Automated Bijections with Combinatorial Exploration

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Outline

Introduction

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- Combinatorics
  - Discrete structures
  - Enumerative combinatorics
  - Permutation patterns
    - First enumeration MacMahon [1]
    - Surge of interest Knuth [2]
    - First bijection Simion and Schmidt [3]

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  - Earliest enumeration example Zeilberger [4]
  - Combinatorial Exploration Bean [5]
  - The translation method Wood and Zeilberger [6]

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  - The translation method Wood and Zeilberger [6]
- My work
  - Use Combinatorial Exploration to find bijections
  - 189 bijections found

Background

Combinatorial class

Definition

A *combinatorial class* is a set $\mathcal{C}$ and a size function $\mathcal{C} \mapsto \mathbb{N} = \{0, 1, 2, \dots\}$ such that

$$\mathcal{C}_n = \{c \in \mathcal{C} \mid \text{size of } c \text{ is } n\}$$

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Example

The set of binary strings B where $|B_n| = 2^n$.

$$B_3 = \{000, 001, 010, 100, 011, 101, 110, 111\}.$$

Bijection

Definition

A bijection between two sets A and B is an invertible function $f : A \mapsto B$, i.e., there exists an inverse $f^{-1} : B \mapsto A$ such that $a = f^{-1}(f(a))$ and $b = f\left(f^{-1}(b)\right)$ for all $a \in A$ and $b \in B$.

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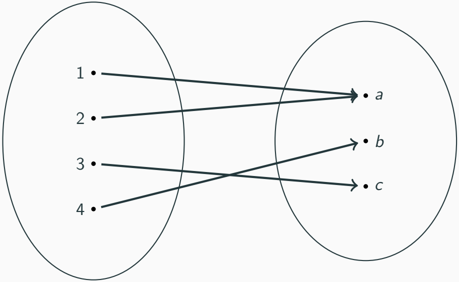

Figure 1: A mapping that is not a bijection as two elements map to the same one.

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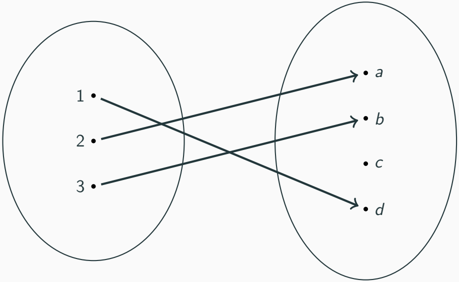

Figure 2: A mapping that is not a bijection as there is an element in the codomain that no element maps to.

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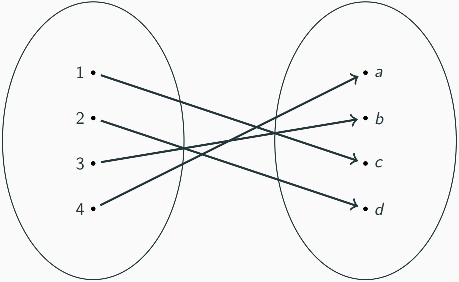

Figure 3: A bijection.

Size preserving bijections

Definition

A bijection f between two combinatorial classes $\mathcal{C}$ and $\mathcal{D}$ is *size preserving* if $|c| = |f(c)|$ for all $c \in \mathcal{C}$.

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Example

Flipping bits in binary strings.

$$\begin{aligned} 0 &\mapsto 1 \\ 0101 &\mapsto 1010 \end{aligned}$$

Counting sequence, generating function and isomorphism

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The *counting sequence* of a combinatorial class $\mathcal{C}$ is the infinite sequence

$$|\mathcal{C}_0|, |\mathcal{C}_1|, |\mathcal{C}_2|, \ldots$$

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The *generating function* of a combinatorial class $\mathcal{C}$ is the power series

$$\sum_{i=0}^{\infty} |\mathcal{C}_i| z^i.$$

Symbolic method

The *symbolic method* in Flajolet and Sedgewick [7] describes how combinatorial classes can be constructed.

- *Combinatorial specifications*
- *Constructors*
- *Atoms*
- *Combinatorial rules*[1]

[1] Not a terminology from Flajolet and Sedgewick [7]

Symbolic method - example

$$(0 \mid 10)* (1 \mid \varepsilon)$$

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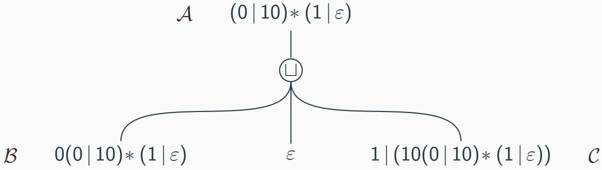
$$\sqcup$$

$$0(0 \mid 10)* (1 \mid \varepsilon) \qquad \varepsilon \qquad 1 \mid (10(0 \mid 10)* (1 \mid \varepsilon))$$

Symbolic method - example



$\mathcal{A} \cong \mathcal{B} \sqcup \{\varepsilon\} \sqcup \mathcal{C}$

$A(z) = 1 + B(z) + C(z)$

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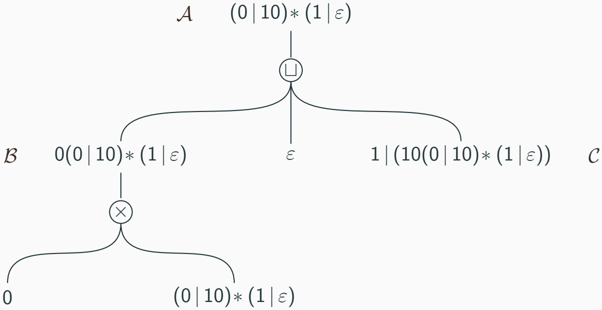

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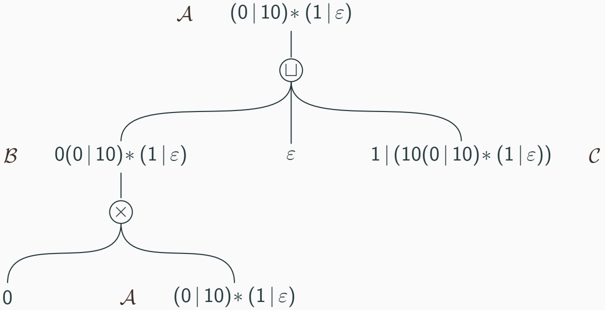

$$\mathcal{A} \cong \mathcal{B} \sqcup \{\varepsilon\} \sqcup \mathcal{C}$$
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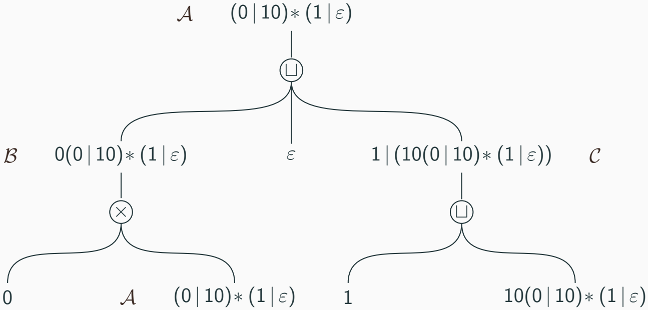
$$A(z) = 1 + B(z) + C(z)$$
$$B(z) = zA(z)$$

Symbolic method - example



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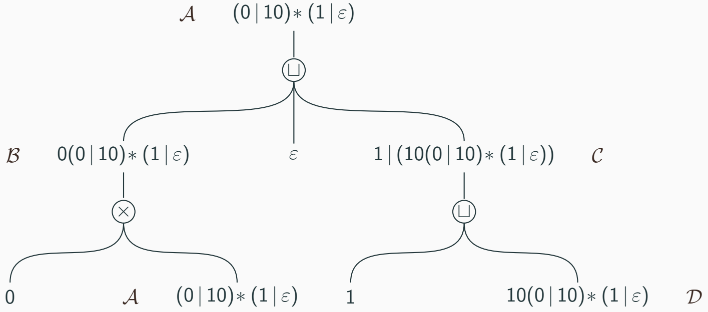

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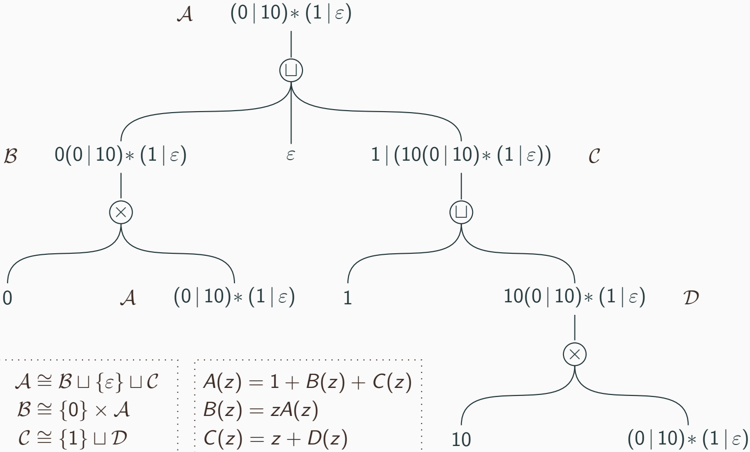
Symbolic method - example



$$\begin{aligned} \mathcal{A} &\cong \mathcal{B} \sqcup \{\varepsilon\} \sqcup \mathcal{C} \\ \mathcal{B} &\cong \{0\} \times \mathcal{A} \\ \mathcal{C} &\cong \{1\} \sqcup \mathcal{D} \end{aligned}$$

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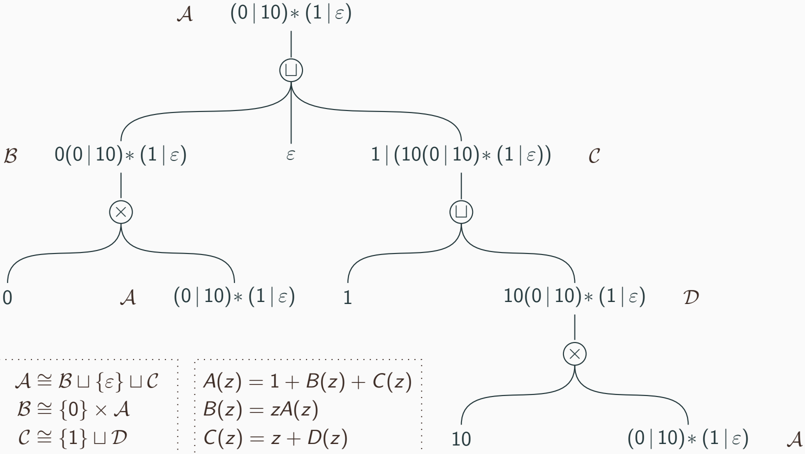

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\mathcal{D} &\cong \{10\} \times \mathcal{A}
\end{aligned}$$

$$\begin{aligned}
A(z) &= 1 + B(z) + C(z) \\
B(z) &= zA(z) \\
C(z) &= z + D(z) \\
D(z) &= z^2 A(z)
\end{aligned}$$

Figure 4: A specification for binary strings avoiding repeated 1's.

Symbolic method - example

We now have a system of equations

$$\begin{cases} A(z) = 1 + B(z) + C(z) \\ B(z) = \qquad zA(z) \\ C(z) = \qquad z + D(z) \\ D(z) = \qquad z^2 A(z) \end{cases}$$

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Taylor series at $z = 0$ is

$$A(z) = 1 + 2z + 3z^2 + 5z^3 + 8z^4 + 13z^5 + 21z^6 + 34z^7 + 55z^8 + \cdots$$

Combinatorial Exploration

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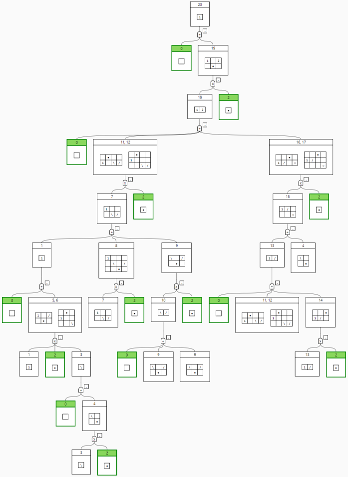

Figure 5: Specification found by Combinatorial Exploration.

Permutation

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Example

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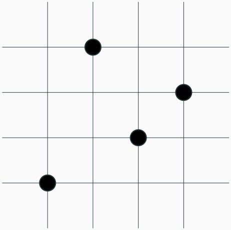

Figure 6: The graphical representation of $\pi = 1423$.

Permutation pattern

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A permutation π *contains* another permutation σ if a subsequence of π has the same relative order as σ, denoted $\sigma \preceq \pi$.

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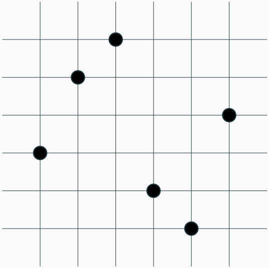

Figure 7: The permutation 356214.

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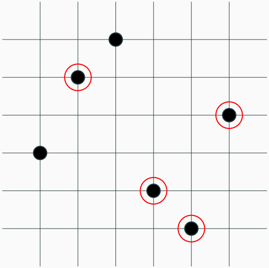

Figure 8: An occurrence of 4213 in 356214.

Permutation class

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The set of permutations

$$\mathrm{Av}(\Pi) = \{\pi \mid \pi \text{ avoids } \Pi\}$$

is a *permutation class*.

Cell

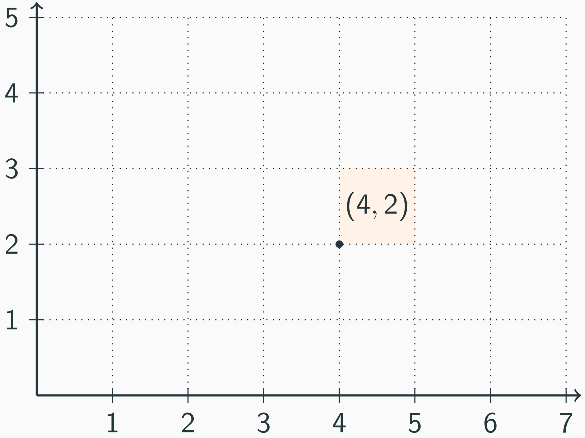

Figure 9: A cell $(c, r) \in \mathbb{N}^2$ defines the region $[c, c+1) \times [r, r+1)$.

Gridded permutation

Definition **(Albert, Atkinson, Bouvel, *et al.* [8])**

A pair (π, P) where

- $\pi = \pi_1 \pi_2 \cdots \pi_n$ is a permutation
- $P = ((c_1, r_1), (c_2, r_2), \ldots, (c_n, r_n))$ is a n-tuple of cells

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is called a *gridded permutation* if

- $i < j$ implies $c_i \leq c_j$
- $\pi_i < \pi_j$ implies $r_i \leq r_j$

for all $1 \leq i, j \leq n$.

Gridded permutation - example

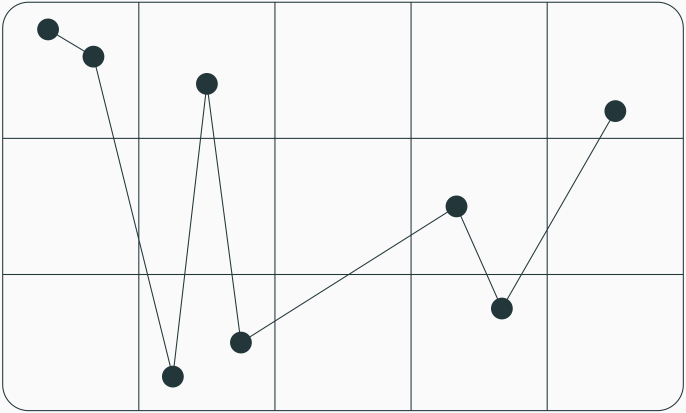

Figure 10: The gridded permutation with $\pi = 87162435$ and $P = ((0,2),(0,2),(1,0),(1,2),(1,0),(3,1),(3,0),(4,2))$.

Gridded permutation - example

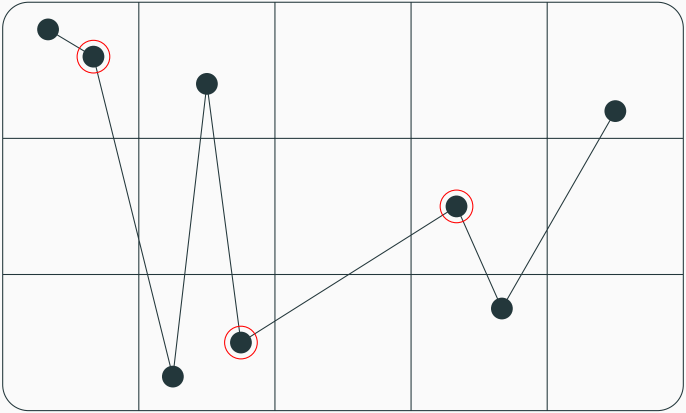

Figure 11: An occurrence of $3^{(0,2)}1^{(1,0)}2^{(3,1)}$ in the gridded permutation $87^{(0,2)}1^{(1,0)}6^{(1,2)}2^{(1,0)}4^{(3,1)}3^{(3,0)}5^{(4,2)}$.

Tiling

Definition **(Bean [5])**

A *tiling* is a triple $\mathcal{T} = ((c, r), \mathcal{O}, \mathcal{R})$ where

- $(c, r) \in \mathbb{Z}^+ \times \mathbb{Z}^+$ is called *dimension*
- $\mathcal{O} \subseteq \mathcal{G}^{(c,r)}$ is called *obstructions*
- $\mathcal{R} = \{\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_k\} \subseteq \left(\mathcal{G}^{(c,r)}\right)^k$ is called *requirements*

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The gridded permutations in $\mathrm{Grid}(\mathcal{T})$ are those in $\mathcal{G}^{(c,r)}$ that

- avoid $\mathcal{O}$
- contain $\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_k$

Tiling - example

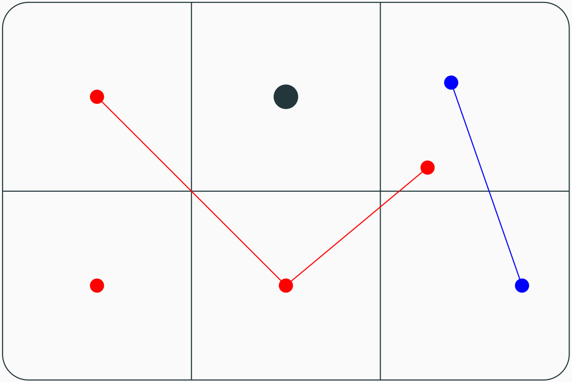

Figure 12: A tiling with $(c, r) = (3, 2)$, $\mathcal{R} = \left\{\left\{1^{(1,1)}\right\}, \left\{2^{(1,1)}1^{(2,0)}\right\}\right\}$ and $\mathcal{O} = \left\{1^{(0,0)}, 12^{(1,1)}, 21^{(1,1)}, 3^{(0,1)}1^{(1,0)}2^{(2,1)}\right\}$.

Parallel Specifications

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Two specifications $\check{\mathcal{C}}$ and $\check{\mathcal{D}}$ are *parallel* if the empty rooted paths in their respective specification graphs are matchable.

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Two specifications are parallel if (recursively from the root) every class pair $\mathcal{C}$ and $\mathcal{D}$

- Both contain a single object which are equal in size.
- There is a recursion at an equal distance to an ancestor for both.
- Their constructors are equivalent and there is a bijection between their children such that we can match them this way.

Special attention to equivalence rules, $\mathcal{C}^{(1)} \cong \mathcal{C}^{(2)}$.

Parallel specifications - example

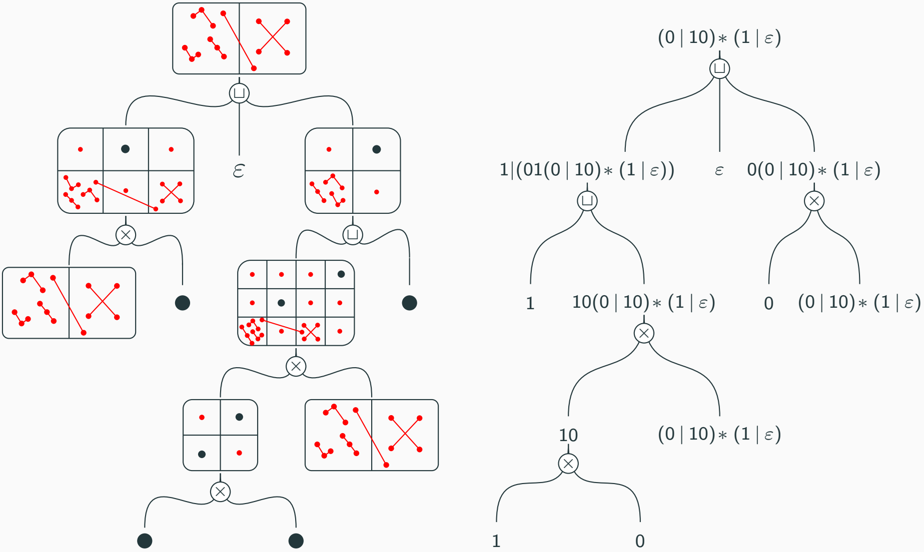
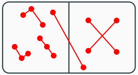


Figure 13: Two parallel specifications.

Parallel specifications - example



$(0 \mid 10)* \,(1 \mid \varepsilon)$

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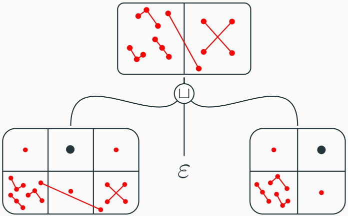

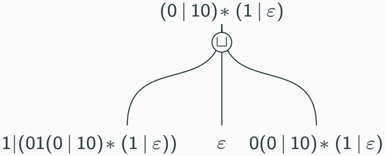

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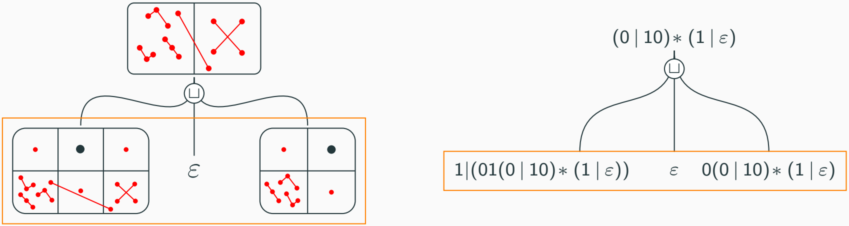

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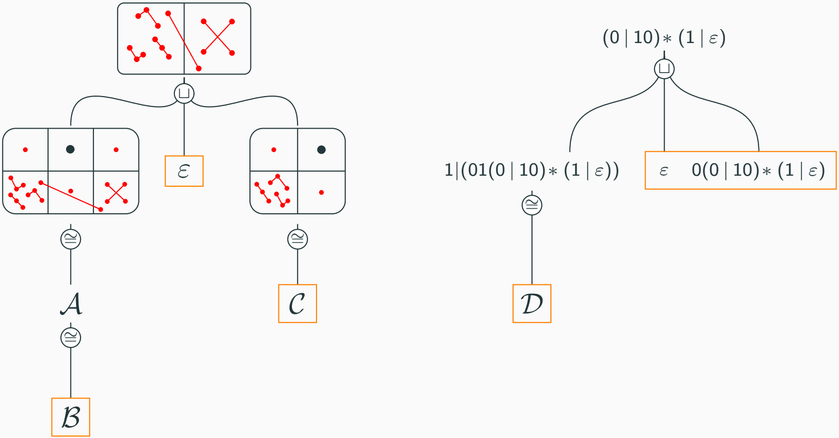

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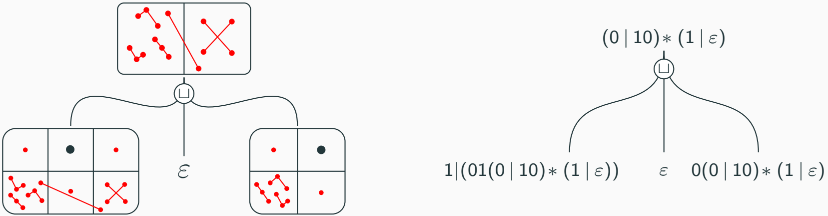

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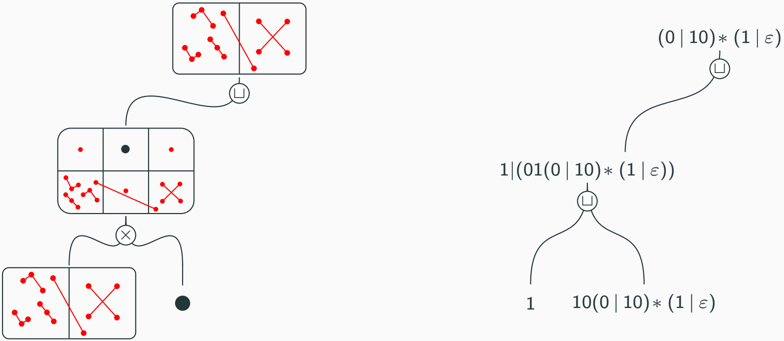

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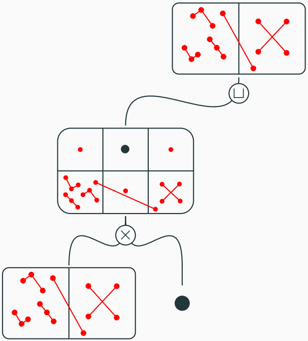

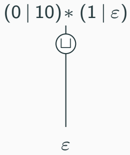

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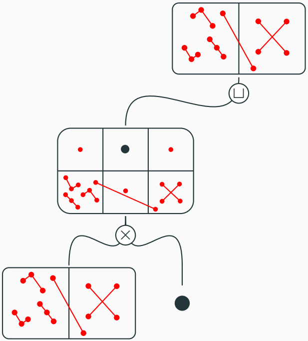

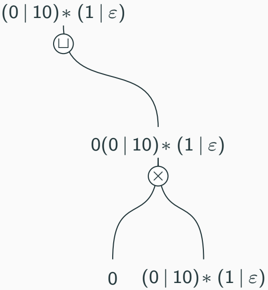

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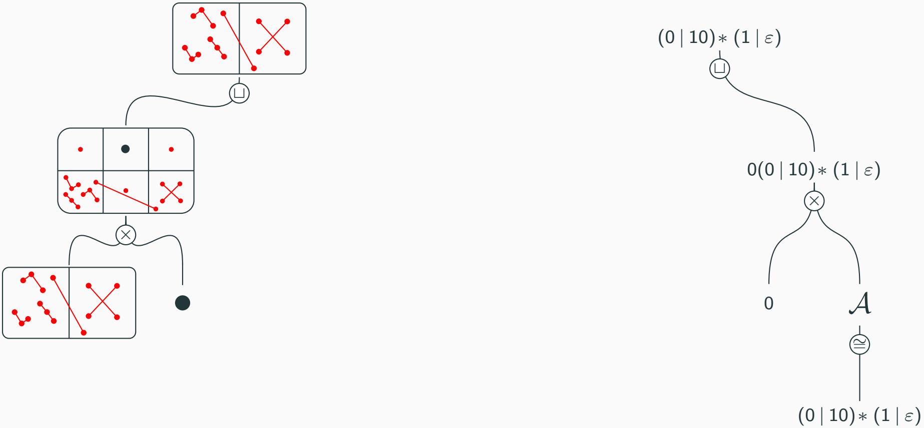

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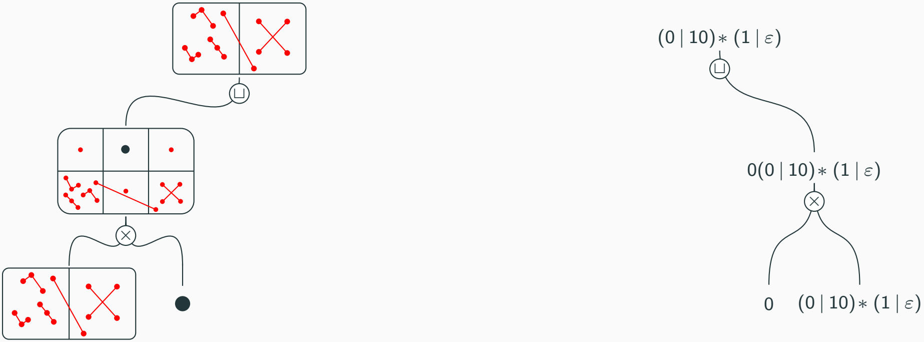

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Parallel specifications - example

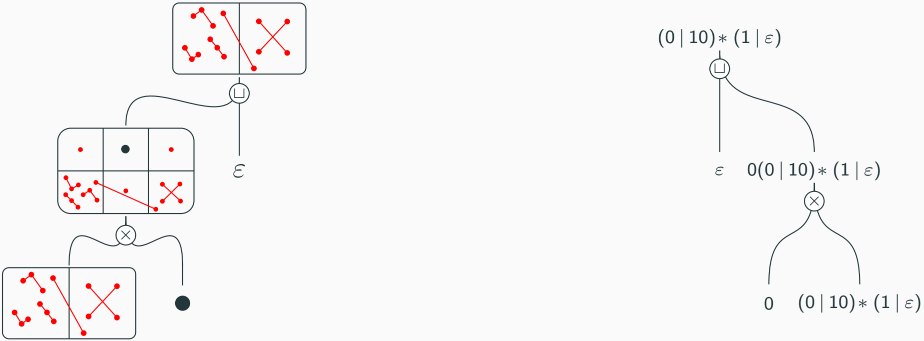


Figure 13: Two parallel specifications.

Parallel specifications - example

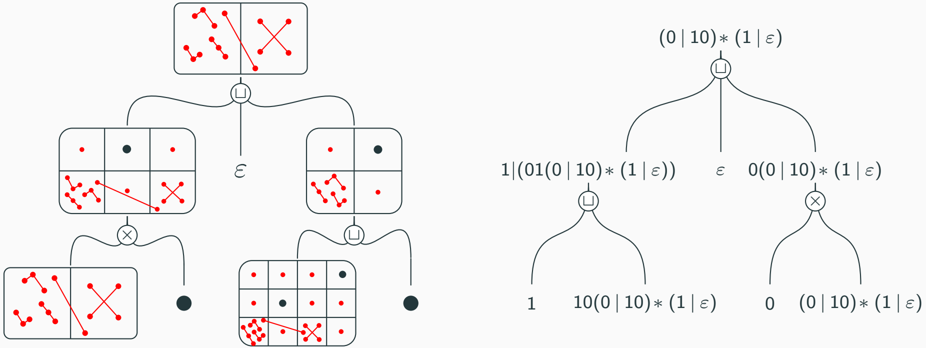
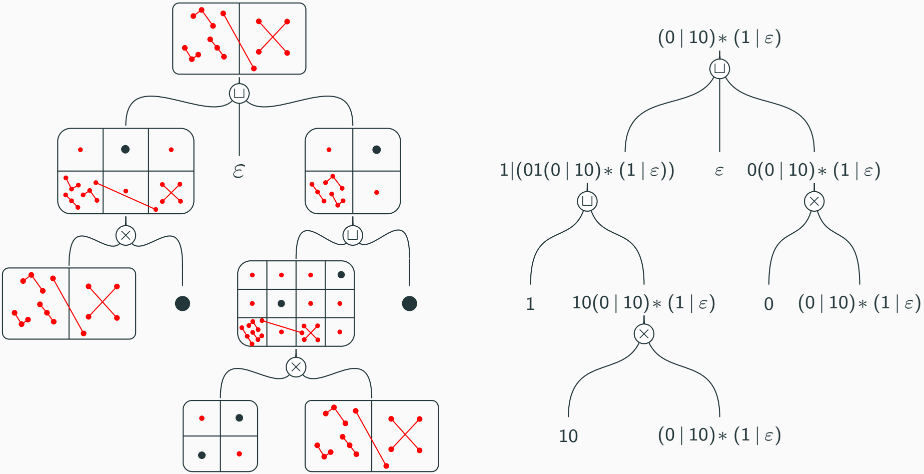


Figure 13: Two parallel specifications.

Parallel specifications - example



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Parallel specifications - example

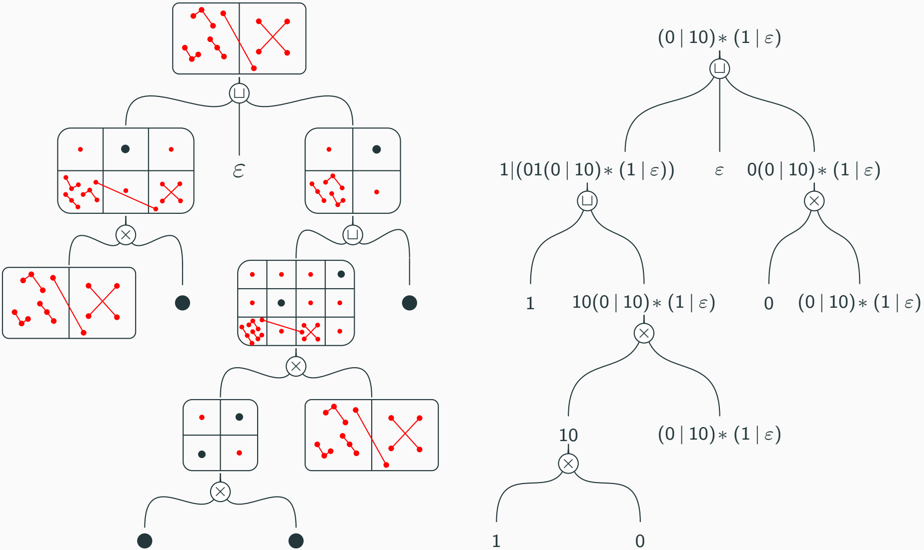


Figure 13: Two parallel specifications.

Parallel Bijections

Parallel bijection

Definition

If $\check{\mathcal{C}}$ and $\check{\mathcal{D}}$ are two parallel specifications with path matching $\gamma_{\check{\mathcal{C}},\check{\mathcal{D}}}$ and root classes $\mathcal{C}$ and $\mathcal{D}$, then their *parallel map* is $\mathfrak{P} : \mathcal{C} \mapsto \mathcal{D}$ where, for any $c \in \mathcal{C}$, we have

$$\mathfrak{P}(c) = \omega_{\check{\mathcal{D}}}^{-1}\left(\left\{\gamma_{\check{\mathcal{C}},\check{\mathcal{D}}}(p) \mid p \in \omega_{\check{\mathcal{C}}}(c)\right\}\right).$$

Parallel bijection - An example for Av (123) and Av (132)

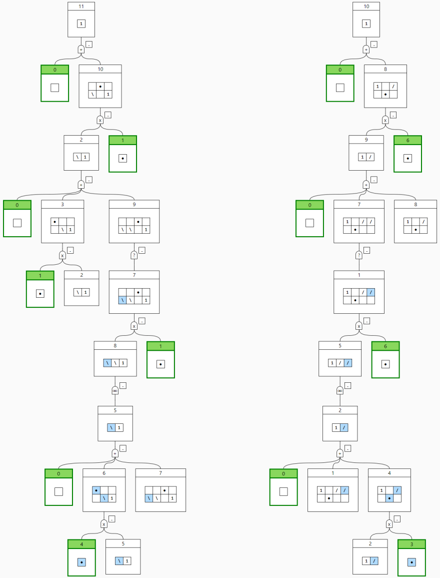


Figure 14: Parallel specifications for Av (123) and Av (132).

Parallel bijection - An example for **Av** (123) and **Av** (132)

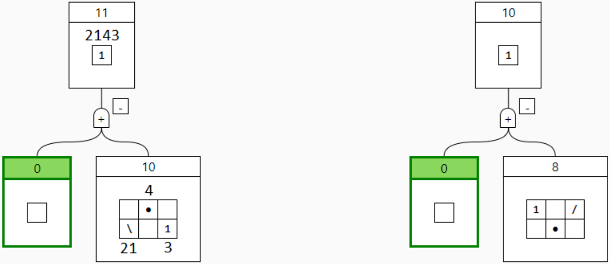


Figure 15: Step 1 - Place topmost point of 2143 in Av (123).

Parallel bijection - An example for Av(123) and Av(132)

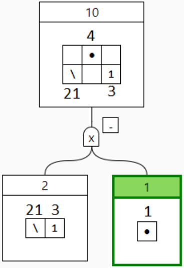


Figure 16: Step 2 - Factor.

Parallel bijection - An example for Av(123) and Av(132)

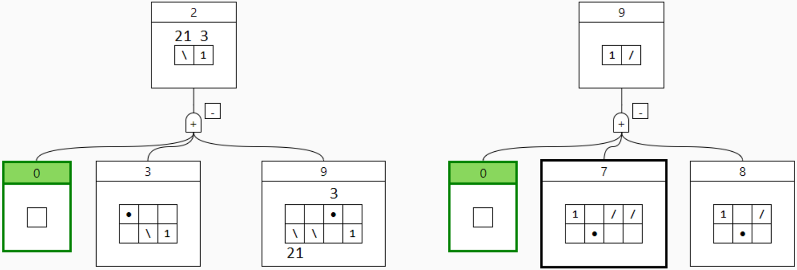


Figure 17: Step 3 - Place topmost in row.

Parallel bijection - An example for Av (123) and Av (132)

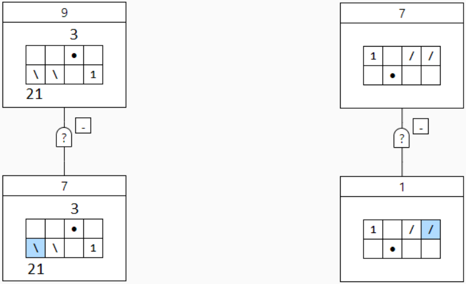


Figure 18: Step 4 - Add assumption in $(0, 0)$.

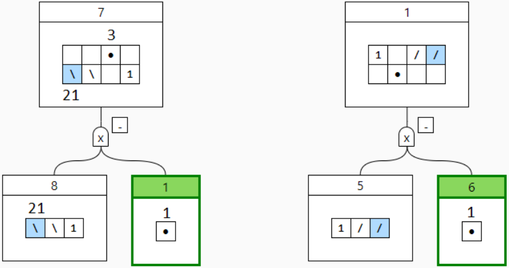


Figure 19: Step 5 - Factor.

Parallel bijection - An example for Av(123) and Av(132)

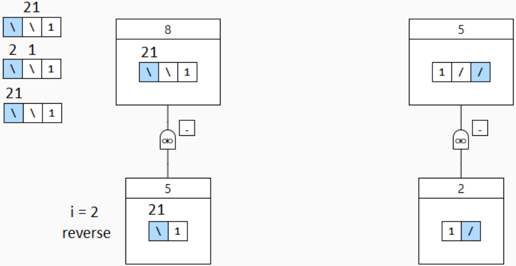


Figure 20: Step 6 - Fuse columns 0 and 1.

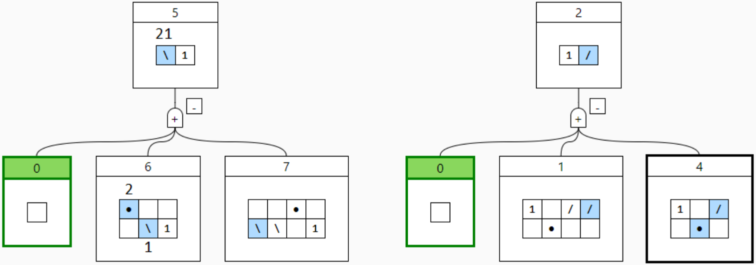


Figure 21: Step 7 - Place topmost in row.

Parallel bijection - An example for Av (123) and Av (132)

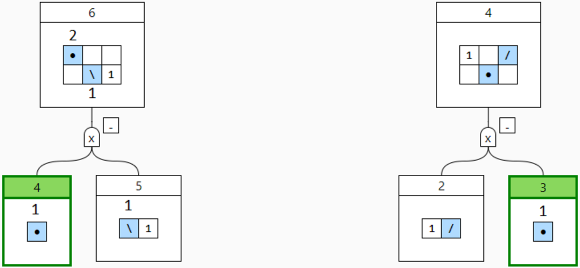


Figure 22: Step 8 - Factor.

Parallel bijection - An example for Av (123) and Av (132)

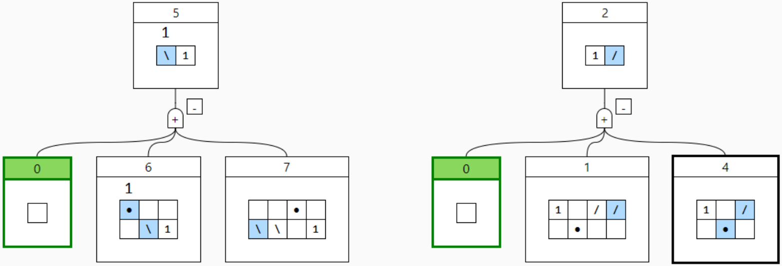


Figure 23: Step 9 - Place topmost in row.

Parallel bijection - An example for Av(123) and Av(132)

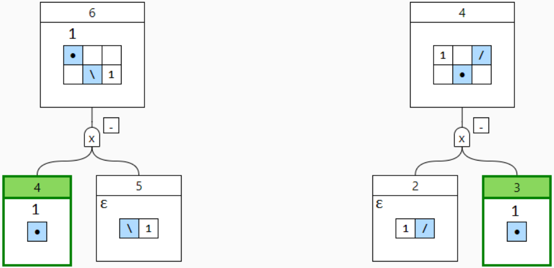


Figure 24: Step 10 - Factor.

Parallel bijection - An example for $\mathbf{Av}(123)$ and $\mathbf{Av}(132)$

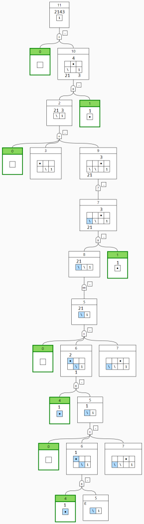


Figure 25: The parse trees we have created.

Parallel bijection - An example for Av(123) and Av(132)

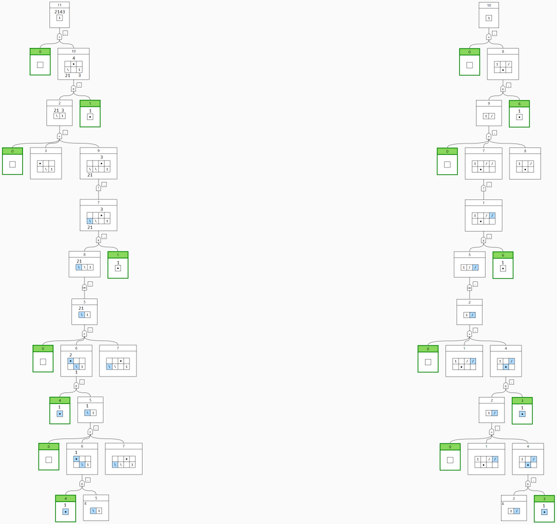


Figure 25: The parse trees we have created.

Parallel bijection - An example for Av(123) and Av(132)

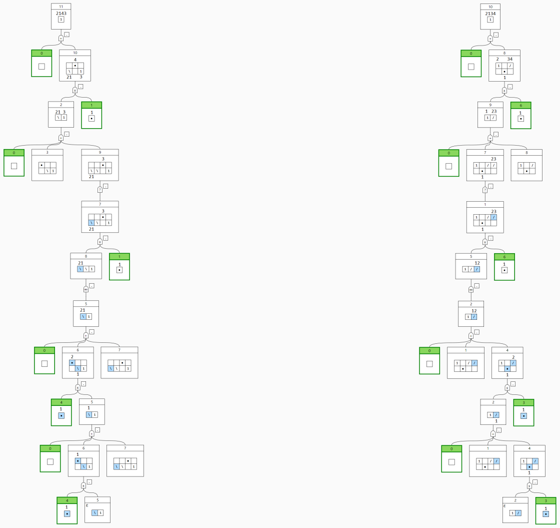
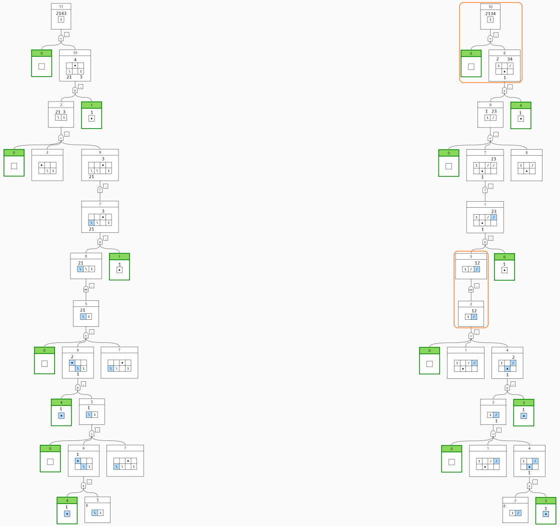


Figure 25: The parse trees we have created.

Parallel bijection - An example for Av(123) and Av(132)



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Parallel bijection - An example for Av (123) and Av (132)

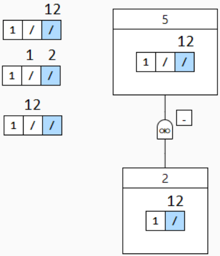

Figure 26: Backward step for fusion.

Parallel bijection - An example for Av (123) and Av (132)

Final step

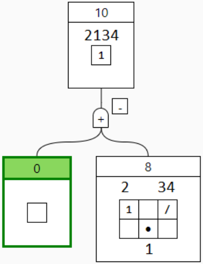

Figure 27: Final step.

Bijection Search

Bijection search

- Expands both universes until both contain specifications
- Gathers all the matched pairs
- Uses dynamic programming and backtracking
- A second tree search attempts to construct parallel specifications from matched pairs
- Detects false positives from recursion

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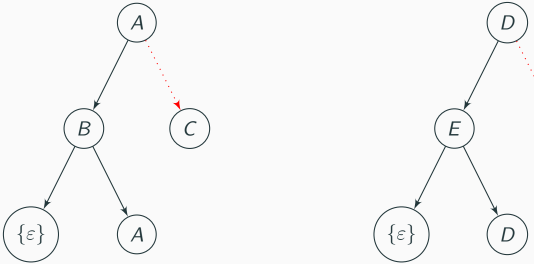

Figure 28: Classes B and E are incorrectly matched because of a recursion to A and D that we fail to matched later.

Results

Results

- Cross domain bijections
  - Words and permutations
- Known Wilf classes
- Experimental classes
  - 11×4
  - 10×4
  - 9×4 (except one class)

A bijection between words and permutations

- $\mathrm{Av}(231, 312, 321)$

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots$$

- Binary strings avoiding 11

$$1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots$$

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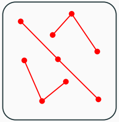

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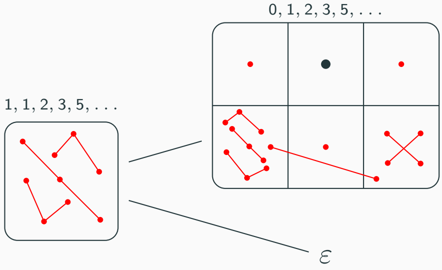

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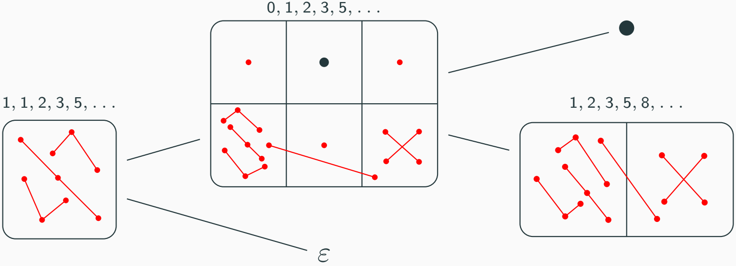

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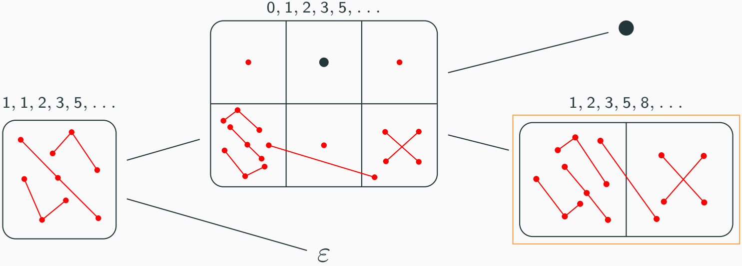

Figure 29: The class that is isomorphic to the binary strings.

Known Wilf classes

- Classes avoiding one, two or three patterns of size 3.
  - All classes in Simion and Schmidt [3].
  - The bijection we found for $\mathrm{Av}(123)$ and $\mathrm{Av}(132)$ seems to agree with theirs.

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Known Wilf classes

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  - The bijection we found for $\mathrm{Av}(123)$ and $\mathrm{Av}(132)$ seems to agree with theirs.
- Classes avoiding one pattern of size 3 and one of size 4.
- $\mathrm{Av}(1234)$, $\mathrm{Av}(1243)$ and $\mathrm{Av}(1432)$.
  - $\mathrm{Av}(2143)$ is missing.

Known Wilf classes, Av(1234) and Av(1243)

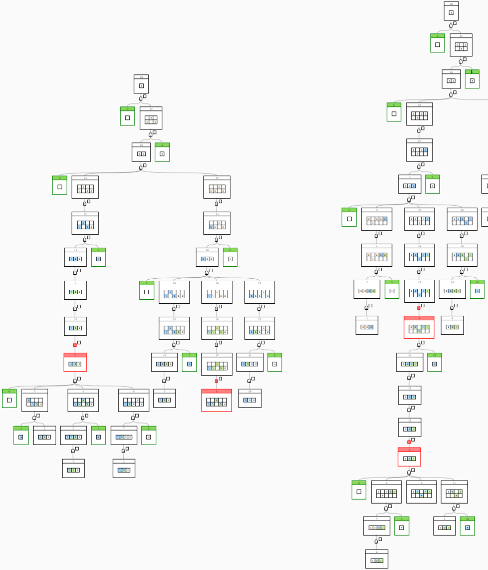

Figure 30: The parallel specifications found for Av(1234) and Av(1243).

Experimental classes

1. $\mathrm{Av}(1243, 1342, 1423, 1432, 2143, 2413, 3142, 3412, 4231)$
2. $\mathrm{Av}(1243, 1342, 1423, 1432, 2413, 2431, 3142, 3412, 4231)$
3. $\mathrm{Av}(1243, 1342, 1423, 2143, 2413, 2431, 3142, 3412, 4231)$
4. $\mathrm{Av}(1243, 1342, 1432, 2143, 2413, 2431, 3142, 3412, 4231)$
5. $\mathrm{Av}(1243, 1342, 2143, 2413, 2431, 3142, 3412, 4132, 4231)$
6. $\mathrm{Av}(1324, 1342, 2143, 2341, 2413, 2431, 3142, 3241, 3412)$

Experimental classes

1. Av (1243, 1342, 1423, 1432, 2143, 2413, 3142, 3412, 4231)
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3. Av (1243, 1342, 1423, 2143, 2413, 2431, 3142, 3412, 4231)
4. Av (1243, 1342, 1432, 2143, 2413, 2431, 3142, 3412, 4231)
5. Av (1243, 1342, 2143, 2413, 2431, 3142, 3412, 4132, 4231)
6. Av (1324, 1342, 2143, 2341, 2413, 2431, 3142, 3241, 3412)

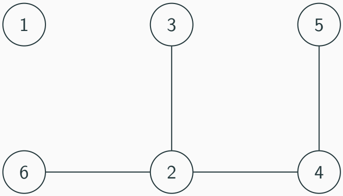

Figure 31: The bijections found for an experimental class.

Conclusion

Conclusion and future work

- First fully automatic bijections (that we know of).
- Offers structural insight.
- Discover equivalence rules.
- Use a search specifically for bijections.
- Continue classifying $n \times 4$ for $n < 9$.

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Thanks for listening

Questions?

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