

# $W^*$ –Algebras have Unique Predual

the Lean OA Team

January 9, 2026

## 0.1 Normality and $\sigma$ –Continuity for Positive Functionals

In what follows, let  $M$  be a (nonzero)  $W^*$ –algebra. Let  $\mathcal{P}(M)$  denote the projection lattice of  $M$ . Let  $\varphi$  be a positive linear functional on  $M$ . We say  $\varphi$  is **normal** if whenever  $(p_\alpha)$  is an increasing net of projections in  $M$  with supremum  $p$ , we have  $\varphi(p_\alpha) \rightarrow \varphi(p)$ . In this section we show that this property is equivalent to  $\sigma(M, M_*)$ –continuity. We say that a linear functional  $\varphi$  on  $M$  is **positive** if  $\varphi(x) \geq 0$  whenever  $x \geq 0$ .

*To do : Net results must be replaced by filter versions.*

### 0.1.1 $\sigma$ –Continuous Implies Normal

This is 1.7.4 in Sakai. Roughly, taking an increasing net of projections, it converges to the sup strongly, hence  $\sigma$ –topology, and the  $\sigma$ –continuity finishes the proof.

### 0.1.2 Normal Implies $\sigma$ –Continuous

**Lemma 1.** *For all self-adjoint  $x \in M$ ,  $\|x\|1 - x \geq 0$ .*

**Lemma 2.** *For all  $p, q \in \mathcal{P}(M)$  such that  $q \leq p$ ,  $p - q \in \mathcal{P}(M)$ .*

**Lemma 3.** *For every increasing bounded net  $(p_\alpha)$  of projections in  $M$ , the supremum  $p$  is a projection in  $M$*

If  $P : \mathcal{P}(M) \rightarrow \text{Prop}$  is a predicate, the usual ordering “ $\leq$ ” on projections induces an order on the set  $\{p \in \mathcal{P}(M) | P(p)\}$ . In what follows there will be no confusion if we also denote this induced order by “ $\leq$ ”.

The following lemma undoubtedly exists in Mathlib in more generality already.

**Lemma 4.** *For every nonzero element  $a \in M$ , there is a  $\sigma(M, M_*)$ –continuous positive linear functional  $\psi$  on  $M$  such that  $\psi(a) \neq 0$ .*

**Lemma 5.** *For every positive normal linear functional  $\varphi$  and nonzero  $p \in \mathcal{P}(M)$  there exists a positive  $\sigma(M, M_*)$ –continuous linear functional  $\psi$  such that  $\varphi(p) < \psi(p)$ .*

**Lemma 6.** *For all positive linear functionals  $\varphi, \psi$  with  $\varphi$  normal and  $\psi$   $\sigma(M, M_*)$ –continuous and every nonzero  $p \in \mathcal{P}(M)$  such that  $\varphi(p) < \psi(p)$ , there exists a nonzero  $p_1 \in \mathcal{P}(M)$  such that  $p_1 \leq p$  and for all nonzero  $q \in \mathcal{P}(M)$  with  $q \leq p_1$ , we have  $\varphi(q) < \psi(q)$ .*

Recall that a compact Hausdorff space is **Stonean** if the closure of every open set is open.

**Lemma 7.** *Let  $K$  be a Stonean space. Then every element  $a$  in  $C(K)$  can be uniformly approximated by finite linear combinations of projections in  $C(K)$ . If  $a \geq 0$  then the coefficients of the approximating linear combinations may be chosen nonnegative.*

**Lemma 8.** *If  $p \in M$  is a projection,  $pMp$  is also a  $W^*$  algebra with identity  $p$ .*

**Lemma 9.** *If  $C$  is any maximal commutative  $C^*$ –subalgebra of the  $W^*$ –algebra  $M$ , its spectrum space (maximal ideal space) is Stonean.*

**Lemma 10.** *Let  $\varphi$  be a normal positive linear functional on  $M$  and consider the predicate  $P : \mathcal{P}(M) \rightarrow \text{Prop}$  defined, for  $p \in \mathcal{P}(M)$ , by “ $M \ni x \mapsto \varphi(xp)$  is  $\sigma(M, M_*)$ –continuous”. If  $(p_\alpha)$  is a chain of projections in  $M$  such that  $P(p_\alpha)$  is true for each  $\alpha$ , then  $P(\sup(p_\alpha))$  is true. Hence by Zorn’s Lemma there is a maximal  $p_0 \in \mathcal{P}(M)$  such that  $P(p_0)$  is true.*

**Lemma 11.** *A linear functional  $\rho$  on  $M$  is  $\sigma$  continuous on the unit sphere (hence  $\sigma$ -continuous) if and only if it is  $s$ -continuous on the unit sphere (hence  $s$ -continuous).*

**Lemma 12.** *A maximal abelian  $*$ -subalgebra of a  $W^*$ -algebra  $M$  is also a  $W^*$ -algebra.*

**Theorem 13.** *Every positive normal linear functional  $\varphi$  on  $M$  is  $\sigma(M, M_*)$ -continuous.*

*Proof.* The claim is obvious for the zero functional. Let  $\varphi$  be a nonzero positive normal linear functional. By Lemma ?? we have a maximal  $p_0 \in \mathcal{P}(M)$  such that  $M \ni x \mapsto \varphi(xp_0)$  is  $\sigma(M, M_*)$ -continuous. Assume for the purposes of finding a contradiction that  $p_0 \neq 1$ . By Lemma ?? there is a  $\sigma(M, M_*)$ -continuous positive functional  $\psi$  on  $M$  such that  $\varphi(1 - p_0) < \psi(1 - p_0)$ . By Lemma ?? there is a nonzero subprojection  $p \leq 1 - p_0$  in  $M$  such that  $\varphi(q)\psi(q)$  for every nonzero  $q \leq p$  in  $M$ . Let  $x \in pMp$  be on the unit sphere. Then  $x^*x$  is positive and hence normal, so the  $C^*$ -subalgebra of  $pMp$  generated by  $x^*x$  and  $p$  is commutative, and is hence contained in a maximal abelian  $*$ -subalgebra  $A$  of  $pMp$ . Now  $A$  is a  $W^*$ -subalgebra of  $pMp$  by Lemma ?? and hence is a maximal commutative  $C^*$ -subalgebra of  $pMp$ . Via the Gelfand Transform,  $A$  is star isomorphic to  $C(K)$ , where  $K$  is Stonean by Lemmas ?? and ??. By Lemma ?? it follows that  $\varphi(a) \leq \psi(a)$  for every  $a \geq 0$  in  $A$ , which holds a fortiori for  $a \geq 0$  in  $C^*(x^*x, p)$ . In particular,  $\varphi(px^*xp) \leq \psi(px^*xp)$ . Therefore,

$$\begin{aligned} |\varphi(x(p_0 + p))| &\leq |\varphi(xp_0)| + |\varphi(xp)| \\ &\leq |\varphi(xp_0)| + \varphi(1)^{1/2} \varphi(px^*xp)^{1/2} \\ &\leq |\varphi(xp_0)| + \varphi(1)^{1/2} \psi(px^*xp)^{1/2}. \end{aligned} \tag{1}$$

Since  $x \mapsto \varphi(xp_0)$  is  $\sigma$ -continuous, it is  $s$ -continuous by Lemma ??. The seminorm  $x \mapsto \psi(px^*xp)^{1/2}$  is a defining seminorm for the  $s$ -topology on  $M$ . It follows that  $x \mapsto \varphi(x(p_0 + p))$  is  $s$ -continuous and therefore  $\sigma$ -continuous. This contradicts the maximality of  $p_0$ , and therefore  $p_0 = 1$  and the result follows.  $\square$