Complex Numbers

IE2107 – Engineering Mathematics II

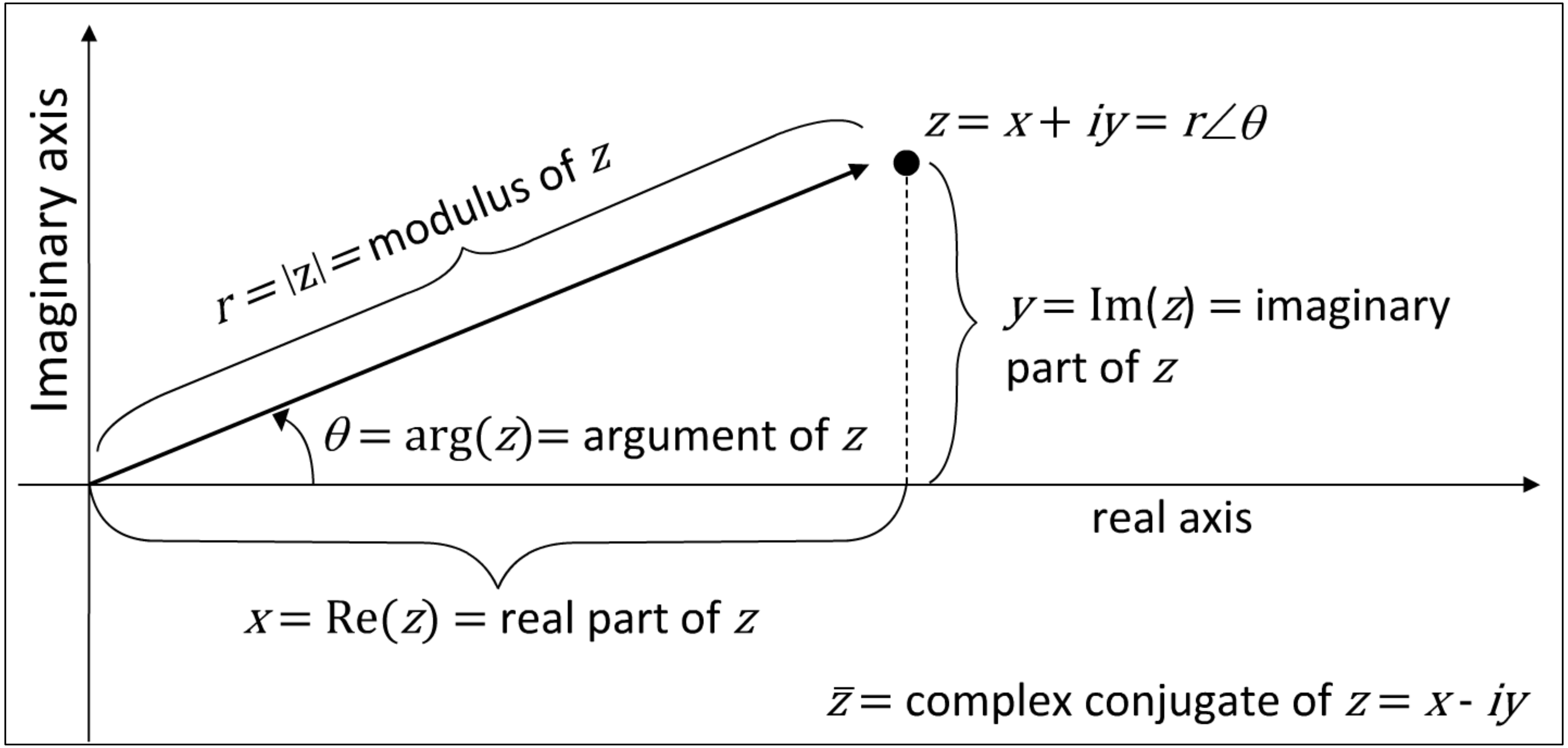
A/P Su Rong

S1-B1b-59, Email: rsu@ntu.edu.sg

At the end of this lesson, you should be able to:

- Define the basics of complex numbers.
- Derive Euler’s Formula and De Moivre’s Formula.
- Derive the complex logarithm and its general power.

A complex number z is defined as $z = x + iy$, where $i = \sqrt{-1}$. Geometrically, a complex number is a point in the complex plane (or the Argand diagram) and can be considered as a vector in the plane. A diagrammatic representation of the complex number is shown below.



Here is an explanation of the equation depicted in the diagram.

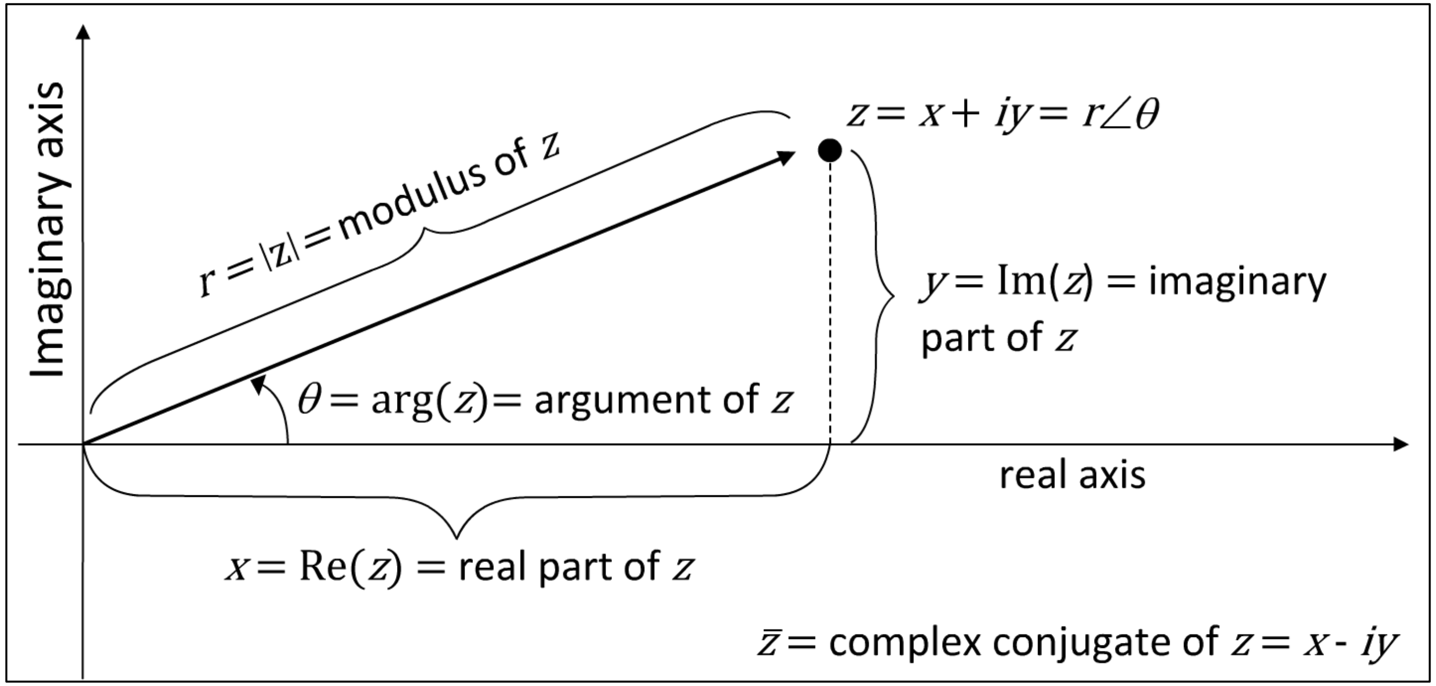
$x = r\cos\theta$, and $y = r\sin\theta$

$$r = |z| = \sqrt{x^2 + y^2} = |\bar{z}| = \sqrt{z\bar{z}}$$

$$\theta = \arg(z) = \arctan\frac{y}{x}\,\text{radians}$$

$$= \mathrm{Arg}(z) + 2n\pi, \qquad n = 0, \pm1, \pm2, \ldots$$

Where, $\mathrm{Arg}(z)$ is the principal value of $\arg(z)$ and satisfies $-\pi < \mathrm{Arg}(z) \leq \pi$



Let us look at an example to understand the concept of complex numbers.

Example 1

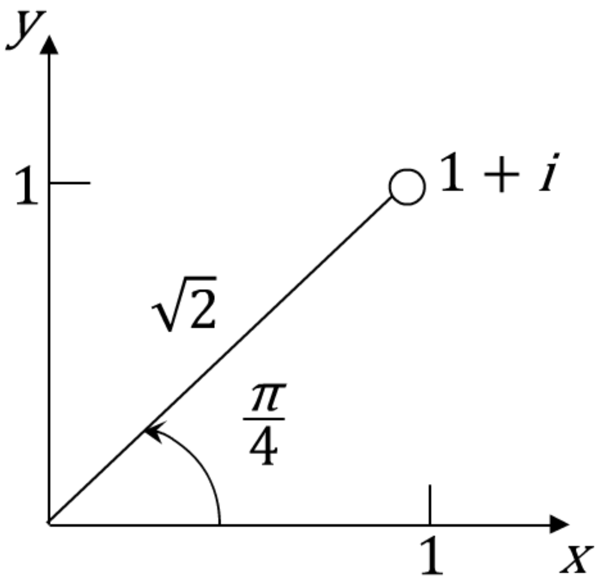
i. Let $z = 1 + i$

Then, $r = |z| = \sqrt{1+1} = \sqrt{2}$

$$\arg z = \arctan\frac{1}{1}$$

$$= \frac{\pi}{4} \pm 2n\pi, n = 0,1,2,\ldots$$

The principal value of the argument is $\frac{\pi}{4}$.



ii. If $z = 1 - i$, then $\arg z = \arctan\frac{-1}{1} = \frac{-\pi}{4} \pm 2n\pi, n = 0,1,2,\ldots$

The principal value of the argument is $\frac{-\pi}{4}$.

From Euler's formula, it can be found that:

$$e^{i\theta} = \cos\theta + i\sin\theta \text{ and } e^{-i\theta} = \cos\theta - i\sin\theta$$

Thus, $\boxed{\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}}$ and $\boxed{\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}}$

From Euler's formula, $e^{i\theta} = \cos\theta + i\sin\theta$, for any real value of θ, the polar form of a complex number can be written as $z = re^{i\theta} = r\angle\theta$.

Let us now look at some Algebraic Rules.

Let $z_1 = x_1 + iy_1 = r_1\angle\theta_1$ and $z_2 = x_2 + iy_2 = r_2\angle\theta_2$

Addition and subtraction $z_1 \pm z_2 = (x_1 \pm x_2) + i(y_1 \pm y_2)$

Multiplication $z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$

Division $\dfrac{z_1}{z_2} = \dfrac{x_1 x_2 + y_1 y_2}{x^2 + y^2} + i\,\dfrac{x_2 y_1 - x_1 y_2}{x^2 + y^2}$

Let us now look at some Algebraic Rules.

Let $z_1 = x_1 + iy_1 = r_1\angle\theta_1$ and $z_2 = x_2 + iy_2 = r_2\angle\theta_2$

Add

It is sometimes more convenient to do multiplication and division in the polar form.

$$z_1 z_2 = r_1 r_2 \angle(\theta_1 + \theta_2),$$

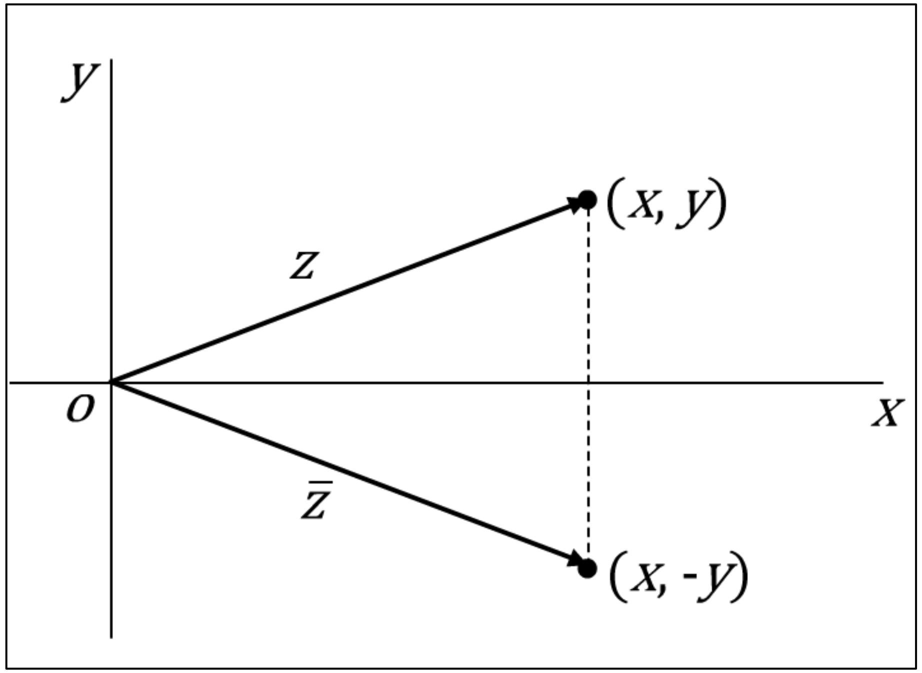
$$\frac{z_1}{z_2} = \frac{r_1\angle\theta_1}{r_2\angle\theta_2} = \frac{r_1}{r_2}\angle(\theta_1 - \theta_2)$$

Division

Let us now understand the complex conjugate of z and its algebraic rules.

In the given equation $z = x + iy$, the complex conjugate of z is defined as $\bar{z} = x - iy$.

Thus, it can be written as:



$$Re(z) = \frac{1}{2}(z + \bar{z}),\ Im(z) = \frac{1}{2i}(z - \bar{z})$$

$$z\bar{z} = x^2 + y^2 = |z|^2,\ \frac{z_1}{z_2} = \frac{z_1\bar{z}_2}{|z_2|^2}$$

$$(\overline{z_1 \pm z_2}) = \bar{z}_1 \pm \bar{z}_2,\ \overline{z_1 z_2} = \bar{z}_1\bar{z}_2,\ \overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$$

Here is the derivation of the De Moivre's formula.

- Let $z = x + iy = r(\cos\theta + i\sin\theta) = r\angle\theta$

- Then, for any integer n,

$$z^n = r^n(\cos\theta + i\sin\theta)^n$$

$$z^n = \underbrace{z.z\ldots.z}_{n} = \underbrace{r.r\ldots.r}_{n}\angle\underbrace{(\theta + \theta + \cdots + \theta)}_{n} = r^n\angle(n\theta)$$

$$= r^n(\cos n\theta + i\sin n\theta)$$

- From the above equation, the De Moivre's formula can be expressed as:
$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ which is useful in deriving certain trigonometric identities.

Let us look at a sample problem to understand the concept of complex numbers.

Sample Problem 1

Find identities for $\cos 2\theta$ and $\sin 2\theta$.

Solution:

$$(\cos\theta + i\sin\theta)^2 = \cos^2\theta - \sin^2\theta + 2i\cos\theta\sin\theta$$

$$= \cos 2\theta + i\sin 2\theta$$

Therefore,

$$\cos 2\theta = \cos^2\theta - \sin^2\theta \text{ and } \sin 2\theta = 2\cos\theta\sin\theta$$

Let us look at another sample problem explaining the concept of complex numbers.

Sample Problem 2

Express $\cos^4\theta$ in terms of multiples of θ.

Solution:

Since $2\cos\theta = e^{i\theta} + e^{-i\theta}$

$$2^4\cos^4\theta = (e^{i\theta} + e^{-i\theta})^4$$

$$= \left(e^{i4\theta} + e^{-i4\theta}\right) + 4\left(e^{i2\theta} + e^{-i2\theta}\right) + 6$$

$$= 2\cos 4\theta + 8\cos 2\theta + 6$$

$$\Rightarrow \cos^4\theta = \frac{1}{8}[\cos 4\theta + 4\cos 2\theta + 3]$$

Consider $z = w^n, n = 1,2, \dots$

For a given $z \neq 0$, the solution of w in the above equation is called the n^{th} root of z and is denoted by $w = \sqrt[n]{z}$.

First, $z = r\angle(\theta + 2k\pi)$.

Next, let $w = R\angle\phi$.

Then, $z = w^n$ gives

$$r\angle(\theta + 2k\pi) = R^n\angle(n\phi).$$

Thus, $R = \sqrt[n]{r}$, and

$$\phi = \frac{\theta + 2k\pi}{n}, \mathrm{k} = 0,1, \dots, (n-1).$$

Consider $z = w^n, n = 1,2, \ldots$

For a given $z \neq$ (
the above equat
root of z and is c

Then, $z = w^n$ g
$r\angle(\theta + 2k\pi) =$

$2k\pi)$.

ϕ.

Thus, $R = \sqrt[n]{r}$, and

$$\phi = \frac{\theta + 2k\pi}{n}, \mathrm{k} = 0,1, \ldots, (n-1).$$

To summarise,

$$w_k = \sqrt[n]{z} = \sqrt[n]{r}\angle\left(\frac{\theta + 2k\pi}{n}\right),$$
$$k = 0,1, \ldots., (n-1)$$

Geometrically, the entire set of roots lies at the vertices of a regular polygon of n sides inscribed in a circle of radius $\sqrt[n]{r}$.

Let us look at an example to understand the concept of roots of complex numbers.

Example 2

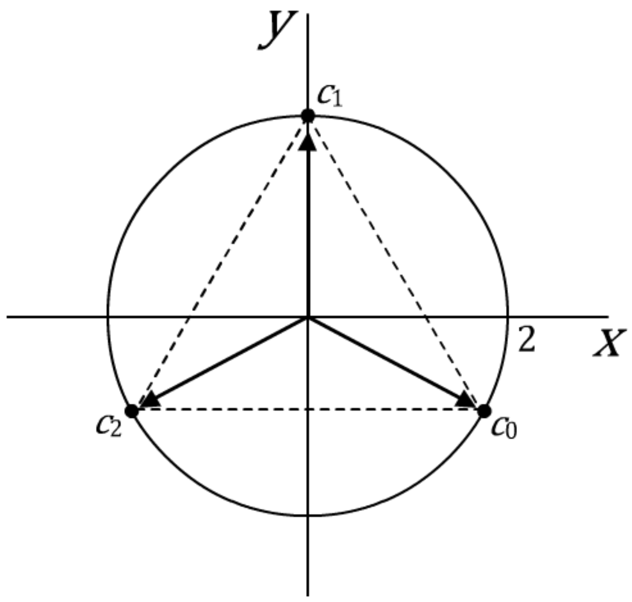
Let us find all values of $(-8i)^{1/3}$, that is, $\sqrt[3]{-8i}$.

First,

$$-8i = 8\angle\left(\frac{-\pi}{2} + 2k\pi\right), k = 0, \pm1, \pm2, \ldots$$

The desired roots are:

$$w_k = 2\angle\left(\frac{-\pi}{6} + \frac{2k\pi}{3}\right), k = 0,1,2$$

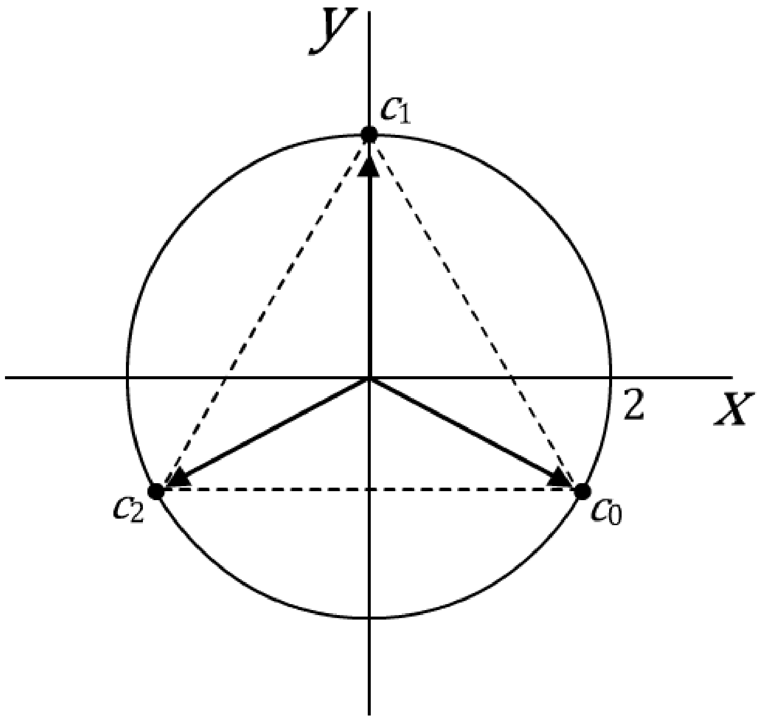


Let us look at an example to understand the concept of roots of complex numbers.

Example 2 (contd.)

The roots lie at the vertices of an equilateral triangle, inscribed in the circle $|z| = 2$ and are equally spaced around that circle every ${}^{2\pi}/_{3}$ radians, starting with the principal root

$$w_0 = 2\angle\left(\frac{-\pi}{6}\right) = \sqrt{3} - i.$$



NANYANG TECHNOLOGICAL UNIVERSITY

Let us now define the exponential function.

The exponential function e^z is defined as:

$$e^z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n = e^x \ (\cos y + i \sin y).$$

If $x = 0$, then the Euler formula becomes: $e^{iy} = \cos y + i \sin y$.

Hence, the polar form of a complex number may be written as

$z = r(\cos\theta + i\sin\theta) = re^{i\theta}$.

If $z = e^{ix} = \cos x + i \sin x$, then

$$\sin x = \frac{1}{2i}\left(e^{ix} - e^{-ix}\right) = \frac{1}{2i}(z - \bar{z}),$$

$$\cos x = \frac{1}{2}\left(e^{ix} + e^{-ix}\right) = \frac{1}{2}(z + \bar{z}).$$

It is also geometrically obvious that $e^{i\pi} = -1, e^{-i\pi/2} = -i$ and $e^{-i4\pi} = 1$.

The natural logarithm of $z = x + iy$ is denoted by $\ln z$ and is defined as the inverse of the exponential function.

Since, $w = \ln z$ is defined for $z \neq 0$ by the relation $e^w = z$.

So, if $z = re^{i\theta}, r > 0$, then $\ln z = \ln r + i\theta$.

Note that the complex logarithm is infinitely many-valued.

The general power of a complex number, z^c, can be derived as follows:

Let $y = z^c, \Rightarrow \ln y = c\ln z, \Rightarrow y = z^c = e^{c\ln z}, z \neq 0$.

Let us look at a sample problem to understand the concept of complex logarithm.

Sample Problem 3

i) Evaluate $\ln(3-4i)$.

ii) Solve $\ln z = -2 - \frac{3}{2}i$.

Solution:

i) $\ln(3-4i) = \ln|3-4i| + i \arg(3-4i)$

$$= 1.609 - i(0.927 \pm 2n\pi), n = 0,1,\ldots$$

Principal value: When $n = 0$

ii) $z = e^{-2-\frac{3}{2}i} = e^{-2}e^{-i\frac{3}{2}} = e^{-2}\left(\cos\frac{3}{2} - i\sin\frac{3}{2}\right)$

$$= 0.010 - i\,0.135$$

Here is another sample problem explaining the concept of complex logarithm.

Sample Problem 4

Find the principal value of $(1+i)^i$.

Solution:

Let $y=(1+i)^i$. Then, $\ln y = i\ln(1+i)$, or $y=e^{i\ln(1+i)}$

Hence, $(1+i)^i = e^{i\ln(1+i)}$

But, $\ln(1+i) = \ln\left(\sqrt{2}e^{i(\pi/4+2k\pi)}\right)$

$$= \ln\sqrt{2} + i\left({}^{\pi}/_{4} + 2k\pi\right), k = 0, \pm 1, \ldots$$

and the principal value is when $k=0$.

Therefore, $e^{i\ln(1+i)} = e^{i(\ln\sqrt{2}+i\pi/4)} = e^{-\frac{\pi}{4}+i(\ln\sqrt{2})}$

Summary

Key points discussed in this lesson:

- A complex number z is defined as $z = x + iy$, where $i = \sqrt{-1}$. Geometrically, a complex number is a point in the complex plane (or the Argand diagram) and can be considered as a vector in the plane.

- In the given complex number $z = x + iy$, the complex conjugate of z is defined as $\bar{z} = x - iy$.

- From Euler's Formula $e^{i\theta} = \cos\theta + i\sin\theta$, and $e^{-i\theta} = \cos\theta - i\sin\theta$. Then, $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$ and $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$.

Key points discussed in this lesson:

- For complex number $z = x + iy = r(\cos\theta + i\sin\theta) = r\angle\theta$. The De Moivre's formula is given as: $(\cos\theta + i\sin\theta)^n = \cos n\theta + i \sin n\theta$.
- The exponential function e^z is defined as: $e^z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n = e^x\,(\cos y + i \sin y)$.
- The natural logarithm of $z = x + iy$ is denoted by $\ln z$ and is defined as the inverse of the exponential function.