

Cubical Abstract Machine

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1 Machine structure

M, A	$::= a \mid \mathbb{X}[\vec{r}; \vec{M}] \mid \text{bool} \mid \text{wbool} \mid \mathbb{S}^1 \mid \langle M, N \rangle \mid \text{fst}(M) \mid \text{snd}(M) \mid \lambda x.M \mid M(N) \mid \langle x \rangle M \mid M @ r$ $\text{if}(M; N; N') \mid \text{if}_{x.A}(M; N; N')$ $\text{coe}_{x.A}^{r \rightsquigarrow r'}(M) \mid \text{hcom}_A^{r \rightsquigarrow r'}(M; \xi \hookrightarrow y.N) \mid \text{fcom}^{r \rightsquigarrow r'}(M; \xi \hookrightarrow y.N)$ $\text{com}_{x.A}^{r \rightsquigarrow r'}(M; \xi \hookrightarrow y.N) \mid \dots$	(terms)
K	$::= \text{coe}_{\square}^{r \rightsquigarrow r'}(M) \mid \square(N) \mid \text{fst}(\square) \mid \text{snd}(\square) \mid \square @ r \mid \text{hcom}_{\square}^{r \rightsquigarrow r'}(M; \xi \hookrightarrow y.N)$ $\text{if}(\square; M; N) \mid \text{if}_{x.A}(\square; M; N) \mid \dots$	(continuations)
C	$::= M \triangleleft \mathcal{E}$	(closures)
\mathcal{E}	$::= (\emptyset, \gamma, \psi)$	(environments)
\emptyset	$::= \frac{\mathbb{X} \hookrightarrow \vec{x}; \vec{a}.M \triangleleft \mathcal{E}}{\quad}$	(meta env.)
γ	$::= \frac{a \hookrightarrow \vec{C}}{\quad}$	(object env.)
ψ	$::= \frac{x \hookrightarrow \vec{r}}{\quad}$	(dimension env.)
f	$::= (K, \mathcal{E}, \Psi)$	(frames)
π	$::= \cdot \mid f :: \pi$	(stacks)
\mathcal{C}	$::= \langle M \triangleleft \mathcal{E} \parallel \pi \rangle$	(states)

This machine is similar to both Krivine's environment machine and the CEK machine. To be honest, Jon Sterling is not really sure of what delineates these from each other, except that they usually seem to be call-by-name and call-by-value respectively. Ours is call-by-name.

Stack frames and machine states A stack frame (K, \mathcal{E}, Ψ) represents a continuation whose hole ranges over a binder of Ψ dimensions. For a stack π , let $\vec{\Psi}$ be the aggregation of all the dimension bindings Ψ mentioned in π ; then the extension of a stack $(K, \mathcal{E}, \Phi) :: \pi$ is wellformed when \mathcal{E} is an environment for $\mathbf{FD}(K) \setminus \vec{\Psi}$; moreover, K is allowed to mention the dimensions in $\vec{\Psi}$.

A machine state $\langle M \triangleleft \mathcal{E} \parallel \pi \rangle$, where $\vec{\Psi}$ is the aggregation of dimension bindings in π , is wellformed when \mathcal{E} is an environment for $\mathbf{FD}(M) \setminus \vec{\Psi}$; moreover, M is allowed to mention the dimensions in $\vec{\Psi}$.

2 Selected transition rules

We define a transition judgment $\mathcal{C} \succ_{\mu} \mathcal{C}'$ with μ ranging over the following possible modes:

1. \boxplus , denoting a cubically stable transition
2. \bullet , denoting a non-cubically stable transition

Moreover, if $\mathcal{C} \succ_{\boxplus} \mathcal{C}'$ then $\mathcal{C} \succ_{\bullet} \mathcal{C}'$.

2.1 Variables

$$\frac{\mathcal{E}(a) \equiv M \triangleleft \mathcal{E}'}{\langle a \triangleleft \mathcal{E} \parallel \pi \rangle \succ_{\mu} \langle M \triangleleft \mathcal{E}' \parallel \pi \rangle} \quad \frac{\mathcal{E}(\bar{x}) \equiv \bar{x}; \bar{a}.N \triangleleft \mathcal{E}' \quad \mathcal{E}'' \triangleq [\mathcal{E}', \bar{a} \hookrightarrow M \triangleleft \mathcal{E}, \bar{x} \hookrightarrow \mathcal{E}(r)]}{\langle \bar{x}[\bar{r}; \bar{M}] \triangleleft \mathcal{E} \parallel \pi \rangle \succ_{\mu} \langle N \triangleleft \mathcal{E}'' \parallel \pi \rangle}$$

2.2 Kan operations

$$\frac{}{\langle \text{hcom}_A^{r \rightsquigarrow r'}(M; \bar{\xi} \hookrightarrow y.\bar{N}) \triangleleft \mathcal{E} \parallel \pi \rangle \succ_{\mu} \langle A \triangleleft \mathcal{E} \parallel (\text{hcom}_{\square}^{r \rightsquigarrow r'}(M; \bar{\xi} \hookrightarrow y.\bar{N}), \mathcal{E}, []) :: \pi \rangle}$$

$$\frac{}{\langle \text{coe}_{x.A}^{r \rightsquigarrow r'}(M) \triangleleft \mathcal{E} \parallel \pi \rangle \succ_{\mu} \langle A \triangleleft \mathcal{E} \parallel (\text{coe}_{\square}^{r \rightsquigarrow r'}(M), \mathcal{E}', [x]) :: \pi \rangle}$$

$$\frac{}{\langle \text{com}_{x.A}^{r \rightsquigarrow r'}(M; \bar{\xi}_i \hookrightarrow y.N_i) \triangleleft \mathcal{E} \parallel \pi \rangle \succ_{\mu} \langle \text{hcom}_A^{r \rightsquigarrow r'}(\text{coe}_{y.A}^{r \rightsquigarrow r'}(M); \bar{\xi}_i \hookrightarrow y.\text{coe}_{y.A}^{y \rightsquigarrow r'}(N_i)) \triangleleft [\mathcal{E}, x \hookrightarrow r'] \parallel \pi \rangle}$$

$$\frac{\mathcal{E} \models r = r'}{\langle \text{fcom}^{r \rightsquigarrow r'}(M; \bar{\xi} \hookrightarrow y.\bar{N}) \triangleleft \mathcal{E} \parallel \pi \rangle \succ_{\mu} \langle M \triangleleft \mathcal{E} \parallel \pi \rangle}$$

$$\frac{\mathcal{E} \models r \# r' \quad \mathcal{E} \models r_i \# r'_i (\forall i < j) \quad \mathcal{E} \models r_j = r'_j}{\langle \text{fcom}^{r \rightsquigarrow r'}(M; r_i = r'_i \hookrightarrow y.N_i) \triangleleft \mathcal{E} \parallel \pi \rangle \succ_{\bullet} \langle N_j \triangleleft [\mathcal{E}, y \hookrightarrow \mathcal{E}(r')] \parallel \pi \rangle}$$

2.3 Dependent function types

$$\frac{}{\langle M(N) \triangleleft \mathcal{E} \parallel \pi \rangle \succ_{\mu} \langle M \triangleleft \mathcal{E} \parallel (\square(N), \mathcal{E}, []) :: \pi \rangle}$$

$$\frac{}{\langle \lambda a.M \triangleleft \mathcal{E} \parallel (\square(N), \mathcal{E}', []) :: \pi \rangle \succ_{\mu} \langle M \triangleleft [\mathcal{E}, a \hookrightarrow N \triangleleft \mathcal{E}'] \parallel \pi \rangle}$$

$$\frac{\mathcal{E}'' \triangleq [\mathcal{E}', \bar{x} \hookrightarrow a.B \triangleleft \mathcal{E}]}{\langle (x : A) \rightarrow B \triangleleft \mathcal{E} \parallel (\text{hcom}_{\square}^{r \rightsquigarrow r'}(M; \bar{\xi}_i \hookrightarrow y.N_i), \mathcal{E}', []) :: \pi \rangle \succ_{\mu} \langle \lambda a.\text{hcom}_{\bar{x}[a]}^{r \rightsquigarrow r'}(M(a); \bar{\xi}_i \hookrightarrow y.N_i(a)) \triangleleft \mathcal{E}'' \parallel \pi \rangle}$$

$$\frac{\mathcal{E}'' \triangleq [\mathcal{E}', \bar{x} \hookrightarrow x; b.B \triangleleft [\mathcal{E}, a \hookrightarrow \text{coe}_{x.\bar{x}[x]}^{r' \rightsquigarrow x}(b) \triangleleft [\mathcal{E}', \bar{z} \hookrightarrow x.A \triangleleft \mathcal{E}]], \bar{y} \hookrightarrow x.A \triangleleft \mathcal{E}]}{\langle (a : A) \rightarrow B \triangleleft \mathcal{E} \parallel (\text{coe}_{\square}^{r \rightsquigarrow r'}(M), \mathcal{E}', [x]) :: \pi \rangle \succ_{\mu} \langle \lambda a.\text{coe}_{x.\bar{x}[x]; a]}^{r \rightsquigarrow r'}(M(\text{coe}_{x.\bar{y}[x]}^{r' \rightsquigarrow r}(a))) \triangleleft \mathcal{E}'' \parallel \pi \rangle}$$

2.4 Dependent pair types

$$\frac{}{\langle \text{fst}(M) \triangleleft \mathcal{E} \parallel \pi \rangle \succ_{\mu} \langle M \triangleleft \mathcal{E} \parallel (\text{fst}(\square), \mathcal{E}, []) :: \pi \rangle} \quad \frac{}{\langle \text{snd}(M) \triangleleft \mathcal{E} \parallel \pi \rangle \succ_{\mu} \langle M \triangleleft \mathcal{E} \parallel (\text{snd}(\square), \mathcal{E}, []) :: \pi \rangle}$$

$$\frac{}{\langle \langle M, N \rangle \triangleleft \mathcal{E} \parallel (\text{fst}(\square), \mathcal{E}', []) :: \pi \rangle \succ_{\mu} \langle M \triangleleft \mathcal{E} \parallel \pi \rangle} \quad \frac{}{\langle \langle M, N \rangle \triangleleft \mathcal{E} \parallel (\text{snd}(\square), \mathcal{E}', []) :: \pi \rangle \succ_{\mu} \langle N \triangleleft \mathcal{E} \parallel \pi \rangle}$$

$$\frac{L_s \triangleq \text{hcom}_{\bar{x}}^{r \rightsquigarrow s}(\text{fst}(M); \bar{\xi}_i \hookrightarrow y.\text{fst}(N_i)) \quad R \triangleq \text{com}_{z.\bar{y}[z]}^{r \rightsquigarrow r'}(\text{snd}(M); \bar{\xi}_i \hookrightarrow y.\text{snd}(N_i)) \quad \mathcal{E}''' \triangleq [\mathcal{E}', \bar{x} \hookrightarrow .A \triangleleft \mathcal{E}] \quad \mathcal{E}'' \triangleq [\mathcal{E}'', \bar{y} \hookrightarrow z.B \triangleleft [\mathcal{E}, a \hookrightarrow L_z \triangleleft \mathcal{E}''']]}{\langle (a : A) \times B \triangleleft \mathcal{E} \parallel (\text{hcom}_{\square}^{r \rightsquigarrow r'}(M; \bar{\xi}_i \hookrightarrow y.N_i), \mathcal{E}', []) :: \pi \rangle \succ_{\mu} \langle \langle L_{r'}, R \rangle \triangleleft \mathcal{E}''' \parallel \pi \rangle}$$

$$\frac{\mathcal{E}'' \triangleq [\mathcal{E}', \bar{x} \hookrightarrow x.A \triangleleft \mathcal{E}, \bar{y} \hookrightarrow x.B \triangleleft [\mathcal{E}, a \hookrightarrow \text{coe}_{y.\bar{x}[y]}^{r \rightsquigarrow x}(\text{snd}(M)) \triangleleft [\mathcal{E}, \bar{x} \hookrightarrow x.A \triangleleft \mathcal{E}]]]}{\langle (a : A) \times B \triangleleft \mathcal{E} \parallel (\text{coe}_{\square}^{r \rightsquigarrow r'}(M), \mathcal{E}', [x]) :: \pi \rangle \succ_{\mu} \langle \langle \text{coe}_{x.\bar{x}[x]}^{r \rightsquigarrow r'}(\text{fst}(M)), \text{coe}_{x.\bar{y}[x]}^{r \rightsquigarrow r'}(\text{snd}(M)) \rangle \triangleleft \mathcal{E}'' \parallel \pi \rangle}$$

2.5 Path types

$$\begin{array}{c}
\overline{\langle M @ r \triangleleft \mathcal{E} \parallel \pi \rangle \succ_{\mu} \langle M \triangleleft \mathcal{E} \parallel (\Box @ r, \mathcal{E}, []) :: \pi \rangle} \quad \overline{\langle \langle x \rangle M \triangleleft \mathcal{E} \parallel (\Box @ r, \mathcal{E}', []) :: \pi \rangle \succ_{\mu} \langle M \triangleleft [\mathcal{E}, x \hookrightarrow \mathcal{E}'(r)] \parallel \pi \rangle} \\
\\
\overline{H \triangleq \text{hcom}_{\mathbb{X}[x]}^{r \rightsquigarrow r'}(M @ x; \overline{x = \epsilon \hookrightarrow y.y_{\epsilon}}, \overline{\xi_i \hookrightarrow y.N_i @ x}) \quad \mathcal{E}'' \triangleq [\mathcal{E}', \mathbb{X} \hookrightarrow x.A \triangleleft \mathcal{E}, y_{\epsilon} \hookrightarrow .P_{\epsilon} \triangleleft \mathcal{E}]} \\
\overline{\langle \text{path}_{x.A}(P_0; P_1) \triangleleft \mathcal{E} \parallel (\text{hcom}_{\Box}^{r \rightsquigarrow r'}(M; \overline{\xi_i \hookrightarrow y.N_i}), \mathcal{E}', []) :: \pi \rangle \succ_{\mu} \langle \langle x \rangle H \triangleleft \mathcal{E}'' \parallel \pi \rangle} \\
\\
\overline{\mathcal{E}'' \triangleq [\mathcal{E}', \mathbb{X} \hookrightarrow x.A \triangleleft \mathcal{E}, y_{\epsilon} \hookrightarrow .P_{\epsilon} \triangleleft \mathcal{E}]} \\
\overline{\langle \text{path}_{x.A}(P_0; P_1) \triangleleft \mathcal{E} \parallel (\text{coe}_{\Box}^{r \rightsquigarrow r'}(M), \mathcal{E}', [x]) :: \pi \rangle \succ_{\mu} \langle \langle x \rangle \text{com}_{y.\mathbb{X}[y]}^{r \rightsquigarrow r'}(M @ x; \overline{x = \epsilon \hookrightarrow y.y_{\epsilon}}) \triangleleft \mathcal{E}'' \parallel \pi \rangle}
\end{array}$$

2.6 Natural numbers

TODO

2.7 Booleans

$$\begin{array}{c}
\overline{\langle \text{if}(M; T; F) \triangleleft \mathcal{E} \parallel \pi \rangle \succ_{\mu} \langle M \triangleleft \mathcal{E} \parallel (\text{if}(\Box; T; F), \mathcal{E}, []) :: \pi \rangle} \quad \overline{\langle \text{true} \triangleleft \mathcal{E} \parallel (\text{if}(\Box; T; F), \mathcal{E}', []) :: \pi \rangle \succ_{\mu} \langle T \triangleleft \mathcal{E}' \parallel \pi \rangle} \\
\\
\overline{\langle \text{false} \triangleleft \mathcal{E} \parallel (\text{if}(\Box; T; F), \mathcal{E}', []) :: \pi \rangle \succ_{\mu} \langle F \triangleleft \mathcal{E}' \parallel \pi \rangle} \quad \overline{\langle \text{bool} \triangleleft \mathcal{E} \parallel (\text{coe}_{\Box}^{r \rightsquigarrow r'}(M), \mathcal{E}', [x]) :: \pi \rangle \succ_{\mu} \langle M \triangleleft \mathcal{E}' \parallel \pi \rangle} \\
\\
\overline{\langle \text{bool} \triangleleft \mathcal{E} \parallel (\text{hcom}_{\Box}^{r \rightsquigarrow r'}(M; \overline{\xi \hookrightarrow y.N}), \mathcal{E}', []) :: \pi \rangle \succ_{\mu} \langle M \triangleleft \mathcal{E}' \parallel \pi \rangle}
\end{array}$$

2.8 Weak booleans

TODO, if, if-fcom, etc.

$$\begin{array}{c}
\overline{\langle \text{if}_{x.A}(M; T; F) \triangleleft \mathcal{E} \parallel \pi \rangle \succ_{\mu} \langle M \triangleleft \mathcal{E} \parallel (\text{if}_{x.A}(\Box; T; F), \mathcal{E}, []) :: \pi \rangle} \\
\\
\overline{\langle \text{true} \triangleleft \mathcal{E} \parallel (\text{if}_{x.A}(\Box; T; F), \mathcal{E}', []) :: \pi \rangle \succ_{\mu} \langle T \triangleleft \mathcal{E}' \parallel \pi \rangle} \quad \overline{\langle \text{false} \triangleleft \mathcal{E} \parallel (\text{if}_{x.A}(\Box; T; F), \mathcal{E}', []) :: \pi \rangle \succ_{\mu} \langle F \triangleleft \mathcal{E}' \parallel \pi \rangle} \\
\\
\overline{H \triangleq \text{fcom}^{r \rightsquigarrow z}(M; \overline{r_i = r'_i \hookrightarrow y.N_i}) \quad \mathcal{E} \models r \# r' \quad \mathcal{E} \models r_i \# r'_i (\forall i) \quad \mathcal{E}'' \triangleq [\mathcal{E}', \mathbb{X} \hookrightarrow z.A \triangleleft [\mathcal{E}', a \hookrightarrow H \triangleleft \mathcal{E}], y \hookrightarrow .M \triangleleft \mathcal{E}, \mathbb{Z}_i \hookrightarrow y.N_i \triangleleft \mathcal{E}]} \\
\overline{\langle \text{fcom}^{r \rightsquigarrow r'}(M; \overline{r_i = r'_i \hookrightarrow y.N_i}) \triangleleft \mathcal{E}' \parallel (\text{if}_{x.A}(\Box; T; F), \mathcal{E}', []) :: \pi \rangle} \\
\overline{\succ_{\mu} \langle \text{com}_{z.\mathbb{X}[z]}^{\mathcal{E}(r) \rightsquigarrow \mathcal{E}(r')}(\text{if}_{a.A}(y; T; F); \overline{\mathcal{E}(r_i) = \mathcal{E}(r'_i) \hookrightarrow y.\text{if}_{a.A}(\mathbb{Z}_i[y]; T; F)}) \triangleleft \mathcal{E}'' \parallel \pi \rangle} \\
\\
\overline{\langle \text{wbool} \triangleleft \mathcal{E} \parallel (\text{hcom}_{\Box}^{r \rightsquigarrow r'}(M; \overline{\xi \hookrightarrow y.N}), \mathcal{E}', []) :: \pi \rangle \succ_{\mu} \langle \text{fcom}^{r \rightsquigarrow r'}(M; \overline{\xi \hookrightarrow y.N}) \triangleleft \mathcal{E}' \parallel \pi \rangle} \\
\\
\overline{\langle \text{wbool} \triangleleft \mathcal{E} \parallel (\text{coe}_{\Box}^{r \rightsquigarrow r'}(M), \mathcal{E}', [x]) :: \pi \rangle \succ_{\mu} \langle M \triangleleft \mathcal{E}' \parallel \pi \rangle}
\end{array}$$

2.9 Circle

TODO, add hcom, coe, rec-fcom, etc.

$$\begin{array}{c}
\frac{\mathcal{E} \models r = \epsilon}{\langle \text{loop}_r \triangleleft \mathcal{E} \parallel \pi \rangle \succ_\mu \langle \text{base} \triangleleft \mathcal{E} \parallel \pi \rangle} \quad \frac{}{\langle \mathbb{S}^1 \text{rec}_{a.A}(M; P; x.L) \triangleleft \mathcal{E} \parallel \pi \rangle \succ_\mu \langle M \triangleleft \mathcal{E} \parallel (\mathbb{S}^1 \text{rec}_{a.A}(\square; P; x.L), \mathcal{E}, []) :: \pi \rangle} \\
\\
\frac{}{\langle \text{base} \triangleleft \mathcal{E} \parallel (\mathbb{S}^1 \text{rec}_{a.A}(\square; P; x.L), \mathcal{E}', []) :: \pi \rangle \succ_\mu \langle P \triangleleft \mathcal{E}' \parallel \pi \rangle} \\
\\
\frac{\mathcal{E} \models r = w}{\langle \text{loop}_r \triangleleft \mathcal{E} \parallel (\mathbb{S}^1 \text{rec}_{a.A}(\square; P; x.L), \mathcal{E}', []) :: \pi \rangle \succ_\bullet \langle L \triangleleft [\mathcal{E}', x \hookrightarrow w] \parallel \pi \rangle} \\
\\
\frac{H \triangleq \text{fcom}^{r \rightsquigarrow z}(M; \overline{r_i = r'_i \hookrightarrow y.N_i}) \quad \mathcal{E} \models r \# r' \quad \mathcal{E} \models r_i \# r'_i \ (\forall i) \quad \mathcal{E}'' \triangleq [\mathcal{E}', x \hookrightarrow z.A \triangleleft [\mathcal{E}', a \hookrightarrow H \triangleleft \mathcal{E}], y \hookrightarrow .M \triangleleft \mathcal{E}, \overline{z_i \hookrightarrow y.N_i \triangleleft \mathcal{E}}]}{\langle \text{fcom}^{r \rightsquigarrow r'}(M; \overline{r_i = r'_i \hookrightarrow y.N_i}) \triangleleft \mathcal{E}' \parallel (\mathbb{S}^1 \text{rec}_{x.A}(\square; P; x.L), \mathcal{E}', []) :: \pi \rangle \succ_\mu \langle \text{com}_{z.x[z]}^{\mathcal{E}(r) \rightsquigarrow \mathcal{E}(r')}(\mathbb{S}^1 \text{rec}_{a.A}(y; P; x.L); \overline{\mathcal{E}(r_i) = \mathcal{E}(r'_i) \hookrightarrow y. \mathbb{S}^1 \text{rec}_{a.A}(z_i[y]; P; x.L)}) \triangleleft \mathcal{E}'' \parallel \pi \rangle}
\end{array}$$

Unloading the machine We can unload the machine at any time; this is useful if we are computing an open term and hit a variable.

$$\frac{}{\langle M \triangleleft \mathcal{E} \parallel \cdot \rangle \Longrightarrow M[\mathcal{E}]} \quad \frac{\langle K[\Psi.M[\mathcal{E}]] \triangleleft \mathcal{E}' \parallel \pi \rangle \Longrightarrow N}{\langle M \triangleleft \mathcal{E} \parallel (K[\square], \mathcal{E}', \Psi) :: \pi \rangle \Longrightarrow N}$$

But using Favonia's Thought, we can come up with something more efficient. This naïve version is rather appalling.