Complex Integration

IE2107 – Engineering Mathematics II

At the end of this lesson, you should be able to:

- Explain the line integrals of complex functions.
- Explain Cauchy’s Integral Theorem and Cauchy’s Integral Formula.

1. A **real definite integral** $\int_a^b f(x)dx$ means that the function $f(x)$ is integrated along the x-axis from a to b, and the integrand $f(x)$ is defined for each point between a and b.

2. In the case of a **complex definite integral**, or **line integral**, $\int_C f(z)dz$ means that the integration is done along the curve C (in a given direction) in the complex plane and the integrand $f(z)$ is defined for each point on C. 'C' is called the **contour** or **path of integration**.

3. If C is a closed contour, the complex line integral is sometimes denoted by $\oint_C f(z)dz$.

4. If C is on the real axis, then, $z = x$, and the complex line integral becomes a real definite integral.

A contour or path of integration on the complex plane can be represented in the following form.

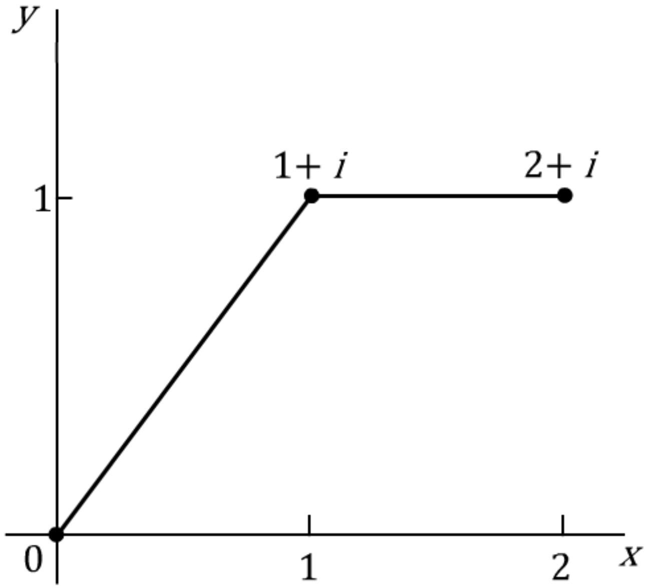
$$z(t) = x(t) + iy(t), a \leq t \leq b$$

Where, t is the real parameter.
This establishes a continuous mapping of the interval $a \leq t \leq b$ into the xy-plane or the z-plane, and the direction of the path is according to the increasing values of t.

For example,

The path in the figure on the right can be represented by:

$$z = \begin{cases} x + ix, 0 \leq x \leq 1 \\ x + i, 1 \leq x \leq 2 \end{cases}$$



Let us take a look at the following sample problem to understand the concept of line integral.

Sample Problem 1

Evaluate $\int_C \bar{z}dz$, where C is given by:

$$x = 3t,\ y = t^2, -1 \leq t \leq 4$$

Solution:

As $z = x + iy$, $z(t) = 3t + it^2$, and $dz(t) = (3 + i2t)dt$

Therefore,

$$\int_C \bar{z}dz = \int_{-1}^{4} (3t - it^2)(3 + i2t)dt$$

$$= \int_{-1}^{4} (2t^3 + 9t)dt + i\int_{-1}^{4} 3t^2 dt = 195 + i65$$

Let us take a look at another sample problem to understand the concept of line integral.

Sample Problem 2

Evaluate $\oint_C \frac{1}{z}\,dz$, where C is the unit circle in the complex plane, counter-clockwise.

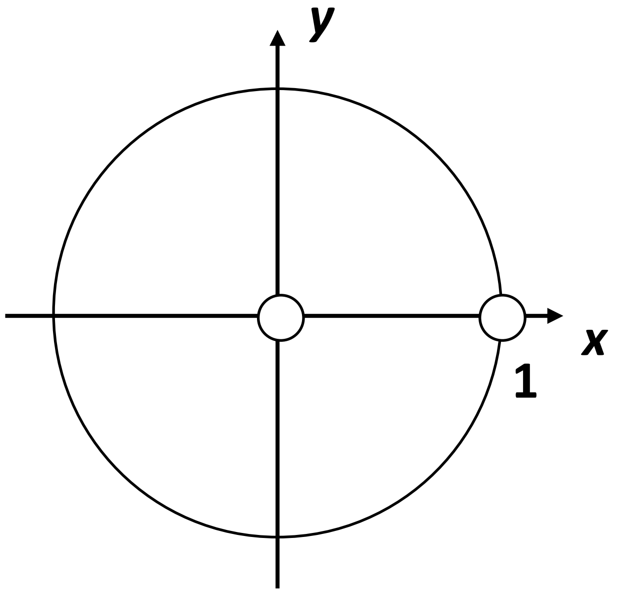
Solution:

The path C can be represented by:

$z(t) = \cos t + i\sin t = e^{it},\ 0 \le t \le 2\pi$

And, $dz(t) = ie^{it}dt = izdt$

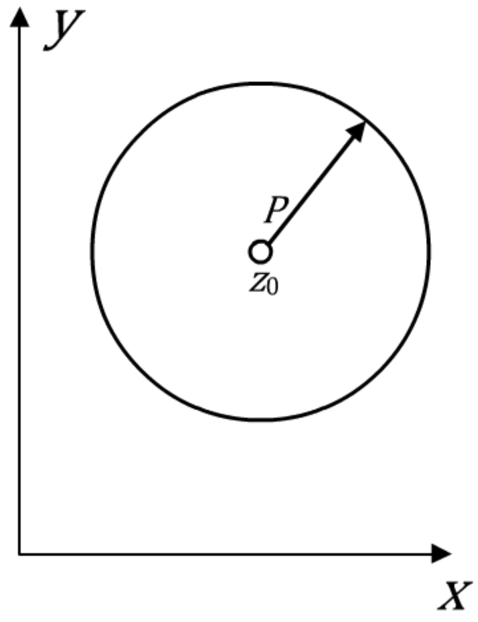
Hence, $\oint_C \frac{1}{z}\,dz = i\int_0^{2\pi} dt = 2\pi i$



Here is another sample problem explaining the concept of line integral.

Sample Problem 3

Evaluate $\int_C (z - z_0)^m dz$, where C is a counter-clockwise circle of radius ρ with centre at z_0.



Solution:

The path is represented as:

$$z(\theta) = z_0 + \rho e^{i\theta}, 0 \leq \theta \leq 2\pi$$

Then,

$$(z - z_0)^m = \rho^m e^{im\theta} \text{ and } dz = i\rho e^{i\theta} d\theta$$

Here is another sample problem explaining the concept of line integral.

Solution (contd.):

Hence,

$$\int_C (z - z_0)^m dz = \int_0^{2\pi} \rho^m e^{im\theta} i\rho e^{i\theta} d\theta$$

$$= i\rho^{m+1} \int_0^{2\pi} e^{i(m+1)\theta} d\theta$$

$$= \begin{cases} 2\pi i & m = -1 \\ 0 & m \neq -1, m \text{ integer} \end{cases}$$

The following theorem provides a practical method to evaluate a complex line integral.

Theorem 1: Let C be a piecewise smooth path, represented by $z = z(t)$, where $a \leq t \leq b$. Let $f(z)$ be a continuous function on C. Then,

$$\boxed{\int_C f(z)dz = \int_a^b f[z(t)]\frac{dz}{dt}dt}$$

There are three basic properties of complex line integrals.

1. **Linearity** $\int_C [k_1 f_1(z) + k_2 f_2(z)]dz = k_1 \int_C f_1(z)dz + k_2 \int_C f_2(z)dz$

2. **Subdivision of Path** $\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz$

3. **Sense of Integration** $\int_{z_1}^{z_2} f(z)dz = -\int_{z_2}^{z_1} f(z)dz$

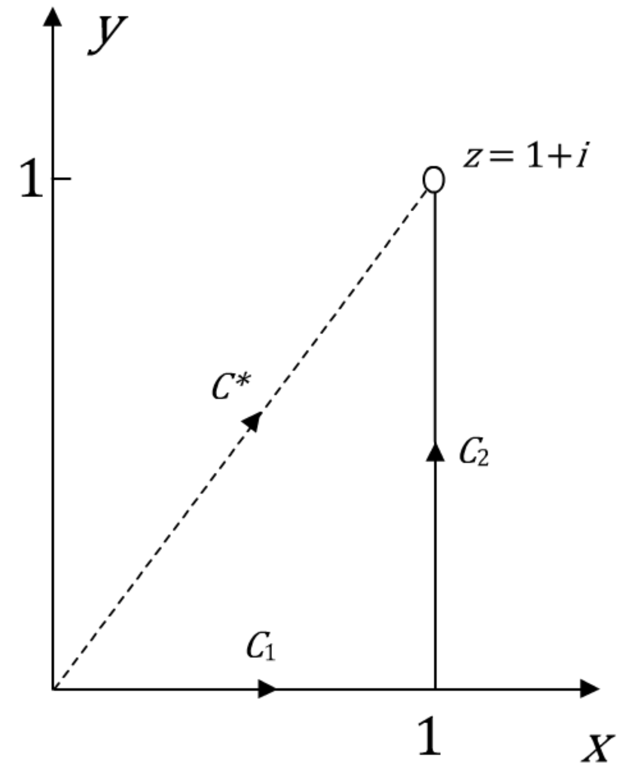
The given sample problem demonstrates the use of the basic properties of line integrals.

Sample Problem 4a

Evaluate $\int_0^{1+i} \mathrm{Re}\, z\, dz$ along:

(A) C^*

(B) C_1 and C_2



The given sample problem demonstrates the use of the basic properties of line integrals.

Solution:

(A) Along C^*, z is represented by:

$$z(t) = t + it, 0 \leq t \leq 1$$

Which gives, $dz = (1 + i)dt$

Hence,

$$\int_0^{1+i} \mathrm{Re}\, z\, dz = \int_0^1 t(1 + i)dt = \frac{1}{2}(1 + i)$$

The given sample problem demonstrates the use of the basic properties of line integrals.

Solution (contd.):

(B) Along C_1: $z(t) = t, 0 \leq t \leq 1$ and $d(z) = dt$

Along C_2: $z(t) = 1 + it, 0 \leq t \leq 1$ and $d(z) = idt$

Hence,

$$\int_0^{1+i} \mathrm{Re}\, z\, dz = \int_{C_1} + \int_{C_2} = \int_0^1 t\, dt + \int_0^1 1 \cdot i\, dt = \frac{1}{2} + i$$

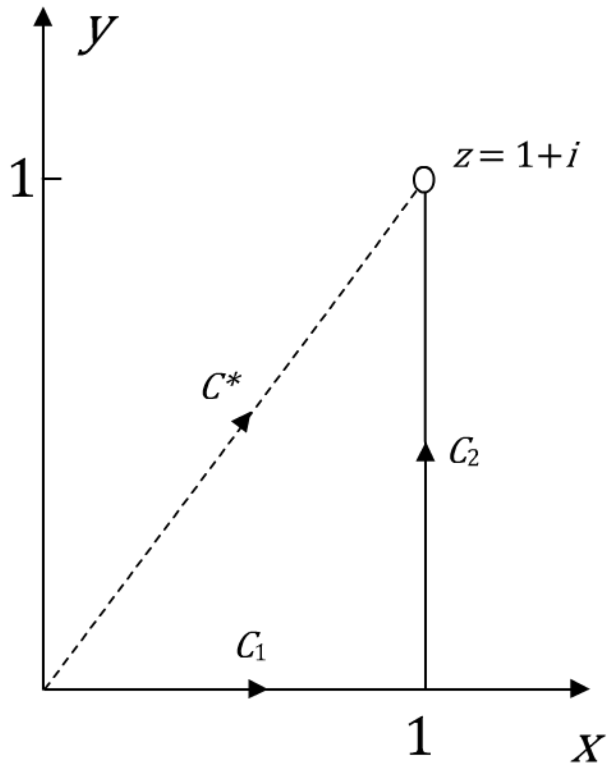
The given sample problem demonstrates the use of the basic properties of line integrals.

Sample Problem 4b

Evaluate $\int_0^{1+i} z\,dz$ along:

(A) C^*

(B) C_1 and C_2



The given sample problem demonstrates the use of the basic properties of line integrals.

Solution:

(A) Along C^*, z is represented by:

$$z(t) = t + it, 0 \leq t \leq 1$$

Which gives, $dz = (1 + i)dt$

Hence,

$$\int_0^{1+i} z\, dz = \int_0^1 (t + it)(1 + i)dt$$

$$= \int_0^1 (t - t + i2t)dt = it^2\Big|_0^1 = i$$

The given sample problem demonstrates the use of the basic properties of line integrals.

Solution (contd.):

(B) Along C_1: $z(t) = t, 0 \le t \le 1$ and $d(z) = dt$

Along C_2: $z(t) = 1 + it, 0 \le t \le 1$ and $d(z) = idt$

Hence,

$$\int_0^{1+i} z\,dz = \int_{C_1} + \int_{C_2} = \int_0^1 t\,dt + \int_0^1 (1+it)\cdot idt = i$$

Here is another sample problem demonstrating the use of the basic properties of line integrals.

Sample Problem 5

Evaluate $\int_C \bar{z} dz$ from $z = 0$ to $z = 4 + 2i$ along the curve given by the line $z = 0$ to $z = 2i$ and then the line from $z = 2i$ to $z = 4 + 2i$.

Solution:

Along $z = 0$ to $z = 2i$:

$$z(t) = 0 + it,\ 0 \le t \le 2 \text{ and } d(z) = idt$$

Along $z = 2i$ to $z = 4 + 2i$:

$$z(t) = t + 2i,\ 0 \le t \le 4 \text{ and } d(z) = dt$$

Here is another sample problem demonstrating the use of the basic properties of line integrals.

Solution (contd.):

$$\int_C \bar{z}dz = \int_0^2 tdt + \int_0^4 (t-2i)dt$$

$$= 2 + \int_0^4 tdt - 2i\int_0^4 dt$$

$$= 2 + \left[\frac{t^2}{2}\right]_0^4 - 8i$$

$$= 2 + 8 - 8i = 10 - 8i$$

Here is another sample problem demonstrating the use of the basic properties of line integrals.

Sample Problem 6

Evaluate $\int_C \bar{z} dz$ from $z = 0$ to $z = 4 + 2i$, where C is a parabola given by $x = y^2$.

Solution:

$$z(t) = t^2 + it,\ 0 \leq t \leq 2 \text{ and } d(z) = (2t + i)dt$$

$$\int_C \bar{z} dz = \int_0^2 (t^2 - it)(2t + i)dt$$

$$= \int_0^2 (2t^3 - it^2 + t)dt = 10 - {}^{8}\!/_{3}\, i$$

Complex Integration > Simple Closed Path and Simply Connected Domain

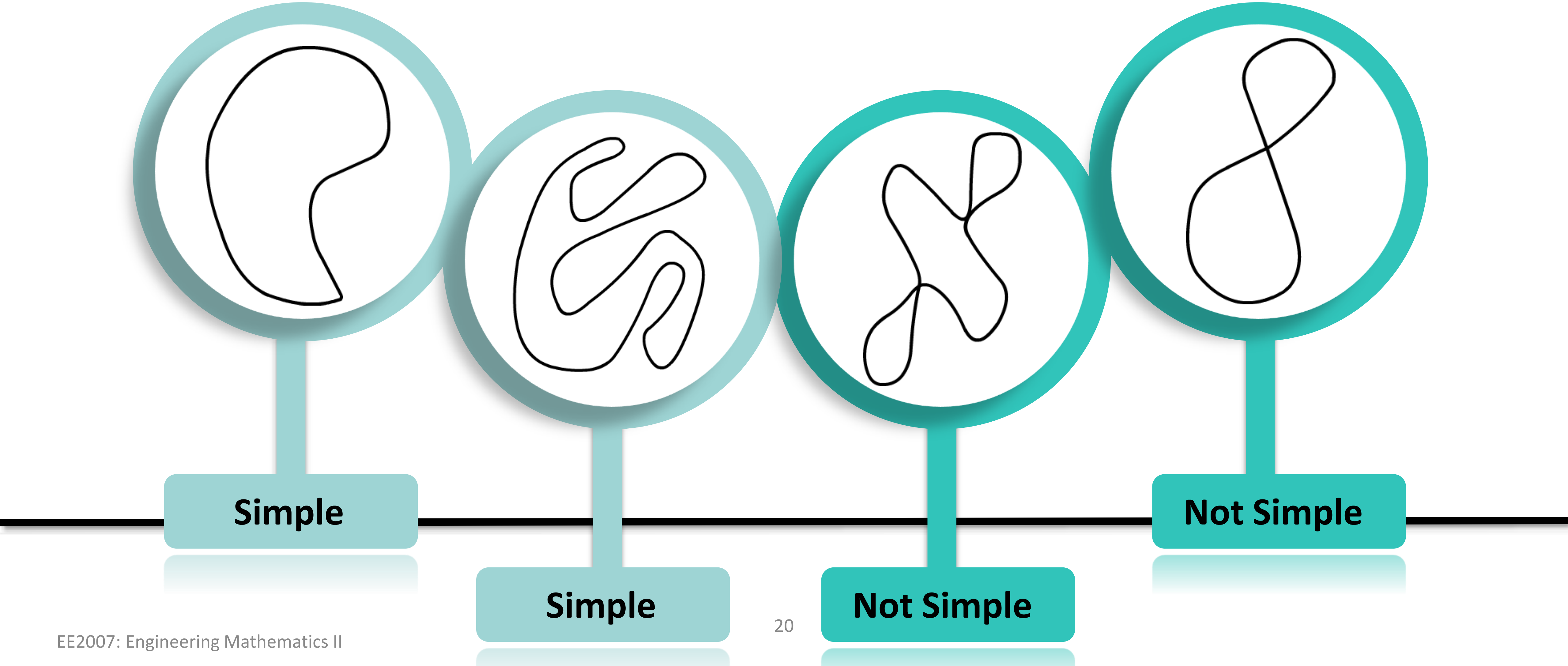
A line integral of $f(z)$ in the complex plane may not always depend on the choice of the path itself. Sometimes, the integrals evaluated turn out to be zero or $2\pi i$.

1. Under what condition will the integral be independent of the path?
2. Under what condition will the integral be zero?
3. Is there something special about the value $2\pi i$?

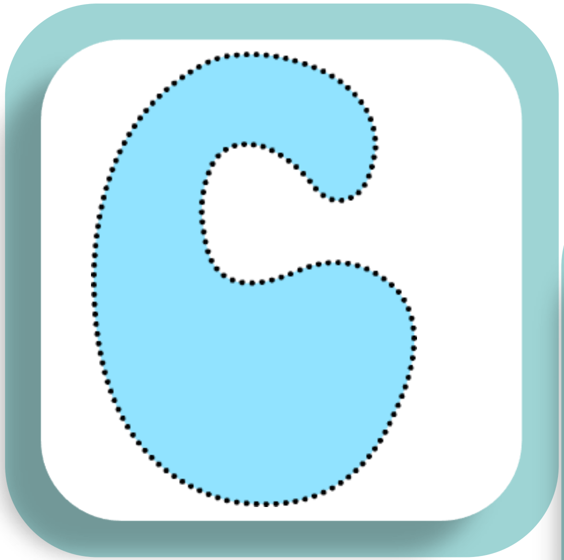
To answer these questions, you need to know about the:

- Concept of Simple Closed Path and Simply Connected Domain
- Cauchy's Integral Theorem

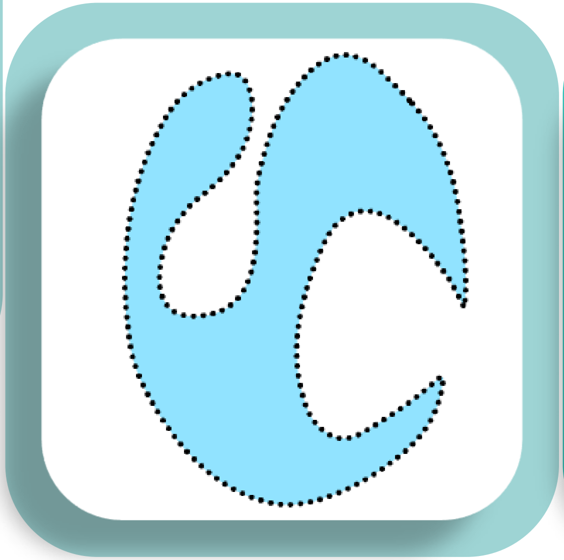
A simple closed path is a path that does not intersect or touch itself.



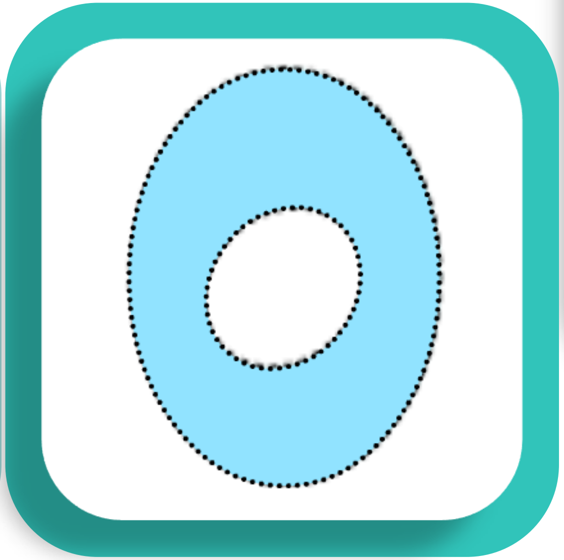
A simply connected domain D in the complex plane is a domain such that every simple closed path in D encloses only points of D.



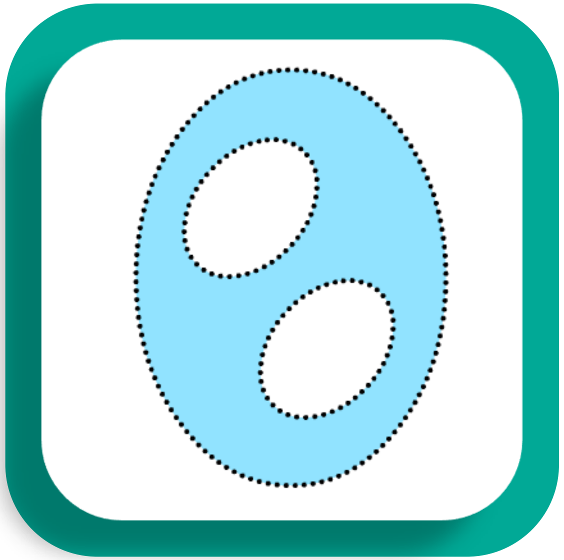
Simply Connected



Simply Connected

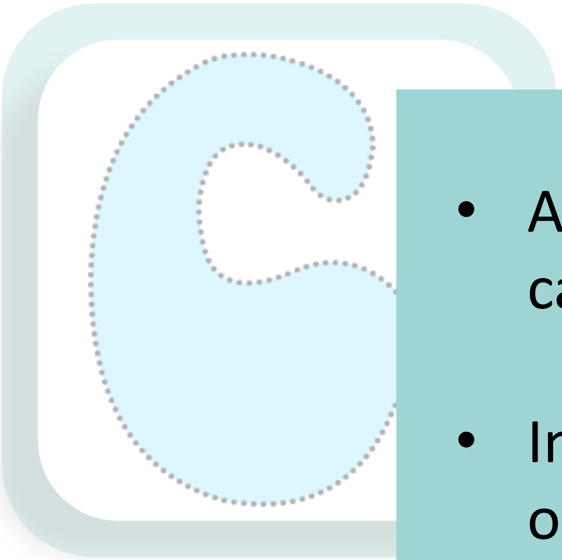


Doubly Connected



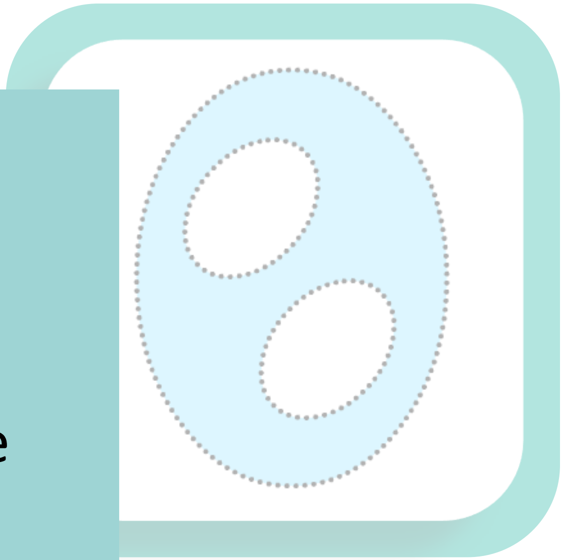
Triply Connected

A simply connected domain D in the complex plane is a domain such that every simple closed path in D encloses only points of D.



Simply Connected

- A domain that is not simply connected is called **multiply connected**.
- Intuitively, a simply connected domain is the one which does not have any ‘holes’ in it, while a multiply connected domain is the one which does.



Triply Connected

Simply Connected

Doubly Connected

Cauchy's Integral Theorem is an important theorem describing the line integrals of analytic functions in a complex plane.

Theorem 2: If $f(z)$ is analytic in a simply connected domain D, then for every simple closed path C in D,

$$\boxed{\int_C f(z)dz = 0}$$

For example, $\int_C e^z dz = 0$, $\int_C \cos z\, dz = 0$, and $\int_C z^n dz = 0; n = 0, 1, \ldots$

for any closed path as these functions are **entire**, that is, analytic for all z.

And, $\int_C \frac{1}{z^2+4} dz = 0$ where, C is a unit circle.

Although the integrand is not analytic at $z = \pm 2i$, these points are not enclosed by C.

Let us try to understand the condition under which the line integral of a complex function would be independent of the path of integration.

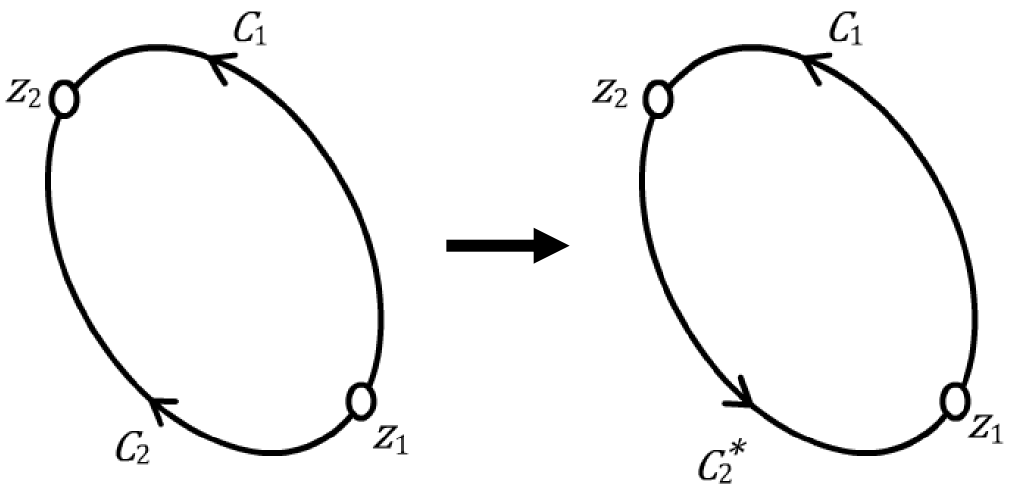
Theorem 3: If $f(z)$ is analytic in a simply connected domain D, then the integral of $f(z)$ is independent of the path in D.

Proof: Let z_1 and z_2 be any two points in D. Consider two paths C_1 and C_2 in D from z_1 to z_2 as shown. Let us reverse the direction of the path C_2 and denote it by C_2^*.
Now, according to Cauchy's theorem,

$$\int_{C_1} f dz + \int_{C_2^*} f dz = 0$$

Thus,

$$\int_{C_1} f dz = -\int_{C_2^*} f dz = \int_{C_2} f dz$$



The most important consequence of Cauchy's Integral Theorem is Cauchy's integral formula. This formula is useful to evaluate integrals of the following form.

$$\int_C \frac{f(z)}{(z-z_0)^m}\,dz \qquad \text{where, } m = 1, 2, 3, \dots$$

Theorem 4: Let $f(z)$ be analytic in a simply connected domain D. Then, for any point z_0 in D and any simple closed path C in D that encloses z_0.

$$\int_C \frac{f(z)}{(z-z_0)}\,dz = 2\pi i f(z_0)$$

In general,

$$\int_C \frac{f(z)}{(z-z_0)^m}\,dz = \frac{2\pi i}{(m-1)!} f^{(m-1)}(z_0) \quad \text{where, } m = 1, 2, 3, \dots$$

Note: The integration is being taken counter-clockwise. Refer to the textbook for the proof of the theorem.

The following sample problem shows how Cauchy's integral formula is used to solve complex line integrals.

Sample Problem 7

Evaluate $\int_C \dfrac{e^z}{(z-2)}\,dz$

Solution:

$$\int_C \frac{e^z}{(z-2)}\,dz = 2\pi i e^z\Big|_{z=2}$$

$$= 2\pi i e^2$$

The following sample problem shows how Cauchy's integral formula is used to solve complex line integrals.

Sample Problem 8

Evaluate $\int_C \frac{z^3-6}{2z-i}\,dz$, where C is a unit circle in counter- clockwise direction.

Solution:

Since C encloses $z = \frac{i}{2}$

$$\int_C = \int_C \frac{z^3-6}{2(z-i/2)}\,dz = \pi i(z^3-6)\Big|_{z=i/2} = \pi i\left[\frac{-i}{8}-6\right]$$

The following sample problem shows how Cauchy's integral formula is used to solve complex line integrals.

Sample Problem 9

Evaluate $\int_C \frac{\cos z}{(z-\pi i)^2}\,dz$, where C is any contour enclosing $z=\pi i$ in counter-clockwise direction.

Solution:

$$\int_C \frac{\cos z}{(z-\pi i)^2}\,dz = 2\pi i \frac{d}{dz}\cos z\Big|_{z=\pi i} = -2\pi i \sin(\pi i)$$

The following sample problem shows how Cauchy's integral formula is used to solve complex line integrals.

Sample Problem 10

Evaluate $\int_C \frac{z^4 - 3z^2 + 6}{(z+i)^3} dz$, where C is any contour enclosing $z = -i$ in counter-clockwise direction.

Solution:

$$\int_C = \frac{2\pi i}{2!} \frac{d^2}{dz^2}(z^4 - 3z^2 + 6)\Big|_{z=-i} = \pi i(12z^2 - 6)\Big|_{z=-i} = -18\pi i$$

The following sample problem shows how Cauchy's integral formula is used to solve complex line integrals.

Sample Problem 11

Evaluate $\int_C \frac{1}{z^2+1} dz; C: |z| = 3$, in counter-clockwise direction.

Solution:

The integrand is not analytic at $z = \pm i$ which are inside C. Cauchy's formula applies to only one singular point inside C. Therefore, use partial fraction decomposition and apply Cauchy's formula.

The following sample problem shows how Cauchy's integral formula is used to solve complex line integrals.

Solution (contd.):

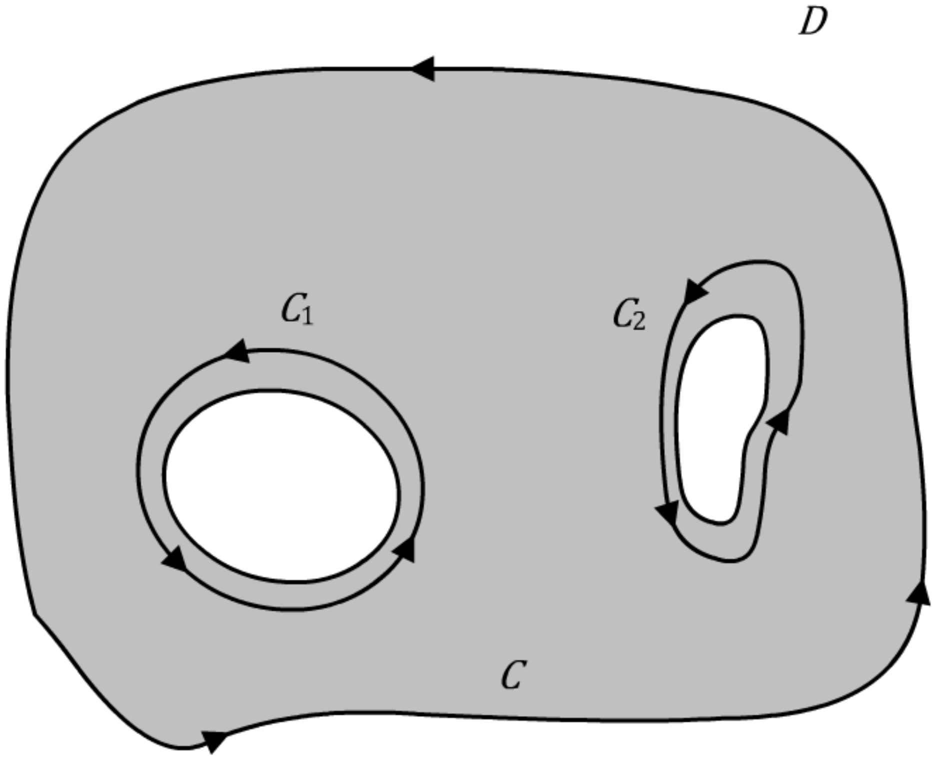
$$\oint_C \frac{dz}{z^2+1} = \oint_C \frac{dz}{(z+i)(z-i)}$$

$$= \frac{1}{2i}\oint_C \left[\frac{1}{z-i} - \frac{1}{z+i}\right] dz$$

$$= \frac{1}{2i}[2\pi i - 2\pi i] = 0$$

Suppose $C, C_1, C_2, \ldots, C_n$ are simple closed curves with a positive orientation such that $C_1, C_2, \ldots, C_n$ are interior to C. However, regions interior to C_k, where $k = 1, 2, \ldots, n$, have no points in common with each other. Now, if f is analytic on each contour and at each point interior to C but exterior to all the C_k, then,

$$\oint_C f(z)dz = \sum_{k=1}^{n} \oint_{C_k} f(z)\, dz$$



Here is a sample problem showcasing how Cauchy's theorem for multiply connected domains can be used to integrate complex functions.

Sample Problem 12

Evaluate $\oint_C \frac{1}{z^2+1} dz$; where C is the circle $|z| = 3$ in counter-clockwise direction.

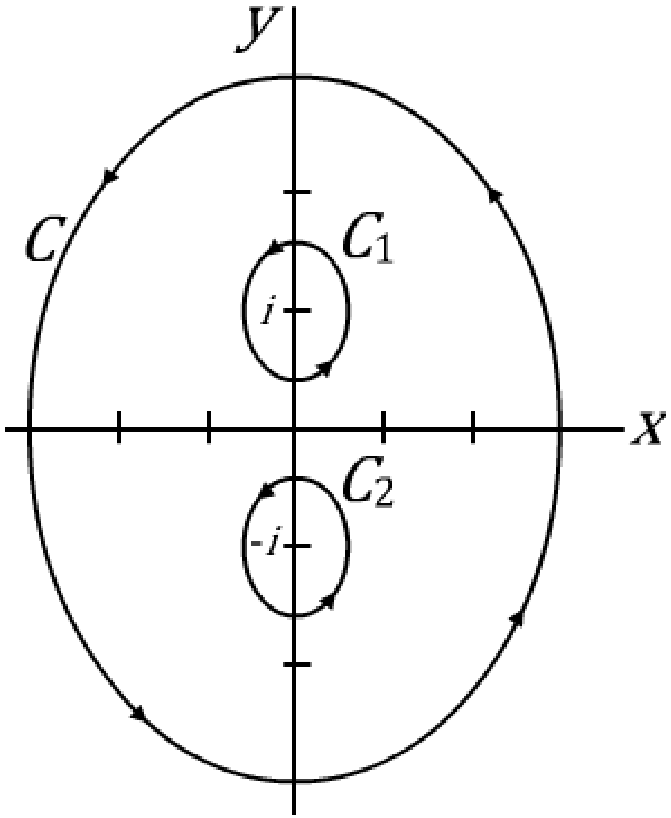
Solution:

The integrand $\frac{1}{z^2+1}$ is not analytic at $z = \pm i$. Both of these points lie within the contour C.

Here is a sample problem showcasing how Cauchy’s theorem for multiply connected domains can be used to integrate complex functions.

Solution (contd.):

Introduce C_1 and C_2 as shown in the figure to exclude these points and then, use Cauchy’s theorem on this multiply connected domain.



Here is a sample problem showcasing how Cauchy’s theorem for multiply connected domains can be used to integrate complex functions.

Solution (contd.):

$$\oint_C \frac{dz}{z^2+1} = \oint_C \frac{dz}{(z+i)(z-i)}$$

$$= \oint_{C_1} \frac{1/(z+i)}{(z-i)} dz + \oint_{C_2} \frac{1/(z-i)}{(z+i)} dz$$

$$= 2\pi i \left[\frac{1}{z+i}\right]\Bigg|_{z\,=\,i} + 2\pi i \left[\frac{1}{z-i}\right]\Bigg|_{z\,=\,-i}$$

$$= 2\pi i \frac{1}{2i} + 2\pi i \frac{1}{-2i} = 0$$

Let us see how real integrals are evaluated using complex functions.

$$\int_0^{2\pi} F(\cos\theta\,,\sin\theta)d\theta$$

Where, $F(\cos\theta\,,\sin\theta)d\theta$ is a real function of $\cos\theta$ and $\sin\theta$ and is finite on the interval of integration.

Basic Idea

Let $z = e^{i\theta}$. This gives,

$\cos\theta = \dfrac{z+\bar{z}}{2}, \sin\theta = \dfrac{z-\bar{z}}{2i}$, and $dz = ie^{i\theta}d\theta \Rightarrow d\theta = \dfrac{1}{iz}dz$

This allows to convert $F(\cos\theta\,,\sin\theta)d\theta$ into $f(z)$, and the integration interval of $0 \le \theta \ \le 2\pi$ is changed to a unit circle.

Thus, $\displaystyle\int_0^{2\pi} F(\cos\theta\,,\sin\theta)d\theta = \oint_C f(z)\frac{1}{iz}dz,$ C: unit circle, counter-clockwise direction.

The sample problem given below helps us understand the concept of evaluating real integrals using complex functions.

Sample Problem 13

Evaluate $\displaystyle\int_0^{2\pi} \frac{d\theta}{\sqrt{2}-\cos\theta}$

Solution:

Let $z = e^{i\theta}$. Substituting in the given equation gives,

$\cos\theta = \dfrac{1}{2}\left[z + \dfrac{1}{z}\right]$ and $d\theta = \dfrac{dz}{iz}$

The sample problem given below helps us understand the concept of evaluating real integrals using complex functions.

Solution (contd.):

The real integral becomes:

$$\oint_C \frac{dz/iz}{\sqrt{2}-\frac{1}{2}\left[z+\frac{1}{z}\right]} = \frac{-2}{i}\oint_C \frac{dz}{\left(z-(\sqrt{2}+1)\right)\left(z-(\sqrt{2}-1)\right)}$$

Where, C is a unit circle in counter-clockwise direction.

The sample problem given below helps us understand the concept of evaluating real integrals using complex functions.

Solution (contd.):

The integrand has simple pole at $z = \sqrt{2} - 1$ inside C and $z = \sqrt{2} + 1$ outside C. Hence, using Cauchy's integral formula, the integral is:

$$\frac{-2}{i} \oint_C \frac{dz}{\left(z - \left(\sqrt{2} + 1\right)\right)\left(z - \left(\sqrt{2} - 1\right)\right)}$$

$$= \frac{-2}{i} \oint_C \frac{\frac{1}{\left(z - \left(\sqrt{2} + 1\right)\right)}}{\left(z - \left(\sqrt{2} - 1\right)\right)} dz = \frac{-2}{i}(2\pi i) \left. \frac{1}{\left(z - \left(\sqrt{2} + 1\right)\right)} \right|_{z = \sqrt{2} - 1} = 2\pi$$

Complex integration can be used to evaluate improper integrals of rational functions.

$$\int_{-\infty}^{\infty} f(x)\,dx = \oint_{\mathrm{UHP}} f(z)dz = \sum_{k=1}^{n} \oint_{C_k \text{ in UHP}} f(z)dz, \text{ if}$$

1. $f(x) = \dfrac{p(x)}{q(x)}$ is a real function with no common factors between $p(x)$ and $q(x)$, and $q(x) \neq 0$ for all real x.

2. Degree of $q(x) \geq$ Degree of $p(x) + 2$

For example, $f(x) = \dfrac{1}{1+x^4}$ satisfies the above conditions but $f(x) = \dfrac{x^3}{1+x^4}$ does not.

Let's see how improper integrals of rational fractions are evaluated.

Sample Problem 14

Show that $\displaystyle\int_0^\infty \frac{dx}{1+x^4} = \frac{\pi}{2\sqrt{2}}$

Solution:

First, check that $f(x)$ satisfies the assumptions. Now, consider

$f(z) = \dfrac{1}{1+z^4}$ which has four simple poles at $z = e^{\pi i/4}$, $e^{3\pi i/4}$, $e^{-3\pi i/4}$, and $e^{-\pi i/4}$.

Let's see how improper integrals of rational fractions are evaluated.

Solution (contd.):

Only the first two poles, that is, $e^{\pi i/4}$ and $e^{3\pi i/4}$, lie inside the UHP. The corresponding complex integral is:

$$\oint_{\mathrm{UHP}} \frac{1}{1+z^4}\,dz$$

$$= \oint_{\mathrm{UHP}} \frac{1}{(z-e^{\pi i/4})(z-e^{3\pi i/4})(z-e^{-3\pi i/4})(z-e^{-\pi i/4})}\,dz$$

Let's see how improper integrals of rational fractions are evaluated.

Solution (contd.):

$$\oint_{\text{UHP}} \frac{1}{1+z^4}\,dz$$

$$= \oint_{C_1} \frac{\frac{1}{(z-e^{3\pi i/4})(z-e^{-3\pi i/4})(z-e^{-\pi i/4})}}{(z-e^{\pi i/4})}\,dz$$

$$+ \oint_{C_2} \frac{\frac{1}{(z-e^{\pi i/4})(z-e^{-3\pi i/4})(z-e^{-\pi i/4})}}{(z-e^{3\pi i/4})}\,dz$$

Let's see how improper integrals of rational fractions are evaluated.

Solution (contd.):

$$\oint_{\mathrm{UHP}} \frac{1}{1+z^4}\,dz = 2\pi i\left[-\frac{1}{4}e^{\pi i/4} + \frac{1}{4}e^{-\pi i/4}\right]$$

Now, since $\dfrac{1}{1+x^4}$ is even,

$$\int_0^{\infty} \frac{dx}{1+x^4} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{1+x^4}$$

$$= \frac{-\pi i}{4}\left[e^{i\pi/4} - e^{-i\pi/4}\right] = \frac{\pi}{2}\sin\frac{\pi}{4} = \frac{\pi}{2\sqrt{2}}$$

Summary

Key points discussed in this lesson:

- In the case of a complex definite integral, or line integral, $\int_C f(z)dz$ means that the integration is done along the curve C (in a given direction) in the complex plane and the integrand $f(z)$ is defined for each point on C. 'C' is called the contour or path of integration.

- A line integral of $f(z)$ in the complex plane may not always depend on the choice of the path itself. Sometimes, the integrals evaluated turn out to be zero or $2\pi i$.

- Cauchy's Integral Theorem is an important theorem describing the line integrals of analytic functions in a complex plane.

Key points discussed in this lesson:

- The most important consequence of Cauchy's Integral Theorem is Cauchy's integral formula. This formula is useful to evaluate integrals of the following form.

  $$\int_C \frac{f(z)}{(z-z_0)^m}\,dz\,,\, m = 1, 2, 3, \ldots$$

- The formula $\int_{-\infty}^{\infty} f(x)\,dx = \oint_{\mathrm{UHP}} f(z)dz = \sum_{k=1}^{n} \oint_{C_k \text{ in UHP}} f(z)dz$ holds true if:
  - $f(x) = p(x)/q(x)$ is a real function with no common factors between $p(x)$ and $q(x)$, and $q(x) \neq 0$ for all real x.
  - Degree of $q(x) \geq$ Degree of $p(x) + 2$