Cubical Abstract Machine

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\begin{array}{rcll}
M, A & ::= & a \mid \mathsf{coe}_{x.A}^{r \rightsquigarrow r'}(M) \mid \ldots & \text{(terms)} \\
K & ::= & \mathsf{coe}_{\square}^{r \rightsquigarrow r'}(M) \mid \ldots & \text{(continuations)} \\
C & ::= & M \triangleleft \mathcal{E} & \text{(closures)} \\
\mathcal{E} & ::= & (\mathfrak{d}, \gamma, \psi) & \text{(environments)} \\
\mathfrak{d} & ::= & \overline{\mathbb{x} \hookrightarrow \vec{x}; \vec{a}.M \triangleleft \mathcal{E}} & \text{(meta env.)} \\
\gamma & ::= & \overline{a \hookrightarrow C} & \text{(object env.)} \\
\psi & ::= & \overline{x \hookrightarrow r} & \text{(dimension env.)} \\
f & ::= & (K, \mathcal{E}, \Psi) & \text{(frames)} \\
\pi & ::= & \cdot \mid f :: \pi & \text{(stacks)} \\
\mathcal{C} & ::= & \langle M \triangleleft \mathcal{E} \parallel \pi \rangle & \text{(states)}
\end{array}
$$

Stack frames and machine states A stack frame $(\ K, \mathcal{E}, \Psi\ )$ represents a continuation whose whole binds Ψ dimensions. For a stack π, let $\vec{\Psi}$ be the aggregation of all the dimension bindings Ψ mentioned in π; then the extension of a stack $(K, \mathcal{E}, \Phi) :: \pi$ is wellformed when $\mathcal{E}$ is an environment for $\mathbf{FD}(K) \setminus \vec{\Psi}$; moreover, K is allowed to mention the dimensions in $\vec{\Psi}$.

A machine state $\langle M \triangleleft \mathcal{E} \parallel \pi \rangle$, where $\vec{\Psi}$ is the aggregation of dimension bindings in π, is wellformed when $\mathcal{E}$ is an environment for $\mathbf{FD}(M) \setminus \vec{\Psi}$; moreover, M is allowed to mention the dimensions in $\vec{\Psi}$.

Selected transition rules We define a transition judgment $\mathcal{C} \succ_{\mu} \mathcal{C}'$ with μ ranging over the following possible modes:

1. $\Cube$, denoting a cubically stable transition
2. $\bullet$, denoting a non-cubically stable transition

Moreover, if $\mathcal{C} \succ_{\Cube} \mathcal{C}'$ then $\mathcal{C} \succ_{\bullet} \mathcal{C}'$.

$$
\begin{aligned}
\langle a \triangleleft (\gamma, \psi) \parallel \pi \rangle &\succ_{\Cube} \langle \gamma(a) \triangleleft (\gamma, \psi) \parallel \pi \rangle && (1) \\
\big\langle M(N) \triangleleft \mathcal{E} \,\big\|\, \pi \big\rangle &\succ_{\Cube} \big\langle M \triangleleft \mathcal{E} \,\big\|\, (\square(N), \mathcal{E}, []) :: \pi \big\rangle && (2) \\
\big\langle \lambda a.M \triangleleft \mathcal{E} \,\big\|\, (\square(N), \mathcal{E}', []) :: \pi \big\rangle &\succ_{\Cube} \langle M \triangleleft [\mathcal{E}, a \hookrightarrow N \triangleleft \mathcal{E}] \parallel \pi \rangle && (3) \\
\langle \mathsf{fst}(M) \triangleleft \mathcal{E} \parallel \pi \rangle &\succ_{\Cube} \big\langle M \triangleleft \mathcal{E} \,\big\|\, (\mathsf{fst}(\square), \mathcal{E}, []) :: \pi \big\rangle && (4) \\
\langle \mathsf{snd}(M) \triangleleft \mathcal{E} \parallel \pi \rangle &\succ_{\Cube} \big\langle M \triangleleft \mathcal{E} \,\big\|\, (\mathsf{snd}(\square), \mathcal{E}, []) :: \pi \big\rangle && (5) \\
\big\langle \langle M, N \rangle \triangleleft \mathcal{E} \,\big\|\, (\mathsf{fst}(\square), \mathcal{E}', []) :: \pi \big\rangle &\succ_{\Cube} \langle M \triangleleft \mathcal{E} \parallel \pi \rangle && (6) \\
\big\langle \langle M, N \rangle \triangleleft \mathcal{E} \,\big\|\, (\mathsf{snd}(\square), \mathcal{E}', []) :: \pi \big\rangle &\succ_{\Cube} \langle N \triangleleft \mathcal{E} \parallel \pi \rangle && (7)
\end{aligned}
$$

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\langle \mathsf{loop}_r \triangleleft (\gamma, \psi) \parallel \pi \rangle \succ_{\Cube} \langle \mathsf{base} \triangleleft (\gamma, \psi) \parallel \pi \rangle \ \mathbf{when}\ r\psi = \epsilon \tag{8}
$$

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\tag{9}
$$

$$\langle \mathsf{coe}_{x.A}^{r \rightsquigarrow r'}(M) \triangleleft \mathcal{E} \parallel \pi \rangle \succ_{\boxdot} \left\langle A \triangleleft \mathcal{E} \,\middle\|\, (\mathsf{coe}_{\square}^{r \rightsquigarrow r'}(M), \mathcal{E}', [x]) :: \pi \right\rangle \tag{10}$$

$$\left\langle \mathsf{bool} \triangleleft \mathcal{E} \,\middle\|\, (\mathsf{coe}_{\square}^{r \rightsquigarrow r'}(M), \mathcal{E}', [x]) :: \pi \right\rangle \succ_{\boxdot} \langle M \triangleleft \mathcal{E}' \parallel \pi \rangle \tag{11}$$

The rules for coercions are more difficult, but using second-order substitutions we can account for them without needing to force closures.

$$\left\langle (a : A) \to B \triangleleft \mathcal{E} \,\middle\|\, (\mathsf{coe}_{\square}^{r \rightsquigarrow r'}(M), \mathcal{E}', [x]) :: \pi \right\rangle \succ_{\boxdot} \left\langle \lambda a.\mathsf{coe}_{x.\mathbb{x}[x;a]}^{r \rightsquigarrow r'}(M(\mathsf{coe}_{x.\mathbb{y}[x]}^{r' \rightsquigarrow r}(a))) \triangleleft \mathcal{E}'' \,\middle\|\, \pi \right\rangle$$
$$\textbf{where } \mathcal{E}'' = [\mathcal{E}', \mathbb{x} \hookrightarrow x; a.B \triangleleft [\mathcal{E}, a \hookrightarrow \mathsf{coe}_{x.\mathbb{z}[x]}^{r' \rightsquigarrow x}(a) \triangleleft [\mathcal{E}', \mathbb{z} \hookrightarrow x.A \triangleleft \mathcal{E}]], \mathbb{y} \hookrightarrow x.A \triangleleft \mathcal{E}]$$

$$\left\langle (a : A) \times B \triangleleft \mathcal{E} \,\middle\|\, (\mathsf{coe}_{\square}^{r \rightsquigarrow r'}(M), \mathcal{E}', [x]) :: \pi \right\rangle \succ_{\boxdot} \left\langle \langle \mathsf{coe}_{x.\mathbb{x}[x]}^{r \rightsquigarrow r'}(\mathsf{fst}(M)), \mathsf{coe}_{x.\mathbb{y}[x]}^{r \rightsquigarrow r'}(\mathsf{snd}(M)) \rangle \triangleleft \mathcal{E}'' \,\middle\|\, \pi \right\rangle$$
$$\textbf{where } \mathcal{E}'' = [\mathcal{E}', \mathbb{x} \hookrightarrow x.A \triangleleft \mathcal{E}, \mathbb{y} \hookrightarrow x.B \triangleleft [\mathcal{E}, a \hookrightarrow \mathsf{coe}_{y.\mathbb{x}[y]}^{r \rightsquigarrow x}(\mathsf{snd}(M)) \triangleleft [\mathcal{E}, \mathbb{x} \hookrightarrow x.A \triangleleft \mathcal{E}]]]$$

Unloading the machine We can unload the machine at any time; this is useful if we are computing an open term and hit a variable.

$$\frac{}{\langle M \triangleleft \mathcal{E} \parallel \cdot \rangle \Longrightarrow M[\mathcal{E}]} \qquad \frac{\langle K[\Psi.M[\mathcal{E}]] \triangleleft \mathcal{E}' \parallel \pi \rangle \Longrightarrow N}{\left\langle M \triangleleft \mathcal{E} \,\middle\|\, (K[\square], \mathcal{E}', \Psi) :: \pi \right\rangle \Longrightarrow N}$$