

# Cubical Abstract Machine

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## 1 Machine structure

$M, A$	$::= a \mid \mathbb{X}[\vec{r}; \vec{M}] \mid \text{bool} \mid \text{wbool} \mid \mathbb{S}^1 \mid \langle M, N \rangle \mid \text{fst}(M) \mid \text{snd}(M) \mid \lambda x.M \mid M(N) \mid \langle x \rangle M \mid M @ r$ $\text{coe}_{x.A}^{r \rightsquigarrow r'}(M) \mid \text{hcom}_A^{r \rightsquigarrow r'}(M; \xi \hookrightarrow y.N) \mid \text{fcom}^{r \rightsquigarrow r'}(M; \xi \hookrightarrow y.N)$ $\text{com}_{x.A}^{r \rightsquigarrow r'}(M; \xi \hookrightarrow y.N) \mid \dots$	(terms)
$K$	$::= \text{coe}_{\square}^{r \rightsquigarrow r'}(M) \mid \square(N) \mid \text{fst}(\square) \mid \text{snd}(\square) \mid \square @ r \mid \text{hcom}_{\square}^{r \rightsquigarrow r'}(M; \xi \hookrightarrow y.N) \mid \dots$	(continuations)
$C$	$::= M \triangleleft \mathcal{E}$	(closures)
$\mathcal{E}$	$::= (\emptyset, \gamma, \psi)$	(environments)
$\emptyset$	$::= \mathbb{X} \hookrightarrow \vec{x}; \vec{a}.M \triangleleft \mathcal{E}$	(meta env.)
$\gamma$	$::= a \hookrightarrow C$	(object env.)
$\psi$	$::= x \hookrightarrow r$	(dimension env.)
$f$	$::= (K, \mathcal{E}, \Psi)$	(frames)
$\pi$	$::= \cdot \mid f :: \pi$	(stacks)
$\mathcal{C}$	$::= \langle M \triangleleft \mathcal{E} \parallel \pi \rangle$	(states)

This machine is similar to both Krivine's environment machine and the CEK machine. To be honest, Jon Sterling is not really sure of what delineates these from each other, except that they usually seem to be call-by-name and call-by-value respectively. Ours is call-by-name.

**Stack frames and machine states** A stack frame  $(K, \mathcal{E}, \Psi)$  represents a continuation whose hole ranges over a binder of  $\Psi$  dimensions. For a stack  $\pi$ , let  $\vec{\Psi}$  be the aggregation of all the dimension bindings  $\Psi$  mentioned in  $\pi$ ; then the extension of a stack  $(K, \mathcal{E}, \Phi) :: \pi$  is wellformed when  $\mathcal{E}$  is an environment for  $\mathbf{FD}(K) \setminus \vec{\Psi}$ ; moreover,  $K$  is allowed to mention the dimensions in  $\vec{\Psi}$ .

A machine state  $\langle M \triangleleft \mathcal{E} \parallel \pi \rangle$ , where  $\vec{\Psi}$  is the aggregation of dimension bindings in  $\pi$ , is wellformed when  $\mathcal{E}$  is an environment for  $\mathbf{FD}(M) \setminus \vec{\Psi}$ ; moreover,  $M$  is allowed to mention the dimensions in  $\vec{\Psi}$ .

## 2 Selected transition rules

We define a transition judgment  $\mathcal{C} \succ_{\mu} \mathcal{C}'$  with  $\mu$  ranging over the following possible modes:

1.  $\boxtimes$ , denoting a cubically stable transition
2.  $\bullet$ , denoting a non-cubically stable transition

Moreover, if  $\mathcal{C} \succ_{\boxtimes} \mathcal{C}'$  then  $\mathcal{C} \succ_{\bullet} \mathcal{C}'$ .

### 2.1 Variables

$$\frac{\mathcal{E}(a) \equiv M \triangleleft \mathcal{E}'}{\langle a \triangleleft \mathcal{E} \parallel \pi \rangle \succ_{\boxtimes} \langle M \triangleleft \mathcal{E}' \parallel \pi \rangle} \quad \frac{\mathcal{E}(\mathbb{X}) \equiv \vec{x}; \vec{a}.N \triangleleft \mathcal{E}' \quad \mathcal{E}'' \triangleq [\mathcal{E}', a \hookrightarrow M \triangleleft \mathcal{E}, x \hookrightarrow \mathcal{E}(r)]}{\langle \mathbb{X}[\vec{r}; \vec{M}] \triangleleft \mathcal{E} \parallel \pi \rangle \succ_{\boxtimes} \langle N \triangleleft \mathcal{E}'' \parallel \pi \rangle}$$

## 2.2 Kan operations

$$\begin{array}{c}
\overline{\langle \text{hcom}_A^{r \rightsquigarrow r'}(M; \bar{\xi} \hookrightarrow y.\bar{N}) \triangleleft \mathcal{E} \parallel \pi \rangle \succ_{\square} \langle A \triangleleft \mathcal{E} \parallel (\text{hcom}_{\square}^{r \rightsquigarrow r'}(M; \bar{\xi} \hookrightarrow y.\bar{N}), \mathcal{E}, [] :: \pi \rangle} \\
\overline{\langle \text{coe}_{x.A}^{r \rightsquigarrow r'}(M) \triangleleft \mathcal{E} \parallel \pi \rangle \succ_{\square} \langle A \triangleleft \mathcal{E} \parallel (\text{coe}_{\square}^{r \rightsquigarrow r'}(M), \mathcal{E}', [x]) :: \pi \rangle} \\
\overline{\langle \text{com}_{x.A}^{r \rightsquigarrow r'}(M; \bar{\xi}_i \hookrightarrow y.N_i) \triangleleft \mathcal{E} \parallel \pi \rangle \succ_{\square} \langle \text{hcom}_A^{r \rightsquigarrow r'}(\text{coe}_{y.A}^{r \rightsquigarrow r'}(M); \bar{\xi}_i \hookrightarrow y.\text{coe}_{y.A}^{y \rightsquigarrow r'}(N_i)) \triangleleft [\mathcal{E}, x \hookrightarrow r'] \parallel \pi \rangle} \\
\overline{\mathcal{E} \models r = r'} \\
\overline{\langle \text{fcom}^{r \rightsquigarrow r'}(M; \bar{\xi} \hookrightarrow y.\bar{N}) \triangleleft \mathcal{E} \parallel \pi \rangle \succ_{\square} \langle M \triangleleft \mathcal{E} \parallel \pi \rangle} \\
\overline{\mathcal{E} \models r \# r' \quad \mathcal{E} \models r_i \# r'_i \ (\forall i < j) \quad \mathcal{E} \models r_j = r'_j} \\
\overline{\langle \text{fcom}^{r \rightsquigarrow r'}(M; r_i = r'_i \hookrightarrow y.N_i) \triangleleft \mathcal{E} \parallel \pi \rangle \succ_{\bullet} \langle N_j \triangleleft [\mathcal{E}, y \hookrightarrow \mathcal{E}(r')] \parallel \pi \rangle}
\end{array}$$

## 2.3 Dependent function types

$$\begin{array}{c}
\overline{\langle M(N) \triangleleft \mathcal{E} \parallel \pi \rangle \succ_{\square} \langle M \triangleleft \mathcal{E} \parallel (\square(N), \mathcal{E}, [] :: \pi \rangle} \\
\overline{\langle \lambda a.M \triangleleft \mathcal{E} \parallel (\square(N), \mathcal{E}', [] :: \pi \rangle \succ_{\square} \langle M \triangleleft [\mathcal{E}, a \hookrightarrow N \triangleleft \mathcal{E}'] \parallel \pi \rangle} \\
\overline{\mathcal{E}'' \triangleq [\mathcal{E}', x \hookrightarrow a.B \triangleleft \mathcal{E}]} \\
\overline{\langle (x : A) \rightarrow B \triangleleft \mathcal{E} \parallel (\text{hcom}_{\square}^{r \rightsquigarrow r'}(M; \bar{\xi}_i \hookrightarrow y.N_i), \mathcal{E}', [] :: \pi \rangle \succ_{\square} \langle \lambda a.\text{hcom}_{x[a]}^{r \rightsquigarrow r'}(M(a); \bar{\xi}_i \hookrightarrow y.N_i(a)) \triangleleft \mathcal{E}'' \parallel \pi \rangle} \\
\overline{\mathcal{E}'' \triangleq [\mathcal{E}', x \hookrightarrow x; b.B \triangleleft [\mathcal{E}, a \hookrightarrow \text{coe}_{x.x[x]}^{r' \rightsquigarrow x}(b) \triangleleft [\mathcal{E}', z \hookrightarrow x.A \triangleleft \mathcal{E}]], y \hookrightarrow x.A \triangleleft \mathcal{E}]} \\
\overline{\langle (a : A) \rightarrow B \triangleleft \mathcal{E} \parallel (\text{coe}_{\square}^{r \rightsquigarrow r'}(M), \mathcal{E}', [x]) :: \pi \rangle \succ_{\square} \langle \lambda a.\text{coe}_{x.x[x;a]}^{r \rightsquigarrow r'}(M(\text{coe}_{x.y[x]}^{r' \rightsquigarrow r}(a))) \triangleleft \mathcal{E}'' \parallel \pi \rangle}
\end{array}$$

## 2.4 Dependent pair types

$$\begin{array}{c}
\overline{\langle \text{fst}(M) \triangleleft \mathcal{E} \parallel \pi \rangle \succ_{\square} \langle M \triangleleft \mathcal{E} \parallel (\text{fst}(\square), \mathcal{E}, [] :: \pi \rangle} \quad \overline{\langle \text{snd}(M) \triangleleft \mathcal{E} \parallel \pi \rangle \succ_{\square} \langle M \triangleleft \mathcal{E} \parallel (\text{snd}(\square), \mathcal{E}, [] :: \pi \rangle} \\
\overline{\langle \langle M, N \rangle \triangleleft \mathcal{E} \parallel (\text{fst}(\square), \mathcal{E}', [] :: \pi \rangle \succ_{\square} \langle M \triangleleft \mathcal{E} \parallel \pi \rangle} \quad \overline{\langle \langle M, N \rangle \triangleleft \mathcal{E} \parallel (\text{snd}(\square), \mathcal{E}', [] :: \pi \rangle \succ_{\square} \langle N \triangleleft \mathcal{E} \parallel \pi \rangle} \\
\overline{L_s \triangleq \text{hcom}_{x}^{r \rightsquigarrow s}(\text{fst}(M); \bar{\xi}_i \hookrightarrow y.\text{fst}(N_i)) \quad R \triangleq \text{com}_{z.y[z]}^{r \rightsquigarrow r'}(\text{snd}(M); \bar{\xi}_i \hookrightarrow y.\text{snd}(N_i))} \\
\overline{\mathcal{E}'' \triangleq [\mathcal{E}', x \hookrightarrow .A \triangleleft \mathcal{E}] \quad \mathcal{E}''' \triangleq [\mathcal{E}'', y \hookrightarrow z.B \triangleleft [\mathcal{E}, a \hookrightarrow L_z \triangleleft \mathcal{E}''']] \\
\overline{\langle (a : A) \times B \triangleleft \mathcal{E} \parallel (\text{hcom}_{\square}^{r \rightsquigarrow r'}(M; \bar{\xi}_i \hookrightarrow y.N_i), \mathcal{E}', [] :: \pi \rangle \succ_{\square} \langle \langle L_{r'}, R \rangle \triangleleft \mathcal{E}''' \parallel \pi \rangle} \\
\overline{\mathcal{E}'' \triangleq [\mathcal{E}', x \hookrightarrow x.A \triangleleft \mathcal{E}, y \hookrightarrow x.B \triangleleft [\mathcal{E}, a \hookrightarrow \text{coe}_{y.x[y]}^{r \rightsquigarrow x}(\text{snd}(M)) \triangleleft [\mathcal{E}, x \hookrightarrow x.A \triangleleft \mathcal{E}]]} \\
\overline{\langle (a : A) \times B \triangleleft \mathcal{E} \parallel (\text{coe}_{\square}^{r \rightsquigarrow r'}(M), \mathcal{E}', [x]) :: \pi \rangle \succ_{\square} \langle \langle \text{coe}_{x.x[x]}^{r \rightsquigarrow r'}(\text{fst}(M)), \text{coe}_{x.y[x]}^{r \rightsquigarrow r'}(\text{snd}(M)) \rangle \triangleleft \mathcal{E}'' \parallel \pi \rangle}
\end{array}$$

## 2.5 Path types

$$\begin{array}{c}
\overline{\langle M @ r \triangleleft \mathcal{E} \parallel \pi \rangle \succ_{\square} \langle M \triangleleft \mathcal{E} \parallel (\square @ r, \mathcal{E}, []) :: \pi \rangle} \\
\\
\overline{\langle \langle x \rangle M \triangleleft \mathcal{E} \parallel (\square @ r, \mathcal{E}', []) :: \pi \rangle \succ_{\square} \langle M \triangleleft [\mathcal{E}, x \hookrightarrow \mathcal{E}'(r)] \parallel \pi \rangle} \\
\\
\frac{H \triangleq \text{hcom}_{\mathbb{Z}[x]}^{r \rightsquigarrow r'}(M @ x; \overline{x = \epsilon \hookrightarrow y.\overline{y}_\epsilon}, \overline{\xi_i \hookrightarrow y.N_i @ x}) \quad \mathcal{E}'' \triangleq [\mathcal{E}', \mathbb{Z} \hookrightarrow x.A \triangleleft \mathcal{E}, \overline{y_\epsilon \hookrightarrow .P_\epsilon \triangleleft \mathcal{E}}]}{\langle \text{path}_{x.A}(P_0; P_1) \triangleleft \mathcal{E} \parallel (\text{hcom}_{\square}^{r \rightsquigarrow r'}(M; \overline{\xi_i \hookrightarrow y.N_i}), \mathcal{E}', []) :: \pi \rangle \succ_{\square} \langle \langle x \rangle H \triangleleft \mathcal{E}'' \parallel \pi \rangle} \\
\\
\frac{\mathcal{E}'' \triangleq [\mathcal{E}', \mathbb{Z} \hookrightarrow x.A \triangleleft \mathcal{E}, \overline{y_\epsilon \hookrightarrow .P_\epsilon \triangleleft \mathcal{E}}]}{\langle \text{path}_{x.A}(P_0; P_1) \triangleleft \mathcal{E} \parallel (\text{coe}_{\square}^{r \rightsquigarrow r'}(M), \mathcal{E}', [x]) :: \pi \rangle \succ_{\square} \langle \langle x \rangle \text{com}_{y.\mathbb{Z}[y]}^{r \rightsquigarrow r'}(M @ x; \overline{x = \epsilon \hookrightarrow y.\overline{y}_\epsilon}) \triangleleft \mathcal{E}'' \parallel \pi \rangle}
\end{array}$$

## 2.6 Natural numbers

TODO

## 2.7 Booleans

TODO, add if, etc.

$$\begin{array}{c}
\overline{\langle \text{bool} \triangleleft \mathcal{E} \parallel (\text{coe}_{\square}^{r \rightsquigarrow r'}(M), \mathcal{E}', [x]) :: \pi \rangle \succ_{\square} \langle M \triangleleft \mathcal{E}' \parallel \pi \rangle} \\
\\
\overline{\langle \text{bool} \triangleleft \mathcal{E} \parallel (\text{hcom}_{\square}^{r \rightsquigarrow r'}(M; \overline{\xi \hookrightarrow y.N}), \mathcal{E}', []) :: \pi \rangle \succ_{\square} \langle M \triangleleft \mathcal{E}' \parallel \pi \rangle}
\end{array}$$

## 2.8 Weak booleans

TODO, add hcom, if, if-fcom, etc.

$$\overline{\langle \text{wbool} \triangleleft \mathcal{E} \parallel (\text{hcom}_{\square}^{r \rightsquigarrow r'}(M; \overline{\xi \hookrightarrow y.N}), \mathcal{E}', []) :: \pi \rangle \succ_{\square} \langle \text{fcom}^{r \rightsquigarrow r'}(M; \overline{\xi \hookrightarrow y.N}) \triangleleft \mathcal{E}' \parallel \pi \rangle}$$

## 2.9 Circle

TODO, add hcom, coe, rec-fcom, etc.

$$\begin{array}{c}
\frac{\mathcal{E} \models r = \epsilon}{\langle \text{loop}_r \triangleleft \mathcal{E} \parallel \pi \rangle \succ_{\square} \langle \text{base} \triangleleft \mathcal{E} \parallel \pi \rangle} \\
\\
\overline{\langle \mathbb{S}^1 \text{rec}_{a.A}(M; P; x.L) \triangleleft \mathcal{E} \parallel \pi \rangle \succ_{\square} \langle M \triangleleft \mathcal{E} \parallel (\mathbb{S}^1 \text{rec}_{a.A}(\square; P; x.L), \mathcal{E}, []) :: \pi \rangle} \\
\\
\overline{\langle \text{base} \triangleleft \mathcal{E} \parallel (\mathbb{S}^1 \text{rec}_{a.A}(\square; P; x.L), \mathcal{E}', []) :: \pi \rangle \succ_{\square} \langle P \triangleleft \mathcal{E}' \parallel \pi \rangle} \\
\\
\frac{\mathcal{E} \models r = w}{\langle \text{loop}_r \triangleleft \mathcal{E} \parallel (\mathbb{S}^1 \text{rec}_{a.A}(\square; P; x.L), \mathcal{E}', []) :: \pi \rangle \succ_{\bullet} \langle L \triangleleft [\mathcal{E}', x \hookrightarrow w] \parallel \pi \rangle}
\end{array}$$

**Unloading the machine** We can unload the machine at any time; this is useful if we are computing an open term and hit a variable.

$$\frac{}{\langle M \triangleleft \mathcal{E} \parallel \cdot \rangle \Longrightarrow M[\mathcal{E}]} \qquad \frac{\langle K[\Psi.M[\mathcal{E}]] \triangleleft \mathcal{E}' \parallel \pi \rangle \Longrightarrow N}{\langle M \triangleleft \mathcal{E} \parallel (K[\square], \mathcal{E}', \Psi) :: \pi \rangle \Longrightarrow N}$$

But using Favonia's Thought, we can come up with something more efficient. This naïve version is rather appalling.