Operator Algebras in Lean

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0.1 Introduction

0.2 Continuous functional calculus

Definition 1 (Continuous functional calculus)**.** A $*$-R-algebra is said to have a continuous functional calculus for elements satisfying a predicate p if, for each a satisfying p, there is a $*$-homomorphism $\phi_a : C(\sigma_R(a), R) \to A$ sending the identity function to a, and which is a closed embedding. Moreover, $\sigma_R(a)$ is compact and nonempty, and ϕ_a satisfies the spectral mapping property (i.e., $\sigma_R(\phi_a(f)) = f(\sigma_R(a))$).

Definition 2 (Non-unital continuous functional calculus)**.** A non-unital $*$-R-algebra is said to have a non-unital continuous functional calculus for elements satisfying a predicate p if, for each a satisfying p, there is a non-unital $*$-homomorphism $\phi_a : C(\sigma'_R(a), R)_0 \to A$ (here $C(\sigma'_R(a), R)_0$ is the collection of functions vanishing at zero on the quasispectrum) sending the identity function to a, and which is a closed embedding. Moreover, $\sigma'_R(a)$ is compact (it's always nonempty because it contains 0), and ϕ'_a satisfies the spectral mapping property (i.e., $\sigma'_R(\phi'_a(f)) = f(\sigma'_R(a))$).

Definition 3. Given $a \in A$ satisfying p and $f : R \to R$ continuous on $\sigma_R(a)$, we define $f(a) := \phi'_a(f)$ (and we give it a junk value of zero when either a does not satisfy p or f is not continuous on the spectrum).

Definition 4. Given $a \in A$ satisfying p and $f : R \to R$ continuous on $\sigma_R(a)$ and $f(0) = 0$, we define $f(a) := \phi'_a(f)$ (and we give it a junk value of zero when and of the conditions on a and f are not met).

Theorem 5. *For every normal element a in a unital C^*-algebra A there is a $*$-isomorphism between $C(\sigma(a), \mathbb{C})$ and the C^*-subalgebra of A generated by a.*

Proof. Use the Gelfand transform. □

Theorem 6. *Every unital C^*-algebra has a continuous functional calculus for normal elements.*

Proof. Compose the $*$-isomorphism of Theorem 5, which is an isometry because its an isomorphism of C^*-algebras, with the inclusion of $C_1^*(a)$ (the unital C^*-subalgebra generated by a) into A. The latter is also an isometry and therefore a closed embedding. □

Theorem 7. *Every unital $*$-algebra A with a continuous functional calculus for normal elements over $\mathbb{C}$ has a continuous functional calculus for self-adjoint elements over $\mathbb{R}$.*

Proof. Since self-adjoint elements are normal, the continuous functional calculus for normal elements over $\mathbb{C}$ with its spectral mapping property guarantees that the $\mathbb{C}$-spectrum of $a \in A$ normal is actually contained in $\mathbb{R}$, and so coincides with the $\mathbb{R}$-spectrum of a. Therefore, the map which sends $f \in C(\sigma_{\mathbb{R}}(a), \mathbb{R})$ to $\hat{f} \in C(\sigma_{\mathbb{C}}(a), \mathbb{C})$ is a $*$-homomorphism, and composing it with ϕ_a yields the desired $*$-homomorphism for the contnuous functional calculus over $\mathbb{R}$ for self-adjoint elements. □

Theorem 8. *Every unital $*$-algebra A which is a $*$-ordered ring (i.e., nonnegative elements are those of the form x^*x) with the property that nonnegative elements have nonnegative spectrum, and with a continuous functional calculus for self-adjoint elements over $\mathbb{R}$ has a continuous functional calculus for self-adjoint elements over $\mathbb{R}_{\geq 0}$.*

Proof. Omitted. □

0.3 Products of nonnegative elements are nonnegative