Cubical Abstract Machine

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\begin{array}{rcll}
M, A & ::= & a \mid \mathsf{coe}_{x.A}^{r\rightsquigarrow r'}(M) \mid \ldots & \text{(terms)} \\
K & ::= & \mathsf{coe}_{\square}^{r\rightsquigarrow r'}(M) \mid \ldots & \text{(continuations)} \\
C & ::= & M \triangleleft \mathcal{E} & \text{(closures)} \\
\mathcal{E} & ::= & (\mathfrak{d}, \gamma, \psi) & \text{(environments)} \\
\mathfrak{d} & ::= & \overline{\mathbb{x} \hookrightarrow \vec{x}; \vec{a}.M \triangleleft \mathcal{E}} & \text{(meta env.)} \\
\gamma & ::= & \overline{a \hookrightarrow C} & \text{(object env.)} \\
\psi & ::= & \overline{x \hookrightarrow r} & \text{(dimension env.)} \\
f & ::= & (K, \mathcal{E}, \Psi) & \text{(frames)} \\
\pi & ::= & \cdot \mid f :: \pi & \text{(stacks)} \\
\mathcal{C} & ::= & \langle M \triangleleft \mathcal{E} \parallel \pi \rangle & \text{(states)}
\end{array}
$$

Stack frames and machine states A stack frame $(\ K, \mathcal{E}, \Psi\ )$ represents a continuation whose whole binds Ψ dimensions. For a stack π, let $\vec{\Psi}$ be the aggregation of all the dimension bindings Ψ mentioned in π; then the extension of a stack $(K, \mathcal{E}, \Phi) :: \pi$ is wellformed when $\mathcal{E}$ is an environment for $\mathbf{FD}(K) \setminus \vec{\Psi}$; moreover, K is allowed to mention the dimensions in $\vec{\Psi}$.

A machine state $\langle M \triangleleft \mathcal{E} \parallel \pi \rangle$, where $\vec{\Psi}$ is the aggregation of dimension bindings in π, is wellformed when $\mathcal{E}$ is an environment for $\mathbf{FD}(M) \setminus \vec{\Psi}$; moreover, M is allowed to mention the dimensions in $\vec{\Psi}$.

Selected transition rules We define a transition judgment $\mathcal{C} \succ_{\mu} \mathcal{C}'$ with μ ranging over the following possible modes:

1. $\text{\Cube}$, denoting a cubically stable transition
2. $\bullet$, denoting a non-cubically stable transition

Moreover, if $\mathcal{C} \succ_{\text{\Cube}} \mathcal{C}'$ then $\mathcal{C} \succ_{\bullet} \mathcal{C}'$.

$$
\frac{\mathcal{E} \equiv (\mathfrak{d}, \gamma, \psi)}{\langle a \triangleleft \mathcal{E} \parallel \pi \rangle \succ_{\text{\Cube}} \langle \gamma(a) \triangleleft \mathcal{E} \parallel \pi \rangle}
$$

$$
\frac{\mathcal{E} \equiv (\mathfrak{d}, \gamma, \psi) \qquad \mathfrak{d}(\mathbb{x}) \equiv \vec{x}; \vec{a}.N \qquad \gamma' \triangleq (\gamma, \overline{a \hookrightarrow M \triangleleft \mathcal{E}}) \qquad \psi' \triangleq (\psi, \overline{x \hookrightarrow r\psi})}{\langle \mathbb{x}[\vec{r}; \vec{M}] \triangleleft \mathcal{E} \parallel \pi \rangle \succ_{\text{\Cube}} \langle N \triangleleft [(\mathfrak{d}, \gamma', \psi')] \parallel \pi \rangle}
$$

$$
\frac{}{\left\langle M(N) \triangleleft \mathcal{E} \,\middle\|\, \pi \right\rangle \succ_{\text{\Cube}} \left\langle M \triangleleft \mathcal{E} \,\middle\|\, (\square(N), \mathcal{E}, []) :: \pi \right\rangle}
$$

$$
\frac{}{\left\langle \lambda a.M \triangleleft \mathcal{E} \,\middle\|\, (\square(N), \mathcal{E}', []) :: \pi \right\rangle \succ_{\text{\Cube}} \langle M \triangleleft [\mathcal{E}, a \hookrightarrow N \triangleleft \mathcal{E}] \parallel \pi \rangle}
$$

$$
\frac{}{\langle \mathsf{fst}(M) \triangleleft \mathcal{E} \parallel \pi \rangle \succ_{\text{\Cube}} \left\langle M \triangleleft \mathcal{E} \,\middle\|\, (\mathsf{fst}(\square), \mathcal{E}, []) :: \pi \right\rangle}
\qquad
\frac{}{\langle \mathsf{snd}(M) \triangleleft \mathcal{E} \parallel \pi \rangle \succ_{\text{\Cube}} \left\langle M \triangleleft \mathcal{E} \,\middle\|\, (\mathsf{snd}(\square), \mathcal{E}, []) :: \pi \right\rangle}
$$

$$\frac{}{\left\langle \langle M, N\rangle \triangleleft \mathcal{E} \,\middle\|\, (\mathsf{fst}(\square), \mathcal{E}', []) :: \pi \right\rangle \succ_{\Box} \langle M \triangleleft \mathcal{E} \parallel \pi\rangle} \qquad \frac{}{\left\langle \langle M, N\rangle \triangleleft \mathcal{E} \,\middle\|\, (\mathsf{snd}(\square), \mathcal{E}', []) :: \pi \right\rangle \succ_{\Box} \langle N \triangleleft \mathcal{E} \parallel \pi\rangle}$$

$$\frac{r\psi \equiv \epsilon}{\langle \mathsf{loop}_r \triangleleft (\mathfrak{T}, \gamma, \psi) \parallel \pi\rangle \succ_{\Box} \langle \mathsf{base} \triangleleft (\mathfrak{T}, \gamma, \psi) \parallel \pi\rangle}$$

$$\frac{}{\langle \mathsf{coe}^{r \rightsquigarrow r'}_{x.A}(M) \triangleleft \mathcal{E} \parallel \pi\rangle \succ_{\Box} \left\langle A \triangleleft \mathcal{E} \,\middle\|\, (\mathsf{coe}^{r \rightsquigarrow r'}_{\square}(M), \mathcal{E}', [x]) :: \pi \right\rangle}$$

$$\frac{}{\left\langle \mathsf{bool} \triangleleft \mathcal{E} \,\middle\|\, (\mathsf{coe}^{r \rightsquigarrow r'}_{\square}(M), \mathcal{E}', [x]) :: \pi \right\rangle \succ_{\Box} \langle M \triangleleft \mathcal{E}' \parallel \pi\rangle}$$

The rules for coercions are more difficult, but using second-order substitutions we can account for them without needing to force closures.

$$\frac{\mathcal{E}'' \triangleq [\mathcal{E}', \mathbbm{x} \hookrightarrow x; a.B \triangleleft [\mathcal{E}, a \hookrightarrow \mathsf{coe}^{r' \rightsquigarrow x}_{x.\mathbbm{z}[x]}(a) \triangleleft [\mathcal{E}', \mathbbm{z} \hookrightarrow x.A \triangleleft \mathcal{E}]], \mathbbm{y} \hookrightarrow x.A \triangleleft \mathcal{E}]}{\left\langle (a : A) \to B \triangleleft \mathcal{E} \,\middle\|\, (\mathsf{coe}^{r \rightsquigarrow r'}_{\square}(M), \mathcal{E}', [x]) :: \pi \right\rangle \succ_{\Box} \left\langle \lambda a.\mathsf{coe}^{r \rightsquigarrow r'}_{x.\mathbbm{x}[x;a]}(M(\mathsf{coe}^{r' \rightsquigarrow r}_{x.\mathbbm{y}[x]}(a))) \triangleleft \mathcal{E}'' \,\middle\|\, \pi \right\rangle}$$

$$\frac{\mathcal{E}'' \triangleq [\mathcal{E}', \mathbbm{x} \hookrightarrow x.A \triangleleft \mathcal{E}, \mathbbm{y} \hookrightarrow x.B \triangleleft [\mathcal{E}, a \hookrightarrow \mathsf{coe}^{r \rightsquigarrow x}_{y.\mathbbm{x}[y]}(\mathsf{snd}(M)) \triangleleft [\mathcal{E}, \mathbbm{x} \hookrightarrow x.A \triangleleft \mathcal{E}]]]}{\left\langle (a : A) \times B \triangleleft \mathcal{E} \,\middle\|\, (\mathsf{coe}^{r \rightsquigarrow r'}_{\square}(M), \mathcal{E}', [x]) :: \pi \right\rangle \succ_{\Box} \left\langle \langle \mathsf{coe}^{r \rightsquigarrow r'}_{x.\mathbbm{x}[x]}(\mathsf{fst}(M)), \mathsf{coe}^{r \rightsquigarrow r'}_{x.\mathbbm{y}[x]}(\mathsf{snd}(M))\rangle \triangleleft \mathcal{E}'' \,\middle\|\, \pi \right\rangle}$$

Unloading the machine We can unload the machine at any time; this is useful if we are computing an open term and hit a variable.

$$\frac{}{\langle M \triangleleft \mathcal{E} \parallel \cdot\rangle \Longrightarrow M[\mathcal{E}]} \qquad \frac{\langle K[\Psi.M[\mathcal{E}]] \triangleleft \mathcal{E}' \parallel \pi\rangle \Longrightarrow N}{\left\langle M \triangleleft \mathcal{E} \,\middle\|\, (K[\square], \mathcal{E}', \Psi) :: \pi \right\rangle \Longrightarrow N}$$