Cubical Abstract Machine

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1 Machine structure

$$
\begin{aligned}
M, A \quad ::= \quad & a \mid \mathfrak{x}[\vec{r}; \vec{M}] \mid \mathsf{bool} \mid \mathsf{wbool} \mid \mathbb{S}^1 \mid \langle M, N\rangle \mid \mathsf{fst}(M) \mid \mathsf{snd}(M) \mid \lambda x.M \mid M(N) \mid \langle x\rangle M \mid M \,@\, r \\
& \mathsf{coe}_{x.A}^{r \rightsquigarrow r'}(M) \mid \mathsf{hcom}_A^{r \rightsquigarrow r'}(M; \overline{\xi \hookrightarrow y.N}) \mid \mathsf{fcom}^{r \rightsquigarrow r'}(M; \overline{\xi \hookrightarrow y.N}) \\
& \mathsf{com}_{x.A}^{r \rightsquigarrow r'}(M; \overline{\xi \hookrightarrow y.N}) \mid \ldots \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(terms)} \\[4pt]
K \quad ::= \quad & \mathsf{coe}_{\square}^{r \rightsquigarrow r'}(M) \mid \square(N) \mid \mathsf{fst}(\square) \mid \mathsf{snd}(\square) \mid \square \,@\, r \mid \mathsf{hcom}_{\square}^{r \rightsquigarrow r'}(M; \overline{\xi \hookrightarrow y.N}) \\
& \mathsf{if}(\square; M; N) \mid \ldots \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \text{(continuations)} \\[2pt]
C \quad ::= \quad & M \triangleleft \mathcal{E} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(closures)} \\
\mathcal{E} \quad ::= \quad & (\mathfrak{v}, \gamma, \psi) \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(environments)} \\
\mathfrak{v} \quad ::= \quad & \overline{\mathfrak{x} \hookrightarrow \vec{x}; \vec{a}.M \triangleleft \mathcal{E}} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(meta env.)} \\
\gamma \quad ::= \quad & \overline{a \hookrightarrow C} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(object env.)} \\
\psi \quad ::= \quad & \overline{x \hookrightarrow r} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(dimension env.)} \\
f \quad ::= \quad & (K, \mathcal{E}, \Psi) \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(frames)} \\
\pi \quad ::= \quad & \cdot \mid f :: \pi \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \text{(stacks)} \\
\mathcal{C} \quad ::= \quad & \langle M \triangleleft \mathcal{E} \parallel \pi\rangle \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(states)}
\end{aligned}
$$

This machine is similar to both Krivine's environment machine and the CEK machine. To be honest, Jon Sterling is not really sure of what delineates these from each other, except that they usually seem to be call-by-name and call-by-value respectively. Ours is call-by-name.

Stack frames and machine states   A stack frame $(K, \mathcal{E}, \Psi)$ represents a continuation whose hole ranges over a binder of Ψ dimensions. For a stack π, let $\vec{\Psi}$ be the aggregation of all the dimension bindings Ψ mentioned in π; then the extension of a stack $(K, \mathcal{E}, \Phi) :: \pi$ is wellformed when $\mathcal{E}$ is an environment for $\mathbf{FD}(K) \setminus \vec{\Psi}$; moreover, K is allowed to mention the dimensions in $\vec{\Psi}$.

A machine state $\langle M \triangleleft \mathcal{E} \parallel \pi\rangle$, where $\vec{\Psi}$ is the aggregation of dimension bindings in π, is wellformed when $\mathcal{E}$ is an environment for $\mathbf{FD}(M) \setminus \vec{\Psi}$; moreover, M is allowed to mention the dimensions in $\vec{\Psi}$.

2 Selected transition rules

We define a transition judgment $\mathcal{C} \succ_\mu \mathcal{C}'$ with μ ranging over the following possible modes:

1. $\boxdot$, denoting a cubically stable transition

2. $\bullet$, denoting a non-cubically stable transition

Moreover, if $\mathcal{C} \succ_{\boxdot} \mathcal{C}'$ then $\mathcal{C} \succ_{\bullet} \mathcal{C}'$.

2.1 Variables

$$
\frac{\mathcal{E}(a) \equiv M \triangleleft \mathcal{E}'}{\langle a \triangleleft \mathcal{E} \parallel \pi\rangle \succ_\mu \langle M \triangleleft \mathcal{E}' \parallel \pi\rangle}
\qquad\qquad
\frac{\mathcal{E}(\mathfrak{x}) \equiv \vec{x}; \vec{a}.N \triangleleft \mathcal{E}' \qquad \mathcal{E}'' \triangleq [\mathcal{E}', \overline{a \hookrightarrow M \triangleleft \mathcal{E}}, \overline{x \hookrightarrow \mathcal{E}(r)}]}{\langle \mathfrak{x}[\vec{r}; \vec{M}] \triangleleft \mathcal{E} \parallel \pi\rangle \succ_\mu \langle N \triangleleft \mathcal{E}'' \parallel \pi\rangle}
$$

2.2 Kan operations

$$\langle \mathsf{hcom}_A^{r\rightsquigarrow r'}(M; \overline{\xi \hookrightarrow y.N}) \triangleleft \mathcal{E} \parallel \pi \rangle \succ_\mu \langle A \triangleleft \mathcal{E} \parallel (\mathsf{hcom}_\square^{r\rightsquigarrow r'}(M; \overline{\xi \hookrightarrow y.N}), \mathcal{E}, []) :: \pi \rangle$$

$$\langle \mathsf{coe}_{x.A}^{r\rightsquigarrow r'}(M) \triangleleft \mathcal{E} \parallel \pi \rangle \succ_\mu \langle A \triangleleft \mathcal{E} \parallel (\mathsf{coe}_\square^{r\rightsquigarrow r'}(M), \mathcal{E}', [x]) :: \pi \rangle$$

$$\langle \mathsf{com}_{x.A}^{r\rightsquigarrow r'}(M; \overline{\xi_i \hookrightarrow y.N_i}) \triangleleft \mathcal{E} \parallel \pi \rangle \succ_\mu \langle \mathsf{hcom}_A^{r\rightsquigarrow r'}(\mathsf{coe}_{y.A}^{r\rightsquigarrow r'}(M); \overline{\xi_i \hookrightarrow y.\mathsf{coe}_{y.A}^{y\rightsquigarrow r'}(N_i)}) \triangleleft [\mathcal{E}, x \hookrightarrow r'] \parallel \pi \rangle$$

$$\frac{\mathcal{E} \vDash r = r'}{\langle \mathsf{fcom}^{r\rightsquigarrow r'}(M; \overline{\xi \hookrightarrow y.N}) \triangleleft \mathcal{E} \parallel \pi \rangle \succ_\mu \langle M \triangleleft \mathcal{E} \parallel \pi \rangle}$$

$$\frac{\mathcal{E} \vDash r \# r' \qquad \mathcal{E} \vDash r_i \# r'_i\ (\forall i < j) \qquad \mathcal{E} \vDash r_j = r'_j}{\langle \mathsf{fcom}^{r\rightsquigarrow r'}(M; \overline{r_i = r'_i \hookrightarrow y.N_i}) \triangleleft \mathcal{E} \parallel \pi \rangle \succ_\bullet \langle N_j \triangleleft [\mathcal{E}, y \hookrightarrow \mathcal{E}(r')] \parallel \pi \rangle}$$

2.3 Dependent function types

$$\langle M(N) \triangleleft \mathcal{E} \parallel \pi \rangle \succ_\mu \langle M \triangleleft \mathcal{E} \parallel (\square(N), \mathcal{E}, []) :: \pi \rangle$$

$$\langle \lambda a.M \triangleleft \mathcal{E} \parallel (\square(N), \mathcal{E}', []) :: \pi \rangle \succ_\mu \langle M \triangleleft [\mathcal{E}, a \hookrightarrow N \triangleleft \mathcal{E}'] \parallel \pi \rangle$$

$$\frac{\mathcal{E}'' \triangleq [\mathcal{E}', \varkappa \hookrightarrow a.B \triangleleft \mathcal{E}]}{\langle (x : A) \rightarrow B \triangleleft \mathcal{E} \parallel (\mathsf{hcom}_\square^{r\rightsquigarrow r'}(M; \overline{\xi_i \hookrightarrow y.N_i}), \mathcal{E}', []) :: \pi \rangle \succ_\mu \langle \lambda a.\mathsf{hcom}_{\varkappa[a]}^{r\rightsquigarrow r'}(M(a); \overline{\xi_i \hookrightarrow y.N_i(a)}) \triangleleft \mathcal{E}'' \parallel \pi \rangle}$$

$$\frac{\mathcal{E}'' \triangleq [\mathcal{E}', \varkappa \hookrightarrow x; b.B \triangleleft [\mathcal{E}, a \hookrightarrow \mathsf{coe}_{x.\varkappa[x]}^{r'\rightsquigarrow x}(b) \triangleleft [\mathcal{E}', \mathsf{z} \hookrightarrow x.A \triangleleft \mathcal{E}]], y \hookrightarrow x.A \triangleleft \mathcal{E}]}{\langle (a : A) \rightarrow B \triangleleft \mathcal{E} \parallel (\mathsf{coe}_\square^{r\rightsquigarrow r'}(M), \mathcal{E}', [x]) :: \pi \rangle \succ_\mu \langle \lambda a.\mathsf{coe}_{x.\varkappa[x;a]}^{r\rightsquigarrow r'}(M(\mathsf{coe}_{x.\mathsf{y}[x]}^{r'\rightsquigarrow r}(a))) \triangleleft \mathcal{E}'' \parallel \pi \rangle}$$

2.4 Dependent pair types

$$\langle \mathsf{fst}(M) \triangleleft \mathcal{E} \parallel \pi \rangle \succ_\mu \langle M \triangleleft \mathcal{E} \parallel (\mathsf{fst}(\square), \mathcal{E}, []) :: \pi \rangle \qquad \langle \mathsf{snd}(M) \triangleleft \mathcal{E} \parallel \pi \rangle \succ_\mu \langle M \triangleleft \mathcal{E} \parallel (\mathsf{snd}(\square), \mathcal{E}, []) :: \pi \rangle$$

$$\langle \langle M, N \rangle \triangleleft \mathcal{E} \parallel (\mathsf{fst}(\square), \mathcal{E}', []) :: \pi \rangle \succ_\mu \langle M \triangleleft \mathcal{E} \parallel \pi \rangle \qquad \langle \langle M, N \rangle \triangleleft \mathcal{E} \parallel (\mathsf{snd}(\square), \mathcal{E}', []) :: \pi \rangle \succ_\mu \langle N \triangleleft \mathcal{E} \parallel \pi \rangle$$

$$\frac{L_s \triangleq \mathsf{hcom}_\varkappa^{r\rightsquigarrow s}(\mathsf{fst}(M); \overline{\xi_i \hookrightarrow y.\mathsf{fst}(N_i)}) \qquad R \triangleq \mathsf{com}_{z.\mathsf{y}[z]}^{r\rightsquigarrow r'}(\mathsf{snd}(M); \overline{\xi_i \hookrightarrow y.\mathsf{snd}(N_i)})}{\mathcal{E}'' \triangleq [\mathcal{E}', \varkappa \hookrightarrow .A \triangleleft \mathcal{E}] \qquad \mathcal{E}''' \triangleq [\mathcal{E}'', \mathsf{y} \hookrightarrow z.B \triangleleft [\mathcal{E}, a \hookrightarrow L_z \triangleleft \mathcal{E}'']]}$$
$$\langle (a : A) \times B \triangleleft \mathcal{E} \parallel (\mathsf{hcom}_\square^{r\rightsquigarrow r'}(M; \overline{\xi_i \hookrightarrow y.N_i}), \mathcal{E}', []) :: \pi \rangle \succ_\mu \langle \langle L_{r'}, R \rangle \triangleleft \mathcal{E}''' \parallel \pi \rangle$$

$$\frac{\mathcal{E}'' \triangleq [\mathcal{E}', \varkappa \hookrightarrow x.A \triangleleft \mathcal{E}, \mathsf{y} \hookrightarrow x.B \triangleleft [\mathcal{E}, a \hookrightarrow \mathsf{coe}_{y.\varkappa[y]}^{r\rightsquigarrow x}(\mathsf{snd}(M)) \triangleleft [\mathcal{E}, \varkappa \hookrightarrow x.A \triangleleft \mathcal{E}]]]}{\langle (a : A) \times B \triangleleft \mathcal{E} \parallel (\mathsf{coe}_\square^{r\rightsquigarrow r'}(M), \mathcal{E}', [x]) :: \pi \rangle \succ_\mu \langle \langle \mathsf{coe}_{x.\varkappa[x]}^{r\rightsquigarrow r'}(\mathsf{fst}(M)), \mathsf{coe}_{x.\mathsf{y}[x]}^{r\rightsquigarrow r'}(\mathsf{snd}(M)) \rangle \triangleleft \mathcal{E}'' \parallel \pi \rangle}$$

2.5   Path types

$$\overline{\langle M @ r \triangleleft \mathcal{E} \,\|\, \pi \rangle \succ_\mu \langle M \triangleleft \mathcal{E} \,\|\, (\square @ r, \mathcal{E}, []) :: \pi \rangle} \qquad \overline{\langle \langle x \rangle M \triangleleft \mathcal{E} \,\|\, (\square @ r, \mathcal{E}', []) :: \pi \rangle \succ_\mu \langle M \triangleleft [\mathcal{E}, x \hookrightarrow \mathcal{E}'(r)] \,\|\, \pi \rangle}$$

$$\frac{H \triangleq \mathsf{hcom}_{\varkappa[x]}^{r \rightsquigarrow r'}(M @ x; \overline{x = \epsilon \hookrightarrow y.\mathsf{y}_\epsilon}, \overline{\xi_i \hookrightarrow y.N_i @ x}) \qquad \mathcal{E}'' \triangleq [\mathcal{E}', \varkappa \hookrightarrow x.A \triangleleft \mathcal{E}, \overline{\mathsf{y}_\epsilon \hookrightarrow .P_\epsilon \triangleleft \mathcal{E}}]}{\langle \mathsf{path}_{x.A}(P_0; P_1) \triangleleft \mathcal{E} \,\|\, (\mathsf{hcom}_\square^{r \rightsquigarrow r'}(M; \overline{\xi_i \hookrightarrow y.N_i}), \mathcal{E}', []) :: \pi \rangle \succ_\mu \langle \langle x \rangle H \triangleleft \mathcal{E}'' \,\|\, \pi \rangle}$$

$$\frac{\mathcal{E}'' \triangleq [\mathcal{E}', \varkappa \hookrightarrow x.A \triangleleft \mathcal{E}, \overline{\mathsf{y}_\epsilon \hookrightarrow .P_\epsilon \triangleleft \mathcal{E}}]}{\langle \mathsf{path}_{x.A}(P_0; P_1) \triangleleft \mathcal{E} \,\|\, (\mathsf{coe}_\square^{r \rightsquigarrow r'}(M), \mathcal{E}', [x]) :: \pi \rangle \succ_\mu \langle \langle x \rangle \mathsf{com}_{y.\varkappa[y]}^{r \rightsquigarrow r'}(M @ x; \overline{x = \epsilon \hookrightarrow y.\mathsf{y}_\epsilon}) \triangleleft \mathcal{E}'' \,\|\, \pi \rangle}$$

2.6   Natural numbers

TODO

2.7   Booleans

$$\overline{\langle \mathsf{if}(M; T; F) \triangleleft \mathcal{E} \,\|\, \pi \rangle \succ_\mu \langle M \triangleleft \mathcal{E} \,\|\, (\mathsf{if}(\square; T; F), \mathcal{E}, []) :: \pi \rangle} \qquad \overline{\langle \mathsf{true} \triangleleft \mathcal{E} \,\|\, (\mathsf{if}(\square; T; F), \mathcal{E}', []) :: \pi \rangle \succ_\mu \langle T \triangleleft \mathcal{E}' \,\|\, \pi \rangle}$$

$$\overline{\langle \mathsf{false} \triangleleft \mathcal{E} \,\|\, (\mathsf{if}(\square; T; F), \mathcal{E}', []) :: \pi \rangle \succ_\mu \langle F \triangleleft \mathcal{E}' \,\|\, \pi \rangle} \qquad \overline{\langle \mathsf{bool} \triangleleft \mathcal{E} \,\|\, (\mathsf{coe}_\square^{r \rightsquigarrow r'}(M), \mathcal{E}', [x]) :: \pi \rangle \succ_\mu \langle M \triangleleft \mathcal{E}' \,\|\, \pi \rangle}$$

$$\overline{\langle \mathsf{bool} \triangleleft \mathcal{E} \,\|\, (\mathsf{hcom}_\square^{r \rightsquigarrow r'}(M; \overline{\xi \hookrightarrow y.N}), \mathcal{E}', []) :: \pi \rangle \succ_\mu \langle M \triangleleft \mathcal{E}' \,\|\, \pi \rangle}$$

2.8   Weak booleans

TODO, add hcom, if, if-fcom, etc.

$$\overline{\langle \mathsf{wbool} \triangleleft \mathcal{E} \,\|\, (\mathsf{hcom}_\square^{r \rightsquigarrow r'}(M; \overline{\xi \hookrightarrow y.N}), \mathcal{E}', []) :: \pi \rangle \succ_\mu \langle \mathsf{fcom}^{r \rightsquigarrow r'}(M; \overline{\xi \hookrightarrow y.N}) \triangleleft \mathcal{E}' \,\|\, \pi \rangle}$$

2.9   Circle

TODO, add hcom, coe, rec-fcom, etc.

$$\frac{\mathcal{E} \vDash r = \epsilon}{\langle \mathsf{loop}_r \triangleleft \mathcal{E} \,\|\, \pi \rangle \succ_\mu \langle \mathsf{base} \triangleleft \mathcal{E} \,\|\, \pi \rangle} \qquad \overline{\langle \mathbb{S}^1 \mathsf{rec}_{a.A}(M; P; x.L) \triangleleft \mathcal{E} \,\|\, \pi \rangle \succ_\mu \langle M \triangleleft \mathcal{E} \,\|\, (\mathbb{S}^1 \mathsf{rec}_{a.A}(\square; P; x.L), \mathcal{E}, []) :: \pi \rangle}$$

$$\overline{\langle \mathsf{base} \triangleleft \mathcal{E} \,\|\, (\mathbb{S}^1 \mathsf{rec}_{a.A}(\square; P; x.L), \mathcal{E}', []) :: \pi \rangle \succ_\mu \langle P \triangleleft \mathcal{E}' \,\|\, \pi \rangle}$$

$$\frac{\mathcal{E} \vDash r = w}{\langle \mathsf{loop}_r \triangleleft \mathcal{E} \,\|\, (\mathbb{S}^1 \mathsf{rec}_{a.A}(\square; P; x.L), \mathcal{E}', []) :: \pi \rangle \succ_\bullet \langle L \triangleleft [\mathcal{E}', x \hookrightarrow w] \,\|\, \pi \rangle}$$

Unloading the machine   We can unload the machine at any time; this is useful if we are computing an open term and hit a variable.

$$\frac{}{\langle M \triangleleft \mathcal{E} \parallel \cdot \rangle \Longrightarrow M[\mathcal{E}]}$$

$$\frac{\langle K[\Psi.M[\mathcal{E}]] \triangleleft \mathcal{E}' \parallel \pi \rangle \Longrightarrow N}{\langle M \triangleleft \mathcal{E} \parallel (K[\square], \mathcal{E}', \Psi) :: \pi \rangle \Longrightarrow N}$$

But using Favonia's Thought, we can come up with something more efficient. This naïve version is rather appalling.