Vector Calculus

EE2007 – Engineering Mathematics II

Scalars are quantities that only have a magnitude like mass, speed, and electric field strength.

Many times it is often useful to have a quantity that has not only a magnitude but also a direction; such a quantity is called a vector.

Examples of quantities represented by vectors include velocity, acceleration, and virtually any type of force (frictional, gravitational, electric, magnetic, etc.).

Let us look at an example as to why vectors are useful. Suppose a plane traveling at 300 mph to the north with no wind present encounters a westerly crosswind of 50 mph. Then, the resultant velocity of the plane is the sum of the velocities of the wind and the plane.

At the end of these series of lectures, you should be able to:

- Describe the vector fundamentals. (1 hr)
- Perform vector differentiation. (1 hr)
- Work the “∇“ (del) operator. (2 hrs)
  - grad, div, curl
- Perform Line Integral. (3 hrs)
- Perform Surface Integral. (2 hrs)
- Perform Volume Integral. (1 hr)
- Apply Divergence, Stokes and Green’s Theorems. (2 hrs)

At the end of this lesson, you should be able to:

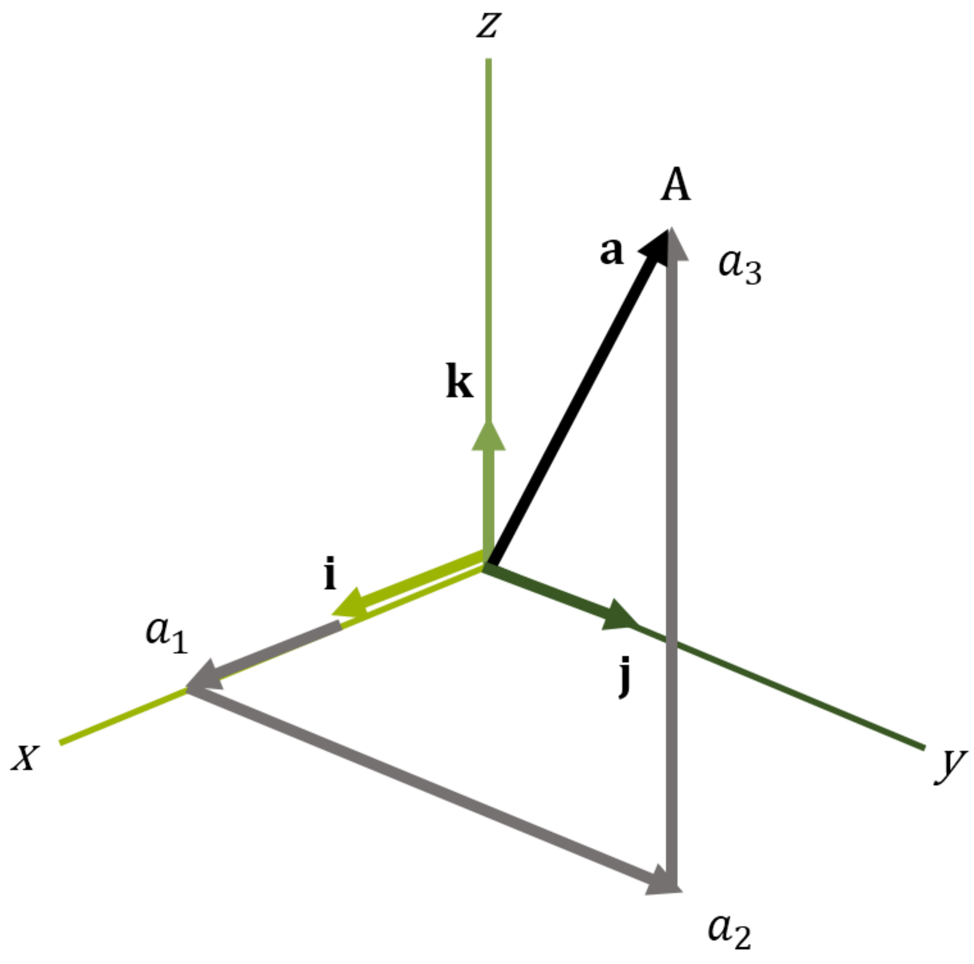
- Describe the fundamentals of vector calculus.

A vector is a quantity that has both magnitude and direction.

For example, $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$

where $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are standard Cartesian unit vectors along *x, y, z* directions.

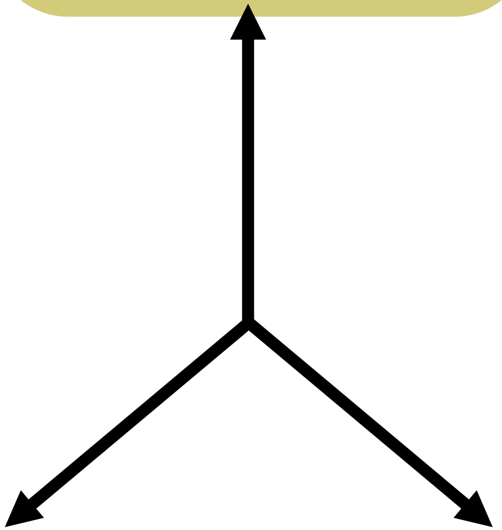
A vector has both direction and magnitude. It points from tail to head. The tail is always at the origin of the coordinate system. The head is at point A with coordinates (a_1, a_2, a_3).



Given below are the coordinates of unit vectors in *x*, *y* and *z* directions respectively.

$\mathbf{k}$ is a unit vector in the *z* direction, it has coordinates (0, 0, 1). So,

$$\mathbf{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$



$\mathbf{i}$ is a unit vector in the *x* direction, it has coordinates (1, 0, 0). So,

$$\mathbf{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$\mathbf{j}$ is a unit vector in the *y* direction, it has coordinates (0, 1, 0). So,

$$\mathbf{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

In the equation $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, $\mathbf{a}$ is a 3 x 1 column vector as shown below.

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$$

$$= a_1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + a_2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + a_3\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} a_1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ a_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ a_3 \end{bmatrix}$$

$$= \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

A 3D (column) vector $\mathbf{v} = [2 \quad 3 \quad -5]^T$ can be expressed in the given form.

$$\mathbf{v} = 2\mathbf{i} + 3\mathbf{j} - 5\mathbf{k}$$

$$\mathbf{v} = 2\begin{bmatrix}1\\0\\0\end{bmatrix} + 3\begin{bmatrix}0\\1\\0\end{bmatrix} - 5\begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$= \begin{bmatrix}2\\0\\0\end{bmatrix} + \begin{bmatrix}0\\3\\0\end{bmatrix} + \begin{bmatrix}0\\0\\-5\end{bmatrix}$$

$$= \begin{bmatrix}2\\3\\-5\end{bmatrix}$$

The transpose of a column vector becomes a row vector. Therefore, it can also be written as:

$$\mathbf{v}^{\mathrm{T}} = [2 \quad 3 \quad -5]$$

Let us look at an example to understand the vector representation.

A vector which represents the position of a particle with coordinates (*x*, *y*, *z*) that is, position vector, can be written as:

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

A vector which represents the force on the particle can be expressed as $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$ where, F_1, F_2, and F_3 are components of $\mathbf{F}$ in the directions of $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ respectively.

Vectors are denoted either by boldfaced alphabets or alphabets with an arrow above. For example: $\mathbf{F}, f, \text{or } \vec{f}$.

Let us look at an example to find out the magnitude of the vector and unit vector.

Example 1

If $\mathbf{a} = 2\mathbf{i} + 1\mathbf{j} + 2\mathbf{k}$, then

$$\|\mathbf{a}\| = \sqrt{2^2 + 1^2 + 2^2} = \sqrt{9} = 3$$

$$\hat{\mathbf{a}} = \frac{\mathbf{a}}{\|\mathbf{a}\|} = \frac{1}{3}(2\mathbf{i} + 1\mathbf{j} + 2\mathbf{k})$$

$$= \left(\frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}\right) = [2/3 \quad 1/3 \quad 2/3]^{\mathrm{T}} \text{ is a unit vector.}$$

In general, given any non-zero vector $\mathbf{v}$, $\hat{\mathbf{v}} = \mathbf{v}/\|\mathbf{v}\|$, is a unit vector in the same direction as $\mathbf{v}$.

Let us look at an example to find out the magnitude of the vector and unit vector.

Example 2

Alternatively $\mathbf{v} = \|\mathbf{v}\|\ \hat{\mathbf{v}}$

Find $\mathbf{v}$ if $\mathbf{v}$ has a length of 5 units in the direction of $2\mathbf{i} + 3\mathbf{j} - 5\mathbf{k}$

$$\hat{\mathbf{v}} = \frac{1}{\sqrt{2^2 + 3^2 + 5^2}}(2\mathbf{i} + 3\mathbf{j} - 5\mathbf{k})$$

$$\hat{\mathbf{v}} = \frac{1}{\sqrt{38}}(2\mathbf{i} + 3\mathbf{j} - 5\mathbf{k})$$

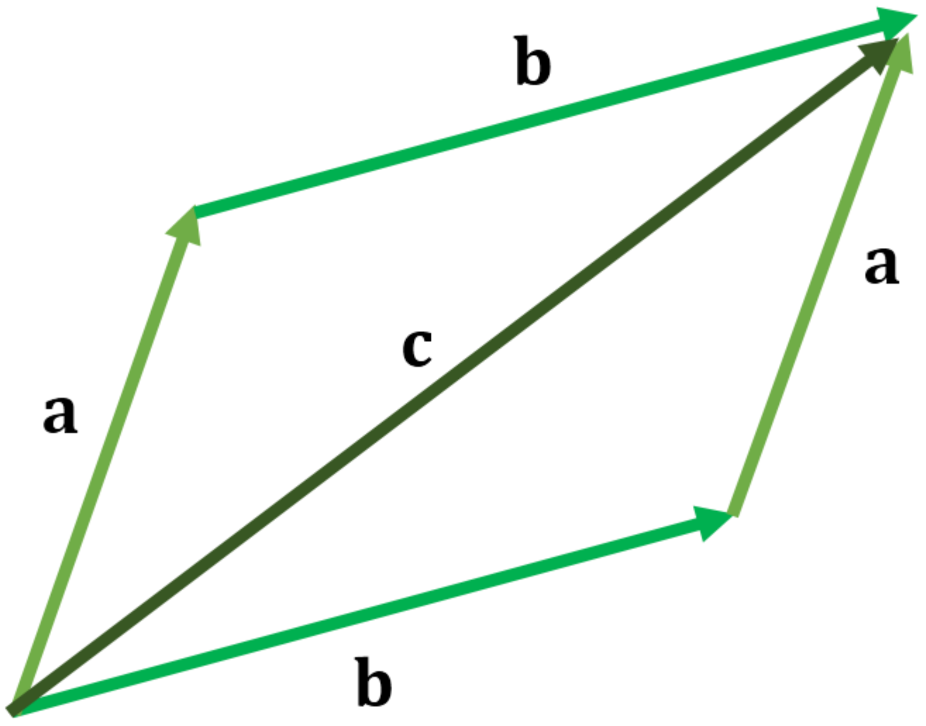
$$\mathbf{v} = 5\left[\frac{1}{\sqrt{38}}(2\mathbf{i} + 3\mathbf{j} - 5\mathbf{k})\right]$$

$$\mathbf{v} = \frac{5}{\sqrt{38}}(2\mathbf{i} + 3\mathbf{j} - 5\mathbf{k})$$

Let us now understand the concept of vector algebra.

For these vectors, $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$

$$\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$$



Vector Addition

$$\mathbf{a} + \mathbf{b} = (a_1 + b_1)\mathbf{i} + (a_2 + b_2)\mathbf{j} + (a_3 + b_3)\mathbf{k}$$
$$= \mathbf{b} + \mathbf{a}$$

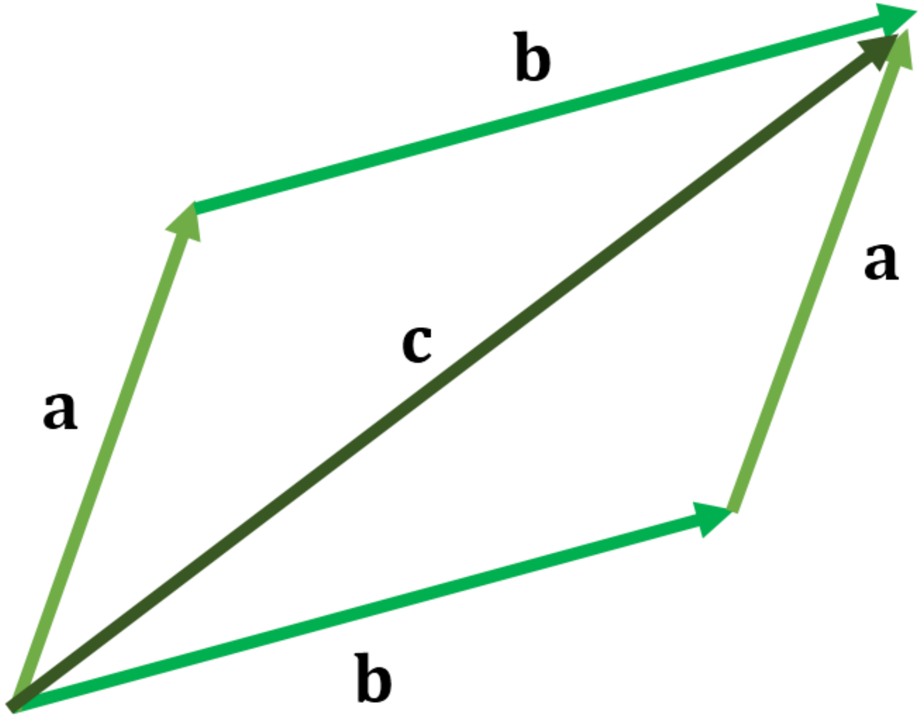
Vector Subtraction

$$\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b})$$

Let us now understand the concept of vector algebra.

For these vectors, $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$

$$\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$$



Scalar Multiplication by a constant *c*

$$c\,\mathbf{a} = c\,a_1\mathbf{i} + c\,a_2\mathbf{j} + c\,a_3\mathbf{k}$$

$$c(\mathbf{a} + \mathbf{b}) = (c\,\mathbf{a} + c\,\mathbf{b})$$

Vector Product

Dot (scalar) product, $\mathbf{a} \cdot \mathbf{b}$ = **scalar quantity**

Cross (vector) product, $\mathbf{a} \times \mathbf{b}$ = **vector quantity**

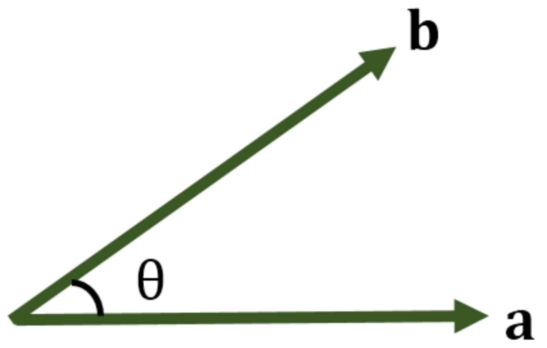
Let us now become familiar with the concept of dot (scalar) product.

The dot product can be expressed as:

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\| \cos\theta \qquad \text{- (1)}$$

$$= (a_1 b_1) + (a_2 b_2) + (a_3 b_3) \qquad \text{- (2)}$$

$$= \mathbf{b} \cdot \mathbf{a} = \text{scalar}$$



Equation (1) gives the geometric meaning of dot product.

Equation (2) shows the algebraic way of computing the dot product.

Let us now become familiar with the concept of dot (scalar) product.

$$\mathbf{a}\cdot\mathbf{b} = (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k})\cdot(b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}\cdot\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

The result of a dot product is a scalar (real value). Thus, the dot product is commutative.

$$\mathbf{a}\cdot\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}\cdot\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = {a_1}^2 + {a_2}^2 + {a_3}^2$$

$$\sqrt{\mathbf{a}\cdot\mathbf{a}} = \|\mathbf{a}\|$$

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Some peculiarities for $\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\| \cos\theta$ are displayed below.

$\mathbf{a} \| \mathbf{b}$ parallel, $\theta = 0$, $\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\|$

Anti-parallel, $\theta = \pi$, $\mathbf{a} \cdot \mathbf{b} = -\|\mathbf{a}\|\|\mathbf{b}\|$

$\mathbf{a} \perp \mathbf{b}$ perpendicular, $\theta = {}^{\pi}/_{2}$, $\mathbf{a} \cdot \mathbf{b} = \mathbf{0}$ that is, **a** is orthogonal to **b**.

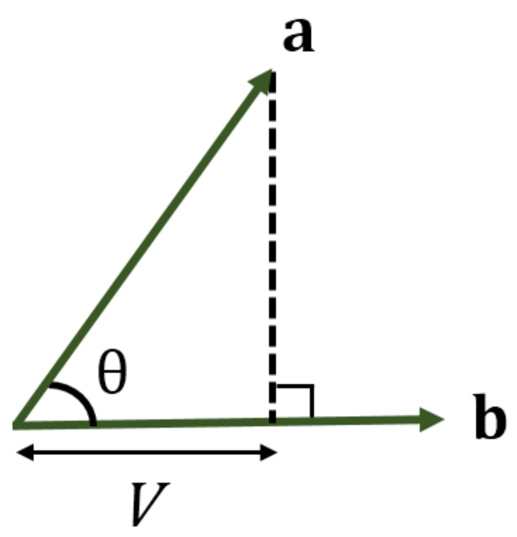
The third property is extremely important. *To show or test that the two vectors are orthogonal to each other, take their dot product. If the result is zero, they are orthogonal.*

Important:

1. $\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = \mathbf{1}$ ($\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are unit vectors)
2. $\mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{i} = \mathbf{0}$ ($\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are mutually orthogonal).

Let us now calculate the component of vector **a** in another direction **b**.

The component of $\mathbf{a}$ in the direction of $\mathbf{b}$ in the given figure is *V*.



Geometrically,

$V = \|\mathbf{a}\| \cos\theta$

Now, $\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\| \cos\theta$

Dividing both sides by $\|\mathbf{b}\|$,

$$\mathbf{a} \cdot \frac{\mathbf{b}}{\|\mathbf{b}\|} = \|\mathbf{a}\| \cos\theta = V$$

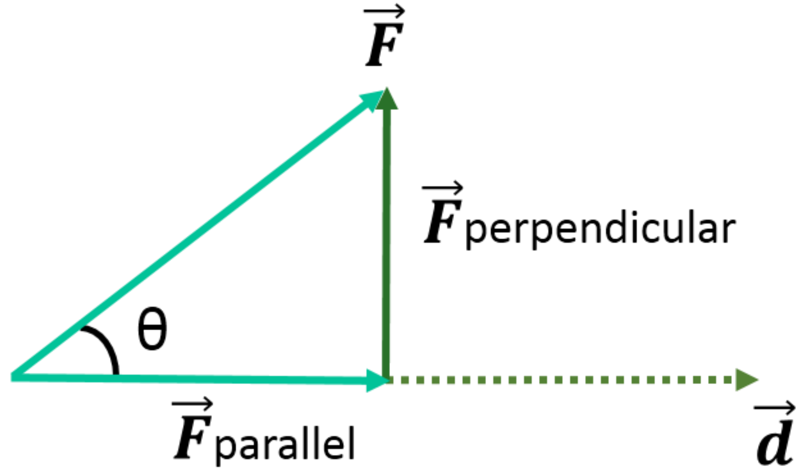
$$V = \mathbf{a} \cdot \hat{\mathbf{b}}$$

Therefore, to obtain the component of $\mathbf{a}$ in the direction of $\mathbf{b}$, just take the dot product of $\mathbf{a}$ and the unit vector in the direction of $\mathbf{b}$.

Let us take a look at an example explaining the dot product concept.

Example 3

Show that force $\mathbf{F}$ is applied on an object which is moved by $\mathbf{d}$ when work is done.



$$
\begin{aligned}
W &= \|\mathbf{F}_{\mathbf{parallel}}\|\|\mathbf{d}\| \\
&= (\|\mathbf{F}\|\cos\theta)\|\mathbf{d}\| \\
&= \|\mathbf{F}\|\|\mathbf{d}\|\cos\theta \\
&= \mathbf{F}\cdot\mathbf{d}
\end{aligned}
$$

Recall that the work done is equal to the distance multiplied by the component of the applied (constant) force, in the same direction as the distance moved.

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Let us solve the following questions.

1. Compute the dot product of $\begin{bmatrix}1 & 2 & 3\end{bmatrix}^{\mathbf{T}}$ and $\mathbf{k}$.
2. What is the length of column vector $\begin{bmatrix}3 & 4 & 5\end{bmatrix}^{\mathbf{T}}$? What about for row vector $\begin{bmatrix}1 & 2 & 3\end{bmatrix}$?
3. Find the angle between the vectors $\begin{bmatrix}1 & 0 & 1\end{bmatrix}^{\mathbf{T}}$ and $\begin{bmatrix}-2 & 3 & 2\end{bmatrix}^{\mathbf{T}}$.
4. What is the resultant vector of the sum of vectors $\begin{bmatrix}1 & 0 & 1\end{bmatrix}^{\mathbf{T}}$, $\mathbf{j}$, and $\begin{bmatrix}-2 & 3 & 2\end{bmatrix}^{\mathbf{T}}$?
5. Suppose $\mathbf{x} \cdot \mathbf{y} = 0$. What is the significance of this?

Let us solve the following questions.

6. What is wrong with this statement : $\mathbf{a} \cdot \mathbf{b} = [1 \quad 1 \quad 0]$?

7. If a constant force $\mathbf{F} = [1 \quad 1 \quad 1]^{\mathbf{T}}$ is applied to an object moving it by $[3 \quad 0 \quad 0]$, what is the work done?

8. Find the component of $[1 \quad 0 \quad 1]^{\mathbf{T}}$ in the direction of $[-2 \quad 3 \quad 2]^{\mathbf{T}}$.

9. What is wrong with the statement: $(\mathbf{a} \cdot \mathbf{b}) \cdot \mathbf{c} = 7$?

Let us now learn about the cross (vector) product.

For the given vectors,

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$$

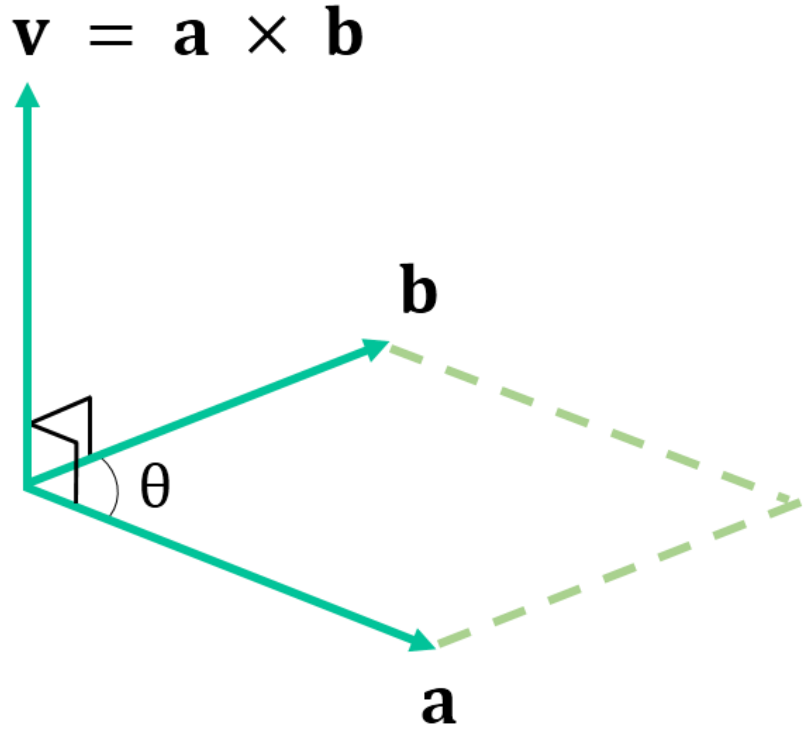
$$\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$$

$$\mathbf{a} \times \mathbf{b}\ (= \text{vector}) = \mathbf{v},$$

The cross (vector) product can be expressed as:

$$\|\mathbf{v}\| = \|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\|\ \|\mathbf{b}\| \sin\theta \qquad - (3)$$

The direction of $\mathbf{v}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$.
$\boldsymbol{\theta}$ is the angle between $\mathbf{a}$ and $\mathbf{b}$.

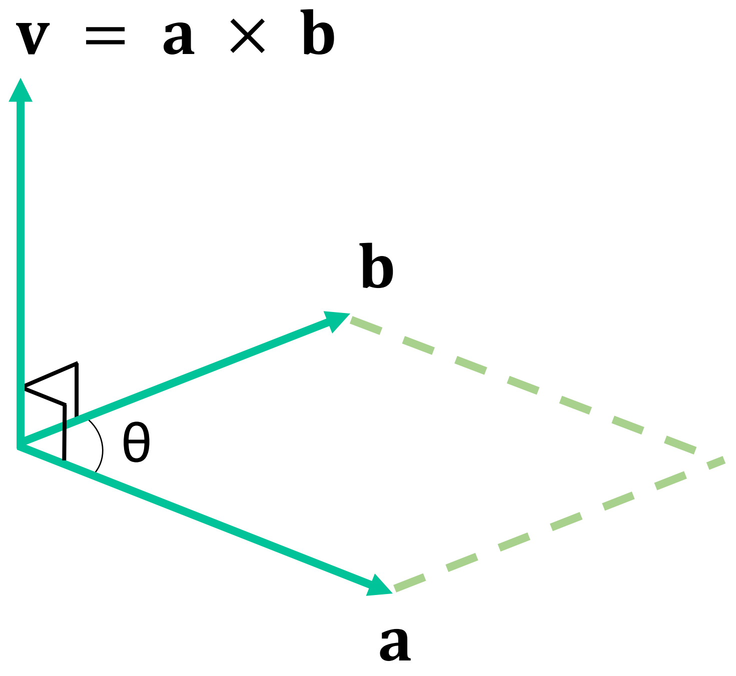


Cross product is valid only for 3D vectors.

Let us define $\hat{\mathbf{n}} = \mathbf{v}/\|\mathbf{v}\|$, where $\hat{\mathbf{n}}$ is the unit vector in the direction of $\mathbf{v}$.

The above relationship can be consolidated by the formula given below.

$$\mathbf{a} \times \mathbf{b} = \|\mathbf{a}\|\,\|\mathbf{b}\| \sin\theta\, \hat{\mathbf{n}} \qquad \text{- (3a)}$$



Where, $\hat{\mathbf{n}}$ is the unit vector that is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$. This also means that $\hat{\mathbf{n}}$ acts as the *normal* to the plane defined by $\mathbf{a}$ and $\mathbf{b}$.

Show how Equation (3) or Equation (3a) can be used to calculate the area of the parallelogram defined by $\mathbf{a}$ and $\mathbf{b}$?

Let us look at an example to learn how to calculate cross (vector) product.

Example 4

$$(3\mathbf{i} - \mathbf{j} + 2\mathbf{k}) \times (\mathbf{i} + 2\mathbf{j} - \mathbf{k})$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -1 & 2 \\ 1 & 2 & -1 \end{vmatrix} = -3\mathbf{i} + 5\mathbf{j} + 7\mathbf{k}$$

For calculating cross (vector) product, note that:

- $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$
- $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} \neq \mathbf{a} \times (\mathbf{b} \times \mathbf{c})$
- $\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = 0$
- $\mathbf{i} \times \mathbf{j} = \mathbf{k},\ \mathbf{j} \times \mathbf{k} = \mathbf{i},\ \mathbf{k} \times \mathbf{i} = \mathbf{j}$

Let us now go through the concept of scalar triple product.

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = (a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k})\cdot[(b_2c_3-b_3c_2)\mathbf{i}+(b_3c_1-b_1c_3)\mathbf{j}+(b_1c_2-b_2c_1)\mathbf{k}]$$

$$= a_1(b_2c_3-b_3c_2)+a_2(b_3c_1-b_1c_3)+a_3(b_1c_2-b_2c_1)$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Let us understand the concept of scalar triple product with an example.

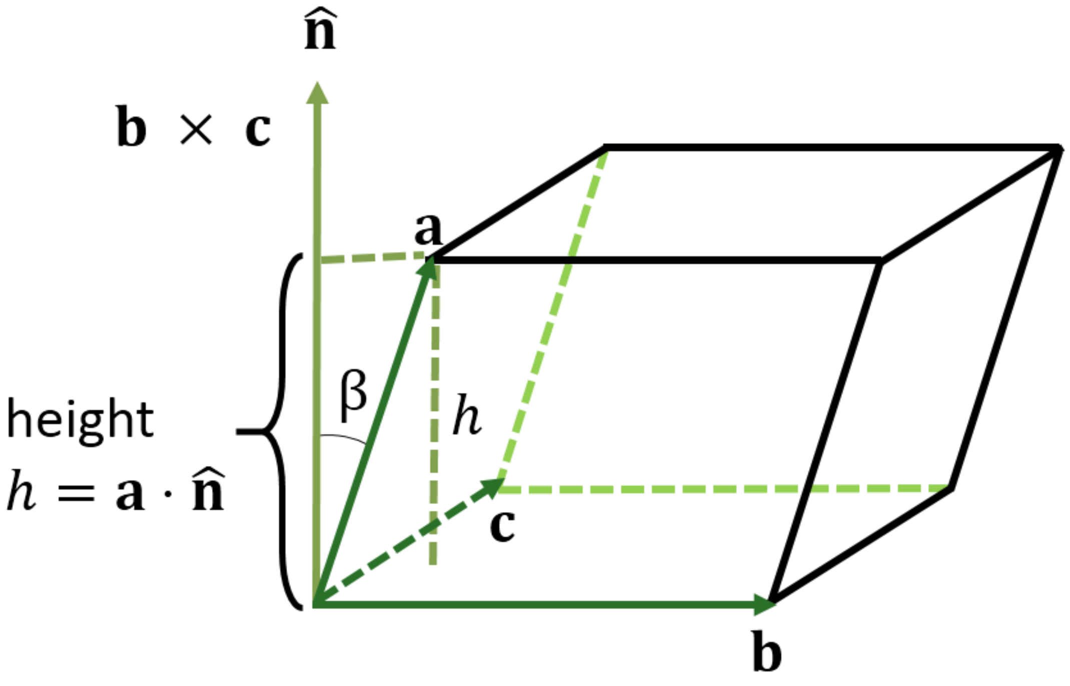
Show that one may interpret the scalar triple product $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ = volume of parallelepiped with $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ as the edges.

Hint: Recall that $||\mathbf{b} \times \mathbf{c}||$ gives the area of the parallelogram and note that the component of $\mathbf{a}$ in the direction of $\mathbf{n}$, the normal to the b-c plane, is h.

Volume = height × base area

$$= \mathbf{a} \cdot \hat{\mathbf{n}} \; ||\mathbf{b} \times \mathbf{c}|| = \mathbf{a} \cdot \underbrace{||\mathbf{b} \times \mathbf{c}||\hat{\mathbf{n}}}$$

$$= \mathbf{a} \cdot \mathbf{b} \times \mathbf{c}$$



It can be seen that $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = -\mathbf{c} \cdot (\mathbf{b} \times \mathbf{a})$.

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = -\begin{vmatrix} c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix} \text{ interchanging rows 1 and 3}$$

$$= -\mathbf{c} \cdot (\mathbf{b} \times \mathbf{a})$$

It can also be shown in the following form:

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$$

Summary

Key points discussed in this lesson:

- A vector is a quantity that has both magnitude and direction.

- The transpose of a column vector can become a row vector.

- Vectors are denoted either by boldfaced alphabets or alphabets with an arrow above.
  For example: $\mathbf{F}$, $\mathbf{f}$, or $\vec{\mathrm{f}}$.

- For a vector, $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, the magnitude (length) of the vector is given as:
  $||\mathbf{a}|| = \sqrt{{a_1}^2 + {a_2}^2 + {a_3}^2}$.

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Key points discussed in this lesson:

- For a vector, $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, the unit vector can be expressed as:

$$\hat{\mathbf{a}} = \frac{1}{\|\mathbf{a}\|}\mathbf{a} = \frac{1}{\sqrt{{a_1}^2 + {a_2}^2 + {a_3}^2}}(a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}).$$

- Geometrically, the dot (scalar) product is expressed as: $\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\| \cos\theta$. The result of a dot product is a scalar (real number). Thus, the dot product is commutative.

- Some useful properties from dot product $\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\| \cos\theta$ are:
  - $\mathbf{a}\|\mathbf{b}$ parallel, θ = 0, $\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\|$
  - Anti-parallel, θ = π, $\mathbf{a} \cdot \mathbf{b} = -\|\mathbf{a}\|\|\mathbf{b}\|$
  - $\mathbf{a} \perp \mathbf{b}$ perpendicular, θ = $^{\pi}/_{2}$, $\mathbf{a} \cdot \mathbf{b} = 0$ that is, **a** is orthogonal to **b**.

Key points discussed in this lesson:

- From the dot product property, following can be derived:
  - $\mathbf{i}\cdot\mathbf{i}=\mathbf{j}\cdot\mathbf{j}=\mathbf{k}\cdot\mathbf{k}=1$ [$\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are unit vectors].
  - $\mathbf{i}\cdot\mathbf{j}=\mathbf{j}\cdot\mathbf{k}=\mathbf{k}\cdot\mathbf{i}=0$ [$\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are mutually orthogonal].

- For vectors, $\mathbf{a}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$, and $\mathbf{b}=b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k}$, the cross (vector) product is expressed as: $\|\mathbf{a}\times\mathbf{b}\|=\|\mathbf{a}\|\,\|\mathbf{b}\|\sin\theta$.

- For calculating cross (vector) product, note that:
  - $\mathbf{a}\times\mathbf{b}=-\mathbf{b}\times\mathbf{a}$
  - $(\mathbf{a}\times\mathbf{b})\times\mathbf{c}\neq\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$
  - $\mathbf{i}\times\mathbf{i}=\mathbf{j}\times\mathbf{j}=\mathbf{k}\times\mathbf{k}=0$
  - $\mathbf{i}\times\mathbf{j}=\mathbf{k},\ \mathbf{j}\times\mathbf{k}=\mathbf{i},\ \mathbf{k}\times\mathbf{i}=\mathbf{j}$

Key points discussed in this lesson:

- The scalar triple product of three vectors **a**, **b**, and **c** is given as:

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

- From scalar triple product you get:
  - $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = -\,\mathbf{c}\cdot(\mathbf{b}\times\mathbf{a}).$
  - $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) = \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b}).$