‘low_thrust’ MATLAB Code

The non-dimensional equations of motion for a low/continuous thrust orbit transfer can be written as:

$$\frac{d^2\rho}{d\tau^2} = \rho\left(\frac{d\theta}{d\tau}\right)^2 - \frac{1}{\rho^2} + \nu_r \tag{1}$$

$$\frac{d}{d\tau}\left(\rho^2\frac{d\theta}{d\tau}\right) = \nu_\theta\rho \tag{2}$$

Expanding the left side of Equation 2, yields:

$$\rho^2\frac{d^2\theta}{d\tau^2} + 2\rho\frac{d\rho}{d\tau}\frac{d\theta}{d\tau} = \nu_\theta\rho \tag{3}$$

Equations 1 and 3 can be re-written as a coupled set of non-linear ordinary differential equations for the unknowns ρ and θ:

$$\left.\begin{aligned} \frac{d^2\rho}{d\tau^2} &= \rho\left(\frac{d\theta}{d\tau}\right)^2 - \frac{1}{\rho^2} + \nu_r \\ \frac{d^2\theta}{d\tau^2} &= -\frac{2}{\rho}\frac{d\rho}{d\tau}\frac{d\theta}{d\tau} + \frac{\nu_\theta}{\rho} \end{aligned}\right\} \tag{4}$$

Using the simplified notation $\varphi' = d\varphi/d\tau$, the coupled non-linear ordinary differential equations become:

$$\left.\begin{aligned} \rho'' &= \rho\theta'^2 - \frac{1}{\rho^2} + \nu_r \\ \theta'' &= -\frac{2}{\rho}\rho'\theta' + \frac{\nu_\theta}{\rho} \end{aligned}\right\} \tag{5}$$

A higher-order ordinary differential equation can always be reduced to a set of first-order differential equations. Therefore Equations 5, the pair of second-order differential equations, can be reduced to a set of 4 first-order differential equations written as:

$$\left.\begin{aligned} \frac{d}{d\tau}\rho' &= \rho\theta'^2 - \frac{1}{\rho^2} + \nu\cos\phi \\ \frac{d}{d\tau}\rho &= \rho' \\ \frac{d}{d\tau}\theta' &= -\frac{2}{\rho}\rho'\theta' + \frac{\nu\sin\phi}{\rho} \\ \frac{d}{d\tau}\theta &= \theta' \end{aligned}\right\} \tag{6}$$

Where $\nu_r = \nu\cos\phi$ and $\nu_\theta = \nu\sin\phi$. The thrust vector angle, ϕ, is defined relative to the radius vector $\vec{r}$.

If the starting point is a circular orbit about a planet, the associated 4 initial conditions are:

$$\rho(0) = \theta'(0) = 1, \;\; \rho'(0) = \theta(0) = 0 \tag{7}$$

Equations 6 can be solved simultaneously using a *fourth-order Runge-Kutta* numerical integration method. The resulting functions determined are $\rho'(\tau)$, $\rho(\tau)$, $\theta'(\tau)$, and $\theta(\tau)$. The functions of interest are $\rho(\tau)$ and $\theta(\tau)$ which map out the orbit in time as the spacecraft spirals outward from a planet.

<u>Implementation</u>

The function file 'derivs.m' contains the code for the differential equations described in Equation 6 above. A mapping of variable names into a single array is used, as described below:

$$\left.\begin{array}{l} y(1) = \rho' \\ y(2) = \rho \\ y(3) = \theta' \\ y(4) = \theta \end{array}\right\} \tag{8}$$

Using the variable substitutions in Equation 8 above, the resulting program lines contained within 'derivs.m' for Equations 6 are as follows:

$$\begin{array}{l} dydx(1) = y(2) * y(3) * y(3) - 1.0/(y(2) * y(2)) + nu * cos(phi) \\ dydx(2) = y(1) \\ dydx(3) = -2.0 * y(1) * y(3)/y(2) + nu * sin(phi)/y(2) \\ dydx(4) = y(3) \end{array}$$

The Runge-Kutta numerical integration scheme in the function file 'rk4.m' is set up for variables x and y, therefore in this problem $x = \tau$, the non-dimensional time variable.

The non-dimensional variables in the problem are:

$$\rho = \frac{r}{r_0}, \quad \tau = \sqrt{\frac{\mu}{r_0^3}}\,t, \quad \nu = \frac{T}{mg} \tag{9}$$

Where r_0 is the radius of the initial circular orbit, μ is the gravitational parameter of the planet, T/m is the thrust to spacecraft mass ratio, and $g = \mu/r_0^2$ is the gravitational acceleration at the initial orbit radius.

Once the numerical results are obtained, dimensional values can be found from the following:

$$r = r_0\rho, \quad t = \sqrt{\frac{r_0^3}{\mu}}\,\tau, \quad \dot{r} = r_0\rho'\sqrt{\frac{\mu}{r_0^3}}, \quad \dot{\theta} = \theta'\sqrt{\frac{\mu}{r_0^3}} \tag{10}$$

Finally, the velocity along the orbit can be found from components:

$$V^2 = \dot{r}^2 + r^2\dot{\theta}^2 \tag{11}$$