Cubical Abstract Machine

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1 Machine structure

$$
\begin{array}{rcll}
M, A & ::= & a \mid \varkappa[\vec{r}; \vec{M}] \mid \mathsf{bool} \mid \mathsf{wbool} \mid \mathbb{S}^1 \mid \langle M, N \rangle \mid \mathsf{fst}(M) \mid \mathsf{snd}(M) \mid \lambda x.M \mid M(N) \mid \langle x \rangle M \mid M \mathbin{@} r & \\
 & & \mathsf{coe}_{x.A}^{r \rightsquigarrow r'}(M) \mid \mathsf{hcom}_{A}^{r \rightsquigarrow r'}(M; \overline{\xi \hookrightarrow y.N}) \mid \mathsf{fcom}^{r \rightsquigarrow r'}(M; \overline{\xi \hookrightarrow y.N}) & \\
 & & \mathsf{com}_{x.A}^{r \rightsquigarrow r'}(M; \overline{\xi \hookrightarrow y.N}) \mid \ldots & \text{(terms)} \\
K & ::= & \mathsf{coe}_{\Box}^{r \rightsquigarrow r'}(M) \mid \Box(N) \mid \mathsf{fst}(\Box) \mid \mathsf{snd}(\Box) \mid \Box \mathbin{@} r \mid \mathsf{hcom}_{\Box}^{r \rightsquigarrow r'}(M; \overline{\xi \hookrightarrow y.N}) & \\
 & & \mathsf{if}(\Box; M; N) \mid \ldots & \text{(continuations)} \\
C & ::= & M \triangleleft \mathcal{E} & \text{(closures)} \\
\mathcal{E} & ::= & (\mho, \gamma, \psi) & \text{(environments)} \\
\mho & ::= & \overline{\varkappa \hookrightarrow \vec{x}; \vec{a}.M \triangleleft \mathcal{E}} & \text{(meta env.)} \\
\gamma & ::= & \overline{a \hookrightarrow C} & \text{(object env.)} \\
\psi & ::= & \overline{x \hookrightarrow r} & \text{(dimension env.)} \\
f & ::= & (K, \mathcal{E}, \Psi) & \text{(frames)} \\
\pi & ::= & \cdot \mid f :: \pi & \text{(stacks)} \\
\mathcal{C} & ::= & \langle M \triangleleft \mathcal{E} \parallel \pi \rangle & \text{(states)}
\end{array}
$$

This machine is similar to both Krivine's environment machine and the CEK machine. To be honest, Jon Sterling is not really sure of what delineates these from each other, except that they usually seem to be call-by-name and call-by-value respectively. Ours is call-by-name.

Stack frames and machine states A stack frame $(K, \mathcal{E}, \Psi)$ represents a continuation whose hole ranges over a binder of Ψ dimensions. For a stack π, let $\vec{\Psi}$ be the aggregation of all the dimension bindings Ψ mentioned in π; then the extension of a stack $(K, \mathcal{E}, \Phi) :: \pi$ is wellformed when $\mathcal{E}$ is an environment for $\mathbf{FD}(K) \setminus \vec{\Psi}$; moreover, K is allowed to mention the dimensions in $\vec{\Psi}$.

A machine state $\langle M \triangleleft \mathcal{E} \parallel \pi \rangle$, where $\vec{\Psi}$ is the aggregation of dimension bindings in π, is wellformed when $\mathcal{E}$ is an environment for $\mathbf{FD}(M) \setminus \vec{\Psi}$; moreover, M is allowed to mention the dimensions in $\vec{\Psi}$.

2 Selected transition rules

We define a transition judgment $\mathcal{C} \succ_{\mu} \mathcal{C}'$ with μ ranging over the following possible modes:

1. $\cube$, denoting a cubically stable transition
2. $\bullet$, denoting a non-cubically stable transition

Moreover, if $\mathcal{C} \succ_{\cube} \mathcal{C}'$ then $\mathcal{C} \succ_{\bullet} \mathcal{C}'$.

2.1 Variables

$$
\frac{\mathcal{E}(a) \equiv M \triangleleft \mathcal{E}'}{\langle a \triangleleft \mathcal{E} \parallel \pi \rangle \succ_{\cube} \langle M \triangleleft \mathcal{E}' \parallel \pi \rangle}
\qquad
\frac{\mathcal{E}(\varkappa) \equiv \vec{x}; \vec{a}.N \triangleleft \mathcal{E}' \qquad \mathcal{E}'' \triangleq [\mathcal{E}', \overline{a \hookrightarrow M \triangleleft \mathcal{E}}, \overline{x \hookrightarrow \mathcal{E}(r)}]}{\langle \varkappa[\vec{r}; \vec{M}] \triangleleft \mathcal{E} \parallel \pi \rangle \succ_{\cube} \langle N \triangleleft \mathcal{E}'' \parallel \pi \rangle}
$$

2.2 Kan operations

$$\frac{}{\langle \mathsf{hcom}_{A}^{r \rightsquigarrow r'}(M; \overline{\xi \hookrightarrow y.N}) \triangleleft \mathcal{E} \parallel \pi \rangle \succ_{\boxdot} \big\langle A \triangleleft \mathcal{E} \,\big\|\, (\mathsf{hcom}_{\square}^{r \rightsquigarrow r'}(M; \overline{\xi \hookrightarrow y.N}), \mathcal{E}, []) :: \pi \big\rangle}$$

$$\frac{}{\langle \mathsf{coe}_{x.A}^{r \rightsquigarrow r'}(M) \triangleleft \mathcal{E} \parallel \pi \rangle \succ_{\boxdot} \big\langle A \triangleleft \mathcal{E} \,\big\|\, (\mathsf{coe}_{\square}^{r \rightsquigarrow r'}(M), \mathcal{E}', [x]) :: \pi \big\rangle}$$

$$\frac{}{\langle \mathsf{com}_{x.A}^{r \rightsquigarrow r'}(M; \overline{\xi_i \hookrightarrow y.N_i}) \triangleleft \mathcal{E} \parallel \pi \rangle \succ_{\boxdot} \langle \mathsf{hcom}_{A}^{r \rightsquigarrow r'}(\mathsf{coe}_{y.A}^{r \rightsquigarrow r'}(M); \overline{\xi_i \hookrightarrow y.\mathsf{coe}_{y.A}^{y \rightsquigarrow r'}(N_i)}) \triangleleft [\mathcal{E}, x \hookrightarrow r'] \parallel \pi \rangle}$$

$$\frac{\mathcal{E} \vDash r = r'}{\langle \mathsf{fcom}^{r \rightsquigarrow r'}(M; \overline{\xi \hookrightarrow y.N}) \triangleleft \mathcal{E} \parallel \pi \rangle \succ_{\boxdot} \langle M \triangleleft \mathcal{E} \parallel \pi \rangle}$$

$$\frac{\mathcal{E} \vDash r \mathrel{\#} r' \qquad \mathcal{E} \vDash r_i \mathrel{\#} r'_i \ (\forall i < j) \qquad \mathcal{E} \vDash r_j = r'_j}{\langle \mathsf{fcom}^{r \rightsquigarrow r'}(M; \overline{r_i = r'_i \hookrightarrow y.N_i}) \triangleleft \mathcal{E} \parallel \pi \rangle \succ_{\bullet} \langle N_j \triangleleft [\mathcal{E}, y \hookrightarrow \mathcal{E}(r')] \parallel \pi \rangle}$$

2.3 Dependent function types

$$\frac{}{\big\langle M(N) \triangleleft \mathcal{E} \,\big\|\, \pi \big\rangle \succ_{\boxdot} \big\langle M \triangleleft \mathcal{E} \,\big\|\, (\square(N), \mathcal{E}, []) :: \pi \big\rangle}$$

$$\frac{}{\big\langle \lambda a.M \triangleleft \mathcal{E} \,\big\|\, (\square(N), \mathcal{E}', []) :: \pi \big\rangle \succ_{\boxdot} \langle M \triangleleft [\mathcal{E}, a \hookrightarrow N \triangleleft \mathcal{E}'] \parallel \pi \rangle}$$

$$\frac{\mathcal{E}'' \triangleq [\mathcal{E}', \mathbb{x} \hookrightarrow a.B \triangleleft \mathcal{E}]}{\big\langle (x : A) \to B \triangleleft \mathcal{E} \,\big\|\, (\mathsf{hcom}_{\square}^{r \rightsquigarrow r'}(M; \overline{\xi_i \hookrightarrow y.N_i}), \mathcal{E}', []) :: \pi \big\rangle \succ_{\boxdot} \big\langle \lambda a.\mathsf{hcom}_{\mathbb{x}[a]}^{r \rightsquigarrow r'}(M(a); \overline{\xi_i \hookrightarrow y.N_i(a)}) \triangleleft \mathcal{E}'' \,\big\|\, \pi \big\rangle}$$

$$\frac{\mathcal{E}'' \triangleq [\mathcal{E}', \mathbb{x} \hookrightarrow x; b.B \triangleleft [\mathcal{E}, a \hookrightarrow \mathsf{coe}_{x.\mathbb{z}[x]}^{r' \rightsquigarrow x}(b) \triangleleft [\mathcal{E}', \mathbb{z} \hookrightarrow x.A \triangleleft \mathcal{E}]], \mathbb{y} \hookrightarrow x.A \triangleleft \mathcal{E}]}{\big\langle (a : A) \to B \triangleleft \mathcal{E} \,\big\|\, (\mathsf{coe}_{\square}^{r \rightsquigarrow r'}(M), \mathcal{E}', [x]) :: \pi \big\rangle \succ_{\boxdot} \big\langle \lambda a.\mathsf{coe}_{x.\mathbb{x}[x;a]}^{r \rightsquigarrow r'}(M(\mathsf{coe}_{x.\mathbb{y}[x]}^{r' \rightsquigarrow r}(a))) \triangleleft \mathcal{E}'' \,\big\|\, \pi \big\rangle}$$

2.4 Dependent pair types

$$\frac{}{\langle \mathsf{fst}(M) \triangleleft \mathcal{E} \parallel \pi \rangle \succ_{\boxdot} \big\langle M \triangleleft \mathcal{E} \,\big\|\, (\mathsf{fst}(\square), \mathcal{E}, []) :: \pi \big\rangle}$$

$$\frac{}{\langle \mathsf{snd}(M) \triangleleft \mathcal{E} \parallel \pi \rangle \succ_{\boxdot} \big\langle M \triangleleft \mathcal{E} \,\big\|\, (\mathsf{snd}(\square), \mathcal{E}, []) :: \pi \big\rangle}$$

$$\frac{}{\big\langle \langle M, N \rangle \triangleleft \mathcal{E} \,\big\|\, (\mathsf{fst}(\square), \mathcal{E}', []) :: \pi \big\rangle \succ_{\boxdot} \langle M \triangleleft \mathcal{E} \parallel \pi \rangle}$$

$$\frac{}{\big\langle \langle M, N \rangle \triangleleft \mathcal{E} \,\big\|\, (\mathsf{snd}(\square), \mathcal{E}', []) :: \pi \big\rangle \succ_{\boxdot} \langle N \triangleleft \mathcal{E} \parallel \pi \rangle}$$

$$\frac{\begin{array}{c} L_s \triangleq \mathsf{hcom}_{\mathbb{x}}^{r \rightsquigarrow s}(\mathsf{fst}(M); \overline{\xi_i \hookrightarrow y.\mathsf{fst}(N_i)}) \qquad R \triangleq \mathsf{com}_{z.\mathbb{y}[z]}^{r \rightsquigarrow r'}(\mathsf{snd}(M); \overline{\xi_i \hookrightarrow y.\mathsf{snd}(N_i)}) \\ \mathcal{E}'' \triangleq [\mathcal{E}', \mathbb{x} \hookrightarrow .A \triangleleft \mathcal{E}] \qquad \mathcal{E}''' \triangleq [\mathcal{E}'', \mathbb{y} \hookrightarrow z.B \triangleleft [\mathcal{E}, a \hookrightarrow L_z \triangleleft \mathcal{E}'']] \end{array}}{\big\langle (a : A) \times B \triangleleft \mathcal{E} \,\big\|\, (\mathsf{hcom}_{\square}^{r \rightsquigarrow r'}(M; \overline{\xi_i \hookrightarrow y.N_i}), \mathcal{E}', []) :: \pi \big\rangle \succ_{\boxdot} \big\langle \langle L_{r'}, R \rangle \triangleleft \mathcal{E}''' \,\big\|\, \pi \big\rangle}$$

$$\frac{\mathcal{E}'' \triangleq [\mathcal{E}', \mathbb{x} \hookrightarrow x.A \triangleleft \mathcal{E}, \mathbb{y} \hookrightarrow x.B \triangleleft [\mathcal{E}, a \hookrightarrow \mathsf{coe}_{y.\mathbb{x}[y]}^{r \rightsquigarrow x}(\mathsf{snd}(M)) \triangleleft [\mathcal{E}, \mathbb{x} \hookrightarrow x.A \triangleleft \mathcal{E}]]]}{\big\langle (a : A) \times B \triangleleft \mathcal{E} \,\big\|\, (\mathsf{coe}_{\square}^{r \rightsquigarrow r'}(M), \mathcal{E}', [x]) :: \pi \big\rangle \succ_{\boxdot} \big\langle \langle \mathsf{coe}_{x.\mathbb{x}[x]}^{r \rightsquigarrow r'}(\mathsf{fst}(M)), \mathsf{coe}_{x.\mathbb{y}[x]}^{r \rightsquigarrow r'}(\mathsf{snd}(M)) \rangle \triangleleft \mathcal{E}'' \,\big\|\, \pi \big\rangle}$$

2.5 Path types

$$\frac{}{\langle M \mathbin{@} r \triangleleft \mathcal{E} \parallel \pi \rangle \succ_{\square} \langle M \triangleleft \mathcal{E} \parallel (\square \mathbin{@} r, \mathcal{E}, []) :: \pi \rangle}$$

$$\frac{}{\langle \langle x \rangle M \triangleleft \mathcal{E} \parallel (\square \mathbin{@} r, \mathcal{E}', []) :: \pi \rangle \succ_{\square} \langle M \triangleleft [\mathcal{E}, x \hookrightarrow \mathcal{E}'(r)] \parallel \pi \rangle}$$

$$\frac{H \triangleq \mathsf{hcom}^{r \rightsquigarrow r'}_{\mathbb{x}[x]}(M \mathbin{@} x; \overline{x = \epsilon \hookrightarrow y.\mathbb{y}_\epsilon}, \overline{\xi_i \hookrightarrow y.N_i \mathbin{@} x}) \qquad \mathcal{E}'' \triangleq [\mathcal{E}', \mathbb{x} \hookrightarrow x.A \triangleleft \mathcal{E}, \overline{\mathbb{y}_\epsilon \hookrightarrow .P_\epsilon \triangleleft \mathcal{E}}]}{\langle \mathsf{path}_{x.A}(P_0; P_1) \triangleleft \mathcal{E} \parallel (\mathsf{hcom}^{r \rightsquigarrow r'}_{\square}(M; \overline{\xi_i \hookrightarrow y.N_i}), \mathcal{E}', []) :: \pi \rangle \succ_{\square} \langle \langle x \rangle H \triangleleft \mathcal{E}'' \parallel \pi \rangle}$$

$$\frac{\mathcal{E}'' \triangleq [\mathcal{E}', \mathbb{x} \hookrightarrow x.A \triangleleft \mathcal{E}, \overline{\mathbb{y}_\epsilon \hookrightarrow .P_\epsilon \triangleleft \mathcal{E}}]}{\langle \mathsf{path}_{x.A}(P_0; P_1) \triangleleft \mathcal{E} \parallel (\mathsf{coe}^{r \rightsquigarrow r'}_{\square}(M), \mathcal{E}', [x]) :: \pi \rangle \succ_{\square} \langle \langle x \rangle \mathsf{com}^{r \rightsquigarrow r'}_{y.\mathbb{x}[y]}(M \mathbin{@} x; \overline{x = \epsilon \hookrightarrow y.\mathbb{y}_\epsilon}) \triangleleft \mathcal{E}'' \parallel \pi \rangle}$$

2.6 Natural numbers

TODO

2.7 Booleans

$$\frac{}{\langle \mathsf{if}(M; T; F) \triangleleft \mathcal{E} \parallel \pi \rangle \succ_{\square} \langle M \triangleleft \mathcal{E} \parallel (\mathsf{if}(\square; T; F), \mathcal{E}, []) :: \pi \rangle}$$

$$\frac{}{\langle \mathsf{true} \triangleleft \mathcal{E} \parallel (\mathsf{if}(\square; T; F), \mathcal{E}', []) :: \pi \rangle \succ_{\square} \langle T \triangleleft \mathcal{E}' \parallel \pi \rangle} \qquad \frac{}{\langle \mathsf{false} \triangleleft \mathcal{E} \parallel (\mathsf{if}(\square; T; F), \mathcal{E}', []) :: \pi \rangle \succ_{\square} \langle F \triangleleft \mathcal{E}' \parallel \pi \rangle}$$

$$\frac{}{\langle \mathsf{bool} \triangleleft \mathcal{E} \parallel (\mathsf{coe}^{r \rightsquigarrow r'}_{\square}(M), \mathcal{E}', [x]) :: \pi \rangle \succ_{\square} \langle M \triangleleft \mathcal{E}' \parallel \pi \rangle}$$

$$\frac{}{\langle \mathsf{bool} \triangleleft \mathcal{E} \parallel (\mathsf{hcom}^{r \rightsquigarrow r'}_{\square}(M; \overline{\xi \hookrightarrow y.N}), \mathcal{E}', []) :: \pi \rangle \succ_{\square} \langle M \triangleleft \mathcal{E}' \parallel \pi \rangle}$$

2.8 Weak booleans

TODO, add hcom, if, if-fcom, etc.

$$\frac{}{\langle \mathsf{wbool} \triangleleft \mathcal{E} \parallel (\mathsf{hcom}^{r \rightsquigarrow r'}_{\square}(M; \overline{\xi \hookrightarrow y.N}), \mathcal{E}', []) :: \pi \rangle \succ_{\square} \langle \mathsf{fcom}^{r \rightsquigarrow r'}(M; \overline{\xi \hookrightarrow y.N}) \triangleleft \mathcal{E}' \parallel \pi \rangle}$$

2.9 Circle

TODO, add hcom, coe, rec-fcom, etc.

$$\frac{\mathcal{E} \vDash r = \epsilon}{\langle \mathsf{loop}_r \triangleleft \mathcal{E} \parallel \pi \rangle \succ_{\square} \langle \mathsf{base} \triangleleft \mathcal{E} \parallel \pi \rangle}$$

$$\frac{}{\langle \mathbb{S}^1\mathsf{rec}_{a.A}(M; P; x.L) \triangleleft \mathcal{E} \parallel \pi \rangle \succ_{\square} \langle M \triangleleft \mathcal{E} \parallel (\mathbb{S}^1\mathsf{rec}_{a.A}(\square; P; x.L), \mathcal{E}, []) :: \pi \rangle}$$

$$\frac{}{\langle \mathsf{base} \triangleleft \mathcal{E} \parallel (\mathbb{S}^1\mathsf{rec}_{a.A}(\square; P; x.L), \mathcal{E}', []) :: \pi \rangle \succ_{\square} \langle P \triangleleft \mathcal{E}' \parallel \pi \rangle}$$

$$\frac{\mathcal{E} \vDash r = w}{\langle \mathsf{loop}_r \triangleleft \mathcal{E} \parallel (\mathbb{S}^1\mathsf{rec}_{a.A}(\square; P; x.L), \mathcal{E}', []) :: \pi \rangle \succ_{\bullet} \langle L \triangleleft [\mathcal{E}', x \hookrightarrow w] \parallel \pi \rangle}$$

Unloading the machine We can unload the machine at any time; this is useful if we are computing an open term and hit a variable.

$$\frac{}{\langle M \triangleleft \mathcal{E} \parallel \cdot \rangle \Longrightarrow M[\mathcal{E}]} \qquad \frac{\langle K[\Psi.M[\mathcal{E}]] \triangleleft \mathcal{E}' \parallel \pi \rangle \Longrightarrow N}{\big\langle M \triangleleft \mathcal{E} \,\big\|\, (K[\Box], \mathcal{E}', \Psi) :: \pi \big\rangle \Longrightarrow N}$$

But using Favonia's Thought, we can come up with something more efficient. This naïve version is rather appalling.