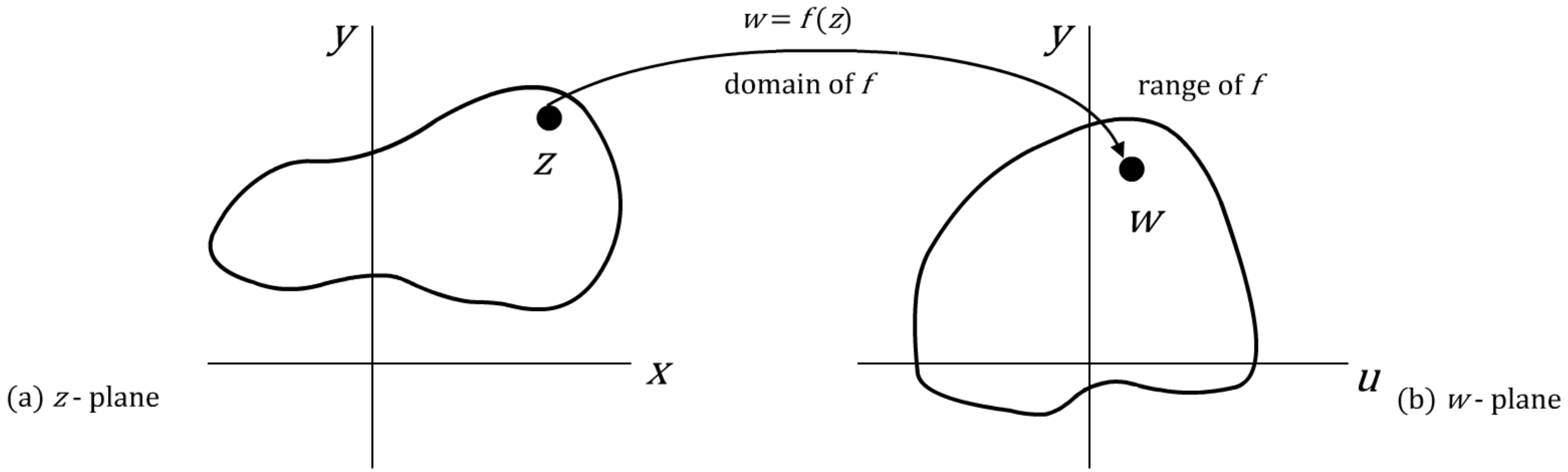
Differentiation of Complex Functions

IE2107 – Engineering Mathematics II

At the end of this lesson, you should be able to:

- Describe the concept of limit and continuity of complex functions.
- Explain the differentiability and analyticity of complex functions.

A complex function f is concerned with complex functions that are differentiable in some domain.



A complex function f is a rule (or mapping) that assigns to every complex number z in a set S, and a complex number w in a set T.

Mathematically, it can be expressed as $w = f(z)$.

The set S is called the domain of f and the set T is called the range of f.

If $z = x + iy$ and $w = u + iv$, then,

$$\boxed{w = f(z) = u(x, y) + iv(x, y)}$$

Let us take a look at a sample problem to understand the concept of complex functions.

Sample Problem 1

Let $w = f(z) = z^2 + 3z$. Find u and v and calculate the value of f at $z = 1 + 3i$.

Solution:

Let $z = x + iy$.

Then, $w = z^2 + 3z$

$$= (x + iy)^2 + 3(x + iy)$$

$$= x^2 - y^2 + i2xy + 3x + i3y$$

Let us take a look at a sample problem to understand the concept of complex functions.

Solution (contd.):

Hence,

$$u = \mathrm{Re}(w) = x^2 - y^2 + 3x$$

$$v = \mathrm{Im}(w) = 2xy + 3y$$

If, $z = x + iy = 1 + i3$

then, $f(z) = u(1,3) + v(1,3) = -5 + i15$

Try using the polar form, $z = r\angle\theta$, and check if you get the same answer.

A function $f(z)$ is said to have the limit L as z approaches a point z_0 if the following conditions are satisfied.

01 $f(z)$ is defined in the neighbourhood of z_0 (except perhaps at z_0 itself).

02 $f(z)$ approaches the same complex number L as $z \to z_0$ from all directions within its neighbourhood.

Mathematically, the limit of a function $f(z)$ can be expressed as:

$$\lim_{z \to z_0} f(z) = L$$

If given $\in$, there exists $\delta > 0$, such that,

$$|f(z) - L| < \in, \forall \; 0 < z - z_0 < \delta$$

The given equation means that the point $f(z)$ can be made arbitrarily close to the point L if the point z is chosen in such a way that it is sufficiently close to, but not equal to the point z_0.

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Examples

1

$$\lim_{z\to\infty}\frac{2z+i}{z+1}=\lim_{z\to\infty}\frac{2+(i/z)}{1+(1/z)}=2$$

2

$$\lim_{z\to\infty}\frac{2z^3-1}{z^2+1}=\lim_{z\to\infty}\frac{2-(1/z^3)}{(1/z)+(1/z^3)}=\lim_{z\to\infty}\frac{2}{0}=\infty$$

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Examples (contd.)

3 $\lim\limits_{z\to\infty} \dfrac{z}{\bar{z}}$ does not exist.

Let $y \to 0$ first and then, let $x \to 0$. In this case,

$$\lim_{x\to 0, y=0} \frac{x+i0}{x-i0} = 1$$

Now, let $x \to 0$ first and then, let $y \to 0$. In this case,

$$\lim_{x=0, y\to 0} \frac{0+iy}{0-iy} = -1$$

As the function does not approach the same value from all directions within its neighbourhood, the limit does not exist.

A function $f(z)$ is said to be continuous at $z = z_0$ if it satisfies the following three conditions.

01	$f(z_0)$ exists
02	$\lim_{z \to z_0} f(z)$ exists
03	$\lim_{z \to z_0} f(z) = f(z_0)$

Note that if condition (3) is true, it implies that conditions (1) and (2) are true as well.

f is said to be a continuous function, if f is continuous for all z in the domain S.

Let us see how to test the continuity of a function with the help of the following sample problem.

Sample Problem 2

Let $f(0) = 0$, and for $z \neq 0$, $f(z) = \mathrm{Re}(z^2)/|z^2|$. Determine whether $f(z)$ is continuous at the origin.

Solution:

$$\lim_{z\to 0} \mathrm{Re}(z^2)/|z^2| = \lim_{z\to 0} \frac{x^2 - y^2}{x^2 + y^2} = \begin{cases} 1 \text{ if } y \to 0 \text{ first} \\ -1 \text{ if } x \to 0 \text{ first} \end{cases}$$

Hence, f is not continuous at the origin.

Let us see how to test the continuity of a function with the help of the following sample problem.

Solution (contd.):

Alternatively, using polar representation,

$$z = re^{i\theta}$$
$$= r\cos\theta + ir\sin\theta$$

$$\lim_{z\to 0} \mathrm{Re}(z^2)/\left|z^2\right| = \lim_{r\to 0} \frac{r^2\cos 2\theta}{r^2} = \cos 2\theta$$

The limit does not exist because it depends on the direction of approach to the origin.

The derivative of a complex function f at a point z_0 is written as $f'(z_0)$ and is defined as:

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$, provided that the limit exists.

Or, by substituting $z = z_0 + \Delta z$

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

For example,

$$\frac{d}{dz}(z^2) = \lim_{\Delta z \to 0} \frac{(z + \Delta z)^2 - z^2}{\Delta z} = \lim_{\Delta z \to 0} (2z + \Delta z) = 2z$$

Thus, $f(z) = z^2$ is differentiable for all z.

The usual differentiation formulae (as in the case of real variables) hold for complex functions. Let us refer to an example.

01 $$\frac{d}{dz}(c) = 0$$

02 $$\frac{d}{dz}(z) = 1$$

03 $$\frac{d}{dz}(z^n) = nz^{n-1}$$

04 $$\frac{d}{dz}\left(2z^2 + i\right)^5 = 5\left(2z^2 + i\right)^4 . 4z = 20z\left(2z^2 + i\right)^4$$

However, care is required for more unusual functions.

Let us take a look at a sample problem to understand the concept of differentiability of complex functions.

Sample Problem 3

Discuss the differentiability of $\bar{z}$.

Solution:

Let $f(z) = \bar{z}$

$$f'(z) = \lim_{\Delta z \to 0} \frac{\overline{z + \Delta z} - \bar{z}}{\Delta z}$$

Using the property $\overline{z + \Delta z} = \bar{z} + \overline{\Delta z}$

$$f'(z) = \lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z}$$

Let us take a look at a sample problem to understand the concept of differentiability of complex functions.

Solution (contd.):

Now, consider $\Delta z = \Delta r e^{i\theta}$. Then, $\Delta z \to 0$ from all directions when $\Delta r \to 0$.

Thus, the limit can be determined as follows:

$$f'(z) = \lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z} = \lim_{\Delta z \to 0} \frac{\Delta r e^{-i\theta}}{\Delta r e^{i\theta}} = e^{-i2\theta}$$

The limit depends on θ, and therefore, it does not exist. Hence, $f(z) = \bar{z}$ is not differentiable anywhere.

A function $f(z)$ is said to be analytic at a point z_0 if its derivative exists not only at z_0, but also in some neighbourhood of z_0.

- A function $f(z)$ is said to be analytic in the domain D if it is analytic at each point in D.
- Hence, analyticity implies differentiability and continuity.
- The point $z = z_0$, where $f(z)$ ceases to be analytic. It is called the singular point or singularity of $f(z)$.

For example,

- $f(z) = z^2$ is analytic everywhere in the complex plane
- $f(z) = \bar{z}$ is not analytic at any point

Cauchy-Riemann (C-R) Equations can be used to test the analyticity of a complex function.

Theorem 1: The complex function $f(z) = u(x, y) + iv(x, y)$ is analytic at a point z_0 if for every point in the neighbourhood of z_0.

1. u, v, and their partial derivatives exist and are continuous.

2. Cauchy-Riemann equations, $u_x = v_y$ and $v_x = -u_y$ are satisfied.

If these two conditions are satisfied in some domain D, then the function is analytic in D.

Derivation of the C-R Equations

The derivative of a complex function f at a point z_0 is given by:

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

$$= \lim_{\Delta x, \Delta y \to 0} \frac{u(x + \Delta x, y + \Delta y) + iv(x + \Delta x, y + \Delta y) - u(x, y) - iv(x, y)}{\Delta x + i\Delta y}$$

Along the x-axis, that is, $\Delta y = 0$,

$$f'(z) = \lim_{\Delta x \to 0} \frac{u(x + \Delta x, y) + iv(x + \Delta x, y) - u(x, y) - iv(x, y)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{u(x + \Delta x, y) - u(x, y) + i(v(x + \Delta x, y) - v(x, y))}{\Delta x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

Derivation of the C-R Equations

Similarly, along the y-axis, that is, $\Delta x = 0$,

$$f'(z) = \lim_{\Delta y \to 0} \frac{u(x, y + \Delta y) + iv(x, y + \Delta y) - u(x, y) - iv(x, y)}{i\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{u(x, y + \Delta y) - u(x, y) + i(v(x, y + \Delta y) - v(x, y))}{i\Delta y} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

For the derivative to exist, the two limits must agree, that is:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

Derivation of the C-R Equations

Thus, the C-R equations are:

$$u_x = v_y \text{ and } v_x = -u_y$$

When $z \neq 0$, the C-R equations in polar coordinates are:

$$u_r = \frac{1}{r} v_\theta \text{ and } v_r = -\frac{1}{r} u_\theta$$

Derivatives of Complex functions

If $f(z) = u(x, y) + iv(x, y)$ and $f'(z)$ exists, then,

$$f'(z) = u_x + iv_x$$

$$= v_y - iu_y$$

$$= u_x - iu_y$$

$$= v_y + iv_x$$

In polar form, if $f(z) = u(r, \theta) + iv\,(r, \theta)$ and $f'(z)$ exists, then,

$$f'(z) = e^{-i\theta}(u_r + iv_r)$$

$$= \frac{1}{r}e^{-i\theta}(v_\theta - iu_\theta)$$

The following sample problem demonstrates how Cauchy-Riemann equations are used to test the analyticity of a complex function.

Sample Problem 4

Verify that $f(z) = \bar{z}$ is not analytic.

Solution:

Using C-R equations,

$$u(x, y) = x \text{ and } v(x, y) = -y$$

Now, $u_y = -v_x = 0$

However, $u_x = 1$ and $v_y = -1$

As the C-R equations are not satisfied, the given function is not analytic.

The following sample problem demonstrates how Cauchy-Riemann equations are used to test the analyticity of a complex function.

Sample Problem 4

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However, $u_x = 1$ and $v_y = -1$

As the C-R equations are not satisfied, the given function is not analytic.

As the function $f(z) = z$ is not differentiable, it can be simply stated that the function is not analytic, without even using the C-R equations.

The following sample problem demonstrates how Cauchy-Riemann equations are used to test the analyticity of a complex function.

Sample Problem 5

Is $f(z) = z^3$ analytic?

Solution:

In general, polynomials of complex variables are analytic. Let's solve the given problem using C-R equations.

$$f(z) = z^3$$

$u(r,\theta) = r^3\cos 3\theta$ and $v(r,\theta) = r^3\sin 3\theta$

The following sample problem demonstrates how Cauchy-Riemann equations are used to test the analyticity of a complex function.

Solution (contd.):

Therefore, $u_r = 3r^2\cos 3\theta$ and $u_\theta = -3r^3\sin 3\theta$

$$v_r = 3r^2\sin 3\theta \text{ and } v_\theta = 3r^3\cos 3\theta$$

As the C-R equations $u_r = \frac{1}{r}v_\theta$ and $v_r = -\frac{1}{r}u_\theta$ are satisfied, and the functions u, v, and their partial derivatives are continuous, the function $f(z) = z^3$ is analytic.

Here is another sample problem that helps us understand how these equations are used to test the analyticity of a complex function.

Sample Problem 6

Discuss the analyticity of the function $f(z) = x^2 + iy^2$.

Solution:

With $u = x^2$ and $v = y^2$: $u_x = 2x$ and $v_y = 2y$

$$v_x = 0 \text{ and } u_y = 0$$

Thus, from C-R equations, $f(z)$ is differentiable only for those values of z that lie along the straight line $x = y$. If z_0 lies on this line, any circle centered at z_0 will contain points for which $f'(z)$ does not exist. Therefore, the given function is not analytic at any point.

Some Common (and Important) Functions

Polynomials, that is, functions of the form, $f(z) = c_0 + c_1 z + c_2 z^2 + \cdots + c_n z^n$ (where c_0, c_1, ..., c_n are complex constants) are analytic in the entire complex plane.

Rational functions, that is, quotient of two polynomials, $f(z) = \dfrac{g(z)}{h(z)}$ are analytic except at points where $h(z) = 0$.

Partial fractions of the form $f(z) = \dfrac{c}{(z - z_0)^m}$, where c and z_0 are complex, and m is a positive integer, are analytic except at z_0.

Summary

Key points discussed in this lesson:

- A complex function f is a rule (or mapping) that assigns to every complex number z in a set S, and a complex number w in a set T.

- A function $f(z)$ is said to have the limit L as z approaches a point z_0 if:
  - $f(z)$ is defined in the neighbourhood of z_0 (except perhaps at z_0 itself)
  - $f(z)$ approaches the same complex number L as $z \to z_0$ from all directions within its neighbourhood

- A function $f(z)$ is said to be continuous at $z = z_0$ if:
  - $f(z_0)$ exists
  - $\lim_{z \to z_0} f(z)$ exists
  - $\lim_{z \to z_0} f(z) = f(z_0)$ exists

Key points discussed in this lesson:

- The derivative of a complex function f at a point z_0 is written as $f'(z_0)$ and is defined as:

  $f'(z_0) = \lim_{z \to z_0} \dfrac{f(z) - f(z_0)}{z - z_0}$, provided that the limit exists.

- A function $f(z)$ is said to be analytic at a point z_0 if its derivative exists not only at z_0, but also in some neighbourhood of z_0.