Reference Point Calculations in MAS

Jon Brodziak

Pacific Islands Fisheries Science Center
1845 Wasp Boulevard, Building 176, Honolulu, HI 96818

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Reference point calculations for the Metapopulation Assessment System are defined for the general case of a total of *P* populations in *D* areas with *G* genders and *A* age groups. The notation for indexing population, area, gender and age variables is p=1…P populations, d=1...D areas, and g=1…G genders. For the age group index, there are two options. First, if the age of recruitment to the population is age-0, then the age index a is a=0…A-1 age groups where age-(A-1) is a plus group comprised of all fish age-(A-1) and older. Second, if the age of recruitment to the population is age-1, then the age index a is a=1…A age groups where age-A is a plus group comprised of all fish age-A and older. Without loss of generality, we will use the second option for indexing ages in this document noting that the calculation of spawning biomass and the resulting recruitment in equilibrium is not affected by a time lag of 1 year for recruitment age. Another convention for notation is that the symbol "~" indicates the equilibrium value of a variable used in the calculation of a reference point. We describe iterative algorithms to calculate spatial reference points for multiple populations in multiple areas.

Calculation of MSY-based reference points

One can use the following numerical search algorithm to calculate Maximum Sustainable Yield (MSY) based reference points. To initialize the calculations, we construct a uniform-spaced vector of fishing mortality rates, denoted by $\underline{F}$, with mesh size δ where $\underline{F} = \left(F_1, F_2, ..., F_{Upper}\right)$ and $F_j = \left(j-1\right)\cdot\delta$ and $F_1 = 0$ and F_{Upper} is a maximal value of fishing mortality, say for example, $F_{Upper} = 3.00$ with $\delta = 0.01$.

The algorithm to calculate MSY-based reference points can be categorized into 8 steps.

Step 1. For each population p, each area d and each gender g, calculate the unfished numbers at age in equilibrium ($\underline{\widetilde{N}}_{U,d,g}^{(p)}$) using Algorithm 1 as

(1) $$\underline{\widetilde{N}}_{U,d,g}^{(p)} = \left(\widetilde{N}_{U,d,g,a=1}^{(p)}, \widetilde{N}_{U,d,g,1}^{(p)}, \ldots, \widetilde{N}_{U,d,g,A}^{(p)}\right)$$

Step 2. For each fishing mortality F_j, each population p, each area d and each gender g, calculate the fished numbers at age in equilibrium ($\underline{\widetilde{N}}_{F,d,g}^{(p)}$) using Algorithm 2 as

(2) $$\underline{\widetilde{N}}_{F_j,d,g}^{(p)} = \left(\widetilde{N}_{F_j,d,g,a=1}^{(p)}, \widetilde{N}_{F_j,d,g,1}^{(p)}, \ldots, \widetilde{N}_{F_j,d,g,A}^{(p)}\right)$$

Step 3. For each fishing mortality F_j, set recruitment strength by population p, each area d and each gender g, as

(3) $$\widetilde{R}_{F_j,d,g}^{(p)} = \widetilde{N}_{F_j,d,g,a=1}^{(p)}$$

Step 4. For each fishing mortality F_j, calculate equilibrium yield per recruit by population, area, and gender as

(4)

$$\widetilde{YPR}\left(F_j\right)_{d,g}^{(p)} = \sum_{a=1}^{A-1}\left\{\frac{\widetilde{W}_{C,d,g,a}^{(p)} \cdot F_j \cdot \widetilde{S}_{d,g,a}^{(p)}}{F_j \cdot \widetilde{S}_{d,g,a}^{(p)} + \widetilde{M}_{d,g,a}^{(p)}}\left(1-\exp\left(-F_j \cdot \widetilde{S}_{d,g,a}^{(p)} - \widetilde{M}_{d,g,a}^{(p)}\right)\right) \cdot \exp\left(-\sum_{k=1}^{a-1}\left(F_j \cdot \widetilde{S}_{d,g,k}^{(p)} + \widetilde{M}_{d,g,k}^{(p)}\right)\right)\right\}$$
$$+ \frac{\widetilde{W}_{C,d,g,A}^{(p)} \cdot F_j \cdot \widetilde{S}_{d,g,A}^{(p)}}{F_j \cdot \widetilde{S}_{d,g,A}^{(p)} + \widetilde{M}_{d,g,A}^{(p)}} \exp\left(-\sum_{k=1}^{A-1}\left(F_j \cdot \widetilde{S}_{d,g,k}^{(p)} + \widetilde{M}_{d,g,k}^{(p)}\right)\right)$$

Step 5. For each fishing mortality F_j, calculate equilibrium yield by population, area, and gender as

(5) $$\widetilde{Y}\left(F_j\right)_{d,g}^{(p)} = \widetilde{R}_{F_j,d,g}^{(p)} \cdot \widetilde{YPR}\left(F_j\right)_{d,g}^{(p)}$$

And calculate equilibrium yield by population and area as

(6) $$\widetilde{Y}\left(F_j\right)_{d}^{(p)} = \sum_{g} \widetilde{Y}\left(F_j\right)_{d,g}^{(p)}$$

And calculate equilibrium yield by population as

(7) $$\widetilde{Y}\left(F_j\right)^{(p)} = \sum_d \widetilde{Y}\left(F_j\right)_d^{(p)}$$

And calculate equilibrium yield by area as

(8) $$\widetilde{Y}\left(F_j\right)_d = \sum_p \widetilde{Y}\left(F_j\right)_d^{(p)}$$

And calculate the equilibrium yield for all populations and areas as

(9) $$\widetilde{Y}\left(F_j\right) = \sum_p \sum_d \widetilde{Y}\left(F_j\right)_d^{(p)}$$

<u>Step 6.</u> Loop over the set of fishing mortalities $F_j \in \underline{F}$, to find $F_{MSY}^{(p)}$ by population *p*, such that $\widetilde{Y}\left(F_{MSY}^{(p)}\right) \geq \widetilde{Y}\left(F_j\right)^{(p)}$ for all $F_j \neq F_{MSY}^{(p)}$ and then set *MSY* by population as

(10) $$MSY^{(p)} = \widetilde{Y}\left(F_{MSY}^{(p)}\right)$$

Loop over the set of fishing mortalities $F_j \in \underline{F}$, to find $F_{MSY,d}$ by area *d*, such that $\widetilde{Y}\left(F_{MSY,d}\right) \geq \widetilde{Y}\left(F_j\right)_d$ for all $F_j \neq F_{MSY,d}$ and then set *MSY* by area as

(11) $$MSY_d = \widetilde{Y}\left(F_{MSY,d}\right)$$

Loop over the set of fishing mortalities $F_j \in \underline{F}$, to find F_{MSY} such that $\widetilde{Y}\left(F_{MSY}\right) \geq \widetilde{Y}\left(F_j\right)$ for all $F_j \neq F_{MSY}$ and then set

(12) $$MSY_{Global} = \widetilde{Y}\left(F_{MSY}\right)$$

<u>Step 7.</u> Loop over the set of fishing mortalities $F_j \in \underline{F}$, to calculate the equilibrium female spawning biomass by population *p* and area *d* as

(13) $$\widetilde{SB}_{F_j,d,female}^{(p)} = \sum_a \widetilde{P}_{M,female,a}^{(p)} \cdot \widetilde{W}_{S,female,a}^{(p)} \cdot \widetilde{N}_{F_j,d,female,a}^{(p)} \cdot \exp\left(-\Delta_S \cdot \widetilde{Z}_{F_j,d,female,a}^{(p)}\right)$$

And calculate equilibrium female spawning biomass by population *p* as

(14) $$\widetilde{SB}_{F_j,female}^{(p)} = \sum_d \widetilde{SB}_{F_j,d,female}^{(p)}$$

And calculate equilibrium female spawning biomass by area d as

(15) $$\widetilde{SB}_{F_j,d,female} = \sum_p \widetilde{SB}^{(p)}_{F_j,d,female}$$

And calculate equilibrium female spawning biomass for all populations p and areas d as

(16) $$\widetilde{SB}_{F_j,female} = \sum_p \sum_d \widetilde{SB}^{(p)}_{F_j,d,female}$$

And set the female spawning biomass to produce MSY by population p as

(17) $$\widetilde{SB}^{(p)}_{MSY} = \widetilde{SB}_{F^{(p)}_{MSY},female}$$

And set the female spawning biomass to produce MSY by area d as

(18) $$\widetilde{SB}_{MSY,d} = \widetilde{SB}_{F_{MSY,d},female}$$

Last set the female spawning biomass to produce MSY as

(19) $$\widetilde{SB}_{MSY_{Global}} = \widetilde{SB}_{F_{MSY},female}$$

<u>Step 8.</u> Loop over the set of fishing mortalities $F_j \in \underline{F}$, to calculate the equilibrium female spawning biomass per recruit by population p and area d as

(20) $$\widetilde{SSBR}\left(F_j\right)^{(p)}_d = \frac{\widetilde{SB}^{(p)}_{F_j,d,female}}{\sum_g \widetilde{R}^{(p)}_{F_j,d,g}}$$

And calculate equilibrium female spawning biomass per recruit by population p as

(21) $$\widetilde{SSBR}\left(F_j\right)^{(p)} = \frac{\sum_d \widetilde{SB}^{(p)}_{F_j,d,female}}{\sum_d \sum_g \widetilde{R}^{(p)}_{F_j,d,g}}$$

And calculate equilibrium female spawning biomass per recruit by area d as

(22) $$\widetilde{SSBR}\left(F_j\right)_d = \frac{\sum_p \widetilde{SB}^{(p)}_{F_j,d,female}}{\sum_p \sum_g \widetilde{R}^{(p)}_{F_j,d,g}}$$

And calculate equilibrium female spawning biomass per recruit for all populations and areas as

$$\widetilde{SSBR}\left(F_j\right) = \frac{\sum_p \sum_d \widetilde{SB}^{(p)}_{F_j,d,female}}{\sum_p \sum_d \sum_g \widetilde{R}^{(p)}_{F_j,d,g}} \tag{23}$$

And set the female spawning biomass per recruit to produce *MSY* by population *p* as

$$\widetilde{SBPR}^{(p)}_{MSY} = \widetilde{SBPR}\left(F^{(p)}_{MSY}\right) \tag{24}$$

And set the female spawning biomass per recruit to produce *MSY* by area *d* as

$$\widetilde{SBPR}_{MSY,d} = \widetilde{SBPR}\left(F_{MSY,d}\right) \tag{25}$$

Last set the female spawning biomass per recruit to produce *MSY* as

$$\widetilde{SBPR}_{MSY_{Global}} = \widetilde{SBPR}\left(F_{MSY}\right) \tag{26}$$

Calculation of $F_{X\%}$ reference points

The fishing mortality that produces a fixed percentage $X\%$ of the unfished spawning biomass is $F_{X\%}$. To calculate $F_{X\%}$, one first needs to start with an estimate of the equilibrium spawning biomass as a function of *F*. We can use the calculations of equilibrium spawning biomass by population, area and for all populations and areas based on equations (14), (15) and (16) in the MSY reference point calculations. Without loss of generality, assume we have the equilibrium female spawning biomass by population ($\widetilde{SB}^{(p)}_{F_j,female}$), area ($\widetilde{SB}_{F_j,d,female}$), and for all populations and areas ($\widetilde{SB}_{F_j,female}$) already calculated for the set of fishing mortalities $F_j \in \underline{F}$. Here is the algorithm to calculate the fishing mortality that produces a fixed percentage $X\%$ of the unfished spawning biomass by population ($F^{(p)}_{X\%}$), area ($F_{X\%,d}$) and for all populations and areas ($F_{X\%}$).

Step 1. For each population and $F_j \in \underline{F}$, calculate the ratio $R^{(p)}_j = \frac{\widetilde{SB}^{(p)}_{F_j,female}}{\widetilde{SB}^{(p)}_{F=0,female}}$ and the difference $\Delta^{(p)}_j = \left|R^{(p)}_j - X_{\%}\right|$ by population.

Step 2. Next find the index $k \in \{1, 2, ..., Upper\}$ that produces the smallest difference $\Delta_k^{(p)} \leq \Delta_j^{(p)} \ \textit{for all} \ j \in \{1, 2, ..., Upper\}$ for each population *p*. Then set the fishing mortality that produces a fixed percentage $X\%$ of the unfished spawning biomass by population along with the associated spawning biomass and fishery yield in each population *p* as

$$F_{X\%}^{(p)} = F_k \text{ and } \widetilde{SB}_{X\%}^{(p)} = \widetilde{SB}_{F_k,female}^{(p)} \text{ and } \tilde{Y}_{X\%}^{(p)} = \tilde{Y}\left(F_k\right)^{(p)} \tag{27}$$

Step 3. For each area and $F_j \in \underline{F}$, calculate the ratio $R_{j,d} = \dfrac{\widetilde{SB}_{F_j,d,female}}{\widetilde{SB}_{F=0,d,female}}$ and the difference $\Delta_{j,d} = \left|R_{j,d} - X_{\%}\right|$ by area.

Step 4. Next find the index $k \in \{1, 2, ..., Upper\}$ that produces the smallest difference $\Delta_{k,d} \leq \Delta_{j,d} \ \textit{for all} \ j \in \{1, 2, ..., Upper\}$ for each area *d*. Then set the fishing mortality that produces a fixed percentage $X\%$ of the unfished spawning biomass along with the associated spawning biomass and fishery yield in each area *d* as

$$F_{X\%,d} = F_k \text{ and } \widetilde{SB}_{X\%,d} = \widetilde{SB}_{F_k,d,female} \text{ and } \tilde{Y}_{X\%,d} = \tilde{Y}\left(F_k\right)_d \tag{28}$$

Step 5. For each $F_j \in \underline{F}$, calculate the global ratio $R_j = \dfrac{\widetilde{SB}_{F_j,female}}{\widetilde{SB}_{F=0,female}}$ for all populations and area and the global difference $\Delta_j = \left|R_j - X_{\%}\right|$.

Step 6. Next find the index $k \in \{1, 2, ..., Upper\}$ that produces the smallest difference $\Delta_k \leq \Delta_j \ \textit{for all} \ j \in \{1, 2, ..., Upper\}$. Then set the fishing mortality that produces a fixed percentage $X\%$ of the unfished spawning biomass for all populations and areas along with the associated global spawning biomass and fishery yield as

$$F_{X\%} = F_k \text{ and } \widetilde{SB}_{X\%} = \widetilde{SB}_{F_k,female} \text{ and } \tilde{Y}_{X\%} = \tilde{Y}\left(F_k\right) \tag{28}$$

Calculate F_{MAX}, the *F* that produces the maximum yield per recruit

The fishing mortality that produces the maximum yield per recruit is F_{MAX}. To calculate F_{MAX}, one first needs to start with an estimate of the equilibrium yield per recruit as a function of *F*. We can use the calculations of equilibrium yield per recruit by population, area and for all populations and areas based on equations (7), (8) and (9) in the MSY reference point calculations. Without loss of generality, assume we have the set of values of the equilibrium yield per recruit by population ($\widetilde{YPR}_{F_j}^{(p)}$), area ($\widetilde{YPR}_{F_j,d}$), and for all populations and areas ($\widetilde{YPR}_{F_j}$) already calculated for the set of fishing mortalities $F_j \in \underline{F}$. Here is the algorithm to calculate the fishing mortality that produces the maximum yield per recruit by population ($F_{MAX}^{(p)}$), area ($F_{MAX,d}$) and for all populations and areas (F_{MAX}).

Step 1. For each population *p*, find the index $k \in \{1,2,...,Upper\}$ that produces the maximum equilibrium yield per recruit such that $\widetilde{YPR}_{F_k}^{(p)} \geq \widetilde{YPR}_{F_j}^{(p)} \; for \; all \; j \in \{1,2,...,Upper\}$. Then set the fishing mortality that produces the maximum yield per recruit by population along with the associated spawning biomass and fishery yield for each population *p* as

(29) $$F_{MAX}^{(p)} = F_k \text{ and } \widetilde{SB}_{F_{MAX}}^{(p)} = \widetilde{SB}_{F_k,female}^{(p)} \text{ and } \tilde{Y}_{F_{MAX}}^{(p)} = \tilde{Y}\left(F_k\right)^{(p)}$$

Step 2. For each area *d*, find the index $k \in \{1,2,...,Upper\}$ that produces the maximum equilibrium yield per recruit such that $\widetilde{YPR}_{F_k,d} \geq \widetilde{YPR}_{F_j,d} \; for \; all \; j \in \{1,2,...,Upper\}$. Then set the fishing mortality that produces the maximum yield per recruit by area along with the associated spawning biomass and fishery yield for each area *d* as

(30) $$F_{MAX,d} = F_k \text{ and } \widetilde{SB}_{F_{MAX},d} = \widetilde{SB}_{F_k,d} \text{ and } \tilde{Y}_{F_{MAX},d} = \tilde{Y}\left(F_k\right)_d$$

Step 3. For each $F_j \in \underline{F}$, find the index $k \in \{1,2,...,Upper\}$ that produces the maximum global equilibrium yield per recruit such that $\widetilde{YPR}_{F_k} \geq \widetilde{YPR}_{F_j} \; for \; all \; j \in \{1,2,...,Upper\}$. Then set the global fishing mortality that produces the global maximum yield per recruit by area along with the associated spawning biomass and fishery yield for each area *d* as

(31) $$F_{MAX} = F_k \text{ and } \widetilde{SB}_{F_{MAX}} = \widetilde{SB}_{F_k} \text{ and } \tilde{Y}_{F_{MAX}} = \tilde{Y}\left(F_k\right)$$

Calculate $F_{0.1}$, the global fishing mortality rate where the slope of the global *YPR* curve is 10% of the value of the slope at the origin, where *F*=0

The reference point $F_{0.1}$ was developed by Gulland and Boerema (1973) and was based on maintaining marginal fishery yield at 10% of the initial fishery CPUE in order to support an economically efficient fishery. This reference point is calculated from the global yield per recruit curve and is the value of *F* that produces 10% of the slope of the yield per recruit curve at the origin.

Step 1. For each population p, area d, and gender g, calculate the unfished equilibrium recruitment $\widetilde{R}_{F=0,d,g}^{(p)}$ as in equation (3) above.

Step 2. Construct the global yield per recruit function $YPR(F)$ as

(32) $$YPR(F) = \frac{\widetilde{R}_{F=0,d,g}^{(p)}}{\sum_{p}\sum_{d}\sum_{g}\widetilde{R}_{F=0,d,g}^{(p)}} \cdot \widetilde{YPR}(F)_{d,g}^{(p)}$$

Where $\widetilde{YPR}(F)_{d,g}^{(p)}$ is defined in equation (4) above.

Step 3. Calculate the derivative of the global yield per recruit curve as a function of fishing mortality F as $\frac{\partial YPR(F)}{\partial F}$ and calculate 10% of the slope at the origin where F=0 as $0.1\frac{\partial YPR(0)}{\partial F}$.

Step 4. For each $F_j \in \underline{F}$, calculate the derivative of the yield per recruit function at F_j as $\frac{\partial YPR(F_j)}{\partial F}$ and the difference $\Delta_j = \left| \frac{\partial YPR(F_j)}{\partial F} - 0.1\frac{\partial YPR(0)}{\partial F} \right|$

Step 5. Find the index $k \in \{1, 2, ..., Upper\}$ that produces the smallest difference $\Delta_k \le \Delta_j \; for \; all \; j \in \{1, 2, ..., Upper\}$ and set $F_{0.1} = F_k$. Then set the spawning biomass and fishery yield at $F_{0.1}$ as $SB_{F_{0.1}} = SB_{Eq}(F_k)$ and $Y_{F_{0.1}} = Y_{Eq}(F_k)$.

Calculate F_{Med}, the global *F* that produces 50% of year classes with stock replacement

The reference point F_{Med} is the fishing mortality rate that produces the value of spawning biomass per recruit equal to the inverse of the median of the observed survival ratios for a stock. This is an empirically-derived reference point that depends on the observed time series of recruitment values and the spawning biomasses that produced them. Let $\underline{R} = (R_1, ..., R_T)$ and $\underline{SB} = (SB_1, ..., SB_T)$ be the observed recruitment and female spawning biomass time series where recruitment and female spawning biomass totals are summed over all population and areas.

Then given the global recruitment and spawning biomass time series, one calculates F_{Med} as

Step 1. Calculate the distribution of observed global survival ratios as the vector $\underline{RS}$ where $\underline{RS} = \left(\frac{R_1}{S_1},...,\frac{R_T}{S_T}\right)$, then calculate the median of this distribution $Median\left(\underline{RS}\right)$ and its inverse $\frac{1}{Median\left(\underline{RS}\right)}$.

Step 2. Based on a uniform grid of fishing mortality rates $\underline{F}$ with mesh size δ where $\underline{F} = \left(F_1, F_2, ..., F_{Upper}\right)$ and $F_j = \left(j-1\right)\cdot\delta$ and F_{Upper} is a maximal value of fishing mortality, then for each $F_j \in \underline{F}$, calculate the global spawning biomass per recruit, $SBPR\left(F_j\right)$ and then calculate the difference between $SBPR\left(F_j\right)$ and the inverse of the median of the global survival ratio as

(33) $$\Delta_j = \left| SBPR\left(F_j\right) - \frac{1}{Median\left(\underline{RS}\right)} \right|$$

Step 3. Find the index $k \in \{1,2,...,Upper\}$ that produces the smallest difference $\Delta_k \le \Delta_j \text{ } for \text{ } all \text{ } j \in \{1,2,...,Upper\}$ and set $F_{Med} = F_k$. Then calculate the global equilibrium spawning biomass at F_{Med} as $SB_{Med} = SB_{Eq}\left(F_k\right)$ given the value of $SBPR\left(F_k\right)$.

Appendix 1. Algorithm to Calculate Unfished Equilibrium Numbers at Age

In this Appendix we provide details of an algorithm to iteratively calculate unfished equilibrium numbers at age, conditioned on the existence of an equilibrium solution. That is, we need to calculate unfished numbers at age in equilibrium by population, area, and gender $\left(\widetilde{N}^{(p)}_{U,d,g,a}\right)$ to determine the values of unfished female spawning biomasses by population and area to calculate the resulting unfished recruitment value. Here note that the unfished numbers at age by population, area, and gender depend on the population movement and recruitment distribution matrices and are needed to compute the unfished spawning biomasses by population and area, which in turn, are needed to implement the recruitment submodels by population and area for the initial fished equilibrium time period and assessment time horizon. That is, this algorithm will determine the values of the unfished equilibrium female spawning biomasses for the recruitment submodels by population and area, which are derived quantities that depend on the unfished recruitment parameters by population and area.

Here we provide the formulas to calculate the unfished numbers at age for *P* populations, *D* areas, *A* ages from *a*=1 to the plus group age *A*, and *G* genders. The inputs for this calculation are:

- The *DxD* matrix of recruitment distribution probabilities from area *k* to area *d* for each population *p* denoted by $\underline{\underline{Q}}^{(p)} = \left(Q^{(p)}_{k \to d}\right)_{DxD}$
- The *Gx1* vector of sex ratio by gender for each population *p* denoted by $\rho^{(p)}_{g}$
- The *Ax1* vectors of natural mortality at age and gender vector for each population *p* denoted by $\underline{M}^{(p)}_{g,a}$
- The *Ax1* vectors of mean spawning weight at age and gender vector for each population *p* denoted by $\underline{W}^{(p)}_{S,g,a}$
- The *Ax1* vectors of probability of maturity at age and gender vector for each population *p* denoted by $\underline{P}^{(p)}_{M,g,a}$
- The *DxD* matrices of movement probabilities from area *k* to area *d* by age and gender for each population *p* denoted by $\underline{\underline{T}}^{(p)}_{k \to d,g,a} = \left(T^{(p)}_{k \to d,g,a}\right)_{DxD}$
- The fraction of the year prior to spawning offset for each population *p* denoted as $\Delta^{(p)}_{S}$

Iteration 1: Calculate the initial unfished numbers at age estimates by population, area, and gender based on unfished recruitment, the recruitment distribution, equilibrium natural mortality and no movement, where $(x)^{[k]}$ denotes the k^{th} iterate of an estimate of a quantity *x*.

i. In general, population recruitment by area and gender is a function of area-specific recruitment production and the recruitment distribution matrix $\underline{\underline{Q}}^{(p)}$. Set age-1 recruits as a function of unfished recruitment by area $\left(R_{U,d}^{(p)}\right)$ and recruitment distribution by area and gender via

(27) $$\left(N_{U,d,g,a=1}^{(p)}\right)^{[1]} = \rho_g^{(p)} \sum_k R_{U,k}^{(p)} \cdot Q_{k \to d}^{(p)}$$

ii. Set age-a survivors by area and gender for true ages a=2 to A-1 via

(28) $$\left(N_{U,d,g,a}^{(p)}\right)^{[1]} = \left(N_{U,d,g,a-1}^{(p)}\right)^{[1]} \cdot \exp\left(-M_{g,a-1}^{(p)}\right)$$

iii. Set the survivors of the age-A plus group by area and gender via

(29) $$\left(N_{U,d,g,A}^{(p)}\right)^{[1]} = \frac{\left(N_{U,d,g,A-1}^{(p)}\right)^{[1]} \cdot \exp\left(-M_{g,A-1}^{(p)}\right)}{1-\exp\left(-M_{g,A}^{(p)}\right)}$$

iv. Set unfished spawning biomass by population, area and gender via

(30) $$\left(SB_{U,d,g}^{(p)}\right)^{[1]} = \sum_a P_{M,g,a}^{(p)} \cdot W_{S,g,a}^{(p)} \cdot \left(N_{U,d,g,a}^{(p)}\right)^{[1]} \cdot \exp\left(-\Delta_S^{(p)} \cdot M_{g,a}^{(p)}\right)$$

Iteration 2: Calculate the next iterate of unfished numbers at age estimates by population, area, and gender based on unfished recruitment, the recruitment distribution, equilibrium survival, movement probabilities by area, age, and gender, and the previous iterate. Unfished recruitment production by area is a function of area-specific spawning biomasses which need to be iteratively calculated to account for the movement probabilities that redistribute fish.

i. Set age-1 recruits as a function of the area-specific stock-recruitment submodel $f_d^{(p)}$ and recruitment distribution by area and gender. This step initiates the recruitment dynamics via

(31) $$\left(R_{U,d}^{(p)}\right)^{[2]} = f_d^{(p)}\left(\left(SB_{U,d,g=female}^{(p)}\right)^{[1]} \mid \underline{\theta}_d^{(p)}, \left(SB_{U,d,g=female}^{(p)}\right)^{[1]}\right) \text{ and}$$

$$\left(N_{F,d,g,a=1}^{(p)}\right)^{[2]} = \rho_g^{(p)} \sum_k Q_{k \to d}^{(p)} \cdot \left(R_{U,k}^{(p)}\right)^{[2]}$$

i. Set age-a survivors for ages a=2 to A-1 by population, area, and gender that did not emigrate plus age-a surviving immigrants from other areas via

(32) $$\left(N_{U,d,g,a}^{(p)}\right)^{[2]} = \sum_k \left(N_{U,k,g,a-1}^{(p)}\right)^{[1]} \cdot \exp\left(-M_{g,a-1}^{(p)}\right) \cdot T_{k\to d,g,a-1}^{(p)}$$

ii. Set age-(A-1) survivors that did not emigrate plus age-(A-1) immigrants from other areas plus age-A group survivors that did not emigrate plus age-A group immigrants from other areas via

(33) $$\left(N_{U,d,g,A}^{(p)}\right)^{[2]} = \sum_k \left(N_{U,k,g,A-1}^{(p)}\right)^{[1]} \cdot \exp\left(-M_{g,A-1}^{(p)}\right) \cdot T_{k\to d,g,A-1}^{(p)} + \sum_k \left(N_{U,k,g,A}^{(p)}\right)^{[1]} \cdot \exp\left(-M_{g,A}^{(p)}\right) \cdot T_{k\to d,g,A}^{(p)}$$

iii. Set unfished spawning biomass by population, area and gender via

(34) $$\left(SB_{U,d,g}^{(p)}\right)^{[2]} = \sum_a P_{M,g,a}^{(p)} \cdot W_{S,g,a}^{(p)} \cdot \left(N_{U,d,g,a}^{(p)}\right)^{[2]} \cdot \exp\left(-\Delta_S^{(p)} \cdot M_{g,a}^{(p)}\right)$$

Iteration j+1: Calculate the (j+1)st iterate of equilibrium fished numbers at age estimates by population, area, and gender based on unfished recruitment, the recruitment distribution, equilibrium survival, movement probabilities by area, age, and gender, and the j^{th} iterate.

i. Set age-0 recruits as a function of the recruitment submodel $f_d^{(p)}$ and recruitment distribution by area and gender via

(35) $$\left(R_{U,d}^{(p)}\right)^{[j+1]} = f_d^{(p)}\left(\left(SB_{U,d,g=female}^{(p)}\right)^{[j]} \mid \underline{\theta}_d^{(p)}, \left(SB_{U,d,g=female}^{(p)}\right)^{[j]}\right)$$ and

(36) $$\left(N_{U,d,g,a=1}^{(p)}\right)^{[j+1]} = \rho_g^{(p)} \sum_k Q_{k\to d}^{(p)} \cdot \left(R_{U,k}^{(p)}\right)^{[j]}$$

ii. Set age-a survivors for true ages a=2 to A-1 by population, area, and gender that did not emigrate plus age-a surviving immigrants from other areas via

(37) $$\left(N_{U,d,g,a}^{(p)}\right)^{[j+1]} = \sum_k \left(N_{U,k,g,a-1}^{(p)}\right)^{[j]} \cdot \exp\left(-M_{g,a-1}^{(p)}\right) \cdot T_{k\to d,g,a-1}^{(p)}$$

iii. Set age-(A-1) survivors that did not emigrate plus age-(A-1) immigrants from other areas plus age-A group survivors that did not emigrate plus age-A group immigrants from other areas via

(38) $$\left(N_{U,d,g,A}^{(p)}\right)^{[j+1]} = \sum_k \left(N_{U,k,g,A-1}^{(p)}\right)^{[j]} \cdot \exp\left(-M_{g,A-1}^{(p)}\right) \cdot T_{k\to d,g,A-1}^{(p)} + \sum_k \left(N_{U,k,g,A}^{(p)}\right)^{[j]} \cdot \exp\left(-M_{g,A}^{(p)}\right) \cdot T_{k\to d,g,A}^{(p)}$$

v. Set unfished spawning biomass by population, area and gender via

$$\left(SB_{U,d,g}^{(p)}\right)^{[j+1]} = \sum_{a} P_{M,g,a}^{(p)} \cdot W_{S,g,a}^{(p)} \cdot \left(N_{U,d,g,a}^{(p)}\right)^{[j+1]} \cdot \exp\left(-\Delta_S^{(p)} \cdot M_{g,a}^{(p)}\right) \tag{39}$$

Continue the iterations until the convergence criteria below is achieved or the maximum number of iterations has been reached.

Convergence Criterion

Calculate the distance between successive sets of unfished equilibrium spawning biomass estimates by population, area and gender, denoted by $\Delta^{[j]}$, by applying the L_1, or least absolute deviations norm, to the set of estimates as

$$\Delta^{[j]} = \sum_{p}\sum_{d}\sum_{g}\left|\left(SB_{U,d,g}^{(p)}\right)^{[j+1]} - \left(SB_{U,d,g}^{(p)}\right)^{[j]}\right| \tag{40}$$

Stop the iterations when the set of unfished spawning biomass estimates have converged. That is, stop when $\Delta^{[j]} < \varepsilon$ for a small positive constant $\varepsilon > 0$.

If the iterations converge, then the outputs are the vectors of unfished numbers at age by population, area, and gender $\underline{\widetilde{N}}_{U,d,g}^{(p)} = \left(\widetilde{N}_{U,d,g,a}^{(p)}\right)_{Ax1}$ in equilibrium along with the unfished spawning biomasses by population, area, and gender $\left(\widetilde{SB}_{U,d,g}^{(p)}\right)$ in equilibrium.

Algorithm 2. Calculate Equilibrium Fished Numbers at Age

Similarly, one needs to calculate fished numbers at age in equilibrium prior to the start of the assessment time horizon by population, area, and gender $\left(\widetilde{N}^{(p)}_{F,d,g,a}\right)$ as a function of population recruitment distribution by area, movement probabilities, and the equilibrium total mortality at age. The equilibrium fished numbers at age by population, area, and gender depend on the population movement and recruitment distribution matrices and are needed to compute the fished equilibrium spawning biomasses by population and area, which in turn, are needed to calculate the equilibrium numbers at by population, area, and gender for the initial fished equilibrium time period to the start of the assessment time horizon.

In this Appendix, we provide the formulas to calculate the unfished numbers at age for *P* populations, *D* areas, *A* ages from *a*=1 to the plus group age *A*, and *G* genders. Similar to the unfished equilibrium calculation, the inputs for fished equilibrium calculation are:

- The *DxD* matrix of recruitment distribution probabilities from area *k* to area *d* for each population *p* denoted by $\underline{\underline{Q}}^{(p)} = \left(Q^{(p)}_{k\rightarrow d}\right)_{DxD}$
- The *Gx1* vector of sex ratio by gender for each population *p* denoted by $\rho^{(p)}_{g}$
- The *Ax1* vectors of total mortality at age by area and gender for each population *p* denoted by $\underline{Z}^{(p)}_{F,d,g,a}$
- The *Ax1* vectors of mean spawning weight at age and gender vector for each population *p* denoted by $\underline{W}^{(p)}_{S,g,a}$
- The *Ax1* vectors of probability of maturity at age and gender vector for each population *p* denoted by $\underline{P}^{(p)}_{M,g,a}$
- The *DxD* matrices of movement probabilities from area *k* to area *d* by age and gender for each population *p* denoted by $\underline{\underline{T}}^{(p)}_{k\rightarrow d,g,a} = \left(T^{(p)}_{k\rightarrow d,g,a}\right)_{DxD}$
- The fraction of the year prior to spawning offset for each population *p* denoted as $\Delta^{(p)}_{S}$
- The vectors of unfished numbers at age by population, area, and gender in equilibrium denoted by $\underline{\widetilde{N}}^{(p)}_{U,d,g} = \left(\widetilde{N}^{(p)}_{U,d,g,a}\right)_{Ax1}$
- The unfished spawning biomasses by population, area, and gender in equilibrium denoted by $\left(\widetilde{SB}^{(p)}_{U,d,g}\right)$

Iteration 1: Calculate the initial equilibrium fished numbers at age estimates by population, area, and gender based on unfished recruitment, the recruitment distribution, equilibrium total mortality and no movement, where $(x)^{[j]}$ denotes the j^{th} iterate of an estimate of a quantity x.

i. In general, population recruitment by area and gender is a function of area-specific recruitment production and the recruitment distribution matrix $\underline{\underline{Q}}^{(p)}$. Set the initial age-1 fished recruits as a function of unfished recruitment by and recruitment distribution by area and gender via

$$\left(N_{F,d,g,a=1}^{(p)}\right)^{[1]} \equiv \left(R_{F,d,g}^{(p)}\right)^{[1]} = \sum_{k} Q_{k\to d}^{(p)} \cdot \widetilde{N}_{U,k,g,a=1}^{(p)} \tag{41}$$

ii. Set initial age-a survivors by area and gender for ages a=2 to A-1 from the initial fished recruits and equilibrium total mortality by area and gender via

$$\left(N_{F,d,g,a}^{(p)}\right)^{[1]} = \left(N_{F,d,g,a-1}^{(p)}\right)^{[1]} \cdot \exp\left(-Z_{F,d,g,a-1}^{(p)}\right) \tag{42}$$

iii. Set initial Age-A group of survivors from the initial fished recruits and equilibrium total mortality by area and gender via

$$\left(N_{F,d,g,A}^{(p)}\right)^{[1]} = \frac{\left(N_{F,d,g,A-1}^{(p)}\right)^{[1]} \cdot \exp\left(-Z_{F,d,g,A-1}^{(p)}\right)}{1-\exp\left(-Z_{F,d,g,A}^{(p)}\right)} \tag{43}$$

iv. Set equilibrium fished spawning biomass by population, area and gender via

$$\left(SB_{F,d,g}^{(p)}\right)^{[1]} = \sum_{a} P_{M,g,a}^{(p)} \cdot W_{S,g,a}^{(p)} \cdot \left(N_{F,d,g,a}^{(p)}\right)^{[1]} \cdot \exp\left(-\Delta_{S}^{(p)} \cdot Z_{F,d,g,a}^{(p)}\right) \tag{44}$$

Iteration 2: Calculate the next iterate of the equilibrium fished numbers at age estimates by population, area, and gender based on the recruitment submodel, the recruitment distribution, equilibrium total mortality, movement probabilities by area, age, and gender, and the previous iterate. Equilibrium fished recruitment production by area is a function of area-specific spawning biomasses which need to be iteratively calculated to account for the movement probabilities that redistribute fish.

ii. Set age-1 recruits as a function of the area-specific stock-recruitment submodel $f_d^{(p)}$ and recruitment distribution by area and gender. This step initiates the recruitment dynamics via

$$\left(R_{F,d}^{(p)}\right)^{[2]} = f_d^{(p)}\left(\left(SB_{F,d,g=female}^{(p)}\right)^{[1]} \mid \underline{\theta}_d^{(p)},\ SB_{U,d,g=female}^{(p)}\right) \text{ and} \tag{45}$$

(46) $$\left(N_{F,d,g,a=1}^{(p)}\right)^{[2]} = \rho_g^{(p)} \sum_k Q_{k\to d}^{(p)} \cdot \left(R_{F,k}^{(p)}\right)^{[2]}$$

iii. Set age-a survivors for true ages a=2 to A-1 by population, area, and gender as survivors that did not emigrate plus age-a surviving immigrants from other areas This step turns on the movement dynamics via

(47) $$\left(N_{F,d,g,a}^{(p)}\right)^{[2]} = \sum_k \left(N_{F,k,g,a-1}^{(p)}\right)^{[1]} \cdot \exp\left(-Z_{F,k,g,a-1}^{(p)}\right) \cdot T_{k\to d,g,a-1}^{(p)}$$

iv. Set age-(A-1) survivors that did not emigrate plus age-(A-1) immigrants from other areas plus age-A group survivors that did not emigrate plus age-A group immigrants from other areas via

(48) $$\begin{aligned}\left(N_{F,d,g,A}^{(p)}\right)^{[2]} &= \sum_k \left(N_{F,k,g,A-1}^{(p)}\right)^{[1]} \cdot \exp\left(-Z_{F,k,g,A-1}^{(p)}\right) \cdot T_{k\to d,g,A-1}^{(p)} \\ &\quad + \sum_k \left(N_{F,k,g,A}^{(p)}\right)^{[1]} \cdot \exp\left(-Z_{F,k,g,A}^{(p)}\right) \cdot T_{k\to d,g,A}^{(p)}\end{aligned}$$

iv. Set equilibrium fished spawning biomass by population, area and gender via

(49) $$\left(SB_{F,d,g}^{(p)}\right)^{[2]} = \sum_a P_{M,g,a}^{(p)} \cdot W_{S,g,a}^{(p)} \cdot \left(N_{F,d,g,a}^{(p)}\right)^{[2]} \cdot \exp\left(-\Delta_S \cdot Z_{F,d,g,a}^{(p)}\right)$$

Iteration j+1: Calculate the (j+1)st iterate of equilibrium fished numbers at age estimates by population, area, and gender based on the recruitment submodels and recruitment distribution, equilibrium total mortality, movement probabilities by area, age, and gender, and the j^{th} iterate.

iv. Set age-1 recruits as a function of the recruitment submodel $f_d^{(p)}$ and recruitment distribution by area and gender via

(50) $$\left(R_{F,d}^{(p)}\right)^{[j+1]} = f_d^{(p)}\left(\left(SB_{F,d,g=female}^{(p)}\right)^{[j]} \mid \underline{\theta}_d^{(p)},\ SB_{U,d,g=female}^{(p)}\right) \text{ and}$$

(51) $$\left(N_{F,d,g,a=1}^{(p)}\right)^{[j+1]} = \rho_g^{(p)} \sum_k Q_{k\to d}^{(p)} \cdot \left(R_{F,k}^{(p)}\right)^{[j]}$$

ii. Set age-a survivors for true ages a=2 to A-1 by population, area, and gender that did not emigrate plus age-a surviving immigrants from other areas via

(52) $$\left(N_{F,d,g,a}^{(p)}\right)^{[j+1]} = \sum_k \left(N_{F,k,g,a-1}^{(p)}\right)^{[j]} \cdot \exp\left(-Z_{F,k,g,a-1}^{(p)}\right) \cdot T_{k\to d,g,a-1}^{(p)}$$

iii. Set age-(A-1) survivors that did not emigrate plus age-(A-1) immigrants from other areas plus age-A group survivors that did not emigrate plus age-A group immigrants from other areas via

(53)
$$\left(N_{F,d,g,A}^{(p)}\right)^{[j+1]} = \sum_k \left(N_{F,k,g,A-1}^{(p)}\right)^{[j]} \cdot \exp\left(-Z_{F,k,g,A-1}^{(p)}\right) \cdot T_{k\to d,g,A-1}^{(p)} + \sum_k \left(N_{F,k,g,A}^{(p)}\right)^{[j]} \cdot \exp\left(-Z_{F,k,g,A}^{(p)}\right) \cdot T_{k\to d,g,A}^{(p)}$$

Set unfished spawning biomass by population, area and gender via

(54)
$$\left(SB_{F,d,g}^{(p)}\right)^{[j+1]} = \sum_a P_{M,g,a}^{(p)} \cdot W_{S,g,a}^{(p)} \cdot \left(N_{F,d,g,a}^{(p)}\right)^{[j+1]} \cdot \exp\left(-\Delta_S \cdot Z_{F,d,g,a}^{(p)}\right)$$

Convergence Criterion

Calculate the distance between successive sets of unfished equilibrium spawning biomass estimates by population, area and gender, denoted by $\Delta^{[j]}$, by applying the least absolute deviations norm to the set of estimates as

(55)
$$\Delta^{[j]} = \sum_p \sum_d \sum_g \left| \left(SB_{F,d,g}^{(p)}\right)^{[j+1]} - \left(SB_{F,d,g}^{(p)}\right)^{[j]} \right|$$

Stop the iterations when the set of unfished spawning biomass estimates have converged. That is, stop when $\Delta^{[j]} < \varepsilon$ for a small positive constant $\varepsilon > 0$.

If the algorithm converges, then one has determined the fished numbers at age by population, area, and gender in equilibrium $\left(\widetilde{N}_{F,d,a,g}^{(p)}\right)$ along with the fished spawning biomass by population, area, and gender in equilibrium $\left(\widetilde{SB}_{F,d,g}^{(p)}\right)$. This population-specific information is used to set the initial conditions at the start (first year) of the initialization time period, prior to the stock assessment time horizon. These initial conditions, along with initial numbers-at-age deviation parameters, determine the population dynamics for the initialization time period.