Numerical Analysis HW3

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November 2022

Problem I

From the definition of s(x). We know that p(x) should satisfy:

$$p(0) = 0 \quad p(1) = 1 \quad p'(1) = -3 \quad p''(1) = 6$$

Use Hermite Interpolation, we have,

$$p(x) = 7x^3 - 18x^2 + 12x$$

To plus, it is not natural since $s''(x) = -36 \neq 0$

Problem II

(a)

Since we need a quadratic spline $s \in S_2^1$, we need two conditions.

(b)

Denote $K_i = f[x_i, x - i + 1]$, the table of divided difference is,

$$\begin{array}{c|ccc} x_i & f_i & & \\ x_i & f_i & m_i & \\ x_{i+1} & f_{i+1} & K_i & \frac{K_i - m_i}{x_{i+1} - x_i} \end{array}$$

Then $p_i(x) = f_i + (x - x_i)m_i + (x - x_i)^2 \frac{K_i - m_i}{x_{i+1} - x_i}$

(c)

From (b), we know that $m_i = p_i'(x_i)$

Problem III

(a)

We have

$$s(-1) = 1, s(0) = 1 + c, s(1) = 1 + 8c$$

and

$$s'(-1) = 0, s'(0) = 3c$$

and

$$s''(1) = 0, s''(0) = 6c$$

from direct computation.
To be a natural cubic spline, we want $s''(1) = s''(-1) = 0$.
Assume that $s_2(x) = \alpha x^3 + \beta x^2 + \gamma x + \theta$, then we need,

$$\begin{cases} \theta = 1 + c \\ \gamma = 3c \\ 2\beta = 6c \\ 6\alpha + 2\beta = 0 \end{cases}$$

Thus $s_2(x) = -cx^3 + 3cx^2 + 3cx + 1 + c$.

(b)

If$s(1) = -1$, we have,

$$-c * 1^3 + 3c * 1^2 + 3c * 1 + 1 + c = -1$$

Thus $c = -\frac{1}{3}$.

Problem IV

(a)

Since $f = \cos\left(\frac{\pi}{2}x\right)$, then $f'(x) = -\frac{\pi}{2}\sin\left(\frac{\pi}{2}x\right)$, $f''(x) = -\left(\frac{\pi}{2}\right)^2\cos\left(\frac{\pi}{2}x\right)$. Thus, we have,

$$\begin{cases} f(-1) = 0,\ f(0) = 1,\ f(1) = 0 \\ f'(-1) = -\frac{\pi}{2},\ f'(0) = 0,\ f'(1) = -\frac{\pi}{2} \\ f''(-1) = 0,\ f''(0) = 1,\ f''(1) = 0 \end{cases}$$

The interpolation of knots -1, 0 ,1 is,

$$s(x) = \begin{cases} -\frac{1}{2}x^3 - \frac{3}{2}x^2 + 1 & \text{if } x \in [-1, 0] \\ \frac{1}{2}x^3 - \frac{3}{2}x^2 + 1 & \text{if } x \in [0, 1] \end{cases}$$

(b)

From (a) we can get,

$$s(x) = \begin{cases} -3x - 3 \text{ if } x \in [-1, 0] \\ 3x - 3 \text{ if } x \in [0, 1] \end{cases}$$

Thus, $\int_{-1}^{1} [s''(x)]^2 \, dx = 6$
(i) If g(x) be the quadratic polynomial, $\int_{-1}^{1} [g''(x)]^2 \, dx = 8 > 6$.
(ii) If $g(x) = cos(\frac{\pi}{2}x), g''(x) = -\frac{\pi^2}{4}cos(\frac{\pi}{2}x)$. Thus $\int_{-1}^{1} [g''(x)]^2 \, dx = \frac{\pi^4}{16} > 6$.

Problem V

(a)

From the textbook, we know that the recursive definition is,

$$B_i^{n+1}(x) = \frac{x - t_{i-1}}{t_{i+n} - t_{i-1}} B_i^n(x) + \frac{t_{i+n+1} - x}{t_{i+n+1} - t_i} B_{i+1}^n(x)$$

and the initial condition is,

$$B_i^0(x) = \begin{cases} 1 & \text{if } x \in (t_{i-1}, t_i] \\ 0 & \text{otherwise} \end{cases}$$

Thus, we have,

$$B_i^2(x) = \begin{cases} \frac{(x-t_{i-1})^2}{(t_{i+1}-t_{i-1})(t_i-t_{i-1})} & \text{if } x \in (t_{i-1}, t_i] \\ \frac{(x-t_{i-1})(t_{i+1}-x)}{(t_{i+1}-t_{i-1})(t_{i+1}-t_i)} + \frac{(t_{i+2}-x)(x-t_i)}{(t_{i+2}-t_i)(t_{i+1}-t_i)} & \text{if } x \in (t_i, t_{i+1}] \\ \frac{(t_{i+2}-x)^2}{(t_{i+2}-t_i)(t_{i+2}-t_{i+1})} & \text{if } x \in (t_{i+1}, t_{i+2}] \\ 0 & \text{otherwise} \end{cases}$$

(b)

We can get $\frac{d}{dx}B_i^2(x)$ from direct computation,

$$\frac{d}{dx}B_i^2(x) = \begin{cases} \frac{2(x-t_{i-1})}{(t_{i+1}-t_{i-1})(t_i-t_{i-1})} & \text{if } x \in (t_{i-1}, t_i] \\ \frac{t_{i-1}+t_{i+1}-2x}{(t_{i+1}-t_{i-1})(t_{i+1}-t_i)} + \frac{t_i+t_{i+2}-2x}{(t_{i+2}-t_i)(t_{i+1}-t_i)} & \text{if } x \in (t_i, t_{i+1}] \\ \frac{-2(t_{i+2}-x)}{(t_{i+2}-t_i)(t_{i+2}-t_{i+1})} & \text{if } x \in (t_{i+1}, t_{i+2}] \\ 0 & \text{otherwise} \end{cases}$$

When $x = t_i$,

$$\lim_{x->t_i^-} \frac{d}{dx}B_i^2(x) = \frac{2}{t_{i+1} - t_{i-1}}, \quad \lim_{x->t_i^+} \frac{d}{dx}B_i^2(x) = \frac{2}{t_{i+1} - t_{i-1}}$$

also, when $x = t_{i+1}$,

$$\lim_{x->t_i^-} \frac{d}{dx}B_i^2(x) = \frac{-2}{t_{i+2} - t_i}, \quad \lim_{x->t_i^+} \frac{d}{dx}B_i^2(x) = \frac{-2}{t_{i+2} - t_i}$$

Thus, it continues at $x = t_i$ and $x = t_{i+1}$.

(c)

When $x \in (t_{i-1}, t_i]$, there is no x^* satisfying $\frac{d}{dx}B_i^2(x^*) = 0$. When $x \in (t_i, t_i i + 1$, we have,

$$x^* = \frac{t_{i+2}t_{i+1} - t_i t_{i-1}}{t_{i+2} + t_{i+1} - t_i - t_{i-1}}$$

satisfying $\frac{d}{dx}B_i^2(x^*) = 0$.

(d)

From the expression of $\frac{d}{dx}B_i^2(x)$ and (c), we know that $B_i^2(x)$ reach it extremes at $x = x^*, t_{i-1}, t_{i+2}$. Since $B_i^2\left(t_{i-1}\right) = B_i^2\left(t_{i+2}\right) = 0,$ and $B_i^2\left(x^*\right) = \frac{t_{i+2}-t_{i-1}}{t_{i+2}+t_{i+1}-t_i-t_{i-1}} < 1$, $B_i^2 \in [0, 1)$.

(e)

Take $i = 0, \ldots, 4$ as an example,

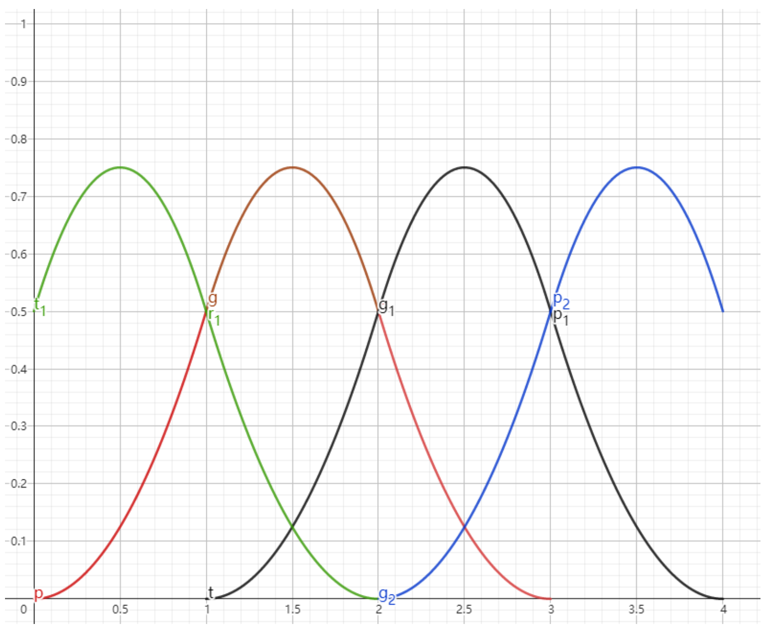

图 1: Plot of $B_i^2(x)$

Problem VI

Let's construct the table of divided difference to verify the Theorem.

1° when $x \in (t_{i-1}, t_i]$,

$$
\begin{array}{c|cccc}
t_{i-1} & 0 & & \\
t_i & (t_i - x)^2 & \frac{(t_i-x)^2}{t_i - t_{i-1}} & \\
t_{i+1} & (t_{i+1} - x)^2 & t_{i+1} + t_i - 2x & \frac{t_{i+1}+t_i-2x-\frac{(t_i-x)^2}{t_i-t_{i-1}}}{t_{i+1}-t_{i-1}} \\
t_{i+2} & (t_{i+2} - x)^2 & t_{i+2} + t_{i+1} - 2x & 1
\end{array}
$$

Thus,

$$
\begin{aligned}
(t_{i+2} - t_{i-1}) [t_{i-1}, t_i, t_{i+1}, t_{i+2}] (t-x)_+^2 &= 1 - \frac{t_i + t_{i+1} - 2x - \frac{(t_i-x)^2}{t_i-t_{i-1}}}{t_{i+1} - t_{i-1}} \\
&= \frac{(x - t_{i-1})(x - t_{i-1})}{(t_{i+1} - ti - 1)(t_i - t_{i-1})} \\
&= B_i^2(x)
\end{aligned}
$$

2° when $x \in (t_i, t_{i+1}]$,

$$
\begin{array}{c|cccc}
t_{i-1} & 0 & & \\
t_i & 0 & 0 & \\
t_{i+1} & (t_{i+1} - x)^2 & \frac{(t_{i+1}-x)^2}{t_{i+1}-t_i} & \frac{(t_{i+1}-x)^2}{(t_{i+1}-t_i)(t_{i+1}-t_i)} \\
t_{i+2} & (t_{i+2} - x)^2 & \frac{(t_{i+2}-x)^2-(t_{i+1}-x)^2}{t_{i+2}-ti+1} & \frac{\frac{(t_{i+2}-x)^2-(t_{i+1}-x)^2}{t_{i+2}-ti+1} - \frac{(t_{i+1}-x)^2}{t_{i+1}-t_i}}{(t_{i+2}-t_i)}
\end{array}
$$

Thus,

$$
\begin{aligned}
(t_{i+2} - t_{i-1}) [t_{i-1}, t_i, t_{i+1}, t_{i+2}] (t-x)_+^2 &= \frac{\frac{(t_{i+2}-x)^2-(t_{i+1}-x)^2}{t_{i+2}-ti+1} - \frac{(t_{i+1}-x)^2}{t_{i+1}-t_i}}{(t_{i+2} - t_i)} - \frac{(t_{i+1} - x)^2}{(t_{i+1} - t_i)(t_{i+1} - t_i)} \\
&= \frac{(x - t_{i-1})(t_{i+1} - x)}{(t_{i+1} - t_{i-1})(t_{i+1} - t_i)} + \frac{(t_{i+2} - x)(x - t_i)}{(t_{i+2} - t_i)(t_{i+1} - t_i)} \\
&= B_i^2(x)
\end{aligned}
$$

3° when $x \in (t_{i+1}, t_{i+2}]$,

$$
\begin{array}{c|cccc}
t_{i-1} & 0 & & \\
t_i & 0 & 0 & \\
t_{i+1} & 0 & 0 & 0 \\
t_{i+2} & (t_{i+2} - x)^2 & \frac{(t_{i+2}-x)^2}{t_{i+2}-ti+1} & \frac{(t_{i+2}-x)^2}{(t_{i+2}-t_i)(t_{i+2}-t_{i+1})}
\end{array}
$$

Thus,

$$
\begin{aligned}
(t_{i+2} - t_{i-1}) [t_{i-1}, t_i, t_{i+1}, t_{i+2}] (t-x)_+^2 &= \frac{(t_{i+2} - x)^2}{(t_{i+2} - t_i)(t_{i+2} - t_{i+1})} \\
&= B_i^2(x)
\end{aligned}
$$

$4°$ when $x \in (-\infty, t_{i-1}]$ or $x \in (t_{i+2}, \infty)$,
it is obvious that $(t_{i+2} - t_{i-1})\,[t_{i-1}, t_i, t_{i+1}, t_{i+2}]\,(t-x)_+^2 = 0$.

Problem VII

According to the theory of derivative of B-Splines, we have,

$$\frac{d}{dx}B_i^n(x) = \frac{nB_i^{n-1}(x)}{t_{i+n-1} - t_{i-1}} - \frac{nB_{i+1}^{n-1}(x)}{t_{i+n} - t_i}$$

Hence,

$$\begin{aligned}
&\int_{t_{i-1}}^{t_{i+n-1}} \frac{B_i^n(x)}{t_{i+n-1} - t_{i-1}}dx - \int_{t_i}^{t_{i+n}} \frac{B_{i+1}^n(x)}{t_{i+n} - t_i}dx \\
=&\int_{t_{i-1}}^{t_{i+n-1}} \left[\frac{B_i^n(x)}{t_{i+n} - t_{i-1}} - \frac{B_{i+1}^n(x)}{t_{i+n} - t_i}\right] dx \\
=&\frac{1}{n}B_i^{n+1}(x)\Big|_{t_{i-1}}^{t_{i+n}} \\
=&0
\end{aligned}$$

Thus, It holds independent of the index i and it has nothing to do with whether the spacing of the knots is uniform

Problem VIII

(a)

Let establish the chart of divided difference,

$$\begin{array}{c|cccc}
x_i & x_i^4 & & \\
x_{i+1} & x_{i+1}^4 & x_{i+1}^3 + x_{i+1}^2 x_i + x_{i+1} x_i^2 + x_i^3 & \\
xi+2 & x_{i+2}^4 & x_{i+2}^3 + x_{i+2}^2 x_{i+1} + x_{i+2} x_{i+1}^2 + x_{i+1}^3 & x_{i+2}^2 + x_{i+1}^2 + x_i^2 + x_i x_{i+1} + x_{i+1} x_{i+2} + x_{i+2} x_i
\end{array}$$

Thus, we have,

$$\tau_2(x_i, x_{i+1}, x_{i+2}) = x_i x_{i+1} + x_{i+1} x_{i+2} + x_{i+2} x_i + x_{i+2}^2 + x_{i+1}^2 + x_i^2 = [x_i, x_{i+1}, x_{i+2}]\,x^4$$

(b)

By the lemma of the recursive definition, we have,

$$\tau_{k+1}(x_1, \cdots, x_n, x_{n+1}) = \tau_{k+1}(x_1, \cdots, x_n) + x_{n+1}\tau_k(x_1, \cdots, x_n, x_{n+1})$$

Thus, we can derive,

$$\begin{aligned}
&(x_{n+1} - x_1)\,\tau_k\,(x_1, \ldots, x_n, x_{n+1}) \\
=&\tau_{k+1}\,(x_1, \ldots, x_n, x_{n+1}) - \tau_{k+1}\,(x_1, \ldots, x_n) - x_1\tau_k\,(x_1, \ldots, x_n, x_{n+1}) \\
=&\tau_{k+1}\,(x_2, \ldots, x_n, x_{n+1}) - \tau_{k+1}\,(x_1, \ldots, x_n)
\end{aligned}$$

By induction,
When n = 0, $\tau_m\,(x_i) = [x_i]\,x^m$ Now assume that the recursive formula is true for every n < m, consider n+1,

$$\begin{aligned}
&\tau_{m-n-1}\,(x_i, \cdots, x_{i+n+1}) \\
=&\frac{\tau_{m-n}\,(x_{i+1}, \cdots, x_{i+n+1}) - \tau_{m-n}\,(x_i, \cdots, x_n)}{x_{i+n+1} - x_i} \\
=&\frac{[x_{i+1}, \cdots, x_{i+n+1}]\,x^m - [x_i, \cdots, x_{i+n}]\,x^m}{x_{i+n+1} - x_i} \\
=&[x_i, \cdots, x_{i+n+1}]\,x^m
\end{aligned}$$

Hence, it's proved.