von Neumann Double Commutant Theorem

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October 2, 2025

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0.1 Introduction

Definition 1 (Blah)**.** The commutant of a set S of operators is the set of $T \in B(\mathcal{H})$ such that $\forall s \in S,\ Ts = sT$.

Definition 2. For $A \subseteq B(\mathcal{H})$, the *weak-operator closure* $\overline{A}^{\mathrm{WOT}}$ is the closure of A in the weak operator topology on $B(\mathcal{H})$.

Definition 3. A $*$-subalgebra $A \subseteq B(\mathcal{H})$ is *self-adjoint* if $T \in A \Rightarrow T^* \in A$.

Theorem 4 (von Neumann 1929)**.** *The weak operator closure of a self-adjoint $*$-subalgebra of $B(\mathcal{H})$ equals its double commutant.*