Vector Calculus

EE2007 – Engineering Mathematics II

At the end of this lesson, you should be able to:

- Perform the vector line integral.
- Perform the vector surface integral.

Vector line integral is the operation of integrating a vector field along a curve in space.

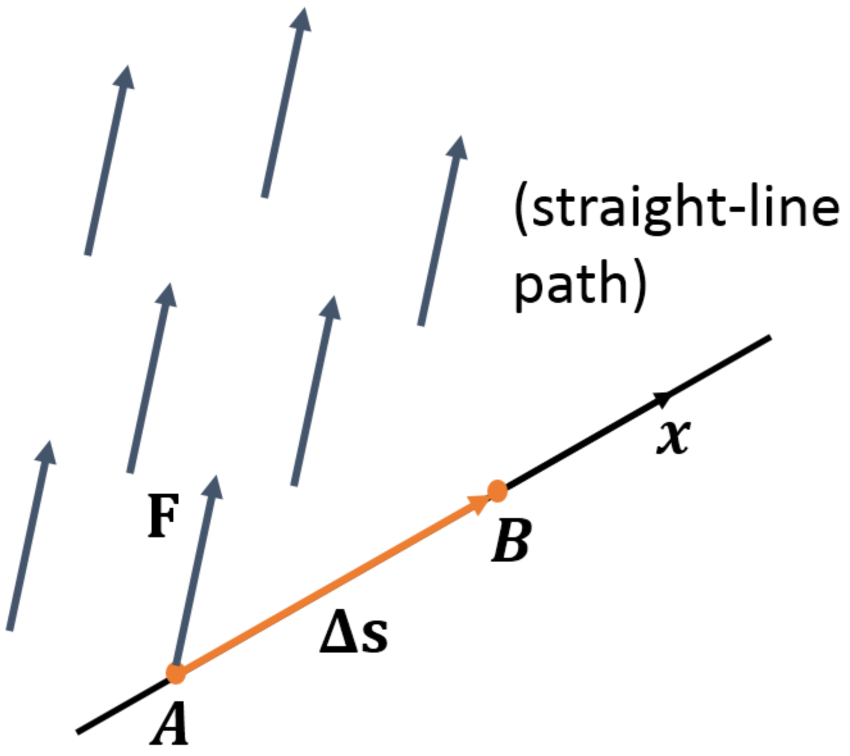
- The curve is expressed in a parametric form by writing x, y, z as functions of a single parameter 't'.
- The parameter 't' can then be used to specify points $(x(t), y(t), z(t))$ along the curve.
- Such integrals have considerable utility in many areas of Engineering and Physics.

The work done W by a force F moving an object is one such example of vector line integrals.

From the given figure, for a constant force field F pushing an object in a straight line, is expressed as:

$$W = \mathbf{F} \cdot \mathbf{r}$$

$$W = \|\mathbf{F}\| \|\mathbf{r}\| \cos\theta$$



If $\mathbf{F}$ is not a constant force and if the object is pushed along a curve instead of a straight line, then use a line integral to calculate the work done.

Let us look at an example to understand the vector line integral.

Example 1

Given $\mathbf{F} = -y\ \mathbf{i}\ - xy\ \mathbf{j}$, and C is the circular arc from A to B, calculate the line integral $\mathbf{F} \cdot d\mathbf{r}$.

The points on the arc can be written as $\mathbf{r} = x\ \mathbf{i} + y\ \mathbf{j}$

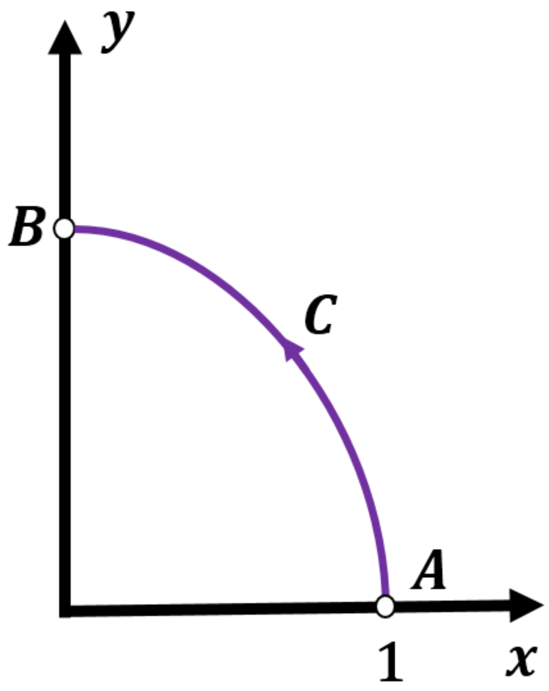
Where, $x^2 + y^2 = 1$

Let us define t such that

$x = \cos t, y = \sin t, 0 \le t \le \pi/2$

Then, $\mathbf{r} = \cos t\ \mathbf{i} + \sin t\ \mathbf{j}$

$$d\mathbf{r} = (-\sin t\ \mathbf{i} + \cos t\ \mathbf{j})\ dt$$



Let us look at an example to understand the vector line integral.

Example 1 (contd.)

$$W = \int_{C} \mathbf{F} \cdot d\mathbf{r}$$

$$= \int_{0}^{\pi/2} (-\sin t\ \mathbf{i} + \cos t \sin t\ \mathbf{j}\ ) \cdot (-\sin t\ \mathbf{i} + \cos t\ \mathbf{j}\ )dt$$

$$= \int_{0}^{\pi/2} (\sin^2 t + \cos^2 t\ \mathrm{sin}t)\mathrm{d}t = \frac{\pi}{4} + \frac{1}{3}$$

Let us look at another example to understand the vector line integral.

Example 2

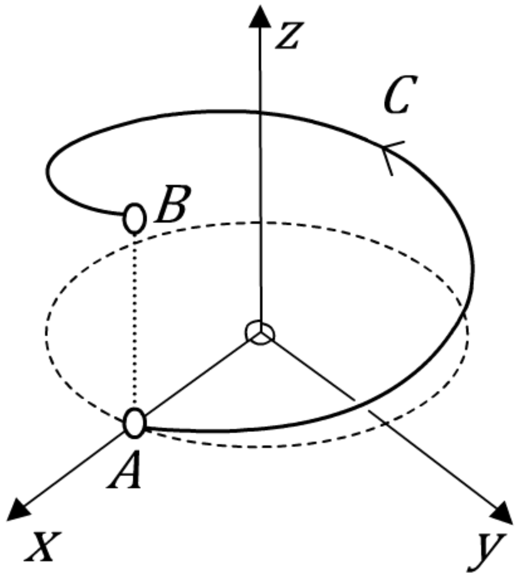
Given $\mathbf{F} = -z\,\mathbf{i} + x\,\mathbf{j} + y\,\mathbf{k}$, and C is the helix given by

$$x = \cos t,\, y = \sin t,\, z = 3t,\, 0 \le t \le 2\pi$$

The points on the helix can be written as:

$$\mathbf{r} = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + 3t\,\mathbf{k}$$

$$d\mathbf{r} = (-\sin t\,\mathbf{i} + \cos t\,\mathbf{j} + 3\mathbf{k})dt$$



Let us look at another example to understand the vector line integral.

Example 2 (contd.)

$$W = \int_{C} \mathbf{F} \cdot d\boldsymbol{r}$$

$$= \int_{P}^{Q} (F_1 \mathbf{i} + F_2 \mathbf{j} + F_3 \mathbf{k}) \cdot (x' \mathbf{i} + y' \mathbf{j} + z' \mathbf{k}) dt$$

$$= \int_{0}^{2\pi} (-3t\, \mathbf{i} + \cos t\, \mathbf{j} + \sin t\, \mathbf{k}) \cdot (-\sin t\, \mathbf{i} + \cos t\, \mathbf{j} + 3\mathbf{k}) dt$$

Let us look at another example to understand the vector line integral.

Example 2 (contd.)

$$W = \int_0^{2\pi} \left(3t\sin t + \cos^2 t + 3\sin t\right)dt$$

$$= -6\pi + \pi + 0$$

$$= -5\pi$$

Here is another example to understand the concept of vector line integral.

Example 3

Given the force $\mathbf{F} = x^2\mathbf{i} + y\mathbf{j} + (xz - y)\mathbf{k}$, find the work done by $\mathbf{F}$ from $(0,0,0)$ to $(1,2,4)$ in:

(i) C_1 a straight line joining the points,

(ii) C_2 along the curve given parametrically by $x = t^2, y = 2t, z = 4t^3$

(i) Parametric equations for the straight line path C_1 are:

$$x = t, y = 2t, z = 4t, 0 \le t \le 1$$

$$\mathrm{d}\boldsymbol{r} = (\mathbf{i} + 2\mathbf{j} + 4\mathbf{k})dt$$

So, W_1 is given by: $\int_{\mathrm{C}_1} \mathbf{F} \cdot d\mathbf{r} = 11/3$

Here is another example to understand the concept of vector line integral.

Example 3 (contd.)

(ii) Along Path C_2 :

$$d\mathbf{r} = \left(2t\,\mathbf{i} + 2\,\mathbf{j} + 12t^2\,\mathbf{k}\right)dt$$

So, W_2 is given by:

$$\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = 7/3$$

In the previous example, it is to note that $W_1 \neq W_2$, which means that work done is dependent on the path C.

Q

Is there a vector field $\mathbf{F}$ such that W is independent of path taken? What is the necessary condition of such a field to yield this path independence property?

The answer is YES. Such a vector field $\mathbf{F}$ is called conservative.

A

Let us understand the meaning of conservative field and prove that the line integration is path independent.

A vector field $\mathbf{F}$ is called conservative if there is a scalar field V such that $\mathbf{F} = \nabla V$ or equivalently if $\nabla \times \mathbf{F} = 0$.

Suppose there exists a scalar field V such that:

$$\mathbf{F} = \nabla V = \frac{\partial V}{\partial x}\mathbf{i} + \frac{\partial V}{\partial y}\mathbf{j} + \frac{\partial V}{\partial z}\mathbf{k}.$$

Then, $\displaystyle\int_{\mathrm{A}}^{\mathrm{B}} \mathbf{F} \cdot \mathrm{d}\boldsymbol{r} = \int_{\mathrm{A}}^{\mathrm{B}} \left(\frac{\partial V}{\partial x}\mathbf{i} + \frac{\partial V}{\partial y}\mathbf{j} + \frac{\partial V}{\partial z}\mathbf{k}\right) \cdot (dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k}).$

Let us understand the meaning of conservative field and prove that the line integration is path independent.

$$\int_{A}^{B} \mathbf{F} \cdot d\mathbf{r} = \int_{A}^{B} \left(\frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz \right) = \int_{A}^{B} dV = V\Big|_{A}^{B} = V(B) - V(A).$$

Since $dV = \left(\frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz \right).$

The line integral is independent of path.

The line integral is given by the values of the scalar field V at the end points, $V(B) - V(A)$.
V is called the potential function of $\mathbf{F}$ and $\mathbf{F}$ is a conservative field.

For a conservative field $\mathbf{F}$, $\mathbf{F} = \nabla V$ and its curl can be shown as $\nabla\times \mathbf{F} = \nabla \times (\nabla V) = 0$ which is derived below.

$$\nabla \times \mathbf{F} = \nabla \times (\nabla V) = \nabla \times \left(\frac{\partial V}{\partial x}\mathbf{i} + \frac{\partial V}{\partial y}\mathbf{j} + \frac{\partial V}{\partial z}\mathbf{k}\right)$$

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{\partial V}{\partial x} & \dfrac{\partial V}{\partial y} & \dfrac{\partial V}{\partial z} \end{vmatrix}$$

$$\nabla \times \mathbf{F} = \left[\frac{\partial^2 V}{\partial y \partial z} - \frac{\partial^2 V}{\partial z \partial y}\right]\mathbf{i} + \left[\frac{\partial^2 V}{\partial z \partial x} - \frac{\partial^2 V}{\partial x \partial z}\right]\mathbf{j} + \left[\frac{\partial^2 V}{\partial x \partial y} - \frac{\partial^2 V}{\partial y \partial x}\right]\mathbf{k} = 0$$

The concept of conservative field can be summarised as below.

There are two ways to check if $\mathbf{F}$ is conservative.

First, find a scalar field V such that $\mathbf{F} = \nabla V$.

Second, equivalently check whether $\nabla \times \mathbf{F} = 0$.

Let us look at an example to understand the conservative field.

Example 4

Given $\mathbf{F} = 2x\,\mathbf{i} + 2y\,\mathbf{j} + 4z\,\mathbf{k}$, check whether $\mathbf{F}$ is conservative, hence or otherwise calculate the line integrals along the two paths from $(0,0,0)$ to $(1,1,1)$.

C_1: $\boldsymbol{r}(t) = t\,\mathbf{i} + t\,\mathbf{j} + t\,\mathbf{k}, 0 \leq t \leq 1$

C_2: $\boldsymbol{r}(t) = t\,\mathbf{i} + t^3\,\mathbf{j} + t^2\,\mathbf{k}, 0 \leq t \leq 1$

To see if $\mathbf{F}$ is conservative, check the curl:

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2x & 2y & 4z \end{vmatrix} = 0.$$ Hence, $\mathbf{F}$ is conservative.

Let us look at an example to understand the conservative field.

Example 4 (contd.)

Along C_1: $\boldsymbol{r}(t) = t\,\mathbf{i} + t\,\mathbf{j} + t\,\mathbf{k}$

That is, $x = t, y = t, z = t, 0 \leq t \leq 1$

$$d\boldsymbol{r}(t) = dt\,\mathbf{i} + dt\,\mathbf{j} + dt\,\mathbf{k}$$

$$d\boldsymbol{r}(t) = (\mathbf{i} + \mathbf{j} + \mathbf{k})dt$$

$$\int_{C_1} \mathbf{F} \cdot \mathrm{d}\boldsymbol{r} = \int_0^1 (2t\,\mathbf{i} + 2t\,\mathbf{j} + 4t\,\mathbf{k}) \,.\, (\mathbf{i} + \mathbf{j} + \mathbf{k})dt$$

$$= \int_0^1 (2t + 2t + 4t\,)dt \;= \frac{8}{2} t^2 \Big|_0^1 = 4$$

Let us look at an example to understand the conservative field.

Example 4 (contd.)

Along C_2: $\boldsymbol{r}(t) = t\,\mathbf{i} + t^3\,\mathbf{j} + t^2\,\mathbf{k}$

That is, $x = t, y = t^3, z = t^2, 0 \le t \le 1$

$$d\boldsymbol{r}(t) = dt\,\mathbf{i} + 3t^2 dt\,\mathbf{j} + 2t dt\,\mathbf{k}$$

$$d\boldsymbol{r}(t) = (\mathbf{i} + 3t^2\mathbf{j} + 2t\mathbf{k})dt$$

$$\int_{C_2} \mathbf{F} \cdot \mathrm{d}\boldsymbol{r} = \int_0^1 \left(2t\,\mathbf{i} + 2t^3\,\mathbf{j} + 4t^2\,\mathbf{k}\right) \cdot (\mathbf{i} + 3t^2\,\mathbf{j} + 2t\,\mathbf{k})dt$$

$$= \int_0^1 \left(2t + 6t^5 + 8t^3\right)dt = t^2 + \frac{6}{6}t^6 + \frac{8}{4}t^4\Big|_0^1 = 4$$

Let us look at an example to understand the conservative field.

Example 4 (contd.)

Once it is found that $\mathbf{F}$ is conservative, the other way to deduce V such that $\mathbf{F} = \nabla V$ is as follows:

$$\mathbf{F} = F_1\,\mathbf{i} + F_2\,\mathbf{j} + F_3\,\mathbf{k} = \frac{\partial V}{\partial x}\mathbf{i} + \frac{\partial V}{\partial y}\mathbf{j} + \frac{\partial V}{\partial z}\mathbf{k}$$

$$\frac{\partial V}{\partial x} = F_1 = 2x, \Rightarrow V = x^2 + f(y,z),$$

$$\frac{\partial V}{\partial y} = F_2 = 2y, \Rightarrow V = y^2 + g(x,z),$$

$$\frac{\partial V}{\partial z} = F_3 = 4z, \Rightarrow V = 2z^2 + h(x,y)$$

Let us look at an example to understand the conservative field.

Example 4 (contd.)

$V = x^2 + y^2 + 2z^2 + c$

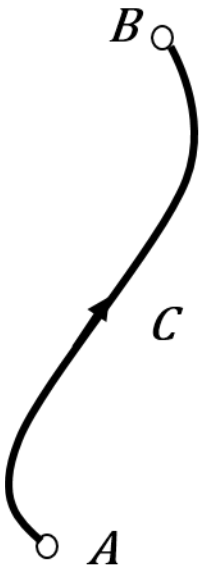
$$\int_{C} \mathbf{F} \cdot d\boldsymbol{r} = \int_{A}^{B} dV = V(B) - V(A)$$

$$= V(1,1,1) - V(0,0,0)$$

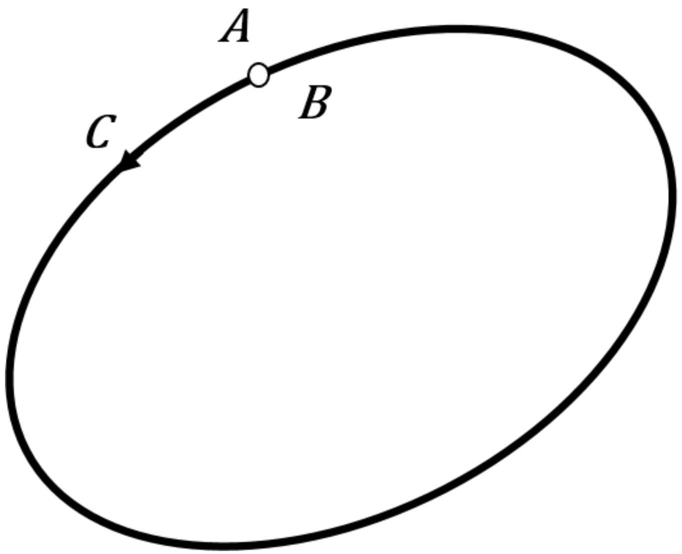
$$= (4 + c) - (0 + c) = 4$$

Where, c is a constant.

The concept of conservative field can be summarised by the given diagrams.



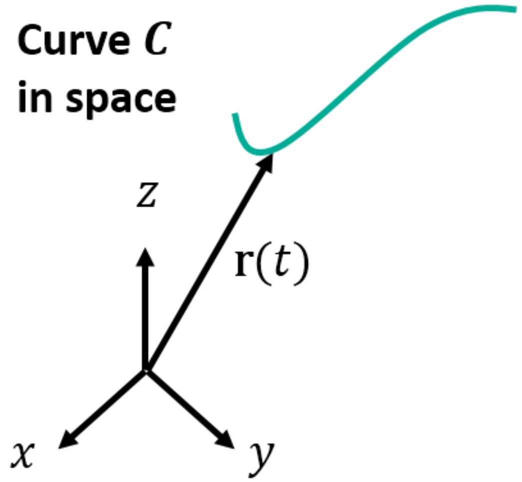
$$\int_A^B \mathbf{F} \cdot d\boldsymbol{r} = V(B) - V(A)$$

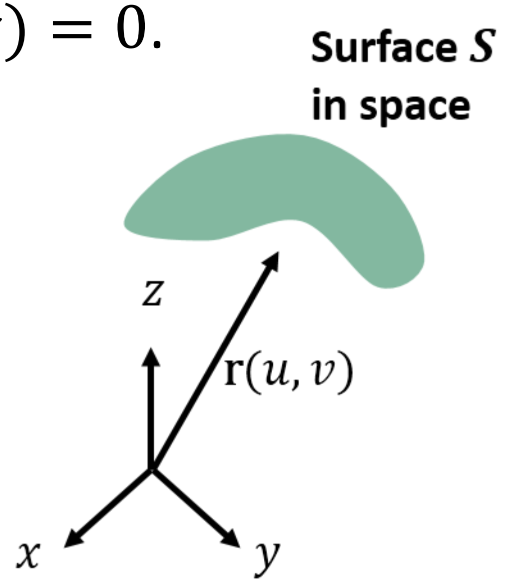


$$\oint_C \mathbf{F} \cdot d\boldsymbol{r} = 0$$

Non-conservative fields are also called as dissipative which can be expressed as $\oint_{C} \mathbf{F} \cdot d\boldsymbol{r} \neq 0$

A surface S in the xyz-space can be represented as $z = f(x, y)$ or $g(x, y, z) = 0$.

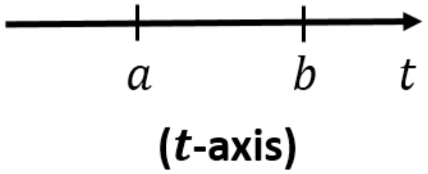




For example, $\mathrm{z} = \sqrt{(\mathrm{a}^2 - \mathrm{x}^2 - \mathrm{y}^2)}$

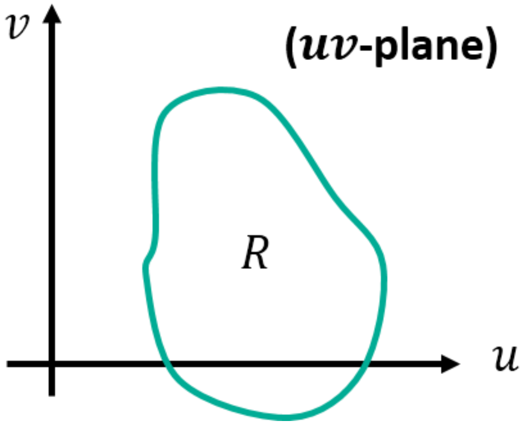
Or $x^2 + y^2 + z^2 - a^2 = 0\,; z \geq 0$

It represents a hemisphere of radius a and centre at the origin.



Just like the parametric representation of a curve in space, a surface can also be represented parametrically.

Surfaces are two-dimensional, hence two parameters are required, which are denoted as u and v.



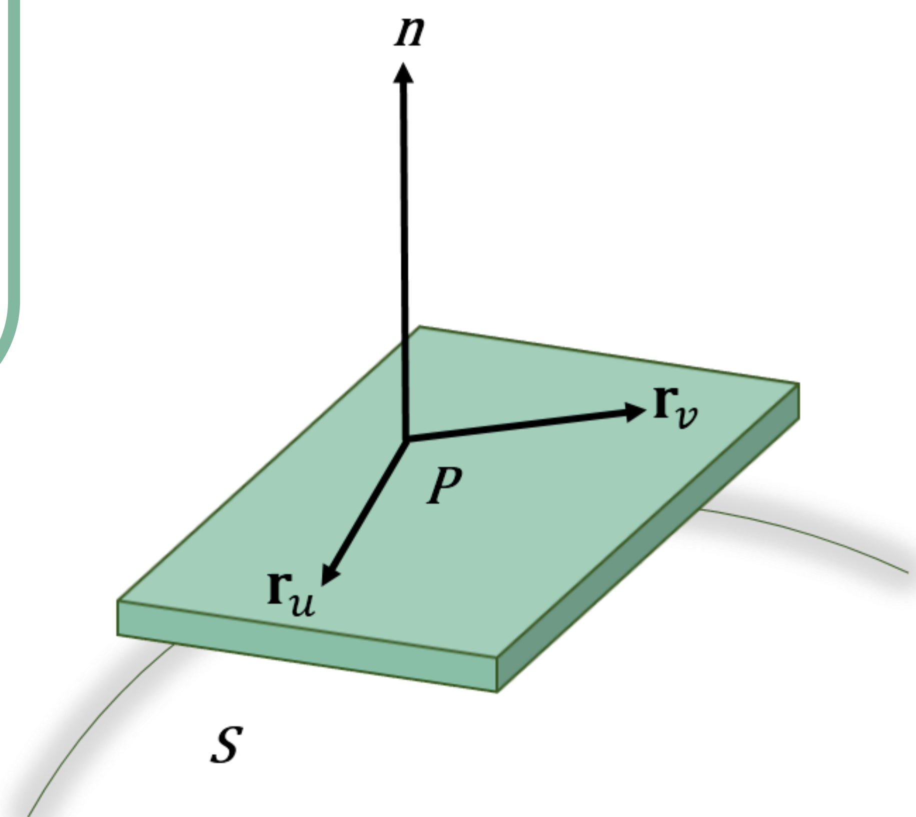
If curve C is represented by $r(t)$, then the tangent to the curve is the derivative $dr(t)/dt$.

From the given parametric representation of the surface S: $\mathbf{r}(u, v)$, the surface normal can be obtained as:

$$\mathbf{N} = \mathbf{r}_{\mathrm{u}} \times \mathbf{r}_{\mathrm{v}}$$

Where, $\mathbf{r}_{\mathrm{u}}$ and $\mathbf{r}_{\mathrm{v}}$ are partial derivatives of $\mathbf{r}(u, v)$ with respect to u and v respectively.

The unit normal vector is $\mathbf{n} = \mathbf{N}/\|\mathbf{N}\|$.



The two ways derived for computing normal to a surface is summarised below.

1. If the surface is specified by $z = f(x, y)$ or $f(x, y, z) = 0$, then grad f is a normal vector.

2. If the surface is specified parametrically, that is, $x = x(u, v)$, $y = y(u, v)$, $z = z(u, v)$, then $\mathbf{r}_u \times \mathbf{r}_v$ is a normal vector.

Unit normal is obtained in both cases by dividing the normal to its length.

Let us look at an example to understand the parametric representation of a circular cylinder.

Example 5

The surface S of a circular cylinder can be shown as:

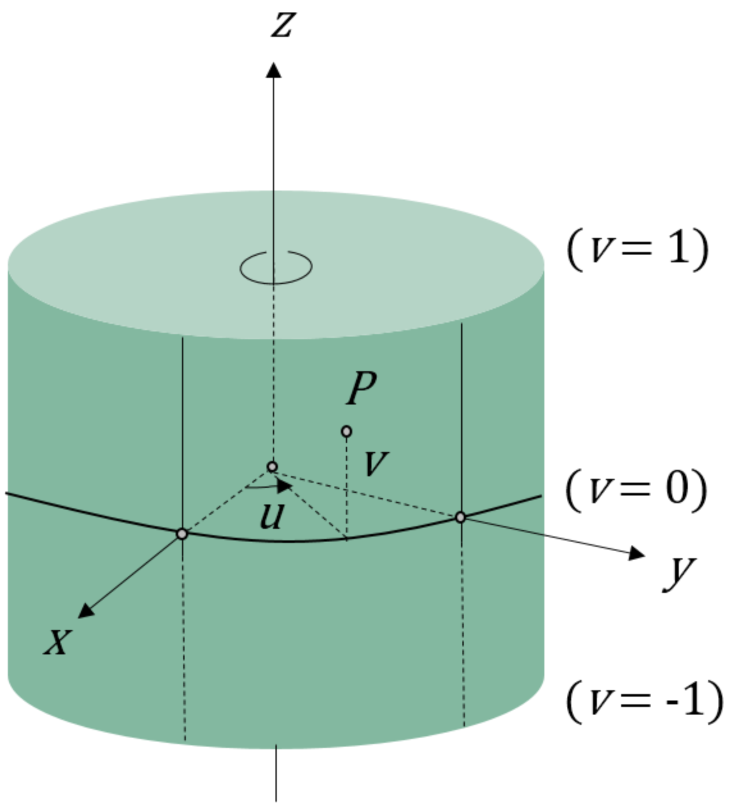
$$x^2 + y^2 = a^2, -1 \leq z \leq 1$$

$$\boldsymbol{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$$

The parameters u and v can be defined as:

$$x = a\cos u;\ y = a\sin u;\ z = v;$$

$$0 \leq u \leq 2\pi;\ -1 \leq v \leq 1$$



Let us look at an example to understand the parametric representation of a circular cylinder.

Example 5 (contd.)

Then, $\mathbf{r}(u,v) = a\cos u\,\mathbf{i} + a\sin u\,\mathbf{j} + v\,\mathbf{k}$

Let $\mathbf{r}_u = \dfrac{\partial \boldsymbol{r}}{\partial u} = -a\sin u\ \ \mathbf{i} + a\cos u\ \ \mathbf{j}$, $\mathbf{r}_v = \dfrac{\partial \boldsymbol{r}}{\partial v} = \mathbf{k}$

A normal to the surface can be given by: $\mathbf{N} = \ \mathbf{r}_u \times \mathbf{r}_v$

$$\mathbf{N} = a\sin u\ \ \mathbf{j} + a\cos u\ \ \mathbf{i} = (x\,\mathbf{i} + y\,\mathbf{j})$$

Check if: $f\ = x^2 + y^2 = a^2$

Then, $\nabla\, f\ = \left(\dfrac{\partial}{\partial x}\mathbf{i} + \dfrac{\partial}{\partial y}\mathbf{j}\right)\left(x^2 + y^2\right) = 2(x\mathbf{i}\ + y\,\mathbf{j})$

Now, let us take a look at an example on the parametric representation of a sphere.

Example 6

Surface S: The position vector $\boldsymbol{r}$ of a point of the sphere S is given by:
$\boldsymbol{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$

Subject to the constraint,

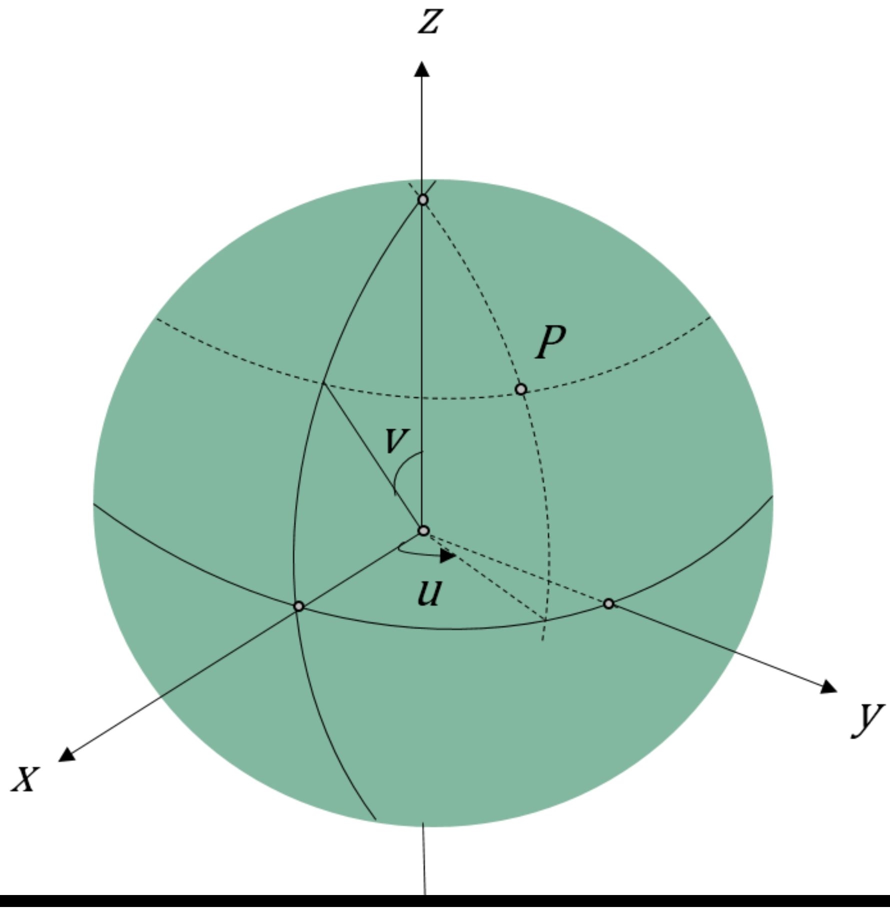
$$x^2 + y^2 + z^2 = a^2,$$

The parameters u and v can be defined as:

$x = a\cos u \sin v\,;\ 0 \le u \le 2\pi$

$y = a\sin u \sin v\,; 0 \le v \le \pi$

$z = a\cos v$



Now, let us take a look at an example on the parametric representation of a sphere.

Example 6 (contd.)

If $\mathbf{r}(u, v) = a\cos u \sin v\,\mathbf{i} + a\sin u \sin v\,\mathbf{j} + a\cos v\,\mathbf{k}$

Then, $\mathbf{r}_u = \dfrac{\partial \boldsymbol{r}}{\partial u} = -a\sin u \sin v\,\mathbf{i} + a\cos u \sin v\,\mathbf{j}$,

$$\mathbf{r}_v = \frac{\partial \boldsymbol{r}}{\partial v} = a\cos u \cos v\,\mathbf{i} + a\sin u \cos v\,\mathbf{j} - a\sin v\,\mathbf{k}$$

A normal to the surface can be found by: $\mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v$

$$\mathbf{N} = -a\sin v(\,a\cos u \sin v\,\mathbf{i} + a\sin u \sin v\,\mathbf{j}\ + a\cos v\,\mathbf{k})$$

$$= -a\sin v(\,x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k})$$

Now, let us take a look at an example on the parametric representation of a sphere.

Example 6 (contd.)

$\mathbf{N}$ is thus in the inward radial direction. For outward pointing normal, take the negative of $\mathbf{N}$. That is, $a \sin v(x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k})$

Check if: $f = x^2 + y^2 + z^2 = a^2$

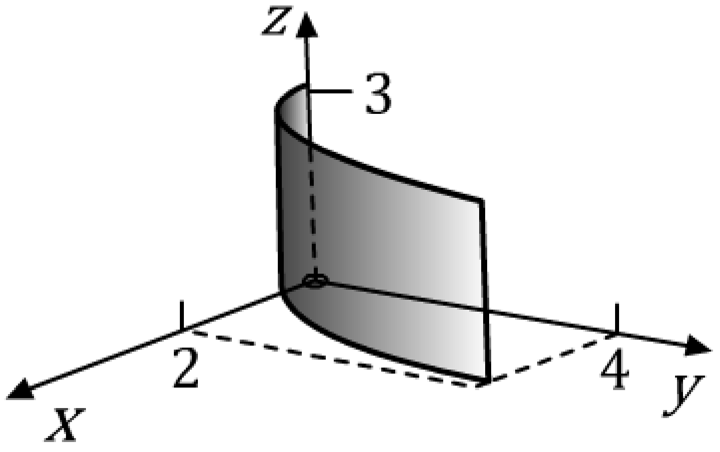
Then, $\nabla f = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)(x^2 + y^2 + z^2) = 2(x\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k})$

which is in outward radial direction.

Here is an example on the parametric representation of a parabolic cylinder.

Example 7

The surface S of a parabolic cylinder can be shown as:



$y = x^2, 0 \leq x \leq 2, 0 \leq z \leq 3$

$\boldsymbol{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

If $\mathbf{r}(u, v) = u\,\mathbf{i} + u^2\,\mathbf{j} + v\,\mathbf{k}$

Then,

$$\mathbf{r}_u = \frac{\partial \boldsymbol{r}}{\partial u} = \mathbf{i} + 2u\,\mathbf{j}, \mathbf{r}_v = \frac{\partial \boldsymbol{r}}{\partial v} = \mathbf{k} \begin{cases} x = u; 0 \leq u \leq 2 \\ y = u^2 \\ z = v; 0 \leq v \leq 3 \end{cases}$$

Here is an example on the parametric representation of a parabolic cylinder.

Example 7 (contd.)

The normal vector can be given as: $\mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v = 2u\,\mathbf{i} - \mathbf{j}$

Check if: $f = y - x^2 = 0$

$$\nabla f = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)(y - x^2)$$

$$= (-2x\,\mathbf{i} - \mathbf{j})$$

Given below is an example to understand the parametric representation of a plane.

Example 8

The surface S of a plane can be shown as:

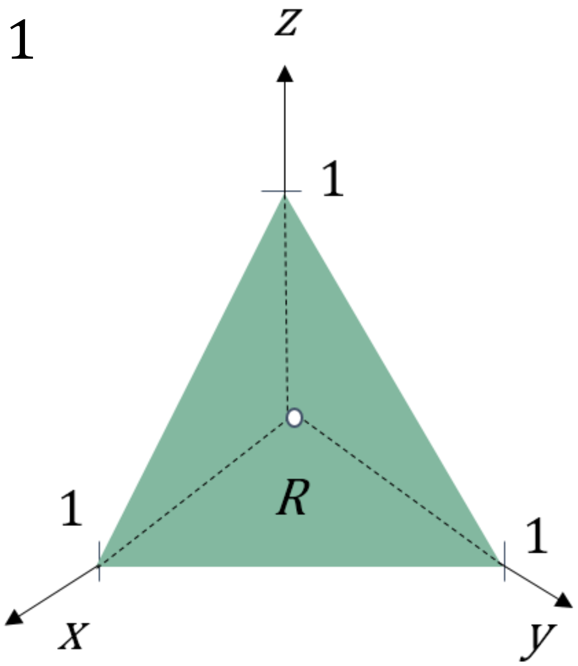
$x + y + z = 1, 0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1$

$\boldsymbol{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

If $\mathbf{r}(u, v) = u\,\mathbf{i} + v\,\mathbf{j} + (1 - u - v)\,\mathbf{k}$

Then,

$$\mathbf{r}_u = \frac{\partial \boldsymbol{r}}{\partial u} = \mathbf{i} - \mathbf{k}, \qquad \mathbf{r}_v = \frac{\partial \boldsymbol{r}}{\partial v} = \mathbf{j} - \mathbf{k} \qquad \begin{cases} x = u; 0 \leq u \leq 1 \\ y = v; 0 \leq v \leq 1 \\ z = 1 - u - v \end{cases}$$



Given below is an example to understand the parametric representation of a plane.

Example 8 (contd.)

The normal vector can be given as: $\mathbf{N} = \mathbf{r}_u \text{ x } \mathbf{r}_v = \mathbf{i} + \mathbf{j} + \mathbf{k}$

Check if: $f = x + y + z = 1$

$$\nabla f = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)(x + y + z)$$

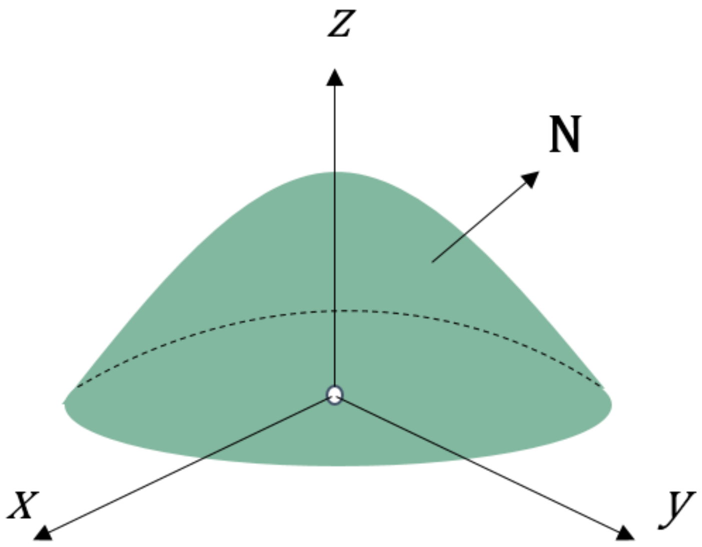
$$= (i + j + k)$$

Let us now look at an example on the parametric representation of a paraboloid.

Example 9

The surface S of a paraboloid can be shown as:

$$z = 1 - (x^2 + y^2), z \geq 0$$



From parameterisation and normal vector:

$$\begin{cases} x = \sqrt{1-v}\cos u \\ y = \sqrt{1-v}\sin u \\ z = v; 0 \leq v \leq 1 \end{cases}$$

$$\mathbf{r}_u = \frac{\partial \boldsymbol{r}}{\partial u} = -\sqrt{1-v}\sin u\,\mathbf{i} + \sqrt{1-v}\cos u\,\mathbf{j},$$

Let us now look at an example on the parametric representation of a paraboloid.

Example 9 (contd.)

$$\mathbf{r}_v = \frac{\partial \boldsymbol{r}}{\partial v} = -\frac{1}{2\sqrt{1-v}}\cos u\,\mathbf{i} - \frac{1}{2\sqrt{1-v}}\sin u\,\mathbf{j} + \mathbf{k}$$

The normal vector can be given as: $\mathbf{N} = \ \mathbf{r}_u \text{ x } \mathbf{r}_v$

$$\mathbf{N} = +\frac{1}{2}\sin^2 u\,\mathbf{k} + \sqrt{1-v}\,\sin u\,\mathbf{j}$$

$$+\frac{1}{2}\cos^2 u\,\mathbf{k} + \sqrt{1-v}\,\cos u\,\mathbf{j}$$

$$= \sqrt{1-v}\,\cos u\,\mathbf{i} + \sqrt{1-v}\,\sin u\,\mathbf{j} + 1/2\ \ \mathbf{k}$$

Now let's get familiarised with the flux integral or flux of F through surface S.

Let $\mathrm{d}A$ be the area of a small patch in a surface S. Then, it can be seen that $\mathbf{n}\mathrm{d}A = \mathbf{N}\,\mathrm{d}u\,\mathrm{d}v$.

Where, $\mathbf{N}$ is a surface normal given by $\mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v$ and $\mathbf{n}$ is the unit normal vector given by $\mathbf{n} = \mathbf{N}/||\mathbf{N}||$.

If the notation $\mathbf{dA}$ can be introduced as a vector normal to the small patch with a magnitude $\mathrm{d}A$, then it can be seen that $\mathbf{dA} = \mathbf{n}\mathrm{d}A = \mathbf{N}\,\mathrm{d}u\,\mathrm{d}v$.

Now let's get familiarised with the flux integral or flux of F through surface S.

Double integral of $\mathbf{F} \cdot \mathbf{dA}$ or $\mathbf{F} \cdot \mathbf{n}\mathrm{d}A$ over S is called a flux integral as it gives the flux (Latin for "flow") of $\mathbf{F}$ through the surface S.

If $\mathbf{F}$ is the velocity of a fluid then the integral gives the amount of fluid that flows through S per unit time.

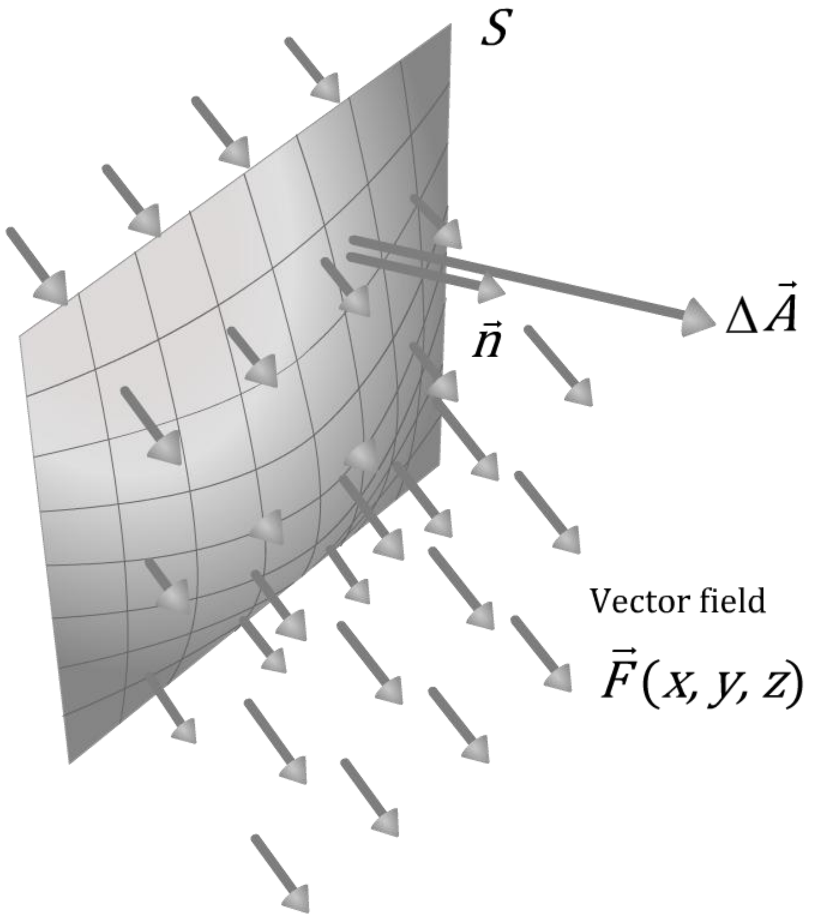
The flux integral of the vector field $\mathbf{F}$ through the oriented surface S is shown below.

$$\iint_S \mathbf{F} \cdot d\mathbf{A} = \lim_{|\Delta \mathrm{A}| \to 0} \sum \mathbf{F} \cdot \Delta\mathbf{A}$$

$$= \iint_S \mathbf{F} \cdot d\mathbf{A}$$

$$= \iint_S \mathbf{F} \cdot \mathbf{n} d\mathbf{A}$$

$$= \iint_R \mathbf{F}(u, v) \cdot \mathbf{N}\, \mathrm{d}u\, \mathrm{d}v$$



Let us take a look at a sample problem to understand the concept of surface integral.

Sample Problem 1

Compute the flux of the vector field $F = \mathbf{i} + xy\,\mathbf{j}$ across the surface given by: $x = u + v$, $y = u - v$, $z = u^2$; $0 \le u \le 1$, $0 \le v \le 1$

Solution:

$$\iint_S \mathbf{F} \cdot \mathbf{N}\, \mathrm{d}u\, \mathrm{d}v = \int_0^1 \int_0^1 \mathbf{i} + xy\,\mathbf{j} \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, \mathrm{d}u\, \mathrm{d}v$$

$$\mathbf{r}(u,v) = (u+v)\mathbf{i} + (u-v)\mathbf{j} + u^2\mathbf{k}$$

$$\mathbf{r}_u = \mathbf{i} + \mathbf{j} + 2u\mathbf{k},\ \mathbf{r}_v = \mathbf{i} - \mathbf{j}$$

Let us take a look at a sample problem to understand the concept of surface integral.

Solution (contd.):

$$\mathbf{F}(u, v) = \mathbf{i} + (u^2 - v^2)\mathbf{j}$$

$$\mathbf{F} \cdot \mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} 1 & u^2 - v^2 & 0 \\ 1 & 1 & 2u \\ 1 & -1 & 0 \end{vmatrix} = 2u^3 - 2uv^2 + 2u$$

$$\int_{\mathbf{0}}^{\mathbf{1}} \int_{\mathbf{0}}^{\mathbf{1}} \left(2u^3 - 2uv^2 + 2u\right) \mathrm{d}u \, \mathrm{d}v = \int_0^1 \left[\frac{1}{2} - v^2 + 1\right] \mathrm{d}v = \frac{7}{6}$$

Let us take a look at another sample problem to understand the concept of surface integral.

Sample Problem 2

Consider the vector field $\boldsymbol{F} = 2x\,\mathbf{i} + 2y\,\mathbf{j}$.

And, consider the surface S:

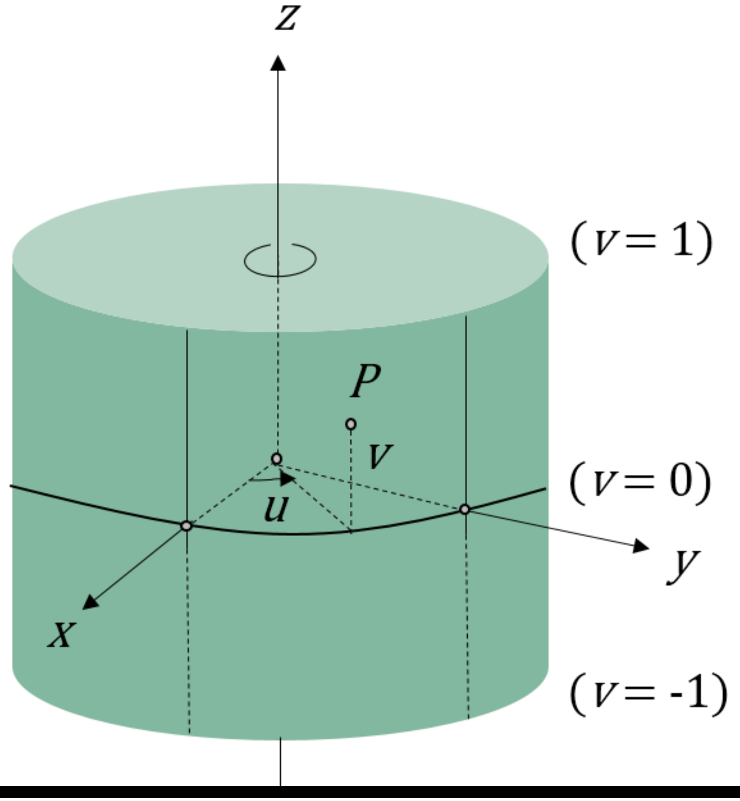
$$x^2 + y^2 = a^2$$

$$-1 \leq z \leq 1$$

Find $\iint_S \mathbf{F} \cdot \mathrm{d}\mathbf{A}$

Parameters are: $\begin{cases} x = a\cos u \\ y = a\sin u\,, 0 \leq u \leq 2\pi \\ z = v \end{cases}$

$\mathbf{N} = a\cos u\,\mathbf{i} + a\sin u\,\mathbf{j}$



Let us take a look at another sample problem to understand the concept of surface integral.

Solution:

$$\iint_S \mathbf{F}\cdot \mathrm{d}\mathbf{A} = \iint_R \mathbf{F}(u,v)\cdot \mathbf{N}(u,v)\,\mathrm{d}u\,\mathrm{d}v$$

$$= \int_{-1}^{1}\int_{0}^{2\pi} (2a\cos u\,\mathbf{i} + 2a\sin u\,\mathbf{j})\cdot(a\cos u\,\mathbf{i} + a\sin u\,\mathbf{j})\,\mathrm{d}u\,\mathrm{d}v$$

$$= \int_{-1}^{1}\int_{0}^{2\pi} \left(2a^2\cos^2 u + 2a^2\sin^2 u\right)\mathrm{d}u\,\mathrm{d}v$$

Let us take a look at another sample problem to understand the concept of surface integral.

Solution (contd.):

$$\iint_S \mathbf{F} \cdot \mathrm{d}\mathbf{A} = \int_{-1}^{1} \int_{0}^{2\pi} \left(2a^2\right) \mathrm{d}u \, \mathrm{d}v$$

$$= 2a^2 \int_{-1}^{1} \mathrm{d}v \int_{0}^{2\pi} \mathrm{d}u$$

$$= 8\pi a^2$$

Here is another sample problem explaining the concept of surface integral.

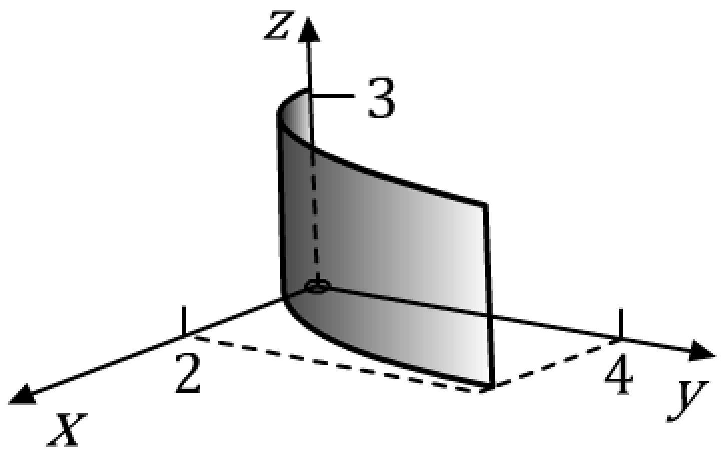
Sample Problem 3

Compute the flux of water through the parabolic cylinder if the velocity vector field is given by $\boldsymbol{F} = 3z^2\mathbf{i} + 6\mathbf{j} + 6xz\mathbf{k}$

And, the surface S is given by:

$y = x^2, 0 \leq x \leq 2, 0 \leq z \leq 3$

$\boldsymbol{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$



Here is another sample problem explaining the concept of surface integral.

Solution:

$$\mathbf{r}(u, v) = u\mathbf{i} + u^2\mathbf{j} + v\mathbf{k};\ 0 \leq u \leq 2,\ 0 \leq v \leq 3$$

$$\mathbf{r}_u = \mathbf{i} + 2u\mathbf{j} + 0\mathbf{k};\ r_v = +\mathbf{k}$$

$$\mathbf{N} = 2u\mathbf{i} - 1\mathbf{j} + 0\mathbf{k};\ \boldsymbol{F}(u, v) = 3v^2\mathbf{i} + 6\mathbf{j} + 6uv\mathbf{k}$$

Here is another sample problem explaining the concept of surface integral.

Solution(contd.):

$$\iint_S \mathbf{F} \cdot \mathbf{n} dA = \iint_R \mathbf{F}(u, v) \cdot \mathbf{N}(u, v) du\, dv$$

$$= \int_0^3 \int_0^2 \left(3v^2\mathbf{i} + 6\mathbf{j} + 6uv\mathbf{k}\right) \cdot (2u\mathbf{i} - \mathbf{j})\, du\, dv$$

$$= 72$$

Summary

Key points discussed in this lesson:

Vector line integral is the operation of integrating a vector field along a curve in space. Such integrals have considerable utility in many areas of Engineering and Physics.

A vector field $\mathbf{F}$ is called conservative if there is a scalar field V such that $\mathbf{F} = \nabla V$ or equivalently if $\nabla \times \mathbf{F} = 0$.

The line integral is independent of path and is given by the values of the scalar field V at the end points, $V(B) - V(A)$ where, V is called the potential function of $\mathbf{F}$, and $\mathbf{F}$ is a conservative field.

Key points discussed in this lesson:

- Non conservative fields are also called as dissipative as $\oint_C \mathbf{F} \cdot d\boldsymbol{r} \neq 0$.
- From the given parametric representation of the surface S: $\mathbf{r}(u, v)$ you could obtain the surface normal as $\mathbf{N} = \mathbf{r}_{\mathrm{u}} \times \mathbf{r}_{\mathrm{v}}$, where, $\mathbf{r}_{\mathrm{u}}$ and $\mathbf{r}_{\mathrm{v}}$ are partial derivatives of $\mathbf{r}(u, v)$ with respect to u and v respectively. The unit normal vector is given as: $\mathbf{n} = \mathbf{N}/||\mathbf{N}||$.
- The two ways derived for computing normal to a surface is summarised below.
  - If the surface is specified by $z = f(x, y)$ or $f(x, y, z) = 0$ then grad f is a normal vector.
  - If the surface is specified parametrically, that is, $x = x(u, v)$, $y = y(u, v)$, $z = z(u, v)$, then $\mathbf{r}_{\mathrm{u}}$ x $\mathbf{r}_{\mathrm{v}}$ is a normal vector.

  Unit normal is obtained in both cases by dividing the normal to its length.

Key points discussed in this lesson:

- Double integral of $\mathbf{F} \cdot \mathbf{dA}$ or $\mathbf{F} \cdot \mathbf{n dA}$ over S is called a flux integral as it gives the flux of $\boldsymbol{F}$ through the surface S.

- The flux integral of the vector field $\mathbf{F}$ through the oriented surface S is given by:
$\iint_S \mathbf{F} \cdot \mathbf{dA} = \iint_R \mathbf{F}(u, v) \cdot \mathbf{N} \, \mathrm{du} \, \mathrm{dv}$