Algorithm 1 Third-Order Reverse Mode Accumulation

Input: *Tape*
$\nabla f = \bar{w}[m] = 0$
$\nabla^2 f = \bar{h}[m][m] = 0$
$\nabla^3 f = \bar{d}[m][m][m] = 0$

$S = \{w_m\}$
$\bar{w}[m] = 1$
for $i = m$ to 1 **do**
  $w_i \cup S_i$
  $S_i = \{\}$
  $\bar{w}_i = \bar{w}[i]$
  $\bar{w}[i] = 0$
  $\bar{h}_{ii} = \bar{h}[i][i]$
  $\bar{h}[i][i] = 0$
  $\bar{d}_{iii} = \bar{d}[i][i][i]$
  $\bar{d}[i][i][i] = 0$
  n = size(variable set)

  for $j = 1$ to n **do**
    $\bar{H}[i][j] = \bar{h}[i][j]$
    $\bar{h}[i][j] = 0$
    for $k = 1$ to n **do**
      $\bar{D}[i][j][k] = \bar{d}[i][j][k]$
      $\bar{d}[i][j][k] = 0$
    end for
  end for

  for $j = 1$ to n **do**
    $\bar{w}[j] += \frac{\partial \phi_i}{\partial x_i} \bar{w}[i]$
    for $k = 1$ to n **do**
      $temp = \bar{h}[i][k] \frac{\partial f}{\partial x_j} + \bar{h}[i][j] \frac{\partial f}{\partial x_k} + \bar{h}[i][i] \frac{\partial f}{\partial x_k} \frac{\partial f}{\partial x_j} + w \frac{\partial^2 f}{\partial x_j \partial x_k}$
      if $temp \neq 0$ **then**
        $\bar{h}[j][k] += temp$
        $j \in S_i$
        $k \in S_i$
      end if
      for $l = 1$ to n **do**
        $temp = \frac{\partial \phi_i}{\partial x_j} \bar{D}[i][k][l] + \frac{\partial \phi_i}{\partial x_k} \bar{D}[i][j][l] + \frac{\partial \phi_i}{\partial x_l} \bar{D}[i][j][k] +$
          $\frac{\partial \phi_i}{\partial x_j} \frac{\partial \phi_i}{\partial x_k} \bar{D}[i][i][l] + \frac{\partial \phi_i}{\partial x_j} \frac{\partial \phi_i}{\partial x_l} \bar{D}[i][i][k] + \frac{\partial \phi_i}{\partial x_k} \frac{\partial \phi_i}{\partial x_l} \bar{D}[i][i][j] +$
          $\frac{\partial \phi_i}{\partial x_j} \frac{\partial \phi_i}{\partial x_k} \frac{\partial \phi_i}{\partial x_l} \bar{d}_{iii} +$
          $\frac{\partial \phi_i}{\partial x_j} \frac{\partial \phi_i}{\partial x_k} \bar{H}[i][l] + \frac{\partial \phi_i}{\partial x_j} \frac{\partial \phi_i}{\partial x_l} \bar{H}[i][k] + \frac{\partial \phi_i}{\partial x_k} \frac{\partial \phi_i}{\partial x_l} \bar{H}[i][j] +$
          $[\frac{\partial \phi_i}{\partial x_j} \frac{\partial^2 \phi_i}{\partial x_k \partial x_l} + \frac{\partial^2 \phi_i}{\partial x_k} \frac{\partial^2 \phi_i}{\partial x_j \partial x_l} + \frac{\partial^2 \phi_i}{\partial x_l} \frac{\partial^2 \phi_i}{\partial x_j \partial x_k}] \bar{h}_{ii} +$
          $\bar{w}_i \frac{\partial^3 \phi_i}{\partial x_j \partial x_k \partial x_l}$
        if $temp \neq 0$ **then**
          $\bar{d}[j][k][l] += temp$
          $j \in S_i$
          $k \in S_i$
          $l \in S_i$
        end if
      end for
    end for
  end for
  if i **then** > 1
    Push statement level variables to entry[i-1].
  end if
end for