Cubical Abstract Machine

Jon Sterling and Kuen-Bang Hou (Favonia)

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\begin{array}{rcll}
M, A & ::= & a \mid \varkappa[\vec{r}; \vec{M}] \mid \mathsf{coe}_{x.A}^{r \rightsquigarrow r'}(M) \mid \ldots & \text{(terms)} \\
K & ::= & \mathsf{coe}_{\square}^{r \rightsquigarrow r'}(M) \mid \square(N) \mid \mathsf{fst}(\square) \mid \ldots & \text{(continuations)} \\
C & ::= & M \triangleleft \mathcal{E} & \text{(closures)} \\
\mathcal{E} & ::= & (\Upsilon, \gamma, \psi) & \text{(environments)} \\
\Upsilon & ::= & \overline{\varkappa \hookrightarrow \vec{x}; \vec{a}.M \triangleleft \mathcal{E}} & \text{(meta env.)} \\
\gamma & ::= & \overline{a \hookrightarrow C} & \text{(object env.)} \\
\psi & ::= & \overline{x \hookrightarrow r} & \text{(dimension env.)} \\
f & ::= & (K, \mathcal{E}, \Psi) & \text{(frames)} \\
\pi & ::= & \cdot \mid f :: \pi & \text{(stacks)} \\
\mathcal{C} & ::= & \langle M \triangleleft \mathcal{E} \parallel \pi \rangle & \text{(states)}
\end{array}
$$

This machine is similar to both Krivine's environment machine and the CEK machine. To be honest, Jon Sterling is not really sure of what delineates these from each other, except that they usually seem to be call-by-name and call-by-value respectively. Ours is call-by-name.

Stack frames and machine states A stack frame ($K, \mathcal{E}, \Psi$) represents a continuation whose whole binds Ψ dimensions. For a stack π, let $\vec{\Psi}$ be the aggregation of all the dimension bindings Ψ mentioned in π; then the extension of a stack $(K, \mathcal{E}, \Phi) :: \pi$ is wellformed when $\mathcal{E}$ is an environment for $\mathbf{FD}(K) \setminus \vec{\Psi}$; moreover, K is allowed to mention the dimensions in $\vec{\Psi}$.

A machine state $\langle M \triangleleft \mathcal{E} \parallel \pi \rangle$, where $\vec{\Psi}$ is the aggregation of dimension bindings in π, is wellformed when $\mathcal{E}$ is an environment for $\mathbf{FD}(M) \setminus \vec{\Psi}$; moreover, M is allowed to mention the dimensions in $\vec{\Psi}$.

Selected transition rules We define a transition judgment $\mathcal{C} \succ_{\mu} \mathcal{C}'$ with μ ranging over the following possible modes:

1. $\boxdot$, denoting a cubically stable transition
2. $\bullet$, denoting a non-cubically stable transition

Moreover, if $\mathcal{C} \succ_{\boxdot} \mathcal{C}'$ then $\mathcal{C} \succ_{\bullet} \mathcal{C}'$.

$$
\frac{\mathcal{E} \equiv (\Upsilon, \gamma, \psi)}{\langle a \triangleleft \mathcal{E} \parallel \pi \rangle \succ_{\boxdot} \langle \gamma(a) \triangleleft \mathcal{E} \parallel \pi \rangle}
$$

$$
\frac{\mathcal{E} \equiv (\Upsilon, \gamma, \psi) \qquad \Upsilon(\varkappa) \equiv \vec{x}; \vec{a}.N \qquad \gamma' \triangleq (\gamma, \overline{a \hookrightarrow M \triangleleft \mathcal{E}}) \qquad \psi' \triangleq (\psi, \overline{x \hookrightarrow r\psi})}{\langle \varkappa[\vec{r}; \vec{M}] \triangleleft \mathcal{E} \parallel \pi \rangle \succ_{\boxdot} \langle N \triangleleft [(\Upsilon, \gamma', \psi')] \parallel \pi \rangle}
$$

$$
\frac{}{\left\langle M(N) \triangleleft \mathcal{E} \parallel \pi \right\rangle \succ_{\boxdot} \left\langle M \triangleleft \mathcal{E} \parallel (\square(N), \mathcal{E}, []) :: \pi \right\rangle}
$$

$$
\frac{}{\left\langle \lambda a.M \triangleleft \mathcal{E} \parallel (\square(N), \mathcal{E}', []) :: \pi \right\rangle \succ_{\boxdot} \langle M \triangleleft [\mathcal{E}, a \hookrightarrow N \triangleleft \mathcal{E}] \parallel \pi \rangle}
$$

$$\frac{}{\langle \mathsf{fst}(M) \triangleleft \mathcal{E} \parallel \pi \rangle \succ_{\boxdot} \langle M \triangleleft \mathcal{E} \parallel (\mathsf{fst}(\Box), \mathcal{E}, []) :: \pi \rangle} \qquad \frac{}{\langle \mathsf{snd}(M) \triangleleft \mathcal{E} \parallel \pi \rangle \succ_{\boxdot} \langle M \triangleleft \mathcal{E} \parallel (\mathsf{snd}(\Box), \mathcal{E}, []) :: \pi \rangle}$$

$$\frac{}{\langle \langle M, N \rangle \triangleleft \mathcal{E} \parallel (\mathsf{fst}(\Box), \mathcal{E}', []) :: \pi \rangle \succ_{\boxdot} \langle M \triangleleft \mathcal{E} \parallel \pi \rangle} \qquad \frac{}{\langle \langle M, N \rangle \triangleleft \mathcal{E} \parallel (\mathsf{snd}(\Box), \mathcal{E}', []) :: \pi \rangle \succ_{\boxdot} \langle N \triangleleft \mathcal{E} \parallel \pi \rangle}$$

$$\frac{\mathcal{E} \equiv (\mathbb{\gamma}, \gamma, \psi) \qquad r\psi \equiv \epsilon}{\langle \mathsf{loop}_r \triangleleft \mathcal{E} \parallel \pi \rangle \succ_{\boxdot} \langle \mathsf{base} \triangleleft \mathcal{E} \parallel \pi \rangle}$$

$$\frac{}{\langle \mathbb{S}^1\mathsf{rec}_{a.A}(M; P; x.L) \triangleleft \mathcal{E} \parallel \pi \rangle \succ_{\boxdot} \langle M \triangleleft \mathcal{E} \parallel (\mathbb{S}^1\mathsf{rec}_{a.A}(\Box; P; x.L), \mathcal{E}, []) :: \pi \rangle}$$

$$\frac{}{\langle \mathsf{base} \triangleleft \mathcal{E} \parallel (\mathbb{S}^1\mathsf{rec}_{a.A}(\Box; P; x.L), \mathcal{E}', []) :: \pi \rangle \succ_{\boxdot} \langle P \triangleleft \mathcal{E}' \parallel \pi \rangle}$$

$$\frac{\mathcal{E} \equiv (\mathbb{\gamma}, \gamma, \psi) \qquad r\psi \equiv w}{\langle \mathsf{loop}_r \triangleleft \mathcal{E} \parallel (\mathbb{S}^1\mathsf{rec}_{a.A}(\Box; P; x.L), \mathcal{E}', []) :: \pi \rangle \succ_{\bullet} \langle L \triangleleft [\mathcal{E}', x \hookrightarrow w] \parallel \pi \rangle}$$

The rules for coercions are more difficult, but using second-order substitutions we can account for them without needing to force closures.

$$\frac{}{\langle \mathsf{coe}_{x.A}^{r \rightsquigarrow r'}(M) \triangleleft \mathcal{E} \parallel \pi \rangle \succ_{\boxdot} \langle A \triangleleft \mathcal{E} \parallel (\mathsf{coe}_{\Box}^{r \rightsquigarrow r'}(M), \mathcal{E}', [x]) :: \pi \rangle}$$

$$\frac{}{\langle \mathsf{bool} \triangleleft \mathcal{E} \parallel (\mathsf{coe}_{\Box}^{r \rightsquigarrow r'}(M), \mathcal{E}', [x]) :: \pi \rangle \succ_{\boxdot} \langle M \triangleleft \mathcal{E}' \parallel \pi \rangle}$$

$$\frac{\mathcal{E}'' \triangleq [\mathcal{E}', \mathbb{x} \hookrightarrow x; a.B \triangleleft [\mathcal{E}, a \hookrightarrow \mathsf{coe}_{x.\mathbb{z}[x]}^{r' \rightsquigarrow x}(a) \triangleleft [\mathcal{E}', \mathbb{z} \hookrightarrow x.A \triangleleft \mathcal{E}]], \mathbb{y} \hookrightarrow x.A \triangleleft \mathcal{E}]}{\langle (a : A) \to B \triangleleft \mathcal{E} \parallel (\mathsf{coe}_{\Box}^{r \rightsquigarrow r'}(M), \mathcal{E}', [x]) :: \pi \rangle \succ_{\boxdot} \langle \lambda a.\mathsf{coe}_{x.\mathbb{x}[x;a]}^{r \rightsquigarrow r'}(M(\mathsf{coe}_{x.\mathbb{y}[x]}^{r' \rightsquigarrow r}(a))) \triangleleft \mathcal{E}'' \parallel \pi \rangle}$$

$$\frac{\mathcal{E}'' \triangleq [\mathcal{E}', \mathbb{x} \hookrightarrow x.A \triangleleft \mathcal{E}, \mathbb{y} \hookrightarrow x.B \triangleleft [\mathcal{E}, a \hookrightarrow \mathsf{coe}_{y.\mathbb{x}[y]}^{r \rightsquigarrow x}(\mathsf{snd}(M)) \triangleleft [\mathcal{E}, \mathbb{x} \hookrightarrow x.A \triangleleft \mathcal{E}]]]}{\langle (a : A) \times B \triangleleft \mathcal{E} \parallel (\mathsf{coe}_{\Box}^{r \rightsquigarrow r'}(M), \mathcal{E}', [x]) :: \pi \rangle \succ_{\boxdot} \langle \langle \mathsf{coe}_{x.\mathbb{x}[x]}^{r \rightsquigarrow r'}(\mathsf{fst}(M)), \mathsf{coe}_{x.\mathbb{y}[x]}^{r \rightsquigarrow r'}(\mathsf{snd}(M)) \rangle \triangleleft \mathcal{E}'' \parallel \pi \rangle}$$

Unloading the machine We can unload the machine at any time; this is useful if we are computing an open term and hit a variable.

$$\frac{}{\langle M \triangleleft \mathcal{E} \parallel \cdot \rangle \Longrightarrow M[\mathcal{E}]} \qquad \frac{\langle K[\Psi.M[\mathcal{E}]] \triangleleft \mathcal{E}' \parallel \pi \rangle \Longrightarrow N}{\langle M \triangleleft \mathcal{E} \parallel (K[\Box], \mathcal{E}', \Psi) :: \pi \rangle \Longrightarrow N}$$