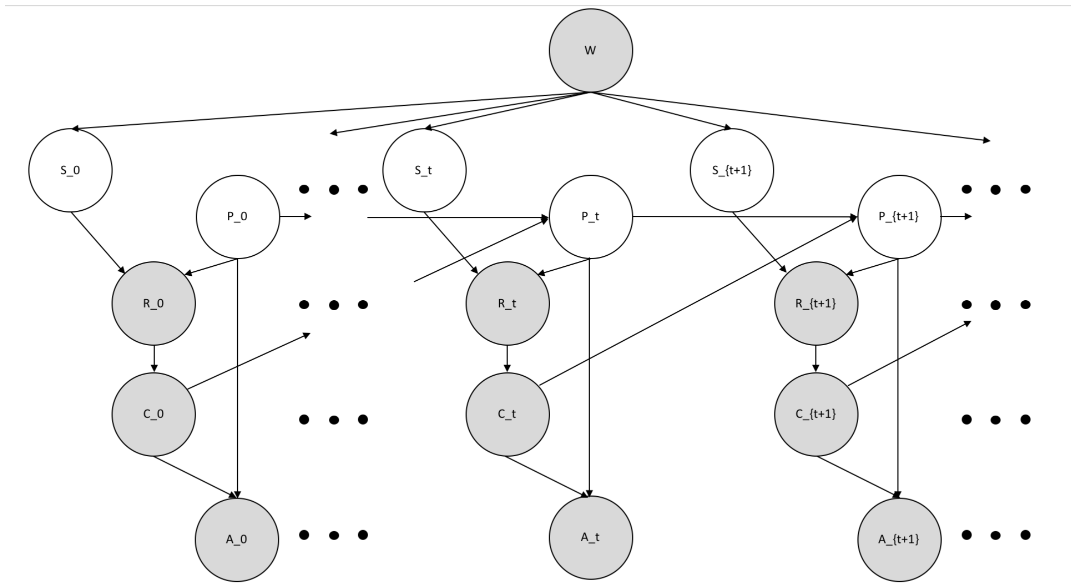
CS 188 Spring 2022	Introduction to Artificial Intelligence	**Written HW 7 Sol.**

Solutions for HW 7 (Written)

Q1. [27 pts] Quadcopter: Pilot

In this question, we look at the same setup as the previous homework, but we now look at it from the perspective of Paul, a quadcopter pilot who can **observe W (weather), R (reading of position), C (control from the pilot), and A (smart alarm warning)**. As before, suppose weather (W) does not change throughout the quadcopter's flight.



(a) Forward Algorithm: The real deal

(i) [2 pts] Now that the only hidden states are S_t and P_t, is this graph a well-behaving HMM (where $E_{t+1} \perp\!\!\!\perp E_t \mid X_{t+1}$ and $X_{t+1} \perp\!\!\!\perp E_t \mid X_t$, recall that X is the hidden variable and E is the evidence variable)? Please explain your reasoning.

No, it is not a well-behaving HMM, because the observed state C_t can affect the hidden state P_{t+1} (in the next time-step). Also, there is one over-arching observed state (the weather) that affects every time step.

(ii) [4 pts] What is the time-elapsed **prediction** formula from time-step t to time-step $t+1$? Be sure to include all hidden states and observed states, and show how to assemble the formula from the conditional probability tables corresponding to the graph. Denote $B(S_t, P_t) = P(S_t, P_t \mid W, R_{0:t}, C_{0:t}, A_{0:t})$. Find $B'(S_{t+1}, P_{t+1}) = P(S_{t+1}, P_{t+1} \mid W, R_{0:t}, C_{0:t}, A_{0:t})$. Hint: refer to the "Filtering algorithm" lecture slides.

- ○ $B'(S_{t+1}, P_{t+1}) = \max_{s_t} \sum_{p_t} P(S_{t+1} \mid W) * P(P_{t+1} \mid p_t, c_t) * B(S_t, P_t)$
- ○ $B'(S_{t+1}, P_{t+1}) = \sum_{s_t} \sum_{p_t} P(S_{t+1} \mid S_t) * P(P_{t+1} \mid p_t) * B(S_t, P_t)$
- ○ $B'(S_{t+1}, P_{t+1}) = \sum_{s_t} \sum_{p_t} P(R_{t+1} \mid S_{t+1}, P_{t+1}) P(S_{t+1} \mid S_t) * P(P_{t+1} \mid p_t) * B(S_t, P_t)$
- ○ $B'(S_{t+1}, P_{t+1}) = \sum_{s_t} \sum_{p_t} P(S_{t+1} \mid S_t) * P(P_{t+1} \mid p_t) P(P_{t+1} \mid c_t) * B(S_t, P_t)$
- ● $B'(S_{t+1}, P_{t+1}) = \sum_{s_t} \sum_{p_t} P(S_{t+1} \mid W) * P(P_{t+1} \mid p_t, c_t) * B(S_t, P_t)$
- ○ $B'(S_{t+1}, P_{t+1}) = \sum_{s_t} \max_{p_t} P(S_{t+1} \mid W) * P(P_{t+1} \mid p_t, c_t) * B(S_t, P_t)$

$B(S_t, P_t) = P(S_t, P_t \mid W, R_{0:t}, C_{0:t}, A_{0:t})$
$B'(S_{t+1}, P_{t+1})$
$= P(S_{t+1}, P_{t+1} \mid W, R_{0:t}, C_{0:t}, A_{0:t})$
$= \sum_{s_t} \sum_{p_t} P(S_{t+1}, P_{t+1}, s_t, p_t \mid W, R_{0:t}, C_{0:t}, A_{0:t})$
$= \sum_{s_t} \sum_{p_t} P(S_{t+1} \mid W) * P(P_{t+1} \mid p_t, c_t)) * P(s_t, p_t \mid W, R_{0:t}, C_{0:t}, A_{0:t})$
$= \sum_{s_t} \sum_{p_t} P(S_{t+1} \mid W) * P(P_{t+1} \mid p_t, c_t)) * B(S_t, P_t)$

(iii) [4 pts] How do we include the observation **update** at time-step $t+1$? Be sure to include all hidden states and observed states, and show how to assemble the update from the conditional probability tables corresponding to the graph. Denote $B'(S_{t+1}, P_{t+1}) = P(S_{t+1}, P_{t+1} \mid W, R_{0:t}, C_{0:t}, A_{0:t})$, find $B(S_{t+1}, P_{t+1})$

○ $B(S_{t+1}, P_{t+1}) = P(R_{t+1} \mid S_{t+1}, P_{t+1}) * P(C_{t+1} \mid R_{t+1}) * P(A_{t+1} \mid C_{t+1}) * B'(S_{t+1}, P_{t+1})$
○ $B(S_{t+1}, P_{t+1}) = P(R_{t+1} \mid S_{t+1}, P_{t+1}) * P(C_{t+1} \mid R_{t+1}) * B'(S_{t+1}, P_{t+1})$
○ $B(S_{t+1}, P_{t+1}) = P(R_{t+1} \mid S_{t+1}, P_{t+1}) * P(C_{t+1} \mid R_{t+1}) * P(A_{t+1} \mid C_{t+1}, P_{t+1}) * B'(S_{t+1}, P_{t+1})$
● $B(S_{t+1}, P_{t+1}) \propto P(R_{t+1} \mid S_{t+1}, P_{t+1}) * P(C_{t+1} \mid R_{t+1}) * P(A_{t+1} \mid C_{t+1}, P_{t+1}) * B'(S_{t+1}, P_{t+1})$
○ $B(S_{t+1}, P_{t+1}) \propto P(R_{t+1} \mid S_{t+1}, P_{t+1}) * P(C_{t+1} \mid R_{t+1}) * B'(S_{t+1}, P_{t+1})$

$B'(S_{t+1}, P_{t+1}) = P(S_{t+1}, P_{t+1} \mid W, R_{0:t}, C_{0:t}, A_{0:t})$
By observation update,
$B(S_{t+1}, P_{t+1}) = P(S_{t+1}, P_{t+1} \mid W, R_{0:t+1}, C_{0:t+1}, A_{0:t+1})$
$\propto P(S_{t+1}, P_{t+1}, R_{t+1}, C_{t+1}, A_{t+1} \mid W, R_{0:t}, C_{0:t}, A_{0:t})$
$= P(R_{t+1} \mid S_{t+1}, P_{t+1}) * P(C_{t+1} \mid R_{t+1}) * P(A_{t+1} \mid C_{t+1}, P_{t+1}) * P(S_{t+1}, P_{t+1} \mid W, R_{0:t}, C_{0:t}, A_{0:t})$
$= P(R_{t+1} \mid S_{t+1}, P_{t+1}) * P(C_{t+1} \mid R_{t+1}) * P(A_{t+1} \mid C_{t+1}, P_{t+1}) * B'(S_{t+1}, P_{t+1})$

(b) Consider a simpler scenario where we only track the 2D position (x, y) of the quadcopter. Paul, the pilot, wants to infer the quadcopter's true position P as accurately as possible.

- x, y **each** can take on values $\in \{0, 1, 2\}$.
- We have four controls: forward, backward, left, and right.
- Let variable E_R be Paul's estimate of the current position, and this variable depends on the reading R. The utility is based on the difference between the estimate of current position E_R and the actual position P: $U(P, E_R) = -||P - E_R||_x - ||P - E_R||_y$, in dollars.
- We consider only one time step. In that time step the **reading R is** $(1, 0)$ and that the weather is cloudy.
- Under cloudy weather, the signal strength can take on 2 values with equal probability: weak and strong. The signal strengths correspond to the following errors in readings:
  - Weak: The reading R returns a random number (for each position element) sampled uniformly from the domain of possible positions.
  - Strong: The reading R is identical to the true position.

Answer the following questions:

(i) [2 pts] Among the hidden variables S and P, Which variable should intuitively have the greatest VPI? Explain your answer. You should not do any calculations for this part.

P would be the best possible variable to observe, since our utility captures how well we can guess P.

Paul's coworker offers to tell him the signal strength (S) in exchange for some cash.

(ii) [3 pts] Suppose the signal strength is strong. Given the current reading R, what is the Maximum Expected Utility after knowing this information of S?

If the reading is strong, then it's the same as the true position. In this case, we would be able to achieve $MEU(S = strong) = 0$ by guessing $(1, 0)$. .

(iii) [3 pts] Suppose the signal strength is weak. Given the current reading R, what is the Maximum Expected Utility after knowing this information of S?

If the reading is weak, then the reading is random, and we gain no information about P. Then, we can achieve $MEU(S = weak)$ by guessing our position is in the middle square at $(1, 1)$, which will give us utility $MEU(S = weak) = -\frac{1}{9}(0 + 1 + 1 + 1 + 1 + 2 + 2 + 2 + 2) = -\frac{4}{3}$. (If we chose a corner square, we would get expected utility -2. If we had chosen any other square, we would get expected utility $-\frac{5}{3}$.)

(iv) [3 pts] Considering the possibility of both signal strength, how much should Paul should pay to know this information of S?

$VPI(S) = MEU(S) - MEU(\varnothing)$.

First, we will calculate $MEU(\varnothing)$. We will make our guess (E) of the position to be equal to our reading R. When we don't know the signal strength, with $\frac{1}{2}$ probability, we will incur no penalty because $R = P$. With $\frac{1}{2} * \frac{8}{9}$ probability, the actual position is a different position. If we are in square $(0, 0), (2, 0), (1, 1)$, we will receive utility -1. If we are in square $(0, 1), (1, 2), (2, 1)$, we achieve utility -2. If we are in $(0, 2), (2, 2)$, we achieve utility -3. Our expected utility is then $MEU(\varnothing) = \frac{1}{2} * 0 + \frac{1}{2}\frac{1}{9}(-(1+1+1+2+2+2+3+3)) = -\frac{5}{6}$.

Now, we will calculate $MEU(S)$ using the fact that each signal strength is possible with equal probability, so $MEU(S) = \sum_s P(S = s) MEU(S = s)$.

$MEU(S) = \frac{1}{2}\sum_s MEU(S = s) = \frac{1}{2}(-\frac{4}{3} + 0) = -\frac{2}{3}$.

$VPI(S) = MEU(S) - MEU(\varnothing) = (-\frac{2}{3}) - (-\frac{5}{6}) = \frac{1}{6} = 0.167$.
You should pay your coworker up to \$0.167.

(c) **(i)** [3 pts] Suppose your coworker only tells you the signal strength with probability q, and with probability $1 - q$, they don't tell you the signal strength even after payment. How much would you be willing to pay in this scenario? Your result should contain q.

Let us denote S' as the response we get from our coworker.
With probability q, we can achieve the previously calculated $MEU(S)$ of $-\frac{2}{3}$.
With probability $1-q$ our coworker doesn't tell us anything, and we can only achieve the value $MEU(\varnothing) = -\frac{5}{6}$.

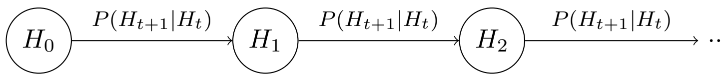
$MEU(S') = q * (-\frac{2}{3}) + (1 - q)(-\frac{5}{6}) = -\frac{5}{6} + \frac{q}{6}$.
$VPI(S') = MEU(S') - MEU(\varnothing) = -\frac{5}{6} + \frac{q}{6} + \frac{5}{6} = \frac{q}{6}$.
We would pay the highest cent amount less than $\frac{q}{6}$.

(ii) [3 pts] How much would you pay to know the true position (P)?

$VPI(P) = MEU(P) - MEU(\varnothing)$.
$MEU(\varnothing) = -\frac{5}{6}$.
We also note that $MEU(P) = 0$ because we can just report the known position.
Then, $VPI(P) = \frac{5}{6} = 0.833$.
Thus, you should pay at most \$0.83.

Q2. [8 pts] MangoBot Human Detector

Your startup company MangoBot wants to build robots that delivers packages on the road. One core module of the robot's software is to detect whether a human is standing in front of it. We model the presence of humans with a Markov model:



where $H_t \in \{0, 1\}$ corresponds to a human being absent or present respectively. The initial distribution and the transition probabilities are given as follows:

H_0	$P(H_0)$
0	p
1	$1-p$

H_t	H_{t+1}	$P(H_{t+1}|H_t)$
0	0	0.9
0	1	0.1
1	0	0.8
1	1	0.2

(a) Express the following quantities in terms of p:

(i) [1 pt] $P(H_1 = 1) = -0.1p + 0.2$
$P(H_1 = 1) = P(H_1 = 1, H_0 = 0) + P(H_1 = 1, H_0 = 1) = P(H_0 = 0)P(H_1 = 1|H_0 = 0) + P(H_0 = 1)P(H_1 = 1|H_0 = 1)$
$= 0.2(1-p) + 0.1p = 0.2 - 0.1p$

(ii) [1 pt] $\lim_{t\to\infty} P(H_t = 0) = 8/9$
As $t \to \infty$, the system converges to the stationary distribution π, which satisfies $\pi = T^\top \pi$, where T is the transition probability matrix $\begin{bmatrix} 0.9 & 0.1 \\ 0.8 & 0.2 \end{bmatrix}$. Assume $\pi = [q \quad (1-q)]^\top$ and solve for q in $\pi = T^\top \pi$ gives $q = 8/9$.

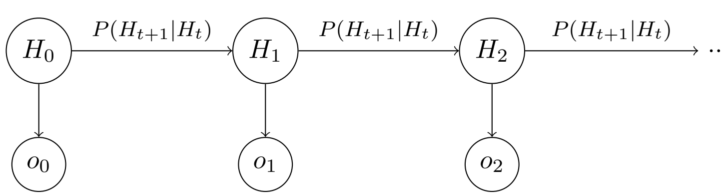
(b) The first-order Markov assumption in the model above can be inaccurate in real-world situations. Some potential ways to improve the model are listed below. For each option, determine whether it is possible to rewrite the process as a first-order Markov process, potentially with a different state representation.

(i) [1 pt] H_t depends not only on H_{t-1} but also on H_{t-2}. ● Yes ○ No
We can make the state $S_t = \{H_t, H_{t-1}\}$, then S_{t+1} is conditionally independent of S_{t-1} given S_t for all t, which means the sequence S_t satisfies the Markov property.

(ii) [1 pt] H_t depends not only on H_{t-1} but also on $H_{t-2}, H_{t-3}, \cdots, H_{t-k}$ for some fixed $k \geq 3$. ● Yes ○ No
Same basic argument except now S_t keeps a history of the k preceding time steps and the sequence S_t satisfies the Markov property. Note that the state-space size is now exponential in k.

(iii) [1 pt] H_t depends not only on H_{t-1} but also on $H_{t-2}, H_{t-3}, \cdots, H_1, H_0$. ○ Yes ● No
Since now H_t depend on a variable number of previous time steps, the trick we used in the previous two parts doesn't work any more. There is no finite state representation that maintains the Markov property.

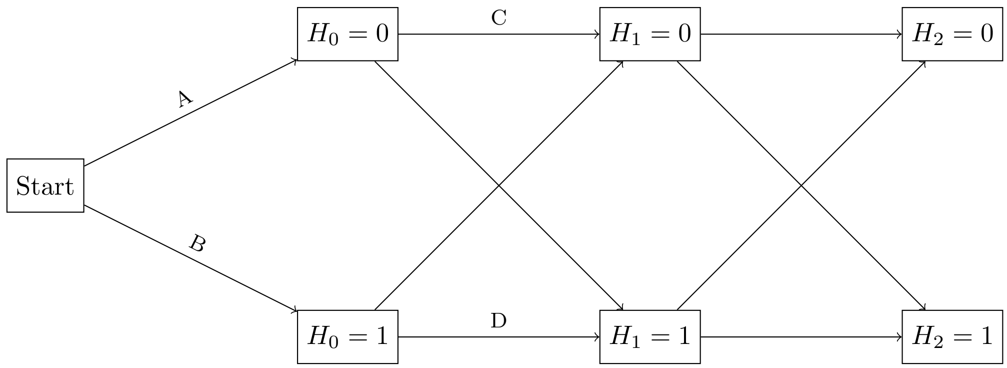
To make things simple, we stick to the original first-order Markov chain formulation. To make the detection more accurate, the company built a sensor that returns an observation O_t each time step as a noisy measurement of the unknown H_t. The new model is illustrated in the figure, and the relationship between H_t and O_t is provided in the table below.



H_t	O_t	$P(O_t \mid H_t)$
0	0	0.8
0	1	0.2
1	0	0.3
1	1	0.7

(c) Based on the observed sensor values $o_0, o_1, \cdots, o_t$, we now want the robot to find the most likely sequence $H_0, H_1, \cdots, H_t$ indicating the presence/absence of a human up to the current time.

(i) [1 pt] Suppose that $[o_0, o_1, o_2] = [0, 1, 1]$ are observed. The following "trellis diagram" shows the possible state transitions. Fill in the values for the arcs labeled A, B, C, and D with the product of the transition probability and the observation likelihood for the destination state. The values may depend on p.



A: $0.8p$; B: $0.3(1-p)$; C: $0.9 * 0.2 = 0.18$; D: $0.2 * 0.7 = 0.14$.

(ii) [1 pt] There are two possible most likely state sequences, depending on the value of p. Complete the following (Write the sequence as "x,y,z" (without quotes), where x, y, z are either 0 or 1):
Hint: it might be helpful to complete the labelling of the trellis diagram above.

- When $p <$ $\boxed{0.25}$, the most likely sequence H_0, H_1, H_2 is $\boxed{1,0,0}$.
- Otherwise, the most likely sequence H_0, H_1, H_2 is $\boxed{0, 0, 0}$.

After filling out the full trellis diagram, we can easily observe that the sequence with largest probability given $H_0 = 0$ is (0, 0, 0) and the sequence with largest probability given $H_0 = 1$ is (1, 0, 0). (To see what is the sequence with largest probability, we run search from $H_0 = 0$ to either $H_2 = 0$ or $H_0 = 1$, but instead of adding the costs we multiply the probabilities.) Therefore the two possible most likely state sequences are (0, 0, 0), with probability $0.8p * 0.18 * 0.18$, and (1, 0, 0), with probability $0.3(1-p) * 0.16 * 0.18$. Setting up the equation $0.8p * 0.18 * 0.18 = 0.3(1-p) * 0.16 * 0.18$ gives $p = 0.25$ to be the threshold. Note that this is a bit counter-intuitive since the observations suggest the exact opposite thing. However, in this problem the transition probabilities for 0 to 0 and 1 to 0 are so large that they dominates the computation.

(d) [1 pt] True or False: For a fixed p value and observations $\{o_0, o_1, o_2\}$ in general, H_1^*, the most likely value for H_1, is always the same as the value of H_1 in the most likely sequence H_0, H_1, H_2. ○ True ● False
The maximum likelihood estimation (MLE) for a single variable is in general not the same as the value of that variable in the most likely sequence estimation (MLSE). For example, in this problem, when $p = 0.3$, the MLE for H_0 is 1, but the value of H_0 in the MLSE is 1. There exists similar examples for H_1.