1 Learning Objectives

(1) Final review

2 Notes

5.2: The Definite Integral

Definition 2.1 (Partition)**.** Let $[a, b]$ be an interval. A **partition** of $[a, b]$ is a collection of points $x_0, \ldots, x_n$ satisfying:

$$a = x_0 < x_1 \cdots < x_n = b$$

The **mesh** of a partition P is the largest subinterval in P:

$$\text{mesh}(P) = \max_{1 \leq i \leq n} x_i - x_{i-1}$$

Definition 2.2 (Refinement of a partition)**.** Let P, Q be two partitions of an interval $[a, b]$. We say Q is a **refinement** of P if $P \subset Q$. We also say P is coarser than Q.

Definition 2.3 (Sample points and tagged partition)**.** Given a partition $P : a = x_0 < \cdots < x_n = b$, a set of **sample points** is a collection of points $c_1, \ldots, c_n$ such that

$$c_i \in [x_{i-1}, x_i]$$

for each i.

A partition along with a set of sample points is called a **tagged partition**.

Definition 2.4 (Riemann sum)**.** Given a function $f : [a, b] \to \mathbb{R}$ and a tagged partition (P, C) of $[a, b]$, the **Riemann sum** of f with tagged partition (P, C) is

$$R(f, P, C) = \sum_{i=1}^{n} f(c_i)(x_i - x_{i-1})$$

Definition 2.5 (Integrable function/Definite integral)**.** We say a function $f : [a, b] \to \mathbb{R}$ is **integrable** if

$$L := \lim_{\substack{\text{mesh}(P) \to 0 \\ (P,C) \text{ tagged partition}}} R(f, P, C)$$

exists. In this case, we call this limit the **definite integral** of f over $[a, b]$, write

$$L = \int_a^b f(x)dx$$

Theorem 2.1 (Continuous functions are Riemann integrable)**.** Functions which are continuous except at finitely many jump discontinuities on and interval $[a, b]$ are integrable over $[a, b]$.

Theorem 2.2 (Properties of definite integrals)**.** (1) Linearity

(2) reversing limits of integration

(3) integral over point

(4) breaking into two integrals via subinterval

(5) monotonicity

5.3: The Indefinite Integral

Definition 2.6 (Antiderivative)**.** A function F is the **antiderivative** of f on an open interval (a, b) if $F'(x) = f(x)$ for all $x \in (a, b)$.

Remark 2.1. Antiderivatives are unique up to a constant.

Definition 2.7. The **indefinite integral** is a just general antiderivative:

$$\int f(x)dx = F(x) + C$$

5.4: Fundamental Theorem of Calculus

Theorem 2.3 (The Fundamental Theorem of Calculus)**.** Assume $a < b$ and f is continuous on $[a, b]$. If F is an antiderivative of f over (a, b), then

$$\int_a^b f(x)dx = F(b) - F(a)$$

Remark 2.2. The FTC connects antiderivatives and definite integration.

5.5: FTC, part II

Theorem 2.4 (FTC, part II)**.** Let f be continuous on an open interval I. If $a \in I$, then

$$\frac{d}{dx}\int_a^x f(t)dt = f(x)$$

Remark 2.3. Both parts of the FTC show that integration and derivatives are inverses of each other (c.f, addition and subtraction).

3 Exercises

Exercise 3.1 (Midterm Question)**.** A person with height H walks away from a 2-m lamppost such that the tip of their shadow moves twice as fast as they do. What is their height?

Exercise 3.2 (Coarsest and finest partitions)**.** What is the coarsest partition of $[0, 1]$? Is there a finest partition?

Exercise 3.3 (Mesh and refinements)**.** If Q is a refinement of P, how are their meshes related?

Exercise 3.4 (Integrating a constant)**.** Show that $\int_a^b c dx = c(b - a)$

Proof. Show that any Riemann sum is $c(b - a)$. □

Exercise 3.5 (Power Rule for integrals)**.** Prove the power rule for integrals:

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C$$

Exercise 3.6 (FTC)**.** Evaluate the integral using FTC:

$$\int_1^8 x^{4/3} dx$$