Cubical Abstract Machine

June 29, 2017

$$
\begin{array}{rcll}
M, A & ::= & a \mid \mathsf{coe}_{x.A}^{r \rightsquigarrow r'}(M) \mid \ldots & \text{(terms)} \\
K & ::= & \mathsf{coe}_{\square}^{r \rightsquigarrow r'}(M) \mid \ldots & \text{(continuations)} \\
C & ::= & M \triangleleft \mathcal{E} & \text{(closures)} \\
\mathcal{E} & ::= & (\gamma, \psi) & \text{(environments)} \\
\gamma & ::= & \overline{a \hookrightarrow C} & \text{(object env.)} \\
\psi & ::= & \overline{x \hookrightarrow r} & \text{(dimension env.)} \\
f & ::= & (K, \mathcal{E}, \Psi) & \text{(frames)} \\
\pi & ::= & \cdot \mid f :: \pi & \text{(stacks)} \\
\mathcal{C} & ::= & \langle M \triangleleft \mathcal{E} \parallel \pi \rangle & \text{(states)}
\end{array}
$$

We will write $M[\mathcal{E}]$ for substituting the free dimensions and variables of a term M using an environment $\mathcal{E}$.

Stack frames and machine states A stack frame $(K, \mathcal{E}, \Psi)$ represents a continuation whose whole binds Ψ dimensions. For a stack π, let $\vec{\Psi}$ be the aggregation of all the dimension bindings Ψ mentioned in π; then the extension of a stack $(K, \mathcal{E}, \Phi) :: \pi$ is wellformed when $\mathcal{E}$ is an environment for $\mathbf{FD}(K) \setminus \vec{\Psi}$; moreover, K is allowed to mention the dimensions in $\vec{\Psi}$.

A machine state $\langle M \triangleleft \mathcal{E} \parallel \pi \rangle$, where $\vec{\Psi}$ is the aggregation of dimension bindings in π, is wellformed when $\mathcal{E}$ is an environment for $\mathbf{FD}(M) \setminus \vec{\Psi}$; moreover, M is allowed to mention the dimensions in $\vec{\Psi}$.

Selected transition rules

$$\langle a \triangleleft (\gamma, \psi) \parallel \pi \rangle \succ \langle \gamma(a) \triangleleft (\gamma, \psi) \parallel \pi \rangle \tag{1}$$

$$\big\langle M(N) \triangleleft \mathcal{E} \parallel \pi \big\rangle \succ \big\langle M \triangleleft \mathcal{E} \,\big\|\, (\square(N), \mathcal{E}, []) :: \pi \big\rangle \tag{2}$$

$$\big\langle \lambda a.M \triangleleft \mathcal{E} \,\big\|\, (\square(N), \mathcal{E}', []) :: \pi \big\rangle \succ \langle M \triangleleft [\mathcal{E}, a \hookrightarrow N \triangleleft \mathcal{E}] \parallel \pi \rangle \tag{3}$$

$$\langle \mathsf{fst}(M) \triangleleft \mathcal{E} \parallel \pi \rangle \succ \big\langle M \triangleleft \mathcal{E} \,\big\|\, (\mathsf{fst}(\square), \mathcal{E}, []) :: \pi \big\rangle \tag{4}$$

$$\langle \mathsf{snd}(M) \triangleleft \mathcal{E} \parallel \pi \rangle \succ \big\langle M \triangleleft \mathcal{E} \,\big\|\, (\mathsf{snd}(\square), \mathcal{E}, []) :: \pi \big\rangle \tag{5}$$

$$\big\langle \langle M, N \rangle \triangleleft \mathcal{E} \,\big\|\, (\mathsf{fst}(\square), \mathcal{E}', []) :: \pi \big\rangle \succ \langle M \triangleleft \mathcal{E} \parallel \pi \rangle \tag{6}$$

$$\big\langle \langle M, N \rangle \triangleleft \mathcal{E} \,\big\|\, (\mathsf{snd}(\square), \mathcal{E}', []) :: \pi \big\rangle \succ \langle N \triangleleft \mathcal{E} \parallel \pi \rangle \tag{7}$$

$$\langle \mathsf{loop}_r \triangleleft (\gamma, \psi) \parallel \pi \rangle \succ \langle \mathsf{base} \triangleleft (\gamma, \psi) \parallel \pi \rangle \ \textbf{when}\ r\psi = \epsilon \tag{8}$$

$$\langle \mathsf{coe}_{x.A}^{r \rightsquigarrow r'}(M) \triangleleft \mathcal{E} \parallel \pi \rangle \succ \big\langle A \triangleleft \mathcal{E} \,\big\|\, (\mathsf{coe}_{\square}^{r \rightsquigarrow r'}(M), \mathcal{E}', [x]) :: \pi \big\rangle \tag{9}$$

$$\big\langle \mathsf{bool} \triangleleft \mathcal{E} \,\big\|\, (\mathsf{coe}_{\square}^{r \rightsquigarrow r'}(M), \mathcal{E}', [x]) :: \pi \big\rangle \succ \langle M \triangleleft \mathcal{E}' \parallel \pi \rangle \tag{10}$$

$$\big\langle (a : A) \to B \triangleleft \mathcal{E} \,\big\|\, (\mathsf{coe}_{\square}^{r \rightsquigarrow r'}(M), \mathcal{E}', [x]) :: \pi \big\rangle \succ \big\langle \lambda b.\mathsf{coe}_{x.B[\mathcal{E}, a \hookrightarrow \mathsf{coe}_{x.A}^{r' \rightsquigarrow x}(b) \triangleleft \mathcal{E}]}^{r \rightsquigarrow r'}(M(\mathsf{coe}_{x.A[\mathcal{E}]}^{r' \rightsquigarrow r}(b))) \triangleleft \mathcal{E}' \parallel \pi \big\rangle \tag{11}$$

The dynamics of coercion on a functional type are unfortunate: it is necessary to force the substitution $\mathcal{E}$ on the types A and B, since we are “suspending” the coercion and placing it underneath a λ-abstraction. The same problem occurs for pair types. This suggests that we may ultimately need some kind of explicit substitution in our calculus.

$$\Big\langle (a:A)\times B \triangleleft \mathcal{E} \,\Big\|\, (\mathsf{coe}_{\Box}^{r\rightsquigarrow r'}(M), \mathcal{E}', [x]) :: \pi \Big\rangle \succ \Big\langle \langle \mathsf{coe}_{x.A[\mathcal{E}]}^{r\rightsquigarrow r'}(\mathsf{fst}(M)), \mathsf{coe}_{x.B[\mathcal{E}, a\hookrightarrow \mathsf{coe}_{x.A[\mathcal{E}]}^{r\rightsquigarrow x}(\mathsf{fst}(M))\triangleleft\mathcal{E}']}^{r\rightsquigarrow r'}(\mathsf{snd}(M))\rangle \triangleleft \mathcal{E}' \,\Big\|\, \pi \Big\rangle \tag{12}$$

Unloading the machine We can unload the machine at any time; this is useful if we are computing an open term and hit a variable.

$$\frac{}{\langle M \triangleleft \mathcal{E} \,\|\, \cdot \rangle \Longrightarrow M[\mathcal{E}]} \qquad\qquad \frac{\langle K[\Psi.M[\mathcal{E}]] \triangleleft \mathcal{E}' \,\|\, \pi \rangle \Longrightarrow N}{\Big\langle M \triangleleft \mathcal{E} \,\Big\|\, (K[\Box], \mathcal{E}', \Psi) :: \pi \Big\rangle \Longrightarrow N}$$