Solutions to Discussion Week 4:

Solution to Exercise 15.1-1

We can verify that $T(n) = 2^n$ is a solution to the given recurrence by the substitution method. We note that for $n = 0$, the formula is true since $2^0 = 1$. For $n > 0$, substituting into the recurrence and using the formula for summing a geometric series yields

$$\begin{aligned} T(n) &= 1 + \sum_{j=0}^{n-1} 2^j \\ &= 1 + (2^n - 1) \\ &= 2^n . \end{aligned}$$

Solution to Exercise 15.1-2

Here is a counterexample for the "greedy" strategy:

length i	1	2	3	4
price p_i	1	20	33	36
p_i / i	1	10	11	1

Let the given rod length be 4. According to a greedy strategy, we first cut out a rod of length 3 for a price of 33, which leaves us with a rod of length 1 of price 1. The total price for the rod is 34. The optimal way is to cut it into two rods of length 2 each fetching us 40 dollars.

Making Change

- Input: n denominations of coins $1 = v_1 < v_2 < \ldots < v_n$
- Problem: Make change for amount of money C using as few coins as possible.
- Note: all of v_i and C are positive integers.

Making Change – Dynamic Programming

$M(j)$ = the minimum number of coins required to make change for an amount of money j.

$$M(j) = min_i\{M(j - v_i)\} + 1$$

The smallest number of coins required to make j, is the smallest number required to make $j - v_i$, plus one.

(Aside: How would you prove it?)

To calculate, start from the smallest amount, 1, and build up a table M.

Making Change Example

$v_1 = 1$
$v_2 = 3$
$v_3 = 4$
$C = 10$

Making Change Example

$$v_1 = 1$$
$$v_2 = 3$$
$$v_3 = 4$$
$$C = 10$$

M[i] $\boxed{1}$

Making Change Example

$$
\begin{aligned}
v_1 &= 1 \\
v_2 &= 3 \\
v_3 &= 4 \\
C &= 10 \\
M[2] &= M[1] + 1
\end{aligned}
$$

M[i] $\boxed{1} \leftarrow \boxed{2}$

Making Change Example

$$
\begin{aligned}
v_1 &= 1 \\
v_2 &= 3 \\
v_3 &= 4 \\
C &= 10 \\
M[3] &= 1
\end{aligned}
$$

M[i] $\boxed{1}$ $\boxed{2}$ $\boxed{1}$

Making Change Example

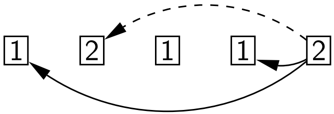
$v_1 = 1$
$v_2 = 3$
$v_3 = 4$
$C = 10$
$M[4] = 1$

M[i] $\boxed{1}$ $\boxed{2}$ $\boxed{1}$ $\boxed{1}$

Making Change Example

$$v_1 = 1$$
$$v_2 = 3$$
$$v_3 = 4$$
$$C = 10$$
$$M[5] = M[4] + 1$$

M[i]



Making Change Example

$v_1 = 1$
$v_2 = 3$
$v_3 = 4$
$C = 10$
$M[6] = M[3] + 1$

M[i] $\boxed{1}$ $\boxed{2}$ $\boxed{1}$ $\boxed{1}$ $\boxed{2}$ $\boxed{2}$

Making Change Example

$v_1 = 1$
$v_2 = 3$
$v_3 = 4$
$C = 10$
$M[7] = M[3] + 1$

M[i] $\boxed{1}$ $\boxed{2}$ $\boxed{1}$ $\boxed{1}$ $\boxed{2}$ $\boxed{2}$ $\boxed{2}$

Making Change Example

$v_1 = 1$
$v_2 = 3$
$v_3 = 4$
$C = 10$
$M[8] = M[4] + 1$

M[i] $\boxed{1}$ $\boxed{2}$ $\boxed{1}$ $\boxed{1}$ $\boxed{2}$ $\boxed{2}$ $\boxed{2}$ $\boxed{2}$

Making Change Example

$v_1 = 1$
$v_2 = 3$
$v_3 = 4$
$C = 10$
$M[9] = M[6] + 1$ **or** $M[9] = M[8] + 1$ **or** $M[9] = M[5] + 1$

M[i] $\boxed{1}$ $\boxed{2}$ $\boxed{1}$ $\boxed{1}$ $\boxed{2}$ $\boxed{2}$ $\boxed{2}$ $\boxed{2}$ $\boxed{3}$

Making Change Example

$v_1 = 1$
$v_2 = 3$
$v_3 = 4$
$C = 10$
$M[10] = M[6] + 1$ **or** $M[10] = M[7] + 1$

M[i] $\boxed{1}$ $\boxed{2}$ $\boxed{1}$ $\boxed{1}$ $\boxed{2}$ $\boxed{2}$ $\boxed{2}$ $\boxed{2}$ $\boxed{3}$ $\boxed{3}$