Numerical Analysis HW2

数学与应用数学 2002 王锦宸

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1.1 Determine $\xi(x)$ explicitly

$$l_1(x) = \frac{x-2}{1-2}, \quad l_2(x) = \frac{x-1}{2-1}$$

$$p_1(f;x) = 1 * \frac{x-2}{1-2} + \frac{1}{2} * \frac{x-1}{2-1}$$

Besides,

$$f''(x) = \frac{2}{x^3}$$

Thus,

$$\frac{1}{x} - (-\frac{x}{2} + \frac{3}{2}) = \frac{1}{(\xi(x))^3}(x-1)(x-2)$$

$$\xi(x) = \sqrt[3]{2x}$$

1.2 Extend the domain of ξ continuously from (x_0, x_1) to $[x_0, x_1]$.Find max $\xi(x)$, min$\xi(x)$, and max $f''(\xi(x))$.

Since $\xi(x)$ is monotonically increasing on $[1, 2]$,

$$max\,\xi(x) = \xi(2) = \sqrt[3]{4}, \quad min\,\xi(x) = \xi(1) = \sqrt[3]{2}$$

Since $f''(\xi(x)) = \frac{1}{x}$ is monotonically decreasing on $[1, 2]$,

$$max\,f''(\xi(x)) = f''(\xi(1)) = 1$$

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Since $f_i > 0$ for i = 0,1,...,n, let $p_0(x_i) = \sqrt{f_i}$ for i = 0,1,...,n. By the interpolation theorem, we can uniquely determine a polynomial of degree $\leqq n$. Let $p(x) = (p_0(x))^2$, p(x) is a polynomial of degree $\leqq 2n$ that are non-negative on the real line and $p(x_i) = f_i$ for i = 0,1,...,n.

3

3.1 Prove by introduction

when n = 1,

$$f[t,t+1] = e^{t+1} - e^t = \frac{(e-1)^1}{1!}e^t$$

Assume that (when n = k),

$$f[t,t+1,\dots,t+k] = \frac{(e-1)^k}{k!}e^t$$

when n = k+1,

$$\begin{aligned}
f[t,t+1,\dots,t+k+1] &= \frac{f[t+1,t+2,\dots,t+k+1] - f[t,t+1,\dots,t+k]}{t+k+1-t} \\
&= \frac{\frac{(e-1)^k}{k!}e^{t+1} - \frac{(e-1)^k}{k!}e^t}{k+1} \\
&= \frac{\frac{(e-1)^{k+1}}{k!}e^t + \frac{(e-1)^k}{k!}e^t - \frac{(e-1)^k}{k!}e^t}{k+1} \\
&= \frac{(e-1)^{k+1}}{(k+1)!}e^t
\end{aligned}$$

By induction,

$$\forall x \in R,\ f[t,t+1,\dots,t+k+1] = \frac{(e-1)^{k+1}}{(k+1)!}e^t$$

3.2 Is ξ located to the left or to the right of the midpoint?

From 3.1,

$$f[0,1,\dots,n] = \frac{(e-1)^n}{n!}e^0 = \frac{(e-1)^n}{n!}.$$

From Corollary 2.2 we know,

$$\exists\ \xi \in (0,n) \quad s.t. \quad f[0,1,\dots,n] = \frac{1}{n!}f^{(n)}(\xi).$$

Thus,

$$\begin{aligned}
f^{(n)}(\xi) &= (e-1)^n. \\
f(x) &= e^(x). \\
f^{(n)}(x) &= e^x. \\
e^\xi &= (e-1)^n \\
\xi &= n\ln(e-1)
\end{aligned}$$

Since $n\ln(e-1) \geqq \frac{n}{2}$, $\quad \xi$ is located to the right of the midpoint.

4

4.1 Use the Newton formula to obtain $p_3(f;x)$

Since we have,

x	0	1	3	4
$f(x)$	5	3	5	12

we can construct the following table of divided difference,

0	5			
1	3	-2		
3	5	1	1	
4	12	7	2	1/4

Thus,

$$p_3(f;x) = 5 - 2x + x(x-1) + \frac{1}{4}x(x-1)(x-3) = \frac{1}{4}x^3 - \frac{9}{4}x + 5$$

4.2 Find an approximate value for the location x_{min} of the minimum.

Let $p_3' = \frac{3}{4}x^2 - \frac{9}{4} = 0$, then $x = \pm\sqrt{3}$.
Then when $x \in (1, \sqrt{3})$, p is monotonic decreasing, while $x \in (\sqrt{3}, 3)$, p is increasing. That is, $f_{\text{min}} = p(\sqrt{3}) = 5 - \frac{3\sqrt{3}}{2} \approx 2.402$ and $x_{min} \approx 1.732$.

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5.1 Compute f [0,1,1,1,2,2]

Since $f = x^7$, then the table of divided differences can be established as follow,

0	0					
1	1	1				
1	1	7	6			
1	1	7	21	15		
2	128	127	120	99	42	
2	128	448	321	201	102	30

Thus, f[0,1,1,1,2,2] = 30

5.2 Determine ξ

Since $f^{(5)}(x) = 2520x^2 = 30$, then $x = \frac{1}{2\sqrt{21}} \approx 0.1091$.

6

6.1 Estimate f(2) using Hermite interpolation.

We can obtain the table of divided differences,

$$
\begin{array}{c|cccccc}
0 & 1 & & & & \\
1 & 2 & 1 & & & \\
1 & 2 & \text{-1} & \text{-2} & & \\
3 & 0 & \text{-1} & 0 & 2\,/\,3 & \\
3 & 0 & 0 & 1\,/\,2 & 1\,/\,4 & \text{-5/36} \\
 & & & & &
\end{array}
$$

Thus,

$$p(x) = 1 + x - 2x(x-1) + \frac{2}{3}x(x-1)^2 - \frac{5}{36}x(x-1)^2(x-3)$$

$$f(2) \approx p(2) = \frac{11}{18}$$

6.2 Estimate the maximum possible error of the above answer.

Since N = 2 + 1 + 1 = 4, by Theorem 2.35, we have,

$$
\begin{aligned}
f(x) - p_5(f;x) &= \frac{f^{(N+1)}(\xi)}{(N+1)!}\prod_{i=0}^{k}(x-x_i)^{m_i+1} \\
&= \frac{f^{(5)}(\xi)}{5!}x(x-1)^2(x-3)^2
\end{aligned}
$$

Thus, $|f(2) - p(2)| \leqq \frac{M}{60}$.

7

Let's prove it by induction.
When k = 1,

$$
\begin{aligned}
\triangle^1 f(x) &= f(x+h) - f(x) \\
&= 1!h^1 f[x, x+h]
\end{aligned}
$$

Assume that(when k = n),

$$\triangle f(x) = k!h^k f[x_0, x_1, \ldots, x_k]$$

when k = n + 1,

$$\begin{aligned}
\triangle^{k+1} f(x) &= \triangle^k f(x+h) - \triangle^k f(x) \\
&= k!h^k f[x_1, x_2, \ldots, x_{k+1}] - k!h^k f[x_0, x_1, \ldots, x_k] \\
&= k!h^k (f[x_1, x_2, \ldots, x_{k+1}] - f[x_0, x_1, \ldots, x_k] \\
&= k!h^k (f[x_0, x_1, \ldots, x_{k+1}])(x_{k+1} - x_0) \\
&= k!h^k (f[x_0, x_1, \ldots, x_{k+1}])(k+1)h \\
&= (k+1)!h^{k+1} (f[x_0, x_1, \ldots, x_{k+1}])
\end{aligned}$$

The proof for backward difference $\bigtriangledown$ is similar.

8

Let's prove by induction.
When n = 1,

$$\begin{aligned}
\frac{\partial}{\partial x_0} f[x_0, x_1] &= \frac{\partial}{\partial x_0} \left(\frac{f(x_0) - f(x_1)}{x_0 - x_1} \right) \\
&= \frac{f'(x_0)(x_0 - x_1) - (f(x_0) - f(x_1))}{(x_0 - x_1)^2} \\
&= \frac{f[x_0, x_1] - f'(x_0)}{x_1 - x_0} \\
&= f[x_0, x_0, x_1]
\end{aligned}$$

Assume that it holds for n = k,
When n = k + 1,

$$\begin{aligned}
\frac{\partial}{\partial x_0} f[x_0, x_1, \ldots, x_{k+1}] &= \frac{\partial}{\partial x_0} \left(\frac{f[x_1, \ldots, x_{k+1}] - f[x_0, x_1, \ldots, x_k]}{x_{k+1} - x_0} \right) \\
&= \frac{-f[x_0, x_0, x_1, \ldots, x_k](x_{k+1} - x_0) + (f[x_1, \ldots, x_{k+1}] - f[x_0, x_1, \ldots, x_k])}{(x_{k+1} - x_0)^2} \\
&= \frac{f[x_0, x_1, \ldots, x_{k+1}] - f[x_0, x_0, x_1, \ldots, x_k]}{x_{k+1} - x_0} \\
&= f[x_0, x_0, x_1, \ldots, x_{k+1}]
\end{aligned}$$

9 A min-max problem

Let $t = \frac{2x-(a+b)}{b-a} \quad (t \in [-1, 1])$, that is, $x = \frac{t(b-a)+(a+b)}{2}$.
Thus, we have $q(t) = p\left(\frac{t(b-a)+(a+b))}{2}\right) = p(x)$, in which the coefficient of t^n is $a_0 \left(\frac{b-a}{2}\right)^n$.
By Theorem 2.44(Chebyshev),

$$\forall\, q \in \tilde{P}_n, \quad max_{t\in[-1,1]} \left| \frac{q(t)}{a_0 \left(\frac{b-a}{2}\right)^n} \right| \geqq max_{t\in[-1,1]} \left| \frac{T_n(t)}{2^{n-1}} \right|$$

that is,

$$min\ max_{t\in[-1,1]}\left|\frac{q(t)}{a_0\left(\frac{b-a}{2}\right)^n}\right| = max_{t\in[-1,1]}\left|\frac{T_n(t)}{2^{n-1}}\right|$$

Therefore,

$$\begin{aligned} min\ max_x \in [a,b]p(x) &= min\ max_t \in [-1,1]q(t) \\ &= \frac{1}{2^n}a_0\left(\right)^n \\ &= a_0\frac{(b-a)^n}{2^{2n-1}} \end{aligned}$$

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First, we know $||\hat{p_n}||_\infty = \frac{1}{T_n(a)}$
By the property of T_n we have,

$$\hat{p}_n(x)\left(x'_k\right) = \frac{(-1)^k}{T_n(a)} \quad \text{for} \quad x'_k = \cos\frac{k}{n}\pi, k = 0, 1, \ldots, n$$

Suppose that $\exists \quad p \in \mathcal{P}_n^a, \quad s.t. \quad ||p||_\infty < \frac{1}{|T_n(a)|}$.
Consider the polynomial $Q(x) = \frac{1}{|T_n(a)|}T_n(x) - p(x)$.

$$Q(x'_k) = \frac{(-1)^k}{|T_n(a)|} - p(x_{k'}), \quad k = 0, 1, \ldots, n.$$

Obviously, Q(x) has alternating signs at $x'_0, x'_1, \ldots, x'_n$. Hence Q(x) must have n zeros. However, by the construction of Q(x), the degree of Q(x) is at most n - 1. Therefore, $Q(x) \equiv 0$, that is, $||p||_\infty = \frac{1}{|T_n(a)|}$, which is contradict to the assumption.
Therefore,

$$\forall p \in P_n^a, \quad \|\hat{p}_n\|_\infty \leqslant \|p\|_\infty$$

11 Prove Lemma 2.48

2.50(a)

Since $t \in (0,1)$, every factor of this polynomial is positive. Hence it holds.

2.50)b)

By the Binomial Theorem,

$$1 = (t + (1-t))^n = \sum_{k=0}^{n}\left(\begin{array}{c} n \\ k \end{array}\right)t^k(1-t)^{n-k} = \sum_{k=0}^{n} b_{n,k}(t)$$

2.50(c)

Derive on both sides of the equation below,

$$(p+q)^n = \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) p^k q^{n-k}$$

we have,

$$n(p+q)^{n-1} = \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) kp^{k-1} q^{n-k}$$

Multiple both sides p times, we have,

$$np(p+q)^{n-1} = \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) kp^{k} q^{n-k}$$

Then we let p = t and q = 1 - t, we have,

$$np = \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) kt^{k} (1-t)^{n-k} = \sum_{k=0}^{n} kb_{n,k}(t)$$

2.50(d)

Take derivative on the both sides of the following equation,

$$np(p+q)^{n-1} = \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) kp^{k} q^{n-k}$$

We have,

$$n(p+q)^{n-1} + n(n-1)p(p+q)^{n-2} = \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) k^2 p^{k-1} q^{n-k}$$

Multiple both sides p times, we have,

$$np(p+q)^{n-1} + n(n-1)p^2(p+q)^{n-2} = \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) k^2 p^{k} q^{n-k}$$

let p = t and q = 1 - t, we have,

$$nt + n(n-1)t^2 = \sum_{k=0}^{n} k^2 b_{n,k}(t)$$

By the result of 2.50(b) and 2.50(c), we have,

$$\begin{aligned} \sum_{k=0}^{n} (k-nt)^2 b_{n,k}(t) &= \sum_{k=0}^{n} k^2 b_{n,k}(t) - 2nt \sum_{k=0}^{n} kb_{n,k}(t) + (nt)^2 \sum_{k=0}^{n} b_{n,k}(t) \\ &= nt + n(n-1)t^2 - 2(nt)^2 + (nt)^2 = nt - nt^2 = nt(1-t) \end{aligned}$$