CS 325 - Homework 4

1. Let X and Y be two decision problems. Suppose we know that X reduces to Y. Which of the following can we infer? Explain

   a. If Y is NP-complete then so is X.
   b. If X is NP-complete then so is Y.
   c. If Y is NP-complete and X is in NP then X is NP-complete.
   d. If X is NP-complete and Y is in NP then Y is NP-complete.
   e. X and Y can't both be NP-complete.
   f. If X is in P, then Y is in P.
   g. If Y is in P, then X is in P.

2. Consider the problem COMPOSITE: given an integer *y*, does *y* have any factors other than one and itself? For this exercise, you may assume that COMPOSITE is in NP, and you will be comparing it to the well-known NP-complete problem SUBSET-SUM: given a set S of *n* integers and an integer target *t*, is there a subset of *S* whose sum is exactly *t*?
Clearly explain whether or not each of the following statements follows from that fact that COMPOSITE is in NP and SUBSET-SUM is NP-complete:

   a. SUBSET-SUM $\leq_p$ COMPOSITE.
   b. If there is an $O(n^3)$ algorithm for SUBSET-SUM, then there is a polynomial time algorithm for COMPOSITE.
   c. If there is a polynomial algorithm for COMPOSITE, then P = NP.
   d. If P $\neq$ NP, then ***no*** problem in NP can be solved in polynomial time.

3. Two well-known NP-complete problems are 3-SAT and TSP, the traveling salesman problem. The 2-SAT problem is a SAT variant in which each clause contains at most two literals. 2-SAT is known to have a polynomial-time algorithm. Is each of the following statements true or false? Justify your answer.

   a. 3-SAT $\leq_p$ TSP.
   b. If P $\neq$ NP, then 3-SAT $\leq_p$ 2-SAT.
   c. If P $\neq$ NP, then no NP-complete problem can be solved in polynomial time.

4. LONG-PATH is the problem of, given (G, u, v, k) where G is a graph, u and v vertices and k an integer, determining if there is a simple path in G from u to v of length at least k. Show that LONG-PATH is NP-complete.

5. Graph-Coloring. Mapmakers try to use as few colors as possible when coloring countries on a map, as long as no two countries that share a border have the same color. We can model this problem with an undirected graph G = (V,E) in which each vertex represents a country and vertices whose respective countries share a border are adjacent. A k-coloring is a function c: V -> {1, 2, … , k} such that c(u) $\neq$ c(v) for every edge (u,v) $\in$ E. In other words the number 1, 2, .., k represent the k colors and adjacent vertices must have different colors. The graph-coloring problem is to determine the minimum number of colors needed to color a given graph.

a. State the graph-coloring problem as a decision problem K-COLOR. Show that your decision problem is solvable in polynomial time if and only of the graph-coloring problem is solvable in polynomial time.

b. It has been proven that 3-COLOR is NP-complete by using a reduction from SAT. Use the fact that 3-COLOR is NP-complete to prove that 4-COLOR is NP-complete.