PROCESOS ESTOCÁSTICOS EN TIEMPO CONTINUO

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1. Exercise 1

A Poisson process with rate $\lambda > 0$ can be defined as a counting process $\{N(t) : t \geq 0\}$ with the following properties:

(a) $N(0) = 0$.

(b) $N(t)$ has independent and stationary increments.

(c) Let $\Delta N(t) = N(t + \Delta t) - N(t)$ with $\Delta t > 0$. The following relations hold:

$$\mathbb{P}[\Delta N(t) = 0] \quad = 1 - \lambda\Delta t + o(\Delta t), \tag{1}$$

$$\mathbb{P}[\Delta N(t) = 1] \quad = \lambda\Delta t + o(\Delta t), \tag{2}$$

$$\mathbb{P}[\Delta N(t) \geq 2] \quad = o(\Delta t). \tag{3}$$

From this definition show that

$$\mathbb{P}[N(t) = n] = \frac{1}{n!}\lambda^n t^n e^{-\lambda t}. \tag{4}$$

To this end, set up a system of differential equations for the quantities $\mathbb{P}[N(t) = 0]$, and $\mathbb{P}[N(t) = n]$ with $n \geq 1$. Then verify that Eq. 4 satisfies the differential equations derived. For instance, the differential equation for $\mathbb{P}[N(t) = 0]$ can be derived from the fact that

$$\mathbb{P}[N(t + \Delta t) = 0] = \mathbb{P}[N(t) = 0]\mathbb{P}[\Delta N(t) = 0] \tag{5}$$

Using Eq. 1, we obtain

$$\mathbb{P}[N(t + \Delta t) = 0] = \mathbb{P}[N(t) = 0] - \mathbb{P}[N(t) = 0]\lambda\Delta t + o(\Delta t) \tag{6}$$

The corresponding differential equation is obtained in the limit $\Delta t \to 0^+$

$$\frac{d}{dt}\mathbb{P}[N(t) = 0] = -\lambda \mathbb{P}[N(t) = 0] \tag{7}$$

The solution of this differential equation for the initial condition $\mathbb{P}[N(0) = 0] = 1$ is

$$\mathbb{P}[N(t) = 0] = e - \lambda t \tag{8}$$

Illustrate the validity of the derivation by comparing the empirical distribution obtained in a simulation of the Poisson process and the theoretical distribution of $labeleq : 1[N(t) = n]$ given by Eq. (4) for the values $\lambda = 10, t = 2$.

Answer: Let's start by interpreting $\mathbb{P}[N(t + \Delta t) = n]$ as a sum of the different possible outcomes' probabilities. Those outcomes are:

- There are no events happening between t and Δt, meaning $N(t) = n$ and $\Delta N(t) = 0$.
- There is one event happening between t and Δt, meaning $N(t) = n - 1$ and $\Delta N(t) = 1$.
- There are two events...
- All events are happening between t and Δt, meaning $N(t) = 0$ and $\Delta N(t) = n$

For clarity purposes, we will define $P_{t;n} = \mathbb{P}[N(t) = n]$ and $P_{\Delta t;n} = \mathbb{P}[\Delta N(t) = n]$

$$P_{t+\Delta t;n} = \sum_{i=0}^{n} P_{t;n-i}P_{t;i} = P_{t;n}P_{\Delta t;0} + P_{t;n-i}P_{\Delta t;1} + \sum_{i=2}^{n} P_{t;n-i}P_{t;i} \tag{9}$$

Given equations 1, 2 and 3, the previous expression results in

$$P_{t+\Delta t;n} = P_{t;n}(1 - \lambda) + P_{t;n-1}\lambda t + o(\Delta t) \tag{10}$$

$$P_{t+\Delta t;n} - P_{t;n} = -\lambda \Delta t P_{t;n} + \lambda \Delta t P_{t;n-1} + o(\Delta t) \tag{11}$$

Each element of the sum tends to 0 when $\Delta t \to 0^+$, obtaining the following ordinary differential equation

$$\frac{d}{dt}P_{t;n} = \lambda(-P_{t;n} + P_{t;n-1}) \tag{12}$$

We can check that the solution to this ODE is $\frac{1}{n!}\lambda^n t^n e^{-\lambda t}$, meaning that its derivative with respect to t is equal to the previous expression and the initial condition $P_{0;0} = 0$ is fulfilled.

$$\begin{aligned}
\frac{d}{dt}\left(\frac{1}{n!}\lambda^n t^n e^{-\lambda t}\right) &= \frac{\lambda^n}{n!}\left(nt^{n-1}e^{-\lambda t} - \lambda t^n e^{-\lambda t}\right) \\
&= \frac{1}{(n-1)!}\lambda^n t^{n-1}e^{-\lambda t} - \frac{1}{n!}\lambda^{n+1}t^n e^{-\lambda t} \\
&= \lambda\left(\frac{1}{(n-1)!}\lambda^{n-1}t^{n-1}e^{-\lambda t} - \frac{1}{n!}\lambda^n t^n e^{-\lambda t}\right) \\
&= \lambda\left(P_{n-1} - P_n\right)
\end{aligned} \tag{13}$$

Since $\frac{1}{n!}\lambda^n t^n e^{-\lambda t} = 1$ if $n = t = 0$.

Now we will apply the use of induction to prove the desired equality.

$$\frac{d}{dt}(e^{\lambda t}P_{t;n}) = \lambda e^{\lambda t}P_{t;n} + \lambda e^{\lambda t}(-P_{t;n}`P_{t;n-1}) = \lambda P_{t;n-1}e^{\lambda t} \tag{14}$$

Given the initial proof, we know that $P_{t;0} = e^{-\lambda t}$:

$$\frac{d}{dt}(e^{\lambda t}P_{t;0}) = \frac{d}{dt}(e^{\lambda t} - e^{-\lambda t}) = 0 \tag{15}$$

For $n = 1$

$$\frac{d}{dt}(e^{\lambda t}P_{t;1}) = \lambda P_{t;0}e^{\lambda t} = \lambda e^{-\lambda t}e^{\lambda t} = \lambda \tag{16}$$

Meaning that

$$e^{\lambda t}P_{t;1} = \lambda t \tag{17}$$

For $n = 2$

$$\frac{d}{dt}(e^{\lambda t}P_{t;2)} = \lambda e^{\lambda t}P_{t;1} = \lambda^2 t; e^{\lambda t}P_{t;2} = \int_0^t \lambda^2 t dt = \frac{1}{2}\lambda^2 t^2 \tag{18}$$

We can conclude that, by applying induction we demonstrate:

$$e^{\lambda t}P_{t;n} = \frac{1}{n!}\lambda^n t^n \tag{19}$$

$$P_{t;n} = \frac{1}{n!}\lambda^n t^n e^{-\lambda t} \tag{20}$$

Next, in the attached notebook we will simulate the Poisson process and compare it with the theoretical distribution.

2. Exercise 2

Simulate a Poisson process with $\lambda = 5.0$. From these simulations show for different values of $n = 1, 2, 5, 10$ that the probability density of the n^{th} arrival is

$$f_{S_n}(t) = \frac{1}{(n-1)!}\lambda^n t^{n-1} e^{-\lambda t}. \tag{21}$$

Answer: Backed up from previous knowledge, we will call the given equation Erlang's distribution. See the attached notebook to look at the implemented solution.

3. Exercise 3

Assume we have a sample $\{U_i\}_{i=1}^n$ of n iid $U[0,t]$ random variables. The probability density of the order statistics $\{U_1 < U_2 < ... < U_n\}$ is:

$$f_{\{U_i\}_{i=1}^n}(\{u_i\}_{i=1}^n) = \frac{n!}{t^n}$$

Let $\{N(t); t \geq 0\}$ be a Poisson proccess with rate λ. Show that conditioned on $N(t) = n$, the distribution of arrival times $\{0 < S_1 < S_2 < ... < S_n\}$ coincides with the distribution of order statistics of n iid $U[0,t]$ random variables.

$$f_{\{S_i\}_{i=1}^n|N(t)}(\{u_i\}_{i=1}^n|n) = \frac{n!}{t^n}$$

Hints:

- Use Bayes theorem to calculte the density $f_{\{S_i\}_{i=1}^{n+1}|N(t)}(\{s_i\}_{i=1}^{n+1}|n)$
- Use the fact that $N(t) = n$ if and only if $s_n \leq t < s_{n+1}$.

$$f_{N(t)|\{S_i\}_{i=1}^{n+1}}(n|\{s_i\}_{i=1}^{n+1}) = \begin{cases} 1 & s_n \leq t < s_{n+1} \\ 0 & \text{otherwise} \end{cases}$$

- Focus on the case $s_n \leq t < s_{n+1}$.
- Use the fact that

$$f_{\{S_i\}_{i=1}^{n+1}|N(t)}(\{s_i\}_{i=1}^{n+1}|n) = f_{S_{n+1}|\{S_i\}_{i=1}^n,N(t)}(s_{n+1}|\{s_i\}_{i=1}^n, n) f_{\{S_i\}_{i=1}^n|N(t)}(\{s_i\}_{i=1}^n|n)$$

- Use the memoryless property for $s_{n+1} > t$

$$f_{S_{n+1}|\{S_i\}_{i=1}^{n+1},N(t)}(s_{n+1}|\{s_i\}_{i=1}^n, n) = f_{S_{n+1}|N(t)}(s_{n+1}|n)$$

Answer: Given a sample $\{s_1, s_2, ..., s_n\}$ of arrival times, the probability density $f_{\{S_i\}_{i=1}^n|N(t)}(\{s_i\}_{i=1}^{n+1}|N(t) = n)$ can be computed using the Bayes theorem as:

$$f_{\{S_i\}_{i=1}^n|N(t)}(\{s_i\}_{i=1}^{n+1}|N(t) = n) = \frac{f_{\{S_i\}_{i=1}^n}(N(t) = n|\{s_i\}_{i=1}^{n+1}) f_{\{S_i\}_{i=1}^n}(\{s_i\}_{i=1}^{n+1})}{f(N(t) = n)} = \frac{f(N(t) = n|\{s_i\}_{i=1}^{n+1}) f(\{s_i\}_{i=1}^{n+1})}{f(N(t) = n)} \tag{22}$$

Where, we have eliminated the sub-index of f in order to understand better the equation. In addition, it is evident that $N(t) = n \Leftrightarrow s_n \leq t < s_{n+1}$ which leads to:

$$f(N(t) = n|\{s_i\}_i^{n+1}) = \begin{cases} 1 & s_n \leq t < s_{n+1} \\ 0 & \text{otherwise} \end{cases} \tag{23}$$

Therefore, we will focus only on the cases where $s_n \leq t < s_{n+1}$, since the rest are zero. This looks obvious since the probability of having observed $n+1$ events at times $\{s_i\}_i^{n+1}$ will be non zero if and only if n events have been previously observed. With this,

$$f(\{s_i\}_i^{n+1}|N(t)=n) = \frac{f(\{s_i\}_i^{n+1})}{f(N(t)=n)} = \frac{\lambda^{n+1}e^{-\lambda s_{n+1}}}{\frac{1}{n!}(\lambda t)^n e^{-\lambda t}} = \frac{\lambda n!}{t^n}e^{-\lambda(s_{n+1}-t)} \tag{24}$$

Now, using the next equivalences:

$$\begin{cases} f(\{s_i\}_i^{n+1}|N(t)=n) = f(s_{n+1}|\{s_i\}_i^{n+1};N(t)=n)f(\{s_i\}_i^{n}|N(t)=n) \\ f(s_{n+1}|\{s_i\}_i^{n};N(t)=n) = f(s_{n+1}|N(t)=n) \end{cases} \tag{25}$$

We can combine these equations in order to have:

$$f(\{s_i\}_i^{n}|N(t)=n) = \frac{f(\{s_i\}_i^{n+1}|N(t)=n)}{f(s_{n+1}|\{s_i\}_i^{n+1};N(t)=n)} = \frac{f(\{s_i\}_i^{n+1}|N(t)=n)}{f(s_{n+1}|N(t)=n} \tag{26}$$

The numerator is the equation given in 24, and the denominator is just the first arrival after time t, given $N(t)=n$.

$$f(\{s_i\}_i^{n}|N(t)=n) = \frac{\lambda n!}{t^n}e^{-\lambda(s_{n+1}-t)}\frac{1}{\lambda e^{-\lambda(s_{n+1}-t)}} = \frac{n!}{t^n} \tag{27}$$

4. Exercise 4

Two teams A and B play a soccer match. The number of goals scored by Team A is modeled by a Poisson proces $N_A(t)$ with rate $\lambda_A = 0{,}02$ goals per minute. The number of goals scored by Team B is modeled by a Poisson process $N_B(t)$ with rate $\lambda = 0{,}03$ goals per minute. The two processes are assumed to be independent. Let $N(t)$ be the total number of goals in the game up to and including time t. The game lasts for 90 minutes.

(a) Find the probability that no goals are scored.

(b) Find the probability that at least two goals are scored in the game.

(c) Find the probability of the final score being Team A:1, Team B:2.

(d) Find the probability that they draw.

(e) Find the probability that Team B scores the first goal.

Confirm your results by writing a Python program that simulates the processes and estimate the answers from the simulation.

Note: In this problem, the series representation of the modified Bessel function of order ν can be useful

$$I_\nu(x) = \sum_{n=0}^{\infty} \frac{1}{n!\Gamma(n+\nu+1)} \left(\frac{x}{2}\right)^{(2n+\nu)}$$

Answer:

We can use the fact that the sum of two independen Poisson processes is also a Poisson process, whose rate is the sum of the rates of both previous Poisson processes. This is: $N(t) = Pois(\lambda = 0{,}05)$.

(a) The probability that no goals are scored is:

$$P(N(t=90)=0) = \frac{(0{,}05 \cdot 90)^0 e^{-0{,}05 \cdot 90}}{0!} \approx 0{,}0111 \tag{28}$$

(b) The probability that at least two goals are scored is the opposite of being scored one or two goals.

$$\begin{gathered} P(N(t=90) \geq 2) = 1 - P(N(t=90)=0) - P(N(t=90)=1) = \\ 1 - \frac{(0{,}05 \cdot 90)^0 e^{-0{,}05 \cdot 90}}{0!} - \frac{(0{,}05 \cdot 90)^1 e^{-0{,}05 \cdot 90}}{1!} \approx 0{,}9389 \end{gathered} \tag{29}$$

(c) In this case, we will separate both distributions in order to obtain the composed distribution by simply multiplying them, since they are independent:

$$\begin{gathered} P(N_1(t=90)=1, N_2(t=90)=2) = P(N_1(t=90)=1)(N_1(t=90)=2) = \\ \frac{(0{,}02 \cdot 90)^1 e^{-0{,}02 \cdot 90}}{1!} \frac{(0{,}03 \cdot 90)^2 e^{-0{,}03 \cdot 90}}{2!} \approx 0{,}0729 \end{gathered} \tag{30}$$

(d) The probability that they draw is the sum of the probabilities of all possible draws, where each draw can be computed by multiplying both probabilities.

$$\begin{gathered} P(draw) = \sum_{i=0}^{\infty} P(N_1(t=90)=i) \cdot P(N_2(t=90)=i) = \\ \sum_{i=0}^{\infty} \frac{(0{,}02 \cdot 90)^i e^{-0{,}02 \cdot 90}}{i!} \frac{(0{,}03 \cdot 90)^i e^{-0{,}03 \cdot 90}}{i!} = e^{-4{,}5} \sum_{i=0}^{\infty} \frac{(90^2 \cdot 6 \cdot 10^{-4})^i}{(i!)^2} = 0{,}1793 \end{gathered} \tag{31}$$

In order to calculate this infinite sum, we can use *mpmath* package from python, obtaining the previous result.

(e) In order to obtain the probability that the team B scores the first goal, we have to use conditional probabilities. On the one hand, we will define X as the number of goals scored by team A before team B scores, and Y will give the time that will need the team B to score. In this way, we have to compute the probability of X=0 given Y.

As we know, the probability will follow the next Poisson distribution, with mean $0{,}02t$:

$$P(X=n|Y=t) = \frac{(0{,}02t)^n e^{-0{,}02t}}{n!} \tag{32}$$

The distribution of Y, which follows a Poisson process, is given by:

$$P(Y = t) = 0{,}03e^{-0{,}03t} \tag{33}$$

Given these two function, we can now calculate the asked probability, given the condition that the match must be no longer than 90 minutes:

$$\begin{aligned} P(X = 0|Y \in [0, 90]) = \int_0^{90} P(X = 0|Y = t)P(Y = t)dt = \\ \int_0^{90} 0{,}03e^{-0{,}05t}dt = -\frac{0{,}03}{0{,}05}e^{-0{,}05t}\,\Big]_0^{90} \approx 0{,}5933 \end{aligned} \tag{34}$$

Note: See the attached notebook to look at the implemented simulations.

5. Exercise 5

Consider the process $X(t) = Z\sqrt{t}$ for $t \geq 0$ with the same value of Z for all t

(a) Show that the distribution of the process at time t is the same as that of a Wiener process: $X(t) \sim N(0, \sqrt{t})$.

(b) What is the mathematical property that allows us to prove that this process is no Brownian?

Answer.

(a) If $Z(t) \sim \mathcal{N}(0, 1)$ and $X(t) = Z\sqrt{t}$, we are multiplying the normal distribution by the constant $\sqrt{t}$. A distribution resulting of multiplying $\mathcal{N}(0, 1)$ by a constant will not change the mean value of the distribution, but the resulting standard deviation will be the previous one multiplied by the constant $\sqrt{t}$. This gives the final distribution:

$$X(t) = Z\sqrt{t} = \mathcal{N}(0, 1)\sqrt{t} = \mathcal{N}(0, \sqrt{t}) \tag{35}$$

(b) In order to prove that this process is not Brownian, we have to define first what a Brownian motion looks alike. A Brownian process has the next distribution: $\rho(x, t) \sim \mathcal{N}(0, \sigma^2)$. One of the main properties of these processes is that they are stochastic processes with stationary independent increments. This condition means that if $0 \leq s_1 < t_1 \leq s_2 < t_2$ then $W_{t_1} - W_{s_1}$ and $W_{t_2} - W_{s_2}$ are independent random variables, for W a Brownian process. In this case, given $0 \leq s_1 < t_1 \leq s_2 < t_2$, our distribution will be $W_{t_1} - W_{s_1} = Z(\sqrt{t_1} - \sqrt{s_1})$ and $W_{t_2} - W_{s_2} = Z(\sqrt{t_2} - \sqrt{s_2})$, which are clearly not independent. Therefore, $X(t)$ cannot be Brownian.

6. Exercise 6

Consider the Wiener (standard Brownian) process $W(t)$ in $[0, 1]$.

(a) From the property of independent increments,

$$\mathbb{E}[(W(t_2)-W(t_1))(W(s_2)-W(s_1))] = \mathbb{E}[(W(t_2)-W(t_1))]\mathbb{E}[(W(s_2)-W(s_1))], t_2 \geq t_1 \geq s_2 \geq s_1 \geq 0,$$

show that the autocovariances are given by

$$\gamma(t, s) = \mathbb{E}[W(t)W(s)] = min(s, t),$$

both for $s > t$ and for $t > s$.

(b) Illustrate this property by simulating a Wiener process in $[0, 1]$ and making a plot of the sample estimate and the theoretical values of $\gamma(t, 0{,}25)$ as a function of $t\epsilon[0, 1]$

Answer.

(a) We are able to show this property simply choosing two arbitrary s, t such that $0 < s < t$. Substituting $W(t) = (W(t) - W(s)) + W(s)$, we can operate in the following form:

$$\gamma(t, s) = \mathbb{E}[W(s)W(t)] = \mathbb{E}[W(s)(W(t)-W(s))+W(s)] = \mathbb{E}[W(s)(W(t)-W(s))]+\mathbb{E}[W(s)^2] \quad (36)$$

As in the property of independent increments and given that $W(0) = 0$:

$$\begin{aligned}\mathbb{E}[(W(t) - W(s))(W(s) - W(0))] = \mathbb{E}[(W(t) - W(s))]\mathbb{E}[(W(s) - W(0))] = \\ \mathbb{E}[(W(t) - W(s))]\mathbb{E}[W(s)] = 0\end{aligned} \quad (37)$$

Substituting in the above equation we get that:

$$\gamma(t, s) = \mathbb{E}[W(s)^2] = Var[W(s)] = s = min(s, t) \quad (38)$$

As we chose an arbitary value for both variables s, t, it is possible to demonstrate the case $t < s$ in the same process.

(b) Solution in the attached notebook.

7. Exercise 7

Consider two independent Wiener processes $W(t), W'(t)$. Show that the following processes have the same covariances as the standard Wiener process:

(a) $\rho W(t) + \sqrt{1-\rho^2}W'(t)\ t \geq 0$

(b) $-W(t)\ t \geq 0$

(c) $\sqrt{c}W(t/c);\ t \geq 0, c > 0$

(d) $V(0) = 0; V(t) = tW(1/t);\ t > 0$

Make a plot of the trajectories of the first three processes to illustrate that they are standard Brownian motion processes. Compare the histogram of the final values of the simulated trajectories with the theoretical density function.

Answer.

We're aiming to demonstrate that all processes have the following covariance: $\gamma(s,t) = min(s,t)$.

Given $0 \leq s \leq t$:

(a)

$$\begin{aligned}\mathbb{E}[V_1(t)V_1(s)] = \rho^2\mathbb{E}[W(t)W(s)] + \rho\sqrt{1-\rho^2}\mathbb{E}[W(t)W'(s)] + \rho\sqrt{1-\rho^2}\mathbb{E}[W'(t)W(s)] \\ +(1-\rho^2)[W'(t)W'(s)] = \rho^2 s + (1-\rho^2)s = s = min(s,t)\end{aligned} \tag{39}$$

(b)

$$\mathbb{E}[V_2(t)V_2(s)] = \mathbb{E}[(-W(t))(-W(s))] = \mathbb{E}[(W(t))(W(s))] = s \tag{40}$$

(c)

$$\mathbb{E}[V_3(t)V_3(s)] = c\mathbb{E}[W(t/c)(W(s/c)] = \frac{cs}{c} = s \tag{41}$$

(d)

$$\mathbb{E}[V_4(t)V_4(s)] = ts\mathbb{E}[W(1/t)(W(1/s)] = ts\ min(1/t, 1/s) = \frac{ts}{t} = s \tag{42}$$

Note: The plot of the trajectories is shown in the notebook.

8. Exercise 8 (Extra Point)

Make an animation in Python illustrating the evolution of the distribution of a Brownian motion process starting form x_0:

$$\mathbb{P}(B(t) = x|B(t_0) = x_0).$$

To this end, simulate M trajectories of the process in the interval $[t_0, t_0+T]$ and plot the time evolution of the histogram using as frames a grid of regularly spaced times in that interval. Plot the theoretical form of the density function on the same graph, so that it can be compared with the histogram.

Answer is in the notebook.