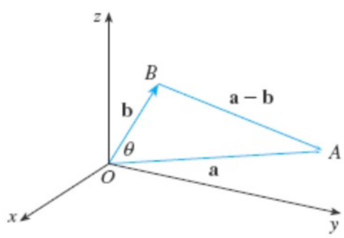
Section 12.3 The Dot Product

So far, we have added two vectors and multiplied a vector by a scalar. Is it possible to multiply two vectors so that their product is a useful quantity? One such product is the dot product, which we will define in this section, and the other is called the cross product, which we will define in the next section.

The dot product between two non zero vectors $\mathbf{a}$ and $\mathbf{b}$ is the number $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta$, where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$, $0 \leq \theta \leq \pi$. If either $\mathbf{a}$ or $\mathbf{b}$ is $\mathbf{0}$, then $\mathbf{a} \cdot \mathbf{b} = 0$

Example 1: Find $\mathbf{a} \cdot \mathbf{b}$ if it is known that $|\mathbf{a}| = 2$, $|\mathbf{b}| = 5$ and $\theta = 60°$.

The dot product between $\mathbf{a}$ and $\mathbf{b}$ can be given a geometric interpretation in terms of the angle, θ, between $\mathbf{a}$ and $\mathbf{b}$. Using Law of Cosines on the triangle below:



$|\mathbf{a} - \mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2|\mathbf{a}||\mathbf{b}| \cos(\theta)$

After we do the algebra above, and use $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta$, we will get the following result:

$$\mathbf{a} \cdot \mathbf{b} = \langle a_1, a_2, a_3 \rangle \cdot \langle b_1, b_2, b_3 \rangle = a_1 b_1 + a_2 b_2 + a_3 b_3$$

Example 2: Find the dot product between the vectors $\langle 1, 5, -2 \rangle$ and $\langle 0, 1, 4 \rangle$.

Example 3: Find the angle between the vectors $\langle 1, 0, -2 \rangle$ and $\langle 2, -1, 3 \rangle$.

Example 4: What is the value of $\mathbf{a} \cdot \mathbf{b}$ if it is known that $\mathbf{a}$ and $\mathbf{b}$ are perpendicular? Parallel?

Example 5: For what value(s) of x are the vectors $\langle x, 1, 2\rangle$ and $\langle 3, 4, x\rangle$ perpendicular?

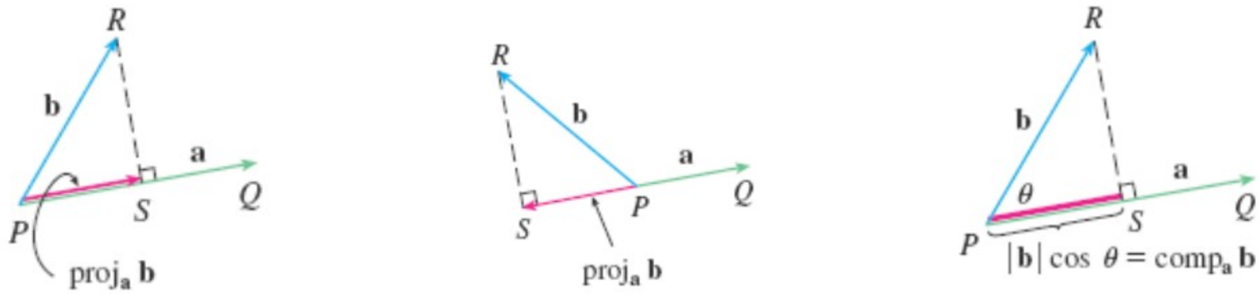
Example 6: The points $A(0, -1, 6)$, $B(2, 1, -3)$ and $C(5, 4, 2)$ form a triangle. Find the angle located at the vertex B.

Example 7: Is the expression $(\mathbf{a} \cdot \mathbf{b}) \cdot \mathbf{c}$ meaningful? If so, is the result a vector or a scalar?

Example 8: Is the expression $|\mathbf{a}|(\mathbf{b} \cdot \mathbf{c})$ meaningful? If so, is the result a vector or a scalar?

Example 9: Is the expression $|\mathbf{a}| \cdot (\mathbf{b} + \mathbf{c})$ meaningful? If so, is the result a vector or a scalar?

Vector and Scalar Projections: Given $\overrightarrow{a} = \langle a_1, a_2, a_3 \rangle$ and $\overrightarrow{b} = \langle b_1, b_2, b_3 \rangle$, we want to project $\overrightarrow{b}$ onto $\overrightarrow{a}$. Think of this as "the vector $\overrightarrow{b}$ in the direction of $\overrightarrow{a}$". Geometrically, to obtain the vector projection of $\overrightarrow{b}$ onto $\overrightarrow{a}$, drop a perpendicular from the end of $\overrightarrow{b}$ onto $\overrightarrow{a}$.



(i) The **Scalar Projection** of $\overrightarrow{b}$ onto $\overrightarrow{a}$ (also called the component of $\overrightarrow{b}$ onto $\overrightarrow{a}$) is:

$$\text{comp}_a b = \frac{\overrightarrow{a} \cdot \overrightarrow{b}}{|\overrightarrow{a}|}$$

(ii) The **Vector Projection** of $\overrightarrow{b}$ onto $\overrightarrow{a}$ is:

$$\text{proj}_a b = \left(\frac{\overrightarrow{a} \cdot \overrightarrow{b}}{|\overrightarrow{a}|} \right) \frac{\overrightarrow{a}}{|\overrightarrow{a}|} = \left(\frac{\overrightarrow{a} \cdot \overrightarrow{b}}{|\overrightarrow{a}|^2} \right) \overrightarrow{a}$$

Example 10: Find the vector and scalar projection of $\langle 1, -1, 3\rangle$ onto $\langle 0, 2, 1\rangle$.

Example 11: If $\mathbf{a} = \langle 2, 1, -2\rangle$, find two different vectors, $\mathbf{b}$, so that $\text{comp}_{\mathbf{a}}\mathbf{b} = 5$.