Vector Calculus

EE2007 – Engineering Mathematics II

At the end of this lesson, you should be able to:

- Perform vector differentiation.
- Describe scalar and vector fields.
- Work with the “∇“ (del) operator - grad, div, curl.

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Velocity $\mathbf{v}$ of the point $\boldsymbol{r}$ is expressed as shown below.

$$\mathbf{v}(t) = \boldsymbol{r}' = \frac{d\boldsymbol{r}}{dt} = \lim_{\Delta t \to 0} \frac{\Delta \boldsymbol{r}}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{\boldsymbol{r}(t + \Delta t) - \boldsymbol{r}(t)}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{[x(t + \Delta t)\mathbf{i} + y(t + \Delta t)\mathbf{j} + z(t + \Delta t)\mathbf{k}] - [x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}]}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{[x(t + \Delta t) - x(t)]\mathbf{i} + [y(t + \Delta t) - y(t)]\mathbf{j} + [z(t + \Delta t) - z(t)]\mathbf{k}}{\Delta t}$$

Velocity $\mathbf{v}$ of the point $\boldsymbol{r}$ is expressed as shown below.

$$= \lim_{\Delta t \to 0} \left[\frac{\Delta x}{\Delta t}\mathbf{i} + \frac{\Delta y}{\Delta t}\mathbf{j} + \frac{\Delta z}{\Delta t}\mathbf{k} \right]$$

$$= \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k}$$

The velocity $v = \boldsymbol{r}'$ is a tangent vector to the curve at the tip of $\boldsymbol{r}$, and points in the direction of increasing t.

$$\boxed{\hat{\mathbf{u}} = \frac{\mathbf{v}}{\|\mathbf{v}\|}}$$

Similarly, acceleration **a** of the point vector ***r*** is expressed as shown below.

$$\mathbf{a}(t) = \mathbf{v}' = \frac{d\mathbf{v}}{dt}$$

$$= \frac{d}{dt}\frac{dx}{dt}\,\mathbf{i} + \frac{d}{dt}\frac{dy}{dt}\,\mathbf{j} + \frac{d}{dt}\frac{dz}{dt}\,\mathbf{k}$$

$$= \frac{d^2x}{dt^2}\,\mathbf{i} + \frac{d^2y}{dt^2}\,\mathbf{j} + \frac{d^2z}{dt^2}\,\mathbf{k}$$

$$= \boldsymbol{r}''$$

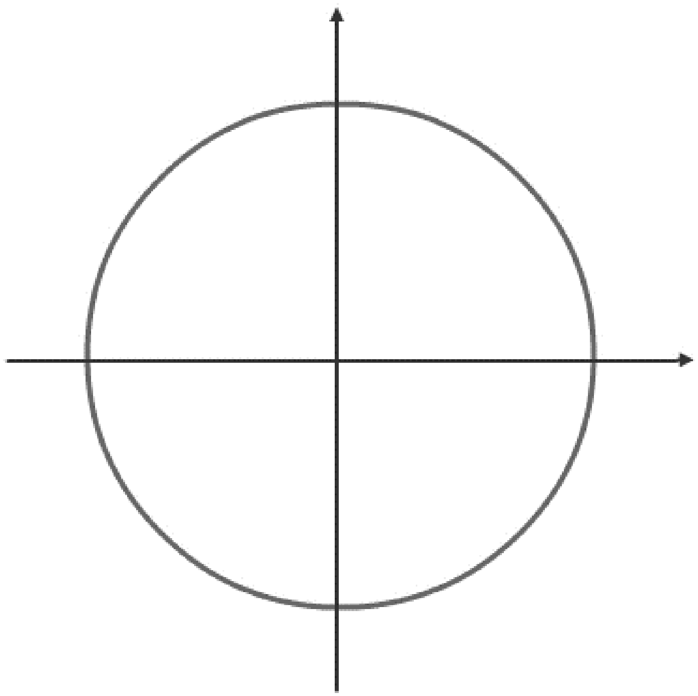
Let’s see how to calculate the velocity and acceleration of a particle from its position vector.

Solution (contd.):

Velocity, $\mathbf{v}(t) = \dfrac{d\mathbf{r}}{dt} = \dfrac{dx}{dt}\,\mathbf{i} + \dfrac{dy}{dt}\,\mathbf{j} + \dfrac{dz}{dt}\,\mathbf{k}$

$$= \frac{d(cost)}{dt}\,\mathbf{i} + \frac{d(\sin t)}{dt}\,\mathbf{j}$$

$$= -sint\;\mathbf{i} + cost\;\mathbf{j}$$



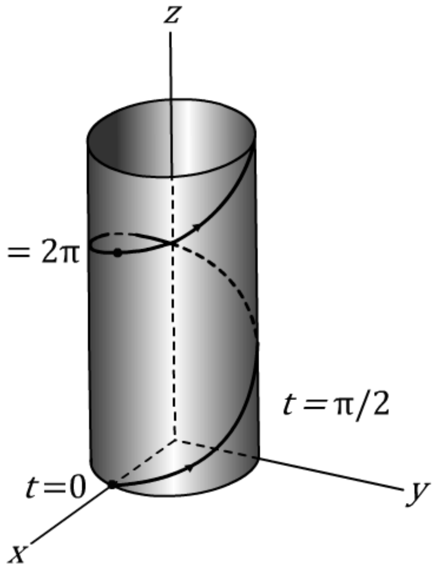
Let’s see how to calculate the velocity and acceleration of a particle from its position vector.

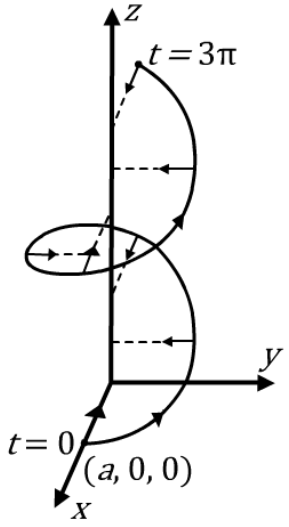
Sample Problem 2

Find the velocity and acceleration for the helical path given by:

$\mathbf{r}(t) = a\cos t\ \mathbf{i} + a\sin t\ \mathbf{j} + bt\ \mathbf{k}$; where a and b are constants.

Solution:





Let's see how to calculate the velocity and acceleration of a particle from its position vector.

Solution (contd.):

Velocity, $\mathbf{v} = -a\sin t\ \mathbf{i} + a\cos t\ \mathbf{j} + b\ \mathbf{k}$

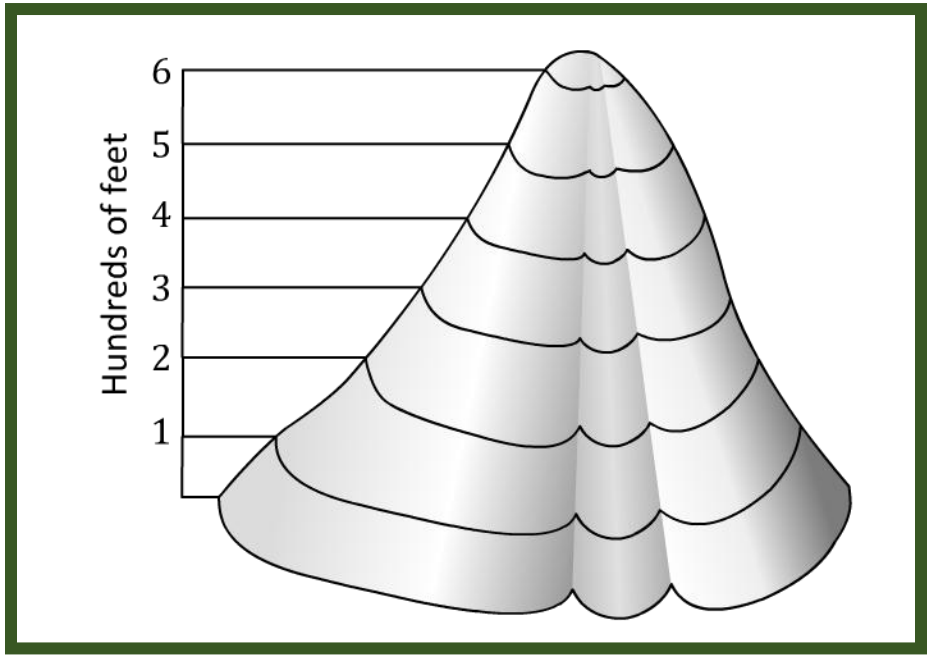
Acceleration, $\mathbf{a} = -a\cos t\ \mathbf{i} - a\sin t\ \mathbf{j}$

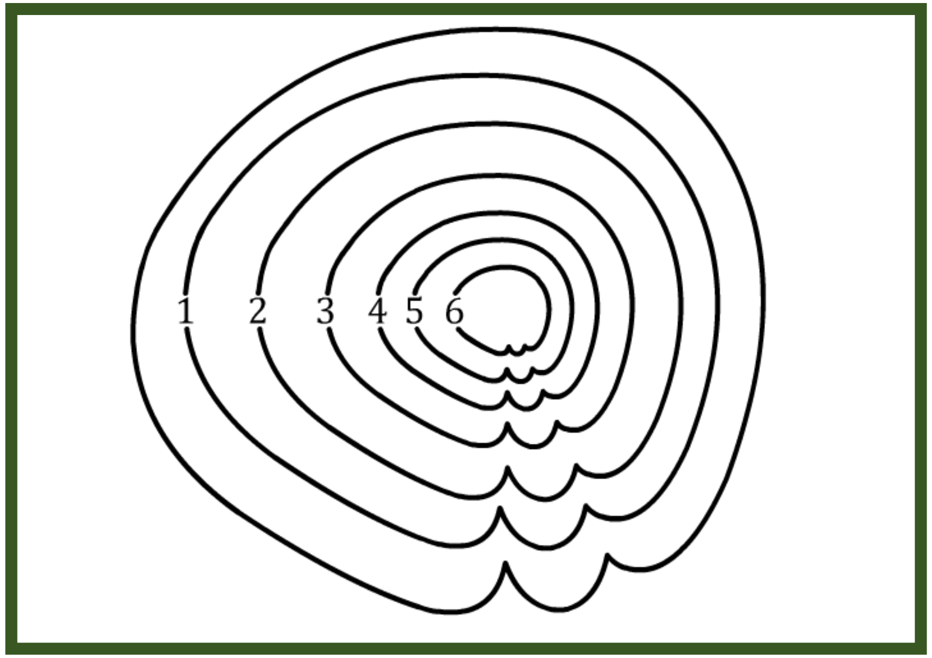
And,

Speed, $\|\mathbf{v}\| = \sqrt{[(-a\sin t)^2 + (a\cos t)^2 + b^2]} = \sqrt{a^2 + b^2}$

Note: The velocity vector is always a tangent vector to the path.

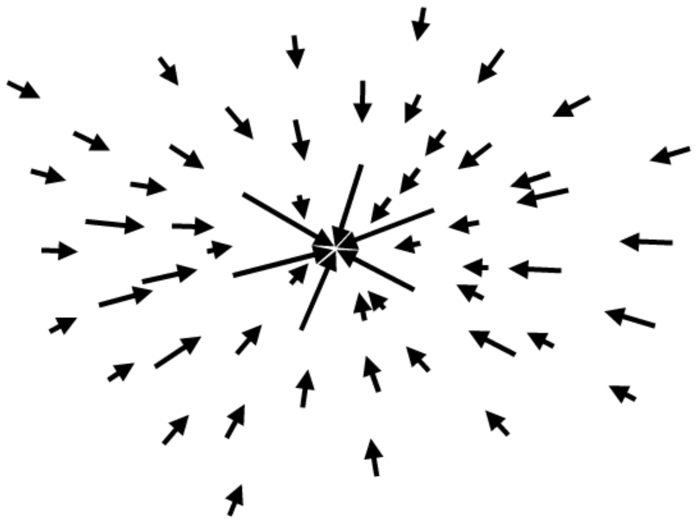
A point in a scalar field possesses a magnitude or value, but no direction. For example, the temperature at a point in a room has a magnitude (say, 20°C). The temperature measured by the thermometer in any direction will have the same magnitude.





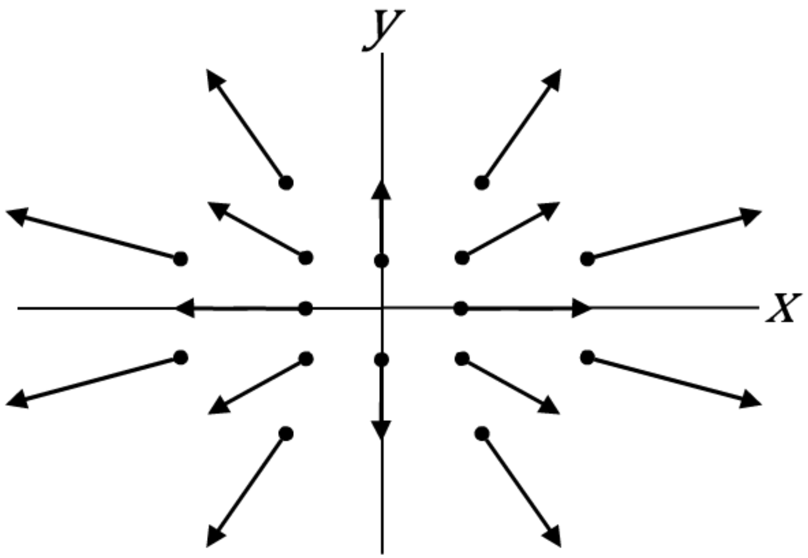
Can you think of more scalar fields?

Let's see a few examples of vector fields.



Gravitational field

$$\mathbf{F} = \frac{GMm}{\|\mathbf{r}^2\|}\left[\frac{-\mathbf{r}}{\|\mathbf{r}\|}\right]$$

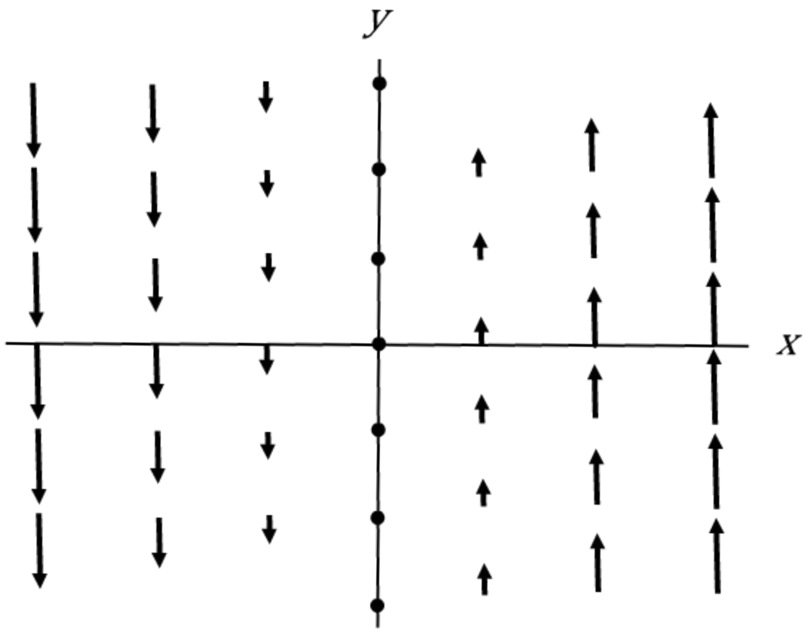


Forces on a plane

$$\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j}$$

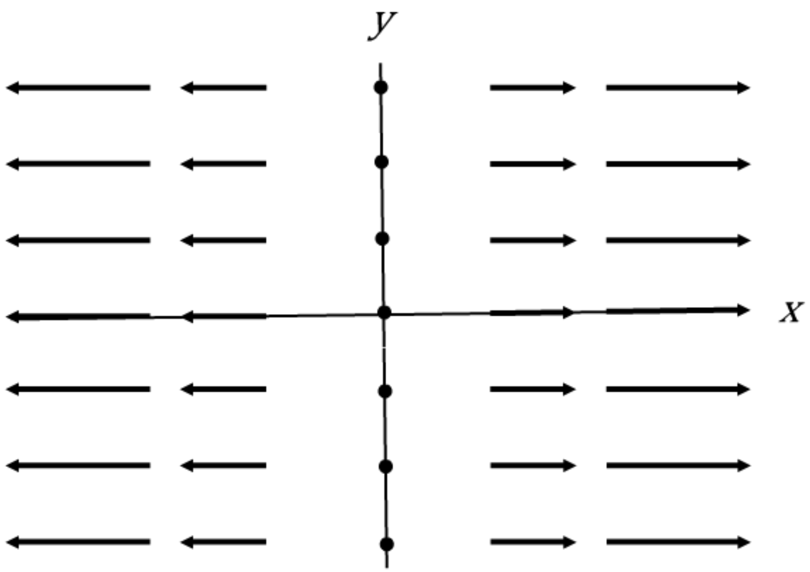
Some examples of vector fields are the earth's gravitational field, force fields, electro-magnetic fields, flow of water in a pipeline, etc.

If the vector field lines flow in the vertical direction, it is known as a vertical flow field, whereas, if the field lines flow in the horizontal direction, it is known as a horizontal flow field.



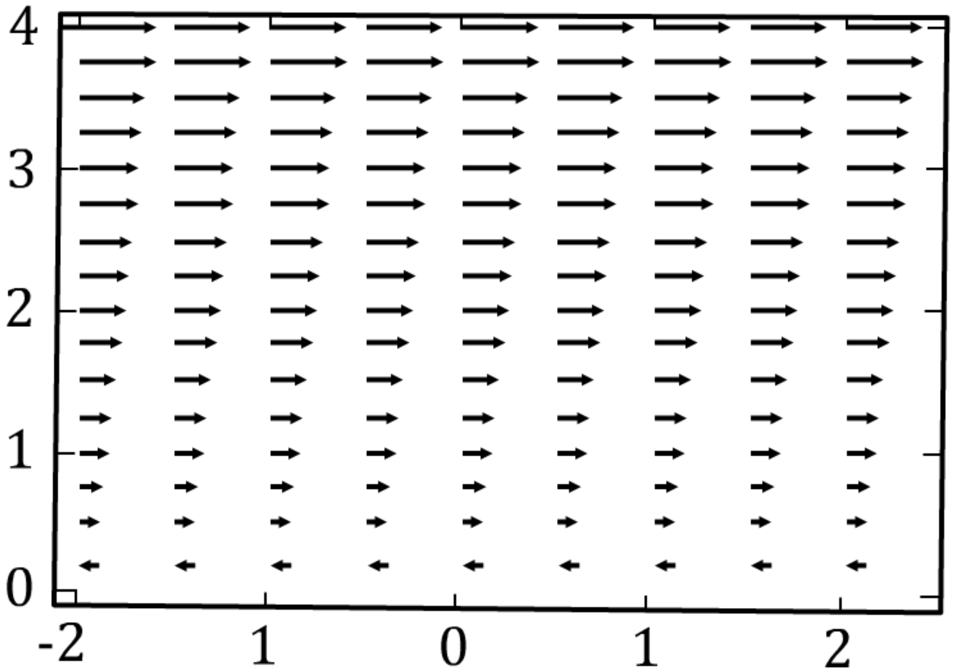
$\mathbf{F} = x\mathbf{j}$

Vertical flow field



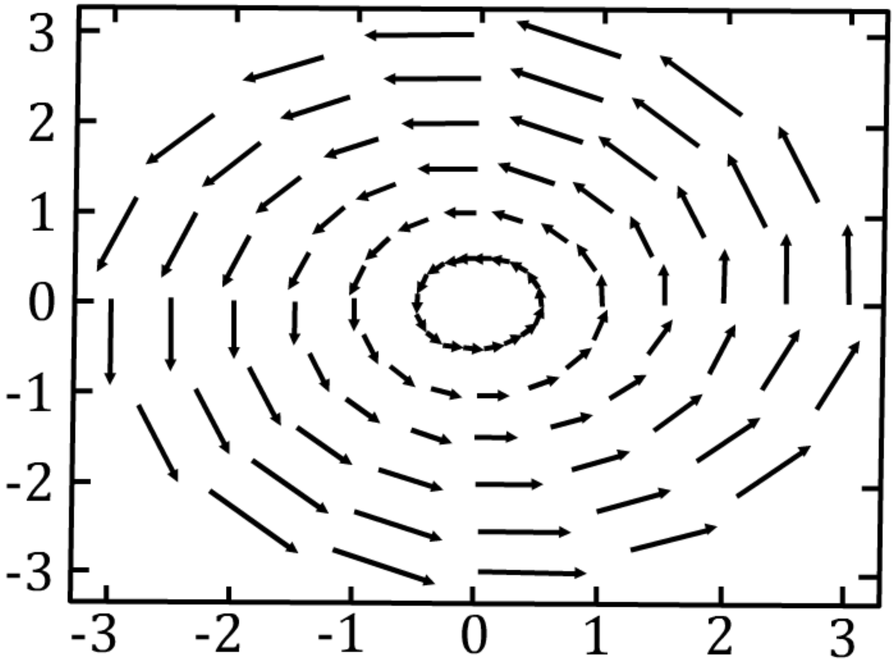
$\mathbf{F} = x\mathbf{i}$

Horizontal flow field



$$\mathbf{F}(x,y) = \frac{1}{5}\sqrt{y}\,\mathbf{i}$$

Horizontal flow with magnitudes as a function of the vertical axis



$$\mathbf{F}(x,y) = -y\mathbf{i} + x\mathbf{j}$$

2D vortex

The del operator is represented as ∇. This operator can be used to operate on scalar fields and vector fields.

The gradient of a scalar field: Grad $f = \nabla f$

$$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}$$

When applied to a scalar field f, yields

$$\nabla f = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right) f$$

$$= \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

Note that f is a scalar and the resulting operation ∇f is a vector.

Do not confuse ∇ with Δ. Δf is usually considered to be a small change in f.

Let us look at an example to understand the del operator.

Example 1

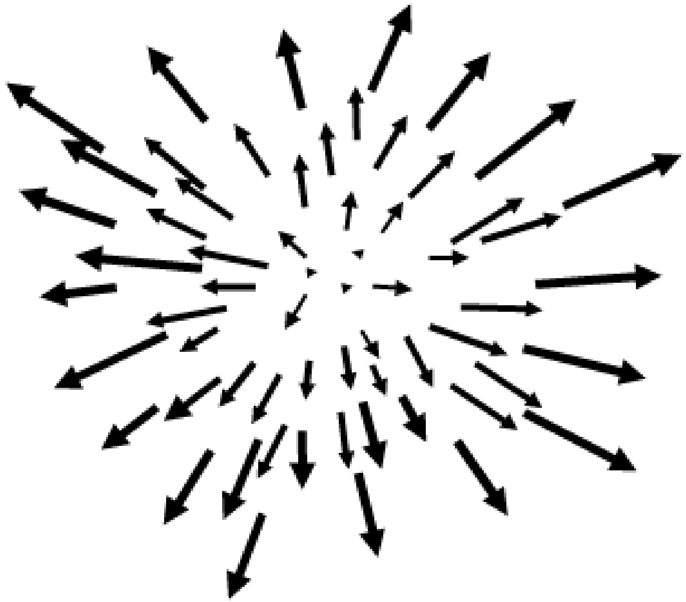
Given that $f(x, y, z)$ is a 3D scalar field :

$$f = x^2 + y^2 + z^2$$

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

$$= 2x\,\mathbf{i} + 2y\,\mathbf{j} + 2z\,\mathbf{k}$$

$$= 2r$$



$\mathbf{F} = 2r$

∇f is a vector field, and is called the gradient of f, or grad f.

Let us now understand some properties of del operator ∇f (grad f).

Consider a scalar field f, differentiable at the point (a, b, c), and $\nabla f(a, b, c)$ is not zero.

- $\nabla f(a, b, c)$ is in the direction of the greatest rate of change of f.
- $\nabla f(a, b, c)$ is perpendicular to the level surface of f at (a, b, c). That is, it is a surface normal at (a, b, c).
- $\|\nabla f(a, b, c)\|$ is the maximum rate of change of f at (a, b, c).

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Level surface (or equipotential surface) of f are points such that $f(x, y, z) =$ c, where c is a constant.

$\|\nabla f(a, b, c)\|$ is the maximum rate of change of f at (a, b, c).

Let us look at another example to understand the concept of del operator.

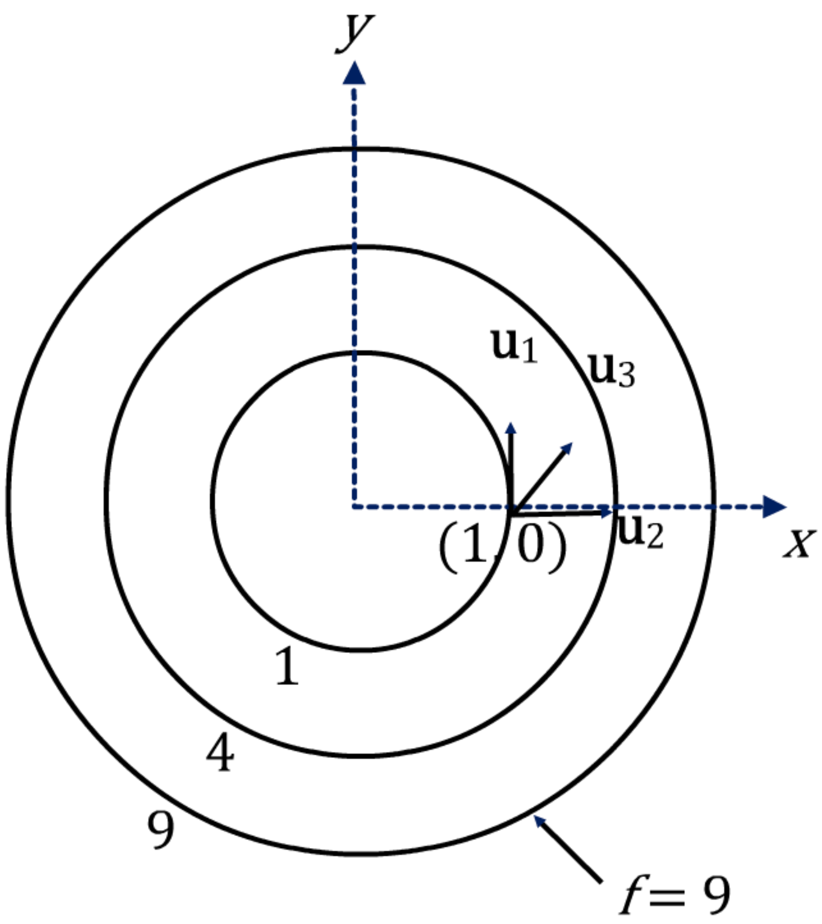
Example 2

Find the height of the stadium.

$$f = x^2 + y^2$$

$$\nabla f = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right) x^2 + y^2$$

$$= 2x\,\mathbf{i} + 2y\,\mathbf{j}$$



Let us now become familiar with the concept of directional derivative.

The magnitude of the rate of change or slope of f in a particular direction, say $\mathbf{u}$, is called the directional derivative.

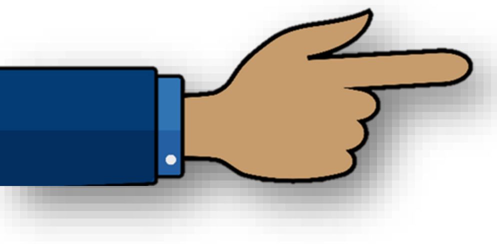
The magnitude of this rate of change is calculated in the following way:

$\nabla f \cdot \mathbf{u}$ (directional derivative of f in the direction $\mathbf{u}$)

where $\mathbf{u}$ is taken to be a unit vector. If $\mathbf{u}$ is not a unit vector, replace it with $\mathbf{u}/\|\mathbf{u}\|$.

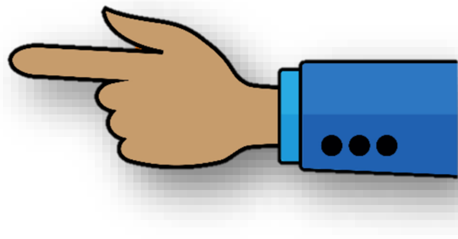
The directional derivative formula has the same use as what you have learnt before: finding the component of one vector in the direction of another vector.

Let us now become familiar with the concept of directional derivative.



In the previous example, you have seen that $\nabla f(x, y) = 2x\,\mathbf{i} + 2y\,\mathbf{j}$. Thus, $\nabla f(1, 0) = 2\mathbf{i}$ is a vector, perpendicular to the contour of f at the point $(1, 0)$, and this direction is the direction of maximum increase of f at the point $(1, 0)$.

Also, this maximum rate of change at the point $(1, 0)$ can be calculated by $\|\nabla f(1,0)\| = \|2\mathbf{i}\| = 2$.
This is the directional derivative of f in the direction of the gradient of f and is the maximum at the point (1, 0).



Here is how you calculate the directional directive for different directions of $\mathbf{u}$ at different point (x, y) to verify the properties of del operator.

At point $(1, 0)$, consider directions $\mathbf{u}_1 = \mathbf{j}, \mathbf{u}_2 = \mathbf{i}, \mathbf{u}_3 = (\mathbf{i} + \mathbf{j})$

Directional derivative along $\mathbf{u}_1$

$$= \nabla f \cdot \mathbf{u}_1 = 2y = 0$$

Directional derivative along $\mathbf{u}_2$

$$= \nabla f \cdot \mathbf{u}_2 = 2x = 2$$

Directional derivative along $\mathbf{u}_3$

$$= \nabla f \cdot \mathbf{u}_3 / \|\mathbf{u}_3\|$$

$$= (2x + 2y)\frac{1}{\sqrt{2}} = \sqrt{2}$$

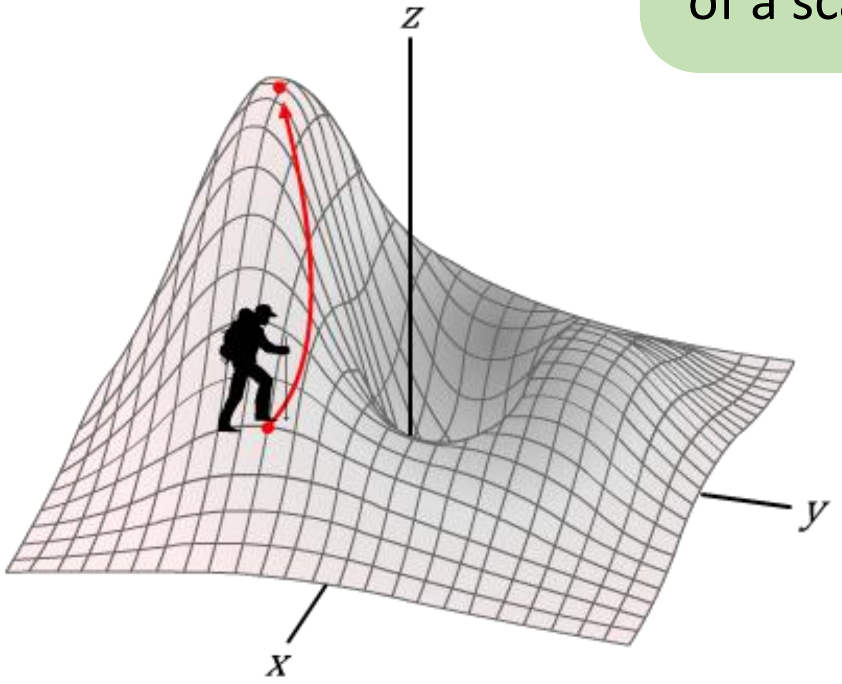
The unit vector perpendicular to the surface of f at (x, y, z), that is, the normal direction to the surface at a point on the surface is given below.

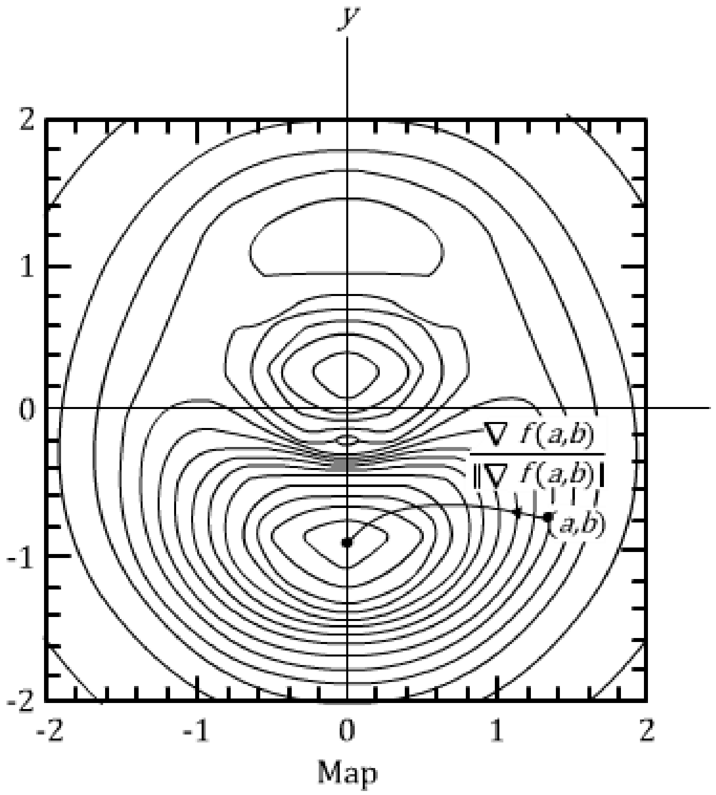
$$\frac{\nabla f\ (x, y, z)}{\|\nabla f\ (x, y, z)\|}$$

Another property of ∇f is that it is normal to the (equipotential) surface of f.

Let us look at an example on how to reach the top of the “hill” by following the path of grad f, that is, the path of maximum rate of change of $\mathbf{f}$.

This technique is known as “hill climbing” or “gradient ascent” method to find the maximum of a scalar function.



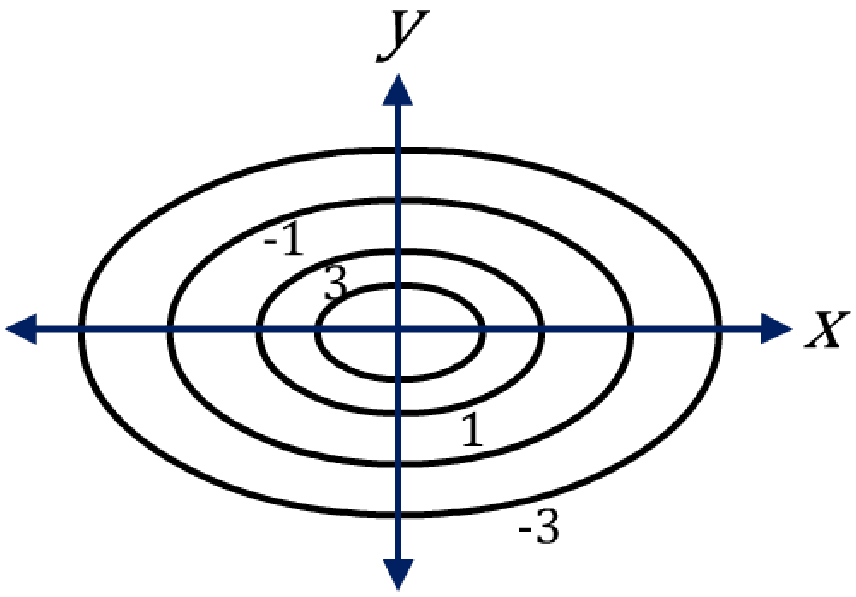


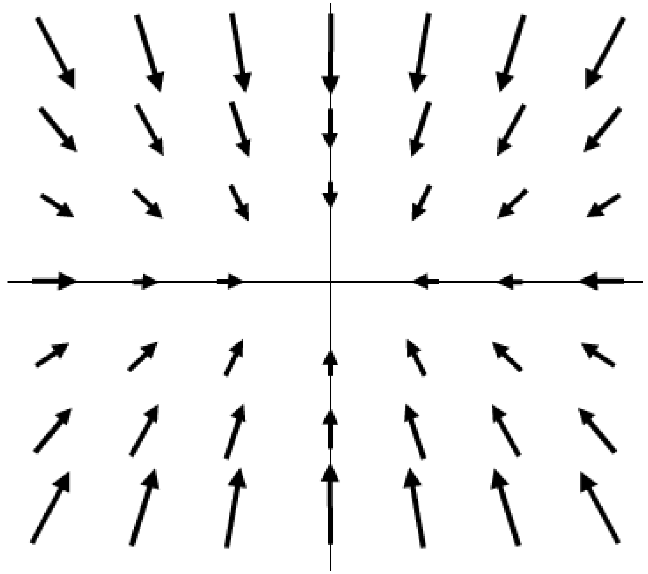
Let us look at an example to understand how del operator works.

Example 3

Given $g(x,y) = 5 - x^2 - 2y^2$

$$\nabla g(x,y) = -2x\mathbf{i} - 4y\mathbf{j}$$



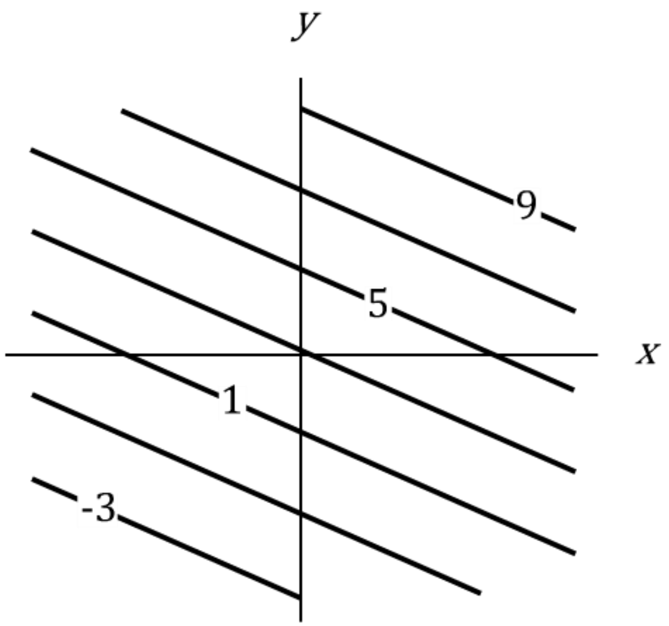


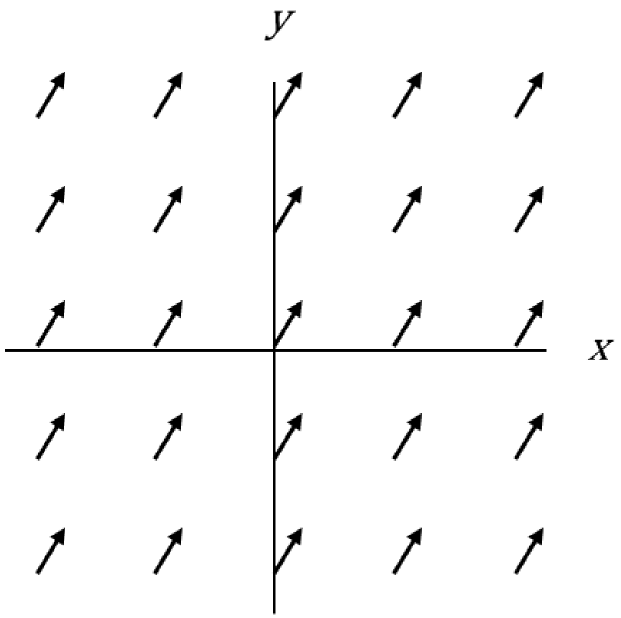
Let us look at another example to understand how del operator functions.

Example 4

Given $h(x, y) = x + 2y + 3$

$$\nabla h(x, y) = \mathbf{i} + 2\mathbf{j}$$





Here is another example to understand how del operator works.

Example 5

Find the directional derivative of $f = x^2yz + 4xz^2$ at $(1, -2, -1)$ in the direction $2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$, and the unit normal to the surface at this point.

$$f = x^2yz + 4xz^2 = 6 \text{ at } (1, -2, -1).$$

$$\nabla f = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)(x^2yz + 4xz^2)$$

$$= (2xyz + 4z^2)\mathbf{i} + (x^2z)\mathbf{j} + (x^2y + 8xz)\mathbf{k}$$

$$= 8\mathbf{i} - \mathbf{j} - 10\mathbf{k} \text{ at } (1, -2, -1)$$

Here is another example to understand how del operator works.

Example 5 (contd.)

The unit vector in the direction of $2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$ is:

$$\hat{\mathbf{u}} = \frac{2\mathbf{i} - \mathbf{j} - 2\mathbf{k}}{\sqrt{2^2 + (-1)^2 + (-2)^2}} = \frac{2}{3}\mathbf{i} - \frac{1}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}$$

The required directional derivative is:

$$\nabla f \cdot \hat{\mathbf{u}} = (8\mathbf{i} - \mathbf{j} - 10\mathbf{k}) \cdot \left(\frac{2}{3}\mathbf{i} - \frac{1}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}\right) = \frac{16}{3} + \frac{1}{3} + \frac{20}{3} = \frac{37}{3}$$

Here is another example to understand how del operator works.

Example 5 (contd.)

The unit normal at $(1, -2, -1)$ is:

$$\frac{\nabla f\ (x, y, z)}{\|\nabla f\ (x, y, z)\|} = \frac{(8\mathbf{i} - \mathbf{j} - 10\mathbf{k})}{\sqrt{8^2 + (-1)^2 + (-10)^2}}$$

$$= \frac{1}{\sqrt{165}}(8\mathbf{i} - \mathbf{j} - 10\mathbf{k})$$

Below are the two possible ways for ∇ to operate on vector field $\mathbf{F}$.

- $\nabla . \mathbf{F}$: This is called the divergence of $\mathbf{F}$, and the result is scalar.
- $\nabla \times \mathbf{F}$: This is called the curl of $\mathbf{F}$, and the result is vector.

For a vector field $\mathbf{F}$, the divergence is given as below.

If the vector field $\mathbf{F}$ is given as $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$

Then, the divergence of $\mathbf{F}$ is given as:

$$\nabla \cdot \mathbf{F} = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right) \cdot (F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k})$$

$$= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

The result is a scalar field and is also written as div $\mathbf{F}$.

Let us look at an example to understand the concept of divergence of a vector.

Example 6

Given: $\mathbf{v} = 3xz\mathbf{i} + 2xy\mathbf{j} - yz^2\mathbf{k}$

The divergence of vector $\mathbf{v}$ is given as:

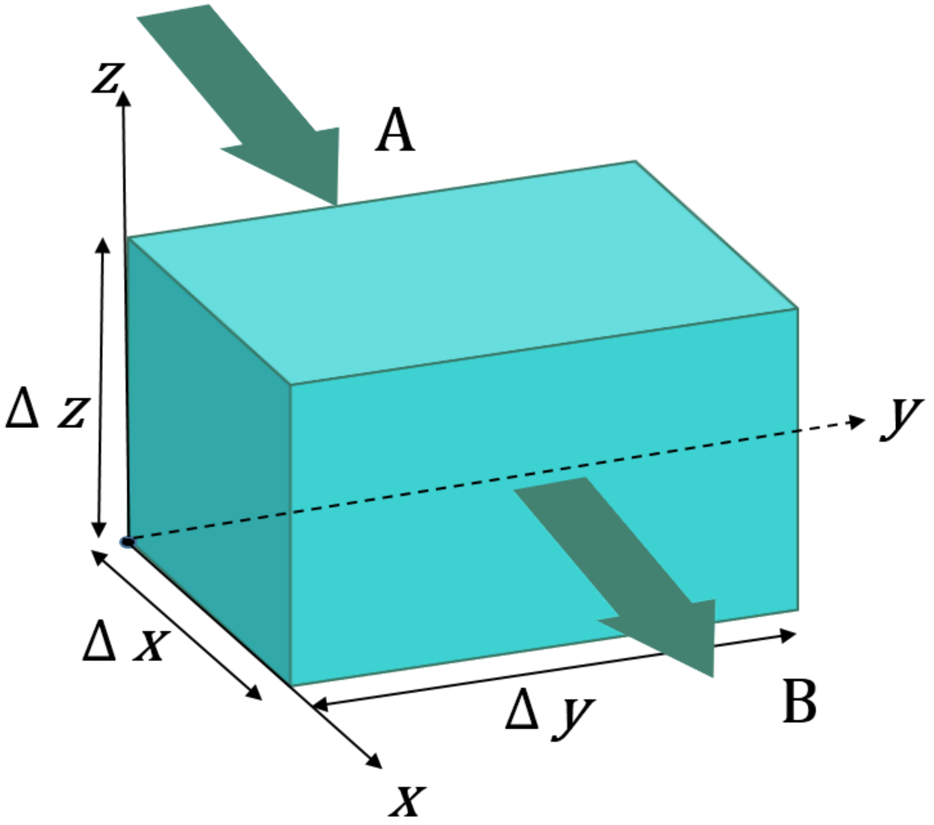
$$\nabla \cdot \mathbf{v} = \frac{\partial(3xz)}{\partial x} + \frac{\partial(2xy)}{\partial y} + \frac{\partial(-yz^2)}{\partial z}$$

$$= 3z + 2x - 2yz$$

The physical interpretation of divergence can be shown in the given figure below.

The divergence (of a vector field) tells the extent to which the field explodes or diverges.

Consider the movement of a fluid, so that its velocity at any point is $\mathbf{v}(x, y, z)$, then the net rate of gain in fluid per unit volume at any point is given by $\nabla \cdot \mathbf{v}$.



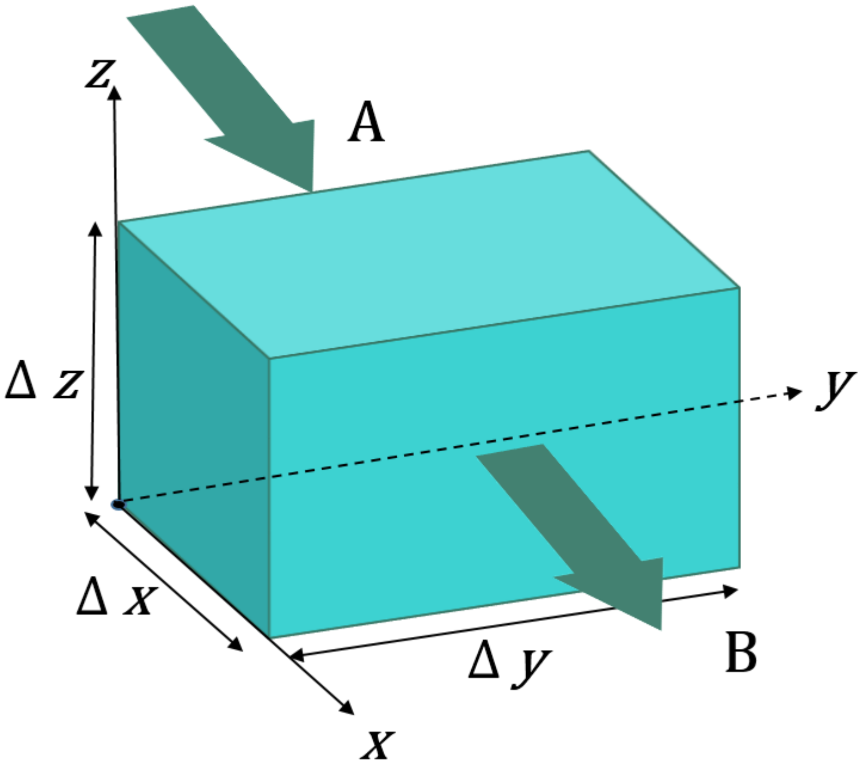
The physical interpretation of divergence can be shown in the given figure below.

If $\nabla \cdot \mathbf{v} = 0$ at a point, then the rate at which the fluid is flowing into that point is equal to the rate at which the fluid is flowing out.

If $\nabla \cdot \mathbf{v} = 0$ everywhere, fluid is incompressible (or solenoidal or divergence free).

If $\nabla \cdot \mathbf{v} > 0$ at a point, more fluid is flowing out than in. For example, the existence of a 'source'.

If $\nabla \cdot \mathbf{v} < 0$ at a point, more fluid is flowing in than out. For example, the existence of a 'sink'.



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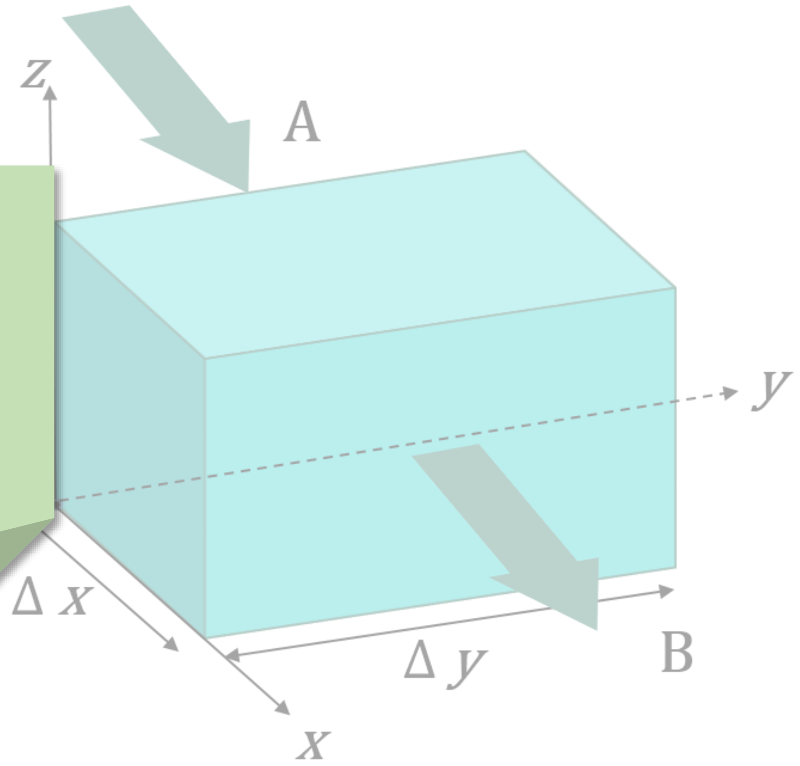
example, the existence of a

The notion of divergence can be extended to any vector field such as magnetic, electric, etc.



If $\nabla \cdot \mathbf{v} < 0$ at a point, more fluid is flowing in than out.
For example, the existence of a ‘sink’.

Let us look at an example to understand the concept of divergence of a vector.

Example 7

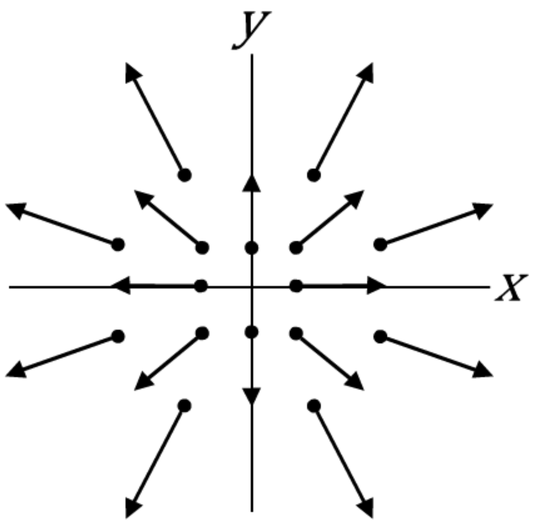
i) $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$

The divergence of vector $\boldsymbol{F}$ is given as:

$$\nabla \cdot \mathbf{F} = \frac{\partial F}{\partial x}\mathbf{i} + \frac{\partial F}{\partial y}\mathbf{j}$$

$$= \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y}$$

$$= 1 + 1 = 2$$



ii) Similarly if, $F = -x\mathbf{i} - y\mathbf{j}$

$$\nabla \cdot \ \mathbf{F} = -2$$

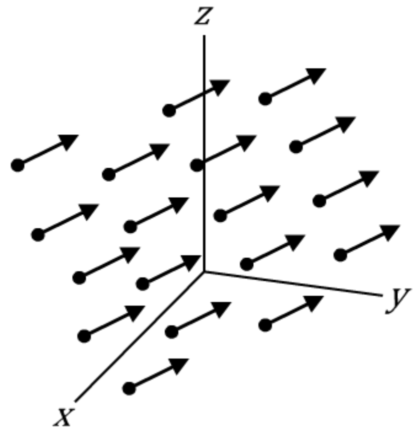
Let us look at an example to understand the concept of divergence of a vector.

Example 8

i) $\mathbf{F} = a$ (constant vector)

The divergence of vector $\mathbf{F}$ is given as:

$$\nabla \cdot \mathbf{F} = 0$$

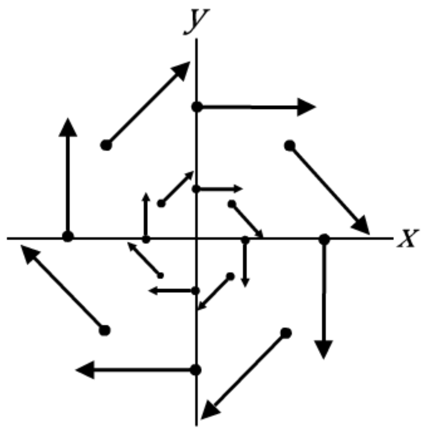


ii) $\mathbf{F} = y\mathbf{i} - x\mathbf{j}$

The divergence of vector $\mathbf{F}$ is given as:

$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}\mathbf{i} \cdot \mathbf{F} + \frac{\partial}{\partial y}\mathbf{j} \cdot \mathbf{F}$$

$$= \frac{\partial y}{\partial x} + \frac{\partial(-x)}{\partial y} = 0$$



For a vector field $\mathbf{F}$, the curl is represented as shown below.

If the vector field $\mathbf{F}$ is expressed as $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$

Then, the curl of $\mathbf{F}$ is expressed as:

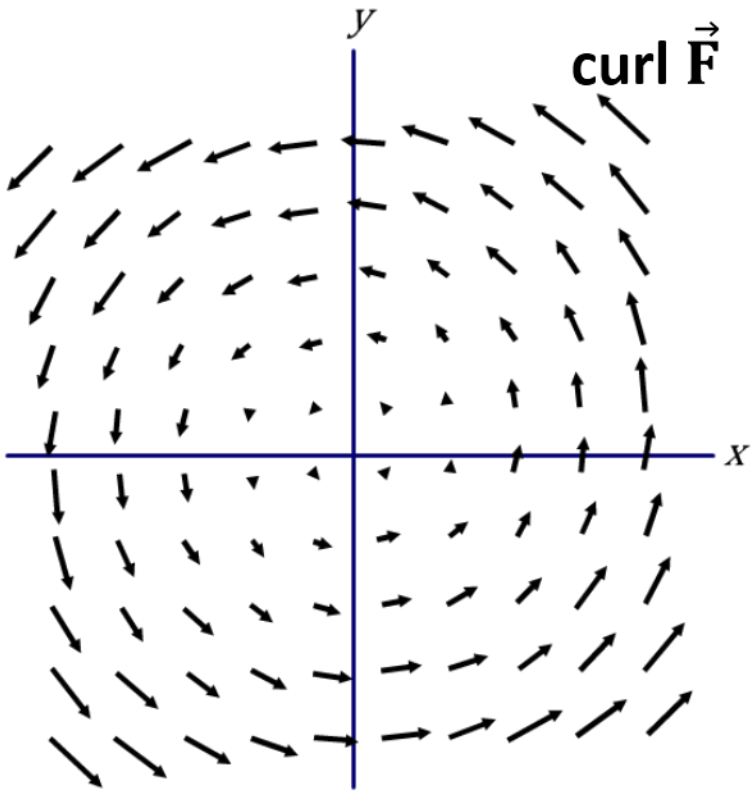
$$\nabla \times \mathbf{F} = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right) \times \left(F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}\right)$$

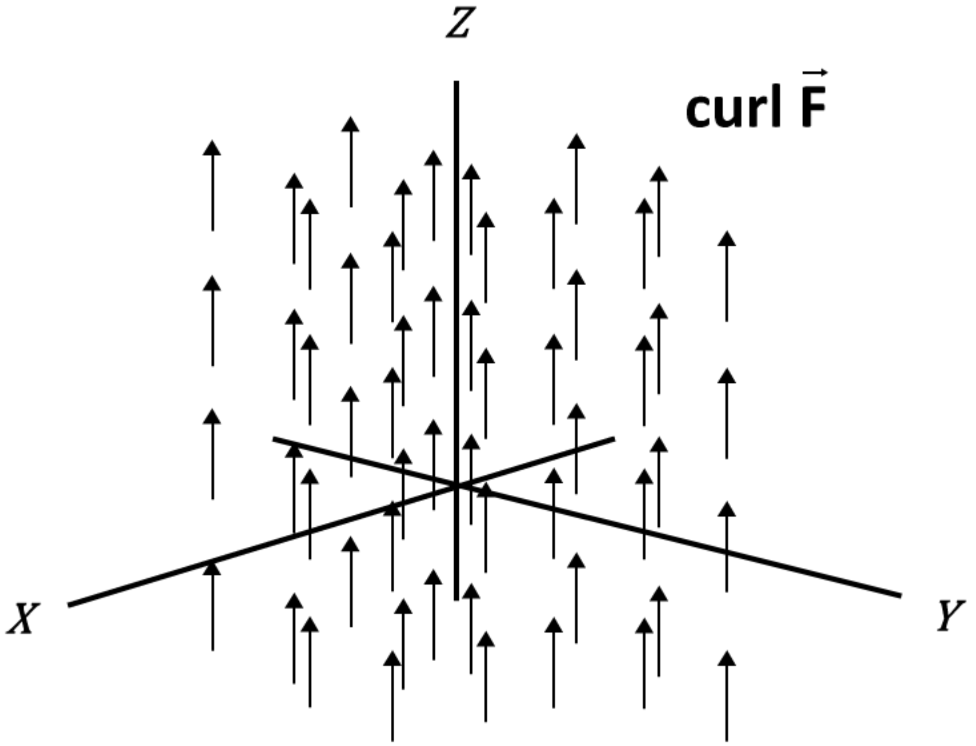
$$= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{k}$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$ which is a vector field.

Let us understand the concept of curl by observing the examples given below.

The curl of a vector is related to the 'circulation' at each point of the vector field.





A curl exists when there is a component of a field whose value is changing in a direction orthogonal to this component.

The result shows that the curl is in the direction orthogonal to both the component and the direction of change.

For example, $\mathbf{F} = 2y^2\mathbf{i}$

$\mathbf{F}$ has a component in the $\mathbf{i}$ direction and its value $(= 2y^2)$ is changing in the $\mathbf{j}$ direction ($\mathbf{j}$ is orthogonal to $\mathbf{i}$). Thus, a curl exists and is equal to $-4\mathbf{yk}$, ($\mathbf{k}$ is orthogonal to both $\mathbf{i}$ and $\mathbf{j}$).

Let us look at an example to understand the curl of a vector.

Example 9

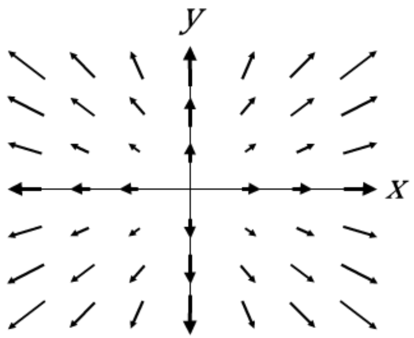
i) $\mathbf{F} = -2y\mathbf{i} + 2x\mathbf{j}$

The curl of vector $\boldsymbol{F}$ is given as:

$$\nabla \times \mathbf{F} = \left(\frac{\partial(0)}{\partial y} - \frac{\partial(2x)}{\partial z}\right)\mathbf{i} - \left(\frac{\partial(0)}{\partial x} - \frac{\partial(-2y)}{\partial z}\right)\mathbf{j} + \left(\frac{\partial(2x)}{\partial x} - \frac{\partial(-2y)}{\partial y}\right)\mathbf{k}$$

$$= 4\mathbf{k}$$

ii) $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$



The curl of vector $\mathbf{F}$ is given as:

$$\nabla \times \mathbf{F} = \left(\frac{\partial(0)}{\partial y} - \frac{\partial y}{\partial z}\right)\mathbf{i} - \left(\frac{\partial(0)}{\partial x} - \frac{\partial x}{\partial z}\right)\mathbf{j} + \left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right)\mathbf{k} = 0$$

Let us look at an example to understand the curl of a vector.

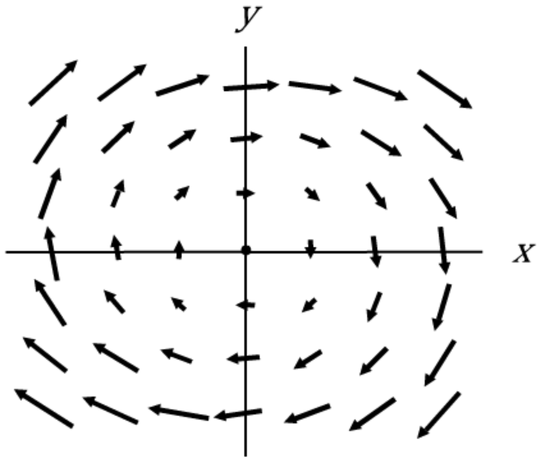
Example 10

Given: $\mathbf{F} = y\mathbf{i} - x\mathbf{j}$

The curl of vector $\mathbf{F}$ is given as:

$$\nabla \times \mathbf{F} = \left(\frac{\partial(0)}{\partial y} - \frac{\partial(-x)}{\partial z}\right)\mathbf{i} + \left(\frac{\partial y}{\partial z} - \frac{\partial(0)}{\partial x}\right)\mathbf{j} + \left(\frac{\partial(-x)}{\partial x} - \frac{\partial(y)}{\partial y}\right)\mathbf{k}$$

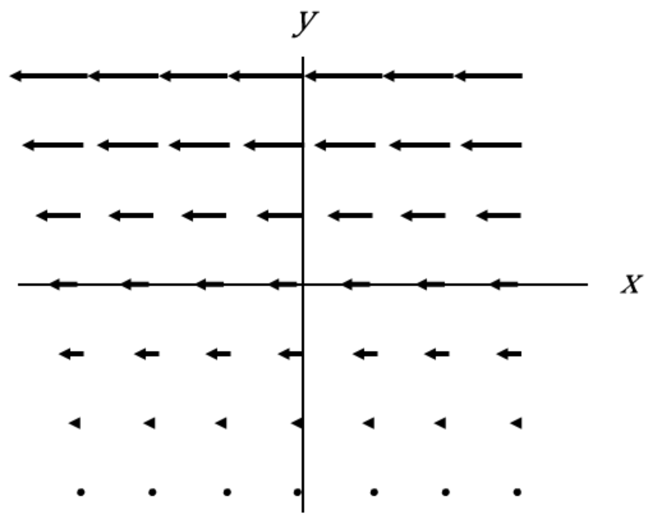
$$= -2\mathbf{k}$$



Let us look at another example to understand the curl of a vector.

Example 11

Given: $\mathbf{F} = -(y+1)\mathbf{i}$



The curl of vector $\mathbf{F}$ is given as:

$$= \nabla \times \mathbf{F}$$

$$= \left(\frac{\partial(0)}{\partial y} - \frac{\partial(0)}{\partial z}\right)\mathbf{i} + \left(\frac{\partial(-y-1)}{\partial z} - \frac{\partial(0)}{\partial x}\right)\mathbf{j} + \left(\frac{\partial(0)}{\partial x} - \frac{\partial(-y-1)}{\partial y}\right)\mathbf{k}$$

$$= \mathbf{k}$$

Let us now get familiar with the concept of Laplacian.

$$\mathrm{div}(\nabla f) = \nabla \cdot (\nabla f) = \nabla^2 f$$

$$\mathrm{div}(\nabla f) = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right) . \left(\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}\right)$$

$$= \frac{\partial^2 f}{x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

$$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) f$$

Let us look at an example to understand the concept of Laplacian.

Example 12

Let $f = x^4y^3 + 1/z$

$$\nabla^2 f = \frac{\partial^2 f}{\partial^2 x} + \frac{\partial^2 f}{\partial^2 y} + \frac{\partial^2 f}{\partial^2 z}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(4x^3y^3\right) = 12x^2y^3$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(x^4 3y^2\right) = 6x^4y$$

$$\frac{\partial^2 f}{\partial z^2} = \frac{\partial}{\partial z}\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}\left(-z^{-2}\right) = 2z^{-3}$$

$$\therefore \nabla^2 f = 12x^2y^3 + 6x^4y + 2z^{-3}$$

Some other useful formulae to be remembered in del operator.

$$\nabla(f\,g) = f\,\nabla g + g\,\nabla f$$

$$\nabla \cdot (f\,\mathrm{v}) = f\,\nabla \cdot \mathrm{v} + \mathrm{v} \cdot \nabla f$$

$$\nabla \times (f\,\mathrm{v}) = f\,\nabla \times \mathrm{v} + \nabla f \times \mathrm{v}$$

$$\nabla \times (\nabla f) = 0$$

$$\nabla \cdot (\nabla \times \mathrm{v}) = 0$$

In the given formulae: f and g must denote scalar functions and **v** must denote a vector function.

Summary

Key points discussed in this lesson:

- The velocity $\mathbf{v}$ of the point $\boldsymbol{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ is given by:

  $$\frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k}$$

- The acceleration $\mathbf{a}$ of the point $\boldsymbol{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ is given by:

  $$\frac{d^2x}{dt^2}\mathbf{i} + \frac{d^2y}{dt^2}\mathbf{j} + \frac{d^2z}{dt^2}\mathbf{k}$$

- A point in a scalar field possesses a magnitude or value, but no direction.

- The del operator is represented as ∇. This operator can be used to operate on scalar fields and vector fields.

Key points discussed in this lesson:

- The gradient of a scalar field: Grad $f = \nabla f$ where, f is a scalar and the resulting operation ∇f is a vector.

- The magnitude of the rate of change or slope of f in a particular direction, say $\mathbf{u}$, is called the directional derivative.

- The two possible ways for ∇ to operate on a vector field $\mathbf{F}$ are:
  - $\nabla \cdot \mathbf{F}$: This is called the divergence of $\mathbf{F}$, and the result is scalar.
  - $\nabla \times \mathbf{F}$: This is called the curl of $\mathbf{F}$, and the result is vector.

Key points discussed in this lesson:

- The divergence (of a vector field) tells the extent to which the field explodes or diverges.
- The curl of a vector is related to the ‘circulation’ at each point of the vector field.
- A curl exists when there is a component of a field whose value is changing in a direction orthogonal to this component.