

# Cubical Abstract Machine

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June 29, 2017

$M, A$	$::= a \mid \text{coe}_{x.A}^{r \rightsquigarrow r'}(M) \mid \dots$	(terms)
$K$	$::= \text{coe}_{\square}^{r \rightsquigarrow r'}(M) \mid \dots$	(continuations)
$C$	$::= M \triangleleft \mathcal{E}$	(closures)
$\mathcal{E}$	$::= (\emptyset, \gamma, \psi)$	(environments)
$\emptyset$	$::= \frac{x \hookrightarrow \{\vec{x}\}[\vec{a}].M \triangleleft \mathcal{E}}{x \hookrightarrow \{\vec{x}\}[\vec{a}].M \triangleleft \mathcal{E}}$	(meta env.)
$\gamma$	$::= \frac{a \hookrightarrow C}{a \hookrightarrow C}$	(object env.)
$\psi$	$::= \frac{x \hookrightarrow r}{x \hookrightarrow r}$	(dimension env.)
$f$	$::= (K, \mathcal{E}, \Psi)$	(frames)
$\pi$	$::= \cdot \mid f :: \pi$	(stacks)
$\mathcal{C}$	$::= \langle M \triangleleft \mathcal{E} \parallel \pi \rangle$	(states)

We will write  $M[\mathcal{E}]$  for substituting the free dimensions and variables of a term  $M$  using an environment  $\mathcal{E}$ .

**Stack frames and machine states** A stack frame  $(K, \mathcal{E}, \Psi)$  represents a continuation whose whole binds  $\Psi$  dimensions. For a stack  $\pi$ , let  $\vec{\Psi}$  be the aggregation of all the dimension bindings  $\Psi$  mentioned in  $\pi$ ; then the extension of a stack  $(K, \mathcal{E}, \Phi) :: \pi$  is wellformed when  $\mathcal{E}$  is an environment for  $\mathbf{FD}(K) \setminus \vec{\Psi}$ ; moreover,  $K$  is allowed to mention the dimensions in  $\vec{\Psi}$ .

A machine state  $\langle M \triangleleft \mathcal{E} \parallel \pi \rangle$ , where  $\vec{\Psi}$  is the aggregation of dimension bindings in  $\pi$ , is wellformed when  $\mathcal{E}$  is an environment for  $\mathbf{FD}(M) \setminus \vec{\Psi}$ ; moreover,  $M$  is allowed to mention the dimensions in  $\vec{\Psi}$ .

**Selected transition rules** We define a transition judgment  $\mathcal{C} \succ_{\mu} \mathcal{C}'$  with  $\mu$  ranging over the following possible modes:

1.  $\boxtimes$ , denoting a cubically stable transition
2.  $\bullet$ , denoting a non-cubically stable transition

Moreover, if  $\mathcal{C} \succ_{\boxtimes} \mathcal{C}'$  then  $\mathcal{C} \succ_{\bullet} \mathcal{C}'$ .

$$\langle a \triangleleft (\gamma, \psi) \parallel \pi \rangle \succ_{\boxtimes} \langle \gamma(a) \triangleleft (\gamma, \psi) \parallel \pi \rangle \quad (1)$$

$$\langle M(N) \triangleleft \mathcal{E} \parallel \pi \rangle \succ_{\boxtimes} \langle M \triangleleft \mathcal{E} \parallel (\square(N), \mathcal{E}, []) :: \pi \rangle \quad (2)$$

$$\langle \lambda a. M \triangleleft \mathcal{E} \parallel (\square(N), \mathcal{E}', []) :: \pi \rangle \succ_{\boxtimes} \langle M \triangleleft [\mathcal{E}, a \hookrightarrow N \triangleleft \mathcal{E}] \parallel \pi \rangle \quad (3)$$

$$\langle \text{fst}(M) \triangleleft \mathcal{E} \parallel \pi \rangle \succ_{\boxtimes} \langle M \triangleleft \mathcal{E} \parallel (\text{fst}(\square), \mathcal{E}, []) :: \pi \rangle \quad (4)$$

$$\langle \text{snd}(M) \triangleleft \mathcal{E} \parallel \pi \rangle \succ_{\boxtimes} \langle M \triangleleft \mathcal{E} \parallel (\text{snd}(\square), \mathcal{E}, []) :: \pi \rangle \quad (5)$$

$$\langle \langle M, N \rangle \triangleleft \mathcal{E} \parallel (\text{fst}(\square), \mathcal{E}', []) :: \pi \rangle \succ_{\boxtimes} \langle M \triangleleft \mathcal{E} \parallel \pi \rangle \quad (6)$$

$$\langle \langle M, N \rangle \triangleleft \mathcal{E} \parallel (\text{snd}(\square), \mathcal{E}', []) :: \pi \rangle \succ_{\boxtimes} \langle N \triangleleft \mathcal{E} \parallel \pi \rangle \quad (7)$$

$$\langle \text{loop}_r \triangleleft (\gamma, \psi) \parallel \pi \rangle \succ_{\square} \langle \text{base} \triangleleft (\gamma, \psi) \parallel \pi \rangle \textbf{ when } r\psi = \epsilon \quad (8)$$

$$(9)$$

$$\langle \text{coe}_{x.A}^{r \rightsquigarrow r'}(M) \triangleleft \mathcal{E} \parallel \pi \rangle \succ_{\square} \langle A \triangleleft \mathcal{E} \parallel (\text{coe}_{\square}^{r \rightsquigarrow r'}(M), \mathcal{E}', [x]) :: \pi \rangle \quad (10)$$

$$\langle \text{bool} \triangleleft \mathcal{E} \parallel (\text{coe}_{\square}^{r \rightsquigarrow r'}(M), \mathcal{E}', [x]) :: \pi \rangle \succ_{\square} \langle M \triangleleft \mathcal{E}' \parallel \pi \rangle \quad (11)$$

$$\langle (a : A) \rightarrow B \triangleleft \mathcal{E} \parallel (\text{coe}_{\square}^{r \rightsquigarrow r'}(M), \mathcal{E}', [x]) :: \pi \rangle$$

$$\succ_{\square}$$

$$\langle \lambda a. \text{coe}_{x.\mathfrak{x}\{x\}}^{r \rightsquigarrow r'}(M(\text{coe}_{x.y\{x\}}^{r' \rightsquigarrow r}(a))) \triangleleft [\mathcal{E}', \mathfrak{x} \hookrightarrow \{x\}[a].B \triangleleft [\mathcal{E}, a \hookrightarrow \text{coe}_{x.\mathfrak{x}[x]}^{r' \rightsquigarrow x}(a) \triangleleft [\mathcal{E}', \mathfrak{z} \hookrightarrow \{x\}.A \triangleleft \mathcal{E}]], y \hookrightarrow \{x\}.A \triangleleft \mathcal{E}] \parallel \pi \rangle$$

$$\langle (a : A) \times B \triangleleft \mathcal{E} \parallel (\text{coe}_{\square}^{r \rightsquigarrow r'}(M), \mathcal{E}', [x]) :: \pi \rangle$$

$$\succ_{\square}$$

$$\langle \langle \text{coe}_{x.\mathfrak{x}\{x\}}^{r \rightsquigarrow r'}(\text{fst}(M)), \text{coe}_{x.y\{x\}}^{r \rightsquigarrow r'}(\text{snd}(M)) \rangle \triangleleft [\mathcal{E}', \mathfrak{x} \hookrightarrow \{x\}.A \triangleleft \mathcal{E}, y \hookrightarrow \{x\}.B \triangleleft [\mathcal{E}, a \hookrightarrow \text{coe}_{y.\mathfrak{x}\{y\}}^{r \rightsquigarrow x}(\text{snd}(M)) \triangleleft [\mathcal{E}, \mathfrak{x} \hookrightarrow \{x\}.A \triangleleft \mathcal{E}]]] \parallel \pi \rangle$$

**Unloading the machine** We can unload the machine at any time; this is useful if we are computing an open term and hit a variable.

$$\frac{}{\langle M \triangleleft \mathcal{E} \parallel \cdot \rangle \Longrightarrow M[\mathcal{E}]} \qquad \frac{\langle K[\Psi.M[\mathcal{E}]] \triangleleft \mathcal{E}' \parallel \pi \rangle \Longrightarrow N}{\langle M \triangleleft \mathcal{E} \parallel (K[\square], \mathcal{E}', \Psi) :: \pi \rangle \Longrightarrow N}$$