

## ELECTROMAGNETIC BOUNDARY CONDITIONS

### OBJECTIVE

The purpose is to demonstrate the redistribution of power at an optical interface. With this set up, Snell's Law can be used to experimentally determine the index of refraction of the semicircular media provided. At the same time, the critical angle of each sample can be found. Then with these values the transmittance and reflectance can be plotted experimentally in order to test the mathematical hypothesis of power redistribution predicted by linear wave theory. This lab should also introduce the ideas diffuse and specular reflection. Also, the finite size of the laser beam cannot be ignored practically, so steps will be taken to account for this size. The appearance of Brewster's Angle will be apparent in the data if taken properly, and it should be determined for each medium.

### APPARATUS

It is a PASCO brand set up equipped with an optics rail, red 650nm diode laser, a polarizer, a primary lens, an angle table with interchangeable semicircular samples, a secondary lens, and a silicon photodetector attached to a wattmeter to measure absolute power. A labeled image is shown below in Figure 1.

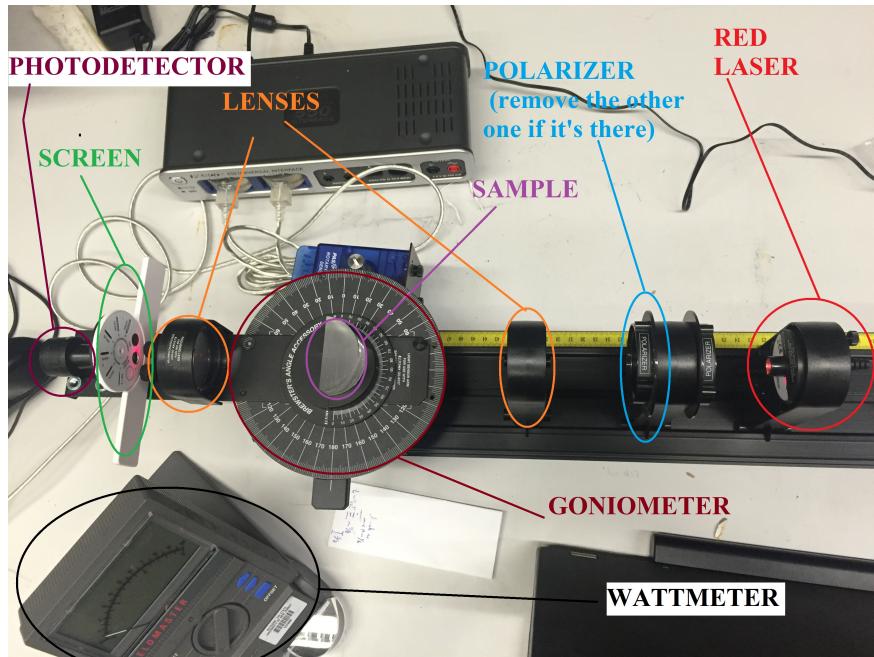


Figure 1: A photo of the experimental setup

### SAFETY

The laser in this lab is not particularly dangerous, but it is good to get in the habit of knowing where the laser beam will end up in an optical setup. A more powerful laser shined in the eyes will focus nearly its full power through the lens of the eye into a tiny point on the retina. This will in short time, depending on the power of the beam, burn the retina and permanently damage the eye. Pretend its a gun and be weary of where it's pointed.

## THEORY IN MATHEMATICS

The full derivation of the boundary conditions, the Fresnel Coefficients, Snell's Law, and Brewster's Angle will be placed in the appendix, and also can be (hopefully) found in any electromagnetism textbook or optics text book. At the moment there will mostly be the statement of the theory's results, along with some of the relevant mathematical ideas. We want to build up from first principles and a few assumptions to a working theory of polarized light passing through the interface between two media. It starts with plain magnetic and electric fields transitioning from one medium into another. One can start with the integral form of Maxwell's Equations and show that an electric field shooting at the surface of the boundary will come out the other side discontinuously due to an induced surface charge, and a magnetic field gliding past the boundary will come out the other side discontinuously due to induced surface currents. This gives the Electromagnetic Boundary Conditions at the surface between the two media:

$$\begin{aligned} (\vec{E}_2 \epsilon_2 - \vec{E}_1 \epsilon_1) \cdot \vec{n} &= \sigma \\ \vec{n} \times (\vec{E}_2 \epsilon_2 - \vec{E}_1 \epsilon_1) &= \vec{0} \\ \vec{n} \times (\vec{B}_2 \mu_2^{-1} - \vec{B}_1 \mu_1^{-1}) &= \vec{K} \\ (\vec{B}_2 \mu_2^{-1} - \vec{B}_1 \mu_1^{-1}) \cdot \vec{n} &= 0 \end{aligned}$$

Where  $\vec{n}$  is the surface normal which points away from the boundary,  $\epsilon$  and  $\mu$  are the electric permittivity and magnetic permeability of the media,  $\vec{K}$  is the induced surface current (k for kurrent maybe), and  $\sigma$  is the induced surface charge (sigma in Greek is said like an "s", but you've lost me on any reason for q and  $\rho$ ). The subscript 1 is for the medium the fields are leaving, and the subscript 2 is for the medium the fields are entering. These equations can be used to study the properties of light at the boundary.

Assume we already have some sinusoidal electrical disturbance in vacuum absent of charges or currents.

$$\vec{E}(\vec{r}, t) = \vec{E}_0 e^{i(\vec{k} \cdot \vec{r} - \omega t)}$$

Where  $\vec{E}_0$  is the maximum amplitude, and the complex exponential modulates its progression in time relative to the position of the wavefront in space. The  $\vec{k}$  in the exponential is the direction of wave propagation, and  $\omega$  is the angular frequency of the wave. We know from Maxwell's equations that this oscillating electric field will create and oscillating magnetic field and vice versa. This is according to the following equations:

$$\begin{aligned} \vec{\nabla} \times \vec{B} &= \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} \\ \vec{\nabla} \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} \end{aligned}$$

These are the differential forms of Ampere's Law and Faraday's Law respectively. We want this electric field to wiggle up and down. This puts it in a line, which certainly fits in a plane. Let's point the electric field on the x-axis. We can continue without talking about any certain direction of propagation just by carrying out the operations. Since it is much easier to curl a known quantity, we go ahead with Faraday's Law. This leads us to the intermediate result:

$$\vec{B} = -(\hat{y} k_z - \hat{z} k_y) \frac{1}{\omega} E_0 e^{i(\vec{k} \cdot \vec{r} - \omega t)}$$

We can find many things in this equation. First, we can choose  $k_z$  and  $k_y$  how we like, and no matter what the magnetic field will be perpendicular to the electric field. This can be shown by dotting both sides of the equation with the  $\hat{x}$  basis vector. Since the cartesian coordinate system is an orthogonal coordinate system, the dot product of any two different basis unit vectors is zero. Thus, the right hand side is zero and the magnetic field is nowhere to be found along the x axis. Then we can choose  $k_z = 0$  so that  $\vec{k}$  only points in the y direction, or we can do a similar thing by setting  $k_y = 0$ . Neither choice is better than the other, but you can see that the choice of  $k_z = 0$  makes the magnetic field in the y direction, and vice versa. Thus  $\vec{k}$  is perpendicular to both  $\vec{E}$  and  $\vec{B}$ , and  $\vec{E}$  and  $\vec{B}$  are perpendicular to each other and lie in the planar wavefront.

For our purposes, we will only need to think of the electric field instead of both the electric and magnetic fields. One reason is because the electric field and magnetic field are related simply. Another reason is because later we will assume that there are no significant magnetic properties of our samples. This sets the magnetic permeability to 1, and alleviates the magnetic field of any special role. Even if none of this was the case, it is convention to speak about the direction of the electric field when speaking of polarization.

There are many different types of polarization states including exotic things such as elliptical polarization. In this lab, we will worry only about linear polarization, and random polarization. **Linear polarization** means the electric field of a purely polarized beam is only defined on a straight line for all positions and all time. If you draw a line on the planar wavefront to symbolize the linearly polarized electric field, it will oscillate up and down this line only.

Any randomly generated light such as directly from a laser, the sun, ceiling lights, a campfire, etc. is **randomly polarized**. This means that there is no particular direction to the electric field. Even though this is the case, we can still make mathematical sense of the random polarization. Because the electric field sits in a plane, the polarization of the wave is some combination of two directions within the plane. We can choose those two directions to be at right angles with respect to each other, so that they don't mix at all. This is much like describing the polarization in terms of its x and y components on an xy plane, although it doesn't have to line up with the axes. In general this means **no matter what direction of the electric field in the plane, it can always be described by the linear combination of two perpendicular arrows**. The way to convince yourself of this is to just draw a straight line on a piece of paper, then make a right triangle out of it such that the first line is the hypotenuse. You now have a randomly oriented line described by two perpendicular lines. Put little arrowheads on them in the proper spots and you now have described a randomly oriented arrow by two perpendicular arrows!

The axis or direction of the polarizer is defined as the unique line through which an electric field line can pass through. This means, when we have randomly polarized light passing through a linear polarizer, only the components of the electric field which are parallel to the axis of the polarizer make it through. This also has the effect of halving the power of the beam. Why a half? Well we have randomly polarized light, which can be abstracted into parallel components to the polarizer axis and perpendicular components to the polarizer axis. Then as the light hits the polarizer, only the parallel components make it through. Because the polarization is random, by the mathematical definition of random no direction should be preferred, so the polarizer is chucking out half of the directions. This statistical argument is akin to having a theoretical bucket of quarters, flipping each one, and throwing out all the tails. Now of course this can be experimentally tested to see if the random polarization hypothesis is correct. Nonetheless, we are not worried about the absolute power of the beam before the polarizer, so we only need to worry about the power of the beam which leaves the polarizer

Now that we have imagined rigorously a sinusoidal linearly polarized beam, it must now hit the target. Our two polarization directions will have different effects at the interface. Thus we split them into two cases. We define the plane of incidence to be the plane made up of the incoming, reflecting and transmitting beams. The polarization states will be given special names relative to this definition. The names are S and P polarization. **P** stands for **parallel** to the plane of incidence, which means it lives in the plane of incidence, and **S** stands for **senkrecht**, which is German for **perpendicular** (having P and P polarization states would become confusing immediately). Take a half of a minute to study Figure 2 on the next page and its pieces to see the differences between S and P.

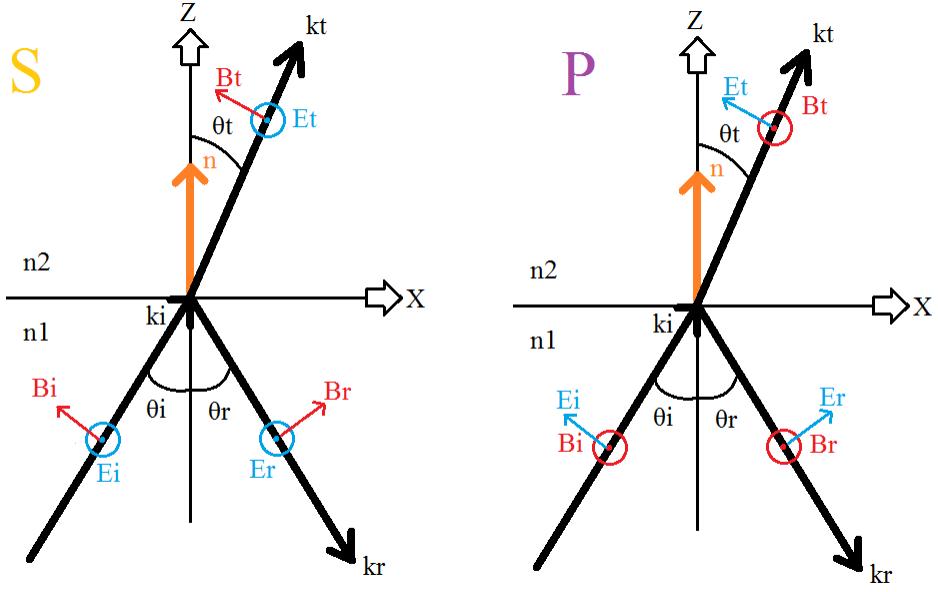


Figure 2: S and P interface configurations. In both cases, the incident beam comes from the bottom left.  $k_i$ ,  $k_r$ , and  $k_t$  are the incident, reflected, and transmitted wave vectors respectively.  $n_1$  and  $n_2$  are the indexes of refraction describing the two different media. Since  $\mu_j = 1$  for all  $j$ ,  $n_j = \frac{1}{\sqrt{\epsilon_j}}$ . Also notice that the wave vector  $\vec{k}$  is in the direction of  $\vec{E} \times \vec{B}$

If we start with our sinusoidal E and B waves in a system without charges or currents, do a bit of trigonometry, take a couple derivatives, solve a few system of equations (one system being the boundary conditions themselves), check limiting cases in our result, and then use the quadratic formula, we can find the following sets of equations. The derivation is long and as was previously stated will be put into the appendix for those who wish to fully understand. For now, the results are good enough. To relate to the diagram,  $n_1 = n_i$  and  $n_2 = n_t$ .

Snell's Law:

$$n_i \sin(\theta_i) = n_t \sin(\theta_t)$$

The Fresnel Coefficients:

$$\begin{aligned} T_s &= \frac{E_{0t}}{E_{0i}} = \frac{2 \cos(\theta_i)}{\cos(\theta_i) + \frac{\mu_i}{\mu_t} \sqrt{\frac{n_t^2}{n_i^2} - \sin^2(\theta_i)}} \\ R_s &= \frac{E_{0r}}{E_{0i}} = \frac{\cos(\theta_i) - \frac{\mu_i}{\mu_t} \sqrt{\frac{n_t^2}{n_i^2} - \sin^2(\theta_i)}}{\cos(\theta_i) + \frac{\mu_i}{\mu_t} \sqrt{\frac{n_t^2}{n_i^2} - \sin^2(\theta_i)}} \\ T_p &= \frac{E_{0t}}{E_{0i}} = \frac{2 \frac{n_t}{n_i} \cos(\theta_i)}{\frac{\mu_i n_t^2}{\mu_t n_i^2} \cos(\theta_i) + \sqrt{\frac{n_t^2}{n_i^2} - \sin^2(\theta_i)}} \\ R_p &= \frac{E_{0r}}{E_{0i}} = \frac{\frac{\mu_i n_t^2}{\mu_t n_i^2} \cos(\theta_i) - \sqrt{\frac{n_t^2}{n_i^2} - \sin^2(\theta_i)}}{\frac{\mu_i n_t^2}{\mu_t n_i^2} \cos(\theta_i) + \sqrt{\frac{n_t^2}{n_i^2} - \sin^2(\theta_i)}} \end{aligned}$$

Brewster's Angle:

$$\theta_B = \arctan\left(\frac{n_t}{n_i}\right)$$

Critical Angle:

$$\theta_C = \arcsin\left(\frac{n_t}{n_i}\right)$$

Where  $E_{0i}$ ,  $E_{0r}$  and  $E_{0t}$  are the incident, reflected, and transmitted beam amplitudes respectively,  $\theta_i$  and  $\theta_t$  are the incident and transmitted angles respectively, and  $n_i$  and  $n_t$  are the incident and transmitted indexes of refraction respectively. One might ask: "where is  $\theta_r$ ?" It is clear in the appendix that  $\theta_i = \theta_r$ . You can also determine this experimentally. Also it should be made clear that it is not as if the index of refraction belongs to the incident and transmitted beams, there is just an incident and transmitted region of sorts.  $T_s$ , and  $R_s$  are the transmissive and reflective coefficients for S polarization, and  $T_p$ , and  $R_p$  are the transmissive and reflective coefficients for P polarization.

These are a lot of equations to think about, but there are some relations between them. The first relationship is between the critical angle and Snell's Law. The critical angle is just the angle at which the transmitted beam is 90 degrees from the surface normal. So using Snell's Law one can derive the critical angle. Since the arcsine function takes arguments from -1 to 1 and nothing outside this range,  $n_i > n_t$  is required for any sort of critical angle. The phenomenon that occurs at the critical angle is called **total internal reflection**, and is one of the phenomenon you will observe in the experiment. Brewster's angle comes from a limiting case of the  $R_p$  coefficient. Suppose that  $R_p = 0$ . Is there a certain angle of incidence where this is achieved? The answer is yes, and this is Brewster's Angle, **but this only applies to the P polarization case**. If applied to the S polarization case, one finds  $n_i = n_t$  to be the only solution, which simply means there is no medium to reflect off of!

The most important goal of this lab is to check the power redistribution due to the dielectric interface. Unfortunately, the Fresnel Coefficients are ratios of electric fields, and not of power. Moreover, when dealing with power, we have to consider the geometry of flux. Derived in the appendix is the radiant flux density, or **irradiance** of sinusoidal electromagnetic waves:

$$I = \frac{v\epsilon}{2} E_0^2$$

Where v is the speed of light in the medium, epsilon is the electric permittivity or dielectric constant, and  $E_0$  is the amplitude of the wave. Irradiance is measured in  $\frac{W}{m^2}$ , so in order to find the absolute power, we need to know the cross sectional area of the beam. Now is where it starts becoming confusing to think about light as a infinitesimal ray. The laser beam has some size, and it has a certain number of photons per unit area. This is what appears as the laser's brightness (and dangerousness!). When this circular, and planar beam of light hits the interface, it will cast some area onto the surface. We can define this cast area as A, and use trigonometry to find the projections on the incoming and outgoing plane waves (see Figure 3 below). We can then take ratios to find new coefficients known as the **reflectance**, r, and **transmittance**, t.

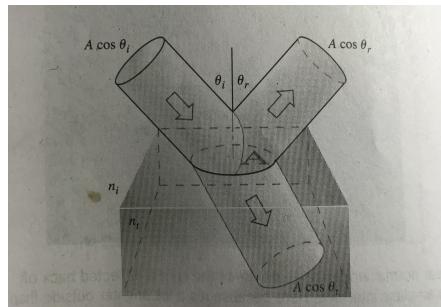


Figure 3: A diagram of how the fluxes projects itself onto a flat surface (Taken from *Optics 4th Ed* by Hecht and Ganesan)

$$r = \frac{I_r A \cos(\theta_r)}{I_i A \cos(\theta_i)} = \frac{I_r}{I_i}$$

$$t = \frac{I_t A \cos(\theta_t)}{I_i A \cos(\theta_i)} = \frac{I_t \cos(\theta_t)}{I_i \cos(\theta_i)}$$

Then, plugging in the expression for I, and using the fact that  $n_j = \frac{c}{v_j}$  where c is the speed of light in vacuum gives us:

$$r_j = R_j^2$$

$$t_j = \left( \frac{n_t \cos(\theta_t)}{n_i \cos(\theta_i)} \right) T_j^2 = \left( \frac{n_t \cos(\arcsin(\frac{n_i}{n_t} \theta_i))}{n_i \cos(\theta_i)} \right) T_j^2$$

Furthermore, it can be shown that:

$$r_j + t_j = 1$$

Where j in the above expression stands for either S or P polarization. The transmittance and reflectance are the quantities to be determined experimentally.

## THEORY IN PRACTICE

The previous theory section touched on the mathematical aspects of the electromagnetic nature of light. Now we can look at further qualitative effects of this phenomenon, and touch more on the optics. The only items we haven't touched on at all are the lenses, and the photodetector. The lenses are in place to make sure that as much of the laser as possible goes to the detector. They can be adjusted at any time to make sure the beam focuses into the detector. This should not affect the results as the detector measures absolute power, and not the size of the beam with respect to its power. While you are adjusting the lenses, you might notice the contrary. The reason that the power measurement changes is because the alignment of the lenses is not perfect, and as you move them they might be rotated slightly. This will either spill some of the beam outside of the detector, or make it hit the side of the detector.

There are a couple more reasons why the full laser might not reach the detector. The first reason is because the light will hit little particulates on the medium and bounce off in all different directions. You may have heard this before as scattering, but it's just an aspect of reflection called **diffuse reflection**. This should not attribute to significant error, but it is the reason you can see the laser from all sides. The type of reflection which cannot be seen from all sides is **specular reflection**. This is an ideal reflection situation. When you have a perfect reflector and a coherent source of light, the reflected beam only goes off in one direction according to the law of reflection (angle of incidence equals the angle of reflection). You can test specular reflection by polishing a piece of glass as best you can, and aiming a laser at it in all different directions. At the angles where the laser reflects away from you, it should get dimmer. If you had an ideal reflector, it would be zero at all angles pointed away from you. This is why diffuse reflection is the phenomenon which lets us view nature with our bare eyes.

When we spoke of reflectance and transmittance we learned that power is redistributed at a dielectric boundary. Practically this means that any new spot where you can see the laser there is a loss of power relative to that interface. What does this mean? Well some of the power loss you would have to pay attention to, and some is lost power of already accounted for beams. This becomes more apparent in Figure 4 on the next page. Also on the next page in Figure 5, there is a picture of the laser lighting up the set up in the dark. In order to back track to the power of the polarized beam, you must take into account the path of the beam. This will be explained again in the procedure set up, but to prepare for that moment: imagine measuring the beam which reflects off the inside of the flat side of the sample. First the laser goes through the curved side where some power is lost, but we concentrate on the transmitted beam. Then when it reflects off the flat surface, we ignore the transmitted beam instead this time. Then as it leaves, we care about the transmitted beam. This makes the analysis a bit more complicated, but it gets rid of significant systematic

error, for the laser can lose a lot of power depending on the angle of incidence. Most of the error will come from the laser bouncing through the sample, but some will come from the lenses. It is okay in the interest of time to not worry about the reflection at the lenses, for a precise analysis requires a measurement of their index of refraction.

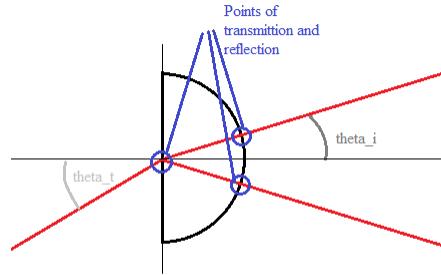


Figure 4: A diagram of the laser entering the sample. Notice that there are three points where transmission and reflection can occur. This is only to first order. In theory, there are infinite reflections and transmissions!

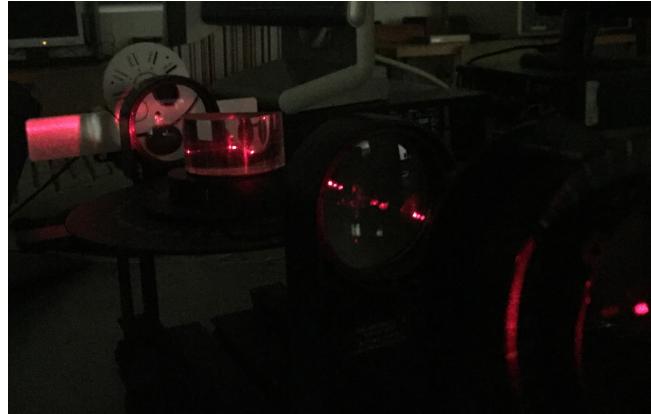


Figure 5: A picture of the setup in the dark. Notice all of the reflections. This means power is redistributed in many directions

Now let's talk about an idealized infinitesimal ray of light traversing through a semicircular piece of dielectric. The purpose of having a semicircular sample is that we only want to worry about refraction at one of the interfaces. Well hows a semicircle going to do that for us? As shown in Figure 6 on the next page, a single ray aimed at the center of the circle will always be perpendicular to a tangent line at the arc. For a semicircular piece, this would be aimed at the psuedocenter, the center that would be if you formed a circle out of the semicircle. The same effect will occur and will, in theory, protect from any need of analyzing refraction at the boundary of the circular arc. Put concisely, because of the geometric properties of a circle, the ray wont bend at the arc. This way we only have to worry about power redistribution there, and not any extra refraction.

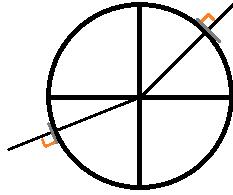


Figure 6: All straight lines emanating from the center will be perpendicular to the tangent line on the circle at their intersection. This means no bending will occur at this point. Can you prove mathematically that this is the case?

The final difficulty all has to do with these "semicircles", and how a finite sized beam goes through them. These semicircular samples are NOT SEMICIRCLES! They are off from semicircular by a small length of arc. The way to tell is a good lesson in circles. First, take anyone of the samples, and draw around the arc. Then, rotate the sample a bit so some of it is on the original arc and some off it is off. Make sure when drawing this arc that it lines up with the original one as close as possible. Then repeat this a third time. Assuming that at least the arc on these samples is a section of a circle, you should now have a circle in front of you. Now, inscribe the sample into the circle by realigning it with the arc, and draw across the flat part. Then, rotate the sample so that one of the corners of the sample touches the corner of the drawing, and draw across the flat part again. If done correctly, you should have a wedge inside of a circle. An example of construction is shown below in Figure 7. This already shows that these aren't semicircular, but if you wish to go farther, take the corner of a notecard to any point on the arc of the circle. Then where the sides of the notecard intersect the circle, draw little dashed lines. You should have drawn two lines. They should appear as crossing the arc. Draw a line connecting these two intersections. Now do the same for any other point on the arc of the circle. The point where the final two lines meet should be the center of the circle. From here you can go ahead and waste hours of your time trying to figure out the relationship between the angle of incidence and the deviation at the arc of the psuedosemicircle. This means that **an arrow leaving the psuedocenter of the psuedosemicircle will not be perpendicular to the arc of the circle**, and bending will occur. This error in reality is only large for large angles, and can be seen most easily in the Snell's Law portion of the lab. You can avoid this by avoiding the larger angles (you will be forced to abandon large angles in the critical angle cases).

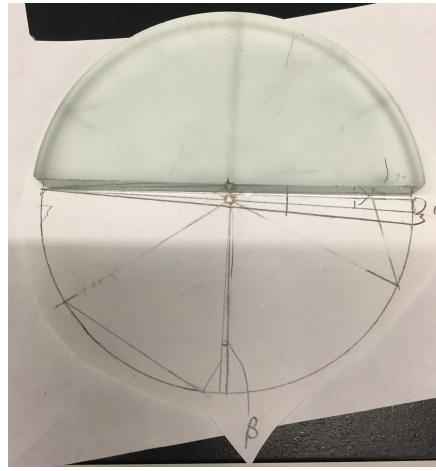


Figure 7: This is what the result of the drawing should be

A more significant form of error is due to the finite size of the beam. Because the center of the beam is pointed at the pseudocenter of the semicircle, the edges of the beam are not. They will then bend and go through the sample as if it were a lens (because it really is!). Because of the geometry of the samples, they are converging lenses no matter if you hit the arc first, or if you hit the flat side first. This is shown

in Figures 8 and 9 below. Even though they are converging lenses, past the focus the rays diverge, and contribute to rapid beam expansion. This is why we have dedicated lenses, so the spread out beam can be focused into the detector. The two photos on the next page in Figures 10 and 11 show the comparison of the beam at different angles, and they show why this effect is significant.

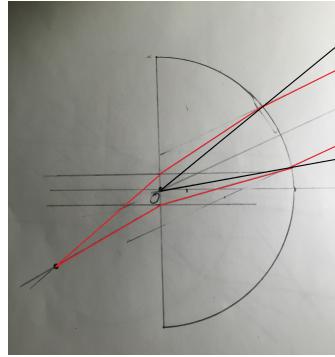


Figure 8: A highlighted sketch of a finite width beam in the **high index to low index** set up. The red lines trace the edges of the laser, while the black lines are to locate the surface normals on the edge of the circle

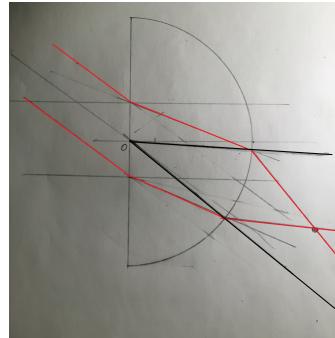


Figure 9: A highlighted sketch of a finite width beam in the **low index to high index** set up. The red lines trace the edges of the laser, while the black lines are to locate the surface normals on the edge of the circle

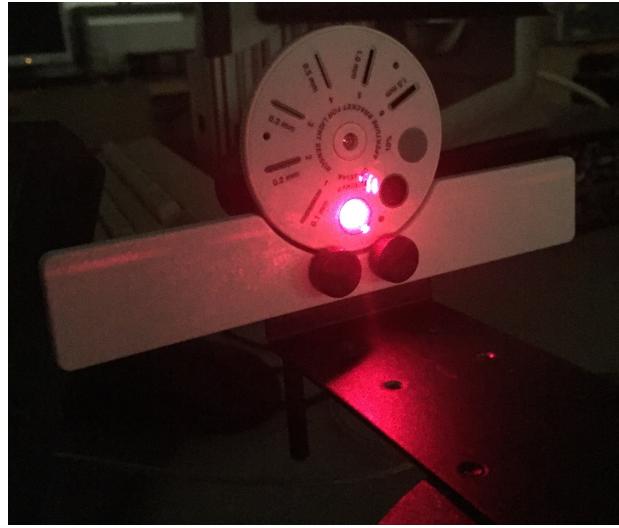


Figure 10: The laser beam through the sample at a small angle. Notice that it too is small.

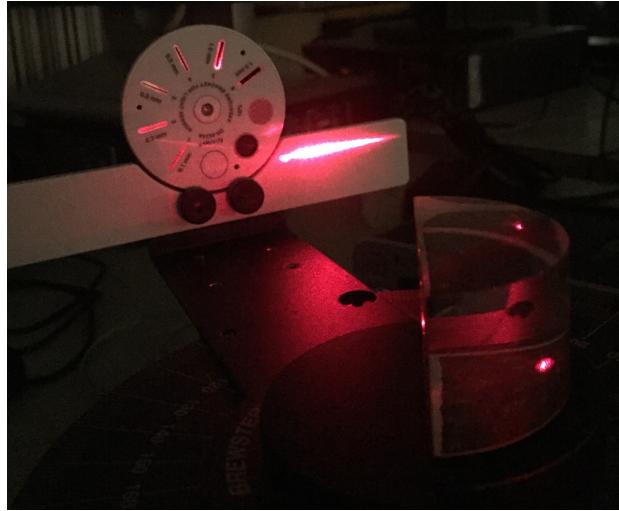


Figure 11: The laser beam through the sample at a large angle. Can you see the problem now?

Also, something to keep in mind about Snell's Law: when going from a lower index to a higher index it should be observed that the ray bends towards the surface normal, i.e the incident angle will be larger than the transmitted angle when going from a lower index to a higher index. The opposite occurs for a higher to lower index transition, as might be expected. If you haven't already discovered this experimentally or mathematically at some point in the past, then this might help you think about what degree of refraction you should expect.

## PROCEDURE

**\*\*If you have any issues with this procedure, try to talk to Phil first.\*\***

- 1) Make sure the optical elements are in order from right to left: laser, polarizer, track lens, sample, magnetic lens. Then align the 180 degree mark on the outer goniometer (angle measurer) to the line on the extended platform. Then align the magnetic stage so that the white line lines up with 90 degrees on the inner

goniometer. Note that there are two white lines, and you can start lining up either, but make sure to put the sample on the higher part of the magnetic stage later.

2) Now with all of the optical elements in place besides the sample, line up the laser. There are two thumbscrews on the laser which control the vertical and horizontal direction of the beam. You can find out very quickly which one does which. Make sure to rotate the screen so that you can see the laser inside of the small white circle on the screen. Try to align it as close to the center of the circle as you can, and feel free to adjust the lenses to focus the beam at that point as best you can.

3) Now add the sample. The important thing is to not only line up the sample on the stage so its pseudocenter is at the center of the circular stage, but also to have the laser still hitting the same spot on the screen while also passing through the pseudocenter of the sample. Doing this by eye is the easiest way, but don't pay attention so much to the sample's relative position on the stage. Since it is not perfectly semicircular, it will not fit there symmetrically as one might expect. Try your best to make sure the laser hits the top of the arc of the semicircle and passes through to the pseudocenter as best you can. **Make sure to record any realignments or lens movements in your lab manual, so you know EXACTLY how it affects your data!**

4) This entire alignment process is the first source of systematic error, but it is not the largest, so as long as you did your best to align it do not worry too much about it. Just make sure not to disturb the sample's relative position on the magnetic stage, and make sure to not disturb any of the optical elements in the middle of each set of measurements. This includes adjusting the thumbscrews on the laser. It is a difficult process to estimate the error induced by any of these mistakes, so be careful!

5) First measure the index of refraction by rotating the sample on the magnetic stage. Keep in mind that the numbers on the goniometers are not in line with the convention derived in the equations. Make sure to account for the fact that according to the derived equations: **zero angle of incidence means the ray is perpendicular to the surface**. Make sure also to find the critical angle where appropriate. Feel free to use Snell's Law to guide you to this angle. Try to take at least 5 measurements, but the more measurements you take the less error you should have. If done perfectly, the highest source of error should be from the goniometer itself. You may have to move the lens in order to figure out where the true focus point of the beam is at larger angles. If you need to do so, move the lens **slowly**, so you can monitor the horizontal deviation of the beam and hopefully reduce it to none.

6) You should have taken a total of four measurements. Two for orientations for each of two samples. Make sure to figure out and understand which orientation corresponds to which index transition. That means to ask yourself: **which orientation corresponds to  $n_{air}$  to  $n_{sample}$  and which is the other way around?**

7) Now it's time to measure the power redistribution. Make sure the beam is aligned and focused as well as possible on the blank screen. Then rotate the screen so that it is open to the beam. Turn on the wattmeter attached to the photodetector. Measure the power of the beam with your sample on the stage. You will need to aim the laser with the thumbscrews to get a maximum power reading. This will be the reference power reading. In subsequent measurements you will rotate the stage, and NOT re-aim the laser to achieve the maximum power reading. Don't forget to be careful if you decide to move the lenses. \*\*A useful way to correlate the power reading with the angular error is to measure the laser as it passes from left to right of the detector. If you decide to do this, be weary of anomalous power readings, as sometimes you can get readings which are many orders of magnitude higher than the beam's power itself! If you can explain this phenomenon concretely (as opposed to an educated guess), then consider it extra credit.

8) **PAY ATTENTION TO THE POLARIZER!** The polarization states of S and P are relative to the sample, but how do you know which way the electric field is facing in the first place? You can either trust what you read here, or check the Polarizer fact sheet in the manual stack at the lab bench. When the polarizer is at zero degrees relative to the top of the optics brace, it is allowing vertically polarized light through. That means, by the time it hits your sample it will be **S polarized**. If you rotate it 90 degrees

then it will come out horizontally polarized and then it will be **P polarized** at the interface. Make sure to keep track of the polarization state in your lab manual.

9) Be weary of which measurements need potential reflectance and transmittance adjustments. You don't have to do this at every measurement, just make a note of it in your lab manual and worry about it later. Also make sure to take many measurements near the critical angle and Brewster's angle. You may guide yourself to these angles by using the formulas. You will have to use the indexes of refraction from the previous part in order to do this systematically. Make sure in this part to **measure BOTH the reflected and transmitted beams**. It may seem obvious while and after reading this, but this is a tedious measurement and your boredom may cause you to forget. **DON'T FORGET TO MEASURE BOTH REFLECTED AND TRANSMITTED BEAMS!** In order to compare your measurements to the reflectance and transmittance, subsequent power readings will have to be divided the reference power. Alternatively, you can multiply the reference power by the theoretical reflectance and transmittance when checking the fit to the data. Both are equivalent. If the alignment is done correctly and the lenses aren't disturbed significantly in the process, the largest source of error should come from the wattmeter and the goniometer. You're going to want to sketch by hand (or just plot directly on the computer) the results you are getting. Compare them to the theoretical curve to make sure that you are on the right track. Make sure that you are NOT using the Fresnel Coefficients when plotting.

10) If all went well then you should now be complete! Estimate the error you have accumulated due to the instrumentation, movement of the optical devices, the shape of the beam, and the location of the sample as you deem fit. It is not expected that you need to do all of this (it is especially not expected that you account for the psuedosemicircular samples), but make sure to understand how each affects your data.

## REPORT

**\*\*These items are supplementary to the requirements on the course website!\*\***

- 1) Make sure to show the linear graph of  $\sin(\theta_i)$  vs  $\sin(\theta_t)$  (or the other way around) for each of the four measurements. The slope of this line is the ratio between the two indexes of refraction. For our purposes, set  $n_{air} = 1$  as the minuscule deviation from one is undetectable in this setup. You do not need to show the table of data if your graph is explicit enough, but it helps your credibility. Always try to be more credible! Make sure to include the details of how each measurement went so you can make sense of your error bars.
- 2) State how you measure the reference power. Make sure to include any other power measurements you made through any configuration of optical elements. Don't forget to label the polarization state on each graph of the transmittance and reflectance. You are free to plot  $t_p$ ,  $t_s$ ,  $r_p$ , and  $r_s$  separately if needed, but you will have less graphs if you have each polarization state grouped with each sample configuration. Make sure to include error bars.

- 3) **IF YOU INCLUDE PHOTOS OF PHENOMENON, DO NOT USE THE ONES IN THIS LAB MANUAL.** Feel free to reuse the difficult to draw diagrams (although an original sketch is recommended), but do not use the same photos here. It is probably unnecessary to use photos in the first place, but photos described in the lab report should be **ORIGINAL!**

## REFERENCES

1. John David Jackson. *Classical Electrodynamics 3rd ed.* John Wiley & Sons, INC., USA 1999
2. Hecht and Ganesan. *Optics 4th ed.* Pearson Education, Freeport New York

Suggested Google Searches: Basis of a Vector Space, Polarizers, Diode Laser, Gaussian Optics, Malus's

Law, Quantum Electrodynamics, Specular Reflection, Diffuse Reflection, Randomness in Probability Theory,  
Circle Theorems

## APPENDIX

The scope of Maxwell's equations for describing physical electromagnetic phenomenon is vast, but most notably it describes the propagation of light as an electromagnetic wave. To see this, one can start with Maxwell's equations in vacuum (absent of charges and currents):

$$\begin{aligned}\vec{\nabla} \times \vec{E} &= -\frac{\partial}{\partial t} \vec{B} \\ \vec{\nabla} \times \vec{B} &= c^{-2} \frac{\partial}{\partial t} \vec{E} \\ \vec{\nabla} \cdot \vec{E} &= 0 \\ \vec{\nabla} \cdot \vec{B} &= 0\end{aligned}$$

Where  $c$  is the speed of light, which is a combination of fundamental constants. First we can take the time derivative of the first and second equations, then we can plug the first equation into the second and the second equation into the first. A small bit of algebra and a vector identity leads us to these results for the electric and magnetic fields:

$$\begin{aligned}-\vec{\nabla}(\vec{\nabla} \cdot \vec{E}) + \vec{\nabla}^2(\vec{E}) &= c^{-2} \frac{\partial^2}{\partial t^2} \vec{E} \\ \vec{\nabla}(\vec{\nabla} \cdot \vec{B}) - \vec{\nabla}^2(\vec{B}) &= -c^{-2} \frac{\partial^2}{\partial t^2} \vec{B}\end{aligned}$$

Since we are absent of electric charges in vacuum, and absent of magnetic charges always, the first terms in each of these equations are zero, and we obtain the wave equations with phase velocity  $c$ :

$$\begin{aligned}\vec{\nabla}^2 \vec{E} &= c^{-2} \frac{\partial^2}{\partial t^2} \vec{E} \\ \vec{\nabla}^2 \vec{B} &= c^{-2} \frac{\partial^2}{\partial t^2} \vec{B}\end{aligned}$$

These equations describe the nature of light as an electromagnetic wave. The interaction of light with media obeys Maxwell's equations entirely. We are now in a position to study what happens when electric and magnetic fields pass through different medium. First, let's introduce the constitutive relations for linear electric and magnetic media.

$$\begin{aligned}\vec{D} &= \epsilon \vec{E} \\ \vec{H} &= \mu^{-1} \vec{B}\end{aligned}$$

Where  $\epsilon$  and  $\mu$  are the dielectric permittivity and magnetic permeability respectively. They describe the intrinsic macroscopic properties of a material, and in general depend on frequency and direction. Frequency dependence corresponds to the frequency of electromagnetic oscillation in the media. Direction dependence occurs in anisotropic media. Since we are using light of essentially one color, we can neglect the frequency dependence. We will also neglect any anisotropic effects and assume that there is no excess charge or current flow intrinsic to our samples. The  $D$  and  $H$  fields are the normal electric and magnetic fields, but for free charges and currents.

First, lets start with this form of Maxwell's Equations for free charges and currents.

$$\begin{aligned}\vec{\nabla} \times \vec{E} &= -\frac{\partial}{\partial t} \vec{B} \\ \vec{\nabla} \times \vec{H} &= \frac{\partial}{\partial t} \vec{D} + \vec{J} \\ \vec{\nabla} \cdot \vec{D} &= \rho \\ \vec{\nabla} \cdot \vec{B} &= 0\end{aligned}$$

It should be noted that since we are doing a linear approximation, the polarization does not curl. Therefore it is appropriate to express the first equation with the electric field. The low intensity of the beam definitely makes a very low magnetic field strength. Thus, even if the materials in question were strongly paramagnetic or diamagnetic, there would be little magnetization, so we will neglect it.

Before examining the mathematics at the boundary, we should have a clear picture of what is going on. Examine the diagram below. It is a bit busy, but careful examination will provide a diagram of the situation. The four fields  $\vec{E}$ ,  $\vec{B}$ ,  $\vec{D}$ , and  $\vec{H}$  are assumed to be general three non zero component fields. We assume that the fields in medium 1 might emerge from medium 2 differently than they entered. We define unit vectors  $\vec{n}$ , and  $\vec{t}$  as the normal vector to the surface, and the tangent vector to the surface respectively. We have constructed the contour C such that the vector  $\vec{t} \times \vec{n}$  points along the line of the contour.

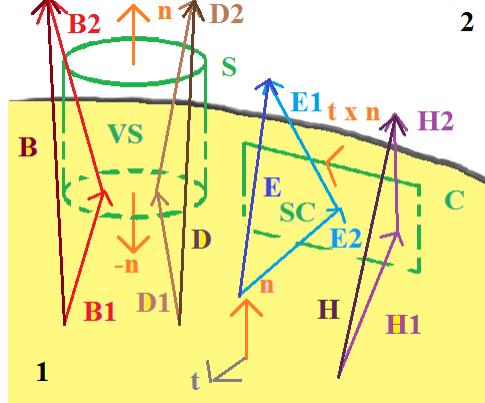


Figure 12: An illustration of electric and magnetic fields passing through two different media

Now we can invoke the integral forms of Maxwell's equations. These can be obtained by Gauss's law, and Green's Theorem for vector fields:

$$\begin{aligned} \oint_S \vec{D} \cdot \vec{n} da_S &= \int_{VS} \rho dV_S \\ \oint_S \vec{B} \cdot \vec{n} da_S &= 0 \\ \oint_C \vec{H} \cdot (\vec{t} \times \vec{n}) dl &= \int_{SC} (\vec{J} + \frac{\partial}{\partial t} \vec{D}) \cdot \vec{t} da_{SC} \\ \oint_C \vec{E} \cdot (\vec{t} \times \vec{n}) dl &= - \int_{SC} (\frac{\partial}{\partial t} \vec{B}) \cdot \vec{t} da_{SC} \end{aligned}$$

Where  $\rho$  and  $\vec{J}$  are the induced volume charge and volume current respectively. We denote S to be the surface of the cylinder, and consequentially the area element  $da_S$  corresponds to this surface area. The volume VS is simply the volume of the cylinder corresponding to the surface of S. C is the contour pointing along  $\vec{t} \times \vec{n}$  with associated area element  $dl$ . SC is the rectangle bounded by this contour with associated area element  $da_{SC}$ . The choice of notation for the tangent unit vector is unfortunate, but hopefully the use of the arrow notation will make it unambiguously different from the variable for time. We adapt this notation for the tangent vector from Jackson Classical Electrodynamics 3rd Edition, where a more concise derivation is provided.

Now, we can imagine shrinking the height of the cylinder to destroy any contribution from its sides. This first takes our volume integral into a surface integral over the surface charge density  $\sigma$ . We also assume the area is infinitesimal, so that  $\vec{D}$ ,  $\vec{B}$  and  $\sigma$  do not vary along the surface. This will make the integral only over the infinitesimal area, and not over the fields or charges. We set  $\vec{D} = \vec{D}_2 + \vec{D}_1$ , and  $\vec{B} = \vec{B}_2 + \vec{B}_1$  as shown in the diagram. Then, we can evaluate the integrals easily by first taking the dot products of  $\vec{D}_1$  and  $\vec{B}_1$  and through the bottom, and  $\vec{D}_2$  and  $\vec{B}_2$  out the top, and then pulling the fields and charges out of their integrals. This leaves us with our first two conditions:

$$\begin{aligned}
(\vec{D}_2 - \vec{D}_1) \cdot \vec{n} \oint_S da_S &= \sigma \oint_S da_S \\
(\vec{D}_2 - \vec{D}_1) \cdot \vec{n} &= \sigma \\
(\vec{B}_2 - \vec{B}_1) \cdot \vec{n} \oint_S da_S &= 0 \\
(\vec{B}_2 - \vec{B}_1) \cdot \vec{n} &= 0
\end{aligned}$$

Thus we learn that the normal component of the electric field is discontinuous across the boundary due to induced surface charge, and the normal component magnetic field is continuous across the boundary in general. We can also imagine shrinking the height of the contour, leaving essentially two antiparallel lines facing in opposite direction by construction. This makes the contribution from tangential components of  $\vec{D}$  and  $\vec{B}$  through the surface zero. A surface current can still make it through, reducing the surface integral for the current density into a line integral of the surface current. As with the cylinder, we make sure the length of the contour is small enough such that  $\vec{E}$  and  $\vec{H}$  do not vary along the contour. We then are left with our second two conditions.

$$\begin{aligned}
\oint_C (\vec{H}_2 - \vec{H}_1) \cdot (\vec{t} \times \vec{n}) dl &= \int_{SC} \vec{J} \cdot \vec{t} da_{SC} \\
\oint_C (\vec{H}_2 - \vec{H}_1) \cdot (\vec{t} \times \vec{n}) dl &= \oint_C \vec{K} \cdot \vec{t} dl \\
(\vec{H}_2 - \vec{H}_1) \cdot (\vec{t} \times \vec{n}) \oint_C dl &= \vec{K} \cdot \vec{t} \oint_C dl \\
\vec{n} \times (\vec{H}_2 - \vec{H}_1) \cdot \vec{t} \oint_C dl &= \vec{K} \cdot \vec{t} \oint_C dl \\
\vec{n} \times (\vec{H}_2 - \vec{H}_1) &= \vec{K} \\
\oint_C (\vec{E}_2 - \vec{E}_1) \cdot (\vec{t} \times \vec{n}) dl &= 0 \\
(\vec{E}_2 - \vec{E}_1) \cdot (\vec{t} \times \vec{n}) \oint_C dl &= 0 \\
\vec{n} \times (\vec{E}_2 - \vec{E}_1) &= 0
\end{aligned}$$

And therefore the tangential component to the magnetic field suffers a discontinuity due to induced surface currents, and the tangential component of the electric field is continuous. A comment on this idea is in order. It seems like any portion of the field which goes through the face of the contour is therefore ignored in general if this were followed naively. The truth is that the orientation of the tangential vector is arbitrary, and hence we have the same case if the contour is rotated 90 degrees. This means the second pair of boundary conditions is actually a set of four equations.

For linear media, these boundary conditions hold in general, and therefore can be applied to our problem with the assumption first that we have a plane wave of electromagnetic radiation passing through one dielectric into another. We follow relatively closely the approach of Jackson:

Incident

$$\begin{aligned}
\vec{E}_i &= \vec{E}_{i0} \exp(i(\vec{k}_i \cdot \vec{r} - \omega t)) \\
\vec{B}_i &= \sqrt{\mu_1 \epsilon_1} \frac{\vec{k}_i \times \vec{E}_i}{k_1}
\end{aligned}$$

Refracted

$$\vec{E}_t = \vec{E}_{t0} \exp(i(\vec{k}_t \cdot \vec{r} - \omega t))$$

$$\vec{B}_t = \sqrt{\mu_2 \epsilon_2} \frac{\vec{k}_t \times \vec{E}_t}{k_2}$$

Reflected

$$\vec{E}_r = \vec{E}_{r0} \exp(i(\vec{k}_r \cdot \vec{r} - \omega t))$$

$$\vec{B}_r = \sqrt{\mu_1 \epsilon_1} \frac{\vec{k}_r \times \vec{E}_r}{k_1}$$

With  $|\vec{k}_i| = |\vec{k}_r| = k_1 = \omega\sqrt{\mu_1\epsilon_1}$ , and  $|\vec{k}_t| = k_2 = \omega\sqrt{\mu_2\epsilon_2}$ . Now, plugging these expressions into our derived boundary conditions gives the following general plane wave boundary conditions:

$$\begin{aligned} [\epsilon_1(\vec{E}_i + \vec{E}_r) - \epsilon_2 \vec{E}_t] \cdot \vec{n} &= 0 \\ [\vec{k}_i \times \vec{E}_i + \vec{k}_r \times \vec{E}_r - \vec{k}_t \times \vec{E}_t] \cdot \vec{n} &= 0 \\ (\vec{E}_i + \vec{E}_r - \vec{E}_t) \times \vec{n} &= \vec{0} \\ [\mu_1^{-1}(\vec{k}_i \times \vec{E}_i + \vec{k}_r \times \vec{E}_r) - \mu_2^{-1} \vec{k}_t \times \vec{E}_t] \times \vec{n} &= \vec{0} \end{aligned}$$

Light exists in many polarization states which in linear cases are superpositions of two orthogonal states, which may be phase dependent. In our experiment, we only consider P and S polarization states. The P state refers to a state where the electric field of the light is in the plane of incidence (the plane spanned by all the wave vectors). The S state is when the electric field is normal to the plane of incidence. A diagram of both is shown below.

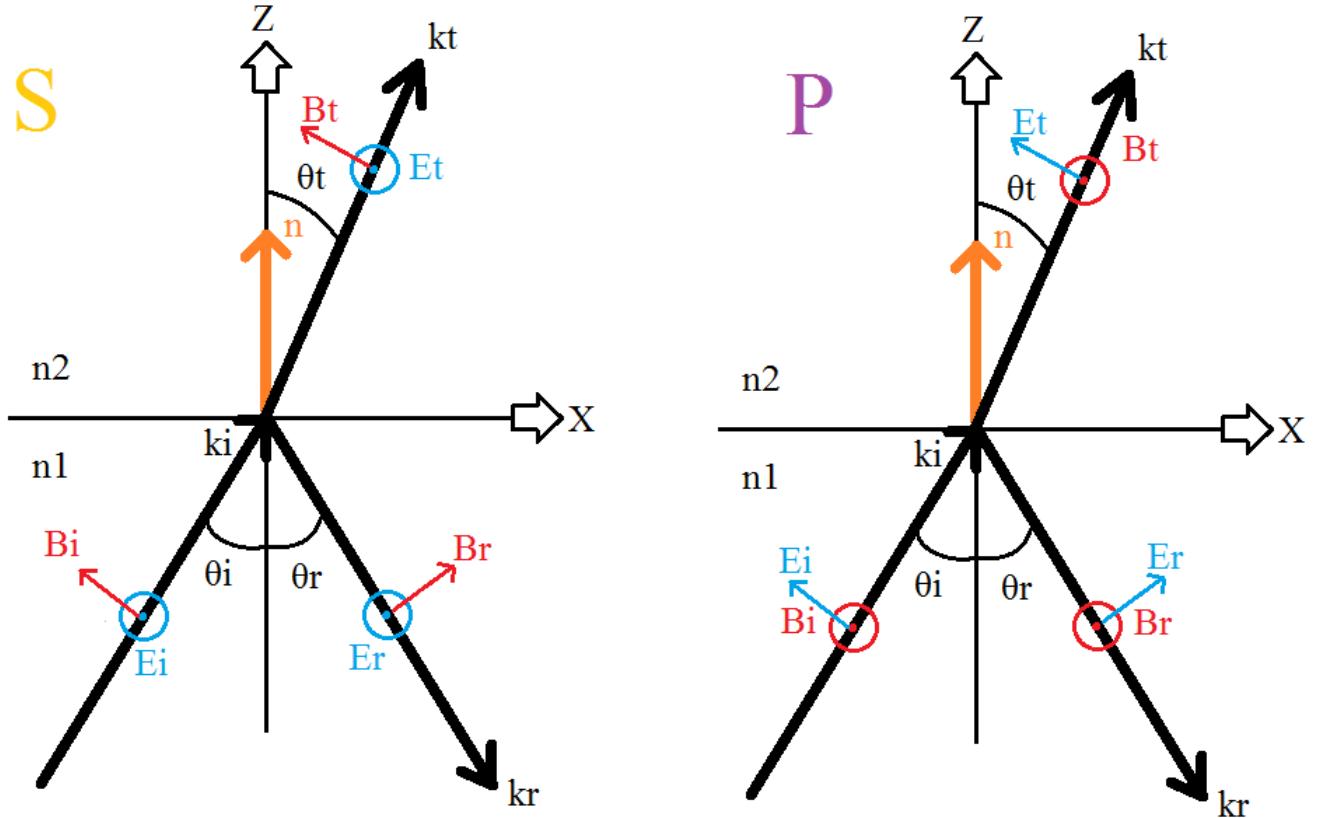


Figure 13: An illustration of S and P polarization states. The left is the S state and the right is the P state. A ring represents a vector parallel to the y axis sticking out towards the reader

First, let's consider the third wave boundary condition in order to derive Snell's Law. This will work with any of the boundary conditions, but the third one is the easiest one to deal with. Plugging in our plane wave constructions yield:

$$\begin{aligned} & (\vec{E}_{i0} \exp(i(\vec{k}_i \cdot \vec{r} - \omega t)) + \vec{E}_{r0} \exp(i(\vec{k}_r \cdot \vec{r} - \omega t)) - \vec{E}_{t0} \exp(i(\vec{k}_t \cdot \vec{r} - \omega t))) \times \vec{n} = \vec{0} \\ & (\vec{E}_{i0} \times \vec{n}) \exp(i(\vec{k}_i \cdot \vec{r} - \omega t)) + (\vec{E}_{r0} \times \vec{n}) \exp(i(\vec{k}_r \cdot \vec{r} - \omega t)) - (\vec{E}_{t0} \times \vec{n}) \exp(i(\vec{k}_t \cdot \vec{r} - \omega t)) = \vec{0} \\ & (\vec{E}_{i0} \times \vec{n}) \exp(i(\vec{k}_i \cdot \vec{r})) + (\vec{E}_{r0} \times \vec{n}) \exp(i(\vec{k}_r \cdot \vec{r})) - (\vec{E}_{t0} \times \vec{n}) \exp(i(\vec{k}_t \cdot \vec{r})) = \vec{0} \end{aligned}$$

Where the time dependence could drop out for a number of reasons, but the simplest reason being that it is a common factor. We wish to equate the arguments of the exponentials to not only obtain Snell's Law, but to show that the angle of incidence is the same as the angle of reflection, and furthermore to drop out the exponentials when we solve for the Fresnel Equations. To show that we can do this, here is a little math:

Suppose

$$A \exp(ax) + B \exp(bx) = C \exp(cx)$$

Clearly if it is true for all  $x$ , then it is true for  $x = 0$ . Thus:

$$A + B = C$$

Taking two derivatives gives:

$$\begin{aligned} Aa \exp(ax) + Bb \exp(bx) &= Cc \exp(cx) \\ Aa^2 \exp(ax) + Bb^2 \exp(bx) &= Cc^2 \exp(cx) \end{aligned}$$

Placing the second equation into the third gives:

$$Aa^2 \exp(ax) + Bb^2 \exp(bx) = c(Aa \exp(ax) + Bb \exp(bx))$$

Plugging in zero again gives:

$$Aa^2 + Bb^2 = c(Aa + Bb)$$

$$Aa(a - c) + Bb(b - c) = 0$$

Clearly if  $A$ ,  $B$ ,  $C$ ,  $a$ ,  $b$  and  $c$  are all assumed to be nonzero, then the only solution of this equation is

$$a = b = c$$

Which is certainly true for

$$ax = bx = cx$$

This leads us back to our original problem giving exactly what we desired:

$$\vec{k}_i \cdot \vec{r} = \vec{k}_r \cdot \vec{r} = \vec{k}_t \cdot \vec{r}$$

Which will be evaluated on the surface of the boundary (at  $z = 0$ ). Using the cosine of the interior angle:

$$k_1 \cos\left(\frac{\pi}{2} - \theta_i\right) = k_1 \cos\left(\frac{\pi}{2} - \theta_r\right) = k_2 \cos\left(\frac{\pi}{2} - \theta_t\right)$$

Right away we see that the solution to the first two is  $\theta_i = \theta_r$ . Giving us our desired confirmation of the law of reflection. Continuing on after a simple sine identity, and the fact that  $k_i = \frac{\omega n_i}{c}$ :

$$n_1 \sin(\theta_i) = n_2 \sin(\theta_t)$$

Where  $n$  is the index of refraction  $n = \frac{c}{v}$ . A direct measurement or a priori knowledge of the phase velocity  $v$  was not made or assumed. You will measure the ratio of  $n_2$  to  $n_1$  in this experiment.

The last thing to find is the relationship between the amplitudes of the three light rays. To do this, we must consider the S and P polarization states separately. Starting with the S state, We can use the third and fourth amplitude boundary conditions since the exponentials all drop out. The third condition is the easiest:

$$\begin{aligned}(\vec{E}_{i0} + \vec{E}_{r0} - \vec{E}_{t0}) \times \vec{n} &= \vec{0} \\ \vec{E}_{i0} + \vec{E}_{r0} - \vec{E}_{t0} &= \vec{0}\end{aligned}$$

Since each wave vector is perpendicular to its wave vector, the fourth equation is also quite simple. The cross product of the wave vector with the electric field brings us to the direction of the magnetic field. The magnetic field has components both normal and tangent to the boundary surface. We are speaking about the tangential component, so we take the cosine of the incident angle and the transmitted angle.

$$\begin{aligned}[\mu_1^{-1}(\vec{k}_i \times \vec{E}_{i0} + \vec{k}_r \times \vec{E}_{r0}) - \mu_2^{-1}\vec{k}_t \times \vec{E}_{t0}] \times \vec{n} &= \vec{0} \\ \sqrt{\frac{\epsilon_1}{\mu_1}}(E_{i0} + E_{r0})\cos(\theta_i) - \sqrt{\frac{\epsilon_2}{\mu_2}}k_t E_{t0}\cos(\theta_t) &= 0\end{aligned}$$

Rearranging yields the ratio of the reflection and transmission coefficients with respect to the incident electric field. It is convenient to express the cosine of the transmission angle in terms of the incident angle using Snell's Law.

$$\begin{aligned}T_s &= \frac{E_{0t}}{E_{0i}} = \frac{2\cos(\theta_i)}{\cos(\theta_i) + \frac{\mu_1}{\mu_2}\sqrt{\frac{n_2^2}{n_1^2} - \sin^2(\theta_i)}} \\ R_s &= \frac{E_{0t}}{E_{0i}} = \frac{\cos(\theta_i) - \frac{\mu_1}{\mu_2}\sqrt{\frac{n_2^2}{n_1^2} - \sin^2(\theta_i)}}{\cos(\theta_i) + \frac{\mu_1}{\mu_2}\sqrt{\frac{n_2^2}{n_1^2} - \sin^2(\theta_i)}}\end{aligned}$$

Now we switch to the P polarization state. This will require the use of the third and fourth boundary conditions again. Through some manipulation we obtain:

$$\begin{aligned}\cos(\theta_i)(E_{0i} - E_{0r}) - \cos(\theta_t)E_{0t} &= 0 \\ \sqrt{\frac{\epsilon_1}{\mu_1}}(E_{0i} + E_{0r}) - \sqrt{\frac{\epsilon_2}{\mu_2}}E_{0t} &= 0\end{aligned}$$

Rearranging and solving for the ratios give us:

$$\begin{aligned}T_p &= \frac{E_{0t}}{E_{0i}} = \frac{2\frac{n_2}{n_1}\cos(\theta_i)}{\frac{\mu_1 n_2}{\mu_2 n_1}\cos(\theta_i) + \sqrt{\frac{n_2^2}{n_1^2} - \sin^2(\theta_i)}} \\ R_p &= \frac{E_{0r}}{E_{0i}} = \frac{\frac{\mu_1 n_2}{\mu_2 n_1}\cos(\theta_i) - \sqrt{\frac{n_2^2}{n_1^2} - \sin^2(\theta_i)}}{\frac{\mu_1 n_2}{\mu_2 n_1}\cos(\theta_i) + \sqrt{\frac{n_2^2}{n_1^2} - \sin^2(\theta_i)}}\end{aligned}$$

These four equations for  $T_s$ ,  $T_p$ ,  $R_s$  and  $R_p$  are known as the Fresnel Equations named after Augustin-Jean Fresnel, and  $T_s$ ,  $T_p$ ,  $R_s$  and  $R_p$  will be referred to as the Fresnel Coefficients. Specifically  $R_s$  and  $R_p$  will be referred to as the reflection coefficients.  $T_s$  and  $T_p$  will be referred to as the transmission coefficients, accordingly. We chose to write them in this form because we will set  $\mu_1 = \mu_2 = 1$  for the materials we are considering, and thus we have the Fresnel Coefficients as a function of the ratio of the index of refraction and the incident angle of the laser.

There are still two things left to derive. First, we suspect from the form of the reflection coefficients that there is some angle where they might equal zero, and thus give zero reflection. This is simply because there are two positive numbers being subtracted from each other. Let's investigate. First, do as we said and set  $R_s = 0$ . This gives:

$$\frac{n_2}{n_1} = 1$$

Which means for S polarization there is only zero reflection when there is no medium to reflect off. The P polarization reflection coefficient is more difficult to handle, but brings new information to the theory. Setting  $R_p = 0$  gives:

$$\frac{n_2^2}{n_1^2} \cos(\theta_i) = \sqrt{\frac{n_2^2}{n_1^2} - \sin^2(\theta_i)}$$

Now, set  $x = \frac{n_2^2}{n_1^2}$ , and rearrange to obtain a quadratic formula:

$$x^2 - \sec^2(\theta_i)x + \tan^2(\theta_i) = 0$$

The solution is standard:

$$x_{\pm} = \frac{\sec^2(\theta_i) \pm \sqrt{\sec^4(\theta_i) - 4\tan^2(\theta_i)}}{2}$$

We do not wish to solve for x. We want to find the angle of incidence at which the reflection coefficient vanishes. From the very beginning we had a transcendental equation for  $\theta_i$ . We can solve for this angle using the identity:  $\tan^2(\theta_i) + 1 = \sec^2(\theta_i)$

$$\begin{aligned} 2x_{\pm} &= \tan^2(\theta_i) + 1 \pm \sqrt{(\tan^2(\theta_i) + 1)^2 - 4\tan^2(\theta_i)} \\ 2x_{\pm} &= \tan^2(\theta_i) + 1 \pm \sqrt{\tan^4(\theta_i) + 1 + 2\tan^2(\theta_i) - 4\tan^2(\theta_i)} \\ 2x_{\pm} &= \tan^2(\theta_i) + 1 \pm \sqrt{(\tan^2(\theta_i) - 1)^2} \\ 2x_{\pm} &= \tan^2(\theta_i) + 1 \pm (\tan^2(\theta_i) - 1) \end{aligned}$$

If we take the negative solution, we recover the same story where there is no reflection if there is no second medium, but if we take the positive solution we get a different result:

$$\begin{aligned} 2x &= 2\tan^2(\theta_i) \\ \frac{n_2}{n_1} &= \tan(\theta_i) \\ \theta_B &= \arctan\left(\frac{n_2}{n_1}\right) \end{aligned}$$

Where we call this special angle of incidence  $\theta_B$  Brewster's Angle, named after Sir David Brewster.

Now continue softly, we derive the last interesting phenomenon. From Snell's Law, we can calculate another special angle.

$$n_1 \sin(\theta_i) = n_2 \sin(\theta_t)$$

Suppose first that  $\theta_t = \frac{\pi}{2}$ , then:

$$\arcsin\left(\frac{n_2}{n_1}\right) = \theta_c$$

Notice this will only work for  $n_1 > n_2$ , since the arcsine function only takes arguments between -1 and 1. Thus we have the condition for total internal reflection, given by the critical incident angle  $\theta_c$ . This occurs when the transmitted light bends 90 degrees. Thus it will only travel along the edge and not through it.