Vector Volume Integral

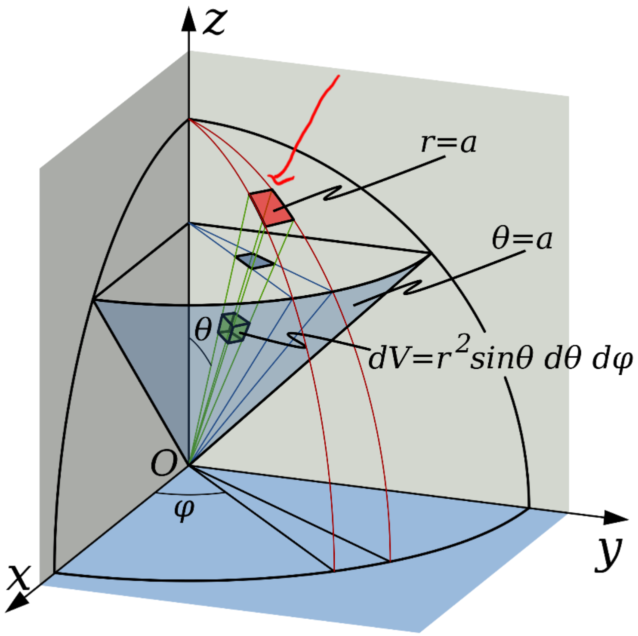
EE2007 – Engineering Mathematics II

At the end of this lesson, you should be able to:

- Perform the volume integrals.

The volume integral of a scalar function f over a prescribed volume V in the Cartesian coordinates is expressed as shown below.

$$\iiint_V f\, dV = \iiint_V f(x, y, z)\, dxdydz$$



Let us look at a sample problem to understand the concept of volume integral.

Sample Problem 1

Let $f = 2(x + y + z)$. Find the volume integral of f over the unit cube.

Solution:

$$\iiint_V f\, dV = \int_0^1 \int_0^1 \int_0^1 2(x + y + z)\, dxdydz$$

$$= \int_0^1 \int_0^1 \left[x^2 + 2x(y + z)\right]_0^1 dydz$$

Let us look at a sample problem to understand the concept of volume integral.

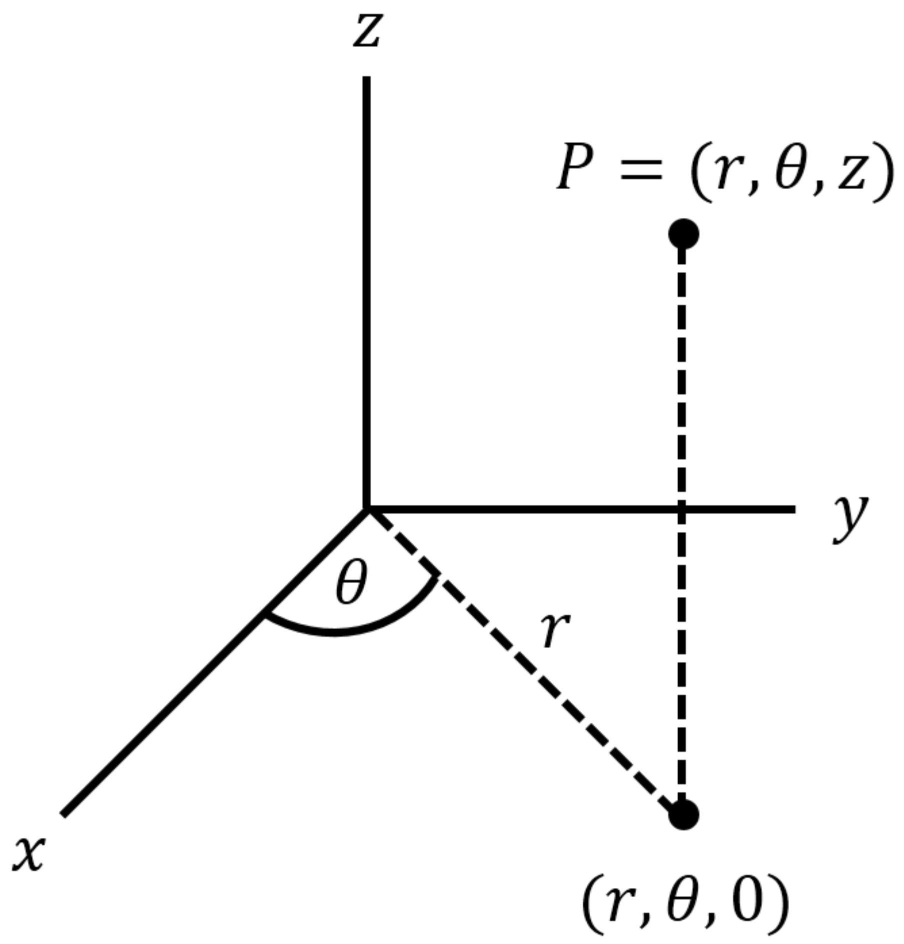
Solution (contd.):

$$\iiint_V f\, dV = \int_0^1 \int_0^1 [1 + 2(y + z)]\, dydz$$

$$= \int_0^1 \left[y + y^2 + 2yz\right]_0^1 dz$$

$$= \int_0^1 [2 + 2z]\, dz = \left[2z + z^2\right]_0^1 = 3$$

The Cartesian coordinates (x, y, z) can be converted into cylindrical coordinates (r, θ, z) as shown below.



$$\begin{cases} \boxed{x = r\cos\theta} \\ \boxed{y = r\sin\theta} \\ \boxed{z = z} \end{cases}$$

$$\boxed{0 \leq r < \infty ; 0 \leq \theta < 2\pi}$$

$$\boxed{x^2 + y^2 = r^2}$$

Using cylindrical coordinates, the terms of the integral can be written in the following form.

$$\int\int\int dV = \int\int\int r\,dr\,d\theta\,dz$$

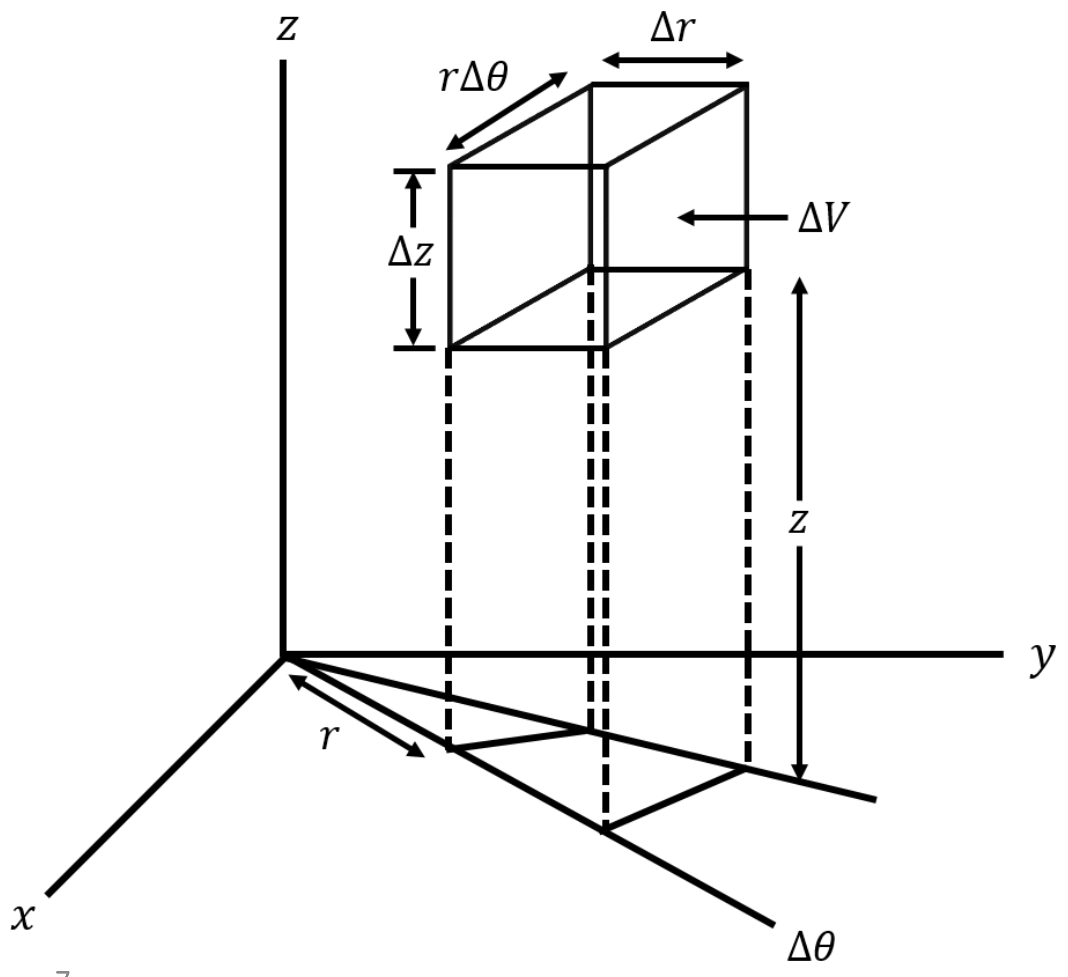
Computing the elemental volume

$$\Delta V \approx (\Delta r)(r\Delta\theta)(\Delta z)$$

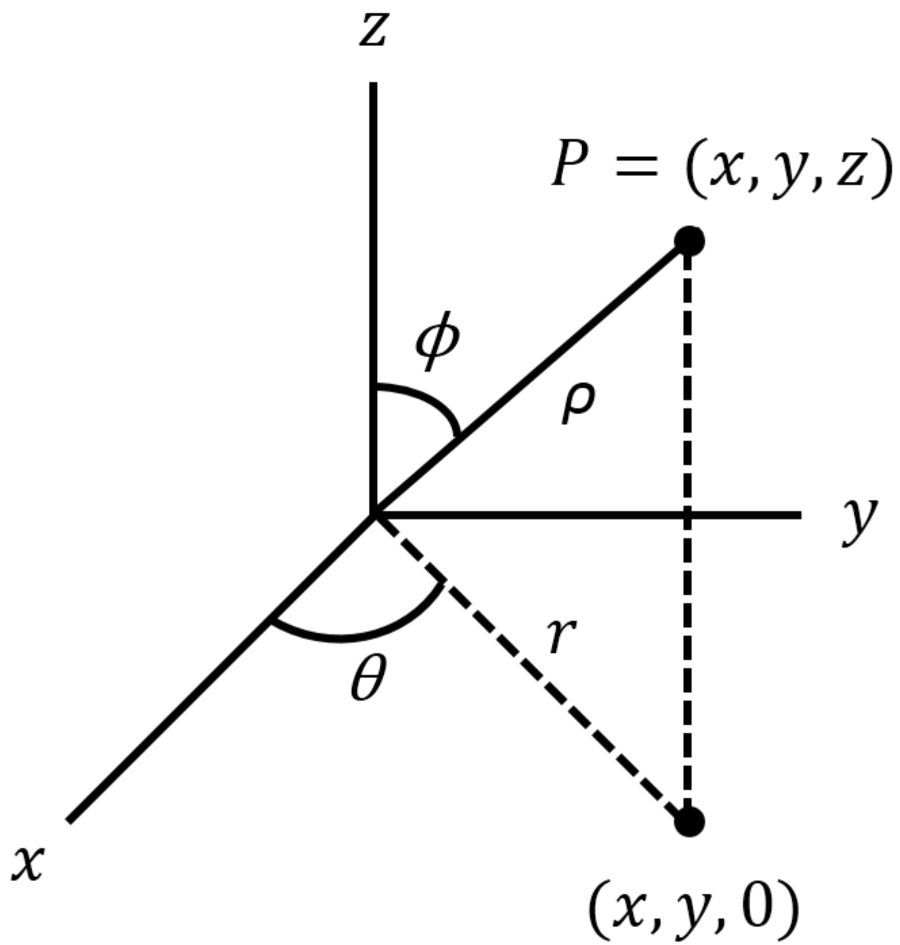
$$\Delta V = \Delta x\,\Delta y\,\Delta z \approx r\,\Delta r\,\Delta\theta\,\Delta z$$

In the limits, it can be written as:

$$dV = dx\,dy\,dz = r\,dr\,d\theta\,dz$$



The Cartesian coordinates can be converted into spherical coordinates.

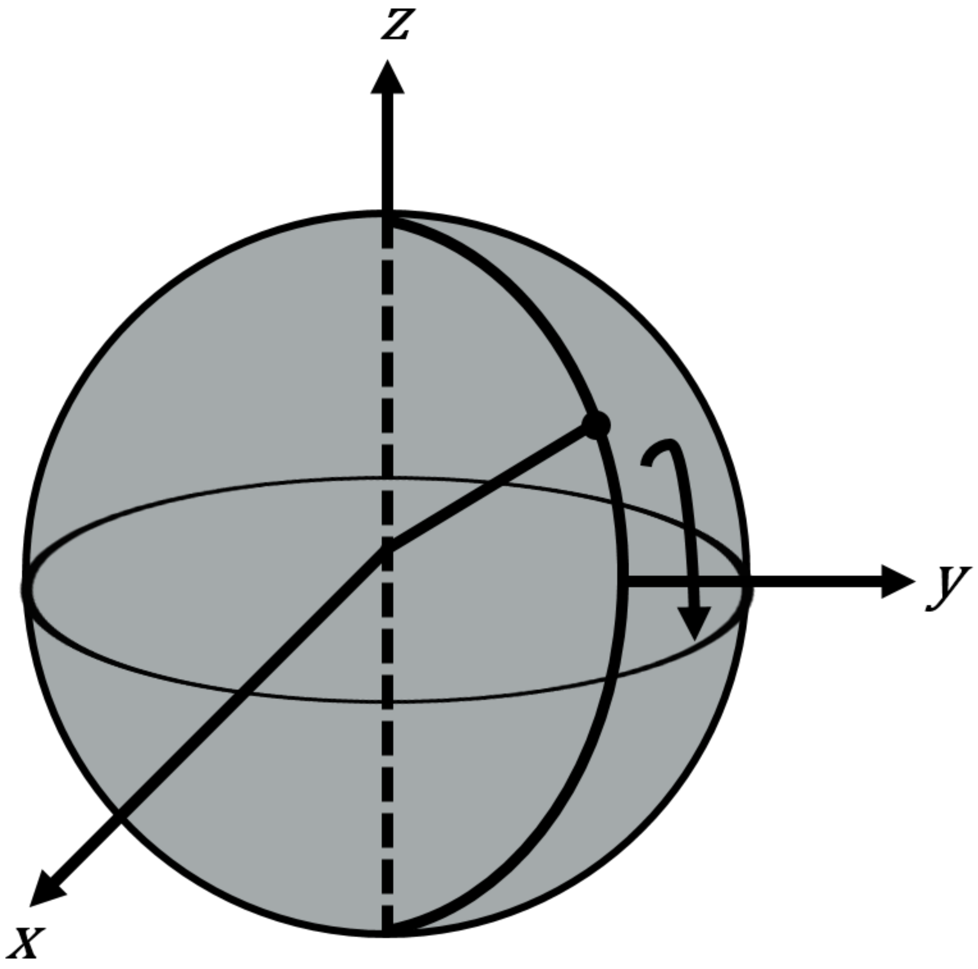


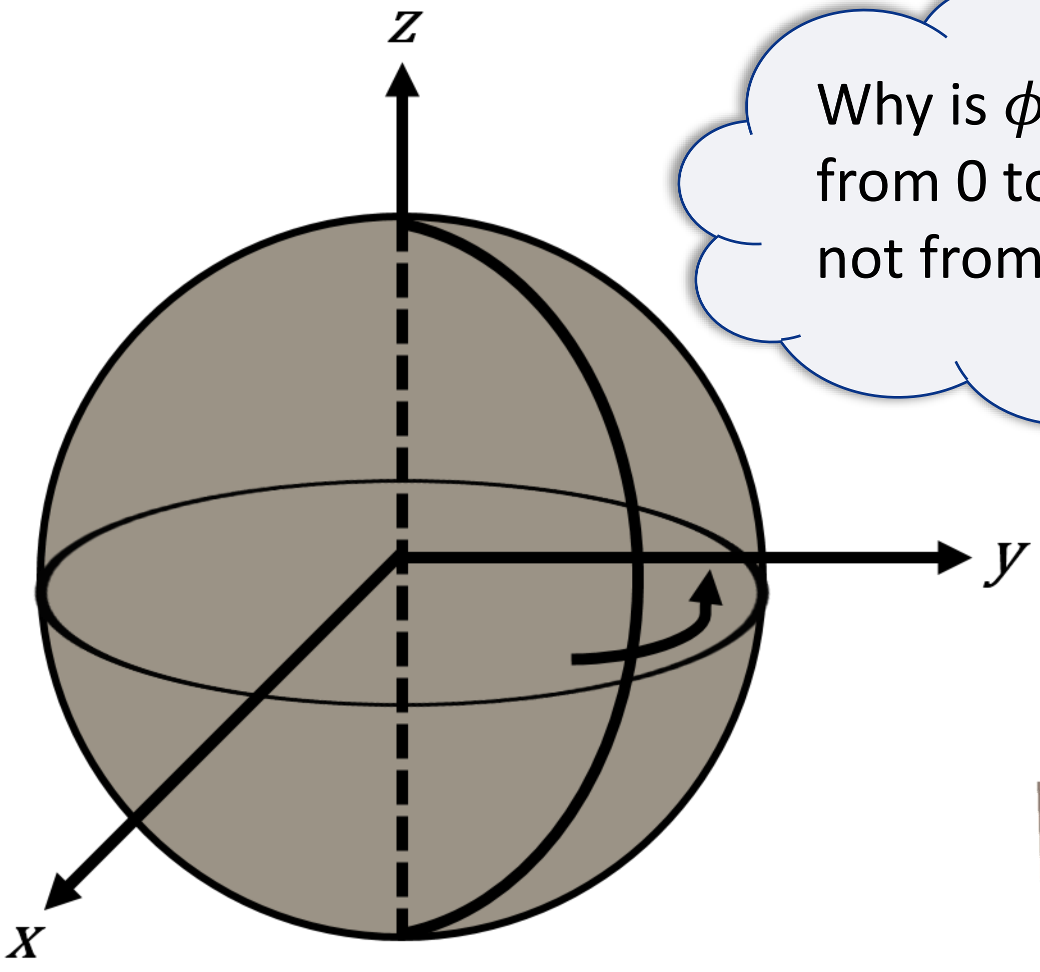
$$\begin{cases} x = \rho\cos\theta\sin\phi \\ y = \rho\sin\theta\sin\phi \\ z = \rho\cos\phi \end{cases}$$

$0 \leq \rho < \infty; 0 \leq \theta < 2\pi$

$0 \leq \phi < \pi$

$x^2 + y^2 + z^2 = \rho^2$





Why is ϕ ranging from 0 to π, and not from 0 to 2π?

Using spherical coordinates, the terms of the integral can be written in the following form.

$$\int\int\int dV = \int\int\int \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$\Delta V \approx (\Delta\rho)(\rho\Delta\phi)(\rho\sin\phi \, \Delta\theta)$$

The following sample problem explains how to evaluate volume integrals in terms of spherical coordinates.

Sample Problem 2

Let $f = z^2(x^2 + y^2 + z^2)^{1/2}$. Find the volume integral of f over the hemisphere given by: $x^2 + y^2 + z^2 = 4$

Solution:

In Cartesian coordinates:

$$\iiint_V f\, dV = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{0}^{\sqrt{4-x^2-y^2}} z^2\sqrt{x^2 + y^2 + z^2}\, dz\, dy\, dx$$

The following sample problem explains how to evaluate volume integrals in terms of spherical coordinates.

Solution (contd.):

In spherical coordinates:

$$\iiint_V f\,dV = \int_0^{2\pi}\int_0^{\pi/2}\int_0^{2}(\rho\cos\phi)^2\rho\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/2}\int_0^{2}\rho^5\cos^2\phi\,\sin\phi\,d\rho\,d\phi\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/2}\left[\frac{1}{6}\rho^6\right]_0^2\cos^2\phi\,\sin\phi\,d\phi\,d\theta$$

The following sample problem explains how to evaluate volume integrals in terms of spherical coordinates.

Solution (contd.):

$$\iiint_V f\,dV = \frac{32}{3}\int_0^{2\pi}\left[-\frac{1}{3}\cos^3\phi\right]_0^{\pi/2} d\theta$$

$$= \frac{32}{9}\int_0^{2\pi} d\theta$$

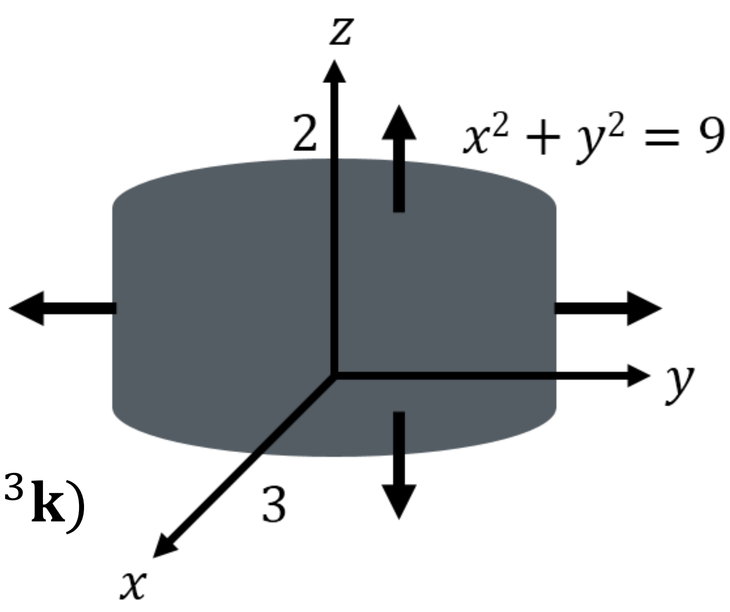
$$= \frac{64}{9}\pi$$

The following sample problem explains how to evaluate volume integrals in terms of cylindrical coordinates.

Sample Problem 3

Let $\mathbf{F} = x^3\,\mathbf{i} + y^3\,\mathbf{j} + z^2\,\mathbf{k}$. Find the volume integral of $\nabla \cdot \boldsymbol{F}$ over the cylinder:

$x^2 + y^2 = 9;\ 0 \le z \le 2$



Solution:

$$\nabla \cdot \mathbf{F} = \left[\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right] \cdot (x^3\mathbf{i} + y^3\mathbf{j} + z^3\mathbf{k})$$

$$= 3x^2 + 3y^2 + 2z$$

The following sample problem explains how to evaluate volume integrals in terms of cylindrical coordinates.

Solution (contd.):

$$\iiint_V \nabla \cdot \mathbf{F}\, dV = \int_0^{2\pi} \int_0^{3} \int_0^{2} \left(3x^2 + 3y^2 + 2z\right) r\, dz\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^{3} \int_0^{2} \left(3r^2 + 2z\right) r\, dz\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^{3} \left[3r^3 z + r z^2\right]_0^2 dr\, d\theta$$

The following sample problem explains how to evaluate volume integrals in terms of cylindrical coordinates.

Solution (contd.):

$$\iiint_V \nabla \cdot \mathbf{F}\, dV = \int_0^{2\pi} \int_0^3 [6r^3 + 4r]\, dr\, d\theta$$

$$= \int_0^{2\pi} \left[\frac{6}{4} r^4 + \frac{4}{2} r^2\right]_0^3 d\theta$$

$$= \int_0^{2\pi} \frac{279}{2}\, d\theta = 279\pi$$

Summary

Vector Volume Integral > Summary

Key points discussed in this lesson:

- The volume integral of a scalar function S over a prescribed volume V in the Cartesian coordinates is given by: $\iiint\limits_V f \cdot dV = f(x, y, z)\; dxdydz$

- The line integral of a conservative vector field is path independent.