Problem Set 3, Written Part

Shortest Paths and Graph Theory

Due: 11:55pm Thursday, November 9, 2017

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1 Listing vertices of a negative cycle (10 marks)

You are given a connected weighted graph $G = (V, E)$. Describe an efficient algorithm that detects whether there is a negative cycle and, if there is a negative cycle, lists the vertices in the cycle. Be sure to explain why the algorithm is correct.

Hint: Start with the Bellman-Ford algorithm.

2 Dijkstra's algorithm and negative edge-weights (10 marks)

In class, we saw a proof of the correctness of Dijkstra's algorithm; the proof can be found in the lecture slides. Explain what part of the proof breaks when edge-weights can be negative.

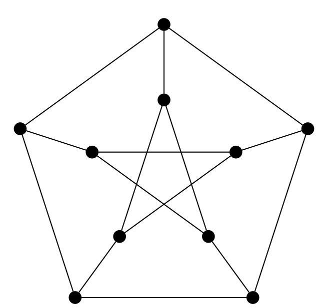
3 Eulerian circuits (10 marks)

A graph is an *even graph* if every vertex has even degree. Prove that every even graph decomposes into cycles. That is, the edges of the graph can be partitioned into sets, and each set corresponds to the edges of a cycle of G. Note that the graph is not necessarily simple, so it can have multiple edges and loops.

4 Graph coloring (10 marks)

(a) Let G be a graph with 5 (non-empty) independent sets, $V_1, V_2, \ldots, V_5$. Suppose that the subgraph induced by taking the vertices of any pair (V_i, V_j) of the independent sets is a complete bipartite graph. What is the chromatic number of G?

(b) Is the graph below 2-colorable? If so, show a proper 2-coloring. If not, explain why not, and show a proper k-coloring for the smallest possible value of k.



* Optional extra credit question: Finding a small coloring *

Suppose you have access to an oracle which, given a graph G, returns the largest independent set of G (in the form of a list of vertices).[1] Using calls to the aforementioned oracle, design an efficient algorithm for finding a $\log_{\chi(G)}(n)$-coloring of a graph G with n vertices, where $\chi(G)$ is the chromatic number of G.

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[1] If there are multiple largest independent sets, it returns an arbitrary one.