Numerical Analysis HW5

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Theoretical Questions

Question 1

Proof:
(1) Positive Definite.

$$\forall u \in C[a,b],\ \langle u, v\rangle = \int_a^b p(x)u(x)\overline{u(x)}dx = \int_a^b p(x)u(x)^2dx > 0 \tag{0.1}$$

(2) Regularity.

$$\langle u, u\rangle = 0 \Leftrightarrow p(x)u(x)^2 = 0 \Leftrightarrow u = 0 \tag{0.2}$$

(3) Additivity.

$$\begin{aligned}\langle u+w, v\rangle &= \int_a^b p(x)(u(x)+w(x))\overline{v(x)}dx \\ &= \int_a^b p(x)u(x)\overline{v(x)}dx + \int_a^b p(x)w(x)\overline{v(x)}dx \\ &= \langle u, v\rangle + \langle w, v\rangle\end{aligned} \tag{0.3}$$

(4) Conjugate Symmetry.

$$\begin{aligned}\langle u, v\rangle &= \int_a^b p(x)u(x)\overline{u(x)}dx \\ &= \overline{\int_a^b \overline{p(x)u(x)\overline{\overline{v(x)}}}dx} \\ &= \overline{\int_a^b p(x)\overline{u(x)}v(x)} \\ &= \overline{\langle v, u\rangle}\end{aligned} \tag{0.4}$$

(5) Norm Positive Definite.

$$\|u\|_2 = \left[\int_a^b \rho(x)|u(x)|^2\,\mathrm{d}x\right]^{1/2} \geq 0 \tag{0.5}$$

(6) Norm Regularity.

$$\|u\|_2 = 0 \Leftrightarrow \rho(x)|u(x)|^2 = 0 \Leftrightarrow |u(x)|^2 = 0 \Leftrightarrow u = 0 \tag{0.6}$$

(7) Norm Inequality.

$$
\begin{aligned}
\forall u, v \in \mathcal{C}[a,b], \quad \|u+v\|_2 &= \left[\int_a^b \rho(x)|u(x)+v(x)|^2 \,\mathrm{d}x\right]^{1/2} \\
&\leq \left[\int_a^b \rho(x)|u(x)|^2 \,\mathrm{d}x\right]^{1/2} + \left[\int_a^b \rho(x)|v(x)|^2 \,\mathrm{d}x\right]^{1/2} \\
&= \|u\|_2 + \|v\|_2
\end{aligned} \tag{0.7}
$$

Question 2

(1)

We let $p(x) = \frac{1}{\sqrt{1-x^2}}$ and the First Chebyshev Polynomial is,

$$
T_n(x) = \cos(n \arccos x), \quad x \in [-1,1], \quad n = 0,1,2,\dots \tag{0.8}
$$

therefore,

$$
\begin{aligned}
\forall m, n \quad \langle T_m, T_n \rangle &= \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} T_m(x)\overline{T_n(x)} \mathrm{d}x \\
&= \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \cos(m \arccos x)\cos(n \arccos x)\mathrm{d}x \\
\text{Let } \theta = \arccos x, & \\
&= \int_0^{\pi} \cos(m\theta)\cos(n\theta)\mathrm{d}\theta \\
&= \frac{1}{2}\int_0^{\pi} [\cos(m+n)\theta + \cos(m-n)\theta]\mathrm{d}\theta
\end{aligned} \tag{0.9}
$$

Thus,

$$
\langle T_m, T_n \rangle = \begin{cases} 0, & m \neq n \\ \pi, & m = n = 0 \\ \frac{\pi}{2}, & m = n \geq 1 \end{cases} \tag{0.10}
$$

Hence, the chebyshev polynomials are orthogonal.

(2)

$$
T_0^*(x) = \sqrt{\frac{1}{\pi}}, \quad T_1^*(x) = \sqrt{\frac{2}{\pi}}x, \quad T_2^*(x) = \sqrt{\frac{2}{\pi}}\left(2x^2 - 1\right). \tag{0.11}
$$

Question 3

(1) Orthonormal Polynomials

According to the previous question, we have orthonormal polynomials,

$$
T_0^*(x) = \sqrt{\frac{1}{\pi}}, \quad T_1^*(x) = \sqrt{\frac{2}{\pi}}x, \quad T_2^*(x) = \sqrt{\frac{2}{\pi}}\left(2x^2 - 1\right). \tag{0.12}
$$

Therefore, the approximation of the circular arc is given as follows,

$$
\begin{aligned}
\varphi(x) &= \langle y, T_0^* \rangle T_0^* + \langle y, T_1^* \rangle T_1^* + \langle y, T_2^* \rangle T_2^* \\
&= \frac{2}{\pi} + 0 - \frac{4}{3\pi}\left(2x^2 - 1\right) \\
&= -\frac{8}{3\pi}x^2 + \frac{10}{3\pi}
\end{aligned}
\tag{0.13}
$$

(2) Normal Equation

Let $\phi(x) = a_0 + a_1 x + a_2 x^2$,

$$
\begin{bmatrix} \langle 1,1\rangle & \langle 1,x\rangle & \langle 1,x^2\rangle \\ \langle x,1\rangle & \langle x,x\rangle & \langle x,x^2\rangle \\ \langle x^2,1\rangle & \langle x^2,x\rangle & \langle x^2,x^2\rangle \end{bmatrix}^T \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \langle y,1\rangle \\ \langle y,x\rangle \\ \langle y,x^2\rangle \end{bmatrix}
\tag{0.14}
$$

that is,

$$
\begin{bmatrix} \pi & 0 & \frac{\pi}{2} \\ 0 & \frac{\pi}{2} & 0 \\ \frac{\pi}{2} & 0 & \frac{3\pi}{8} \end{bmatrix}^T \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ \frac{2}{3} \end{bmatrix}
\tag{0.15}
$$

Solving these linear equations, we have,

$$
a_0 = \frac{10}{3\pi},\ a_1 = 0,\ a_2 = \frac{-8}{3\pi}
\tag{0.16}
$$

Therefore,

$$
\phi(x) = -\frac{8}{3\pi}x^2 + \frac{10}{3\pi}
\tag{0.17}
$$

Question 4

(1)

Let the input $u_1 = 1, u_2 = x, u_3 = x^2$, by Gram-Schmidt,

$$
\begin{aligned}
&v_1 = u_1 \\
&u_1^* = v_1 / \left\| v_1 \right\|_2 = v_1/\sqrt{12} = \frac{1}{2\sqrt{3}} \\
&v_2 = u_2 - \langle u_2, u_1^* \rangle u_1^* = x - \frac{13}{2} \\
&u_2^* = v_2 / \left\| v_2 \right\|_2 = \frac{1}{\sqrt{143}}\left(x - \frac{13}{2}\right) \\
&v_3 = u_3 - \langle u_3, u_1^* \rangle u_1^* - \langle u_3, u_2^* \rangle u_2^* = x^2 - \frac{325}{6} - 13 \times \left(x - \frac{13}{2}\right) = x^2 - 13x + \frac{91}{3} \\
&u_3^* = v_3 / \left\| v_3 \right\|_2 = \frac{\sqrt{3}}{\sqrt{4004}}\left(x^2 - 13x + \frac{91}{3}\right)
\end{aligned}
\tag{0.18}
$$

(2)

By corollary 5.25, the best approximation $\hat{\varphi(x)}$ is the projection on the orthonormal basis.

$$
\begin{aligned}
\hat{\varphi}(x) &= \langle y, u_1^* \rangle u_1^* + \langle y, u_2^* \rangle u_2^* + \langle y, u_3^* \rangle u_3^* \\
&= \frac{831}{\sqrt{3}} u_1^* + \frac{589}{\sqrt{143}} u_2^* + \frac{12068\sqrt{3}}{\sqrt{4004}} u_3^* \\
&\approx 9.042x^2 - 113.4266x + 386.0013
\end{aligned}
\tag{0.19}
$$

(3)

Orthonormal Polynomials :

For different y_is, the orthonormal basis $\{u_1^*, u_2^*, u_3^*\}$ can be reused, while the computation of the projection can not. So we only need to recompute $\hat{\varphi}(x) = \langle y, u_1^* \rangle u_1^* + \langle y, u_2^* \rangle u_2^* + \langle y, u_3^* \rangle u_3^*$.

Normal Equation:

For different y_is, the coefficient matrix can be reused, but we have to recompute the vectors righthand, which requires us to compute n inner matrices and to solve the linear equations.

The advantage of orthonormal polynomials over normal equations:

(1) The orthonormal basis can be reused in different approximation.

(2) When using **ORTHONORMAL POLYNOMIALS**, we only need to recompute n inner products, while using **NORMAL EQUATION** requires us to additionally solve linear equation, which increases the amount of calculation and the condition number.

Programming Problems

MATLAB is utilized for this part.

Program 1

The input is x, y and we can find the coefficient in the output a.

$$
a = [2.1757, 2.6704, -0.2384]^T \tag{0.20}
$$

Thus, the best approximation is,

$$
\hat{\varphi(x)} = 2.1752 + 2.6704x - 0.2384x^2 \tag{0.21}
$$

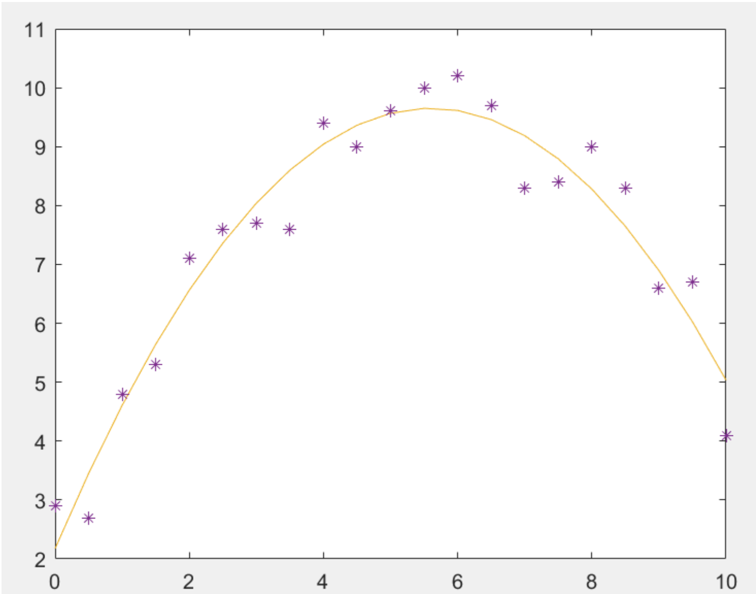

Figure 1: The approximation by Normal Equation

Program 2

After QR Factorization for the Vandermonde Matrix and solving $R_1 * x = c$, we also have,

$$a = [2.1757, 2.6704, -0.2384]^T \tag{0.22}$$

Thus, the best approximation is,

$$\hat{\varphi(x)} = 2.1752 + 2.6704x - 0.2384x^2 \tag{0.23}$$

which is the same as in Program 1, verifying the correctness.

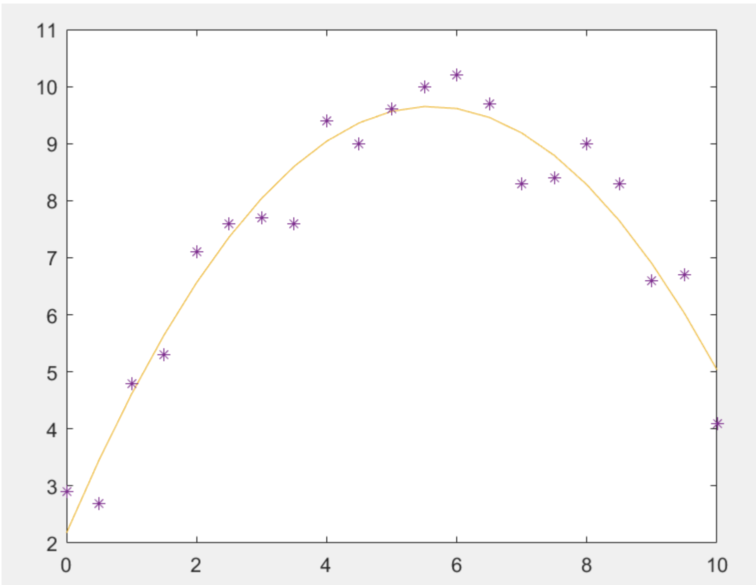

Figure 2: The approximation by QR

Besides,
the condition number of G is 4.585153881696407e+04,
the condition number of R1 is 2.141297242723767e+02.

Apparently, the condition number of the Gram matrix is larger than the condition number of the matrix R_1 corresponding to the QR Factorization.