W^*–Algebras have Unique Predual

the Lean OA Team

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0.1 Normality and σ–Continuity for Positive Functionals

In what follows, let M be a (nonzero) W^*–algebra. Let $\mathcal{P}(M)$ denote the projection lattice of M. Let φ be a positive linear functional φ on M. We say φ is **normal** if whenever (p_α) is an increasing net of projections in M with supremum p, we have $\varphi(p_\alpha) \to \varphi(p)$. In this section we show that this property is equivalent to $\sigma(M, M_*)$–continuity. We say that a linear functional φ on M is **positive** if $\varphi(x) \geq 0$ whenever $x \geq 0$.

To do : Net results must be replaced by filter results.

0.1.1 σ–Continuous Implies Normal

This is 1.7.4 in Sakai. Roughly, taking an increasing net of projections, it converges to the sup strongly, hence σ-topology, and the σ-continuity finishes the proof.

0.1.2 Normal Implies σ–Continuous

Lemma 1. *For all self-adjoint $x \in M$, $\|x\|1 - x \geq 0$.*

Lemma 2. *For all $p, q \in \mathcal{P}(M)$ such that $q \leq p$, $p - q \in \mathcal{P}(M)$.*

Lemma 3. *For every increasing bounded net (p_α) of projections in M, the supremum p is a projection in M*

If $P : \mathcal{P}(M) \to \text{Prop}$ is a predicate, the usual ordering "$\leq$" on projections induces an order on the set $\{p \in \mathcal{P}(M) | P(p)\}$. In what follows there will be no confusion if we also denote this induced order by "$\leq$".

The following lemma undoubtedly exists in Mathlib in more generality already.

Lemma 4. *For every nonzero element $a \in M$, there is a $\sigma(M, M_*)$–continuous positive linear functional ψ on M such that $\psi(a) \neq 0$.*

Lemma 5. *For every positive normal linear functional φ and nonzero $p \in \mathcal{P}(M)$ there exists a positive $\sigma(M, M_*)$–continuous linear functional ψ such that $\varphi(p) < \psi(p)$.*

Lemma 6. *For all positive linear functionals φ, ψ with φ normal and ψ $\sigma(M, M_*)$–continuous and every nonzero $p \in \mathcal{P}(M)$ such that $\varphi(p) < \psi(p)$, there exists a nonzero $p_1 \in \mathcal{P}(M)$ such that $p_1 \leq p$ and for all nonzero $q \in \mathcal{P}(M)$ with $q \leq p_1$, we have $\varphi(q) < \psi(q)$.*

Recall that a compact Hausdorff space is **Stonean** if the closure of every open set is open.

Lemma 7. *Let K be a Stonean space. Then every element a in $C(K)$ can be uniformly approximated by finite linear combinations of projections in $C(K)$. If $a \geq 0$ then the coefficients of the approximating linear combinations may be chosen nonnegative.*

Lemma 8. *If $p \in M$ is a projection, pMp is also a W^* algebra with identity p.*

Lemma 9. *If C is any maximal commutative C^*–subalgebra of the W^*–algebra M, its spectrum space (maximal ideal space) is Stonean.*

Lemma 10. *Let φ be a normal positive linear functional on M and consider the predicate $P : \mathcal{P}(M) \to$ Proj defined, for $p \in \mathcal{P}(M)$, by "$M \ni x \mapsto \varphi(xp)$ is $\sigma(M, M_*)$–continuous". If (p_α) is a chain of projections in M such that $P(p_\alpha)$ is true for each α, then $P(\sup(p_\alpha))$ is true. Hence by Zorn's Lemma there is a maximal $p_0 \in \mathcal{P}(M)$ such that $P(p_0)$ is true.*

Lemma 11. *A linear functional ρ on M is σ continuous on the unit sphere (hence σ–continuous) if and only if it is s–continuous on the unit sphere (hence s–continuous).*

Lemma 12. *A maximal abelian $*$–subalgebra of a W^*–algebra M is also a W^*–algebra.*

Theorem 13. *Every positive normal linear functional φ on M is $\sigma(M, M_*)$–continuous.*

Proof. The claim is obvious for the zero functional. Let φ be a nonzero positive normal linear functional. By Lemma 10 we have a maximal $p_0 \in \mathcal{P}(M)$ such that $M \ni x \mapsto \varphi(xp_0)$ is $\sigma(M, M_*)$–continuous. Assume for the purposes of finding a contradiction that $p_0 \neq 1$. By Lemma 5 there is a $\sigma(M, M_*)$–continuous positive functional ψ on M such that $\varphi(1 - p_0) < \psi(1 - p_0)$. By Lemma 6 there is a nonzero subprojection $p \leq 1 - p_0$ in M such that $\varphi(q)\psi(q)$ for every nonzero $q \leq p$ in M. Let $x \in pMp$ be on the unit sphere. Then x^*x is positive and hence normal, so the C^*–subalgebra of pMp generated by x^*x and p is commutative, and is hence contained in a maximal abelian $*$–subalgebra A of pMp. Now A is a W^*–subalgebra of pMp by Lemma 12 and hence is a maximal commutative C^*–subalgebra of pMp. Via the Gelfand Transform, A is star isomorphic to $C(K)$, where K is Stonean by Lemmas 8 and 9. By Lemma 7 it follows that $\varphi(a) \leq \psi(a)$ for every $a \geq 0$ in A, which holds a fortiori for $a \geq 0$ in $C^*(x^*x, p)$. In particular, $\varphi(px^*xp) \leq \psi(px^*xp)$. Therefore,

$$\begin{aligned} |\varphi(x(p_0 + p))| &\leq |\varphi(xp_0)| + |\varphi(xp)| \\ &\leq |\varphi(xp_0)| + \varphi(1)^{1/2}\varphi(px^*xp)^{1/2} \\ &\leq |\varphi(xp_0)| + \varphi(1)^{1/2}\psi(px^*xp)^{1/2}. \end{aligned} \tag{1}$$

Since $x \mapsto \varphi(xp_0)$ is σ–continuous, it is s-continuous by Lemma 11. The seminorm $x \mapsto \psi(px^*xp)^{1/2}$ is a defining seminorm for the s–topology on M. It follows that $x \mapsto \varphi(x(p_0 + p))$ is s-continuous and therefore σ–continuous. This contradicts the maximality of p_0, and therefore $p_0 = 1$ and the result follows. $\square$