is decreasing when f' is negative. These intervals are

$$\left(-\infty, \frac{-1 - \sqrt{5}}{2} \right) \cup \left(\frac{-1 + \sqrt{5}}{2}, +\infty \right)$$

In particular, $x = \frac{-1-\sqrt{5}}{2}$ is a local minimum and $x = \frac{-1+\sqrt{5}}{2}$ is a local maximum. $\square$