von Neumann Double Commutant Theorem

Jon Bannon   Jireh Loreaux

October 2, 2025

Contents

0.1 Introduction

[Blah] The commutant of a set S of operators is the set of $T \in B(\mathcal{H})$ such that $\forall s \in S,\ Ts = sT$.

For $A \subseteq B(\mathcal{H})$, the *weak-operator closure* $\overline{A}^{\text{WOT}}$ is the closure of A in the weak operator topology on $B(\mathcal{H})$.

A $*$-subalgebra $A \subseteq B(\mathcal{H})$ is *self-adjoint* if $T \in A \Rightarrow T^* \in A$.

0.2 Main Theorem

[von Neumann 1929] The weak operator closure of a self-adjoint $*$-subalgebra of $B(\mathcal{H})$ equals its double commutant.